Deformation of Complex Structures

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Abstract

In the first several sections, the basic idea is to view a deformation as complex manifolds fibred over a base space, which leads to the concept of the complex analytic family. The basic result, that all fibres of a complex analytic family are diffeomorphic, is given. Therefore it turns out that a complex analytic family is just a given compact differentiable manifold with different complex structures parametrizing over a base space. This point of view allows us to treat the deformation of compact complex manifolds from a slightly different angle in the final part. Then we give some examples of complex analytic families. To measure how the complex structures vary with the parameters locally, we develop a way in cohomological level - the infinitesimal deformation, an analogue of the derivative of a function at a certain point. Under certain additional conditions, the vanishment of infinitesimal deformation indeed implies that the complex structures stay the same near some point in the base space. In this sense the infinitesimal deformation can be regarded as taking "derivative" of the complex structures. The next problem is to see if there exists a complex analytic family such that it has the given infiniesimal deformation. This is not always possible since the obstructions may occur, and thus a necessary condition for the existence is given. In the last section, as another approach to the theory of deformation of complex structures, we consider the variation of a family of integrable almost complex structures over a differentiable manifold, which is studied by a power series expansion of this variation. The Maurer-Cartan equation, deduced from the integrability condition, is equivalent to several recursive equations for the coefficients of this power series expansion. These equations can not always be solved. However on a Calabi-Yau manifold, a formal solution of these coefficients exists due to the Tian-Todorov lemma.

Keywords: complex structures, deformation.

Contents

1 Introduction

Classifying all complex structures leads to the theory of moduli spaces: the set of all complex structures on a given differentiable manifold comes itself with a natural differentiable and in fact complex structure. This article gives a first step discussion, the theory of deformation, about certain local aspects of this general classification, following K. Kodaira[] and D. Huybrechts[].

The idea of deformation of Riemann surfaces, the one dimensional complex manifolds, originated from Riemann who calculated the number of effective parameters where the deformation depends. But it is not until D. Spencer and Kodaira's famous work[] on the deformation of compact complex manifolds was developed that the deformation of higher dimensional cases started to be in focus. Their primitive idea is that, since a compact complex manifold M is compoed of a finite number of coordinate neighborhoods patched together, its deformation would be a shift in the patches. But this approach seems to be not so effective[], and thus here comes another point of view.

Given a family $\{M_t\}$ of complex structures with a underlying compact differentiable manifold, M_t depends (holomorphically) on a parameter t in a base space $B \subseteq \mathbb{C}^m$. The union $\mathscr{M} = \cup_t M_t$ can be regarded as a kind of fibre space over B whose structure is given by the complex structures along the fibres. This picture is our starting point. With some further consideration we have the definition of a complex analytic family(cf. Definition 2.1). This is what we call "deformation". Note that in Kodaira's original book[], the concept of a differentiable family, which only requires weaker conditions than that of a complex analytic family, is introduced. But the interesting cases often occur to be complex analytic families. So it is safe, and also convenient, for us to consider complex analytic families only.

2 Complex Analytic Family

2.1 Basic Definitions and Results

Definition 2.1. *Suppose $B \subseteq \mathbb{C}^m$ is a domain, and $\{M_t \mid t \in B\}$ is a family of compact complex manifolds M_t of dimension n depending on $t = (t_1, \ldots, t_m) \in B$. Then $\{M_t \mid t \in B\}$ is said to be a complex analytic family of compact complex manifolds if there is a complex manifolds $\mathscr{M}$ and a surjective holomorphic map $\varpi : \mathscr{M} \to B$ satisfying:*

(i) $\varpi^{-1}(t)$ is a compact complex submanifold of $\mathscr{M}$.

(ii) $M_t = \varpi^{-1}(t)$.

(iii) The rank of the Jacobian of of ϖ is equal to m at every point of $\mathscr{M}$, i.e., ϖ is a submersion.
In this case, we also say that M_t depends holomorphically on t.

First we shall explain what these three conditions mean. Condition (i) and (ii) are obviously necessary, and condition (iii) gives a nice local charaterization of ϖ. Let

$$z = (z^1, \ldots, z^n, z^{n+1}, \ldots, z^{n+m})$$

be a local coordinate on $\mathscr{M}$ and let $(t_1, \ldots, t_m) = \varpi(z)$, then (iii) implies that

$$\operatorname{rank} \frac{\partial(t_1, \ldots, t_m)}{\partial(z^1, \ldots, z^n, z^{n+1}, \ldots, z^{n+m})} = m.$$

Permutating $z^1, \ldots, z^n, z^{n+1}, \ldots, z^{n+m}$ if necessary, we may assume that

$$\operatorname{rank} \frac{\partial(t_1, \ldots, t_m)}{\partial(z^{n+1}, \ldots, z^{n+m})} = m,$$

which means that we can use $(z^1, \ldots, z^n, t_1, \ldots, t_m)$ as a local coordinate on $\mathscr{M}$. Therefore we can choose a system of local coordinates $\{z_1, \ldots, z_j, \ldots\}$ such that each z_j is defined on a domain $\mathscr{U}_j \subseteq \mathscr{M}$ where $\{\mathscr{U}_j \mid j = 1, 2, \ldots\}$ is a locally finite covering of $\mathscr{M}$, and for $p \in \mathscr{U}_j$ with $\varpi(p) = (t_1, \ldots, t_m)$, we have $z_j(p) = (z_j^1, \ldots, z_j^n, t_1, \ldots, t_m)$.

In terms of these coordinates, ϖ is given by $\varpi : (z_j^1, \ldots, z_j^n, t_1, \ldots, t_m) \to (t_1, \ldots, t_m)$. And each M_t naturally carries from $\mathscr{M}$ a system of coordinates: $\{(z_j^1, \ldots, z_j^n) \mid \mathscr{U}_j \cap M_t \neq \emptyset\}$.

If $\mathscr{U}_j \cap \mathscr{U}_k \neq \emptyset$ for some j, k, we denote the coordinate transformation from z_k to z_j by

$$f_{jk} : (z_k^1, \ldots, z_k^n, t) \to (z_j^1, \ldots, z_j^n, t), \text{ where } z_j^\alpha = f_{jk}^\alpha(z_k^1, \ldots, z_k^n, t), \alpha = 1, \ldots, n. \tag{1}$$

Note that $t = (t_1, \ldots, t_m)$ does not change under these coordinate transformations.

Now we can define what is the so-called "deformation".

Definition 2.2. *Let M and N be two compact complex manifolds. N is said to be a deformation of M if M and N belong to the same complex analytic family.*

The trivial example is $(M \times B, B, \pi)$, where π is the natural projection $M \times B \to B$. In $(M \times B, B, \pi)$, for any $s, t \in B$, obviously $M_s = \pi^{-1}(s)$ is biholomorphic to $M_t = \pi^{-1}(t)$, which means that this trivial deformation does not change the complex structure.

So we may wonder in what sense the complex structures stay identical under deformation.

Definition 2.3. *Two complex analytic families $(\mathscr{M}, B, \varpi)$ and $(\mathscr{N}, B, \pi)$ are said to be* holomorphically equivalent *if there is a biholomorphic map $\Psi : \mathscr{M} \to \mathscr{N}$ with $\varpi = \pi \circ \Psi$.*

We say that $(\mathscr{M}, B, \varpi)$ is *trivial* if it is holomorphically equivalent to $(M \times B, B, \pi)$. Obiously in this case the complex structure of M_t is independent of t globally. Here "global" means that it is true for all $t \in B$.

Analogously, $(\mathscr{M}, B, \varpi)$ is said to be *locally trivial*, if for each $t \in B$ there is a subdomain $U \subseteq B$ containing t, such that the complex analytic family $(\mathscr{M}_U = \varpi^{-1}(U), U, \varpi)$, obtained by the restriction of $(\mathscr{M}, B, \varpi)$ to U, is trivial.

Then we can easily see that if $(\mathscr{M}, B, \varpi)$ is said to be locally trivial, then each M_t is biholomorphically isomorphic to a fixed $M = \varpi_0$, which means that the complex structure of M_t does not vary with t.

One important fact is that the underlying differentiable structure does not vary under deformation, which is the folloing theorem:

Theorem 2.1. *In a complex analytic family $(\mathscr{M}, B, \varpi)$, all fibres of $\varpi : \mathscr{M} \to B$ are diffeomorphic.*

Proof. To prove theorem (2.1), we need the lemma (2.1) below. The proof of it is rather technical but the method is useful. So it'd better to give a detailed argument here.

The basic settings are much like the conditions in the definition of complex analytic families, which allow us to apply theorem (2.1) to other situations:

Suppose that $\mathscr{M}$ is a differentiable manifold, $B \subseteq \mathbb{R}^m$ is a domain with $0 \in B$. $\varpi : \mathscr{M} \to B$ is a surjective smooth map. They satisfy:

(i) $M_t = \varpi^{-1}(t)$ is a compact differentiable manifold of dimension n for any $t \in B$.

(ii) The rank of the Jacobian of ϖ is equal to m at every point of $\mathscr{M}$.

Then as in the case of complex analytic families, we can find a locally finite open covering $\{\mathscr{U}_j \mid j = 1, 2, \ldots\}$ of $\mathscr{M}$ such that the local coordinates x_j defined over $\mathscr{U}_j$ with the form: $x_j(p) = (x_j^1, \ldots, x_j^n, t_1, \ldots, t_m)$ for $\varpi(p) = (t_1, \ldots, t_m)$.

Take $U := \{t \mid |t_1| < r, \ldots, |t_m| < r\} \subseteq \mathbb{R}^m$, such that the closure of U is contained in B, and let $\pi : M_0 \times U \to U$ be the natural projection.

Lemma 2.1. *There is a diffeomorphism $\Psi : M_0 \times U \to \mathscr{M}_U := \varpi^{-1}(U)$ such that $\varpi \circ \Psi = \pi$.*

Proof. The proof is given by induction on the dimension m of B.

For $m = 1$, $U = (-r, r)$ and $\bar{U} = [-r, r] \subseteq B \subseteq \mathbb{R}$. First we use partition of unity to construct a vector field on $\mathscr{M}$ on each $\mathscr{U}_j$ of the form:

$$\sum_{\alpha=1}^{n} v_j^\alpha(x_j, t_1) \frac{\partial}{\partial x_j^\alpha} + \frac{\partial}{\partial t_1}.$$

In fact, denote the vector field $\frac{\partial}{\partial t_1}$ on $\mathscr{U}_j$ by $(\frac{\partial}{\partial t_1})_j$. Then

$$\left(\frac{\partial}{\partial t_1}\right)_k = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha}{\partial t_1}(x_k, t_1) \frac{\partial}{\partial x_j^\alpha} + \left(\frac{\partial}{\partial t_1}\right)_j.$$

Let $\{\rho_j\}$ be a partition of unity subordinate to $\{\mathscr{U}_j\}$. Then we obtain a C^∞ vector field on $\mathscr{M}$ with the form on each $\mathscr{U}_j$:

$$\sum_k \rho_k \left(\frac{\partial}{\partial t_1}\right)_k = \sum_{\alpha=1}^{n} \sum_{k \neq j} \rho_k \frac{\partial f_{jk}^\alpha}{\partial t_1}(x_k, t_1) \frac{\partial}{\partial x_j^\alpha} + \left(\frac{\partial}{\partial t_1}\right)_j.$$

Let $v_j^\alpha = \sum_{k \neq j} \rho_k \frac{\partial f_{jk}^\alpha}{\partial t_1}$, we obtain the required vector field.

Consider the following system of linear differential equations:

$$\begin{cases} \frac{dx_j^\alpha}{dt} = v_j^\alpha(x_j^1, \ldots, x_j^n, t_1), \quad \alpha = 1, \ldots, n, \\ \frac{dt_1}{dt} = 1. \end{cases}$$

For any $(\xi_i) = (\xi_i^1, \ldots, \xi_i^n) \in M_0 = \varpi^{-1}(0)$, let

$$\begin{cases} x_j^\alpha = x_j^\alpha(\xi_i, t), \quad \alpha = 1, \ldots, n, \\ t_1(t) = t, \end{cases}$$

be the solution of this system of linear differential equations satisfying the initial condition:

$$\begin{cases} x_j^\alpha(\xi_i, 0) = \xi_i^\alpha, \quad \alpha = 1, \ldots, n, \\ t_1(0) = 0, \end{cases}$$

which means that $t \mapsto (x_j(\xi_i, t), t), -r < t < r$ is a smooth curve passing through $\xi_i \in M_0$. By the existence and uniqueness theorem of the solution of systems of ordinary differential equations, for each point $(x_j, t) \in \mathscr{M}_U$ there exists one and only one smooth curve passing through it. Thus the map

$$\Psi : M_0 \times U \to M_U, (\xi_i, t) \mapsto (x_j^1(\xi_i, t), \ldots, x_j^n(\xi_i, t), t)$$

is a diffeomorphism. Clearly $\varpi \circ \Psi(\xi_i, t) = t = \pi(\xi_i, t)$.

For the general case, by induction, suppose the lemma is true for $m - 1$. Put $U_{m-1} = \{(t_1, \ldots, t_{m-1}) \mid |t_1| < 1, \ldots, |t_{m-1}| < r\}$, and $U_m = \{t_m \mid |t_m| < r\}$, then $U = U_{m-1} \times U_m$. Since by induction hypothesis $\varpi^{-1}(U_{m-1}) = M_0 \times U_{m-1}$, all we have to show is that $\varpi^{-1}(U) = \varpi^{-1}(U_{m-1}) \times U_m$.

Define $\varpi_m(p) = t_m$ for $p \in \mathscr{M}$ and $\pi_m : \varpi^{-1}(U_{m-1}) \times U_m \to U_m$ the natural projection. As before, we can construct a vector field on $\mathscr{M}$ of the form

$$\sum_{\alpha=1}^n v_j^\alpha(x_j^1, \ldots, x_j^n, t_1, \ldots, t_m) \frac{\partial}{\partial x_j^\alpha} + \frac{\partial}{\partial t_m}.$$

And we obtain the corresponding system of differential equations

$$\begin{cases} \frac{dx_j^\alpha}{dt} = v_j^\alpha(x_j^1, \ldots, x_j^n, t_1, \ldots, t_m), \quad \alpha = 1, \ldots, n, \\ \frac{dt_v}{dt} = 0, \quad v = 1, \ldots, m-1, \\ \frac{dt_m}{dt} = 1, \end{cases}$$

Therefore we have a diffeomorphism $\Psi_m : \varpi^{-1}(U_{m-1}) \times U_m \to \varpi^{-1}(U)$ with $\varpi_m \circ \Psi_m = \pi_m$. Consequently $\varpi^{-1}(U) = \varpi^{-1}(U_{m-1}) \times U_m = M_0 \times U$ as desired. $\square$

With this lemma, we can prove theorem(2.1) easily. In fact, the complex analytic family $(\mathscr{M}, B, \varpi)$ obviously satisfies the conditions in the lemma. Thus for any $a \in B$, there exists a neighborhood $U(a)$ of a such that $\varpi^{-1}(U(a)) = M_a \times U(a)$.

For any two points $a, b \in B$, we can find a path, denoted by L, connecting a and b. Then we can choose a finite open cover $\{U_j \mid j = 1, \ldots, N\}$ of the path L such that $a \in U_1$, $b \in U_N$, $U_{j-1} \cap U_j \neq \emptyset$ and $\varpi^{-1}(U_j) = M_{c_j} \times U_j$ for some $c_j \in U_j \cap L$. Thus $M_a = M_{c_1} = \cdots = M_{c_N} = M_b$. $\square$

Remark: This theorem implies that a complex analytic family is the same as a given compact differentiable manifold with different complex structures M_t (holomorphically) parametrizing over a base space B. Thus the definition of a complex analytic family is exactly what we want to define "deformation" as stated in section 1.

2.2 Examples

Example 1 Let $w \in \mathbb{C}$ with $\mathrm{Im} w > 0$, and $G_w := \{mw + n \mid m, n \in \mathbb{Z}\}$. Then G_w acts on $\mathbb{C}$ by parallel translation: $(mw + n) \cdot z = z + mw + n$. As well known, $C_w := \mathbb{C}/G_w$, which is also called an elliptic curve, is a compact complex Riemann surface.

Claim: The set $\{C_w \mid \mathrm{Im} w > 0\}$ forms a complex analytic family. In fact, let $\mathbb{H}^+$ be the upper half plane of $\mathbb{C}$, $G := \{g_{mn} : \mathbb{C} \times \mathbb{H}^+ \to \mathbb{C} \times \mathbb{H}^+, (z, w) \mapsto (z + mw + n, w) \mid m, n \in \mathbb{Z}\}$. Then G acts on $\mathbb{C} \times \mathbb{H}^+$ as a group of automorphism in a nice way, such that $\mathscr{M} := \mathbb{C} \times \mathbb{H}^+/G$ is a complex manifold. The natural projection $\mathbb{C} \times \mathbb{H}^+ \to \mathbb{H}^+, (z, w) \mapsto w$ induces a surjective holomorphic map $\varpi : \mathscr{M} \to \mathbb{H}^+$ since the projection $\mathbb{C} \times \mathbb{H}^+ \to \mathbb{H}^+$ commutes with each g_{mn}. We have $\varpi^{-1}(w) = \mathbb{C} \times \{w\}/G \cong \mathbb{C}/G_w = C_w$. We can use (z, w) as local coordinates on $\mathscr{M}$, then locally ϖ is the projection $(z, w) \mapsto w$. Thus the rank of the Jocobian matrix of ϖ is equal to 1. We conclude that $\{C_w \mid w \in \mathbb{H}^+\}$ is indeed a complex analytic family.

Next, we consider when we have $C_w \cong C_{w'}$ for some $w, w' \in \mathbb{H}^+$.

The following fact is useful: $w_1 = aw + b, w_2 = cw + d$ with $a, b, c, d \in \mathbb{Z}$ generate G_w if and only if $ad - bc = \pm 1$. Interchanging w_1 and w_2 if necessary, we may assume that $ad - bc = 1$. Define the coordinate transformation $\varphi : \mathbb{C} \to \mathbb{C}, z \mapsto \frac{z}{w_2}$, then $w' := \varphi(w_1) = \frac{w_1}{w_2}, 1 = \varphi(w_2)$. Clearly $\mathrm{Im} w' > 0$, $\varphi(G_w) = G_{w'}$ and therefore φ induces a biholomorphic map from C_w to $C_{w'}$.

Conversely, if $C_w \cong C_{w'}$, let f be the biholomorphic map, then f is lifted to be a biholomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{C}$. Thus $\tilde{f}$ must be a linear map $z \mapsto z' = \alpha z + \beta$ with $\alpha \neq 0$. Then $\tilde{f}(G_w) = G_{w'}$ and we have $\tilde{f}(0) = \beta \in G_{w'}$. Since 1 and w generate G_w, $\tilde{f}(1) = \alpha + \beta$ and $\tilde{f}(w) = \alpha w + \beta$ must generate $G_{w'}$. But $\beta \in G_{w'}$, we see that αw and α generate G_w. Thus $w' = a\alpha w + b\alpha$, and $1 = c\alpha w + d\alpha$ for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. We have $w' = \frac{w'}{1} = \frac{aw+b}{cw+d}$.

Hence, $C_w \cong C_{w'}$ if and only if

$$w' = \frac{aw + b}{cw + d}, \qquad a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1. \tag{2}$$

Let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Γ acts on $\mathbb{H}^+$ by the formula in (2), and $\mathbb{H}^+/\Gamma$ is biholomorphic to $\mathbb{C}$. This biholomorphism induces a Γ-invariant holomorphic function $J(w)$ defined on $\mathbb{H}^+$. According to the above argument, C_w and $C_{w'}$ are biholomorphic if and only if $J(w) = J(w')$, which means that the complex structure of C_w varies "continuously" as w moves in $\mathbb{H}^+$.

Example 2 Given $W = \mathbb{C}^2 - \{0\}$, $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$. Let $g_t, t \in \mathbb{C}$ be an automorphism of W given by

$$g_t : (z_1, z_2) \mapsto (\alpha z_1 + t z_2, \alpha z_2), \text{i.e. } g_t \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Set $G_t = \{g_t^m \mid m \in \mathbb{Z}\}$. Then G_t also acts on W in a nice way such that $M_t := W/G_t$ is complex manifold with dimenson 2. In particular, $M_0 = W/G_0$ is the so-called Hopf manifold.

Claim: $\{M_t \mid t \in \mathbb{C}\}$ forms a complex analytic family. In fact, the argument is much like the argument in Example 1. The key point is to construct the total space $\mathscr{M}$ containing all M_t.

Here $\mathscr{M} := W \times \mathbb{C}/G$, where G is a cyclic group generated by the automorphism g of $W \times \mathbb{C}$:

$$g \cdot \begin{pmatrix} z_1 \\ z_2 \\ t \end{pmatrix} = \begin{pmatrix} \alpha & t & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ t \end{pmatrix}$$

Now we consider how the complex structure of M_t varies as t moves in $\mathbb{C}$.

Let $U = \mathbb{C} - \{0\}$. Then the restriction $(\mathscr{M}_U, U, \varpi_U)$ is trivial. In fact, we can use coordinate transformation on $W \times U$:

$$\begin{pmatrix} w_1 \\ w_2 \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ t \end{pmatrix}$$

In terms of this new coordinate (w_1, w_2, t), g is represented as

$$g \cdot \begin{pmatrix} w_1 \\ w_2 \\ t \end{pmatrix} = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ t \end{pmatrix}.$$

Thus $\mathscr{M}_U = W \times U/G \cong (W/G_1) \times U = M_1 \times U$, i.e. $(\mathscr{M}_U, U, \varpi_U) = (M_1 \times U, U, \pi)$, which means that M_t has the same complex structure as that of M_1 for $t \neq 0$.

However, the complex structure of M_0 is different from that of M_t with $t \neq 0$. To prove this, the idea is to compare the number of linearly independent holomorphic vector fields on M_t. For details, see(). In fact, $\dim H^0(M_0, TM_0) = 4$ but $\dim H^0(M_t, TM_t) = 2$ for $t \neq 0$. Thus the complex structure of M_t "jumps" at $t = 0$.

3 Infinitesimal Deformation

The basic setting is the same as that of subsection(2.1). Suppose that $(\mathscr{M}, B, \varpi)$ is a complex analytic family, $0 \in B \subseteq \mathbb{C}^m$ is a domain. Let $\{z_1, \ldots, z_j, \ldots\}$ be an arbitrary system of local coordinates of $\mathscr{M}$ with $\mathscr{U}_j$ the domain of z_j.

Identifying $p \in \mathscr{U}_j$ with $z_j(p) = (z_j, t) = (z_j^1, \ldots, z_j^n, t_1, \ldots, t_m)$, we consider $\mathscr{U}_j$ as a subset in $\mathbb{C}^n \times B$. For $t \in B$, $M_t = \varpi^{-1}(t)$. Then

$$\mathscr{U}_j \cap M_t = U_{jt} \times t \cong U_{jt} \subseteq \mathbb{C}^n.$$

Thus we obtain an open cover $\mathfrak{U}_t$ of M_t. Since M_t is compact, $\mathfrak{U}_t$ can be chosen to be a finite cover where we omit some U_{jt} if it is empty.

If $\mathscr{U}_j \cap \mathscr{U}_k \cap M_t \neq \emptyset$, then we have the coordinate tranformation as in subsection(2.1)

$$z_j^\alpha = f_{jk}^\alpha(z_k^1, \ldots, z_k^n, t), \alpha = 1, \ldots, n,$$

or write $z_j = f_{jk}(z_k, t)$ for short. Then on $\mathscr{U}_i \cap \mathscr{U}_j \cap \mathscr{U}_k \cap M_t \neq \emptyset$, we have similarly

$$z_i = f_{ij}(z_j, t) = f_{ik}(z_k, t), \quad z_j = f_{jk}(z_k, t).$$

Hence this is of the form:

$$f_{ik}(z_k, t) = f_{ij}(f_{jk}(z_k, t), t). \tag{3}$$

Now we are looking for some useful criterions to see whether the complex structure of $M_t = \varpi^{-1}(t)$ actually depends on t. In calculus by differentiating a differentiable function $f(t)$ of a real variable t, we know whether $f(t)$ is a constant independent of t or not. Analogically in our case we expect to differentiate the equation (3) in t.

Let $\frac{\partial}{\partial t} := \sum_{\lambda=1}^{m} c_\lambda \frac{\partial}{\partial t_\lambda}$, $c_\lambda \in \mathbb{C}$ be a holomorphic tangent vector. Taking differentiation in equation (3) to both two sides with respect to t, we have

$$\begin{aligned}\frac{\partial f_{ik}^{\alpha}(z_k,t)}{\partial t} &= \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t} \\ &= \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t}\end{aligned}$$

Using holomorphic vector fields, we can rewrite this equality as:

$$\begin{aligned}\sum_{\alpha=1}^{n} \frac{\partial f_{ik}^{\alpha}(z_k,t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}} &= \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}} + \sum_{\alpha,\beta=1}^{n} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \frac{\partial}{\partial z_i^{\alpha}} \\ &= \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}} + \sum_{\beta=1}^{n} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t} \frac{\partial}{\partial z_j^{\beta}}.\end{aligned}$$

Introduce the vector field

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t} \frac{\partial}{\partial z_j^{\alpha}}.$$

Then the above formular is

$$\begin{aligned}\theta_{ik}(t) &= \theta_{ij}(t) + \theta_{jk}(t) \\ \text{i.e.}\quad \theta_{ij}(t) &- \theta_{ik}(t) + \theta_{jk}(t) = 0\end{aligned}$$

Since $f_{kk}^{\alpha} = z_k^{\alpha}$, we have $\theta_{kk}(t) = 0$. This implies that $\{\theta_{jk}(t)\}$ is a 1-cocycle, i.e. $\{\theta_{jk}(t)\} \in Z^1(\mathfrak{U}_t, TM_t)$, where TM_t is the holomorphic tangent bundle of M_t.

Definition 3.1. *$\theta(t) \in H^1(\mathfrak{U}_t, TM_t) = H^1(M_t, TM_t)$, called the infinitesimal deformation of M_t along $\frac{\partial}{\partial t}$, is defined to be the cohomology class determined by $\{\theta_{jk}(t)\}$. We also denote $\theta(t)$ by $\frac{\partial M_t}{\partial t}$.*

Of course we need to verify that this definition is well-defined.

Lemma 3.1. *$\frac{\partial M_t}{\partial t} = \theta(t)$ does not depend on the choice of systems of local coordinates.*

Proof. First we show that $\frac{\partial M_t}{\partial t}$ does not change under the refinement of the open cover $\mathfrak{U} = \{\mathscr{U}_j\}$.

Let $\mathfrak{V} = \{\mathscr{V}_\lambda\}$, $\mathscr{V}_\lambda \subseteq \mathscr{U}_{j(\lambda)}$ be a refinement of $\mathfrak{U}$ and define local coordinates $\hat{z}_\lambda$ on $\mathscr{V}_\lambda$ to be the restriction of $z_{j(\lambda)}$ to $\mathscr{V}_\lambda$. Then $\hat{\theta}_{\lambda\mu}(t)$, obtained by means of $\{\hat{z}_\lambda\}$, is exactly the restriction of $\theta_{j(\lambda)j(\mu)}$. Thus $\{\hat{\theta}_{\lambda\mu}(t)\}$ and $\{\theta_{jk}(t)\}$ determine the same cohomology class in $H^1(M_t, TM_t)$.

Now it suffices to show that: Given another local coordinates $\{(w_j, t)\}$ on each $\mathscr{U}_j$, the infinitesimal deformation $\eta(t)$ defined by $\{(w_j, t)\}$ coincides with $\theta(t)$.

Let the coordinate transformation of $\{(w_j, t)\}$ be

$$(w_j, t) = (h_{jk}(w_k, t), t).$$

Then $\eta(t)$ is the cohomology class of the 1-cocycle $\{\eta_{jk}(t)\}$ given by

$$\eta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial h_{jk}^{\alpha}(w_k, t)}{\partial t} \frac{\partial}{\partial w_j^{\alpha}}.$$

Let the transformation between $\{(w_j, t)\}$ and $\{(z_j, t)\}$ be $w_j^{\alpha} = g_j^{\alpha}(z_j, t)$, for $\alpha = 1, \ldots, n$ Then on $\mathscr{U}_j \cap \mathscr{U}_k \neq \emptyset$, we have

$$g_j^{\alpha}(z_j, t) = w_j^{\alpha} = h_{jk}^{\alpha}(w_k, t) = h_{jk}^{\alpha}(g_k(z_k, t), t),$$

Putting $z_j = f_{jk}(z_k, t)$, the above formula is

$$g_j^{\alpha}(f_{jk}(z_k, t), t) = h_{jk}^{\alpha}(g_k(z_k, t), t).$$

Differentiating both two sides with respect to t we can obtain:

$$\sum_{\beta=1}^{n} \frac{\partial w_j^{\alpha}}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}}{\partial t} + \frac{\partial g_j^{\alpha}}{\partial t} = \sum_{\beta=1}^{n} \frac{\partial w_j^{\alpha}}{\partial w_k^{\beta}} \frac{\partial g_k^{\beta}}{\partial t} + \frac{\partial h_{jk}^{\alpha}}{\partial t}.$$

Multiply $\frac{\partial}{\partial w_j^{\alpha}}$ and take the summation $\sum_{\alpha=1}^{n}$:

$$\sum_{\beta=1}^{n} \frac{\partial f_{jk}^{\beta}}{\partial t} \frac{\partial}{\partial z_j^{\beta}} + \sum_{\alpha=1}^{n} \frac{\partial g_j^{\alpha}}{\partial t} \frac{\partial}{\partial w_j^{\alpha}} = \sum_{\beta=1}^{n} \frac{\partial g_k^{\beta}}{\partial t} \frac{\partial}{\partial w_k^{\beta}} + \sum_{\alpha=1}^{n} \frac{\partial h_{jk}^{\alpha}}{\partial t} \frac{\partial}{\partial w_j^{\alpha}}.$$

Let $\theta_j(t) = \sum_{\alpha} \frac{\partial g_j^{\alpha}}{\partial t} \frac{\partial}{\partial w_j^{\alpha}}$. Then $\{\theta_j(t)\} \in C^0(\mathfrak{U}_t, TM_t)$ and

$$\begin{aligned} \theta_{jk}(t) - \eta_{jk}(t) &= \theta_k(t) - \theta_j(t), \\ \text{i.e.} \quad \{\theta_{jk}(t)\} - \{\eta_{jk}(t)\} &= \delta\{\theta_j(t)\}. \end{aligned}$$

Therefor $\eta(t)$ coincides with $\theta(t)$. □

Remark: As a result we can find a suitable open cover of M_t to define $\{\theta_{jk}\}$. Since M_t is compact, we can choose finitely many $\mathscr{U}_j, j = 1, \ldots, l$ such that $\{\mathscr{U}_j, j = 1, \ldots, l\}$ covers M_t and $z_j(\mathscr{U}_j) = U_j \times I_j$ where U_j is a polydisk in $\mathbb{C}^n$ and $0 \in I_j \subset B$ is a domain. Take t = 0. Identifying $\mathscr{U}_j$ with $U_j \times I_j$ by z_j, then $M_0 \subseteq \cup_{j=1}^{l} U_j \times I_j$. Putting $I = \cap_{j=1}^{l} I_j$ we have

$$M_I = \varpi^{-1}(I) = \cup_{j=1}^{l} U_j \times I.$$

Later we shall frequently use this open cover.

Define a $\mathbb{C}$-linear map:

$$\rho_t : T_t B \to H^1(M_t, TM_t), \frac{\partial}{\partial t} \mapsto \frac{\partial M_t}{\partial t}.$$

Take $t = 0$ and $M = M_0 = \varpi^{-1}(0)$. The map $\rho_0 : T_0 B \to H^1(M, TM)$ is called the *Kodaira-Spencer map* and the class $\rho_0(v) \in H^1(M, TM)$ is the *Kodaira-Spencer class* associated with the tangent vector $v \in T_0 B$.

There is another way to define the Kodaira-Spencer map ρ_0. For $x \in M$, we have a surjective linear map $\varpi_* : T_x\mathscr{M} \to T_0 B$ with kernel being $T_x M$. Thus we obtain a short exact sequence of the form:

$$0 \to TM \to T\mathscr{M}|_M \to T_0 B \otimes \mathscr{O}_M \to 0.$$

The boundary map $\delta^* : T_0B \to H^1(M, TM)$ is exactly the same as ρ_0. In fact, since ϖ is of the form $(z_j^1, \ldots, z_j^n, t) \to t$ on each $\mathscr{U}_j \cap M \neq \emptyset$, the Jacobian matrix of ϖ is an $m \times (m+n)$ matrix of the form $(*|I_m)$, where I_m is the $m \times m$ identity matrix. Thus it sends $(\frac{\partial}{\partial t})_j$, the vector field $\frac{\partial}{\partial t}$ on $\mathscr{U}_j \cap M \neq \emptyset$, to $\frac{\partial}{\partial t} \in T_0B$. Since

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_j &= \sum_{\alpha=1}^{n} \frac{\partial z_k^\alpha}{\partial t} \frac{\partial}{\partial z_k^\alpha} + \left(\frac{\partial}{\partial t}\right)_k \\ &= \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_k^\alpha} + \left(\frac{\partial}{\partial t}\right)_k \end{aligned}$$

on $\mathscr{U}_j \cap \mathscr{U}_k \cap M \neq \emptyset$, we immediately see that $\delta\{(\partial/\partial t)_j\} = \{\theta_{jk}(t)\}$. Hence $\delta^*(\partial/\partial t) = \theta(t)$, i.e. $\delta^* = \rho_0$. This way seems to be more elegent and hides tedious details of the construction. But the explicit formula for $\frac{\partial M_t}{\partial t}$ is useful as we will see.

We hope that in some sense $\frac{\partial M_t}{\partial t}$ is the “derivative” of the complex structure of M_t along $\frac{\partial}{\partial t}$.

Proposition 3.1. *If $(\mathscr{M}, B, \varpi)$ is locally trivial, then $\rho_t \equiv 0$ for all t.*

Proof. For any sufficiently small domain $t \in U \subseteq B$, $(\mathscr{M}_U, U, \varpi)$ is equivalent to $(M \times U, U, \pi)$. Then we can calculate $\frac{\partial M_t}{\partial t}$ by means of local coordinates on $M \times U$.

Let $\{u_\lambda = (w_\lambda, t)\}$ be local coordinates on $M \times U$, where $\{w_\lambda\}$ is a system of local coordinates of M independent of t. The coordinate transformation for $\{u_\lambda\}$ is of the form:

$$(w_\lambda, t) = (h_{\lambda\mu}(w_\mu), t),$$

where $h_{\lambda\mu}(w_\mu)$ is independent of t. We obtain

$$\theta_{\lambda\mu}(t) = \sum_{\alpha=1}^{n} \frac{\partial h_{\lambda\mu}^\alpha(w_\mu)}{\partial t} \frac{\partial}{\partial w_\lambda^\alpha} = 0.$$

Thus $\frac{\partial M_t}{\partial t} = 0$, i.e. $\rho(t) \equiv 0$. □

If $\frac{\partial M_t}{\partial t}$ is truly the derivative of the complex structure of M_t, then $\rho_t \equiv 0$ for all t must imply the local triviality of $(\mathscr{M}, B, \varpi)$. At least this is true under certain additional assumptions:

Theorem 3.1. *If dim $H^1(M_t, TM_t)$ is independent of $t \in B$ and $\rho_t \equiv 0$ for all t, then $(\mathscr{M}, B, \varpi)$ is locally trivial.*

Proof. The proof is based on the method used in theorem(2.1). Use induction on $m = \dim B$.

First consider the case $m = 1$. Take a sufficiently small disk D with $0 \in D \subseteq B \subseteq \mathbb{C}^m = \mathbb{C}$ and $\mathscr{M}_D = \varpi^{-1}(D) = \cup_{j=1}^{l} U_j \times D$. That $\rho_t \equiv 0$ implies that $\theta(t) = \frac{dM_t}{dt} \equiv 0$. By definition, $\theta(t)$ is represented by the 1-cocycle $\{\theta_{jk}(t)\} \in Z^1(\mathscr{U}_t, TM_t)$ where $\mathscr{U}_t = \{U_j \times t\}$ is the open cover of M_t and $\theta_{jk}(t) = \sum_\alpha \frac{\partial f_{jk}^\alpha}{\partial t} \frac{\partial}{\partial z_j^\alpha}$. Then $\theta(t) = 0$ means that there is a 0-cochain $\{\theta_j(t)\} \in C^0(\mathscr{U}_t, TM_t)$ such that

$$\theta_{jk}(t) = \theta_k(t) - \theta_j(t), \text{with } \theta_j(t) := \sum_{\alpha=1}^{n} \theta_j^\alpha(z_j, t) \frac{\partial}{\partial z_j^\alpha}.$$

We denote the vector field $\frac{\partial}{\partial t}$ on $U_k \times D$ by $(\frac{\partial}{\partial t})_k$. Then

$$\left(\frac{\partial}{\partial t}\right)_k = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}}{\partial t} \frac{\partial}{\partial z_j^{\alpha}} + \left(\frac{\partial}{\partial t}\right)_j$$

$$i.e. \quad \theta_{jk}(t) = \left(\frac{\partial}{\partial t}\right)_k - \left(\frac{\partial}{\partial t}\right)_j .$$

This implies that

$$-\theta_k(t) + \left(\frac{\partial}{\partial t}\right)_k = -\theta_j(t) + \left(\frac{\partial}{\partial t}\right)_j .$$

Thus we can define a vector field v on $\mathscr{M}_D$ of the form

$$v = -\theta_j(t) + \frac{\partial}{\partial t} = -\sum_{\alpha=1}^{n} \theta_j^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}} + \frac{\partial}{\partial t} \quad \text{on } U_j \times D.$$

However, a priori v may not be a holomorphic vector field on $\mathscr{M}_D$. Though $\theta_j(t)$ is holomorphic on $U_j \times D \cap U_k \times D$ with respect to z_j, $\theta_j(t)$ may not be holomorphic in t. Hence we have to consider the following problem: Can $\theta_j(t)$ be chosen so that $\theta_j^{\alpha}(z_j, t)$ for $\alpha = 1, \ldots, n$, is holomorphic in z_j and t? The answer is yes, even for $t \in D \subseteq \mathbb{C}^m, m \geq 1$, due to the following lemma, where the additional assumptions is required:

Lemma 3.2. *If $dimH^1(M_t, TM_t)$ is independent of $t \in D$, then we can choose a 0-cochain $\{\theta_j(t)\}$ with $\{\theta_{jk}(t)\} = \delta\{\theta_j(t)\}$ such that $\theta_j^{\alpha}(z_j, t)$ is holomorphic in z_j and t where $\theta_j(t) = \sum_{\alpha=1}^{n} \theta_j^{\alpha}(z_j, t)(\frac{\partial}{\partial z_j^{\alpha}})$.*

The proof of this lemma is rather technical and is based on the theory of elliptic partial differential operators on manifolds, see().

Therefore we can assume that v is a holomorphic vector field on $\mathscr{M}_D$. Consider the systems of differential equations:

$$\begin{cases} \frac{dz_j^{\alpha}}{ds} = -\theta_j^{\alpha}(z_j, t), \alpha = 1, \ldots, n, \\ \frac{dt}{ds} = 1, \end{cases}$$

As in theorem(2.1), by solving these equations we can obtain a *biholomorphic*(not only just diffeomorphic) map:

$$\Psi : M_0 \times D \to M_D, (\xi_i, t) \mapsto (z_j^1(\xi_i, t), \ldots, z_j^n(\xi_i, t), t)$$

for $(\xi_i) \in M_0$. For other points $0 \neq c \in B$, the prove is the same. Thus we conclude that $\mathscr{M}$ is locally trivial.

The proof of the case that $m > 1$ is basically the same as that of theorem(2.1), and we shall omit it.

□

The last part of this section is to develop the "chain rule" for $\frac{\partial M_t}{\partial t}$.

Suppose we have a holomorphic map $h : D \subseteq \mathbb{C}^r \to B \subseteq \mathbb{C}^m, h(s) = t$ where $D \subseteq \mathbb{C}^r$ is a domain. Define $\Pi : \mathscr{M} \times D \to B \times D, (p, s) \mapsto (\varpi(p), s)$. Then $(\mathscr{M} \times D, B \times D, \Pi)$ is a complex analytic family with $\Pi^{-1}(t, s) = M_t \times s$.

The graph of h: $G = \{(h(s), s) \in B \times D \mid s \in D\}$ is a submanifold of $B \times D$, and $\mathscr{N} = \Pi^{-1}(G)$ is a submanifold of $\mathscr{M} \times D$. Then $(\mathscr{N}, G, \Pi)$ a complex analytic family over the parameter space G. Identifying G with D via the projection $P : B \times D \to D$, we obtain the complex analytic family $(\mathscr{N}, D, \pi)$ where $\pi = P \circ \Pi$. And $\pi^{-1}(s) = \Pi^{-1}(h(s), s) = M_{h(s)} \times s = M_{h(s)}$. Thus $\{M_{h(s)} \mid s \in D\}$ form a complex analytic family of $(\mathscr{N}, D, \pi)$. We call $(\mathscr{N}, D, \pi)$ is the complex analytic family induced from $(\mathscr{M}, B, \varpi)$ by the holomorphic map $h : D \to B$.

Theorem 3.2. *For any $\partial/\partial s \in T_s D$, the infiniesimal deformation of $M_{h(s)}$ along $\partial/\partial s$ is given by*

$$\frac{\partial M_{h(s)}}{\partial s} = \sum_{\lambda=1}^{m} \frac{\partial t_\lambda}{\partial s} \frac{\partial M_t}{\partial t_\lambda}.$$

Proof. This can be easily seen from the following formula:

$$\frac{\partial f_{jk}^\alpha(z_k, h(s))}{\partial s} = \sum_{\lambda=1}^{m} \frac{\partial t_\lambda}{\partial s} \cdot \frac{\partial f_{jk}^\alpha(z_k, t_1, \ldots, t_m)}{\partial t_\lambda}.$$

□

Remark: Theorem(3.1) contains an additional assumption that $\dim H^1(M_t, TM_t)$ is independent of t. The following example shows that this assumption is also necessary in some sense.

Let $(\mathscr{M}, \mathbb{C}, \varpi)$ be the complex analytic family given in the example 2 of section(2.2). Then M_0 and $M_t (t \neq 0)$ have different complex structures but for $U = \mathbb{C} - 0$, $(\mathscr{U}, U, \varpi)$ is trivial, which implies that $dM_t/dt = 0$ for $t \neq 0$.

Now consider the complex analytic family $(\mathscr{N}, \mathbb{C}, \pi)$ induced from $(\mathscr{M}, \mathbb{C}, \varpi)$ by the holomorphic map $t = h(s) = s^2$. Then

$$\frac{dM_{s^2}}{ds} = \frac{dt}{ds} \frac{dM_t}{dt} = 2s \frac{dM_t}{dt} \equiv 0,$$

i.e. $\rho_s \equiv 0$. But $\pi^{-1}(0) = M_0$ and $\pi^{-1}(s) = M_{s^2}$ with $s \neq 0$ are not isomorphic as complex manifolds. We see that though $\rho_s \equiv 0$, $(\mathscr{N}, \mathbb{C}, \pi)$ is not locally trivial. And by an explicit computation, see(), we have

$$\dim H^1(M_t, TM_t) = \begin{cases} 4, & t = 0 \\ 2, & t \neq 0 \end{cases}.$$

4 Obstructions

In this section, we always assume that $(\mathscr{M}, B, \varpi)$ with $0 \in B \subseteq \mathbb{C}$ is a complex analytic family, such that $\varpi^{-1}(0) = M$, $\varpi^{-1}(t) = M_t$ and $\frac{dM_t}{dt}|_{t=0} \in H^1(M, TM)$. Given $\theta \in H^1(M, TM)$, our question is whether there is a complex analytic family $(\mathscr{M}, B, \varpi)$ such that $\frac{dM_t}{dt}|_{t=0} = \theta$.

As before, we can take a small disk D such that $0 \in D \subseteq B$, and represent $\mathscr{M}_D = \varpi^{-1}(D)$ with the form $\mathscr{M}_D = \cup_{j=1}^{l} U_j \times D$. By identifying $U_j \times 0$ with $U_j \subseteq \mathbb{C}^n$, let $\mathfrak{U} = \{U_j\}$ be an open cover of M.

On $U_i \cap U_j \neq \emptyset$, we have the vector field

$$\theta_{jk}(t) = \sum_{\alpha} \theta_{jk}^\alpha(z_j, t) \frac{\partial}{\partial z_j^\alpha},$$

where $\theta^\alpha_{jk}(z_j,t)=\frac{\partial f^\alpha_{jk}}{\partial t}(z_k,t), z_k=f_{kj}(z_j,t)$. We write $(\frac{\partial}{\partial t})_j$ to denote the differentiation of a function of $z^1_j,\dots,z^n_j,t$ with respect to t. Then:

$$\left(\frac{\partial}{\partial t}\right)_j=\sum_{\beta=1}\left(\frac{\partial z^\beta_i}{\partial t}\right)_j\frac{\partial}{\partial z^\beta_i}+\left(\frac{\partial}{\partial t}\right)_i \quad \text{where} \left(\frac{\partial z^\beta_i}{\partial t}\right)_j=\frac{\partial f^\beta_{ij}(z_j,t)}{\partial t}=\theta^\beta_{ij}(z_i,t).$$

On $U_i\cap U_j\cap U_k\neq\emptyset$, we already have

$$\theta_{ij}(t)-\theta_{ik}(t)+\theta_{jk}(t)=0.$$

Spelling out componentwise, this is equivalent to

$$\theta^\alpha_{ij}(z_i,t)-\theta^\alpha_{ik}(z_i,t)+\sum_{\beta=1}^n\frac{\partial z^\alpha_i}{\partial z^\beta_j}\theta^\beta_{jk}(z_j,t)=0,\quad \text{where}\quad \alpha=1,\dots,n. \tag{4}$$

Differentiate both two sides by $(\frac{\partial}{\partial t})_j$, and let $\dot\theta_{ik}(t)=\sum_\alpha\frac{\partial\theta^\alpha_{ik}(z_i,t)}{\partial t}\frac{\partial}{\partial z^\alpha_i}$. We obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_j\theta^\alpha_{ik}(z_i,t)&=\sum_{\beta=1}^n\theta^\beta_{ij}(z_i,t)\frac{\partial}{\partial z^\beta_i}\theta^\alpha_{ik}(z_i,t)+\frac{\partial\theta^\alpha_{ik}(z_i,t)}{\partial t}\\&=\theta_{ij}(t)\cdot\theta^\alpha_{ik}(z_i,t)+\dot\theta_{ik}(z_i,t).\end{aligned}$$

Similarly,

$$\left(\frac{\partial}{\partial t}\right)_j\theta^\alpha_{ij}(z_i,t)=\theta_{ij}(t)\cdot\theta^\alpha_{ij}(z_i,t)+\dot\theta^\alpha_{ij}(z_i,t)$$

and

$$\left(\frac{\partial}{\partial t}\right)_j\sum_{\beta=1}^n\frac{\partial z^\alpha_i}{\partial z^\beta_j}\theta^\beta_{jk}(z_j,t)=\theta_{jk}(t)\cdot\theta^\alpha_{ik}(z_i,t)+\sum_\beta\frac{\partial z^\alpha_i}{\partial z^\beta_j}\dot\theta^\beta_{jk}(z_j,t).$$

Therefore, differentiating equation(4) with respect to t, we have

$$\theta_{ij}(t)\cdot\theta^\alpha_{ik}(z_i,t)+\dot\theta^\alpha_{ik}(z_i,t)=\theta_{ij}(t)\cdot\theta^\alpha_{ij}(z_i,t)+\dot\theta^\alpha_{ij}(z_i,t)+\theta_{jk}(t)\cdot\theta^\alpha_{ik}(z_i,t)+\sum_\beta\frac{\partial z^\alpha_i}{\partial z^\beta_j}\dot\theta^\beta_{jk}(z_j,t).$$

Again by equation(4), this is equivalent to:

$$\dot\theta^\alpha_{ij}(z_i,t)-\dot\theta^\alpha_{ik}(z_i,t)+\sum_\beta\frac{\partial z^\alpha_i}{\partial z^\beta_j}\dot\theta^\beta_{jk}(z_j,t)=\theta_{ij}(t)\cdot\sum_\beta\frac{\partial z^\alpha_i}{\partial z^\beta_j}\theta^\beta_{jk}(z_j,t)-\theta_{jk}(t)\cdot\theta^\alpha_{ij}(z_i,t).$$

Multiply the above formula by $\frac{\partial}{\partial z^\alpha_i}$ and take summation over α. Then using Lie brackets we finally obtain:

$$\delta\{\dot\theta_{ij}(t)\}=\dot\theta_{ij}(t)-\dot\theta_{ik}(t)+\dot\theta_{jk}(t)=[\theta_{ij}(t),\theta_{jk}(t)]. \tag{5}$$

For convenient, we omit the variable t and let $\zeta_{ijk}:=[\theta_{ij},\theta_{jk}]$. Then $\{\zeta_{ijk}\}$ is a 2-cocycle, i.e. $\{\zeta_{ijk}\}\in Z^1(\mathfrak{U}_t,TM_t)$, and therefore defines a cohomology class of $\zeta\in H^1(\mathfrak{U}_t,TM_t)$.

ζ is uniquely determined by the cohomology classes θ of $\{\theta_{ij}\}$. In fact, if $\theta_{ij}=\theta_j-\theta_i$, then

$$\zeta_{ijk}=[\theta_j-\theta_i,\theta_k-\theta_j]=[\theta_j,\theta_k]-[\theta_i,\theta_k]+[\theta_i,\theta_j]=\delta\{[\theta_i,\theta_j]\}.$$

Thus we can write $\zeta=[\theta,\theta]$. Then the equation (5) is equivalent to $\zeta=[\theta,\theta]=0$. Let $t=0$, we obtain the following necessary condition:

Theorem 4.1. *Suppose that M is a given compact complex manifold and $\theta \in H^1(M, TM)$. If there exists a complex analytic family $(\mathscr{M}, B, \varpi)$ such that $\varpi^{-1}(0) = M$ and $\frac{dM_t}{dt}|_{t=0} = \theta$, then $[\theta, \theta] = 0$.*

On the other hand, if $[\theta, \theta] \neq 0$, there exists no such a deformation M_t with $M_0 = M$ and $\frac{dM_t}{dt}|_{t=0}$. In this sense, we call $[\theta, \theta] \in H^2(M, TM)$ the *obstruction* to deformation of M.

5 The Deformation of Almost Complex Structures

In previous sections, the deformation is considered as a complex manifold fibred over a base space. Here we turn to another different angle.

Recall that an almost complex structure I over a differentiable manifold M is a vector bundle endomorphism $I : TM \to TM$ with $I^2 = -id$. (M, I) is called an almost complex manifold. I is called integrable if and only if $d\alpha = \partial\alpha + \bar{\partial}\alpha$ for all $\alpha \in \mathcal{A}^{\bullet}_{\mathbb{C}}(M)$. Clearly, if M is a complex manifold, then the complex structure of M induces an integrable almost complex structure. Conversely, there is a theorem proven by Newlander and Nierenberg, which says that any integrable almost complex structure is induced by exactly one complex structure. Thus there is a one-to-one correspondence between complex structures and integrable almost complex structures. Complex manifolds and differentiable manifolds endowed with an integrable complex structure are describing the same geometry object. Hence it makes sense to consider the deformation of (integrable) almost complex structures.

We say that (M, I) and (M', I') are isomorphic if there exists a diffeomorphism $F : M \to M'$ such that $dF \circ I = I' \circ dF$. Let $\mathcal{A}_c := \{I \mid I \text{ is an integrable almost complex structure on } M\}$, and let $\text{Diff}(M)$ be the diffeomorphism group of M. Then $\text{Diff}(M)$ acts on $\mathcal{A}_c(M)$ with the action given by

$$\text{Diff}(M) \times \mathcal{A}_c(M) \to \mathcal{A}_c(M), (F, I) \mapsto dF \circ I \circ (dF)^{-1}$$

Then the set of isomorphism classes of complex structures on M is the quotient $\mathcal{A}_c(M)/\text{Diff}(M)$. Thus to describe how a family of almost complex structures deform in the space consisting of all almost complex structures on M, we also need to divide out by the action of $\text{Diff}(M)$. We deal with these questions by means of an infiniesimal deformation of almost complex structures, which is studied by a power series expansion with the coeffients of it being under consideration.

First, let I be an arbitary almost complex structure on M, and $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ the decomposition corresponding to I. Suppose $\{I(t)\}$ is a continuous[1] one-parameter family of almost complex structures on M with $I(0) = I$. Then we have decompositions $T_{\mathbb{C}}M = T^{1,0}_tM \oplus T^{0,1}_tM$ for each $I(t)$. Here we require that t varies on an infinitesimal level, i.e., on a small enough open neigbourhood, such that $T^{0,1}_tM \subseteq T_{\mathbb{C}}M \to T^{0,1}M$ is an isomorphism. Then for small t, $T^{1,0}M \cap T^{0,1}_tM = 0$. Such a small open neighbourhood exists at least for a compact manifold[2]. Thus we always assume that M is compact.

With the above assumptions, the deformation $I(t)$ of I is totally described by a map

$$\phi(t) : T^{0,1}M \to T^{1,0}_tM, \quad \phi(t) = -\text{pr}_{T^{1,0}_tM} \circ j,$$

[1]The space consisting of all almost complex structures on M is a nice space, see(), so we can talk about the continuity of the map $t \to I(t)$.

[2]see().

where $\mathrm{pr}_{T_t^{1,0}M} : T_{\mathbb{C}}M \to T_t^{1,0}M$ is the projection and $j : T^{0,1}M \to T_{\mathbb{C}}M$ is the inclusion. Then $\phi(0) = 0$ and $(\mathrm{id} + \phi(t))(T^{0,1}M) = T_t^{0,1}M$.

Now we consider the power series expansion of $\phi(t)$:

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$$

Then $\phi_0 = 0$, and since TX is isomorphic to $T^{1,0}M$ as complex vector bundles, the higher order coefficients $\phi_{i>0}$ are viewed as elements in $\mathcal{A}^{0,1}(TX)$.

Next the integrability condition is put into the deformation $I(t)$. Assume that I is integrable and (M, I) becomes a complex manifold. Let $X = (M, I)$ and its holomorphic tangent bundle is denoted by TX. We also want that $I(t)$ is integrable. The integrability condition for $I(t)$, $[T_t^{0,1}M, T_t^{0,1}M] \subseteq T_t^{0,1}M$ (see), can be rephrased in terms of $\phi(t)$.

We define a "Lie bracket"

$$[\quad,\quad] : \mathcal{A}^{0,p}(TX) \times \mathcal{A}^{0,q}(TX) \to \mathcal{A}^{0,p+q}(TX)$$

by takeing the usual Lie bracket in TX and the exterior product in the form part. In local coordinates, this means for $\alpha_1 = \sum d\bar{z}^I \otimes \xi_{Ik}\frac{\partial}{\partial z^k} \in \mathcal{A}^{0,p}(TX)$ and $\alpha_2 = \sum d\bar{z}^J \otimes \eta_{Jl}\frac{\partial}{\partial z^l} \in \mathcal{A}^{0,q}(TX)$,

$$[\alpha_1, \alpha_2] = \sum_{I,J,k,l} (d\bar{z}^I \wedge d\bar{z}^J) \otimes \left[\xi_{Ik}\frac{\partial}{\partial z^k}, \eta_{Jl}\frac{\partial}{\partial z^l}\right].$$

This is well-defined. Note that we consider $\phi(t) \in \mathcal{A}^{0,1}(TX)$. The operator $\bar{\partial} : \mathcal{A}^{0,1}(TX) \to \mathcal{A}^{0,2}(TX)$ acts on $\phi(t)$.

Lemma 5.1. *The integrability condition $[T_t^{0,1}M, T_t^{0,1}M] \subseteq T_t^{0,1}M$ is equivalent to the* Maurer-Cartan equation

$$\bar{\partial}\phi(t) + [\phi(t), \phi(t)] = 0. \tag{6}$$

Proof. This can be proved by using local coordinates. Write $\phi = \phi(t) = \sum \phi_{ij} d\bar{z}^i \otimes \frac{\partial}{\partial z^j}$. Since $(\mathrm{id} + \phi(t))(T^{0,1}M) = T_t^{0,1}M$, the integrability condition is equivalent to

$$\left[\frac{\partial}{\partial \bar{z}^i} + \phi(\frac{\partial}{\partial \bar{z}^i}), \frac{\partial}{\partial \bar{z}^k} + \phi(\frac{\partial}{\partial \bar{z}^k})\right] \in T_t^{0,1}M.$$

Spell it out and use $[\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k}] = 0$, we have

$$\sum_j \left[\frac{\partial}{\partial \bar{z}^i}, \phi_{kj}\frac{\partial}{\partial z^j}\right] + \sum_j \left[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right] + \sum_{j,l} \left[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\right] \in T_t^{0,1}M.$$

The first two terms can be written as:

$$\begin{aligned}
&\sum_j \left[\frac{\partial}{\partial \bar{z}^i}, \phi_{kj}\frac{\partial}{\partial z^j}\right] + \sum_j \left[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right] \\
=&\sum_j \frac{\partial \phi_{kj}}{\partial \bar{z}^i}\frac{\partial}{\partial z^j} + \sum_j \left(-\frac{\partial \phi_{ij}}{\partial \bar{z}^k}\frac{\partial}{\partial z^j}\right) = \sum_j \left(\frac{\partial \phi_{kj}}{\partial \bar{z}^i} - \frac{\partial \phi_{ij}}{\partial \bar{z}^k}\right)\frac{\partial}{\partial z^j} \\
=&\left(\sum_{\alpha,\beta,\gamma}\left(\frac{\partial \phi_{\alpha\beta}}{\partial \bar{z}^\gamma} d\bar{z}^\gamma \wedge d\bar{z}^\alpha\right) \otimes \frac{\partial}{\partial z^\beta}\right)\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k}\right) = (\bar{\partial}\phi)\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k}\right).
\end{aligned}$$

The last term:

$$\begin{aligned}
&\sum_{j,l}\left[\phi_{ij}\frac{\partial}{\partial z^j},\phi_{kl}\frac{\partial}{\partial z^l}\right]\\
=&\sum_{\alpha,\beta,j,l}\left(d\bar z^\alpha\wedge d\bar z^\beta\right)\otimes\left[\phi_{\alpha j}\frac{\partial}{\partial z^j},\phi_{\beta l}\frac{\partial}{\partial z^l}\right]\left(\frac{\partial}{\partial\bar z^i},\frac{\partial}{\partial\bar z^k}\right)\\
=&[\phi,\phi]\left(\frac{\partial}{\partial\bar z^i},\frac{\partial}{\partial\bar z^k}\right).
\end{aligned}$$

Thus the integrability condition is equivalent to

$$(\bar\partial\phi+[\phi,\phi])\left(\frac{\partial}{\partial\bar z^i},\frac{\partial}{\partial\bar z^k}\right)\in T_t^{0,1}M.$$

Hence $(\bar\partial\phi+[\phi,\phi])\in\mathcal{A}^{0,2}(T^{1,0}M\cap T_t^{0,1}M)$. But for small t, $T^{1,0}M\cap T_t^{0,1}M=0$. Therefore $\bar\partial\phi+[\phi,\phi]=0$.

□

Replace ϕ in (6) by the power series and obtain a system of equations:

$$\begin{aligned}
0&=\bar\partial\phi_1\\
0&=\bar\partial\phi_2+[\phi_1,\phi_1]\\
&\vdots\\
0&=\bar\partial\phi_k+\sum_{0<i<k}[\phi_i,\phi_{k-i}].
\end{aligned}\tag{7}$$

The first equation $\bar\partial\phi_1=0$ implies that $\phi_1\in\mathcal{A}^{0,1}(TX)$ is a cloed form and thus defines an element $[\phi_1]\in H^1(X,TX)$. We call $[\phi_1]\in H^1(X,TX)$ the *Kodaira-Spencer class* of the deformation $\{I(t)\}$.

Remark: In order to identify isomorphic (first order) deformations, the actions of Diff(M) on an infinitesimal level have to be taken into account. In fact, let $\{F_t\}$ be a family of diffeomorphisms of M. Suppose that $I_t=dF_t\circ I\circ(dF_t)^{-1}$ are the induced almost complex structures by F_t. Then the first order deformation ϕ_1 of $\{I_t\}$ is $\bar\partial$-exact, see(), i.e. $[\phi_1]=0\in H^1(X,TX)$. This implies that the deformation induced by the action of Diff(M) is indeed an isomorphic deformation. Hence we obtain the following proposition:

Proposition 5.1. *There is a one-to-one correspondence between all first order deformations of X and elements of $H^1(X,TX)$.*

In deformation theory, our main task is to "integrate" the given first order deformation $v\in H^1(X,TX)$, i.e. to find $\{I(t)\}$ such that its Kodaira-Spencer class is v.

In general this is not possible. For example, the obstructions can occur at order two: No matter how we choose ϕ_1 with $[\phi_1]=v\in H^1(X,TX)$, $[\phi_1,\phi_1]$ may not be $\bar\partial-$exact, i.e. we can not find ϕ_2 such that $\bar\partial\phi_2=-[\phi_1,\phi_1]$.

Note that $\bar\partial[a_1,a_2]=[\bar\partial a_1,a_2]+(-1)^p[a_1,\bar\partial a_2]$ for all $a_1\in\mathcal{A}^{0,p}(TX)$, $a_2\in\mathcal{A}^{0,q}(TX)$, which can be checked by using local coordinates. Thus the bracket induces a map $H^p(X,TX)\times H^q(X,TX)\to H^{p+q}(X,TX)$.

Take $p=q=1$, and therefore we get a necessary condition to integrate $v\in H^1(X,TX)$:

Corollary 5.1. *A first-order deformation $v \in H^1(X, TX)$ cannot be integrated if $[v, v] \in H^2(X, TX)$ does not vanish.*

In some special cases, we can integrate $v \in H^1(X, TX)$ if the underlying manifold X is nice. As an example, we then show how to solve the recursive system of equation (7) when X is a Calabi-Yau manifold, which by definition is a compact Kähler manifold of dimension n with trivial canonical bundle $\bigwedge^n \Omega_X \cong \mathcal{O}_X$. Then X admits a holomorphic volume form, i.e. a nowhere nondegenerate trivializing section $\Omega \in H^0(X, \bigwedge^n \Omega_X)$.

Define a map:

$$\eta : \bigwedge^p TX \to \Omega_X^{n-p}, \eta(v_1 \wedge \ldots \wedge v_p) = i_{v_1} \ldots i_{v_p}(\Omega),$$

where i_v is the contraction: $i_v(\alpha)(w_1, \ldots, w_k) = \alpha(v, w_1, \ldots, w_k)$. Suppose locally $\Omega = f dz^1 \wedge \ldots \wedge dz^n$, then

$$\eta\left(g\frac{\partial}{\partial z^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial z^{i_p}}\right) = (-1)^{(\sum_\alpha i_\alpha)-p} fg dz^1 \wedge \ldots \wedge \widehat{dz^{i_1}} \wedge \ldots \wedge \widehat{dz^{i_p}} \wedge \ldots \wedge dz^n$$

Then η is an isomorphism since f is invertible. And thus η induces an isomorphism: $\mathcal{A}^{0,q}(\bigwedge^p TX) \xrightarrow{\cong} \mathcal{A}^{n-p,q}(X)$ by

$$d\bar{z}^{i_1} \wedge \ldots \wedge d\bar{z}^{i_q} \otimes (v_1 \wedge \ldots \wedge v_p) \mapsto d\bar{z}^{i_1} \wedge \ldots \wedge d\bar{z}^{i_q} \otimes (\eta(v_1 \wedge \ldots \wedge v_p)).$$

We shall denote this isomorphism still by $\eta : \mathcal{A}^{0,q}(\bigwedge^p TX) \xrightarrow{\cong} \mathcal{A}^{n-p,q}(X)$.

Definition 5.1. *The operator $\Delta : \mathcal{A}^{0,q}(\wedge^p TX) \to \mathcal{A}^{0,q}(\wedge^{p-1} TX)$ is given by the composition of*

$$\mathcal{A}^{0,q}(\wedge^p TX) \xrightarrow{\eta} \mathcal{A}^{n-p,q}(X) \xrightarrow{\partial} \mathcal{A}^{n-p+1,q}(X) \xrightarrow{\eta^{-1}} \mathcal{A}^{0,q}(\wedge^{p-1} TX)$$

i.e. $\Delta := \eta^{-1} \circ \partial \circ \eta$.

Lemma 5.2. $\Delta \circ \bar{\partial} = -\bar{\partial} \circ \Delta$.

Proof. Suppose $\Omega = f dz^1 \wedge \ldots \wedge dz^n$ locally, where f is holomorphic. Let I, J be multi-indices. Then

$$\begin{aligned} \bar{\partial}(\eta(g d\bar{z}^I \otimes \frac{\partial}{\partial z^J})) &= \pm \bar{\partial}(f g d\bar{z}^I) \otimes dz^{\{1,\ldots,n\}-J} \\ = \pm f \cdot \bar{\partial}(g d\bar{z}^I) \otimes dz^{\{1,\ldots,n\}-J} &= \eta(\bar{\partial}(g d\bar{z}^I \otimes \frac{\partial}{\partial z^J}), \end{aligned}$$

i.e. $\bar{\partial} \circ \eta = \eta \circ \bar{\partial}$, here we use the fact that $\bar{\partial} f = 0$. Thus,

$$\begin{aligned} \bar{\partial}(\Delta(\alpha)) &= \bar{\partial}\eta^{-1}\partial\eta(\alpha) = \eta^{-1}(\bar{\partial}\partial\eta)(\alpha) \\ &= -\eta^{-1}(\partial\bar{\partial}\eta)(\alpha) = -\eta^{-1}(\partial\eta(\bar{\partial}(\alpha))) \\ &= -\Delta(\bar{\partial}(\alpha)) \end{aligned}$$

□

Lemma 5.3. (Tian-Todorov lemma) *If $\alpha \in \mathcal{A}^{0,p}(TX)$ and $\beta \in \mathcal{A}^{0,q}(TX)$, then*

$$(-1)^p[\alpha, \beta] = \Delta(\alpha \wedge \beta) - \Delta(\alpha) \wedge \beta - (-1)^{p+1}\alpha \wedge \Delta(\beta).$$

For proof, see()

Corollary 5.2. *Suppose that for $i = 1, 2$, $\alpha_i \in \mathcal{A}^{0,q_i}(TX)$, $\eta(\alpha_i) \in \mathcal{A}^{n-1,q_i}(X)$ is ∂-closed. Then $\eta[\alpha_1, \alpha_2]$ is ∂-exact. More precisely, $\eta[\alpha_1, \alpha_2] = (-1)^{q_1}\partial\eta(\alpha_1 \wedge \alpha_2)$.*

Proof. First, $\Delta(\alpha_i) = \eta^{-1}\partial(\eta(\alpha_i)) = 0$. Then by Tian-Todorov lemma, $(-1)^{q_1}[\alpha_1, \alpha_2] = \Delta(\alpha_1 \wedge \alpha_2)$. Acting on both two sides with η, we get $(-1)^{q_1}\eta[\alpha_1, \alpha_2] = \eta\Delta(\alpha_1 \wedge \alpha_2) = \partial\eta(\alpha_1 \wedge \alpha_2)$. □

Using the *Hodge decompositions*

$$\mathcal{A}^{0,1}(TX) = \bar{\partial}\mathcal{A}^0(TX) \oplus \mathcal{H}^{0,1}(TX) \oplus \bar{\partial}^*\mathcal{A}^{0,2}(TX),$$
$$\mathcal{A}^{n-1,1}(X) = \bar{\partial}\mathcal{A}^{n-1,0}(X) \oplus \mathcal{H}^{n-1,1}(X) \oplus \bar{\partial}^*\mathcal{A}^{n-1,2}(X),$$

with respect to a Kähler metric on X and the above results, we finally obtain

Theorem 5.1. *Let X be a Calabi-Yau manifold and $v \in H^1(X, TX)$. Then there exists a formal power series $\phi_1 t + \phi_2 t^2 + \dots$ with $\phi_i \in \mathcal{A}^{0,1}(TX)$ satisfying the Maurer-Cartan equations*

$$\bar{\partial}\phi_1 = 0, \quad \bar{\partial}\phi_k = -\sum_{0<i<k} [\phi_i, \phi_{k-i}],$$

such that $[\phi_1] = v$ and $\eta(\phi_i) \in \mathcal{A}^{n-1,1}(X)$ is ∂-exact for all $i > 1$.

Proof. (i) First we choose a suitable $\phi_1 \in \mathcal{A}^{0,1}(TX)$ representing $v \in H^1(X, TX)$ such that ϕ_2 is what we want.

Since $\bar{\partial} \circ \eta = \eta \circ \bar{\partial}$, $\eta : \mathcal{A}^{0,1}(TX) \to \mathcal{A}^{n-1,1}(X)$ take $\bar{\partial}$-closed forms to $\bar{\partial}$-closed forms, $\bar{\partial}$-exact forms to $\bar{\partial}$-exact forms. Then by the above Hodge decompositions, we can find a lift ϕ_1 of v such that $\eta(\phi_1) \in \mathcal{A}^{n-1,1}(X)$ is harmonic. Of course $\eta(\phi_1)$ is ∂-closed and, a priori, $\bar{\partial}$-closed. Thus $\eta[\phi_1, \phi_1]$ is $\bar{\partial}$-closed, and according to the corollary (5.2), we see that $\eta[\phi_1, \phi_1]$ is ∂-exact.

Applying the $\partial\bar{\partial}$-lemma (see), which says that on a compact Kähler manifold if a d-closed (p,q)-form is ∂-exact, then it is also $\partial\bar{\partial}$-exact, we find $\gamma \in \mathcal{A}^{n-2,1}(X)$ with $-\eta[\phi_1, \phi_1] = \bar{\partial}\partial\gamma$. Put $\phi_2 := \eta^{-1}(\partial\gamma)$. Then $\eta(\phi_2) = \partial\gamma$ and

$$\bar{\partial}\phi_2 = \eta^{-1}\bar{\partial}\partial(\gamma) = -\eta^{-1}(\eta[\phi_1, \phi_1]) = -[\phi_1, \phi_1].$$

(ii) Now suppose that $\phi_2, \dots, \phi_{k-1} (k > 2)$ have been found as claimed by the theorem, we want to construct ϕ_k.

Again, by Corollary (5.2), $\eta[\phi_i, \phi_{k-i}]$ is ∂-exact for $0 < i < k$ and thus $\sum_{0<i<k} \eta[\phi_i, \phi_{k-i}]$ is ∂-exact.

We need to check that $\sum_{0<i<k} \eta[\phi_i, \phi_{k-i}]$ is also $\bar{\partial}$-closed. One has

$$\begin{aligned}
\bar{\partial}(\sum_{0<i<k} \eta[\phi_i, \phi_{k-i}]) &= \sum_{0<i<k} ([\bar{\partial}\phi_i, \phi_{k-i}] - [\phi_i, \bar{\partial}\phi_{k-i}]) \\
&= -\sum_{0<i<k} \left(\sum_{0<j<i} [[\phi_j, \phi_{i-j}], \phi_{k-i}] - \sum_{0<l<k-i} [\phi_i, [\phi_l, \phi_{k-i-l}]] \right) \\
&= -\sum_{0<i<k} \sum_{0<j<i} [[\phi_j, \phi_{i-j}], \phi_{k-i}] + \sum_{0<i<k} \sum_{0<l<i} [\phi_{k-i}, [\phi_l, \phi_{i-l}]] = 0.
\end{aligned}$$

Note that we use the fact that for $\alpha \in \mathcal{A}^{0,p}(TX), \beta \in \mathcal{A}^{0,q}(TX)$, $[\alpha, \beta] = (-1)^{pq+1}[\beta, \alpha]$. Thus $\sum_{0<i<k} \eta[\phi_i, \phi_{k-i}]$ is indeed $\bar{\partial}$-closed.

Similar to part (i), by $\partial\bar{\partial}$-lemma, we can choose $\gamma' \in \mathcal{A}^{n-2,1}(X)$ such that

$$\bar{\partial}\partial\gamma' = -\sum_{0<i<k} \eta[\phi_i, \phi_{k-i}].$$

Therefore, we choose $\phi_k = \eta^{-1}(\partial\gamma')$.

$\square$

Remark: So far we have not said anything about the convergence of the solution. There is a standard procedure to turn any formal solution into a convergent one. But this is far beyond this article. For the details we have to refer back to

6 Conclusions and Further Development

Often our notion of families and deformation used so far is not flexible enough. [1]

References

[1] X.-C. Yin, X. Yin, K. Huang, and H.-W. Hao, "Robust text detection in natural scene images," *IEEE transactions on pattern analysis and machine intelligence*, vol. 36, no. 5, pp. 970–983, 2014.