S&DS242 HW7

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Problem 1

(a) We have that

$$\mu = \int_\theta^\infty x e^{x-\theta} dx = 1 + \theta$$

so we get the method of moments estimate by setting $\bar{X} = 1 + \theta$, which gives us $\theta = \bar{X} - 1$.

(b) Notice that if $\theta < \min X_i$, we get 0 likelihood. Therefore we can constrain ourselves to looking for $\arg\max_\theta L(\theta)$ where $\theta \leq \min X_i$. Applying this constraint, we get that

$$\begin{aligned}
\arg\max_\theta L(\theta) &= \arg\max_\theta l(\theta) \\
&= \arg\max_\theta \sum_i \log e^{-(X_i - \theta)} \\
&= \arg\max_\theta \sum_i \theta - X_i \\
&= \arg\max_\theta n\theta - \sum_i X_i \\
&= \arg\max_\theta \theta
\end{aligned}$$

Therefore $\hat{\theta}_{MLE} = \min_i X_i$.

Problem 2

(a) Reasonable estimators are $\hat{p} = \frac{\sum_i X_i}{n}$ and $\hat{q} = \frac{\sum_i Y_i}{m}$. A plug-in estimator for the log-odds ratio is

$$g(\hat{p}, \hat{q}) = \log \left(\frac{\frac{\hat{p}}{1-\hat{p}}}{\frac{\hat{q}}{1-\hat{q}}} \right)$$

(b) To simplify our calculations, write

$$g(p, q) = \log p - \log(1-p) - \log q + \log(1-q).$$

Applying the Taylor expansion we get that

$$g(\hat{p},\hat{q}) \approx +\frac{\hat{p}-p}{p(1-p)} + \frac{\hat{q}-q}{q(1-q)}$$

By the CLT, we get that $\sqrt{n}(\hat{p}-p) \to \mathcal{N}(0, p(1-p))$ and $\sqrt{m}(\hat{q}-q) \to \mathcal{N}(0, q(1-q))$. Asymptotically, both error terms are independent Gaussians, so their sum is a Gaussian with appropriate parameters. Thus in the limit

$$g(\hat{p},\hat{q}) \sim \mathcal{N}\left(g(p,q), \frac{1}{np(1-p)} + \frac{1}{mq(1-q)}\right).$$

Problem 3

(a) We have that

$$\frac{\sqrt{n}(\hat{p}-p)}{\sqrt{\hat{p}(1-\hat{p}}} \to \mathcal{N}(0,1)$$

as $n \to \infty$. Therefore our confidence interval is

$$\hat{p} - sqrt\frac{\hat{p}(1-\hat{p})}{n} z(0.025) \leq p \leq \hat{p} + \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} z(0.025)$$

(b) We are inside the interval in question whenever

$$n(\hat{p}-p)^2 \leq p(1-p)z(\alpha/2)^2,$$

or equivalently when

$$(n + z(\alpha/2)^2)p^2 - (2n\hat{p} + z(\alpha/2)^2)p + n\hat{p}^2 \leq 0.$$

Notice the left-hand side is a quadratic equation with positive leading coefficient, so applying the quadratic formula we get a confidence interval of

$$\frac{\hat{p} + \frac{z(0.025)^2}{2n} \pm z(0.025)\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z(0.025)^2}{4n^2}}}{1 + \frac{z(0.025)^2}{n}}$$

hw7

April 21, 2022

0.1 Problem 3 Part C

[1]:
```
import numpy as np
from scipy.stats import norm
```

[2]:
```
num_trials = int(1e4)
ps = [0.1, 0.3, 0.5]
ns = [10, 40, 100]

sims={}
for p in ps:
    sims[str(p)] = {}
    for n in ns:
        sims[str(p)][n] = np.random.binomial(n, p, (num_trials))

sims["0.3"][40]
```

[2]:
```
array([17, 10, 16, ..., 15,  8, 12])
```

[3]:
```
def get_part_a_interval(phats, p, n):
    z = norm.ppf(1-0.025)

    delta = np.sqrt(phats*(1-phats)/n) * z

    return phats - delta, phats + delta

print("part A interval lower and upper: ", get_part_a_interval(np.array([0.11,␣
 ↪0.12]), 0.115, 1000))

def get_part_b_interval(phats, p, n):
    z = norm.ppf(1-0.025)

    center = phats + z**2 / (2*n)

    delta = z * np.sqrt(phats*(1-phats)/n + z**2/(4*n**2))

    normalizer = 1 + z**2/n

    return (center-delta)/normalizer, (center+delta)/normalizer
```

```
print("part B interval lowers and uppers", get_part_b_interval(np.array([0.11,␣
↪0.12]), 0.115, 1000))
```

```
part A interval lower and upper:  (array([0.09060725, 0.09985905]),
array([0.12939275, 0.14014095]))
part B interval lowers and uppers (array([0.09207937, 0.10129926]),
array([0.1309055 , 0.14160908]))
```

[4]:
```
def report_coverage(sims, p, n, get_interval):
    phats = sims / n
    lower, upper = get_interval(phats, p, n)

    covered = np.logical_and(p > lower, p < upper).astype(int)

    return np.mean(covered)
```

[5]:
```
for p in ps:
    for n in ns:
        a_coverage = report_coverage(sims[str(p)][n], p, n, get_part_a_interval)
        print(f"n={n}, p={p}, part a coverage: {a_coverage}",)
        b_coverage = report_coverage(sims[str(p)][n], p, n, get_part_b_interval)
        print(f"n={n}, p={p}, part b coverage: {b_coverage}",)
```

```
n=10, p=0.1, part a coverage: 0.6427
n=10, p=0.1, part b coverage: 0.9304
n=40, p=0.1, part a coverage: 0.9129
n=40, p=0.1, part b coverage: 0.9381
n=100, p=0.1, part a coverage: 0.9314
n=100, p=0.1, part b coverage: 0.9348
n=10, p=0.3, part a coverage: 0.8462
n=10, p=0.3, part b coverage: 0.9266
n=40, p=0.3, part a coverage: 0.93
n=40, p=0.3, part b coverage: 0.9461
n=100, p=0.3, part a coverage: 0.9474
n=100, p=0.3, part b coverage: 0.9356
n=10, p=0.5, part a coverage: 0.8931
n=10, p=0.5, part b coverage: 0.9782
n=40, p=0.5, part a coverage: 0.9232
n=40, p=0.5, part b coverage: 0.9647
n=100, p=0.5, part a coverage: 0.9462
n=100, p=0.5, part b coverage: 0.9462
```

The test from part (B) yields much better coverage for small values of n, especially when p is small

1 Problem 4

[6]:
```
from scipy.special import gamma, digamma, polygamma
from matplotlib import pyplot as plt
from tqdm import tqdm
```

1.1 Part A

Below we define the function $f(\alpha)$ from lecture 13. This function has the property that $f(\alpha) = 0$ iff α is the MLE. We also define its derivative

$$f'(\alpha) = \frac{1}{\alpha} - \frac{\Gamma''(\alpha)\Gamma(\alpha) - \Gamma'(\alpha)^2}{\Gamma(\alpha)^2}$$

Note that `polygamma(1, z)` is the trigamma function.

[7]:
```
def f(alpha, x):
    term_1 = np.log(alpha)
    term_2 = -digamma(alpha)
    term_3 = -np.log(np.mean(x))
    term_4 = np.mean(np.log(x))
    return term_1 + term_2 + term_3 + term_4

def f_prime(alpha):
    return 1/alpha - polygamma(1, alpha)
```

[8]:
```
def update(alpha, x):
    return max(alpha - f(alpha, x)/f_prime(alpha), 1e-10)
```

1.2 Part B

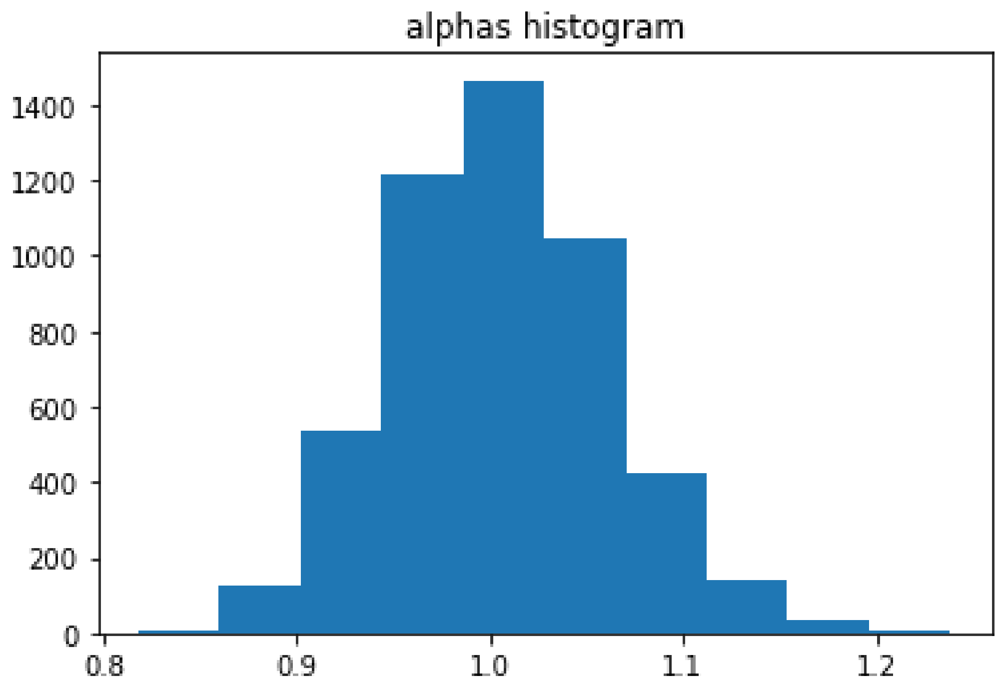
[9]:
```
# numpy's gamma distribution takes shape and scale parameters,
# rather than shape and rate.
alphas = []
betas = []
for _ in tqdm(range(5000)):
    x = np.random.gamma(1, 1/2, (500))
    x_bar = np.mean(x)
    alpha_init = x_bar**2 / np.mean((x-x_bar)**2)
    alpha = alpha_init
    for _ in range(100):
        alpha = update(alpha, x)
    alphas.append(alpha)
    betas.append(alpha/x_bar)


print(f"alpha mean: {np.mean(alphas)}")
print(f"alpha variance: {np.var(alphas)}")
plt.cla()
```

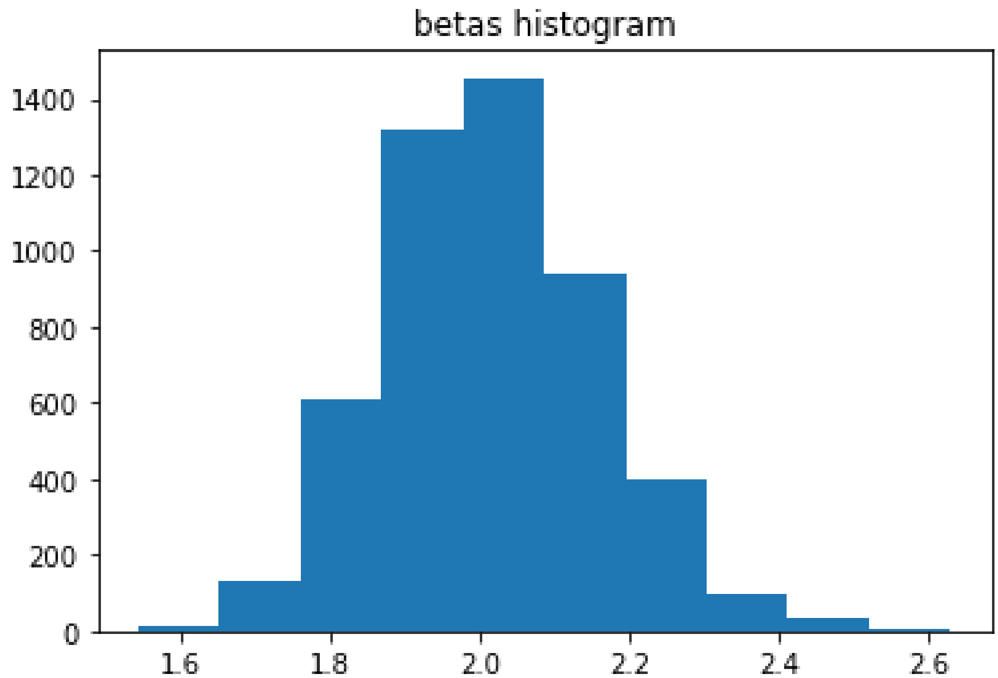
```python
plt.hist(alphas)
plt.title("alphas histogram")
plt.show()
print(f"beta mean: {np.mean(betas)}")
print(f"beta variance: {np.var(betas)}")
plt.hist(betas)
plt.title("betas histogram")
plt.show()
print(f"alpha and beta covariance: {np.cov(np.stack((alphas, betas)))[0,1]}")
```

```
100%|| 5000/5000 [00:19<00:00,
259.54it/s]

alpha mean: 1.0053403088002046
alpha variance: 0.003136007404175372
```




```
beta mean: 2.014524881226936
beta variance: 0.020349896243377113
```




```
alpha and beta covariance: 0.006217366169198483
```

1.3 Part C

The value of $\frac{1}{n}I(1,2)^{-1}$ is

[10]:
```
alpha = 1
beta = 2
constant = 1/500 * 1/(alpha*polygamma(1,alpha)-1)
mat = np.array([[alpha, beta], [beta, beta**2*polygamma(1, alpha)]])
cov_matrix = constant * mat
print(cov_matrix)
```

```
[[0.00310109 0.00620218]
 [0.00620218 0.02040437]]
```

Our simulations very closely match the approximate distribution of $\mathcal{N}((1,2), \frac{1}{n}I(1,2)^{-1})$