MATH242 HW3

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Using 3 late days.

problem1

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1 Problem 1

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1.1 Part A

Two reasonable test statistics are

$$T_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$T_2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - 6)^2$$

where $n = 6115$

[2]:
```
sample = np.array([7, 45, 181, 478, 829, 1112, 1343, 1033,
                670, 286, 104, 24, 3])

t1_sample = np.sum(np.arange(13)*sample)/6115

t2_sample = np.sum(((np.arange(13)-6)**2) * sample)/6115

print(f"T_1 sample value: {t1_sample}")
print(f"T_2 sample value: {t2_sample}")
```

```
T_1 sample value: 5.769419460343418
T_2 sample value: 3.542436631234669
```

1.2 Part B

[3]:
```
null_samples = np.random.binomial(12, .5, (1000, 6115))
```

The exterior of the red lines in the two following histograms is the rejection region for a two-sided $\alpha = 0.05$ significance test. The green line is the sample value of the test statistic

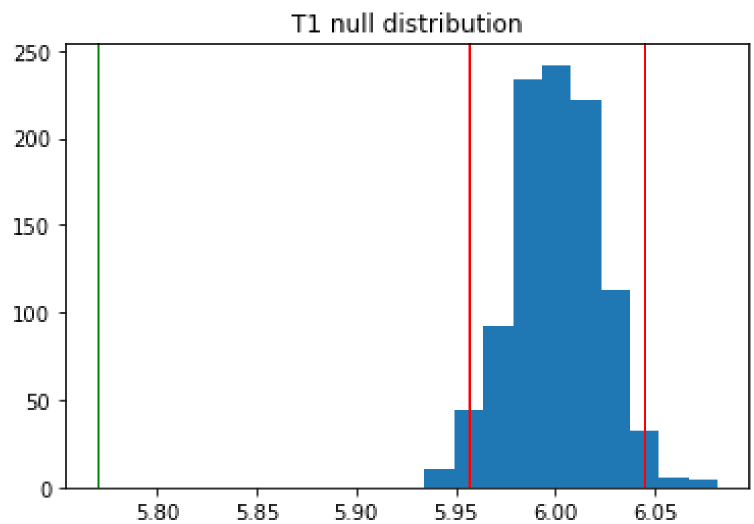
[4]:
```
# Histogram and rejection region
t1_null = np.sort(np.mean(null_samples, axis=1))
plt.hist(t1_null)
```

```
plt.title("T1 null distribution")

t1_lower_reject = (t1_null[25]+t1_null[24])/2
plt.axvline(t1_lower_reject, color='r', linestyle='solid', linewidth=1)

t1_upper_reject = (t1_null[975]+t1_null[974])/2
plt.axvline(t1_upper_reject, color='r', linestyle='solid', linewidth=1)


# p value computation
plt.axvline(t1_sample, color='g', linestyle='solid', linewidth=1)
insert_idx = np.searchsorted(t1_null, t1_sample)
p_value = (min(insert_idx, 1000-insert_idx)*2)/1000
plt.show()
print(f"p value is: {p_value}")
```




```
p value is: 0.0
```

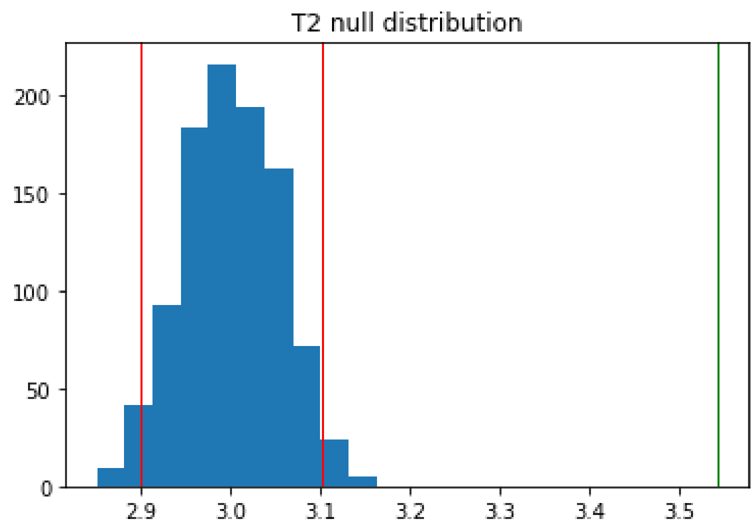
[5]:
```
# T2
t2_null = np.sort(np.mean((null_samples - 6)**2, axis=1))
plt.hist(t2_null)
plt.title("T2 null distribution")

t2_lower_reject = (t2_null[25]+t2_null[24])/2
```

```
plt.axvline(t2_lower_reject, color='r', linestyle='solid', linewidth=1)

t2_upper_reject = (t2_null[975]+t2_null[974])/2
plt.axvline(t2_upper_reject, color='r', linestyle='solid', linewidth=1)

# p value computation
plt.axvline(t2_sample, color='g', linestyle='solid', linewidth=1)
insert_idx = np.searchsorted(t2_null, t2_sample)
p_value = (min(insert_idx, 1000-insert_idx)*2)/1000
plt.show()
print(f"p value is: {p_value}")
```




```
p value is: 0.0
```

For both test statistics, we get approximate p-values of 0. This means in both cases we can reject H_0 at significance level $\alpha = 0.05$.

1.3 Part C

One reason the null hypothesis may not hold is because of selection bias introduced by only considering families that have 12 children. For example, suppose each family i adopts the strategy of having children until they reach n_i boys where n_i is typically less than six. Then the families that reach twelve children will have less than 50% boys.

Problem 2

(a) Suppose H_0. Then $X \sim \mathcal{N}(50, \sigma^2 = 25)$. The probability of Type I error is $P(|X - 50| > 10) = P(|z| > 2) = 0.05$.

(b) Suppose $X \sim \text{Binom}(100, 4/5)$. Then we can approximate $X \sim \mathcal{N}(80, 16)$. Therefore

$$
\begin{aligned}
1 - \beta &= P(\text{reject } H_0) \\
&= P(|X - 50| > 10) \\
&= P\left(\frac{X - 80}{4} > -5 \text{ or } \frac{X - 80}{4} < -15/2\right) \\
&= P(z > -5 \text{ or } z < -15/2) \\
&\approx 1
\end{aligned}
$$

Problem 3

Since the hypotheses are simple, by the Neyman-Pearson lemma the most powerful test is the likelihood ratio test. Then

$$
\begin{aligned}
\alpha &= P_{H_0}(L(X) < c) \\
&= P_{H_0}\left(\frac{1}{2X} < c\right) \\
&= P_{H_0}\left(\frac{1}{2c} < X\right) \\
&= 1 - \frac{1}{2c}.
\end{aligned}
$$

Solving for c gives us $c = 9/5$. Therefore the power of the test is

$$
\begin{aligned}
1 - \beta &= P_{H_1}(L(X) < c) \\
&= P_{H_1}\left(\frac{1}{2c} < X\right) \\
&= \int_{\frac{1}{2c}}^{1} 2x \mathrm{d}x = 0.19
\end{aligned}
$$

Problem 4

(a) We have that

$$
\prod_{i=1}^{n} f_0(X_i) = \frac{1}{\sigma_i\sqrt{2\pi}} \prod_{i=1}^{n} e^{\frac{1}{2}\left(\frac{x}{\sigma_i}\right)^2}.
$$

Therefore

$$\begin{aligned}
L(X) &= \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 - \left(\frac{X_i}{\sigma_1}\right)^2\right) \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left(\frac{1}{2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)} \sum_{i=1}^n X_i^2\right).
\end{aligned}$$

Since $\sigma_0^2 < \sigma_1^2$, the likelihood-ratio $L(X)$ is a decreasing function of $\sum_{i=1}^n X_i^2$. Thus setting our test statistic to $T = \sum_{i=1}^n X_i^2$ to perform the likelihood ratio test we need to find c such that

$$P_{H_0}(T > c) = \alpha.$$

Since $\frac{1}{\sigma_0^2}T \sim \chi_n^2$, we can choose $c = \sigma_0^2 \chi_n^2(\alpha)$. So our rejection region is $T > \sigma_0^2 \chi_n^2(\alpha)$.

(b) The distribution of $\frac{1}{\sigma_1^2}T$ under H_1 is χ_n^2 (this is equivalent to knowing the distribution of T under H_1). We then get

$$\begin{aligned}
1 - \beta &= P_{H_1}(T > \sigma_0^2 \chi_n^2(\alpha)) \\
&= P_{H_1}\left(\frac{1}{\sigma_1^2}T > \frac{\sigma_0^2}{\sigma_1^2}\chi_n^2(\alpha)\right) \\
&= 1 - F\left(\frac{\sigma_0^2}{\sigma_1^2}\chi_n^2(\alpha)\right).
\end{aligned}$$