MATH242 HW1

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Because I joined the course late, my due date for this assignment was Sunday February 13th. On top of this, I used 2 late days.

Problem 1

We have $X = \sum_{i=1}^n X_i$ where the $X_i \sim \text{Bernoulli}(p)$ and the X_i are independent. We have that

$$\begin{aligned}\mathbb{E}(e^{tX_i}) &= pe^t + (1-p)e^0 \\ &= 1 + p(e^t - 1).\end{aligned}$$

Therefore

$$\begin{aligned}M_X(t) &= \prod_{i=1}^n M_{X_i}(t) \\ &= \left(1 + p(e^t - 1)\right)^n .\end{aligned}$$

Problem 2

(a) Letting $a_1, a_2 \in \mathbb{R}$ we see that

$$a_1X_1 + a_2X_2 = (a_1c_1 + a_2c_2)Z_1 + (a_1d_1 + a_2d_2)Z_2 + a_1e_1 + a_2e_2,$$

which is normal.

(b) Letting

$$\begin{aligned}c_1 &= \sigma_1 \\ d_1 &= 0 \\ e_1 &= \mu_1 \\ c_2 &= \rho\sigma_2 \\ d_2 &= \sigma_2\sqrt{1-\rho^2} \\ e_2 &= \mu_2\end{aligned}$$

we get the desired result.

Problem 3

(a) We have that

$$\mathbb{E}\left[\frac{f(X_i)}{g(X_i)}\right] = \int_a^b \frac{f(x)}{g(x)}g(x)dx = \int_a^b f(x)dx$$

therefore

$$\mathbb{E}\left[\frac{1}{n}\sum\frac{f(X_i)}{g(X_i)}\right] = \frac{1}{n}\sum\mathbb{E}\left[\frac{f(X_i)}{g(X_i)}\right] = \int_a^b f(x)dx.$$

If $\mathrm{Var}[f(X_i)/g(X_i)] < \infty$, then $\mathrm{Var}[\hat{I}_n(f)] = \frac{1}{n}\mathrm{Var}[f(X_i)/g(X_i)]$. Therefore as $n \to \infty$ the variance of $\hat{I}_n(f)$ goes to 0, thus $\hat{I}_n(f) \to \mathbb{E}[\hat{I}_n(f)] = I(f)$ with probability 1.

(b) We have that

$$\begin{aligned}
\mathrm{Var}\left[\frac{f(X_i)}{g(X_i)}\right] &= \mathbb{E}\left[\left(\frac{f(X_i)}{g(X_i)}\right)^2\right] - \mathbb{E}\left[\frac{f(X_i)}{g(X_i)}\right]^2 \\
&= \int_a^b \frac{f(x)^2}{g(x)}dx - \left(\int_a^b f(x)dx\right)^2
\end{aligned}$$

This implies that

$$c_n = \frac{1}{\mathrm{Std}[\hat{I}_n(f)]} = \sqrt{n}\left(\int_a^b \frac{f(x)^2}{g(x)}dx - \left(\int_a^b f(x)dx\right)^2\right)^{-1}.$$

(c) We have that

$$\begin{aligned}
c_n &= \sqrt{1000}\left(\int_0^1 \cos^2(2\pi x)dx - \left(\int_a^b \cos(2\pi x)dx\right)^2\right)^{-1} \\
&= 2\sqrt{1000}
\end{aligned}$$

Therefore our desired probability is $P(|z| > 0.05c_n)$ where $z \sim \mathcal{N}(0,1)$, which is approximately equal to 0.00157.

problem4

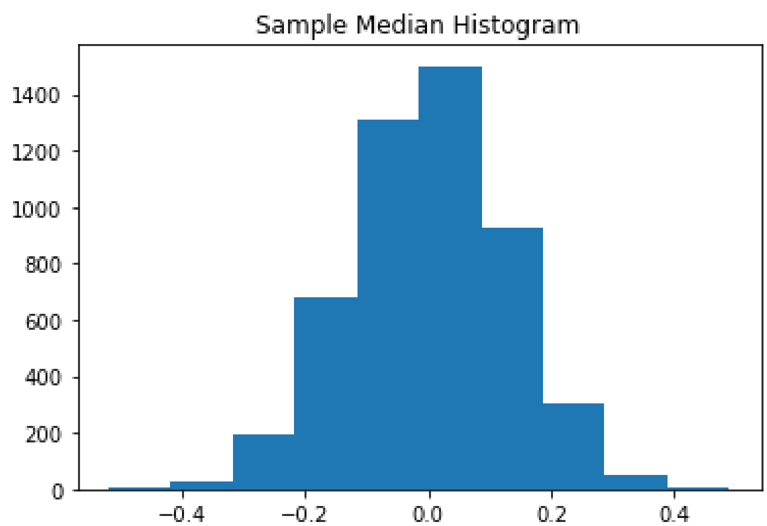
February 15, 2022

0.1 Problem 4

[1]:
```
import numpy as np
from numpy.random import standard_normal
import matplotlib.pyplot as plt
```

[2]:
```
x = standard_normal((5000, 99))
sorted_x = np.sort(x, axis=1)
medians = sorted_x[:, 49]
```

[3]:
```
plt.hist(medians)
plt.title("Sample Median Histogram")
plt.show()
```



[4]:
```
std = np.std(medians)
print("standard deviations of medians: ", std)
```

```
standard deviations of medians:  0.12797205979697948
```

The theoretical standard deviation of the sample means is $\frac{1}{\sqrt{99}} \approx 0.1005$. According to our simulation, the sample median is more variable than the sample mean.