Anisotropic Regularity of the Free-Boundary Problem in Compressible Ideal Magnetohydrodynamics

HANS LINDBLAD* JUNYAN ZHANG†

June 23, 2021

Abstract

We consider 3D free-boundary compressible ideal magnetohydrodynamic (MHD) system under the Rayleigh-Taylor sign condition. It describes the motion of a free-surface perfect conducting fluid in an electro-magnetic field. The local well-posedness was recently proved by Trakhinin and Wang [66] by using Nash-Moser iteration. In this paper, we prove the a priori estimates without loss of regularity for the free-boundary compressible MHD system in Lagrangian coordinates in anisotropic Sobolev space, with more regularity tangential to the boundary than in the normal direction. It is based on modified Alinhac good unknowns, which take into account the covariance under the change of coordinates to avoid the derivative loss; full utilization of the cancellation structures of MHD system, to turn normal derivatives into tangential ones; and delicate analysis in anisotropic Sobolev spaces. Our method is also completely applicable to compressible Euler equations and thus yields an alternative estimate for compressible Euler equations without the analysis of div-curl decomposition or the wave equation in Lindblad-Luo [42], that do not work for compressible MHD. To the best of our knowledge, we establish the first result on the energy estimates without loss of regularity for the free-boundary problem of compressible ideal MHD.

Contents

*Johns Hopkins University, 3400 N Charles St, Baltimore, MD 21218, USA. Email: lindblad@math.jhu.edu
†Johns Hopkins University, 3400 N Charles St, Baltimore, MD 21218, USA. Email: zhang.junyan@jhu.edu

1 Introduction

In this paper, we consider the 3D compressible ideal magnetohydrodynamics (MHD) equations

$$\begin{cases} \rho D_t u = B \cdot \nabla B - \nabla P, \ \ P := p + \frac{1}{2}|B|^2 & \text{in } \mathcal{D}; \\ D_t \rho + \rho(\nabla \cdot u) = 0 & \text{in } \mathcal{D}; \\ D_t B = B \cdot \nabla u - B(\nabla \cdot u), & \text{in } \mathcal{D}; \\ \nabla \cdot B = 0 & \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

describing the motion of a compressible conducting fluid in an electro-magnetic field. Here $\mathcal{D} := \bigcup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$ and $\mathcal{D}_t \subset \mathbb{R}^3$ is the domain occupied by the conducting fluid whose boundary $\partial\mathcal{D}_t$ moves with the velocity of the fluid. $\nabla := (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ is the standard spatial derivative and div $X := \partial_{x_i} X^i$ is the standard divergence for any vector field X. $D_t := \partial_t + u \cdot \nabla$ is the material derivative. Throughout this paper, $X^i = \delta^{li} X_l$ for any vector field X, i.e., we use Einstein summation convention. The fluid velocity $u = (u_1, u_2, u_3)$, the magnetic field $B = (B_1, B_2, B_3)$, the fluid density ρ, the fluid pressure p and the domain $\mathcal{D} \subseteq [0, T] \times \mathbb{R}^3$ are to be determined. Here we consider the isentropic case, and thus the fluid pressure $p = p(\rho)$ should be a given strictly increasing smooth function of the density ρ.

1.1 Initial and boundary conditions and constraints

The boundary conditions of (1.1) are

$$\begin{cases} D_t|_{\partial\mathcal{D}} \in \mathcal{T}(\partial\mathcal{D}) \\ P = 0 & \text{on } \partial\mathcal{D}, \\ B \cdot n = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.2}$$

where $\mathcal{T}(\partial\mathcal{D})$ denotes the tangent bundle of $\partial\mathcal{D}$ and n denotes the unit exterior normal vector to $\partial\mathcal{D}_t$. The first condition in (1.2) means that the boundary moves with the velocity of the fluid. It can be equivalently rewritten as "$V(\partial\mathcal{D}_t) = u \cdot n$ on $\partial\mathcal{D}$" or "$(1, u)$ is tangent to $\partial\mathcal{D}$". The second condition in (1.2) means that outside the fluid region $\mathcal{D}_t$ is the vacuum. The third boundary condition $B \cdot n = 0$ shows that the fluid is a perfect conductor.

Remark. The conditions $\nabla \cdot B = 0$ in $\mathcal{D}$ and $B \cdot n = 0$ on $\partial\mathcal{D}$ are both constraints only for initial data so that the system is not over-determined. They can propagate to any time $t > 0$ if initially hold. See Hao-Luo [28] for details.

We consider the Cauchy problem of (1.1): Given a simply-connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^3$ and the initial data u_0, ρ_0 and B_0 satisfying the constraints $\nabla \cdot B_0 = 0$ in $\mathcal{D}_0$ and $(B_0 \cdot n)|_{\{0\}\times\partial\mathcal{D}_0} = 0$, we want to find a set $\mathcal{D}$, the vector field u, the magnetic field B, and the density ρ solving (1.1) satisfying the boundary conditions (1.2) and the initial data

$$\mathcal{D}_0 = \{x : (0, x) \in \mathcal{D}\}, \quad (u, B, \rho) = (u_0, B_0, \rho_0), \quad \text{in } \{0\} \times \mathcal{D}_0, \tag{1.3}$$

Energy conservation law The free-boundary compressible MHD system (1.1) together with the boundary conditions (1.2) satisfies the following energy conservation law. Let $Q(\rho) = \int_1^\rho p(R)/R^2 dR$, then we use (1.1) to get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}_t}\rho|u|^2\,dx + \frac{1}{2}\int_{\mathcal{D}_t}|B|^2\,dx + \int_{\mathcal{D}_t}\rho Q(\rho)\,dx\right) \\ =& \int_{\mathcal{D}_t}\rho u\cdot D_t u\,dx + \int_{\mathcal{D}_t} B\cdot D_t B\,dx + \int_{\mathcal{D}_t}\rho D_t Q(\rho)\,dx + \frac{1}{2}\int_{\mathcal{D}_t}\rho D_t(1/\rho)|B|^2\,dx \\ =& \int_{\mathcal{D}_t} u\cdot(B\cdot\nabla B)\,dx - \int_{\mathcal{D}_t} u\cdot\nabla P\,dx + \int_{\mathcal{D}_t} B\cdot(B\cdot\nabla u)\,dx - \int_{\mathcal{D}_t}|B|^2(\nabla\cdot u)\,dx \\ &+ \int_{\mathcal{D}_t} p(\rho)\frac{D_t\rho}{\rho}\,dx - \frac{1}{2}\int_{\mathcal{D}_t}\frac{D_t\rho}{\rho}|B|^2\,dx. \end{aligned} \tag{1.4}$$

Integrating by part in the first term in the last equality, this term will cancel with $\int_{\mathcal{D}_t} B\cdot(B\cdot\partial u)\,dx$ because the boundary term and the other interior term vanish due to $B\cdot n|_{\partial\mathcal{D}_t} = 0$ and $\text{div}\, B = 0$. Also we integrate by parts in the second term and then use the continuity equation to get

$$\begin{aligned} -\int_{\mathcal{D}_t} u\cdot\nabla P\,dx =& \int_{\mathcal{D}_t} P(\nabla\cdot u)\,dx - \underbrace{\int_{\partial\mathcal{D}_t}(u\cdot N)P\,dS}_{=0} = -\int_{\mathcal{D}_t} p\frac{D_t\rho}{\rho}\,dx + \frac{1}{2}\int_{\mathcal{D}_t}|B|^2(\nabla\cdot u)\,dx \\ =& -\int_{\mathcal{D}_t} p\frac{D_t\rho}{\rho}\,dx + \int_{\mathcal{D}_t}|B|^2(\nabla\cdot u)\,dx - \frac{1}{2}\int_{\mathcal{D}_t}|B|^2(\nabla\cdot u)\,dx \\ =& -\int_{\mathcal{D}_t} p\frac{D_t\rho}{\rho}\,dx + \int_{\mathcal{D}_t}|B|^2(\nabla\cdot u)\,dx + \frac{1}{2}\int_{\mathcal{D}_t}\frac{D_t\rho}{\rho}|B|^2\,dx. \end{aligned} \tag{1.5}$$

where dS is the surface measure of $\partial\mathcal{D}_t$.

Summing up (1.4) and (1.5), one can get the energy conservation

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}_t}\rho|u|^2\,dx + \frac{1}{2}\int_{\mathcal{D}_t}|B|^2\,dx + \int_{\mathcal{D}_t}\rho Q(\rho)\,dx\right) = 0. \tag{1.6}$$

When $B = \mathbf{0}$, one can see such energy conservation exactly coincides with that of free-boundary compressible Euler equations established in Lindblad-Luo [42].

Equation of state: Isentropic liquid We assume the fluid considered in this paper is an isentropic liquid, i.e., there exists some constant $0 < \rho_1 < \rho_2$ such that $\rho_1 \le \rho \le \rho_2$ as opposed to a gas[1], and the fluid pressure $p = p(\rho)$ is an increasing smooth function of ρ. Next we impose the following natural conditions on $\rho'(p)$ for some fixed constant $A_0 > 1$. See also Lindblad-Luo [42].

$$A_0^{-1} \le |\rho^{(m)}(p)| \le A_0 \quad \text{for } 1 \le m \le 8. \tag{1.7}$$

Rayleigh-Taylor sign condition We also need to impose the Rayleigh-Taylor sign condition

$$-\nabla_n P \ge c_0 > 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.8}$$

where $c_0 > 0$ is a constant and $P := p + \frac{1}{2}|B|^2$ is the total pressure. When $B = \mathbf{0}$, Ebin [16] proved the ill-posedness of the free-boundary incompressible Euler equations when the Rayleigh-Taylor sign condition is violated. For the free-boundary MHD equations, (1.8) is also necessary: Hao-Luo [29] proved that the free-boundary problem of 2D incompressible MHD equations is ill-posed when (1.8) fails. We also note that (1.8) is only required for initial data and it propagates in a short time interval because one can prove it is $C_{t,x}^{0,\frac{1}{4}}$ Hölder continuous by using Morrey's embedding.

Compatibility conditions on initial data To make the initial-boundary value problem (1.1)-(1.3) be well-posed, the initial data has to satisfy certain compatibility conditions on the boundary. In fact, we need to require $P_0|_{\partial\mathcal{D}_0}$=0. Also the constraints on the magnetic field $\text{div}\, B = 0$ and $B\cdot n|_{\partial\mathcal{D}} = \mathbf{0}$ requires that $\nabla\cdot B_0 = 0$ and $B_0\cdot n|_{\{0\}\times\partial\mathcal{D}_0} = \mathbf{0}$. Furthermore, we define the k-th($k \ge 0$) order compatibility condition as follows:

$$D_t^j P|_{\partial\mathcal{D}_0} = 0 \ \text{ at time } t = 0 \ \ \forall 0 \le j \le k. \tag{1.9}$$

[1] In the case of a gas, the boundary condition should be $\rho = 0$.

1.2 History and background

1.2.1 Background in physics

The free-boundary problem (1.1)-(1.3) can be considered as the basic model of the plasma-vacuum free-interface problem which is important in the study of confined plasma both in laboratory and in astro-physical magnetohydrodynamics. The plasma is confined in a vacuum in which there is another magnetic field $\hat{B}$, and there is a free interface $\Gamma(t)$, moving with the motion of plasma, between the plasma region $\Omega_+(t)$ and the vacuum region $\Omega_-(t)$. This model requires that (1.1) holds in the plasma region $\Omega_+(t)$ and the pre-Maxwell system holds in vacuum $\Omega_-(t)$:

$$\nabla \times \hat{B} = \mathbf{0}, \quad \nabla \cdot \hat{B} = 0. \tag{1.10}$$

On the interface $\Gamma(t)$, it is required that there is no jump for the pressure or the normal components of the magnetic fields:

$$B \cdot n = \hat{B} \cdot n = 0, \quad P := p + \frac{1}{2}|B|^2 = \frac{1}{2}|\hat{B}|^2 \tag{1.11}$$

where n is the exterior unit normal to $\Gamma(t)$. Finally, there is a rigid wall W wrapping the vacuum region, on which the following boundary condition holds

$$\hat{B} \times \hat{n} = \mathbf{J} \quad \text{on } W,$$

where $\mathbf{J}$ is the given outer surface current density (as an external input of energy) and $\hat{n}$ is the exterior normal to the rigid wall W. Note that for ideal MHD, the conditions $\operatorname{div} B = 0$ and $B \cdot n = 0$ should also be constraints for initial data instead of imposed conditions. For details we refer to [20, Chapter 4, 6].

Hence, the free-boundary problem (1.1)-(1.3) can be considered as a special case of that plasma-vacuum model that the vacuum magnetic field $\hat{B}$ vanishes. It characterizes the motion of an isolated perfect conducting fluid in an electro-magnetic field.

1.2.2 An overview of previous results

In the past a few decades, there have been numerous studies of the free-boundary inviscid fluids. We start with incompressible Euler equations.

Free-boundary Euler equations The free-boundary Euler equations have been studied intensively by a lot of authors. The first breakthrough in solving the local well-posedness (LWP) for the incompressible irrotational problem for general initial data came in the work of Wu [70, 71] who proved the LWP of 2D and 3D full water wave system. Lannes [35] proved the LWP for finite-depth water wave problem. When the surface tension is not neglected, we refer to Iguchi [32], Ambrose-Masmoudi [3], Ming-Zhang [49] and references therein. In the case of nonzero vorticity, Christodoulou-Lindblad [8] first established the a priori estimates and then Lindblad [38, 39] proved the LWP by using Nash-Moser iteration. Coutand-Shkoller [12, 13] proved the LWP for incompressible Euler equations with or without surface tension and avoid the loss of regularity by introducing tangential smoothing method. We also refer to the related works Zhang-Zhang [75], Alazard-Burq-Zuily [1] for the case of zero surface tension and Shatah-Zeng [57, 58, 59] for nonzero surface tension.

The study of compressible perfect fluid is not quite developed as opposed to the incompressible case. Lindblad [40, 41] established the first LWP result by Nash-Moser iteration. Trakhinin [64] proved the LWP for the non-isentropic case by a hyperbolic approach and Nash-Moser iteration. Lindblad-Luo [42] established the first result of the a priori estimates and the incompressible limit. Then Luo [43] generalized [42] to compressible water wave with vorticity. Later, Ginsberg-Lindblad-Luo [19] proved the LWP for a self-gravitating liquid. Luo-Zhang [46] proved the LWP for a compressible gravity water wave with vorticity. In the case of nonzero surface tension, we refer to Coutand-Hole-Shkoller [10] for the LWP and the vanishing surface tension limit and Disconzi-Luo [15] for the incompressible limit. For the case of a gas, we refer to [11, 14, 33, 47, 26, 31] and references therein.

Free-boundary MHD equations: Incompressible case The study of free-boundary MHD is much more complicated than Euler equations due to the strong coupling between fluid and magnetic field and the failure of irrotational assumption. For the incompressible ideal free-boundary MHD under Rayleigh-Taylor sign condition, Hao-Luo [28] established the Christodoulou-Lindblad [8] type a priori estimates and Gu-Wang [24] proved the LWP. Hao-Luo [30] also proved the LWP for the linearized problem when the fluid region is diffeomorphic to a ball and of large curvature. Luo-Zhang [44] proved the low regularity a priori estimates when the fluid domain is small. We also mention that Lee [36, 37] obtained a local solution via the vanishing viscosity-resistivity limit.

For the full plasma-vacuum model, Gu [21, 22] proved the LWP for the axi-symmetric case under Rayleigh-Taylor sign condition. Hao [27] proved the LWP in the case of $\mathbf{J} = \mathbf{0}$. For the general case, all of the previous results require a non-collinearity condition $|B \times \hat{B}| \geq c_0 > 0$ on the free interface[2]. Under this condition, Morando-Trakhinin-Trebeschi [50] proved LWP for the linearized problem and then Sun-Wang-Zhang [61] proved the LWP for the full plasma-vacuum model. We also note that the study of the full plasma-vacuum model in ideal MHD under Rayleigh-Taylor sign condition is still an open problem when the vacuum magnetic field $\hat{B}$ is non-trivial with $\mathbf{J} \neq \mathbf{0}$. For the incompressible current-vortex sheets, we refer to Coulombel-Morando-Secchi-Trebeschi [9] for the a priori estimates and Sun-Wang-Zhang [60] for the LWP.

For incompressible ideal MHD with surface tension, Luo-Zhang [45] proved the a priori estimates and Gu-Luo-Zhang [23] proved the LWP. For incompressible dissipative MHD with surface tension, we refer to Chen-Ding [4] for the inviscid limit for viscous non-resistive MHD, Wang-Xin [69] for the global well-posedness of the plasma-vacuum model for inviscid resistive MHD around a uniform transversal magnetic field, and Padula-Solonnikov [53] and Guo-Zeng-Ni [25] for viscous and resistive MHD.

Free-boundary MHD equations: Compressible case Compared with compressible Euler equations and incompressible MHD, compressible MHD has an extra coupling between the pressure wave and the magnetic field which makes the analysis completely different. Here we emphasize that there is a normal derivative loss in the div-curl analysis of compressible MHD. On the one hand, the second author [73, 74] recently observed that the magnetic resistivity exactly compensates the derivative loss mentioned above. However, it is still hopeless to derive the vanishing resistivity limit. On the other hand, one can still expect to establish the tame estimates for the linearized equation. Based on this and Nash-Moser iteration, Trakhinin-Wang [66, 67] recently proved the LWP for free-boundary compressible ideal MHD with or without surface tension. We also mention that Chen-Wang [5] and Trakhinin [63] proved the LWP for the current-vortex sheets, and Secchi-Trakhinin [56] proved the LWP for the full plasma-vacuum problem for compressible ideal MHD under the non-collinearity condition. However, there is a big loss of regularity caused by the Nash-Moser iteration. Finding suitable estimates without loss of regularity is still a widely open problem.

In this paper, we prove the a priori estimates without loss of regularity for the free-boundary compressible ideal MHD system in the anisotropic Sobolev spaces. Our proof is based on the modified Alinhac good unknown method, full utilization of the cancellation structure of MHD system and very delicate analysis under the setting of anisotropic Sobolev spaces.

1.3 Reformulation in Lagrangian coordinates and main result

We use Lagrangian coordinates to reduce the free-boundary problem to a fixed-domain problem. We assume $\Omega := \mathbb{T}^2 \times (-1, 1)$ to be the reference domain and $\Gamma := \mathbb{T}^2 \times (\{-1\} \cup \{1\})$ to be the boundary. The coordinates on Ω is $y := (y', y_3) = (y_1, y_2, y_3)$. We define $\eta : [0, T] \times \Omega \to \mathcal{D}$ as the flow map of velocity field u, i.e.,

$$\partial_t \eta(t, y) = u(t, \eta(t, y)), \quad \eta(0, y) = \eta_0(y), \tag{1.12}$$

where η_0 is a diffeomorphism between Ω and $\mathcal{D}_0$. For technical simplicity we assume $\eta_0 = \mathrm{Id}$, i.e., the initial domain is assumed to be $\mathcal{D}_0 = \mathbb{T}^2 \times (-1, 1)$. By chain rule, it is easy to see that the material derivative D_t becomes ∂_t in the (t, y) coordinates and the free-boundary $\partial\mathcal{D}_t$ becomes fixed ($\Gamma = \mathbb{T}^2 \times (\{-1\} \cup \{1\})$). We introduce the Lagrangian variables as follow: $v(t, y) := u(t, \eta(t, y))$, $b(t, y) := B(t, \eta(t, y))$, $q(t, y) := p(t, \eta(t, y))$, $Q(t, y) := P(t, \eta(t, y))$ and $R(t, y) := \rho(t, \eta(t, y))$.

Let $\partial = \partial_y$ be the spatial derivative in Lagrangian coordinates and we define $\mathrm{div}\, Y = \partial_i Y^i$ to be the (Lagrangian) divergence of the vector field Y. We introduce the matrix $A = [\partial\eta]^{-1}$, specifically $A^{li} := \frac{\partial y^l}{\partial x^i}$ where $x^i = \eta^i(t, y)$ is the i-th variable in Eulerian coordinates. From now on, we define $\nabla_A^i = \frac{\partial}{\partial x^i} = A^{li}\partial_l$ to be the covariant derivative in Lagrangian coordinates (or say Eulerian derivative) and $\mathrm{div}_A X := \nabla_A \cdot X = A^{li}\partial_l X_i$ to be the Eulerian divergence mof the vector field X. In the manuscript, we adopt the convention that the Latin indices range over $1, 2, 3$. In addition, since $\eta(0, \cdot) = \mathrm{Id}$, we have $A(0, \cdot) = I$, where I is the identity matrix, and (u_0, B_0, p_0) and (v_0, b_0, q_0) agree respectively.

In terms of η, v, b, q, R, the system (1.1)-(1.8) becomes

[2]The non-collinearity condition enhaces extra 1/2-order regularity of the free-interface than Taylor sign condtion (1.8). Such condition originates from the stabilization condition for the current-vortex sheet model.

$$
\begin{cases}
\partial_t\eta = v & \text{in } [0,T]\times\Omega\\
R\partial_t v = (b\cdot\nabla_A)\,b - \nabla_A Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T]\times\Omega\\
\partial_t R + R\text{div}_A v = 0 & \text{in } [0,T]\times\Omega\\
q = q(R) & \text{in } [0,T]\times\bar{\Omega}\\
\partial_t b = (b\cdot\nabla_A)\,v - b\text{div}_A v & \text{in } [0,T]\times\Omega\\
\text{div}_A b = 0 & \text{in } [0,T]\times\Omega\\
Q = 0, \quad b_i A^{li} N_l = 0 & \text{on } [0,T]\times\Gamma\\
-\frac{\partial Q}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } \{t=0\}\\
(\eta, v, b, q, R)|_{t=0} = (\text{Id}, v_0, b_0, q_0, \rho_0).
\end{cases}
\tag{1.13}
$$

Here $N = (0,0,\pm 1)$ is the unit outer normal of the boundary $\mathbb{T}^2\times\{\pm 1\}$ and $q = q(R)$ is a strictly increasing function of R with $A_0^{-1} \leq q'(R) \leq A_0$ for some constant $A_0 > 1$.

Let $J := \det[\partial\eta]$ and $\hat{A} := JA$. Then we have the Piola's identity

$$\partial_l \hat{A}^{li} = 0, \tag{1.14}$$

and J satisfies

$$\partial_t J = J\text{div}_A v \tag{1.15}$$

which together with $\partial_t R + R\text{div}_A v = 0$ gives that $\rho_0 = RJ$.

Suppose D is the derivative ∂ or ∂_t, then we have the following identity

$$DA^{li} = -A^{lr}\,\partial_k D\eta_r\, A^{ki}. \tag{1.16}$$

Next we express the magnetic field b in terms of b_0 and η in the following Lemma. This is called the "frozen effect of the magnetic field".

Lemma 1.1. We have $b = J^{-1}(b_0\cdot\partial)\eta$.

Proof. Let us first compute the equation of b/R. We have

$$
\begin{aligned}
\partial_t\left(\frac{b}{R}\right) =& \frac{1}{R}\partial_t b + b\partial_t\left(\frac{1}{R}\right) = \frac{1}{R}\partial_t b + b\partial_t\left(\frac{J}{RJ}\right) = \frac{1}{R}\partial_t b + \frac{b}{\rho_0}\partial_t J\\
=& \frac{1}{R}\left((b\cdot\nabla_A)v - b\text{div}_A v\right) + \frac{b}{\rho_0}J\text{div}_A v = \frac{b}{R}\cdot\nabla_A v - \frac{b}{R}\text{div}_A v + \frac{b}{R}\text{div}_A v\\
=& \left(\frac{b}{R}\cdot\nabla_A\right)v.
\end{aligned}
$$

Therefore, invoking (1.16) we have

$$\partial_t\left(\frac{b_i}{R}A^{li}\right) = \partial_t\left(\frac{b_i}{R}\right)A^{li} + \frac{b_i}{R}\partial_t A^{li} = \frac{b_j}{R}\,A^{kj}\,\partial_k v_i\,A^{li} - \frac{b_i}{R}\,A^{lj}\,\partial_k v_j\,A^{ki} = 0,$$

which implies $\frac{b_i}{R}A^{li} = \frac{b_{0i}}{\rho_0}\delta^{li} = \frac{{b_0}^l}{\rho_0}$, i.e., $b_i A^{li} = \frac{{b_0}^l R}{\rho_0} = J^{-1}b_0^l$. Finally, the identity $A^{li}\partial_l\eta_i = 1$ gives us $b_i = J^{-1}b_0^l\partial_l\eta_i = J^{-1}(b_0\cdot\partial)\eta_i$. □

Inserting $\rho_0 = RJ$ and Lemma 1.1 into (1.13), we get the following system

$$
\begin{cases}
\partial_t\eta = v & \text{in } [0,T]\times\Omega\\
R\partial_t v = J^{-1}(b_0\cdot\partial)\left(J^{-1}(b_0\cdot\partial)\eta\right) - \nabla_A Q, \quad Q = q + \frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2 & \text{in } [0,T]\times\Omega\\
\frac{JR'(q)}{\rho_0}\partial_t q + \text{div}_A v = 0 & \text{in } [0,T]\times\Omega\\
q = q(R) \text{ strictly increasing} & \text{in } [0,T]\times\bar{\Omega}\\
\text{div}\, b_0 = 0 & \text{in } [0,T]\times\Omega\\
Q = 0, \quad b_0^3 = 0 & \text{on } [0,T]\times\Gamma\\
-\frac{\partial Q}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } \{t=0\}\\
(\eta, v, q, Q)|_{t=0} = (\text{Id}, v_0, q_0, Q_0), \quad Q_0 = q_0 + \frac{1}{2}|b_0|^2.
\end{cases}
\tag{1.17}
$$

Before stating our result, we should first define the anisotropic Sobolev space $H_*^m(\Omega)$ for $m \in \mathbb{N}^*$. Let $\sigma = \sigma(y_3)$ be a cutoff function on $[-1, 1]$ defined by $\sigma(y_3) = (1 - y_3)(1 + y_3)$. Then we define $H_*^m(\Omega)$ for $m \in \mathbb{N}^*$ as follows

$$H_*^m(\Omega) := \left\{ f \in L^2(\Omega) \middle| (\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3} f \in L^2(\Omega), \ \ \forall i_1 + i_2 + 2i_3 + i_4 \le m \right\},$$

equipped with the norm

$$\|f\|_{H_*^m(\Omega)}^2 := \sum_{i_1+i_2+2i_3+i_4 \le m} \|(\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3} f\|_{L^2(\Omega)}^2.$$

For any multi-index $I := (i_0, i_1, i_2, i_3, i_4) \in \mathbb{N}^5$, we define

$$\partial_*^I := \partial_t^{i_0}(\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3}, \ \ \langle I \rangle := i_0 + i_1 + i_2 + 2i_3 + i_4,$$

and define the **space-time anisotropic Sobolev norm** $\|\cdot\|_{m,*}$ by

$$\|f\|_{m,*}^2 := \sum_{\langle I \rangle \le m} \|\partial_*^I f\|_{L^2(\Omega)}^2 = \sum_{i_0 \le m} \|\partial_t^{i_0} f\|_{H_*^{m-i_0}(\Omega)}^2.$$

We define $f_{(j)} = \partial_t^j f|_{t=0}$ for $j \in \mathbb{N}$. The main results in this manuscript is the following theorem.

Theorem 1.2 (Energy estimates)**.** Let the initial data be $(v_0, b_0, Q_0) \in H_*^8(\Omega)$ such that

- $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H_*^{8-j}(\Omega)$ for $1 \le j \le 8$.
- The compatibility condition holds up to 7-th order, i.e., $Q_{(j)}|_\Gamma = 0$ for $0 \le j \le 7$.

Then there exists some $T_1 > 0$, such that the solution (η, v, Q) to the system (1.17) satisfies the following estimates in $[0, T_1]$

$$\sup_{0 \le t \le T_1} \mathcal{E}(T) \le C(\mathcal{E}(0)), \tag{1.18}$$

under the a priori assumptions on $[0, T_1]$

$$\|J - 1\|_{7,*} \le \frac{1}{4} \tag{1.19}$$

$$-\frac{\partial Q}{\partial N} \ge \frac{3}{4} c_0. \tag{1.20}$$

Here the energy functional $\mathcal{E}(t)$ is defined to be

$$\mathcal{E}(t) := \|\eta(t,\cdot)\|_{8,*}^2 + \|v(t,\cdot)\|_{8,*}^2 + \|J^{-1}(b_0 \cdot \partial)\eta(t,\cdot)\|_{8,*}^2 + \|Q(t,\cdot)\|_{8,*}^2 + \sum_{\langle I \rangle = 8} \left| A^{3i}\partial_*^I \eta_i \right|_0^2, \tag{1.21}$$

and $C(\mathcal{E}(0)) > 0$ denotes a positive constant depending on $\mathcal{E}(0)$.

Remark (On the existence of initial data satisfying the compatibility conditions)**.**
There exists initial data $(v_0, b_0, Q_0) \in H^8(\Omega) \hookrightarrow H_*^8(\Omega)$ satisfying the compatibility conditions up to 7-th order, such that

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{H^{8-j}(\Omega)} \lesssim P(\|v_0\|_{H^8(\Omega)}, \|b_0\|_{H^8(\Omega)}, \|Q_0\|_{H^8(\Omega)}). \tag{1.22}$$

In particular, by the Sobolev embedding $H^{8-j}(\Omega) \hookrightarrow H_*^{8-j}(\Omega)$ for $0 \le j \le 8$, we have

$$\mathcal{E}(0) \lesssim P(\|v_0\|_{H^8(\Omega)}, \|b_0\|_{H^8(\Omega)}, \|Q_0\|_{H^8(\Omega)}). \tag{1.23}$$

Due to the anisotropy of the function space, it is not possible to establish

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{H_*^{8-j}(\Omega)} \lesssim P(\|v_0\|_{H_*^8(\Omega)}, \|b_0\|_{H_*^8(\Omega)}, \|Q_0\|_{H_*^8(\Omega)}). \tag{1.24}$$

See Section 8 for detailed discussion.

1.4 Strategy of the proof

In this part, we introduce the basic strategies and techniques of our proof.

1.4.1 Choice of the function spaces

The compressible MHD system (1.1)-(1.3) is a hyperbolic system with charactersitic boundary conditions and violates the uniform Kreiss-Lopatinskiĭ condition [34]. This usually causes a loss of normal derivative. For certain types of such hyperbolic system, e.g., compressible Euler equations [42, 64], one can control the normal derivatives by the div-curl analysis so that the energy estimates and the LWP can be established in standard Sobolev spaces. However, such div-curl analysis is not applicable to compressible ideal MHD. In fact, taking curl eliminates the symmetry enjoyed by the equations, and there is also a derivative loss in the source term of the wave equation of pressure which is the key to the divergence estimates. For related details, we refer to [73, Section 1.5].

To compensate such derivative loss, Chen [6] first introduced the anisotropic Sobolev spaces H^m_* to study the hyperbolic system with characteristic boundary conditions. Then Yanagisawa-Matsumura [72] established the first LWP result for the fixed-domain problem of compressible ideal MHD in anisotropic Sobolev spaces. Later, [72] was improved by Secchi [54, 55] such that the regularity loss was avoided. On the other hand, Ohno-Shirota [51] constructed an explicit counterexample to show the ill-posedness for the linearized fixed-domain problem for compressible MHD in $H^l (l \geq 2)$.

Hence, the failure of div-curl analysis and the results of the fixed-domain problem [6, 72, 54, 51, 7] motivate us to study the free-boundary compressible ideal MHD system under the setting of anisotropic Sobolev spaces instead of standard Sobolev spaces. However, we emphasize that it is still difficult to directly generalize Secchi [54] to the free-boundary problem due to the following three reasons:

1. The regularity of the boundary is no longer C^∞ as in the case of fixed domain. In fact, the regularity of the free boundary enters to the highest order.
2. The regularity of the flow map is limited. After reducing the free-boundary problem to a fixed-domain problem, the commutator of the covariant derivative and the full derivative cannot be controlled directly.
3. The Eulerian normal velocity $u \cdot n$ does not vanish on the free boundary. However, $u \cdot n = 0$ plays an important role in the proof of [72, 54].

In fact, our analysis in the presenting manuscript is based on the modified Alinhac good unknown method, subtle cancellation structures of MHD system and the utilization of the anisotropy of the function space H^m_*. Here we also emphasize that our strategy is completely applicable to compressible Euler equations just by setting $b_0 = 0$. Our result also gives an alternative energy estimate for compressible Euler equations without the analysis of div-curl decomposition or the wave equation.

1.4.2 Motivation for introducing Alinhac good unknowns

Denote $\partial^I_* = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3}$ with $\langle I\rangle := i_0 + i_1 + i_2 + 2i_3 + i_4 = 8$. For simplicity of the notations, we use $\|\cdot\|_s, |\cdot|_s$ to represent the $H^s(\Omega)$ norm and the $H^s(\Gamma)$ norm respectively. Taking ∂^I_* in the second equation of (1.17), we get

$$\rho_0\partial_t\partial^I_* v = -J\partial^I_*(\nabla_A Q) + (b_0\cdot\partial)\partial^I_* b + [\partial^I_*, (b_0\cdot\partial)]b$$

In the energy estimates, we need to commute ∇_A with ∂^I_* and then integrate by parts. However, the commutator $[\partial^I_*, A^{li}]\partial_l f$ contains the following terms whose $L^2(\Omega)$-norms cannot be controlled in the anisotropic Sobolev space

- $(\partial^I_* A^{li})(\partial_l f)$, which cannot be controlled even in the standard Sobolev spaces when $i_0 = 0$;
- $(\partial^{I-I'}_* A^{li})(\partial^{I'}_*\partial_l f)$, when $l = 1, 2$ since A^{li} consists of $(\overline{\partial}\eta)(\partial_3\eta)$;
- $(\partial^{I'}_* A^{li})(\partial^{I-I'}_*\partial_l f)$, when $l = 3$,

where $f = Q$ or v_i and I' is a multi-index with $\langle I'\rangle = 1$. To overcome such difficulty, we can use the ideas of the Alinhac good unknown method, i.e., we can rewrite $\partial^I_*(\nabla_A Q)$ and $\partial^I_*(\nabla_A\cdot v)$ in terms of the sum of the covariant derivative part and the commutator part satisfying

$$\partial^I_*(\nabla_A Q) = \nabla_A \mathbf{Q} + C(Q), \text{ with } \|\mathbf{Q} - \partial^I_* Q\|_0 + \|\partial_t(\mathbf{Q} - \partial^I_* Q)\|_0 + \|C(Q)\|_0 \leq P(\mathcal{E}(t)), \tag{1.25}$$

$$\partial^I_*(\nabla_A\cdot v) = \nabla_A\cdot\mathbf{V} + C(v), \text{ with } \|\mathbf{V} - \partial^I_* v\|_0 + \|\partial_t(\mathbf{V} - \partial^I_* v)\|_0 + \|C(v)\|_0 \leq P(\mathcal{E}(t)). \tag{1.26}$$

Here $\mathbf{Q}, \mathbf{V}$ are called the "Alinhac good unknowns" of Q, v (The precise expressions will be determined later).

In other words, the above analysis shows that the essential highest order term in $\partial_*^I(\nabla_A f)$ is not the term got by simply commuting ∂_*^I with ∇_A. Instead, the essential highest order term in $\partial_*^I(\nabla_A f)$ is **exactly** the covariant derivative of the Alinhac good unknown of f, and the good unknowns $\mathbf{V}$ and $\mathbf{Q}$ are essentially formed by replacing the derivatives in the Lagragian coordinates ∂_*^I by the covariant derivatives with respect to the Eulerian coordinates expressed in the Lagrangian coordinates. Such crucial fact was first observed by Alinhac [2] and has been widely used for quasilinear hyperbolic system. In the study of free-surface fluid, such method was first implicitly used in the Q-tensor energy introduced by Christodoulou-Lindblad [8] which was later generalized by [28, 42, 43, 17, 73]. See also [48, 68, 24, 46, 74, 18] for the explicit applications.

Under the setting of (1.25)-(1.26), we can do the energy estimates by analyzing the Alinhac good unknowns via their evolution equation instead of the ∂_*^I-differentiated variables.

$$\rho_0\partial_t\mathbf{V} = -J\nabla_A\mathbf{Q} + (b_0\cdot\partial)(\partial_*^I b) + \underbrace{\left(\rho_0\partial_t(\mathbf{V}-\partial_*^I v) - C(Q) + [\partial_*^I, (b_0\cdot\partial)]b\right)}_{=:\mathbf{F}}. \tag{1.27}$$

Taking $L^2(\Omega)$ inner product of (1.27) and $\mathbf{V}$ and then integrating by parts, we can get the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2 = -\int_\Omega (\partial_*^I(J^{-1}(b_0\cdot\partial)\eta))\cdot(b_0\cdot\partial)\mathbf{V}\,dy + \int_\Omega J\mathbf{Q}(\nabla_A\cdot\mathbf{V})\,dy + \int_\Omega J\mathbf{F}\cdot\mathbf{V}\,dy - \int_\Gamma JA^{3i}N_3\mathbf{Q}\mathbf{V}_i\,dy'. \tag{1.28}$$

By direct computation we can prove $\|\mathbf{F}\|_0 \le P(\mathcal{E}(t))$, so it remains to control

$$K_1 := -\int_\Omega (\partial_*^I(J^{-1}(b_0\cdot\partial)\eta))\cdot(b_0\cdot\partial)\mathbf{V}\,dy, \tag{1.29}$$

$$I_1 := \int_\Omega J\mathbf{Q}(\nabla_A\cdot\mathbf{V})\,dy, \tag{1.30}$$

$$IB := -\int_\Gamma JA^{3i}N_3\mathbf{Q}\mathbf{V}_i\,dy', \tag{1.31}$$

where $dy' := dy_1\,dy_2$ is the area unit of the boundary Γ.

1.4.3 Interior estimates and cancellation structure

Below we use "$\cdots$" to represent the terms whose L^2 norms can be directly controlled by $P(\mathcal{E}(t))$. The term K_1 gives the energy of the magnetic field. Recall that the top-order term in $\mathbf{V}$ is $\partial_*^I v = \partial_*^I\partial_t\eta$ which yields

$$\begin{aligned}
K_1 &= -\int_\Omega (\partial_*^I(J^{-1}(b_0\cdot\partial)\eta^i))((b_0\cdot\partial)\partial_*^I\partial_t\eta_i)\,dy + \cdots \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_*^I(J^{-1}(b_0\cdot\partial)\eta)|^2\,dy - \int_\Omega \partial_*^I(J^{-1}(b_0\cdot\partial)\eta^i)\,(J^{-1}(b_0\cdot\partial)\eta_i)\partial_t\partial_*^I J\,dy + \cdots \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_*^I(J^{-1}(b_0\cdot\partial)\eta)|^2\,dy \underbrace{- \int_\Omega J\partial_*^I(J^{-1}(b_0\cdot\partial)\eta^i)\,(J^{-1}(b_0\cdot\partial)\eta_i)\partial_*^I(\mathrm{div}_A v)\,dy}_{=:K_{11}} + \cdots,
\end{aligned}$$

where we use $b = J^{-1}(b_0\cdot\partial)\eta$ and $\partial_t J = J\mathrm{div}_A v$. Note that K_{11} cannot be directly controlled due to the presence of $\partial_*^I(\mathrm{div}_A v)$. Instead, it will be exactly cancelled by another term produced by I_1.

The term I_1 gives the energy of the fluid pressure q and the cancellation structure with K_{11}. Recall that $\nabla_A\cdot\mathbf{V} = \partial_*^I(\mathrm{div}_A v) - C(v)$ and $Q = q + \frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2$. We get

$$\begin{aligned}
I_1 &= \int_\Omega J(\partial_*^I q)\partial_*^I(\mathrm{div}_A v) + \int_\Omega J\left(\partial_*^I\left(\frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2\right)\right)\partial_*^I(\mathrm{div}_A v)\,dy + \cdots \\
&= -\int_\Omega J(\partial_*^I q)\partial_*^I\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) + \int_\Omega J\partial_*^I(J^{-1}(b_0\cdot\partial)\eta^i)(J^{-1}(b_0\cdot\partial)\eta_i)\partial_*^I(\mathrm{div}_A v)\,dy + \cdots \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{J^2R'(q)}{\rho_0}|\partial_*^I q|^2\,dy + (-K_{11}) + \cdots
\end{aligned}$$

Then using $Q = q + \frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2$, we also get the control of the total pressure Q.

1.4.4 Modified Alinhac good unknowns

Before analyzing the boundary integral IB, we have to figure out the precise expressions of the Alinhac good unknowns $\mathbf{V}, \mathbf{Q}$ which can be derived by analyzing $\partial_*^I(\nabla_A f)$ for $f = v_i$ and Q. We will repeatedly use (1.16) in the analysis of commutators. First, for any multi-index I' with $\langle I' \rangle = 1$, we have with the notation $[T, f, g] := T(fg) - T(f)g - fT(g)$

$$\begin{aligned}
\partial_*^I(\nabla_A^i f) =& \nabla_A^i(\partial_*^I f) + (\partial_*^I A^{li})\,\partial_l f + [\partial_*^I, A^{li}, \partial_l f]\\
\overset{(1.16)}{=}& \nabla_A^i(\partial_*^I f) - \partial_*^{I-I'}(A^{lr}\,\partial_*^{I'}\partial_m\eta_r\,A^{mi})\,\partial_l f + [\partial_*^I, A^{li}, \partial_l f]\\
=& \nabla_A^i(\underbrace{\partial_*^I f - \partial_*^I\eta_r A^{lr}\partial_l f}_{=:\partial_*^I f - \partial_*^I\eta\cdot\nabla_A f}) + \partial_*^I\eta_r\,\nabla_A^i(\nabla_A^r f) - ([\partial_*^{I-I'}, A^{lr}A^{mi}]\partial_*^{I'}\partial_m\eta_r)\partial_l f + [\partial_*^I, A^{li}, \partial_l f].
\end{aligned}$$

Under the setting of standard Sobolev spaces, the term $\partial_*^I f - \partial_*^I\eta\cdot\nabla_A f$ is already the standard Alinhac good unknown of f (with respect to ∂_*^I). See also [48, 68, 24, 46, 74, 18]. However, under the setting of anisotropic Sobolev spaces, we still need to analyze the commutators $-([\partial_*^{I-I'}, A^{lr}A^{mi}]\partial_*^{I'}\partial_m\eta_r)\partial_l f$ and $[\partial_*^I, A^{li}, \partial_l f]$ whose $L^2(\Omega)$ norms may not be directly controlled due to the anisotropy of H_*^m.

In particular, as long as ∂_*^I is not the purely non-weighted normal derivative ∂_3^4, the commutator $[\partial_*^I, A^{li}, \partial_l f]$ always contains the term $(\partial_*^{I'}A^{li})(\partial_*^{I-I'}\partial_l f)$ whose $L^2(\Omega)$ norm cannot be controlled when $l = 3$ due to the anisotropy of H_*^m. In fact, we should use different methods to analyze this term for $f = Q$ and $f = v_i$ respectively.

- When $f = v_i$, by using (1.16), we can rewrite this term to be

$$\begin{aligned}
(\partial_*^{I'}A^{li})(\partial_*^{I-I'}\partial_l v_i) =& -(A^{lp}\,\partial_*^{I'}\partial_m\eta_p\,A^{mi})\partial_*^{I-I'}\partial_l v_i = -A^{li}\,\partial_*^{I'}\partial_m\eta_i\,A^{mp}\,\partial_*^{I-I'}\partial_l v_p\\
=& -\nabla_A^i(\partial_*^{I-I'}v_p\,A^{mp}\,\partial_*^{I'}\partial_m\eta_i) + \nabla_A^i(A^{mp}\partial_*^{I'}\partial_m\eta_i)\,\partial_*^{I-I'}v_p.
\end{aligned}$$

  Then we can merge $-\partial_*^{I-I'}v_p\,A^{mp}\,\partial_*^{I'}\partial_m\eta_i$ into the good unknown of v, i.e., the covariant derivative part in (1.26), and merge $\nabla_A^i(A^{mp}\partial_*^{I'}\partial_m\eta_i)\partial_*^{I-I'}v_p$ into the commutator part $C(v)$ in (1.26) because its L^2 norm can be directly controlled.

- When $f = Q$, we invoke (1.16) and the MHD equation $-\nabla_{\hat{A}}Q = \rho_0\partial_t v - (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)$ to get

$$\begin{aligned}
J(\partial_*^{I'}A^{li})(\partial_*^{I-I'}\partial_l Q) =& -(\hat{A}^{lp}\,\partial_*^{I'}\partial_m\eta_p\,A^{mi})(\partial_*^{I-I'}\partial_l Q)\\
=& -(\partial_*^{I'}\partial_m\eta_p\,A^{mi})\,\partial_*^{I-I'}\underbrace{(\hat{A}^{lp}\partial_l Q)}_{=\nabla_{\hat{A}}^p Q} + (\partial_*^{I-I'}\hat{A}^{lp})(\partial_l Q)(\partial_*^{I'}\partial_m\eta_p\,A^{mi}) + \cdots\\
=& (\partial_*^{I'}\partial_m\eta_p\,A^{mi})\partial_*^{I-I'}\left(\rho_0\partial_t v^p - (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta^p)\right)\\
&+ (\partial_*^{I-I'}\hat{A}^{lp})(\partial_l Q)(\partial_*^{I'}\partial_m\eta_p\,A^{mi}) + \cdots
\end{aligned}$$

  Remark. Note that ∂_t and $(b_0\cdot\partial)$ are both tangential derivatives while $\nabla_A Q$ always contains a normal derivative. Such substitution actually makes the order of the derivatives lower with the help of the anisotropy of H_*^m.

  Since $\langle I - I' \rangle = 7$, we have

$$\|(\partial_*^{I'}\partial_m\eta_p\,A^{mi})\partial_*^{I-I'}(\rho_0\partial_t v^p)\|_0 \lesssim \|\partial_*^{I'}\partial_m\eta_p\,A^{mi}\|_{L^\infty}\|\partial_*^{I-I'}(\rho_0\partial_t v^p)\|_0 \lesssim P(\|\eta\|_{7,*})\|\rho_0\|_{7,*}\|v\|_{8,*}. \tag{1.32}$$

  For the term $-\partial_*^{I'}\partial_m\eta_p\,A^{mi}\,\partial_*^{I-I'}\left((b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta_p)\right)$, we need to use $b_0^3|_\Gamma = 0$ to produce a weight function to make $b_0^3\partial_3$ become a weighted normal derivative. By the fundamental theorem of calculus, we know (suppose $y_3 > 0$ without loss of generality)

$$|b_0^3(t, y_3)|_{L^\infty(\mathbb{T}^2)} = \left|0 + \int_1^{y_3}\partial_3 b_0^3(t, \zeta_3)\,d\zeta_3\right|_{L^\infty(\mathbb{T}^2)} \leq (1 - y_3)\|\partial_3 b_0\|_{L^\infty} \lesssim \sigma(y_3)\|\partial_3 b_0\|_{L^\infty},$$

  and thus

$$\begin{aligned}
&\left\|(\partial_*^{I'}\partial_m\eta_p\,A^{mi})\,\partial_*^{I-I'}\left((b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta_p)\right)\right\|_0\\
\lesssim& P(\|\eta\|_{7,*})\left(\|b_0\|_{7,*}\|J^{-1}(b_0\cdot\partial)\eta\|_{8,*} + \|\partial_3 b_0\|_{L^\infty}\|(\sigma\partial_3)\partial_*^{I-I'}(J^{-1}(b_0\cdot\partial)\eta)\|_0\right)\\
\lesssim& P(\|\eta\|_{7,*})\left(\|b_0\|_{7,*}\|J^{-1}(b_0\cdot\partial)\eta\|_{8,*}\right).
\end{aligned}\tag{1.33}$$

In addition, the term $(\partial_*^{I-I'}\hat{A}^{lp})(\partial_l Q)(\partial_*^{I'}\partial_m\eta_p A^{mi})$ can be directly controlled when $l=3$ since $\hat{A}^{3p}$ consists of $(\overline{\partial}\eta)(\overline{\partial}\eta)$ (cf. (2.2)). When $l=1,2$, one should again invoke (1.16) to compute the highest order term and use $\overline{\partial}Q|_\Gamma=0$ to produce a weight function as in (1.33).

Remark. From (1.33), the definition of H_*^m and the fact $\sigma|_\Gamma=0$, the weighted derivative $(\sigma\partial_3)$ plays a similar role as a tangential derivative. In fact, one should consider the weighted derivative $(\sigma\partial_3)$ as a tangential derivative throughout this manuscript.

There are three other terms which need further analysis:

- $e_1 := -\partial_*^{I-I'}(A^{lr}A^{mi})\,(\partial_*^{I'}\partial_m\eta_r\,\partial_l f)$. When $\partial_*^{I-I'}$ does not contain time derivative, the term $\partial_*^{I-I'}(A^{lr}A^{mi})$ cannot be controlled since both A^{1i} and A^{2i} contain $\partial_3\eta$.
- $e_2 := -\partial_*^{I'}(A^{lr}A^{mi})\,(\partial_*^{I-I'}\partial_m\eta_r\,\partial_l f)$. When $\partial_*^{I-I'}$ does not contain time derivative, the term $\partial_*^{I-I'}\partial_m\eta_r$ cannot be controlled when $m=3$ since $\partial_*^{I-I'}\partial_3\eta$ should be controlled by $\|\eta\|_{9,*}$.
- $e_3 := (\partial_*^{I-I'}A^{li})\,(\partial_*^{I'}\partial_l f)$. When $\partial_*^{I-I'}$ does not contain time derivative, the term $\partial_*^{I-I'}A^{li}$ cannot be controlled when $l=1,2$ since A^{1i} and A^{2i} contains $\partial_3\eta$.

  Remark. Since $\partial_t\eta$ (resp. $\partial_t A$) has the same spatial regularity as η (resp. A), the $L^2(\Omega)$-norms of e_1,e_2,e_3 can be directly controlled when $\partial_*^{I-I'}$ contains at least one time derivative.

- When ∂_*^I contains the weighted normal derivative $(\sigma\partial_3)$, we need to analyze the extra terms which are produced when ∂_3 falls on $\sigma(y_3)$. This appears when we commute $(b_0\cdot\partial)$ or ∇_A with ∂_*^I.

We note that these terms can be controlled by similar arguments as in the analysis of $(\partial_*^{I'}A^{li})(\partial_*^{I-I'}\partial_l f)$. In other words, the following three techniques are enough for us to control the remaining terms.

- Modify the definition of Alinhac good unknowns by rewriting the higher order terms to be a covariant derivative plus $L^2(\Omega)$-bounded terms.
- Produce a weight function by using $b_0^3|_\Gamma=0$ and $\overline{\partial}Q|_\Gamma=0$ in order to replace one ∂_3 by $(\sigma\partial_3)$.
- Replace $\nabla_{\hat{A}}Q$ by $-\rho_0\partial_t v+(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)$ in order to make the order of the derivatives lower thanks to the anisotropy of H_*^m.

See Section 4.1.1 for detailed derivation of the modified Alinhac good unknowns and Section 6 for the analysis of weighted derivatives. Therefore, we can write

$$\mathbf{Q}=\partial_*^I Q-\partial_*^I\eta\cdot\nabla_A Q+\Delta_Q, \tag{1.34}$$

$$\mathbf{V}_i=\partial_*^I v_i-\partial_*^I\eta\cdot\nabla_A v_i+(\Delta_v)_i, \tag{1.35}$$

where $\|\Delta_f\|_{1,*}\lesssim P(\mathcal{E}(t))$ and the properties (1.25)-(1.26) still hold.

1.4.5 Boundary estimates and necessity of anisotropy

Invoking (1.34)-(1.35), we have

$$\begin{aligned} IB=&-\int_\Gamma JA^{3i}N_3\mathbf{Q}\mathbf{V}_i\,dy'\\ =&\underbrace{-\int_\Gamma JA^{3i}N_3(\partial_*^I Q)\mathbf{V}_i\,dy'}_{=:IB_0}+\underbrace{\int_\Gamma JA^{3i}N_3(\partial_*^I\eta\cdot\nabla_A Q)\mathbf{V}_i\,dy'}_{=:IB_1}+\cdots, \end{aligned} \tag{1.36}$$

modulo the terms involving Δ_Q and Δ_v which are easier to analyze. The detailed analysis can be found in Section 3.3, 4.2.2 and 5.1.3.

Regularity of the free surface and standard cancellation structure First, IB_1 in (1.36) gives the boundary energy and a cancellation structure enjoyed by the standard Alinhac good unknown arguments as in [48, 68, 24, 46, 74, 18]. In specific, since

$\overline{\partial}_1 Q = \overline{\partial}_2 Q = 0$ on Γ, we have

$$
\begin{aligned}
IB_1 =& \int_\Gamma J\frac{\partial Q}{\partial N}\partial_*^I\eta_k\, A^{3k}\, A^{3i}\, (\partial_*^I\partial_t\eta_i - \partial_*^I\eta_r\, A^{lr}\,\partial_l v_i)\, dy' \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Gamma\left(-J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\partial_*^I\eta_i\right|^2\, dy' + \frac{1}{2}\int_\Gamma \partial_t\left(-J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\partial_*^I\eta_i\right|^2\, dy' \\
&- \int_\Gamma J\frac{\partial Q}{\partial N}(A^{3k}\partial_*^I\eta_k)\,\partial_t A^{3i}\,\partial_*^I\eta_i\, dy' - \int_\Gamma J\frac{\partial Q}{\partial N}(A^{3k}\partial_*^I\eta_k) A^{3i}\partial_*^I\eta_r\, A^{lr}\partial_l v_i\, dy'.
\end{aligned}
\tag{1.37}
$$

Invoking the Rayleigh-Taylor sign condition (1.20), we get the boundary energy $\left|A^{3i}\partial_*^I\eta_i\right|_0^2$ which exactly controls the second fundamental form of the free surface. The second term can be directly controlled thanks to the boundary energy. Then plugging $\partial_t A^{3i} = -A^{3r}\,\partial_l v_r\, A^{li}$ into the third term yields the cancellation with the fourth term.

Remark. The cancellation structure above, enjoyed by the Alinhac good unknown, relies on the fact that $A^{3i}(\partial_*^I\partial_t\eta_i - \partial_*^I\eta_r\, A^{lr}\partial_l\partial_t\eta_i) = \partial_t(A^{3i}\partial_*^I\partial_t\eta_i)$ which can be proved by using (1.16) with $D = \partial_t$. This identity will be repeatedly used to derive similar cancellation structure in the boundary estimates.

Reduction of the normal derivatives and the advantage of the anisotropy When ∂_*^I contains normal derivative, $\partial_*^I Q$ no longer vanishes on Γ. In this case we write $\partial_*^I = \partial_*^{I-e_3}\partial_3$ where the multi-index e_3 is defined by $(i_0, i_1, i_2, i_3, i_4) = (0,0,0,1,0)$ and $\langle I - e_3\rangle = 6$. We shall analyze

$$
IB_0 = \int_\Gamma N_3 J(\partial_*^{I-e_3}\partial_3 Q)(A^{3i}\,\partial_*^{I-e_3}\partial_3 v_i)\, dy' + \int_\Gamma N_3 J(\partial_*^{I-e_3}\partial_3 Q)(\partial_*^{I-e_3}\partial_3\eta_p\, A^{lp}\partial_l v_i)\, dy' =: IB_{01} + IB_{02}.
\tag{1.38}
$$

First, for IB_{01}, we invoke the third equation in (1.17) to replace the normal derivative in $A^{3i}\partial_3 v_i$ by tangential derivative

$$
A^{3i}\,\partial_*^{I-e_3}\partial_3 v_i = \partial_*^{I-e_3}(A^{3i}\partial_3 v_i) + \cdots = -\partial_*^{I-e_3}\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) - \sum_{L=1}^{2}\partial_*^{I-e_3}(A^{Li}\overline{\partial}_L v_i) + \cdots
\tag{1.39}
$$

Note that $\partial_*^{I-e_3}A^{Li} = -A^{Lp}(\partial_*^{I-e_3}\partial_m\eta_p)A^{mi} = -A^{Lp}(\partial_*^{I-e_3}\partial_3\eta_p)A^{3i} - \sum_{M=1}^{2} A^{Lp}(\partial_*^{I-e_3}\overline{\partial}_M\eta_p)A^{Mi}$, in which the contribution of $-A^{Lp}(\partial_*^{I-e_3}\partial_3\eta_p)A^{3i}$ in IB_{01} exactly cancels with the contribution of $l = 1, 2$ in IB_{02}. The highest order term in $\partial_*^{I-e_3}\partial_3\eta_p\, A^{lp}\partial_l v_i$ corresponding to $l = 3$ in IB_{02} is actually $\partial_3\eta_p\,\partial_*^{I-e_3}A^{3p}\,\partial_l v_i = \partial_3\eta_p\,\partial_*^{I-e_3}(J^{-1}\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_p\,\partial_l v_i$ thanks to the identity $A^{3p}\partial_3\eta_p = 1$.

Next we replace $\partial_3 Q$ by tangential derivative of v and $(b_0\cdot\partial)\eta$. Since $A^{3i}\partial_3\eta_i = 1$, we have

$$
\begin{aligned}
J(\partial_*^{I-e_3}\partial_3 Q) =& \partial_3\eta_i\,\hat{A}^{3i}\,(\partial_*^{I-e_3}\partial_3 Q) = \partial_3\eta_i\,\partial_*^{I-e_3}(\hat{A}^{3i}\partial_3 Q) + \cdots \\
=& \partial_3\eta_i\,\partial_*^{I-e_3}(\underbrace{\hat{A}^{li}\partial_l Q}_{=\nabla_{\hat{A}}^i Q}) - \sum_{L=1}^{2}\partial_3\eta_i\,\partial_*^{I-e_3}(\hat{A}^{Li}\overline{\partial}_L Q) + \cdots \\
=& \partial_3\eta_i\,\partial_*^{I-e_3}\left(-\rho_0\partial_t v^i - (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta^i)\right) - \sum_{L=1}^{2}\partial_3\eta_i\,\partial_*^{I-e_3}(\hat{A}^{Li}\overline{\partial}_L Q) + \cdots
\end{aligned}
\tag{1.40}
$$

Note that $\overline{\partial}_L Q|_\Gamma = 0$ for $L = 1, 2$ eliminates the highest order term $\partial_3\eta_i\,(\partial_*^{I-e_3}\hat{A}^{Li})\,\overline{\partial}_L Q$. And $b_0^3|_\Gamma = 0$ implies that $(b_0\cdot\partial)|_\Gamma = b_0^1\overline{\partial}_1 + b_0^2\overline{\partial}_2$ is a tangential derivative on Γ. Combining (1.38)-(1.40), the highest order terms in IB_0 can all be written as the following form

$$
\int_\Gamma N_3(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_*^{I-e_3}\mathfrak{D}g)h\, dy',
$$

where $\mathfrak{D}$ can be $(b_0\cdot\overline{\partial}), \overline{\partial}, \partial_t$(tangential), and f, g can be $\eta, v, q, J^{-1}(b_0\cdot\partial)\eta$, and h consists of the the terms containing at most first-order derivative of η and v. To control such boundary integral, we first rewrite it to the interior thanks to the divergence theorem in y-coordinates, and then integrate $\mathfrak{D}$ by parts

$$
\begin{aligned}
&\int_\Gamma N_3(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_*^{I-e_3}\mathfrak{D}g)h\,dy' \\
=&\int_\Omega(\partial_3\partial_*^{I-e_3}\mathfrak{D}f)(\partial_*^{I-e_3}\mathfrak{D}g)h\,dy+\int_\Omega(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_3\partial_*^{I-e_3}\mathfrak{D}g)h\,dy+\int_\Omega(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_*^{I-e_3}\mathfrak{D}g)\partial_3h\,dy \\
\overset{\mathfrak{D}}{=}&-\int_\Omega(\partial_3\partial_*^{I-e_3}f)(\partial_*^{I-e_3}\mathfrak{D}^2g)h+\int_\Omega(\partial_3\partial_*^{I-e_3}f)(\partial_*^{I-e_3}\mathfrak{D}g)\mathfrak{D}h \\
&-\int_\Omega(\partial_*^{I-e_3}\mathfrak{D}^2f)(\partial_3\partial_*^{I-e_3}g)h+\int_\Omega(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_3\partial_*^{I-e_3}g)\mathfrak{D}h+\int_\Omega(\partial_*^{I-e_3}\mathfrak{D}f)(\partial_*^{I-e_3}\mathfrak{D}g)\partial_3h \\
\lesssim&\|f\|_{8,*}\|g\|_{8,*}\|h\|_3,
\end{aligned}
\tag{1.41}
$$

where the anisotropy of the function space H_*^8 is crucial in the last step because $\langle I-e_3\rangle=6$ allows us to have two more tangential derivatives $\mathfrak{D}^2$. When $\mathfrak{D}$ in (1.41) is ∂_t, this step should be done under time integral. See (3.43) for example.

The analysis of IB_0 above also shows the advantage of using anisotropic Sobolev space as pointed out as an important conclusion in the survey article [7] by Chen who first introduced the anisotropic Sobolev spaces in [6]

> "For the nonlinear hyperbolic system with characteristic boundary conditions, the growth of one normal derivative on the boundary should be compensated by the decrease in regularity of two tangential derivatives. This is one of the advantages of the anisotropic Sobolev space that the standard Sobolev space fails to carry."

In specific, if we start with the estimates of ∂_3^4, then by (1.41) we need the control of $\partial_3^3\mathfrak{D}^2$ where $\mathfrak{D}$ is a tangential derivative. To control the latter one, we need the control of $\partial_3^2\mathfrak{D}^4$ again due to (1.41). Repeatedly, we finally need to derive the estimates of $\mathfrak{D}^8$. In addition, the weighted derivative $(\sigma\partial_3)$ is necessary in the interior estimates, e.g., in (1.33). On the other hand, we also need the control of 4 normal derivatives in order to close the energy estimates of 8 tangential derivatives. So we find that the anisotropic Sobolev space exactly meets all of these requirements in our mechanism of reducing normal derivatives on the boundary.

Finally, the contribution of Δ_Q and Δ_v in IB can be controlled by using the boundary energy $|A^{3i}\partial_*^I\eta_i|_0$ together with either the trace lemma for anisotropic Sobolev spaces (cf. Lemma 2.1) or similar technique as in (1.41). Hence, the control of boundary integral IB is finished.

1.5 Organization of the paper

In Section 2 we record the lemmas which will be repeatedly used in the manuscript. Then we show the control of purely non-weighted normal derivatives in Section 3 and purely tangential derivatives in Section 4. Combining the techniques in Section 3, 4, we conclude the control of general non-weighted derivatives in Section 5. The control weighted derivatives is analyzed in Section 6. Finally, we conclude the a priori estimates in Section 7 and discuss the existence of the initial data satisfying the compatibility conditions in Section 8. Below we list all the notations repeatedly used in this manuscript.

List of Notations:

- $\Omega:=\mathbb{T}^2\times(-1,1)$ and $\Gamma:=\mathbb{T}^2\times(\{-1\}\cup\{1\})$.
- $\|\cdot\|_s$: We denote $\|f\|_s:=\|f(t,\cdot)\|_{H^s(\Omega)}$ for any function $f(t,y)$ on $[0,T]\times\Omega$.
- $|\cdot|_s$: We denote $|f|_s:=|f(t,\cdot)|_{H^s(\Gamma)}$ for any function $f(t,y)$ on $[0,T]\times\Gamma$.
- $\|\cdot\|_{m,*}$: For any function $f(t,y)$ on $[0,T]\times\Omega$, $\|f\|_{m,*}^2:=\sum_{\langle I\rangle\le m}\|\partial_*^If(t,\cdot)\|_{L^2}^2$ denotes the m-th order space-time anisotropic Sobolev norm of f.
- $P(\cdots)$: A generic polynomial in its arguments;
- $\mathcal{P}_0$: $\mathcal{P}_0=P(\mathcal{E}(0))$;
- $[T,f]g:=T(fg)-fT(g)$, and $[T,f,g]:=T(fg)-T(f)g-fT(g)$, where T denotes a differential operator and f,g are arbitrary functions.
- $\overline{\partial}$: $\overline{\partial}=\partial_1,\partial_2$ denotes the spatial tangential derivative.
- $\nabla_A^if:=A^{li}\partial_lf$ denotes the covariant (Eulerian) derivative.
- $X\cdot\nabla_Af$: For any function f and vector field X, such notation denotes the inner-product defined by $X\cdot\nabla_Af:=X_pA^{lp}\partial_lf$.
- $X\cdot\nabla_AY\cdot\nabla_Af$: For any function f and vector field X,Y, such notation denotes the inner-product defined by $X\cdot\nabla_AY\cdot\nabla_Af:=X_pA^{lp}\partial_lY_rA^{mr}\partial_mf$.

□

2 Preliminary lemmas

2.1 Some geometric identities

We record the explicit form of the matrix A which will be repeatedly used.

$$A = J^{-1}\begin{pmatrix} \overline{\partial}_2\eta^2\partial_3\eta^3 - \partial_3\eta^2\overline{\partial}_2\eta^3 & \partial_3\eta^1\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^1\partial_3\eta^3 & \overline{\partial}_2\eta^1\partial_3\eta^2 - \partial_3\eta^1\overline{\partial}_2\eta^2 \\ \partial_3\eta^2\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^2\partial_3\eta^3 & \overline{\partial}_1\eta^1\partial_3\eta^3 - \partial_3\eta^1\overline{\partial}_1\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_1\eta^2 - \overline{\partial}_1\eta^1\partial_3\eta^2 \\ \overline{\partial}_1\eta^2\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^2\overline{\partial}_1\eta^3 & \overline{\partial}_2\eta^1\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^1\overline{\partial}_2\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_2\eta^2 - \overline{\partial}_2\eta^1\overline{\partial}_1\eta^2 \end{pmatrix} \tag{2.1}$$

Moreover, since $\hat{A} = JA$, and in view of (2.1), we can write

$$\hat{A}^{1i} = \epsilon^{ijk}\overline{\partial}_2\eta_j\partial_3\eta_k, \ \ \hat{A}^{2i} = -\epsilon^{ijk}\overline{\partial}_1\eta_j\partial_3\eta_k, \ \ \hat{A}^{3i} = \epsilon^{ijk}\overline{\partial}_1\eta_j\overline{\partial}_2\eta_k. \tag{2.2}$$

Here, ϵ^{ijk} is the sign of the 3-permutation $(ijk) \in S_3$. We will repeatedly use that fact that $\hat{A}^{1,\cdot}, \hat{A}^{2,\cdot}$ consist of the linear combination of $\overline{\partial}\eta \cdot \partial_3\eta$ and $\hat{A}^{3\cdot}$ consists of $\overline{\partial}\eta \cdot \overline{\partial}\eta$.

We also record the following identity: Suppose D is the derivative ∂ or ∂_t, then

$$DA^{li} = -A^{lr}\,\partial_k D\eta_r\,A^{ki}. \tag{2.3}$$

2.2 Anisotropic Sobolev space

We list two preliminary lemmas on the basic properties of anisotropic Sobolev space.

Lemma 2.1 (Trace lemma for anisotropic Sobolev space)**.** Let $m \geq 1,\ m \in \mathbb{N}^*$, then we have the following trace lemma for the anisotropic Sobolev space.

1. If $f \in H_*^{m+1}(\Omega)$, then its trace $f|_\Gamma$ belongs to $H^m(\Gamma)$ and satisfies

$$|f|_m \lesssim \|f\|_{H_*^{m+1}(\Omega)}.$$

2. There exists a linear continuous operator $\mathfrak{R}_T : H^m(\Gamma) \to H_*^{m+1}(\Omega)$ such that $(\mathfrak{R}_T g)|_\Gamma = g$ and

$$\|\mathfrak{R}_T g\|_{H_*^{m+1}(\Omega)} \lesssim |g|_m.$$

Proof. See Ohno-Shizuta-Yanagisawa [52, Theorem 1]. □

Remark. The condition $m \geq 1$ is necessary and analogous result may not hold when $m = 0$. One can see the importance of $m \geq 1$ from (1.41), as a special case, where we need to integrate one tangential derivative by part and thus $m \geq 1$ is necessary.

Lemma 2.2 (Sobolev embedding lemma for anisotropic Sobolev space)**.** We have the following inequalities

$$H^m(\Omega) \hookrightarrow H_*^m(\Omega) \hookrightarrow H^{\lfloor m/2 \rfloor}(\Omega), \ \ \forall m \in \mathbb{N}^*$$
$$\|u\|_{L^\infty} \lesssim \|u\|_{H_*^3(\Omega)}, \ \ \|u\|_{W^{1,\infty}} \lesssim \|u\|_{H_*^5(\Omega)}.$$

Proof. See Trakhinin-Wang [66, Lemma 3.3]. □

3 Control of purely non-weighted normal derivatives

In this section, we prove the following estimates by the standard Alinhac good unknown argument.

Proposition 3.1. The following energy inequality holds

$$\|\partial_3^4 v\|_0^2 + \left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^4 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_3^4\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{3.1}$$

3.1 Evolution equation of Alinhac good unknowns

We first compute the estimates of purely normal derivatives. When $\langle I\rangle = 8$, the purely non-weighted normal derivative should be $\partial_*^I = \partial_3^4$. First we introduce the following Alinhac good unknowns of v and Q with respect to ∂_3^4

$$\mathbf{V}_i := \partial_3^4 v_i - \partial_3^4\eta_p\, A^{lp}\,\partial_l v_i,\quad \mathbf{Q} := \partial_3^4 Q - \partial_3^4\eta_p\, A^{lp}\,\partial_l Q. \tag{3.2}$$

Then we have that for any function f

$$\begin{aligned}
\partial_3^4(\nabla_A^i f) &= \nabla_A^i(\partial_3^4 f) + (\partial_3^4 A^{li})\partial_l f + [\partial_3^4, A^{li}, \partial_l f]\\
&= \nabla_A^i(\partial_3^4 f) - \partial_3^3(A^{lp}\,\partial_3\partial_m\eta_p\, A^{mi})\partial_l f + [\partial_3^4, A^{li}, \partial_l f]\\
&= \nabla_A^i(\underbrace{\partial_3^4 f - \partial_3^4\eta_p\, A^{lp}\,\partial_l f}_{\text{good unknowns}}) + \underbrace{\partial_3^4\eta_p\nabla_A^i(\nabla_A^p f) - ([\partial_3^3, A^{lp}A^{mi}]\partial_3\partial_m\eta_p)\partial_l f + [\partial_3^4, A^{li}, \partial_l f]}_{=:C^i(f)},
\end{aligned} \tag{3.3}$$

and thus

$$\nabla_A\cdot\mathbf{V} = \partial_3^4(\mathrm{div}_A v) - C^i(v_i),\quad \nabla_A\mathbf{Q} = \partial_3^4(\nabla_A Q) - C(Q), \tag{3.4}$$

where the commutator satisfies the estimate

$$\|C(f)\|_4 \lesssim P(\|\eta\|_4)\|f\|_4. \tag{3.5}$$

Now taking ∂_3^4 in the second equation of (1.17) and invoking (3.2) and (3.4), we get the evolution equation of the Alinhac good unknowns

$$R\partial_t\mathbf{V} - J^{-1}(b_0\cdot\partial)\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right) + \nabla_A\mathbf{Q} = \underbrace{\left[R, \partial_3^4\right]\partial_t v + \left[\partial_3^4, J^{-1}(b_0\cdot\partial)\right]b - C(Q) - R\partial_t(\partial_3^4\eta\cdot\nabla_A v)}_{=:\mathbf{F}}. \tag{3.6}$$

Taking $L^2(\Omega)$-inner product of (3.6) and $J\mathbf{V}$ and using $\rho_0 = RJ$, we get the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0\,|\mathbf{V}|^2\;dy = \int_\Omega (b_0\cdot\partial)\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V} - \int_\Omega(\nabla_{\hat{A}}\mathbf{Q})\cdot\mathbf{V} + \int_\Omega J\mathbf{F}\cdot\mathbf{V}. \tag{3.7}$$

3.2 Interior estimates

The third integral on the RHS of (3.7) can be directly controlled

$$\int_\Omega J\mathbf{F}\cdot\mathbf{V} \lesssim \|J\mathbf{F}\|_0\|\mathbf{V}\|_0 \lesssim P(\|\rho_0\|_4, \|b_0\|_4, \|\eta\|_4, \|J^{-1}(b_0\cdot\partial)\eta\|_4, \|Q\|_4, \|v\|_4, \|\partial_t v\|_3)\|\mathbf{V}\|_0. \tag{3.8}$$

The first integral on the RHS of (3.7) gives the energy of magnetic field $b = J^{-1}(b_0\cdot\partial)\eta$ after integrating $(b_0\cdot\partial)$ by parts. Note that $b_0^3|_\Gamma = 0$ and div $b_0 = 0$, there will be no boundary integral. In specific, we have

$$\begin{aligned}
&\int_\Omega (b_0\cdot\partial)\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}\,dy = -\int_\Omega \partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)\mathbf{V}\,dy\\
&= -\int_\Omega \partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)\partial_3^4 v\,dy + \underbrace{\int_\Omega \partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)(\partial_3^4\eta\cdot\nabla_A v)\,dy}_{=:L_1}\\
&= -\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\partial_3^4\partial_t\eta\right)dy + L_1\\
&= -\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\partial_3^4\partial_t(J^{-1}(b_0\cdot\partial)\eta)\,dy \underbrace{-\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\left[J^{-1}(b_0\cdot\partial), \partial_3^4\partial_t\right]\eta\,dy}_{K_1} + L_1\\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\partial_3^4(b_0\cdot\partial)\eta\right|^2\,dy + \frac{1}{2}\int_\Omega \partial_t J\left|\partial_3^4(b_0\cdot\partial)\eta\right|^2\,dy + K_1 + L_1.
\end{aligned} \tag{3.9}$$

The term L_1 can be directly controlled

$$L_1 \lesssim P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4, \|v\|_4\right). \tag{3.10}$$

The term K_1 produces a higher order term when $\partial_3^4\partial_t$ falls on J^{-1}. We invoke $\partial_t J = J\mathrm{div}_A v$ to get

$$
\begin{aligned}
&-\left[J^{-1}(b_0\cdot\partial), \partial_3^4\partial_t\right]\eta\\
=&\partial_3^4\partial_t(J^{-1})\,(b_0\cdot\partial)\eta + \sum_{N=0}^{3}\partial_3^N\partial_t(J^{-1})\,\partial_3^{4-N}b_0\cdot\partial\eta + \sum_{M=0}^{3}\partial_t\left(\partial_3^M(J^{-1}b_0^l)\,\partial_l\partial_3^{4-M}\eta\right)\\
=&-J^{-1}\partial_3^4(\mathrm{div}_A v)\,(b_0\cdot\partial)\eta\\
&+\underbrace{\left([\partial_3^4, J^{-1}]\mathrm{div}_A v\right)(b_0\cdot\partial)\eta + \sum_{N=0}^{3}\partial_3^N\partial_t(J^{-1})(\partial_3^{4-N}b_0^l)(\partial_l\eta) + \sum_{M=0}^{3}\partial_t\left(\partial_3^M(J^{-1}b_0^l)\partial_l\partial_3^{4-M}\eta\right)}_{KL_1},
\end{aligned}
\tag{3.11}
$$

and thus

$$
\begin{aligned}
K_1 =&\underbrace{-\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\mathrm{div}_A v)\,dy}_{K_{11}} + \int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot(KL_1)\\
\lesssim& K_{11} + \|J\|_{L^\infty}\|J^{-1}(b_0\cdot\partial)\eta\|_4\|KL_1\|_0\\
\lesssim& K_{11} + P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4\right).
\end{aligned}
\tag{3.12}
$$

Summarizing (3.9)-(3.12), we get the following estimates

$$
\int_\Omega (b_0\cdot\partial)\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}\,dy \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\partial_3^4(b_0\cdot\partial)\eta\right|^2\,dy + K_{11} + P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4, \|v\|_4\right). \tag{3.13}
$$

We note that the term K_{11} cannot be directly controlled, but will be cancelled by another term produced by $-\int_\Omega(\nabla_{\hat{A}}\mathbf{Q})\cdot\mathbf{V}$.

Next we analyze the second integral on the RHS of (3.7). Integraing by parts and invoking Piola's identity $\partial_l\hat{A}^{li} = 0$, we get

$$
-\int_\Omega(\nabla_{\hat{A}}\mathbf{Q})\cdot\mathbf{V}\,dy = \int_\Omega J\mathbf{Q}(\nabla_A\cdot\mathbf{V})\,dy - \int_\Gamma J\mathbf{Q}A^{li}N_l\mathbf{V}_i\,dy' =: I + IB. \tag{3.14}
$$

Plugging (3.2) and (3.4) as well as $Q = q + \frac{1}{2}|b|^2$ into I, we get

$$
\begin{aligned}
I =& \int_\Omega J\partial_3^4 q\,\partial_3^4(\mathrm{div}_A v) + \int_\Omega J\partial_3^4\left(\frac{1}{2}\left|J^{-1}(b_0\cdot\partial)\eta\right|^2\right)\partial_3^4(\mathrm{div}_A v)\\
&-\int_\Omega \partial_3^4\eta_p\,\hat{A}^{lp}\partial_l Q\,\partial_3^4(\mathrm{div}_A v) - \int_\Omega\partial_3^4 Q\,C(v)\\
=:& I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.15}
$$

The term I_4 can be directly controlled by using (3.5)

$$
I_4 \lesssim \|Q\|_4\|C(v)\|_0 \lesssim P(\|\eta\|_4)\|Q\|_4\|v\|_4. \tag{3.16}
$$

The term I_1 gives the energy of q by invoking $\mathrm{div}_A v = -\dfrac{\partial_t R}{R} = -\dfrac{JR'(q)}{\rho_0}\partial_t q$

$$
\begin{aligned}
I_1 =& -\int_\Omega J\partial_3^4 q\,\partial_3^4\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) = -\int_\Omega \frac{J^2R'(q)}{\rho_0}\partial_3^4 q\,\partial_3^4\partial_t q - \int_\Omega J\partial_3^4 q\left(\left[\partial_3^4, \frac{JR'(q)}{\rho_0}\right]\partial_t q\right)\\
=& -\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{J^2R'(q)}{\rho_0}\left|\partial_3^4 q\right|^2\,dy + \frac{1}{2}\int_\Omega\partial_t\left(\frac{J^2R'(q)}{\rho_0}\right)\left|\partial_3^4 q\right|^2 - \int_\Omega J\partial_3^4 q\left(\left[\partial_3^4, \frac{JR'(q)}{\rho_0}\right]\partial_t q\right)\\
\lesssim& -\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{J^2R'(q)}{\rho_0}\left|\partial_3^4 q\right|^2\,dy + P(\|q\|_{8,*}, \|\rho_0\|_4, \|\eta\|_4).
\end{aligned}
\tag{3.17}
$$

The term I_2 will produce another higher order term to cancel with K_{11}

$$
\begin{aligned}
I_2 =& \underbrace{\int_\Omega J\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\mathrm{div}_A v)}_{\text{exactly cancel with } K_{11}} \\
&+\int_\Omega\sum_{N=1}^{3}\binom{4}{N}J\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\mathrm{div}_A v) \\
=&-K_{11}-\int_\Omega\sum_{N=1}^{3}\binom{4}{N}J\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) \\
=&-K_{11}-\int_\Omega\sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2R'(q)}{\rho_0}\right)\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4\partial_t q \\
&+\int_\Omega\sum_{N=1}^{3}\binom{4}{N}J\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\partial_3^4,\frac{JR'(q)}{\rho_0}\right]\partial_t q\right) \\
=:&-K_{11}+I_{21}+I_{22}.
\end{aligned}
\tag{3.18}
$$

We should control I_{21} by integrating ∂_t by parts under time integral

$$
\begin{aligned}
\int_0^T I_{21} \overset{\partial_t}{=}& \int_0^T\int_\Omega\sum_{N=1}^{3}\binom{4}{N}\partial_t\left(\frac{J^2R'(q)}{\rho_0}\right)\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q \\
&+\int_0^T\int_\Omega\sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2R'(q)}{\rho_0}\right)\partial_t\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q \\
&-\int_\Omega\sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2R'(q)}{\rho_0}\right)\partial_3^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q\Big|_0^T \\
\lesssim& \int_0^T P(\|J^{-1}(b_0\cdot\partial)\eta\|_4,\|\partial_t(J^{-1}(b_0\cdot\partial)\eta)\|_3,\|q\|_4)+\mathcal{P}_0+\|J^{-1}(b_0\cdot\partial)\eta\|_3^2\|\partial_3^4 q\|_0 \\
\lesssim&\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt+\varepsilon\|\partial_3^4 q\|_0^2+\int_0^T\|\partial_t(J^{-1}(b_0\cdot\partial)\eta)\|_3^2\le\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt+\varepsilon\|\partial_3^4 q\|_0^2.
\end{aligned}
\tag{3.19}
$$

Then I_{22} can be directly controlled since at most three ∂_3's fall on $\partial_t q$.

$$
I_{22}\lesssim\|J^{-1}(b_0\cdot\partial)\eta\|_3^2\|q\|_{7,*}.
\tag{3.20}
$$

The term I_3 should also be controlled under time integral. We have

$$
\begin{aligned}
\int_0^T I_3 =& \int_0^T\int_\Omega\frac{JR'(q)}{\rho_0}\partial_3^4\eta_p\hat{A}^{lp}\partial_l Q\,\partial_3^4\partial_t q+\underbrace{\int_0^T\int_\Omega\partial_3^4\eta_p\hat{A}^{lp}\partial_l Q\left[\partial_3^4,\frac{JR'(q)}{\rho_0}\right]\partial_t q}_{L_2} \\
\overset{\partial_t}{=}&-\int_0^T\int_\Omega\partial_t\left(\frac{JR'(q)}{\rho_0}\partial_3^4\eta_p\hat{A}^{lp}\partial_l Q\right)\partial_3^4 q+\int_\Omega\frac{JR'(q)}{\rho_0}\partial_3^4\eta_p\hat{A}^{lp}\partial_l Q\,\partial_3^4 q\Big|_0^T+L_2 \\
\lesssim&\mathcal{P}_0+\left\|\frac{JR'(q)}{\rho_0}A\,\partial Q\right\|_{L^\infty}\|\partial_3^4 q\|_0\|\partial_3^4\eta\|_0+\int_0^T P\left(\|q\|_{8,*},\|\eta\|_4,\|v\|_4,\|\rho_0\|_4\right)\,dt. \\
\lesssim&\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt+\left\|\frac{JR'(q)}{\rho_0}A\,\partial Q\right\|_{L^\infty}\|\partial_3^4 q\|_0\int_0^T\|\partial_3^4 v(t)\|_0\,dt \\
\lesssim&\mathcal{P}_0+P(\mathcal{E}(t))\int_0^T P(\mathcal{E}(t))\,dt,
\end{aligned}
\tag{3.21}
$$

where we use $\partial^4\eta|_{t=0}=0$ in the last step. Summarizing (3.16)-(3.21) and choosing $\varepsilon>0$ suitably small, we get the estimates of I under time integral

$$
\int_0^T I\,dt\lesssim-\frac{1}{2}\int_\Omega\frac{J^2R'(q)}{\rho_0}\left|\partial_3^4 q\right|^2\,dy\Big|_0^T+\mathcal{P}_0+P(\mathcal{E}(t))\int_0^T P(\mathcal{E}(t))\,dt.
\tag{3.22}
$$

3.3 Boundary estimates

To finish the estimates of purely non-weighted normal derivative, it remains to control the boundary integral IB in (3.14) which reads

$$
\begin{aligned}
-\int_\Gamma J\mathbf{Q}A^{li}N_l\mathbf{V}_i\,dy' =& -\int_\Gamma \hat{A}^{3i}N_3\,\partial_3^4 Q\,\mathbf{V}_i\,dy' \\
&+\int_\Gamma \hat{A}^{3i}N_3\partial_3^4\eta_p\,A^{3p}\partial_3 Q\,\partial_3^4 v_i\,dy' - \int_\Gamma \hat{A}^{3i}N_3\partial_3^4\eta_p\,A^{3p}\partial_3 Q\,(\partial_3^4\eta_r\,A^{mr}\,\partial_m v_i)\,dy' \\
=:& IB_0 + IB_1 + IB_2.
\end{aligned}
\tag{3.23}
$$

First, IB_1 will produce the boundary energy with the help of Rayleigh-Taylor sign condition (1.8) and the error terms will be cancelled with IB_2. In specific, we have

$$
\begin{aligned}
IB_1 =& -\int_\Gamma\left(-\frac{\partial Q}{\partial N}\right)JA^{3i}\partial_3^4\eta_p\,A^{3p}\partial_3^4\partial_t\eta_i\,dy' \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Gamma\left(-J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\partial_3^4\eta_i\right|^2\,dy' \\
& -\frac{1}{2}\int_\Gamma\partial_t\left(J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\partial_3^4\eta_i\right|^2\,dy' + \int_\Gamma\left(-J\frac{\partial Q}{\partial N}\right)\partial_t A^{3i}\,\partial_3^4\eta_p\,A^{3p}\partial_3^4\eta_i\,dy' \\
=:& IB_{11} + IB_{12} + IB_{13}.
\end{aligned}
\tag{3.24}
$$

Invoking Rayleigh-Taylor sign condition, we get

$$
\int_0^T IB_{11}\,dt \lesssim -\frac{c_0}{4}\int_\Gamma\left|A^{3i}\partial_3^4\eta_i\right|^2\,dy'\Big|_0^T,
\tag{3.25}
$$

and thus the term IB_{12} can be directly controlled by the boundary energy

$$
IB_{12} \lesssim |\partial_t(J\partial_3 Q)|_{L^\infty}\left|A^{3i}\partial_3^4\eta_i\right|_0^2 \lesssim P(\mathcal{E}(t)).
\tag{3.26}
$$

Then we plug $\partial_t A^{3i} = -A^{3r}\,\partial_m v_r\,A^{mi}$ into IB_{13} to get

$$
IB_{13} = \int_\Gamma\left(\frac{\partial Q}{\partial N}\right)A^{3r}\partial_m v_r\,\hat{A}^{mi}\partial_3^4\eta_p\,A^{3p}\partial_3^4\eta_i\,dy',
\tag{3.27}
$$

and this term exactly cancel with IB_2 if we replace the indices (r,i) by (i,r).

It now remains to control IB_0. We have

$$
IB_0 = -\int_\Gamma N_3 J\,\partial_3^4 Q\,(A^{3i}\partial_3^4 v_i)\,dy' + \int_\Gamma \hat{A}^{3i}N_3\partial_3^4 Q\,\partial_3^4\eta_p\,A^{lp}\partial_l v_i\,dy' =: IB_{01} + IB_{02}.
\tag{3.28}
$$

To control IB_0, we shall differentiate the following relations

$$
A^{3i}\partial_3 v_i = \mathrm{div}_A v - A^{1i}\overline{\partial}_1 v_i - A^{2i}\overline{\partial}_2 v_i = -\frac{JR'(q)}{\rho_0}\partial_t q - A^{1i}\overline{\partial}_1 v_i - A^{2i}\overline{\partial}_2 v_i.
\tag{3.29}
$$

In IB_{01}, we use the relation (3.29) to get

$$
\begin{aligned}
A^{3i}\partial_3^4 v_i =& \partial_3^3(A^{3i}\partial_3 v_i) - \partial_3^3 A^{3i}\,\partial_3 v_i - 3\partial_3^2 A^{3i}\,\partial_3^2 v_i - 3\partial_3 A^{3i}\,\partial_3^3 v_i \\
=& -\partial_3^3\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) - \sum_{L=1}^{2}\partial_3^3(A^{Li}\overline{\partial}_L v_i) - \partial_3^3 A^{3i}\,\partial_3 v_i - 3\partial_3^2 A^{3i}\,\partial_3^2 v_i - 3\partial_3 A^{3i}\,\partial_3^3 v_i,
\end{aligned}
\tag{3.30}
$$

and thus IB_{01} becomes

$$
\begin{aligned}
IB_{01} =& \int_\Gamma N_3 J\,\partial_3^4 Q\,\partial_3^3\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) + \sum_{L=1}^{2}\int_\Gamma N_3 J\,\partial_3^4 Q\,\partial_3^3(A^{Li}\overline{\partial}_L v_i) \\
&+ \int_\Gamma N_3 J\partial_3^4 Q\,\left(\partial_3^3 A^{3i}\,\partial_3 v_i + 3\partial_3^2 A^{3i}\,\partial_3^2 v_i + 3\partial_3 A^{3i}\,\partial_3^3 v_i\right) \\
=:& IB_{011} + IB_{012} + IB_{013}.
\end{aligned}
\tag{3.31}
$$

In IB_{012}, the highest order term contains $\partial_3^3 A^{Li} = \partial_3^4\eta\cdot\overline{\partial}\eta+\cdots$ which cannot be directly controlled. However, this term can produce cancellation with IB_{02}. We have

$$\begin{aligned}\partial_3^3 A^{Li} =&-\partial_3^2(A^{Lp}\,\partial_3\partial_m\eta_p\,A^{mi})\\ =&-A^{Lp}\partial_3^4\eta_p A^{3i}-\sum_{M=1}^{2}A^{Lp}\partial_3^3\overline{\partial}_M\eta_p A^{Mi}-[\partial_3^2,A^{Lp}A^{mi}]\partial_3\partial_m\eta_p,\end{aligned}\tag{3.32}$$

and thus IB_{012} can be written as

$$IB_{012}=-\sum_{L=1}^{2}\int_\Gamma \hat{A}^{3i}N_3\partial_3^4 Q\,\partial_3^4\eta_p\,A^{Lp}\overline{\partial}_L v_i \tag{3.33}$$

$$-\sum_{L=1}^{2}\int_\Gamma N_3 J\partial_3^4 Q\left(\sum_{M=1}^{2}A^{Lp}\partial_3^3\overline{\partial}_M\eta_p\,A^{Mi}+[\partial_3^2,A^{Lp}A^{mi}]\partial_3\partial_m\eta_p\right).\tag{3.34}$$

On the other hand, we write IB_{02} as

$$IB_{02}=\int_\Gamma \hat{A}^{3i}N_3\partial_3^4 Q\,\partial_3^4\eta_p\,A^{3p}\partial_3 v_i\,dy' \tag{3.35}$$

$$+\sum_{L=1}^{2}\int_\Gamma \hat{A}^{3i}N_3\partial_3^4 Q\,\partial_3^4\eta_p\,A^{Lp}\overline{\partial}_L v_i\,dy'.\tag{3.36}$$

Therefore, (3.36) exactly cancels with the main term (3.33) in IB_{012}.

Now it remains to control IB_{011}, IB_{013} and (3.34), (3.35). Invoking the relation

$$\hat{A}^{3i}\partial_3 Q=-\sum_{L=1}^{2}\hat{A}^{Li}\overline{\partial}_L Q-\rho_0\partial_t v^i+(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta^i),\tag{3.37}$$

we get

$$\begin{aligned}\hat{A}^{3i}\partial_3^4 Q =&\partial_3^3(\hat{A}^{3i}\partial_3 Q)-\partial_3^3\hat{A}^{3i}\,\partial_3 Q-3\partial_3^2\hat{A}^{3i}\,\partial_3^2 Q-3\partial_3\hat{A}^{3i}\,\partial_3^3 Q\\ =&\partial_3^3\left(-\rho_0\partial_t v^i+(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)\right)-\sum_{L=1}^{2}\partial_3^3(\hat{A}^{Li}\overline{\partial}_L Q)\\ &-\partial_3^3\hat{A}^{3i}\,\partial_3 Q-3\partial_3^2\hat{A}^{3i}\,\partial_3^2 Q-3\partial_3\hat{A}^{3i}\,\partial_3^3 Q.\end{aligned}\tag{3.38}$$

Note that

- The term $\hat{A}^{3i}$ is of the form $\overline{\partial}\eta\times\overline{\partial}\eta$, so the leading order term in $\partial_3^3 A^{3i}$ should be $(\partial_3^3\overline{\partial}\eta)(\overline{\partial}\eta)$.
- The highest order term in $\partial_3^3(\hat{A}^{Li}\overline{\partial}_L Q)$ is $\partial_3^3\hat{A}^{Li}\,\overline{\partial}_L Q=0$ due to $\overline{\partial}_L Q|_{\Gamma=0}$.
- The highest order term in $\partial_3^3((b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta))$ is $(b_0\cdot\overline{\partial})\partial_3^3(J^{-1}(b_0\cdot\partial)\eta)$ because $b_0^3|_\Gamma=0$ makes $(b_0\cdot\partial)$ tangential on the boundary.

Therefore, we can rewrite $\partial_3^4 Q$ to be the terms of at most 3 normal derivatives and one tangential derivative:

$$\begin{aligned}\partial_3^4 Q =&\underbrace{J^{-1}\hat{A}^{3i}\partial_3\eta_i}_{=1}\partial_3^4 Q=J^{-1}\partial_3\eta_i(\hat{A}^{3i}\partial_3^4 Q)\\ =&J^{-1}\partial_3\eta_i\Bigg(\partial_3^3\left(-\rho_0\partial_t v^i+(b_0\cdot\overline{\partial})(J^{-1}(b_0\cdot\partial)\eta)\right)-\sum_{L=1}^{2}\sum_{N=0}^{2}\binom{3}{N}(\partial_3^N\hat{A}^{Li})(\partial_3^{3-N}\overline{\partial}_L Q)\\ &-\partial_3^3\hat{A}^{3i}\,\partial_3 Q-3\partial_3^2\hat{A}^{3i}\,\partial_3^2 Q-3\partial_3\hat{A}^{3i}\,\partial_3^3 Q\Bigg).\end{aligned}\tag{3.39}$$

In (3.35), we need to rewrite $A^{3p}\partial_3^4\eta_p$ by using $A^{3p}\partial_3\eta_p=1$ in $\bar{\Omega}$ (and thus $\partial_3^3(A^{3p}\partial_3\eta_p)=0$)

$$A^{3p}\partial_3^4\eta_p=-\partial_3^3A^{3p}\,\partial_3\eta_p-3\partial_3^2A^{3p}\,\partial_3^2\eta_p-3\partial_3A^{3p}\,\partial_3^3\eta_p.\tag{3.40}$$

In the light of (3.38)-(3.40), we are able to write IB_{011}, IB_{013} and (3.34), (3.35) in the form of

$$\int_\Gamma N_3(\partial_3^3\mathfrak{D}f)(\partial_3^3\mathfrak{D}g)h\,dy' + \text{lower order terms}, \tag{3.41}$$

where $\mathfrak{D} = \overline{\partial}$ or ∂_t or $b_0\cdot\overline{\partial}$, and f, g can be $\eta, v, q, J^{-1}(b_0\cdot\partial)\eta$, and h contains at most first order derivative of η, v. Then (3.41) can be controlled in the following way

$$\begin{aligned}\int_\Gamma N_3(\partial_3^3\mathfrak{D}f)(\partial_3^3\mathfrak{D}g)h\,dy' =&\left(\int_\Omega(\partial_3^4\mathfrak{D}f)(\partial_3^3\mathfrak{D}g)h - \int_\Omega(\partial_3^3\mathfrak{D}f)(\partial_3^4\mathfrak{D}g)h - \int_\Omega(\partial_3^3\mathfrak{D}f)(\partial_3^3\mathfrak{D}g)(\partial_3 h)\right)\\ \overset{\mathfrak{D}}{=}&-\int_\Omega(\partial_3^4 f)(\partial_3^3\mathfrak{D}^2 g)h - \int_\Omega(\partial_3^4 f)(\partial_3^3\mathfrak{D}g)(\mathfrak{D}h)\\ &+\int_\Omega(\partial_3^3\mathfrak{D}^2 f)(\partial_3^4 g)h + \int_\Omega(\partial_3^3\mathfrak{D}f)(\partial_3^4 g)(\mathfrak{D}h) - \int_\Omega(\partial_3^3\mathfrak{D}f)(\partial_3^3\mathfrak{D}g)(\partial_3 h)\\ \lesssim&(\|\partial_3^4 f\|_0 + \|\partial_3^3\mathfrak{D}^2 f\|_0)(\|\partial_3^4 g\|_0 + \|\partial_3^3\mathfrak{D}^2 g\|_0)\|\partial h\|_{L^\infty} \lesssim \|f\|_{8,*}\|g\|_{8,*}\|h\|_3,\end{aligned} \tag{3.42}$$

which gives the control of IB_{011}, IB_{013} and (3.34), (3.35).

Remark. If we integrate $\mathfrak{D} = \partial_t$ by parts in (3.42) (such term appears in a leading order term $\partial_3^3\partial_t v$ in $\partial_3^4 Q$), then we should proceed the estimate under time integral and also consider the terms like $\int_\Omega(\partial_3^4 f)(\partial_3^3\mathfrak{D}g)h$ which can be controlled by

$$\begin{aligned}&\int_\Omega(\partial_3^4 v)(\partial_3^3\mathfrak{D}g)h \lesssim \varepsilon\|\partial_3^4 v\|_0^2 + \frac{1}{8\varepsilon}\|\partial_3^3\mathfrak{D}g\|_0^4 + \frac{1}{8\varepsilon}\|h\|_{L^\infty}^4\\ \lesssim&\varepsilon\|\partial_3^4 v\|_0^2 + \frac{1}{8\varepsilon}\left(\|g(0)\|_{7,*}^4 + \|h(0)\|_2^4 + \int_0^T \|\partial_3^3\mathfrak{D}\partial_t g(t)\|_0^4 + \|\partial_t h(t)\|_2^4\right)\\ \lesssim&\varepsilon\|\partial_3^4 v\|_0^2 + \mathcal{P}_0 + \int_0^T P(\|g\|_{8,*}, \|h\|_{5,*})\,dt.\end{aligned} \tag{3.43}$$

According to (3.42)-(3.43), we can finalize the estimates of the boundary integral IB as follows

$$IB \lesssim \varepsilon\|\partial_3^4 v\|_0^2 - \frac{c_0}{4}\frac{d}{dt}\int_\Gamma\left|A^{3i}\partial_3^4\eta_i\right|^2\,dy' + P(\mathcal{E}(t)). \tag{3.44}$$

3.4 Energy estimates of purely normal derivatives

Now, (3.44) together with (3.7), (3.8), (3.13), (3.22) gives the estimates of Alinhac good unknowns of v, Q in the case of purely non-weighted normal derivatives

$$\|\mathbf{V}\|_0^2 + \left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^4 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_3^4\eta_i\right|_0^2\Big|_{t=T} \lesssim \varepsilon\|\partial_3^4 v\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{3.45}$$

Finally, by the definition of Alinhac good unknown (3.2) and $\partial_3^4\eta|_{t=0} = \mathbf{0}$, $\partial_3^4 v$ is controlled by

$$\|\partial_3^4 v\|_0^2 \lesssim \|\mathbf{V}\|_0^2 + \|a\partial v\|_{L^\infty}^2\int_0^T\|\partial_3^4 v\|_0^2\,dt \lesssim \|\mathbf{V}\|_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt, \tag{3.46}$$

and thus by choosing $\varepsilon > 0$ sufficiently small, we get

$$\|\partial_3^4 v\|_0^2 + \left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^4 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_3^4\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{3.47}$$

4 Control of purely tangential derivatives

Now we consider the purely tangential derivatives. In this case, the top order derivative becomes $\partial_*^I = \partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_0+i_1+i_2 = 8$. We will prove the following estimates by a modified Alinhac good unknown method.

Proposition 4.1. The following energy inequality holds for any sufficiently small $\varepsilon > 0$

$$\sum_{i_3=i_4=0} \|\partial_*^I v\|_0^2 + \left\|\partial_*^I \left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_*^I\eta_i\right|_0^2\Big|_{t=T} \lesssim \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{4.1}$$

For simplicity, we mainly study the case $i_0 = 0$, i.e., $\partial_*^I = \overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_1 + i_2 = 8$. For sake of clean notations, we denote $\overline{\partial}^8 = \overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$. In fact, most of the steps of the proof in this section are completely applicable to the case of $i_0 > 0$.

4.1 The case of full spatial derivatives

4.1.1 Derivation of "modified Alinhac good unknowns" in anisotropic Sobolev space

We still use Alinhac good unknowns to control the tangential derivatives. However, we cannot directly replace ∂_3^4 by $\overline{\partial}^8$ in (3.2) because the commutator contains the terms like $\overline{\partial}^7\partial\eta$, $\overline{\partial}^7\partial v$ and $\overline{\partial}^7\partial Q$ whose L^2-norm cannot be controlled in H_*^8. In specific, we have

$$\begin{aligned}\overline{\partial}^8(\nabla_A^i f) =&\nabla_A^i(\overline{\partial}^8 f) + (\overline{\partial}^8 A^{li})\partial_l f + [\overline{\partial}^8, A^{li}, \partial_l f]\\ =&\nabla_A^i(\overline{\partial}^8 f) - \overline{\partial}^7(A^{lr}\,\overline{\partial}\partial_m\eta_r\,A^{mi})\partial_l f + [\overline{\partial}^8, A^{li}, \partial_l f]\\ =&\nabla_A^i(\overline{\partial}^8 f - \overline{\partial}^8\eta_r\,A^{lr}\,\partial_l f) + \overline{\partial}^8\eta_r\,\nabla_A^i(\nabla_A^r f) - ([\overline{\partial}^7, A^{lr}A^{mi}]\overline{\partial}\partial_m\eta_r)\partial_l f + [\overline{\partial}^8, A^{li}, \partial_l f].\end{aligned} \tag{4.2}$$

We notice that the $L^2(\Omega)$-norm of the following quantities coming from the last two terms of (4.2) cannot be controlled because $\overline{\partial}^7$ may fall on $a = \partial\eta\times\partial\eta$ and ∂f.

$$\begin{aligned}e_1 := -\overline{\partial}^7(A^{lr}A^{mi})\,\overline{\partial}\partial_m\eta_r\,\partial_l f,\quad e_2 := -7\overline{\partial}(A^{lr}A^{mi})\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f\\ e_3 := 8(\overline{\partial}^7 A^{li})(\overline{\partial}\partial_l f),\quad e_4 := 8(\overline{\partial}A^{li})(\overline{\partial}^7\partial_l f).\end{aligned} \tag{4.3}$$

Here $8\overline{\partial}^7$ means there are 8 terms of the form $\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_1 + i_2 = 7$. We will repeatedly use similar notations throughout the manuscript.

Our idea to overcome this difficulty is mainly based on the following three techniques:

1. Modify the definition of "Alinhac good unknowns": Rewrite these quantities in terms of $\nabla_A^i(\cdots) +$ L^2-bounded terms, and then merge the terms inside the covariant derivative ∇_A^i into the "Alinhac good unknowns".
2. Produce a weighted normal derivative to replace a non-weighted one: There are terms like $(\overline{\partial}^7\partial_3\eta)(\overline{\partial}Q)$. Since $Q|_\Gamma = 0$, we know $\overline{\partial}Q|_\Gamma = 0$. Therefore, we can estimate the L^∞-norm of $\overline{\partial}Q$ by fundamental theorem of calculus: (Suppose $y_3 > 0$ without loss of generality)
$$|\overline{\partial}Q(t,y_3)|_{L^\infty(\mathbb{T}^2)} = \left|0 + \int_1^{y_3}\overline{\partial}\partial_3 Q(t,\zeta_3)d\zeta_3\right|_{L^\infty(\mathbb{T}^2)} \le (1-y_3)\|\overline{\partial}\partial_3 Q\|_{L^\infty} \le \sigma(y_3)\|\overline{\partial}\partial_3 Q\|_{L^\infty},$$
then we move the $\sigma(y_3)$ to $\overline{\partial}^7\partial_3\eta$ to get a weighted normal derivative $(\sigma\partial_3)^1\overline{\partial}^7\eta$ whose L^2-norm can be directly bounded in H_*^8.
3. Replace $\nabla_{\hat{A}}Q$ (contains a normal derivative) by $-\rho_0\partial_t v + (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)$ (only contains tangential derivative) in order to make the order of the derivatives lower thanks to the anisotropy of H_*^m.

Now we analyze these extra terms from the commutator. We start with $8(\overline{\partial}^7 A^{li})(\overline{\partial}\partial_l f)$ and $8(\overline{\partial}A^{li})(\overline{\partial}^7\partial_l f)$ coming from $[\overline{\partial}^8, A^{li}, \partial_l f]$ in (4.2). Since $\overline{\partial}A^{li} = -A^{lp}\,\overline{\partial}\partial_m\eta_p\,A^{mi}$, we have

$$\overline{\partial}^7 A^{li} = -A^{lp}\,\overline{\partial}^7\partial_m\eta_p\,A^{mi} - [\overline{\partial}^6, A^{lp}A^{mi}]\partial_m\eta_p,$$

where the highest order term in $[\overline{\partial}^6, A^{lp}A^{mi}]\partial_m\eta_p$ is $\overline{\partial}^6\partial_m\eta_p$ whose L^2-norm can be directly bounded by $\|\eta\|_{8,*}$. Therefore, we have

$$\begin{aligned}8(\overline{\partial}^7 A^{li})(\overline{\partial}\partial_l f) =& -8(A^{mi}\,\partial_m\overline{\partial}^7\eta_p\,A^{lp})\overline{\partial}\partial_l f - 8([\overline{\partial}^6, A^{lp}A^{mi}]\partial_m\eta_p)\overline{\partial}\partial_l f\\ =& -8\nabla_A^i(\overline{\partial}^7\eta_p\,A^{lp}\overline{\partial}\partial_l f)\underbrace{+8\nabla_A^i(\nabla_A^p\overline{\partial}f)\overline{\partial}^7\eta_p - 8([\overline{\partial}^6, A^{lp}A^{mi}]\partial_m\eta_p)\overline{\partial}\partial_l f}_{=:C_1(f)}\\ =:& -8\nabla_A^i(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}f) + C_1(f),\end{aligned} \tag{4.4}$$

where $C_1(f)$ can be controlled by using $H^{1/2} \hookrightarrow L^3$ and $H^1 \hookrightarrow L^6$ in 3D domain

$$\begin{aligned}
C_1(f) \lesssim & \|a\|_{L^\infty}^2 \|\partial^2 \overline{\partial} f\|_{L^6} \|\overline{\partial}^7 \eta\|_{L^3} + \|a\, \overline{\partial} f\, \partial a\|_{L^\infty} \|\overline{\partial}^7 \eta\|_{L^2} + P(\|\eta\|_{8,*}) \|\overline{\partial} \partial f\|_{L^\infty} \\
\lesssim & \|a\|_{L^\infty}^2 \|\partial^2 \overline{\partial} f\|_1 \|\langle \overline{\partial} \rangle^{1/2} \overline{\partial}^7 \eta\|_0 + P(\|\eta\|_{8,*}) \|\overline{\partial} \partial f\|_{L^\infty} \\
\lesssim & \|a\|_{L^\infty}^2 \|\partial^2 \overline{\partial} f\|_1 \|\overline{\partial}^7 \eta\|_0^{1/2} \|\overline{\partial}^8 \eta\|_0^{1/2} + P(\|\eta\|_{8,*}) \|\overline{\partial} \partial f\|_{L^\infty} \\
\lesssim & P(\|\eta\|_{8,*}) \|f\|_{7,*}.
\end{aligned}$$

The term $8(\overline{\partial} A^{li})(\overline{\partial}^7 \partial_l f)$ should be treated differently in the case of $f = v_i$ and $f = Q$ respectively.

- When $f = v_i$, then this term becomes

$$\begin{aligned}
8(\overline{\partial} A^{li})(\overline{\partial}^7 \partial_l v_i) = & -8A^{lp}\, \overline{\partial} \partial_m \eta_p\, A^{mi} \overline{\partial}^7 \partial_l v_i = -8A^{li}\, \overline{\partial} \partial_m \eta_i\, A^{mp}\, \overline{\partial}^7 \partial_l v_p \\
= & -8\nabla_A^i (\overline{\partial}^7 v_p\, A^{mp}\, \overline{\partial} \partial_m \eta_i) + \underbrace{8\nabla_A^i (\overline{\partial} \partial_m \eta_i\, A^{mp}) \overline{\partial}^7 v_p}_{=:C_2(v)} \\
=: & -8\nabla_A^i (\overline{\partial}^7 v \cdot \nabla_A \overline{\partial} \eta_i) + C_2(v),
\end{aligned} \tag{4.5}$$

  and similarly we have $\|C_2(v)\|_0 \lesssim P(\|\eta\|_{7,*}) \|v\|_{8,*}$.

- When $f = Q$, we cannot mimic the simplification as above. Instead, we need to invoke the MHD equation to replace $\nabla_A Q$ by tangential derivatives. We consider

$$\begin{aligned}
8(J\overline{\partial} A^{li})(\overline{\partial}^7 \partial_l Q) = & -8(\hat{A}^{lp}\, \overline{\partial} \partial_m \eta_p\, A^{mi}) \overline{\partial}^7 \partial_l Q \\
= & -8\overline{\partial}^7 (\hat{A}^{lp} \partial_l Q)\, A^{mi} \overline{\partial} \partial_m \eta_p + 8(\overline{\partial}^7 \hat{A}^{lp})(\partial_l Q)(\overline{\partial} \partial_m \eta_p\, A^{mi}) \\
& + 8 \sum_{N=1}^{6} \binom{7}{N} (\overline{\partial}^N \hat{A}^{lp})(\overline{\partial}^{7-N} \partial_l Q)(\overline{\partial} \partial_m \eta_p\, A^{mi}) \\
= & 8\overline{\partial}^7 \left(\rho_0 \partial_t v^p - (b_0 \cdot \partial)(J^{-1} (b_0 \cdot \partial) \eta^p)\right) A^{mi} \overline{\partial} \partial_m \eta_p + 8(\overline{\partial}^7 \hat{A}^{lp})(\partial_l Q)(\overline{\partial} \partial_m \eta_p\, A^{mi}) \\
& + 8 \sum_{N=1}^{6} \binom{7}{N} (\overline{\partial}^N \hat{A}^{lp})(\overline{\partial}^{7-N} \partial_l Q)(\overline{\partial} \partial_m \eta_p\, A^{mi}) \\
=: & C_{21} + C_{22} + C_{23}.
\end{aligned} \tag{4.6}$$

  The L^2-norm of C_{23} can be directly controlled since the top order derivative is $\overline{\partial}^6 \partial$

$$\|C_{23}\|_0 \lesssim \|\eta\|_{8,*} \|Q\|_{8,*} P(\|\eta\|_{7,*}). \tag{4.7}$$

  The L^2-norm of C_{22} can be directly controlled when $l = 3$ because $\hat{A}^{3p}$ consists of $\overline{\partial} \eta \times \overline{\partial} \eta$. When $l = 1, 2$, we need to invoke the second technique above, i.e., using $\overline{\partial} Q|_\Gamma = 0$ to produce a weight function $\sigma(y_3)$.

$$\begin{aligned}
\|C_{22}\|_0 \lesssim & \|\overline{\partial}^7 \hat{A}^{3p}\|_0 \|\partial_3 Q\, \overline{\partial} \partial \eta\, a\|_{L^\infty} + \sum_{L=1}^{2} \|(\overline{\partial}^7 \hat{A}^{Lp})(\overline{\partial}_L Q)(\overline{\partial} \partial_m \eta_p\, A^{mi})\|_0 \\
\lesssim & P(\|\eta\|_{7,*}) \|\overline{\partial}^8 \eta\|_0 \|Q\|_3 + \sum_{L=1}^{2} \|(\overline{\partial}^7 \hat{A}^{Lp})(\sigma(y_3) \partial_3 \overline{\partial}_L Q)(\overline{\partial} \partial_m \eta_p\, A^{mi})\|_0 \\
\lesssim & P(\|\eta\|_{7,*}) \|\overline{\partial}^8 \eta\|_0 \|Q\|_3 + \sum_{L=1}^{2} \|\sigma \overline{\partial}^7 \hat{A}^{Lp}\|_0 \|(\partial_3 \overline{\partial}_L Q)(\overline{\partial} \partial_m \eta_p\, A^{mi})\|_{L^\infty} \\
\lesssim & P(\|\eta\|_{7,*}) \|Q\|_{7,*} \left(\|\overline{\partial}^8 \eta\|_0 + \|(\sigma \partial_3) \overline{\partial}^7 \eta\|_0\right),
\end{aligned} \tag{4.8}$$

  where we use the fact that $\hat{A}^{Lp}$ consists of $(\partial_3 \eta)(\overline{\partial} \eta)$ in the last step.

Finally, C_{21} can also be directly bounded because the top order derivatives are $\overline{\partial}^7\partial_t$ and $\overline{\partial}^7(b_0\cdot\partial)$. Note that $b_0^3|_\Gamma = 0$ yields the following estimates by using the second technique mentioned above.

$$\|b_0^3\partial_3\overline{\partial}^7(J^{-1}(b_0\cdot\partial)\eta)\|_0 \lesssim \|\partial b_0\|_2\|(\sigma\partial_3)\overline{\partial}^7(J^{-1}(b_0\cdot\partial)\eta)\|_0,$$

and thus

$$C_{21} \lesssim P(\|\eta\|_{7,*})(\|\rho_0\|_{7,*}\|v\|_{8,*} + \|b_0\|_{7,*}\|(b_0\cdot\partial)\eta\|_{8,*}). \tag{4.9}$$

Therefore, we have the estimates for $C_2(Q) := 8\overline{\partial}A^{li}\overline{\partial}^7\partial_l Q$

$$\|C_2(Q)\|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|b_0\|_{7,*}, \|J^{-1}(b_0\cdot\partial)\eta\|_{8,*}, \|\rho_0\|_{7,*}, \|Q\|_{8,*}). \tag{4.10}$$

Next we analyze $-(\overline{\partial}^7(A^{lr}A^{mi})\,\overline{\partial}\partial_m\eta_r)\partial_l f$ coming from $-([\overline{\partial}^7, A^{lr}A^{mi}]\overline{\partial}\partial_m\eta_r)\partial_l f$. There are two terms of top order derivatives:

$$-\overline{\partial}^7(A^{lr}A^{mi})\,\overline{\partial}\partial_m\eta_r\,\partial_l f = -(\overline{\partial}^7A^{lr})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l f - A^{lr}(\overline{\partial}^7A^{mi})\,\overline{\partial}\partial_m\eta_r\,\partial_l f - \sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{lr})(\overline{\partial}^{6-N}A^{mi})\overline{\partial}\partial_m\eta_r\,\partial_l f,$$

where the L^2-norm of the last term can be directly controlled

$$\left\|\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{lr})(\overline{\partial}^{6-N}A^{mi})\overline{\partial}\partial_m\eta_r\,\partial_l f\right\|_0 \lesssim P(\|\eta\|_{8,*})\|f\|_3.$$

Similarly as (4.4), the term $-A^{lr}(\overline{\partial}^7A^{mi})\overline{\partial}\partial_m\eta_r\,\partial_l f$ can be written as the covariant derivatives plus L^2-bounded terms

$$\begin{aligned}
-A^{lr}(\overline{\partial}^7A^{mi})\overline{\partial}\partial_m\eta_r\,\partial_l f =&A^{lr}A^{mp}(\partial_k\overline{\partial}^7\eta_p)A^{ki}\,\overline{\partial}\partial_m\eta_r\,\partial_l f + ([\overline{\partial}^6, A^{mp}A^{ki}]\overline{\partial}\partial_k\eta_p)A^{lr}\,\overline{\partial}\partial_m\eta_r\,\partial_l f\\
=&\nabla_A^i(\overline{\partial}^7\eta_p\,A^{mp}\,\overline{\partial}\partial_m\eta_r\,A^{lr}\,\partial_l f)\\
&\underbrace{-\overline{\partial}^7\eta_p\,\nabla_A^i(A^{mp}\,\overline{\partial}\partial_m\eta_r\,A^{lr}\,\partial_l f) + ([\overline{\partial}^6, A^{mp}A^{ki}]\overline{\partial}\partial_k\eta_p)A^{lr}\,\overline{\partial}\partial_m\eta_r\,\partial_l f}_{=:C_3(f)}\\
=:&\nabla_A^i(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A f) + C_3(f),
\end{aligned} \tag{4.11}$$

where $C_3(f)$ can be directly controlled similarly as $C_1(f)$

$$\|C_3(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial f\|_2.$$

We then compute $-(\overline{\partial}^7A^{lr})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l f$.

- When $f = v_i$: Similarly as in (4.11), we have

$$\begin{aligned}
-(\overline{\partial}^7A^{lr})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l v_i =&A^{lp}(\overline{\partial}^7\partial_k\eta_p)A^{kr}A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l v_i - ([\overline{\partial}^6, A^{lp}A^{kr}]\overline{\partial}\partial_k\eta_p)A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l v_i\\
=&A^{lp}(\overline{\partial}^7\partial_k\eta_p)A^{ki}A^{mr}\,\overline{\partial}\partial_m\eta_i\,\partial_l v_r - ([\overline{\partial}^6, A^{lp}A^{kr}]\overline{\partial}\partial_k\eta_p)A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l v_i\\
=&\nabla_A^i(\overline{\partial}^7\eta_p\,A^{lp}\,\partial_l v_r\,A^{mr}\,\overline{\partial}\partial_m\eta_i)\\
&\underbrace{-\nabla_A^i(A^{lp}A^{mr}\overline{\partial}\partial_m\eta_i\,\partial_l v_r)\overline{\partial}^7\eta_p - ([\overline{\partial}^6, A^{lp}A^{kr}]\overline{\partial}\partial_k\eta_p)A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l v_i}_{=:C_4(v)}\\
=:&\nabla_A^i(\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i) + C_4(v),
\end{aligned} \tag{4.12}$$

  where $C_4(v)$ can be directly controlled similarly as $C_1(f)$

$$\|C_4(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial v\|_2.$$

- When $f = Q$: If $l = 3$, then this term can be directly controlled since $A^{3r} = J^{-1}\overline{\partial}\eta\times\overline{\partial}\eta$ only contains first-order tangential derivatives. If $l = 1, 2$, then we can mimic the treatment of C_{22}, i.e., using $\overline{\partial}_L Q|_\Gamma = 0$ and fundamental theorem of calculus to produce a weight function $\sigma(y_3)$ and move that to $\overline{\partial}^7 A^{lr}$. Define $C_4(Q) := -(\overline{\partial}^7 A^{lr})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_l Q$, then

$$\begin{aligned}\|C_4(Q)\|_0 \lesssim&\|(\overline{\partial}^7 A^{3r})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\partial_3 Q\|_0 + \sum_{L=1}^{2}\|(\overline{\partial}^7 A^{Lr})A^{mi}\,\overline{\partial}\partial_m\eta_r\,\overline{\partial}_L Q\|_0\\ \lesssim&\|\overline{\partial}^8\eta\|_0\|Q\|_3 P(\|\partial\eta\|_2,\|\overline{\partial}\partial\eta\|_2) + \sum_{L=1}^{2}\|\sigma\overline{\partial}^7 A^{Lr}\|_0\|A^{mi}\,\overline{\partial}\partial_m\eta_r\,\overline{\partial}_L\partial_3 Q\|_{L^\infty}\\ \lesssim&\left(\|\overline{\partial}^8\eta\|_0 + \|(\sigma\partial_3)\overline{\partial}^7\eta\|_0\right)P(\|Q\|_3,\|\overline{\partial}Q\|_3,\|\eta\|_{7,*}).\end{aligned}\tag{4.13}$$

Next we analyze $-7\overline{\partial}(A^{lr}A^{mi})\overline{\partial}^7\partial_m\eta_r\,\partial_l f$ coming from $-[\overline{\partial}^7, A^{lr}A^{mi}]\overline{\partial}\partial_m\eta_r\,\partial_l f$. This term cannot be directly controlled when $m = 3$. We should analyze it term by term. First we have

$$\begin{aligned}-7\overline{\partial}(A^{lr}A^{mi})\overline{\partial}^7\partial_m\eta_r\,\partial_l f =& -7\overline{\partial}A^{lr}\,A^{mi}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f - 7A^{lr}\,\overline{\partial}A^{mi}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f\\ =&7A^{lp}\,\partial_k\overline{\partial}\eta_p\,A^{kr}A^{mi}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f + 7A^{lr}A^{mp}\,\partial_k\overline{\partial}\eta_p\,A^{ki}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f.\end{aligned}$$

The first term can be directly rewritten as follows

$$\begin{aligned}7A^{lp}\,\partial_k\overline{\partial}\eta_p\,A^{kr}A^{mi}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f =&7\nabla_i(\overline{\partial}^7\eta_r\,A^{kr}\,\partial_k\overline{\partial}\eta_p\,A^{lp}\,\partial_l f)\underbrace{-7\nabla_A^i(A^{kr}\,\partial_k\overline{\partial}\eta_p\,A^{lp}\,\partial_l f)\overline{\partial}^7\eta_r}_{C_5(f)}\\ =:&7\nabla_A^i(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A f) + C_5(f),\end{aligned}\tag{4.14}$$

where $C_5(f)$ can be similarly controlled as $C_1(f)$

$$\|C_5(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial f\|_3.$$

Then we analyze $7A^{lr}A^{mp}(\partial_k\overline{\partial}\eta_p)A^{ki}(\overline{\partial}^7\partial_m\eta_r)\partial_l f$, which needs different treatment for $f = v_i$ and $f = Q$ respectively.

- When $f = v_i$, we have the following simplification

$$\begin{aligned}&7A^{lr}A^{mp}\partial_k\overline{\partial}\eta_p\,A^{ki}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l v_i = 7A^{lr}A^{mi}\,\partial_k\overline{\partial}\eta_i\,A^{kp}\,\overline{\partial}^7\partial_m\eta_r\,\partial_l v_p\\ =&7\nabla_A^i(\overline{\partial}^7\eta_r\,A^{lr}\,\partial_l v_p\,A^{kp}\,\overline{\partial}\partial_k\eta_i) + \underbrace{7\nabla_A^i(A^{lr}\,\partial_l v_p\,A^{kp}\,\overline{\partial}\partial_k\eta_i)\overline{\partial}^7\eta_r}_{C_6(v)}\\ =:&7\nabla_A^i(\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i) + C_6(v),\end{aligned}\tag{4.15}$$

  and $\|C_6(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|v\|_3$ follows from direct computation.

- When $f = Q$, this term becomes

$$\begin{aligned}C_6(Q) :=& -7A^{lr}(\overline{\partial}A^{mi})(\overline{\partial}^7\partial_m\eta_r)\partial_l Q\\ =& -7\left(\underbrace{\overline{\partial}^7(A^{lr}\partial_m\eta_r)}_{=\overline{\partial}^7\delta_m^l=0} - (\overline{\partial}^7 A^{lr})(\partial_m\eta_r) - \sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{lr})(\overline{\partial}^{7-N}\partial_m\eta_r)\right)\overline{\partial}A^{mi}\,\partial_l Q\\ =&7(\overline{\partial}^7 A^{3r})\partial_m\eta_r\,\overline{\partial}A^{mi}\,\partial_3 Q + \sum_{L=1}^{2}(\overline{\partial}^7 A^{Lr})\partial_m\eta_r\,\overline{\partial}A^{mi}\,\overline{\partial}_L Q\\ &+ 7\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{lr})(\overline{\partial}^{7-N}\partial_m\eta_r)\overline{\partial}A^{mi}\,\partial_l Q\\ =:&C_{61} + C_{62} + C_{63}.\end{aligned}\tag{4.16}$$

  Since $A^{3r} = J^{-1}\overline{\partial}\eta\times\overline{\partial}\eta$, we know the top order term is of the form $\overline{\partial}^8\eta\cdot\overline{\partial}\eta$ and thus C_{61} can be directly controlled

$$\|C_{61}\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial_3 Q\|_2.$$

The term C_{62} can be treated in the same way as $C_4(Q)$ in (4.13) by using $\overline{\partial}_L Q|_\Gamma = 0$ to produce a weight function σ

$$\|C_{62}\|_0 \lesssim (\|(\sigma\partial_3)\overline{\partial}^7\eta\| + \|\overline{\partial}^8\eta\|_0)P(\|\eta\|_{7,*})\|\overline{\partial}\partial_3 Q\|_2 \lesssim P(\|\eta\|_{8,*})\|Q\|_{7,*}.$$

Finally, C_{63} can be directly controlled

$$\|C_{63}\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial Q\|_2,$$

and thus

$$\|C_6(Q)\|_0 \lesssim P(\|\eta\|_{8,*})\|Q\|_{7,*}. \tag{4.17}$$

Now we plug (4.4)-(4.5), (4.10)-(4.17) into (4.2) and define the "modified Alinhac good unknowns" of v and Q with respect to $\overline{\partial}^8$ as

$$\begin{aligned}
\mathbf{V}_i^* := & \overline{\partial}^8 v_i - \overline{\partial}^8\eta\cdot\nabla_A v_i \\
& - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}v_i - 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_i \\
& + \overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i + \overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i \\
& + 7\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i + 7\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i \\
= & \overline{\partial}^8 v_i - \overline{\partial}^8\eta\cdot\nabla_A v_i - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}v_i - 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_i + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i + 8\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i,
\end{aligned} \tag{4.18}$$

and

$$\mathbf{Q}^* := \overline{\partial}^8 Q - \overline{\partial}^8\eta\cdot\nabla_A Q - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}Q + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q. \tag{4.19}$$

Then the modified good unknowns satisfy the following relations

$$\overline{\partial}^8(\mathrm{div}_A v) = \nabla_A\cdot\mathbf{V}^* + \sum_{M=0}^{6} C_M(v), \quad \overline{\partial}^8(\nabla_A Q) = \nabla_A\mathbf{Q}^* + \sum_{M=0}^{6} C_M(Q), \tag{4.20}$$

where $C_0(f)$ comes from the directly controllable terms in the RHS of (4.2)

$$C_0(f) := \overline{\partial}^8\eta_r\,\nabla_A^i(\nabla_A^r f) - \sum_{N=2}^{6}\binom{7}{N}\overline{\partial}^N(A^{lr}A^{mi})\overline{\partial}^{7-N}(\overline{\partial}\partial_m\eta_r)\partial_l f + \sum_{N=2}^{6}\binom{8}{N}(\overline{\partial}^N A^{li})(\overline{\partial}^{8-N}\partial_l f), \tag{4.21}$$

satisfies

$$\|C_0(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|f\|_{8,*},$$

and $C_1 \sim C_6$ are constructed in (4.4)-(4.5), (4.10)-(4.17).

4.2 Energy estimates of purely tangential derivatives

We denote $C^*(f) := C_0(f) + C_1(f) + \cdots + C_6(f)$ and the "extra modification terms" in the modified Alinhac good unknowns by

$$\begin{aligned}
(\Delta_v^*)_i := & -8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}v_i - 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_i + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i + 8\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i, \\
\Delta_Q^* := & -8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}Q + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q.
\end{aligned}$$

Then the modified Alinhac good unknowns become

$$\mathbf{V}^* = \overline{\partial}^8 v - \overline{\partial}^8\eta\cdot\nabla_A v + \Delta_v^*, \quad \mathbf{Q}^* = \overline{\partial}^8 Q - \overline{\partial}^8\eta\cdot\nabla_A Q + \Delta_Q^*.$$

Remark. There are more modification terms in $\mathbf{V}^*$ than in $\mathbf{Q}^*$. The reason is that we can replace $\nabla_A Q$ which contains a normal derivative with tangential derivative ($\partial_t v$ and $(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)$) by invoking the MHD equation. However, similar relation only holds for $\mathrm{div}_A v$ instead of $\nabla_A v$. Therefore, for those terms in the commutators containing v, we have to rewrite them to be the covariant derivatives of the modifition terms plus $L^2(\Omega)$-bounded terms.

It is straightforward to see that the $L^2(\Omega)$ norms of $\Delta_v^*, \Delta_Q^*, \partial_t(\Delta_v^*)$ and $\partial_t(\Delta_Q)$ can be controlled by $P(\mathcal{E}(t))$

$$
\begin{aligned}
\|\partial_t(\Delta_v^*)\|_0 \lesssim &\|\overline{\partial}^7 v\|_0(\|\nabla_A\overline{\partial}v\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A v\|_2) + \|\overline{\partial}^7\partial_t v\|_0\|\nabla_A\overline{\partial}\eta\|_2 \\
&+ \|\overline{\partial}^7\eta\|_0(\|\nabla_A\overline{\partial}\partial_t v\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A\partial_t v\|_2 + \|\nabla_A\overline{\partial}v\|_2\|\nabla_A v\|_2) \\
\lesssim &P(\|\eta\|_{8,*}, \|v\|_{8,*}),
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
\|\partial_t(\Delta_Q^*)\|_0 \lesssim &\|\overline{\partial}^7 v\|_0(\|\nabla_A\overline{\partial}Q\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A Q\|_2) \\
&+ \|\overline{\partial}^7\eta\|_0(\|\nabla_A\overline{\partial}\partial_t Q\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A\partial_t Q\|_2 + \|\nabla_A\overline{\partial}v\|_2\|\nabla_A Q\|_2) \\
\lesssim &P(\|\eta\|_{8,*}, \|v\|_{7,*}, \|Q\|_{8,*}),
\end{aligned} \tag{4.23}
$$

$$
\|\Delta_Q^*\|_0 + \|\Delta_v^*\|_0 \lesssim P(\|\eta\|_{7,*}, \|v\|_{7,*}, \|Q\|_{7,*}). \tag{4.24}
$$

Now we take $\overline{\partial}^8$ in the second equation of compressible MHD system (1.17) to get

$$
R\partial_t(\overline{\partial}^8 v) - J^{-1}(b_0\cdot\partial)\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right) + \overline{\partial}^8(\nabla_A Q) = \left[R, \overline{\partial}^8\right]\partial_t v + \left[\overline{\partial}^8, J^{-1}(b_0\cdot\partial)\right]\left(J^{-1}(b_0\cdot\partial)\eta\right).
$$

Then invoking (4.20) to get

$$
R\partial_t(\overline{\partial}^8 v) - J^{-1}(b_0\cdot\partial)\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right) + \nabla_A\mathbf{Q}^* = \left[R, \overline{\partial}^8\right]\partial_t v + \left[\overline{\partial}^8, J^{-1}(b_0\cdot\partial)\right]\left(J^{-1}(b_0\cdot\partial)\eta\right) - C^*(Q).
$$

Finally, plugging $\mathbf{V}^* = \overline{\partial}^8 v - \overline{\partial}^8\eta\cdot\nabla_A v + \Delta_v^*$ yields the evolution equation of $\mathbf{V}^*$ and $\mathbf{Q}^*$

$$
\begin{aligned}
R\partial_t\mathbf{V}^* - J^{-1}(b_0\cdot\partial)\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right) + \nabla_A\mathbf{Q}^* = &\left[R, \overline{\partial}^8\right]\partial_t v + \left[\overline{\partial}^8, J^{-1}(b_0\cdot\partial)\right]\left(J^{-1}(b_0\cdot\partial)\eta\right) \\
&- C^*(Q) + R\partial_t(-\overline{\partial}^8\eta\cdot\nabla_A v + \Delta_v^*)
\end{aligned} \tag{4.25}
$$

We denote the RHS of (4.25) by $\mathbf{F}^*$. Similarly as in Section 3, we compute the L^2-inner product of (4.25) and $J\mathbf{V}^*$ to get the energy identity

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\mathbf{V}^*|^2\,dy = \int_\Omega (b_0\cdot\partial)\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}^* - \int_\Omega (\nabla_{\hat{A}}\mathbf{Q}^*)\cdot\mathbf{V}^* + \int_\Omega J\mathbf{F}^*\cdot\mathbf{V}^*. \tag{4.26}
$$

4.2.1 Interior estimates

Using (4.22), the third integral on RHS of (4.26) is controlled by direct computation

$$
\int_\Omega J\mathbf{F}^*\cdot\mathbf{V}^* \lesssim \|J\mathbf{F}^*\|_0\|\mathbf{V}^*\|_0 \lesssim P\left(\|(\rho_0, \eta, v, Q, b_0, (b_0\cdot\partial)\eta)\|_{8,*}\right)\|\mathbf{V}^*\|_0. \tag{4.27}
$$

The first integral on RHS of (4.26) can be similarly treated as (3.9)-(3.13) by replacing ∂_3^4 by $\overline{\partial}^8$ and $\|\cdot\|_4$-norm by $\|\cdot\|_{8,*}$-norm. We omit the details and list the result

$$
\int_\Omega (b_0\cdot\partial)\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}^*\,dy \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\overline{\partial}^8((b_0\cdot\partial)\eta)\right|^2\,dy + K_{11}^* + P\left(\|(\eta, v, b_0, (b_0\cdot\partial)\eta)\|_{8,*}\right), \tag{4.28}
$$

where K_{11}^* is defined to be

$$
K_{11}^* := -\int_\Omega J\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_A v)\,dy. \tag{4.29}
$$

Next we analyze the term $-\int_\Omega J\nabla_A\mathbf{Q}\cdot\mathbf{V}$. Integrating by parts and using Piola's identity, we get

$$
-\int_\Omega (\nabla_{\hat{A}}\mathbf{Q}^*)\cdot\mathbf{V}^* = \int_\Omega J\mathbf{Q}(\nabla_A\cdot\mathbf{V}^*) - \int_\Gamma J\mathbf{Q}A^{li}N_l\mathbf{V}_i^*\,dy' =: I^* + IB^*. \tag{4.30}
$$

Invoking (4.18), (4.20) and $Q = q + \frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2$, we get

$$
\begin{aligned}
I^* =& \int_\Omega J\overline{\partial}^8 q\,\overline{\partial}^8(\mathrm{div}_A v) + \int_\Omega J\overline{\partial}^8\left(\frac{1}{2}\left|J^{-1}(b_0\cdot\partial)\eta\right|^2\right)\overline{\partial}^8(\mathrm{div}_A v)\\
&+\int_\Omega(-\overline{\partial}^8\eta_p\,\hat{A}^{lp}\,\partial_l Q+\Delta_Q^*)\overline{\partial}^8(\mathrm{div}_A v) - \int_\Omega \overline{\partial}^8 Q\,C^*(v)\\
=:& I_1^*+I_2^*+I_3^*+I_4^*,
\end{aligned}
\tag{4.31}
$$

where I_4^* can be directly controlled by using the estimates of $C^*(v)$

$$
I_4^* \lesssim \|\overline{\partial}^8 Q\|_0\|C^*(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|\overline{\partial}^8 Q\|_0\|v\|_{8,*}. \tag{4.32}
$$

Similarly, I_2^* produces another higher order term to cancel with K_{11}^*

$$
\begin{aligned}
I_2 =& \underbrace{\int_\Omega J\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_A v)}_{\text{exactly cancel with } K_{11}^*}\\
&+\sum_{N=1}^{7}\binom{8}{N}\int_\Omega J\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_A v)\\
=&-K_{11}^*-\sum_{N=1}^{7}\binom{8}{N}\int_\Omega\frac{J^2R'(q)}{\rho_0}\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8\partial_t q\\
&-\sum_{N=1}^{7}\binom{8}{N}\int_\Omega J\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\overline{\partial}^8,\frac{JR'(q)}{\rho_0}\right]\partial_t q\right)\\
=:&-K_{11}^*+I_{21}^*+I_{22}^*
\end{aligned}
\tag{4.33}
$$

Similarly as in (3.19)-(3.20), the term I_{21}^* should be controlled by integrating ∂_t by parts under time integral and I_{22}^* can be directly controlled. We omit the details

$$
\int_0^T I_{21}^* \lesssim \varepsilon\|\overline{\partial}^8 q\|_0^2+\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt \tag{4.34}
$$

$$
I_{22}^* \lesssim \|J^{-1}(b_0\cdot\partial)\eta\|_{7,*}^2\|q\|_{8,*}. \tag{4.35}
$$

The term I_1^* produces the energy term $\|\overline{\partial}^8 q\|_0^2$ as in (3.17).

$$
I_1^* \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{J^2R'(q)}{\rho_0}|\overline{\partial}^8 q|^2+P(\|q\|_{8,*},\|\rho_0\|_{8,*},\|\eta\|_{8,*}). \tag{4.36}
$$

I_3^* can be controlled by integrating ∂_t by parts under time integral after invoking $\mathrm{div}_A v = -\frac{JR'(q)}{\rho_0}\partial_t q$ and (4.23)-(4.24).

$$
\begin{aligned}
\int_0^T I_3^* =& \int_0^T\int_\Omega\frac{JR'(q)}{\rho_0}(\overline{\partial}^8\eta_p\,\hat{A}^{lp}\,\partial_l Q-\Delta_Q^*)\overline{\partial}^8\partial_t q+\underbrace{\int_0^T\int_\Omega(\overline{\partial}^8\eta_p\,\hat{A}^{lp}\,\partial_l Q-\Delta_Q^*)\left(\left[\overline{\partial}^8,\frac{JR'(q)}{\rho_0}\right]\partial_t q\right)}_{L_2^*}\\
\overset{\partial_t}{=}&-\int_0^T\int_\Omega\partial_t\left(\frac{JR'(q)}{\rho_0}\overline{\partial}^8\eta_p\,\hat{A}^{lp}\,\partial_l Q-\Delta_Q^*\right)\overline{\partial}^8 q+\int_\Omega\frac{JR'(q)}{\rho_0}(\overline{\partial}^8\eta_p\,\hat{A}^{lp}\,\partial_l Q-\Delta_Q^*)\overline{\partial}^8 q\Big|_0^T+L_2^*\\
\lesssim&\mathcal{P}_0+\left(\left\|\frac{JR'(q)}{\rho_0}A\,\partial Q\right\|_{L^\infty}\|\overline{\partial}^8\eta\|_0+\|\Delta_Q^*\|_0\right)\|\overline{\partial}^8 q\|_0+\int_0^T P\left(\|(\eta,v,q,\rho_0)\|_{8,*}\right)\,dt.\\
\lesssim&\mathcal{P}_0+\varepsilon\|\overline{\partial}^8 q\|_0^2+\left\|\frac{JR'(q)}{\rho_0}A\,\partial Q\right\|_{L^\infty}^4+\|\overline{\partial}^8\eta\|_0^4+\|\Delta_Q^*\|_0^2+\int_0^T P(\mathcal{E}(t))\,dt\\
\lesssim&\mathcal{P}_0+\varepsilon\|\overline{\partial}^8 q\|_0^2+\int_0^T\left\|\partial_t\left(\frac{JR'(q)}{\rho_0}A\,\partial Q\right)\right\|_{L^\infty}^4+\|\overline{\partial}^8 v(t)\|_0^4+\|\partial_t(\Delta_Q^*)\|_0^2\,dt+\int_0^T P(\mathcal{E}(t))\,dt\\
\lesssim&\varepsilon\|\overline{\partial}^8 q\|_0^2+\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt,
\end{aligned}
\tag{4.37}
$$

Summarizing (4.31)-(4.37) and choosing $\varepsilon > 0$ to be sufficiently small, we get the estimates of I^* under time integral

$$\int_0^T I^*\,dt \lesssim -\frac{1}{2}\int_\Omega \frac{J^2R'(q)}{\rho_0}\left|\overline{\partial}^8 q\right|^2\,dy\Big|_0^T + \mathcal{P}_0 + \int_0^T P(\mathcal{E}(t))\,dt. \tag{4.38}$$

4.2.2 Boundary estimates

Now it remains to deal with the boundary integral IB^*. Since $Q|_\Gamma = 0$, we know

$$\mathbf{Q}^*|_\Gamma = -\overline{\partial}^8\eta_p\,A^{3p}\,\partial_3 Q + \Delta_Q^*,$$

and

$$\Delta_Q^*|_\Gamma = -8\overline{\partial}^7\eta_p\,A^{3p}\,\overline{\partial}\partial_3 Q + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta_r\,A^{3r}\,\partial_3 Q.$$

Then the boundary integral IB^* reads

$$\begin{aligned} IB^* =& \int_\Gamma \hat{A}^{3i}N_3\overline{\partial}^8\eta_p\,A^{3p}\,\partial_3 Q\,\overline{\partial}^8 v_i\,dy' - \int_\Gamma \hat{A}^{3i}N_3(\overline{\partial}^8\eta_p A^{3p}\partial_3 Q)(\overline{\partial}^8\eta\cdot\nabla_A v_i)\,dy' \\ &- \int_\Gamma \hat{A}^{3i}N_3\Delta_Q^*\,\overline{\partial}^8 v_i\,dy' + \int_\Gamma \hat{A}^{3i}N_3\Delta_Q^*\,\overline{\partial}^8\eta\cdot\nabla_A v_i\,dy' \\ &- \int_\Gamma \hat{A}^{3i}N_3\Delta_Q^*(\Delta_v^*)_i\,dy' + \int_\Gamma \hat{A}^{3i}N_3(\overline{\partial}^8\eta_p\,A^{3p}\,\partial_3 Q)(\Delta_v^*)_i\,dy' \\ =:& IB_1^* + IB_2^* + IB_3^* + IB_4^* + IB_5^* + IB_6^*. \end{aligned} \tag{4.39}$$

Before going to the proof, we would like to state our basic strategy to deal with the boundary control

- IB_1^* together with the Raylor-Taylor sign condition gives the boundary energy $|A^{3i}\overline{\partial}^8\eta_i|_0^2$ and the extra terms can be cancelled by IB_2^*. This step also appears in the study of Euler equations [8, 13, 42, 43, 46] and incompressible MHD [28, 24, 21, 22] and compressible resistive MHD [74]. It actually gives the control of the second fundamental form of the free surface [8].
- IB_3^*: We can write $\overline{\partial}^8 v_i = \overline{\partial}^8\partial_t\eta_i$ and integrate ∂_t by parts. When ∂_t falls on Δ_Q^*, the boundary integral can be directly controlled by using trace lemma. When ∂_t falls on $\hat{A}^{3i}$, such terms exactly cancel with the top order term in IB_4^*.
- IB_5^* and IB_6^*: Direct computation together with the trace lemma gives the control.

We first compute IB_1^*. Similarly as (3.24), we have

$$\begin{aligned} IB_1^* =& -\int_\Gamma\left(-\frac{\partial Q}{\partial N}\right)JA^{3i}\overline{\partial}^8\eta_p\,A^{3p}\,\overline{\partial}^8\partial_t\eta_i\,dy' \\ =& -\frac{1}{2}\frac{d}{dt}\int_\Gamma\left(-J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\overline{\partial}^8\eta_i\right|^2\,dy' \\ &-\frac{1}{2}\int_\Gamma\partial_t\left(J\frac{\partial Q}{\partial N}\right)\left|A^{3i}\overline{\partial}^8\eta_i\right|^2\,dy' + \int_\Gamma\left(-J\frac{\partial Q}{\partial N}\right)\partial_t A^{3i}\,\overline{\partial}^8\eta_p\,A^{3p}\,\overline{\partial}^8\eta_i\,dy' \\ =:& IB_{11}^* + IB_{12}^* + IB_{13}^*, \end{aligned} \tag{4.40}$$

The term IB_{11}^* together with the Rayleigh-Taylor sign condition gives the boundary energy

$$\int_0^T IB_{11}^* \le -\frac{c_0}{4}\left|A^{3i}\overline{\partial}^8\eta_i\right|_0^2\Big|_0^T, \tag{4.41}$$

and IB_{12}^* can be directly controlled by the boundary energy

$$IB_{12}^* \lesssim \left|A^{3i}\overline{\partial}^8\eta_i\right|_0^2\left|\partial_t\left(J\frac{\partial Q}{\partial N}\right)\right|_{L^\infty} \lesssim P(\mathcal{E}(t)). \tag{4.42}$$

Then we plug $\partial_t A^{3i} = -A^{3r}\partial_k v_r A^{ki}$ into IB_{13}^* to get the cancellation structure

$$\begin{aligned} IB_{13}^* =& \int_\Gamma J\frac{\partial Q}{\partial N}A^{3r}\,\partial_k v_r A^{ki}\,\overline{\partial}^8\eta_p\,A^{3p}\overline{\partial}^8\eta_i \\ =& \int_\Gamma J\frac{\partial Q}{\partial N}A^{3i}\,\partial_k v_i\,A^{kr}\,\overline{\partial}^8\eta_p\,A^{3p}\,\overline{\partial}^8\eta_r = -IB_2^* \end{aligned} \tag{4.43}$$

Next we analyze IB_3^*. We write $v_i = \partial_t \eta_i$ and integrate this ∂_t by parts

$$
\begin{aligned}
\int_0^T IB_3^* =& - \int_0^T \int_\Gamma JA^{3i} N_3 \Delta_Q^* \,\overline{\partial}^8 \partial_t \eta_i \, dy' \, dt \\
&\overset{\partial_t}{=} \int_0^T \int_\Gamma J \,\partial_t A^{3i}\, N_3 \Delta_Q^* \,\overline{\partial}^8 \eta_i \, dy' \, dt \\
&\quad - \int_0^T \int_\Gamma A^{3i} N_3 \,\partial_t (J\Delta_Q^*)\, \overline{\partial}^8 \eta_i \, dy' \, dt - \int_\Gamma JA^{3i} N_3 \,\Delta_Q^* \,\overline{\partial}^8 \eta_i \, dy' \Big|_0^T \\
&=: IB_{31}^* + IB_{32}^* + IB_{33}^*.
\end{aligned}
\tag{4.44}
$$

Again, plug $\partial_t A^{3i} = -A^{3r}\partial_k v_r A^{ki}$ into IB_{31}^* to get the cancellation with IB_4^*

$$
\begin{aligned}
IB_{31}^* =& - \int_0^T \int_\Gamma JA^{3r} \,\partial_k v_r \, A^{ki} \, N_3 \, \Delta_Q^* \,\overline{\partial}^8 \eta_i \, dy' \, dt \\
=& - \int_0^T \int_\Gamma JA^{3i} \,\partial_k v_i \, A^{kr} \, N_3 \, \Delta_Q^* \,\overline{\partial}^8 \eta_r \, dy' \, dt = -IB_4^*.
\end{aligned}
\tag{4.45}
$$

For IB_{33}^*, we use the fact that $\overline{\partial}^7 \eta|_{t=0} = 0$ (and thus $\Delta_Q^*|_\Gamma = 0$ when $t = 0$) together with Lemma 2.1 to get

$$
\begin{aligned}
&\int_\Gamma JA^{3i} N_3 \Delta_Q^* \,\overline{\partial}^8 \eta_i \, dy' \Big|_{t=T} = - \int_\Gamma JA^{3i} N_3 (8\overline{\partial}^7 \eta_p \, A^{3p} \,\overline{\partial}\partial_3 Q - 8\overline{\partial}^7 \eta \cdot \nabla_A \overline{\partial}\eta_r \, A^{3r} \,\partial_3 Q) \overline{\partial}^8 \eta_i \, dy' \\
\lesssim & \left| A^{3i} \overline{\partial}^8 \eta_i \right|_0 |J|_{L^\infty} (|A^{3p} \overline{\partial}\partial_3 Q|_{L^\infty} + |(\nabla_A \overline{\partial}\eta_r) A^{3r} \partial_3 Q|_{L^\infty}) \int_0^T |\overline{\partial}^7 v(t)|_0 \, dt \\
\lesssim & \left| A^{3i} \overline{\partial}^8 \eta_i \right|_0 P(\|\eta\|_{8,*}, \|Q\|_{8,*}) \int_0^T \|v(t)\|_{8,*} \, dt.
\end{aligned}
\tag{4.46}
$$

In IB_{32}^*, we invoke the relation (3.37) to get

$$
\begin{aligned}
\partial_t (J\Delta_Q^*)|_\Gamma =& - 8\overline{\partial}^7 v_p \, \hat{A}^{3p} \,\overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v \cdot \nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \partial_3 Q \\
& - 8\overline{\partial}^7 \eta_p \, \partial_t (\hat{A}^{3p} \overline{\partial}\partial_3 Q) + 8\overline{\partial}^7 \eta \cdot \partial_t (\nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \partial_3 Q) \\
=& - 8\overline{\partial}^7 v_p \, \hat{A}^{3p} \,\overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v \cdot \nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \partial_3 Q \\
& - 8\overline{\partial}^7 \eta_p \partial_t \overline{\partial} (\hat{A}^{3p} \partial_3 Q) + 8\overline{\partial}^7 \eta_p \partial_t (\overline{\partial}\hat{A}^{3p} \,\partial_3 Q) + 8\overline{\partial}^7 \eta \cdot \partial_t (\nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \,\partial_3 Q) \\
\overset{(3.37)}{=}& - 8\overline{\partial}^7 v_p \, \hat{A}^{3p} \,\overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v \cdot \nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \partial_3 Q \\
& + 8\overline{\partial}^7 \eta_p \partial_t \overline{\partial} \left(\rho_0 \partial_t v^p - (b_0 \cdot \overline{\partial})(J^{-1}(b_0 \cdot \partial)\eta)^p \right) + 8\overline{\partial}^7 \eta_p \, \partial_t (\overline{\partial}\hat{A}^{3p} \,\partial_3 Q) + 8\overline{\partial}^7 \eta \cdot \partial_t (\nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \partial_3 Q).
\end{aligned}
$$

Then we use $H^{\frac{3}{2}}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, Lemma 2.1 and standard Sobolev trace lemma to get

$$
\begin{aligned}
\left| \partial_t (J\Delta_Q^*)|_\Gamma \right|_0 \lesssim & |\overline{\partial}^7 v|_0 \left(|A \,\overline{\partial}\partial Q|_{L^\infty} + |a \,\overline{\partial}\partial\eta \, A \,\partial Q|_{L^\infty} \right) \\
& + |\overline{\partial}^7 \eta_p|_0 \left(\left| \rho_0 \partial_t^2 v^p + \partial_t (b_0 \cdot \overline{\partial}) b^p \right|_{W^{1,\infty}(\mathbb{T}^2)} + \left| \partial_t (\overline{\partial}\hat{A}^{3p} \,\partial_3 Q) + \partial_t (\nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \,\partial_3 Q) \right|_{L^\infty} \right) \\
\lesssim & \|v\|_{8,*} \|\partial Q\|_3 P(\|\partial\eta\|_3) \\
& + \|\eta\|_{8,*} \left(\|\rho_0 \partial_t^2 v + \partial_t (b_0 \cdot \overline{\partial}) b^p\|_3 + \left\| \partial_t (\overline{\partial}\hat{A}^{3p} \,\partial_3 Q) + \partial_t (\nabla_A \overline{\partial}\eta_r \, \hat{A}^{3r} \,\partial_3 Q) \right\|_2 \right) \\
\lesssim & P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{8,*}, \|b\|_{8,*}, \|\rho_0\|_3),
\end{aligned}
$$

and thus

$$
IB_{32}^* \lesssim \int_0^T \left| A^{3i} \overline{\partial}^8 \eta_i \right|_0 P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{8,*}, \|b\|_{8,*}, \|\rho_0\|_3) \, dt.
\tag{4.47}
$$

From (4.39), we know it suffices to control the product of "error part" IB_5^*

$$
IB_5^* \lesssim |\hat{A}^{3i}|_{L^\infty} |\Delta_Q^*|_\Gamma|_0 |(\Delta_v^*)_i|_0,
$$

and the RHS can be directly controlled by Lemma 2.1 and standard trace lemma

$$\left|\Delta_Q^*|_\Gamma\right|_0 \lesssim \left|\overline{\partial}^7\eta_p\right|_0 \left(|A^{3p}\overline{\partial}\partial_3 Q|_{L^\infty} + |\nabla_A^p\overline{\partial}\eta_r\, A^{3r}\partial_3 Q|_{L^\infty}\right) \lesssim P(\|\eta\|_{8,*}, \|Q\|_{7,*}),$$

$$\begin{aligned}\left|\Delta_v^*\right|_0 \lesssim &|\overline{\partial}^7\eta|_0 \left(|\nabla_A\overline{\partial}v|_{L^\infty} + |\nabla_A\overline{\partial}\eta\cdot\nabla_A v|_{L^\infty} + |\nabla_A v\cdot\nabla_A\overline{\partial}\eta|_{L^\infty}\right) + |\overline{\partial}^7 v|_0|\nabla_A\overline{\partial}\eta|_{L^\infty}\\ \lesssim &P(\|\eta\|_{8,*}, \|v\|_{8,*}).\end{aligned}$$

Therefore,

$$IB_5^* \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}), \tag{4.48}$$

and similarly

$$IB_6^* \lesssim |\hat{A}^{3i}\partial_3 Q|_{L^\infty}|A^{3p}\overline{\partial}^8\eta_p|_0|(\Delta_v^*)_i|_0. \tag{4.49}$$

Summarizing (4.39)-(4.49) gives the control of the boundary integral

$$\int_0^T IB^* \lesssim -\frac{c_0}{4}\left|A^{3i}\overline{\partial}^8\eta_i\right|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{4.50}$$

Combining (4.26), (4.27), (4.28), (4.38) and (4.50) and choosing $\varepsilon > 0$ in (4.34) to be suitably small, we get the following energy inequality

$$\|\mathbf{V}^*\|_0^2 + \left\|\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\overline{\partial}^8 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\overline{\partial}^8\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{4.51}$$

Finally, invoking (4.18), we get the $\overline{\partial}^8$-estimates of v

$$\|\overline{\partial}^8 v\|_0 \lesssim \|\mathbf{V}^*\|_0 + \|\overline{\partial}^8\eta\|_0\|\nabla_A v_i\|_{L^\infty} + \|\overline{\partial}^7\eta\|_0\left(\|\nabla_A\overline{\partial}v\|_{L^\infty} + \|\nabla_A\overline{\partial}\eta\cdot\nabla_A v\|_{L^\infty}\right) + \|\overline{\partial}^7 v\|_0\|\nabla_A\overline{\partial}\eta\|_{L^\infty}.$$

Since $\partial^m\eta|_{t=0} = 0$ for any $m \geq 2, m\in\mathbb{N}^*$, we know

$$\|\overline{\partial}^8 v\|_0 \lesssim \|\mathbf{V}^*\|_0 + P(\|v\|_{7,*}, \|\eta\|_{7,*})\int_0^T P(\|v\|_{8,*}), \tag{4.52}$$

and thus

$$\|\overline{\partial}^8 v\|_0^2 + \left\|\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\overline{\partial}^8 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\overline{\partial}^8\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{4.53}$$

4.3 The case of one time derivative $\overline{\partial}^7\partial_t$

If we replace $\partial_*^I = \overline{\partial}^8$ by $\overline{\partial}^7\partial_t$, then most of steps in the proof above are still applicable because we do not integrate the derivative(s) in $\mathfrak{D}^8$ by parts. However, we still need to do the following modifications due to the presence of time derivative.

4.3.1 Extra difficulty: non-vanishing initial data of $\partial_*^I\eta$

If $\partial_*^I = \overline{\partial}^7\partial_t$, then we can no longer derive $\overline{\partial}^7\partial_t\eta|_{t=0} = \mathbf{0}$ from $\eta|_{t=0} = \mathrm{Id}$ due to the presence of time derivative and $\partial_t\eta = v$. This property is used in the analysis of IB_{33}^* and the control of the difference between $\mathbf{V}^*$ and $\partial_*^I v$. Before we analyze the analogues of IB_{33}^* and (4.52) in the case of $\partial_*^I = \overline{\partial}^7\partial_t$, we have to find out the precise form of the modified Alinhac good unknowns when $\partial_*^I = \overline{\partial}^7\partial_t$.

4.3.2 The modified Alinhac good unknowns

Recall the "extra modification terms" Δ_Q^*, Δ_v^* in (4.25) come from the bad terms (4.3). Now we replace $\overline{\partial}^8$ by $\overline{\partial}^7\partial_t$. In e_1, e_2, e_3 in (4.3), if we replace $\overline{\partial}^7$ by $\overline{\partial}^6\partial_t$ (i.e., the time derivative falls on the higher order term), then their L^2 norms can be directly controlled since $\partial_t a$ has the same spatial regularity as a. Therefore, the remaining quantities whose L^2-norms cannot be directly controlled in the case of $\partial_*^I = \overline{\partial}^7\partial_t$ are

$$\begin{aligned} e_1 &:= -\overline{\partial}^7(A^{lr}A^{mi})\partial_t\partial_m\eta_r\,\partial_l f, \quad e_2 := -7\partial_t(A^{lr}A^{mi})\,\overline{\partial}^7\partial_m\eta_r\,\partial_l f \\ e_3 &:= 8(\overline{\partial}^7A^{li})\partial_t\partial_l f, \quad e_4 := (\partial_t A^{li})(\overline{\partial}^7\partial_l f) + 7(\overline{\partial}A^{li})(\overline{\partial}^6\partial_t\partial_l f). \end{aligned} \tag{4.54}$$

Then the corresponding Alinhac good unknowns now becomes (with the abuse of terminology)

$$\mathbf{V}^* = \overline{\partial}^7\partial_t v - \overline{\partial}^7\partial_t\eta\cdot\nabla_A v + \Delta_v^*, \quad \mathbf{Q}^* = \overline{\partial}^7\partial_t Q - \overline{\partial}^7\partial_t\eta\cdot\nabla_A Q + \Delta_Q^*., \tag{4.55}$$

where

$$\begin{aligned} (\Delta_v^*)_i &:= -8\overline{\partial}^7\eta\cdot\nabla_A\partial_t v_i - 8\overline{\partial}^7 v\cdot\nabla_A v_i + 16\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A v_i, \\ \Delta_Q^* &:= -8\overline{\partial}^7\eta\cdot\nabla_A\partial_t Q + 8\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A Q, \end{aligned} \tag{4.56}$$

and

$$\overline{\partial}^7\partial_t(\mathrm{div}_A v) = \nabla_A\cdot\mathbf{V}^* + C^*(v), \quad \overline{\partial}^7\partial_t(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^*(Q), \tag{4.57}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathcal{E}(t))\|f\|_{8,*}.$$

Now, the analogue of IB_{33}^* becomes the following quantity (recall such term comes from the product of Δ_Q and $\overline{\partial}^7\partial_t v$

$$\int_\Gamma JA^{3i}N_3(8\overline{\partial}^7\eta_p\,A^{3p}\,\partial_t\partial_3 Q - 8\overline{\partial}^7\eta\cdot\nabla_A\partial_t\eta_r\,A^{3r}\,\partial_3 Q)\overline{\partial}^7\partial_t\eta_i\,dy', \tag{4.58}$$

and we can still use $\overline{\partial}^7\eta|_{t=0} = \mathbf{0}$.

The analogue of (4.52) now needs some small modifications

$$\begin{aligned} \|\overline{\partial}^7\partial_t v\|_0 \lesssim& \|\mathbf{V}^*\|_0 + \|\overline{\partial}^7 v\|_0\|\nabla_A v\|_{L^\infty} + \|\overline{\partial}^7\eta\|_0\left(8\|\nabla_A\partial_t v\|_{L^\infty} + 16\|\nabla_A v\|_{L^\infty}^2\right) \\ \lesssim& \|\mathbf{V}^*\|_0 + \|\overline{\partial}^7 v\|_0^2 + \|\nabla_A v\|_2^2 + \|\nabla_A\overline{\partial}v\|_2^2 + \left(8\|\nabla_A\partial_t v\|_{L^\infty} + 16\|\nabla_A v\|_{L^\infty}^2\right)\int_0^T\|\overline{\partial}^7 v\|_0\,dt \\ \lesssim& \|\mathbf{V}^*\|_0^2 + \mathcal{P}_0 + \int_0^T P\left(\|\overline{\partial}^7\partial_t v\|_0, \|\partial_t\partial v\|_2, \|\partial_t\partial\overline{\partial}v\|_2, \|\partial\overline{\partial}\eta\|_2^2\right) + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt \\ \lesssim& \|\mathbf{V}^*\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \end{aligned} \tag{4.59}$$

The remaining analysis should follow in the same way as in Section 4.2, so we omit those details. The result is

$$\|\overline{\partial}^7\partial_t v\|_0^2 + \left\|\overline{\partial}^7\partial_t\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\overline{\partial}^7\partial_t q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\overline{\partial}^7\partial_t\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{4.60}$$

4.4 The case of 2~7 time derivatives

If the number of time derivatives in ∂_*^I is between 2 and 7, i.e,. ∂_*^I contains at least one spatial and two time derivatives, we can still mimic most steps in Section 4.3. In this case we write $\partial_*^I = \mathfrak{D}^6\partial_t^2$ where $\mathfrak{D} = \overline{\partial}$ or ∂_t and $\mathfrak{D}^6$ contains at least one $\overline{\partial}$.

The extra time derivatives allow us to eliminate most of the "extra modification terms" in the modified Alinhac good unknowns as in (4.25), (4.55)-(4.56) and thus much simplify the analysis of Alinhac good unknowns and tne boundary control. The reason is that the L^2-norm of the analogues of $e_1 \sim e_3$ in (4.3) can be directly controlled in the case of $\mathfrak{D}^8 = \mathfrak{D}^6\partial_t^2$. In specific, we have

$$\begin{aligned} \mathfrak{D}^6\partial_t^2(\nabla_A^i f) =& \nabla_A^i(\mathfrak{D}^6\partial_t^2 f) + (\mathfrak{D}^6\partial_t^2 A^{li})\partial_l f + [\mathfrak{D}^6\partial_t^2, A^{li}, \partial_l f] \\ =& \nabla_A^i(\mathfrak{D}^6\partial_t^2 f) - \mathfrak{D}^6\partial_t(A^{lr}\,\partial_t\partial_m\eta_r\,A^{mi})\partial_l f + [\mathfrak{D}^6\partial_t^2, A^{li}, \partial_l f] \\ =& \nabla_A^i(\mathfrak{D}^6\partial_t^2 f - \mathfrak{D}^6\partial_t^2\eta_r\,A^{lr}\partial_l f) + \underbrace{\mathfrak{D}^6\partial_t^2\eta_r\,\nabla_A^i(\nabla_A^r f) - ([\mathfrak{D}^6\partial_t, A^{lr}A^{mi}]\partial_t\partial_m\eta_r)\partial_l f}_{C_0(f)} + [\mathfrak{D}^6\partial_t^2, A^{li}, \partial_l f] \end{aligned} \tag{4.61}$$

and

$$\|C_0(f)\|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*})\|f\|_{8,*}.$$

Therefore, the analogous analysis of $C_1, C_3 \sim C_6$ in Section 4.1 are no longer needed here. The only problematic term is $-2(\partial_t A^{li})(\mathfrak{D}^6\partial_t\partial_l f) - 6(\mathfrak{D}A^{li})(\mathfrak{D}^5\partial_t^2\partial_l f)$ which comes from $[\mathfrak{D}^6\partial_t^2, A^{li}, \partial_l f]$. By mimicing the treatment of $C_2(Q)$ and $C_2(v)$ in (4.5)-(4.6), we can define the modified Alinhac good unknowns in the case of $\partial_*^I = \partial_t^N\overline{\partial}^{8-N}$ $(2 \le N \le 7)$ as the following

$$\mathbf{V}^* = \mathfrak{D}^6\partial_t^2 v - \mathfrak{D}^6\partial_t^2\eta\cdot\nabla_A v + \Delta_v^*,\ \ \mathbf{Q}^* = \mathfrak{D}^6\partial_t^2 Q - \mathfrak{D}^6\partial_t^2\eta\cdot\nabla_A Q, \tag{4.62}$$

where

$$(\Delta_v^*)_i := -6\mathfrak{D}^5\partial_t^2 v\cdot\nabla_A\mathfrak{D}\eta_i - 2\mathfrak{D}^6\partial_t v\cdot\nabla_A v_i \tag{4.63}$$

and

$$\mathfrak{D}^6\partial_t^2(\mathrm{div}_A v) = \nabla_A\cdot\mathbf{V}^* + C^*(v),\quad \mathfrak{D}^6\partial_t^2(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^*(Q), \tag{4.64}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathcal{E}(t))\|f\|_{8,*}.$$

In this case, $\Delta_Q^* = 0$, and thus the boundary integrals IB_3^*, IB_4^*, IB_5^* all vanish. The analogues of IB_1^*, IB_2^*, IB_6^* in this case can still be controlled in the same way as in Section 4.2. In the control of the difference between $\mathbf{V}^*$ and $\mathfrak{D}^6\partial_t^2$, we have by (4.62)-(4.63) that

$$\begin{aligned}\|\mathfrak{D}^6\partial_t^2 v\|_0 \lesssim&\|\mathbf{V}^*\|_0 + \|\mathfrak{D}^6\partial_t v\|_0\|\nabla_A v\|_{L^\infty} + \|\mathfrak{D}^5\partial_t^2 v\|_0\|\nabla_A\mathfrak{D}\eta\|_{L^\infty}\\ \lesssim&\|\mathbf{V}^*\|_0 + \|\mathfrak{D}^6\partial_t v\|_0^2 + \|\nabla_A v\|_2^2 + \|\mathfrak{D}^5\partial_t^2 v\|_0^2 + \|\nabla_A\mathfrak{D}\eta\|_2^2\\ \lesssim&\|\mathbf{V}^*\|_0 + \mathcal{P}_0 + \int_0^T P\left(\|\mathfrak{D}^6\partial_t^2 v\|_0, \|\mathfrak{D}^5\partial_t^3 v\|_0, \|\partial\mathfrak{D}v\|_2\right)dt \lesssim \|\mathbf{V}^*\|_0 + \mathcal{P}_0 + \int_0^T P(\mathcal{E}(t))\,dt\end{aligned} \tag{4.65}$$

The remaining analysis should follow in the same way as in Section 4.2 and 4.3 so we omit the details. The result is

$$\|\mathfrak{D}^6\partial_t^2 v\|_0^2 + \left\|\mathfrak{D}^6\partial_t^2\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\mathfrak{D}^6\partial_t^2 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\mathfrak{D}^6\partial_t^2\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt, \tag{4.66}$$

where $\mathfrak{D}^6$ contains at least one spatial derivative $\overline{\partial}$.

4.5 The case of full time derivatives

In the case of full time derivatives, the modified Alinhac good unknown is still defined similarly as in (4.62)-(4.64):

$$\mathbf{V}^* = \partial_t^8 v - \partial_t^8\eta\cdot\nabla_A v + \Delta_v^*,\ \ \mathbf{Q}^* = \partial_t^8 Q - \partial_t^8\eta\cdot\nabla_A Q, \tag{4.67}$$

where

$$(\Delta_v^*)_i := -8\partial_t^7 v\cdot\nabla_A v_i \tag{4.68}$$

and

$$\partial_t^8(\mathrm{div}_A v) = \nabla_A\cdot\mathbf{V}^* + C^*(v),\quad \partial_t^8(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^*(Q), \tag{4.69}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathcal{E}(t))\|f\|_{8,*}.$$

Extra difficulty: trace lemma is no longer applicable When $\partial_*^I = \partial_t^8$, there are terms of the form $\partial_t^7 v$ in the boundary integrals. In the case of full time derivative, one cannot apply Lemma 2.1 to control $|\partial_t^7 v|_0$. This difficulty appears in the estimates of the analogue of IB_6^*. Instead, we need to write IB_6^* in terms of interior integrals by using a similar technique in (3.42).

$$\begin{aligned}IB_6^* =& -8\int_\Gamma \hat{A}^{3i}N_3\,\partial_t^8\eta_p\,A^{3p}\partial_3 Q\,\partial_t^7 v_r\,A^{lr}\partial_l v_i\,dy' = -8\int_\Gamma \hat{A}^{3i}N_3\,\partial_t^7 v_p\,A^{3p}\partial_3 Q\,\partial_t^7 v_r\,A^{lr}\partial_l v_i\,dy'\\ =& -8\int_\Omega \hat{A}^{3i}\partial_3\partial_t^7 v_p\,A^{3p}\partial_3 Q\,\partial_t^7 v_r\,A^{lr}\partial_l v_i\,dy - 8\int_\Omega \hat{A}^{3i}\partial_t^7 v_p\,A^{3p}\partial_3 Q\,\partial_3\partial_t^7 v_r\,A^{lr}\partial_l v_i\,dy\\ &- 8\int_\Omega \partial_t^7 v_p\,\partial_t^7 v_r\,\partial_3(\hat{A}^{3i}A^{3p}\partial_3 Q\,A^{lr}\partial_l v_i)\,dy\\ =:& IB_{61}^* + IB_{62}^* + IB_{63}^*.\end{aligned} \tag{4.70}$$

The term IB^*_{63} can be directly controlled

$$IB^*_{63} \lesssim P(\|\partial_t^7 v\|_0, \|\partial v\|_3, \|\partial Q\|_3, \|a\|_3) \lesssim P(\|v\|_{8,*}, \|Q\|_{8,*}, \|\eta\|_4). \tag{4.71}$$

The term IB^*_{61}, IB^*_{62} should be controlled by integrating ∂_t by parts under time integral.

$$\begin{aligned}
\int_0^T IB^*_{61} =& -8\int_0^T \int_\Omega \hat{A}^{3i}\partial_3\partial_t^7 v_p\, A^{3p}\partial_3 Q\, \partial_t^7 v_r\, A^{lr}\partial_l v_i\, dy\, dt \\
\overset{\partial_t}{=}& -8\int_\Omega \hat{A}^{3i}\partial_3\partial_t^6 v_p\, A^{3p}\partial_3 Q\, \partial_t^7 v_r\, A^{lr}\partial_l v_i\, dy \\
&+8\int_0^T \int_\Omega \hat{A}^{3i}\partial_3\partial_t^6 v_p\, A^{3p}\partial_3 Q\, \partial_t^8 v_r\, A^{lr}\partial_l v_i\, dy\, dt \\
&+8\int_0^T \int_\Omega \hat{A}^{3i}\partial_3\partial_t^6 v_p\, \partial_t^7 v_r\, \partial_t(A^{3p}\partial_3 Q\, A^{lr}\partial_l v_i)\, dy\, dt \\
\lesssim& \|\partial_3\partial_t^6 v\|_0 \|\partial_t^7 v\|_0 P(\|a\|_{L^\infty}, \|\partial Q\|_{L^\infty}, \|\partial v\|_{L^\infty}) \\
&+\int_0^T \|\partial_3\partial_t^6 v\|_0 \left(\|\partial_t^8 v\|_0 P(\|a\|_{L^\infty}, \|\partial Q\|_{L^\infty}, \|\partial v\|_{L^\infty}) + \|\partial_t^7 v\|_0 \|A\cdot\partial_t(a\cdot\partial Q\cdot a\cdot\partial v)\|_{L^\infty}\right) \\
\lesssim& \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + \int_0^T P\left(\|\partial_t^8 v\|_0, \|(\partial_t a, \partial_t\partial Q, \partial_t\partial v\|_{L^\infty}^2\right) + \int_0^T P(\|v\|_{8,*}, \|Q\|_{7,*}, \|\eta\|_3)\, dt \\
\le& \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + \int_0^T P(\mathcal{E}(t))\, dt,
\end{aligned} \tag{4.72}$$

IB^*_{62} can be controlled in the same way, so we omit the details. Summarizing the estimates above, we get the energy inequality of the full time derivatives

$$\|\partial_t^8 v\|_0^2 + \left\|\partial_t^8\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_t^8 q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_t^8\eta_i\right|_0^2\Big|_{t=T} \lesssim \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\, dt, \tag{4.73}$$

which together with (4.53), (4.60), (4.66) concludes the proof of Proposition 4.1.

5 Control of mixed non-weighted derivatives

The case of mixed non-weighted derivatives correspond to $\partial_*^I = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$ with $1 \le i_3 \le 3,\ i_4 = 0$. In this case, the modified Alinhac good unknowns introduced in Section 4 are still needed when commuting ∂_*^I with ∇_A. On the other hand, the highest order term $\partial_*^I Q$ no longer vanishes on the boundary due to the presence of normal derivatives, so we need to use the method in Section 3 to deal with the boundary integral. Therefore, we should combine the methods in Section 3 and Section 4 to get the control of mixed non-weighted derivatives. The result of this section is

Proposition 5.1. The following energy inequality holds for sufficiently small $\varepsilon > 0$

$$\sum_{1\le i_3\le 3,\ i_4=0} \|\partial_*^I v\|_0^2 + \left\|\partial_*^I\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_*^I\eta_i\right|_0^2\Big|_{t=T} \lesssim \varepsilon\|\partial_3^4 v\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\, dt. \tag{5.1}$$

5.1 Purely spatial derivatives

We still start with the control of purely spatial derivatives. Let $N = 1, 2, 3$ and we consider $\partial_*^I = \partial_3^N\overline{\partial}^{8-2N}$.

5.1.1 The modified Alinhac good unknowns

Similarly as in Section 4.1.1, we have

$$
\begin{aligned}
\partial_3^N\overline{\partial}^{8-2N}(\nabla_A^i f) =&\nabla_A^i(\partial_3^N\overline{\partial}^{8-2N}f)+(\partial_3^N\overline{\partial}^{8-2N}A^{li})\partial_l f+[\partial_3^N\overline{\partial}^{8-2N},A^{li},\partial_l f]\\
=&\nabla_A^i(\partial_3^N\overline{\partial}^{8-2N}f)-\partial_3^N\overline{\partial}^{7-2N}(A^{lr}\,\overline{\partial}\partial_m\eta_r\,A^{mi})\partial_l f+[\partial_3^N\overline{\partial}^{8-2N},A^{li},\partial_l f]\\
=&\nabla_A^i(\partial_3^N\overline{\partial}^{8-2N}f-\partial_3^N\overline{\partial}^{8-2N}\eta_r\,A^{lr}\,\partial_l f)+(\partial_3^N\overline{\partial}^{8-2N}\eta_r)\nabla_A^i(\nabla_A^r f)\\
&-([\partial_3^N\overline{\partial}^{7-2N},A^{lr}A^{mi}]\overline{\partial}\partial_m\eta_r)\partial_l f+[\partial_3^N\overline{\partial}^{8-2N},A^{li},\partial_l f],
\end{aligned}
\tag{5.2}
$$

where the last line still contains the terms whose $L^2(\Omega)$-norms cannot be directly bounded under the setting of anisotropic Sobolev space $H_*^8(\Omega)$. The reason is that $\partial_3^N\overline{\partial}^{7-2N}$ may fall on $A=\partial\eta\times\partial\eta$ and $\partial_l f$. The following quantities are exactly these terms.

$$
\begin{aligned}
&e_1^\sharp:=-\partial_3^N\overline{\partial}^{7-2N}(A^{lr}A^{mi})(\overline{\partial}\partial_m\eta_r)\partial_l f,\quad e_2^\sharp:=-(7-2N)\overline{\partial}(A^{lr}A^{mi})(\partial_3^N\overline{\partial}^{7-2N}\partial_m\eta_r)\partial_l f,\\
&e_3^\sharp:=(8-2N)(\partial_3^N\overline{\partial}^{7-2N}A^{li})(\overline{\partial}\partial_l f),\quad e_4^\sharp:=(8-2N)(\overline{\partial}A^{li})(\partial_3^N\overline{\partial}^{7-2N}\partial_l f).
\end{aligned}
\tag{5.3}
$$

One can mimic the derivation of (4.18) and (4.19) to define the "modified Alinhac good unknowns" of v and Q with respect to $\partial_3^N\overline{\partial}^{8-2N}$ to be

$$
\begin{aligned}
\mathbf{V}_i^\sharp:=&\partial_3^N\overline{\partial}^{8-2N}v_i-\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_i\\
&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}v_i-(8-2N)\partial_3^N\overline{\partial}^{7-2N}v\cdot\nabla_A\overline{\partial}\eta_i\\
&+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i,
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
\mathbf{Q}^\sharp:=&\partial_3^N\overline{\partial}^{8-2N}Q-\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A Q\\
&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}Q+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q.
\end{aligned}
\tag{5.5}
$$

Then $\mathbf{V}^\sharp$ and $\mathbf{Q}^\sharp$ satisfy the following relations

$$
\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v)=\nabla_A\cdot\mathbf{V}^\sharp+C^\sharp(v),\quad \partial_3^N\overline{\partial}^{8-2N}(\nabla_A Q)=\nabla_A\mathbf{Q}^*+C^\sharp(Q),
\tag{5.6}
$$

where the commutator $C^\sharp$ satisfies

$$
\|C^\sharp(f)\|_0\lesssim P(\mathcal{E}(t))\|f\|_{8,*}.
\tag{5.7}
$$

Denote $\Delta_v^\sharp$ and $\Delta_Q^\sharp$ to be

$$
\begin{aligned}
(\Delta_v^\sharp)_i:=&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}v_i-(8-2N)\partial_3^N\overline{\partial}^{7-2N}v\cdot\nabla_A\overline{\partial}\eta_i\\
&+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_i+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_i,\\
\Delta_Q^\sharp:=&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}Q+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q.
\end{aligned}
$$

Then we can derive the evolution equation of $\mathbf{V}^\sharp$ and $\mathbf{Q}^\sharp$

$$
\begin{aligned}
&R\partial_t\mathbf{V}^\sharp-J^{-1}(b_0\cdot\partial)\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)+\nabla_A\mathbf{Q}^\sharp\\
=&[R,\partial_3^N\overline{\partial}^{8-2N}]\partial_t v+\left[J^{-1}(b_0\cdot\partial),\partial_3^N\overline{\partial}^{8-2N}\right]\left(J^{-1}(b_0\cdot\partial)\eta\right)\\
&+C^\sharp(Q)+R\partial_t(-\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v+\Delta_v^\sharp).
\end{aligned}
\tag{5.8}
$$

Denote the RHS of (5.8) to be $\mathbf{F}^\sharp$, then direct computation yields that

$$
\|\mathbf{F}^\sharp\|_0\lesssim P\left(\|\eta\|_{8,*},\|v\|_{8,*},\|Q\|_{8,*}\right).
$$

Now we take $L^2(\Omega)$ inner product of (5.8) and $J\mathbf{V}^\sharp$ to get the following energy identity

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega\rho_0|\mathbf{V}^\sharp|^2\,dy=\int_\Omega(b_0\cdot\partial)\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\mathbf{V}^\sharp-\int_\Omega(\nabla_{\hat{A}}Q)\cdot\mathbf{V}^\sharp+\int_\Omega J\mathbf{F}^\sharp\cdot\mathbf{V}^\sharp.
\tag{5.9}
$$

5.1.2 Interior estimates

The last integral on RHS of (5.9) is directly controlled

$$\int_\Omega J\mathbf{F}^\sharp \cdot \mathbf{V}^\sharp \lesssim \int_\Omega \|\mathbf{F}^\sharp\|_0\|\mathbf{V}^\sharp\|_0. \tag{5.10}$$

Then for the first term on RHS of (5.9) we integrate $(b_0\cdot\partial)$ by parts to produce the energy of magnetic field. Again, there is one term which cannot be directly controlled but will cancel with another term produced by $-\int_\Omega(\nabla_{\hat{A}}Q)\cdot\mathbf{V}^\sharp$. The proof follows in the same way as (3.13) so we omit the details.

$$\begin{aligned}&\int_\Omega (b_0\cdot\partial)\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\mathbf{V}^\sharp\\ \lesssim& -\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\partial_3^N\overline{\partial}^{8-2N}(J^{-1}(b_0\cdot\partial)\eta)\right|^2 + K_{11}^\sharp + P\left(\|(\eta,v,b_0,(b_0\cdot\partial))\|_{8,*}\right),\end{aligned} \tag{5.11}$$

where

$$K_{11}^\sharp := -\int_\Omega J\partial_3^N\overline{\partial}^{8-2N}(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v)\,dy. \tag{5.12}$$

Next we analyze the term $-\int_\Omega(\nabla_{\hat{A}}Q)\cdot\mathbf{V}^\sharp$. Integrating by parts and using Piola's identity $\partial_l\hat{A}^{li}=0$, we get

$$-\int_\Omega(\nabla_{\hat{A}}Q)\cdot\mathbf{V}^\sharp = \int_\Omega J\mathbf{Q}^\sharp(\nabla_A\cdot\mathbf{V}^\sharp) - \int_\Gamma J\mathbf{Q}^\sharp A^{li}N_l\mathbf{V}_i^\sharp\,dy' =: I^\sharp + IB^\sharp. \tag{5.13}$$

Plugging (5.6) into $I^\sharp$, we get

$$\begin{aligned}I^\sharp =& \int_\Omega J\partial_3^N\overline{\partial}^{8-2N}q\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v) + \int_\Omega J\partial_3^N\overline{\partial}^{8-2N}\left(\frac{1}{2}\left|J^{-1}(b_0\cdot\partial)\eta\right|^2\right)\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v)\\ &+\int_\Omega\left(-(\partial_3^N\overline{\partial}^{8-2N}\eta_p)\hat{A}^{lp}\,\partial_l Q + \Delta_Q^\sharp\right)\,\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v) - \int_\Omega(\partial_3^N\overline{\partial}^{8-2N}Q)C^\sharp(v)\\ =:& I_1^\sharp + I_2^\sharp + I_3^\sharp + I_4^\sharp,\end{aligned} \tag{5.14}$$

where $I_4^\sharp$ can be directly controlled by using the estimates of $C^\sharp(v)$

$$I_4^\sharp \lesssim \|\partial_3^N\overline{\partial}^{8-2N}Q\|_0\|C^\sharp(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial_3^N\overline{\partial}^{8-2N}Q\|_0\|v\|_{8,*}. \tag{5.15}$$

The term $I_1^\sharp$ produces the energy of fluid pressure

$$I_1^\sharp \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{J^2R'(q)}{\rho_0}\left|\partial_3^N\overline{\partial}^{8-2N}q\right|^2 + P(\|q\|_{8,*},\|\rho_0\|_{8,*},\|\eta\|_{8,*}). \tag{5.16}$$

Similarly as in (4.33), the term $I_2^\sharp$ produces the cancellation with $K_{11}^\sharp$.

$$\begin{aligned}I_2^\sharp =& \underbrace{\int_\Omega J\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v)}_{\text{exactly cancel with } K_{11}^\sharp}\\ &+\sum_{1\le N_1+N_2=8}\binom{N}{N_1}\binom{8-2N}{N_2}\int_\Omega J\partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^N\overline{\partial}^{8-2N}(\mathrm{div}_A v)\\ =& -K_{11}^\sharp\\ &-\sum_{1\le N_1+N_2=8}\binom{N}{N_1}\binom{8-2N}{N_2}\int_\Omega \frac{J^2R'(q)}{\rho_0}\partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)(\partial_3^N\overline{\partial}^{8-2N}\partial_t q)\\ &-\sum_{1\le N_1+N_2=8}\binom{N}{N_1}\binom{8-2N}{N_2}\int_\Omega J\partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\overline{\partial}^8,\frac{JR'(q)}{\rho_0}\right]\partial_t q\right)\\ =:& -K_{11}^\sharp + I_{21}^\sharp + I_{22}^\sharp\end{aligned} \tag{5.17}$$

and by direct computation we have

$$\int_0^T I_{21}^\sharp \lesssim \varepsilon\|\partial_3^N\overline{\partial}^{8-2N}q\|_0^2+\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt \tag{5.18}$$

$$I_{22}^\sharp \lesssim \|J^{-1}(b_0\cdot\partial)\eta\|_{7,*}^2\|q\|_{8,*}. \tag{5.19}$$

Then $I_3^\sharp$ can be controlled by integrating ∂_t by parts under time integral after invoking $\text{div}_A v=-\frac{JR'(q)}{\rho_0}\partial_t q$. The proof is similar to (4.37) so we do not repeat the proof.

$$\int_0^T I_3^\sharp \lesssim \varepsilon\|\overline{\partial}^{8-2N}\partial_3^N q\|_0^2+\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt. \tag{5.20}$$

Summarizing (5.14)-(5.20) and choosing $\varepsilon>0$ sufficiently small, we get the interior estimates

$$\int_0^T I^\sharp\,dt \lesssim -\frac{1}{2}\int_\Omega \frac{J^2R'(q)}{\rho_0}\left|\partial_3^N\overline{\partial}^{8-2N}q\right|^2\,dy\Big|_0^T+\mathcal{P}_0+\int_0^T P(\mathcal{E}(t))\,dt. \tag{5.21}$$

Therefore, it suffices to analyze the boundary integral $IB^\sharp$.

5.1.3 Boundary estimates

Invoking (5.4)-(5.5), the boundary integral now reads

$$\begin{aligned}
IB^\sharp=&-\int_\Gamma \mathbf{Q}^\sharp JA^{3i}N_3\mathbf{V}_i^\sharp\,dy'=-\int_\Gamma JA^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}Q)\mathbf{V}_i^\sharp\,dy'\\
&+\int_\Gamma \hat{A}^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}\eta_p)A^{3p}\partial_3Q\,\partial_3^N\overline{\partial}^{8-2N}v_i\,dy'\\
&-\int_\Gamma \hat{A}^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}\eta_p\,A^{3p}\partial_3Q)(\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_i)\,dy'\\
&-\int_\Gamma \hat{A}^{3i}N_3(\Delta_Q^\sharp)(\partial_3^N\overline{\partial}^{8-2N}v_i)\,dy'+\int_\Gamma \hat{A}^{3i}N_3(\Delta_Q^\sharp)\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_i\,dy'\\
&-\int_\Gamma \hat{A}^{3i}N_3\Delta_Q^\sharp(\Delta_v^\sharp)_i\,dy'+\int_\Gamma \hat{A}^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}\eta_p\,A^{3p}\,\partial_3Q)(\Delta_v^\sharp)_i\,dy'\\
=:&IB_0^\sharp+IB_1^\sharp+IB_2^\sharp+IB_3^\sharp+IB_4^\sharp+IB_5^\sharp+IB_6^\sharp.
\end{aligned} \tag{5.22}$$

To control $IB^\sharp$, we only need to combine the techniques used in Section 3.3 and Section 4.2.2:

- $IB_1^\sharp, IB_2^\sharp$ together with the Rayleigh-Taylor sign condition produces the boundary energy $|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i|_0^2$, similarly as IB_1+IB_2 in Section 3.3 and $IB_1^*+IB_2^*$ Section 4.2.2.
- The term $IB_0^\sharp$ is the analogue of IB_0 in Section 3.3 and can be controlled with similar method as in Section 3.3.
- $IB_3^\sharp\sim IB_6^\sharp$ are the analogues of $IB_3^*\sim IB_6^*$ in Section 4.2.2. These terms can be controlled exactly in the same way as $IB_3^*\sim IB_6^*$.

First, $IB_1^\sharp$ and $IB_2^\sharp$ give the boundary energy with the help of Rayleigh-Taylor sign condition. The proof is exactly the same as in Section 3.3 and Section 4.2.2 by merely replacing ∂_3^4 or $\overline{\partial}^8$ with $\partial_3^N\overline{\partial}^{8-2N}$, so we do not repeat the computations here.

$$\int_0^T IB_1^\sharp+IB_2^\sharp=-\frac{c_0}{4}\left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0^2\Big|_0^T+\int_0^T P(\mathcal{E}(t))\,dt. \tag{5.23}$$

We then analyze $IB_0^\sharp$. Invoking (5.4), we have

$$\begin{aligned}
IB_0^\sharp=&-\int_\Gamma N_3(J\partial_3^N\overline{\partial}^{8-2N}Q)(A^{3i}\,\partial_3^N\overline{\partial}^{8-2N}v_i)\,dy'+\int_\Gamma JA^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}Q)(\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_i)\,dy'\\
&-\int_\Gamma JA^{3i}N_3(\partial_3^N\overline{\partial}^{8-2N}Q)(\Delta_v^\sharp)_i\,dy'\\
=:&IB_{01}^\sharp+IB_{02}^\sharp+IB_{03}^\sharp.
\end{aligned} \tag{5.24}$$

We note that $IB^\sharp_{01}$ and $IB^\sharp_{02}$ are the analogues of IB_{01} and IB_{02} in Section 3.3, so we do not repeat all the details here. The extra term $IB^\sharp_{03}$ can be directly controlled (cf. (5.31) below).

We differentiate the continuity equation (3.29) by $\partial_3^N\overline{\partial}^{8-2N}$ to simplify the top order term containing v in $IB^\sharp_{01}$:

$$
\begin{aligned}
A^{3i}\partial_3^N\overline{\partial}^{8-2N}v_i =& -\partial_3^{N-1}\overline{\partial}^{8-2N}\left(\frac{JR'(q)}{\rho_0}\partial_t q\right)-\sum_{L=1}^{2}\partial_3^{N-1}\overline{\partial}^{8-2N}(A^{Li}\overline{\partial}_L v_i)\\
&-\sum_{N_1+N_2\geq 1, N_1\leq N-1}\binom{N-1}{N_1}\binom{8-2N}{N_2}\left(\partial_3^{N_1}\overline{\partial}^{N_2}A^{3i}\right)\left(\partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}v_i\right),
\end{aligned}
\tag{5.25}
$$

where the term contains $\partial_3^{N-1}\overline{\partial}^{8-2N}A^{Li}=\partial_3^N\overline{\partial}^{8-2N}\eta\times\overline{\partial}\eta+L.O.T.$ which cannot be controlled on the boundary. Invoking (2.3) with $D=\overline{\partial}$, we expand this problematic term to be

$$
\begin{aligned}
(\partial_3^{N-1}\overline{\partial}^{8-2N}A^{Li})\overline{\partial}_L v_i =& -\left(\partial_3^{N-1}\overline{\partial}^{7-2N}(A^{Lp}\,\overline{\partial}\partial_m\eta_p\,A^{mi})\right)\overline{\partial}_L v_i\\
=& -A^{Lp}\,\partial_3^N\overline{\partial}^{8-2N}\eta_p\,A^{3i}\overline{\partial}_L v_i-\sum_{M=1}^{2}A^{Lp}(\partial_3^{N-1}\overline{\partial}^{8-2N}\overline{\partial}_M\eta_p)A^{Mi}\overline{\partial}_L v_i-([\partial_3^{N-1}\overline{\partial}^{7-2N},A^{Lp}A^{mi}]\overline{\partial}\partial_m\eta_p)\overline{\partial}_L v_i.
\end{aligned}
\tag{5.26}
$$

On the other hand, in $IB^\sharp_{02}$ we have

$$
A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_i=A^{3i}\sum_{L=1}^{2}\partial_3^N\overline{\partial}^{8-2N}\eta_p A^{Lp}\overline{\partial}_L v_i+A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_p A^{3p}\partial_3 v_i,
\tag{5.27}
$$

where the first term exactly cancels with the first term in the RHS of (5.26). In fact, this is the analogue of (3.33)-(3.36) by merely replacing ∂_3^4 with $\partial_3^N\overline{\partial}^{8-2N}$. Thus we get the cancellation of the top order terms in $IB^\sharp_{01}$ and $IB^\sharp_{02}$.

The second term in (5.27) could be treated similarly as in (3.40) by invoking $A^{3p}\partial_3\eta_p=1$

$$
\partial_3^N\overline{\partial}^{8-2N}\eta_p A^{3p}=\underbrace{\partial_3^{N-1}\overline{\partial}^{8-2N}(\partial_3\eta_p A^{3p})}_{=0}-(\partial_3^{N-1}\overline{\partial}^{8-2N}A^{3p})\partial_3\eta_p-(\overline{\partial}A^{3p})(\partial_3^N\overline{\partial}^{7-2N}\eta_p)+\text{lower order terms.}
\tag{5.28}
$$

To control $IB^\sharp_0$, we still need to analyze $\partial_3^N\overline{\partial}^{8-2N}Q$. Following the aruments in (3.37)-(3.39) and replacing ∂_3^4 with $\partial_3^N\overline{\partial}^{8-2N}$, we can reduce one normal derivative of Q to one tangential derivative of v and $(b_0\cdot\partial)\eta$ via

$$
\begin{aligned}
\partial_3^N\overline{\partial}^{8-2N}Q=&J^{-1}\partial_3\eta_i\left(\rho_0\partial_3^{N-1}\overline{\partial}^{8-2N}\partial_t v^i+(b_0\cdot\overline{\partial})\partial_3^{N-1}\overline{\partial}^{8-2N}(J^{-1}(b_0\cdot\partial)\eta^i)\right)-\sum_{L=1}^{2}\hat{A}^{Li}(\partial_3^{N-1}\overline{\partial}^{8-2N}\overline{\partial}_L Q)\\
&-(N-1)(\partial_3^{N-1}\overline{\partial}^{8-2N}\hat{A}^{3i})(\partial_3 Q)+\text{ lower order terms.}
\end{aligned}
\tag{5.29}
$$

Plugging the expression of $\Delta^\sharp_v$ and (5.25)-(5.29) into (5.24), we find that every highest order term in $IB^\sharp_0$ must be one of the following forms

$$
\begin{aligned}
K^\sharp_1&:=\int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D}f)(\partial_3^{N-1}\overline{\partial}^{9-2N}g)(\partial h)r\,dy',\\
K^\sharp_2&:=\int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D}f)(\partial_3^{N}\overline{\partial}^{7-2N}g)(\partial\overline{\partial}h)r\,dy',\\
K^\sharp_3&:=\int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D}f)(\partial_3^{N}\overline{\partial}^{7-2N}g)(\partial h)r\,dy',
\end{aligned}
$$

where $\mathfrak{D}=\overline{\partial}$ or ∂_t or $(b_0\cdot\overline{\partial})$, the functions f,g,h can be $\eta,v,Q,J^{-1}(b_0\cdot\partial)\eta$, and r contains at most one derivative of η, v or Q. We note that the term $K^\sharp_2$ comes from $IB^\sharp_{03}$ where $\Delta^\sharp_v$ contributes to $\partial_3^N\overline{\partial}^{7-2N}g\cdot\partial\overline{\partial}h\cdot r$.

Since $1\leq N\leq 3$, we know $7-2N\geq 1$ and thus we can directly apply lemma 2.1 to control $K^\sharp_1\sim K^\sharp_3$.

$$
\begin{aligned}
K^\sharp_1\lesssim&|\partial_3^{N-1}\mathfrak{D}f|_{8-2N}|\partial_3^{N-1}g|_{9-2N}|\partial h\,r|_{L^\infty}\lesssim\|\partial_3^{N-1}\mathfrak{D}f\|_{H_*^{9-2N}}\|\partial_3^{N-1}g\|_{H_*^{10-2N}}\|\partial h\,r\|_{H^2}\\
\lesssim&\|f\|_{2(N-1)+1+(9-2N),*}\|g\|_{2(N-1)+(10-2N),*}\|h\|_3\|r\|_2=\|f\|_{8,*}\|g\|_{8,*}\|h\|_3\|r\|_2.
\end{aligned}
\tag{5.30}
$$

and

$$\begin{aligned}K_2^\sharp \lesssim & |\partial_3^{N-1}\mathfrak{D}f|_{8-2N}|\partial_3^N g|_{7-2N}|(\partial\overline{\partial}h)r|_{L^\infty} \lesssim |\partial_3^{N-1}\mathfrak{D}f|_{8-2N}|\partial_3^N g|_{7-2N}|(\partial\overline{\partial}h)r|_{1.5}\\ \lesssim & \|\partial_3^{N-1}\mathfrak{D}f\|_{H_*^{9-2N}}\|\partial_3^N g\|_{H_*^{8-2N}}\|(\partial\overline{\partial}h)r\|_2 \lesssim \|f\|_{8,*}\|g\|_{8,*}\|h\|_{7,*}\|r\|_2,\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}K_3^\sharp \lesssim & |\partial_3^{N-1}\mathfrak{D}f|_{8-2N}|\partial_3^N g|_{7-2N}|(\partial h)r|_{L^\infty} \lesssim |\partial_3^{N-1}\mathfrak{D}f|_{8-2N}|\partial_3^N g|_{7-2N}|(\partial h)r|_{1.5}\\ \lesssim & \|\partial_3^{N-1}\mathfrak{D}f\|_{H_*^{9-2N}}\|\partial_3^N g\|_{H_*^{8-2N}}\|(\partial h)r\|_2 \lesssim \|f\|_{8,*}\|g\|_{8,*}\|h\|_3\|r\|_2.\end{aligned} \tag{5.32}$$

One can use either trace lemma or similar techniques as in (3.41)-(3.43) to analyze the remaining terms which are all of lower order than $K_1^\sharp \sim K_3^\sharp$. This completes the control of $IB_0^\sharp$.

The analysis of $IB_3^\sharp \sim IB_6^\sharp$ can be proceeded exactly in the same way as $IB_3^* \sim IB_6^*$. Since these quantities involving the modification terms $\Delta_Q^\sharp, \Delta_v^\sharp$ are of lower order, we do not repeat the details again. We can finally prove that

$$\begin{aligned}\int_0^T IB_3^\sharp + IB_4^\sharp \, dt \lesssim & \int_0^T \left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0 P\left(\|(\eta,v,b)\|_{8,*}, \|Q\|_{8,*}, \|\rho_0\|_3\right) dt, \\ & + \left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0 P(\|\eta\|_{8,*}, \|Q\|_{8,*}) \int_0^T \|v(t)\|_{8,*}\, dt \end{aligned} \tag{5.33}$$

$$IB_5^\sharp \lesssim |\hat{A}^{3i}|_{L^\infty}|\Delta_Q^\sharp|_0|(\Delta_v^\sharp)_i|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}), \tag{5.34}$$

$$IB_6^\sharp \lesssim |\hat{A}^{3i}\partial_3 Q|_{L^\infty}|A^{3p}\partial_3^N\overline{\partial}^{8-2N}\eta_p|_0|(\Delta_v^\sharp)_i|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}). \tag{5.35}$$

Summarizing the estimates above, we get the control of the boundary integral

$$\int_0^T IB^\sharp \lesssim -\frac{c_0}{4}\left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\, dt. \tag{5.36}$$

Combining (5.9)-(5.11), (5.21), (5.36) and choosing $\varepsilon > 0$ in (5.18) to be suitably small, we get the following inequality

$$\|\mathbf{V}^\sharp\|_0^2 + \left\|\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^N\overline{\partial}^{8-2N}q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\, dt. \tag{5.37}$$

Finally, invoking (5.4), we get the $\partial_3^N\overline{\partial}^{8-2N}$ $(N=1,2,3)$-estimates of v

$$\|\partial_3^N\overline{\partial}^{8-2N}v\|_0 \lesssim \|\mathbf{V}^\sharp\|_0 + \|\partial_3^N\overline{\partial}^{8-2N}\eta\|_0\|\nabla_A v_i\|_{L^\infty} + \|\partial_3^N\overline{\partial}^{7-2N}\eta\|_0\left(\|\nabla_A\overline{\partial}v\|_{L^\infty} + \|\nabla_A\overline{\partial}\eta\cdot\nabla_A v\|_{L^\infty}\right) + \|\partial_3^N\overline{\partial}^{7-2N}v\|_0\|\nabla_A\overline{\partial}\eta\|_{L^\infty}.$$

Since $\partial^m\eta|_{t=0} = 0$ for any $m \geq 2, m \in \mathbb{N}^*$, we know

$$\|\partial_3^N\overline{\partial}^{8-2N}v\|_0 \lesssim \|\mathbf{V}^\sharp\|_0 + P(\|v\|_{7,*}, \|\eta\|_{7,*})\int_0^T P(\|v\|_{8,*}), \tag{5.38}$$

and thus

$$\|\partial_3^N\overline{\partial}^{8-2N}v\|_0^2 + \left\|\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^N\overline{\partial}^{8-2N}q\|_0^2 + \frac{c_0}{4}\left|A^{3i}\partial_3^N\overline{\partial}^{8-2N}\eta_i\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\, dt. \tag{5.39}$$

5.2 Control of time derivatives

In the case of $\partial_*^I = \partial_3^N\overline{\partial}^{8-2N-k}\partial_t^k$ for $1 \leq k \leq 8-2N$, most of steps in the proof are still applicable. However, the presence of time derivative(s) could simplify the "modified Alinhac good unknowns". We note that most of the modifications are essentially similar to Section (4.3) $\sim$ Section 4.5, so we no longer repeat the details.

5.2.1 One time derivative

When $k=1$, the modified Alinhac good unknowns can be defined by replacing $8\overline{\partial}^7$ by $(8-2N)\partial_3^N\overline{\partial}^{7-2N}$ in Section 4.3.2, i.e.,

$$\mathbf{V}^\sharp=\partial_3^N\overline{\partial}^{7-2N}\partial_t v-\partial_3^N\overline{\partial}^{7-2N}\partial_t\eta\cdot\nabla_A v+\Delta_v^\sharp,\ \ \mathbf{Q}^\sharp=\partial_3^N\overline{\partial}^{7-2N}\partial_t Q-\partial_3^N\overline{\partial}^{7-2N}\partial_t\eta\cdot\nabla_A Q+\Delta_Q^\sharp.,\tag{5.40}$$

where

$$\begin{aligned}(\Delta_v^\sharp)_i:=&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\partial_t v_i-(8-2N)\partial_3^N\overline{\partial}^{7-2N}v\cdot\nabla_A v_i+(16-4N)\partial_3^N\overline{\partial}^{7-2N}\cdot\nabla_A v\cdot\nabla_A v_i,\\ \Delta_Q^\sharp:=&-(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\partial_t Q+(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A v\cdot\nabla_A Q,\end{aligned}\tag{5.41}$$

and

$$\partial_3^N\overline{\partial}^{7-2N}\partial_t(\mathrm{div}_A v)=\nabla_A\cdot\mathbf{V}^\sharp+C^\sharp(v),\quad \partial_3^N\overline{\partial}^{7-2N}\partial_t(\nabla_A Q)=\nabla_A\mathbf{Q}^\sharp+C^\sharp(Q),\tag{5.42}$$

with

$$\|C^\sharp(f)\|_0\lesssim P(\mathcal{E}(t))\|f\|_{8,*}.$$

The difference between $\partial_3^N\overline{\partial}^{7-2N}v$ and $\mathbf{V}^\sharp$ should be controlled in the same way as (4.59) by replacing $\overline{\partial}^7$ with $\partial_3^N\overline{\partial}^{7-2N}$

$$\|\partial_3^N\overline{\partial}^{7-2N}\partial_t v\|_0\lesssim\|\mathbf{V}^*\|_0^2+\mathcal{P}_0+P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt,\tag{5.43}$$

and thus

$$\begin{aligned}&\|\partial_3^N\overline{\partial}^{7-2N}\partial_t v\|_0^2+\left\|\partial_3^N\overline{\partial}^{7-2N}\partial_t\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2+\|\partial_3^N\overline{\partial}^{7-2N}\partial_t q\|_0^2+\frac{c_0}{4}\left|A^{3i}\partial_3^N\overline{\partial}^{7-2N}\partial_t\eta_i\right|_0^2\Big|_{t=T}\\ \lesssim&\mathcal{P}_0+P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt.\end{aligned}\tag{5.44}$$

5.2.2 2~(7-2N) time derivatives

When $2\le k\le 7-2N$, we can mimic the analysis in Section 4.4: We just need to replace $\mathfrak{D}^6\partial_t^2$ by $\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2$ where $\mathfrak{D}$ denotes $\overline{\partial}$ or ∂_t and $\mathfrak{D}^{6-2N}$ contains at least one $\overline{\partial}$. The analogous problematic term becomes $-2(\partial_t A^{li})(\partial_3^N\mathfrak{D}^{6-2N}\partial_t\partial_l f)-(6-2N)(\mathfrak{D}A^{li})(\partial_3^N\mathfrak{D}^{5-2N}\partial_t^2\partial_l f)$ which comes from $[\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2,A^{li},\partial_l f]$. Following (4.62)-(4.64), we can similarly define

$$\mathbf{V}^\sharp=\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 v-\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta\cdot\nabla_A v+\Delta_v^\sharp,\ \ \mathbf{Q}^\sharp=\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 Q-\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta\cdot\nabla_A Q,\tag{5.45}$$

where

$$(\Delta_v^\sharp)_i:=-(6-2N)\partial_3^N\mathfrak{D}^{5-2N}\partial_t^2 v\cdot\nabla_A\mathfrak{D}\eta_i-2\partial_3^N\mathfrak{D}^{6-2N}\partial_t v\cdot\nabla_A v_i\tag{5.46}$$

and

$$\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2(\mathrm{div}_A v)=\nabla_A\cdot\mathbf{V}^\sharp+C^\sharp(v),\quad \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2(\nabla_A Q)=\nabla_A\mathbf{Q}^*+C^\sharp(Q),\tag{5.47}$$

with

$$\|C^\sharp(f)\|_0\lesssim P(\mathcal{E}(t))\|f\|_{8,*}.$$

Again we have $\Delta_Q^\sharp$ in this case, and thus the analogues of $IB_3^\sharp\sim IB_5^\sharp$ all vanish. The boundary integrals $IB_0^\sharp,IB_1^\sharp,IB_2^\sharp,IB_6^\sharp$ and the interior terms can be controlled in the same way as Section 5.1. Finally, one has

$$\begin{aligned}&\|\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 v\|_0^2+\left\|\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2+\|\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 q\|_0^2+\frac{c_0}{4}\left|A^{3i}\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta_i\right|_0^2\Big|_{t=T}\\ \lesssim&\mathcal{P}_0+P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt,\end{aligned}\tag{5.48}$$

where $\mathfrak{D}^{6-2N}$ contains at least one spatial derivative $\overline{\partial}$.

5.2.3 Full time derivatives

When $\partial_*^I = \partial_3^N \partial_t^{8-2N}$ for $N = 1, 2, 3$, there is not tangential spatial derivative on the boundary and thus Lemma 2.1 is no longer applicable. In this case, the modified Alinhac good unknowns become

$$\mathbf{V}^\sharp = \partial_3^N \partial_t^{8-2N} v - \partial_3^N \partial_t^{8-2N} \eta \cdot \nabla_A v + \Delta_v^\sharp, \quad \mathbf{Q}^\sharp = \partial_3^N \partial_t^{8-2N} Q - \partial_3^N \partial_t^{8-2N} \eta \cdot \nabla_A Q, \tag{5.49}$$

where

$$(\Delta_v^\sharp)_i := -(8-2N) \partial_3^N \partial_t^{8-2N} v \cdot \nabla_A v_i \tag{5.50}$$

and

$$\partial_3^N \partial_t^{8-2N} (\mathrm{div}_A v) = \nabla_A \cdot \mathbf{V}^\sharp + C^\sharp(v), \quad \partial_3^N \partial_t^{8-2N} (\nabla_A Q) = \nabla_A \mathbf{Q}^\sharp + C^\sharp(Q), \tag{5.51}$$

with

$$\|C^\sharp(f)\|_0 \lesssim P(\mathcal{E}(t)) \|f\|_{8,*}.$$

The proof follows in the same way as Section 4.5 after replacing ∂_t^7 by ∂_t^{7-2N} and the coefficient 8 by $(8-2N)$. So we no longer repeat the details. Finally, we can get

$$\begin{aligned} &\|\partial_3^N \partial_t^{8-2N} v\|_0^2 + \left\| \partial_3^N \partial_t^{8-2N} \left(J^{-1} (b_0 \cdot \partial) \eta \right) \right\|_0^2 + \|\partial_3^N \partial_t^{8-2N} q\|_0^2 + \frac{c_0}{4} \left| A^{3i} \partial_3^N \partial_t^{8-2N} \eta_i \right|_0^2 \Bigg|_{t=T} \\ \lesssim &\varepsilon \|\partial_3^{N+1} \partial_t^{6-2N} v\|_0^2 + \mathcal{P}_0 + P(\mathcal{E}(T)) \int_0^T P(\mathcal{E}(t))\, dt, \end{aligned} \tag{5.52}$$

which together with (5.39), (5.44), (5.48) concludes the proof of Proposition 5.1.

6 Control of weighted normal derivatives

Now we consider the most general case $\partial_*^I = \partial_t^{i_0} (\sigma\partial_3)^{i_4} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3}$ with $i_1 + i_2 + 2i_3 + i_4 = 8$ and $i_4 > 0$. The presence of the weighted normal derivatives $(\sigma\partial_3)^{i_4}$ makes the following difference from the non-weighted case.

1. Extra terms are produced when we commute ∂_*^I with ∂_3 because σ is a function of y_3. Once ∂_3 falls on the weight function, we will lose a weight and $(\sigma\partial_3)$ becomes ∂_3, which causes a loss of derivative. This appears when we commute ∂_*^I with ∇_A^i that falls on Q or v_i and commute ∂_*^I with $(b_0 \cdot \partial)$.
2. There is no boundary integral because the weight function σ vanishes on Γ.

To overcome the difficulty mentioned above, we can again use the techniques, similar with those in the previous sections.

- Invoke the MHD equation and the continuity equation to replace $\nabla_A Q$ and $\mathrm{div}_A v$ by tangential derivatives.
- Produce a weight funtion by using $b_0^3|_\Gamma = 0$ and $\overline{\partial} Q|_\Gamma = 0$.
- In particular, if ∂_*^I does not contain time derivative, we need to add an extra modification term in the good unknown of v.

First we analyze $[(b_0 \cdot \partial), \partial_t^{i_0} (\sigma\partial_3)^{i_4} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3}] f$. Compared with the non-weighted case, we need to control the extra term $b_0^3 \partial_3 (\sigma^{i_4})\, (\partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+i_4} f) = i_4 b_0^3 (\partial_3 \sigma) \left(\partial_t^{i_0} (\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+1} \right) f$. Using $b_0^3|_\Gamma = 0$, one can produce a weight function σ as in (4.9). Therefore

$$\left\| b_0^3 (\partial_3 \sigma) \left(\partial_t^{i_0} (\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+1} f \right) \right\|_0 \lesssim \|\partial_3 b_0\|_{L^\infty} \|(\sigma\partial_3) \partial_t^{i_0} (\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} f\|_0 \leq \|b_0\|_3 \|f\|_{8,*}.$$

Next we analyze the commutator between $\partial_*^I = \partial_t^{i_0} (\sigma\partial_3)^{i_4} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3}$ and $\nabla_A f$. Compared with the non-weighted case, we shall analyze an extra term C_σ below. In specific, one has

$$\begin{aligned} &\partial_t^{i_0} (\sigma\partial_3)^{i_4} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} (A^{li} \partial_l f) \\ =&\sigma^{i_4} \partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+i_4} (A^{li} \partial_l f) \\ =&\sigma^{i_4} \left(A^{li} \partial_l (\partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+i_4}) f \right) + \underbrace{\sigma^{i_4} [\partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+i_4}, A^{li}] \partial_l f}_{\mathring{C}} \\ =&A^{li} \partial_l \left(\sigma^{i_4} \partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+i_4} f \right) - (i_4 \partial_3 \sigma) \underbrace{A^{3i} \left((\sigma\partial_3)^{i_4-1} \partial_t^{i_0} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+1} f \right)}_{C_\sigma} + \mathring{C}. \end{aligned} \tag{6.1}$$

The term $\mathring{C}$ consists of the commutators produced in the same way as the non-weighted case. It can be analyzed in the same way as in previous sections by just considering $(\sigma\partial_3)$ as a tangential derivative on the boundary. As for the extra term, we do the following computation

$$\begin{aligned}
&A^{3i}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}f\right)\\
=&(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3i}\partial_3 f)-\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3},A^{3i}\right]\partial_3 f\\
=:&C_1^\sigma(f)+C_2^\sigma(f).
\end{aligned}\tag{6.2}$$

Note that $i_0+i_1+i_2+i_4=8-2i_3$. We know the leading order terms in C_2^σ are $\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{3i}\right)f$ and $(\mathfrak{D}A^{3i})(\mathfrak{D}^{6-2i_3}\partial_3^{i_3+1}f)$, where $\mathfrak{D}$ represents any one of $(\sigma\partial_3),\partial_t,\overline{\partial}_1,\overline{\partial}_2$. Recall that A^{3i} cosists of $\overline{\partial}\eta\cdot\overline{\partial}\eta$. This shows that the highest order term in $\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{3i}\right)\partial_3 f$ is $(\mathfrak{D}^{8-2i_3}\partial_3^{i_3}\eta)(\overline{\partial}\eta)f$ whose $L^2(\Omega)$ norm can be directly controlled by $\|\eta\|_{8,*}\|\overline{\partial}\eta\|_{L^\infty}\|\partial_3 f\|_{L^\infty}$. As for the second term, we have

$$\|(\mathfrak{D}A^{3i})(\mathfrak{D}^{6-2i_3}\partial_3^{i_3+1}f)\|_0\lesssim\|(\mathfrak{D}\overline{\partial}\eta)(\overline{\partial}\eta)\|_{L^\infty}\|f\|_{8,*}.$$

Therefore, C_2^σ can be directly controlled.

The control of C_1^σ is more complicated. We should use the structure of MHD system (1.17) to replace one normal derivative by one tangential derivative.

$$A^{3i}\partial_3 Q=-\sum_{L=1}^{2}A^{Li}\overline{\partial}_L Q-R\partial_t v^i+J^{-1}(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)\tag{6.3}$$

$$A^{3i}\partial_3 v_i=\mathrm{div}_A v-A^{1i}\overline{\partial}_1 v_i-A^{2i}\overline{\partial}_2 v_i=-\frac{JR'(q)}{\rho_0}\partial_t q-\sum_{L=1}^{2}A^{Li}\overline{\partial}_L v_i\tag{6.4}$$

When $f=Q$, we plug (6.3) into $C_1^\sigma(Q)$ to get

$$\begin{aligned}
C_1^\sigma(Q)=&(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3i}\partial_3 Q)\\
=&-\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{Li}\overline{\partial}_L Q)\\
&-(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(R\partial_t v^i)+(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\left(J^{-1}(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)\right)\\
=:&C_{11}^\sigma(Q)+C_{12}^\sigma(Q)+C_{13}^\sigma(Q).
\end{aligned}\tag{6.5}$$

When $f=v_i$, we plug (6.4) into $C_1^\sigma(v)$ to get

$$\begin{aligned}
C_1^\sigma(v)=&(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3i}\partial_3 v_i)\\
=&-\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{Li}\overline{\partial}_L v_i)-(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\left(\frac{JR'(q)}{\rho_0}\partial_t q\right)\\
=:&C_{11}^\sigma(v)+C_{12}^\sigma(v).
\end{aligned}\tag{6.6}$$

The terms $C_{12}^\sigma(Q)$ and $C_{12}^\sigma(v)$ can be directly controlled. Note that $i_0+i_1+i_2+(i_4-1)=7-2i_3$, so

$$\|C_{12}^\sigma(Q)\|_0\lesssim\|R\|_{7,*}\|v\|_{8,*}\lesssim\|q\|_{7,*}\|v\|_{8,*},\tag{6.7}$$

$$\|C_{12}^\sigma(v)\|_0\lesssim\|\rho_0\|_{7,*}\|q\|_{8,*}.\tag{6.8}$$

Using $b_0^3|_\Gamma=0$, one can produce a weight function σ as in (4.9) when all the derivatives fall on $J^{-1}(b_0\cdot\partial)\eta$.

$$\begin{aligned}
\|C_{13}^\sigma(Q)\|_0\lesssim&\|J^{-1}(b_0\cdot\partial)(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(J^{-1}(b_0\cdot\partial)\eta)\|_0\\
&+\left\|\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3},J^{-1}(b_0\cdot\partial)\right](J^{-1}(b_0\cdot\partial)\eta)\right\|_0\\
\lesssim&\|\partial_3(J^{-1}b_0)\|_{L^\infty}\|(\sigma\partial_3)^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(J^{-1}(b_0\cdot\partial)\eta)\|_0\\
\lesssim&\|b_0\|_{7,*}\|J^{-1}(b_0\cdot\partial)\eta\|_{8,*}.
\end{aligned}\tag{6.9}$$

As for C^{σ}_{11}, the highest order term can be merged into the modified Alinhac good unknowns. One has

$$\begin{aligned}
C^{\sigma}_{11}(f) :=& -\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{Li}\overline{\partial}_L f)\\
=& -\sum_{L=1}^{2}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{Li}\right)\overline{\partial}_L f\underbrace{-\sum_{L=1}^{2}\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3},A^{Li}\right]\overline{\partial}_L f}_{C^{\sigma}_{111}(f)}\\
=& -\sum_{L=1}^{2}A^{Lp}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_k\eta_p\right)A^{ki}\overline{\partial}_L f\underbrace{-\sum_{L=1}^{2}\left(\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3},A^{Lp}A^{ki}\right]\partial_k\eta_p\right)\overline{\partial}_L f}_{C^{\sigma}_{112}(f)}+C^{\sigma}_{111}(f).
\end{aligned}\tag{6.10}$$

Since $i_0+i_1+i_2+i_4 = 8-2i_3$, one can directly control the $L^2(\Omega)$-norms of $C^{\sigma}_{111}(f),\ C^{\sigma}_{112}(f)$ by $P(\|\eta\|_{8,*})\|f\|_{8,*}$. For the first term in the RHS of (6.10), one can proceed in the following ways

- $f = Q$: Since $\overline{\partial}_L Q|_{\Gamma} = 0$, one can produce a weight function as in (4.13) and thus

$$\begin{aligned}
&\left\|A^{Lp}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_k\eta_p\right)A^{ki}\overline{\partial}_L Q\right\|_0\\
\lesssim&\sum_{M=1}^{2}\left\|A^{Lp}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\overline{\partial}_M\eta_p\right)A^{Mi}\overline{\partial}_L Q\right\|_0+\|A^{Lp}A^{3i}\overline{\partial}\partial_3 Q\|_{L^\infty}\|(\sigma\partial_3)^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p\|_0\\
\lesssim& P(\|\eta\|_3)\|Q\|_{7,*}\|\eta\|_{8,*}.
\end{aligned}\tag{6.11}$$

- $f = v_i$: When ∂_*^I contains time derivative ($i_0 > 0$), then it can be directly controlled due to $\partial_t\eta = v$

$$\left\|A^{Lp}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_k\eta_p\right)A^{ki}\overline{\partial}_L v_i\right\|_0=\left\|A^{Lp}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_k v_p\right)A^{ki}\overline{\partial}_L v_i\right\|_0\lesssim P(\|\eta\|_3)\|v\|_{5,*}\|v\|_{8,*}.\tag{6.12}$$

  If $i_0 = 0$, then it can be written in the form of covariant derivative plus a controllable term.

$$\begin{aligned}
&-A^{Lp}\left((\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_k\eta_p\right)A^{ki}\overline{\partial}_L v_i\\
=&-A^{ki}\partial_k\left((\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p A^{Lp}\overline{\partial}_L v_i\right)\\
&+\underbrace{A^{3i}(\partial_3\sigma)\left((i_4-1)(\sigma\partial_3)^{i_4-2}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}\eta_p\right)A^{Lp}\overline{\partial}_L v_i+\nabla_A^i(A^{Lp}\overline{\partial}_L v_i)\left((\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p\right)}_{C^{\sigma}_{113}(v_i)}\\
=&-\nabla_A^i\left((\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p A^{Lp}\overline{\partial}_L v_i\right)+C^{\sigma}_{113}(v_i).
\end{aligned}\tag{6.13}$$

  We note that the first term in $C^{\sigma}_{113}(f)$ appears when ∂_k ($k = 3$) falls on the weight function and $i_4 \geq 2$ and can also be directly controlled by $P(\|\eta\|_{8,*})\|f\|_{8,*}$.

Next we merge the covariant derivative terms in C_σ into the modified Alinhac good unknowns, i.e., for $\partial_*^I = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$ we define

$$\mathbf{V}_i^{\sigma} := \begin{cases}\partial_*^I v_i - \partial_*^I\eta\cdot\nabla_A v_i + (\Delta_v^{\sigma})_i & i_0 \geq 1\\ \partial_*^I v_i - \partial_*^I\eta\cdot\nabla_A v_i + (\Delta_v^{\sigma})_i + \sum\limits_{L=1}^{2}\left((i_4\partial_3\sigma)(\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p\right)A^{Lp}\overline{\partial}_L v_i, & i_0 = 0,\end{cases}\tag{6.14}$$

and

$$\mathbf{Q}^{\sigma} := \partial_*^I Q - \partial_*^I\eta\cdot\nabla_A Q + \Delta_Q^{\sigma}.\tag{6.15}$$

Then one has

$$\partial_*^I(\nabla_A\cdot v) = \nabla_A\cdot\mathbf{V}^{\sigma} + C^{\sigma}(v),\tag{6.16}$$

$$\partial_*^I(\nabla_A Q) = \nabla_A\mathbf{Q}^{\sigma} + C^{\sigma}(Q),\tag{6.17}$$

with $\|C^\sigma(f)\|_0 \lesssim P(\mathcal{E}(t))\|f\|_{8,*}$. Here the "extra modification terms" Δ_v^σ and Δ_Q^σ comes from the analysis of $\mathring{C}$ in (6.1) whose precise expressions can be derived in the same way as Section 4 ~ Section 5. The term $\left((i_4\partial_3\sigma)(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_p\right)A^{Lp}\overline{\partial}_L f$ comes from (6.1) and (6.13). Finally, the commutator $C^\sigma(f)$ consists of the commutator part in $\mathring{C}$, $C_{111}^\sigma(f) \sim C_{113}^\sigma(f), C_{12}^\sigma(f)$ and $C_{13}^\sigma(Q)$.

Recall that $\sigma|_\Gamma = 0$ and $\overline{\partial}Q|_\Gamma = 0$ imply $\mathbf{Q}^\sigma|_\Gamma = 0$. Therefore the boundary integral $\int_\Gamma N_3\hat{A}^{3i}\mathbf{Q}^\sigma\mathbf{V}_i^\sigma\,dy'$ vanishes. Hence, we can get the following estimates for $\partial_*^I := \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$

$$\|\partial_*^I v\|_0^2 + \left\|\partial_*^I\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2\bigg|_{t=T} \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt. \tag{6.18}$$

7 A priori estimates of the compressible MHD system

7.1 Finalizing the energy estimates

Combining the L^2-energy conservation (1.6) with (3.1), (4.1), (5.1) and (6.18), and then choosing $\varepsilon > 0$ to be suitably small, we finally get the following energy inequality

$$\mathcal{E}(T) - \mathcal{E}(0) \lesssim \mathcal{P}_0 + P(\mathcal{E}(T))\int_0^T P(\mathcal{E}(t))\,dt \tag{7.1}$$

under the a priori asuumptions (1.19)-(1.20). By the Gronwall-type inequality in Tao [62, Chapter 1.3], one can find some $T_1 > 0$ depending only on the initial data, such that

$$\sup_{0\le t\le T_1} \mathcal{E}(t) \le C(\mathcal{E}(0)), \tag{7.2}$$

where $C(\mathcal{E}(0))$ denotes a positive constant depending on $\mathcal{E}(0)$. This completes the a priori estimates of (1.17).

7.2 Justification of the a priori assumptions

It suffices to justify the a priori assumptions (1.19)-(1.20). First, invoking $\partial_t J = J\mathrm{div}_A v$ and $J|_{t=0} = 1$, we get

$$\|J-1\|_{7,*} \le \int_0^T \|J\mathrm{div}_A v\|_{7,*}\,dt \lesssim \int_0^T P(\|\partial\eta\|_{L^\infty})\|\partial_t q\|_{7,*}\,dt \le \int_0^T P(\|\partial\eta\|_{L^\infty})\|q\|_{8,*}\,dt.$$

Therefore choosing $T > 0$ to be sufficiently small yields (1.19). The Rayleigh-Taylor sign condition in $[0, T_1]$ can be justified by proving $\partial Q/\partial N$ is a Hölder-continuous function in t and y variables. In specific, from the energy estimates we know that

$$\frac{\partial Q}{\partial N} \in L^\infty([0,T]; H^{\frac{5}{2}}(\Gamma)),\ \partial_t\left(\frac{\partial Q}{\partial N}\right) \in L^\infty([0,T]; H^{\frac{3}{2}}(\Gamma)).$$

By using the 2D Sobolev embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^4(\Gamma)$ and Morrey's embedding $W^{1,4} \hookrightarrow C^{0,\frac{1}{4}}$ in 3D domain, we get the Hölder continuity of the Rayleigh-Taylor sign

$$\frac{\partial Q}{\partial N} \in W^{1,\infty}([0,T]; H^{\frac{3}{2}}(\Gamma)) \hookrightarrow W^{1,\infty}([0,T]; W^{1,4}(\Gamma)) \hookrightarrow W^{1,4}([0,T]\times\Gamma) \hookrightarrow C_{t,x}^{0,\frac{1}{4}}([0,T]\times\Gamma).$$

Therefore, (1.20) holds in a positive time interval provided that $-\dfrac{\partial Q_0}{\partial N} \ge c_0 > 0$ holds initially. Theorem 1.2 is proved.

8 Initial data satisfying the compatibility conditions

Define $f_{(j)} := \partial_t^j f|_{t=0}$ to be the initial data of $\partial_t^j f$ for $j \in \mathbb{N}$. Finally, we need to prove the existence of initial data satisfying the following properties:

- The compatibility conditions (1.9) up to 7-th order.

- The constraints $\nabla \cdot B_0 = 0$, $B_0 \cdot n|_{\{0\}\times\partial\mathcal{D}_0} = 0$ and the Rayleigh-Taylor sign condition (1.8) on $\{0\}\times\partial\mathcal{D}_0$.
- The norms of the initial datum of the time derivatives of (v, b, Q) can be controlled by the norms of initial data (v_0, b_0, Q_0).

We note that the compatibility conditions up to order m can be expressed in Lagrangian coordinates by using the formal power series solution to (1.17) in t:

$$\hat{v}(t,y) = \sum v_{(j)}(y)\frac{t^j}{j!},\ \hat{b}(t,y) = \sum b_{(j)}(y)\frac{t^j}{j!}, \hat{Q}(t,y) = \sum Q_{(j)}(y)\frac{t^j}{j!},$$

satisfying $Q_{(j)}|_\Gamma = 0$ for $j = 0, 1, \cdots, m$. Since we study the solutions in (anisotropic) Sobolev spaces, such compatibility conditions have to be expressed in a weak form

$$Q_{(j)}(y) \in H^1_0(\Omega),\ \ 0 \le j \le m. \tag{8.1}$$

From $(v_0, b_0, Q_0) \in H^8_*(\Omega)$ and the system (1.17), one can only get $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H^{8-2j}_*(\Omega)$ for $0 \le j \le 4$. To guarantee $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H^{8-j}_*(\Omega)$ and $Q_{(j)} \in H^1_0(\Omega)$, the initial data should be constructed in the standard Sobolev space $H^8(\Omega)$ with

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{8-j} \lesssim P(\|v_0\|_8, \|b_0\|_8, \|Q_0\|_8),$$

instead of in the anisotropic Sobolev space $H^8_*(\Omega)$. In Trakhinin-Wang [66, Lemma 4.1], they have proved the existence of such initial data, so we do not repeat the proof here.

On the one hand, by Lemma 2.2 we know $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H^{8-j}(\Omega) \hookrightarrow H^{8-j}_*(\Omega)$ which satisfies our requirement and implies $\mathcal{E}(0) \lesssim P(\|v_0\|_8, \|b_0\|_8, \|Q_0\|_8)$. On the other hand, if we directly construct the initial data $(v_0, b_0, q_0) \in H^8_*(\Omega)$ such that $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H^{8-j}_*(\Omega)$, then it is not clear in which sense the boundary conditions and the compatibility conditions are satisfied. For example, $Q_{(7)} \in H^1_*(\Omega)$, but the trace of such function in that space has no meaning in general. This also explains why we require $Q_{(7)} \in H^1_0(\Omega)$ in (8.1). See also Secchi [55, Theorem 2.1] and Ginsberg-Lindblad-Luo [19, Section 2.3] for the related discussion. Therefore, the initial data (v_0, b_0, Q_0) has to be constructed in the standard Sobolev space $H^8(\Omega)$.

Acknowledgement

Hans Lindblad was supported in part by Simons Foundation Collaboration Grant 638955. Junyan Zhang would like to thank Professor Tao Wang for helpful discussion.

References

[1] Alazard, T., Burq, N., Zuily, C. *On the Cauchy problem for gravity water waves.* Invent. Math., 198(1), 71–163. 2014.

[2] Alinhac, S. *Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels.(French. English summary) [Existence of rarefaction waves for multidimensional hyperbolic quasilinear systems].* Communications in Partial Differential Equations, 14(2), 173-230, 1989.

[3] Ambrose, D., Masmoudi, N. *The zero surface tension limit of two-dimensional water waves.* Comm. Pure Appl. Math., 58(10), 1287-1315, 2005.

[4] Chen, P., Ding, S. *Inviscid limit for the free-boundary problems of MHD equations with or without surface tension.* arXiv: 1905.13047, preprint, 2019.

[5] Chen, G.-Q., Wang, Y.-G. *Existence and Stability of Compressible Current-Vortex Sheets in Three-Dimensional Magnetohydrodynamics.* Arch. Ration. Mech. Anal., 187(3), 369-408, 2008.

[6] Chen, S.-X. *Initial boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary.* Translated from Chin. Ann. Math. 3(2), 222–232 (1982). Front. Math. China 2(1), 87–102 (2007).

[7] Chen, S.-X. *Multidimensional nonlinear systems of conservation laws.* (in Chinese) SCIENTIA SINICA Mathematica, 43(4), 317-332, 2013.

[8] Christodoulou, D., Lindblad, H. *On the motion of the free surface of a liquid.* Comm. Pure Appl. Math., 53(12), 1536–1602, 2000.

[9] Coulombel, J.-F., Morando, A., Secchi, P., Trbeschi, P. *A priori Estimates for 3D Incompressible Current-Vortex Sheets.* Comm. Math. Phys., 311(1), 247-275, 2012.

[10] Coutand, D., Hole, J., Shkoller, S. *Well-Posedness of the Free-Boundary Compressible 3-D Euler Equations with Surface Tension and the Zero Surface Tension Limit.* SIAM J. Math. Anal., 45(6), 3690-3767, 2013.

[11] Coutand, D., Lindblad, H., Shkoller, S. *A priori estimtes for the free-boundary 3D compressible Euler equations in physical vacuum.* Comm. Math. Phys., 296(2), 559-587, 2010.

[12] Coutand, D., Shkoller, S. *Well-posedness of the free-surface incompressible Euler equations with or without surface tension.* J. Amer. Math. Soc., 20(3), 829–930, 2007.

[13] Coutand, D., Shkoller, S. *A simple proof of well-posedness for the free-surface incompressible Euler equations.* Discrete and Continuous Dynamical Systems (Series S), 3(3), 429-449, 2010.

[14] Coutand, D., Shkoller, S. *Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum.* Arch. Ration. Mech. Anal., 206(2), 515-616, 2012.

[15] Disconzi, M. M. and Luo, C. *On the incompressible limit for the compressible free-boundary Euler equations with surface tension in the case of a liquid.* Arch. Ration. Mech. Anal., 237(2), 829-897, 2020.

[16] Ebin, D. G. *The equations of motion of a perfect fluid with free boundary are not well posed.* Communications in Partial Differential Equations, 12(10), 1175–1201, 1987.

[17] Ginsberg, D. *A priori estimates for a relativistic liquid with free surface boundary.* J. Hyperbolic Differ. Eq., 16(03), 401-442, 2019.

[18] Ginsberg, D., Lindblad, H. *On the local well-posedness for the relativistic Euler equations for an isolated liquid body.* Preprint, 2021.

[19] Ginsberg, D., Lindblad, H., Luo, C. *Local well-posedness for the motion of a compressible, self-gravitating liquid with free surface boundary.* Arch. Ration. Mech. Anal., 236(2), 603-733, 2020.

[20] Goedbloed, H., Keppens, R. and Poedts, S. *Magnetohydrodynamics of Laboratory and Astrophysical plasmas.* Cambridge University Press, 2020.

[21] Gu, X. *Well-posedness of axially symmetric incompressible ideal magnetohydrodynamic equations with vacuum under the non-collinearity condition.* Comm. Pure & Appl. Anal., 18(2), 569-602, 2019.

[22] Gu, X. *Well-posedness of axially symmetric incompressible ideal magnetohydrodynamic equations with vacuum under the Rayleigh-Taylor sign condition.* arXiv: 1712.02152, preprint, 2017.

[23] Gu, X., Luo, C., Zhang, J. *Local Well-posedness of the Free-Boundary Incompressible Magnetohydrodynamics with Surface Tension.* arxiv: 2105.00596, preprint, 2021.

[24] Gu, X., Wang, Y. *On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations.* J. Math. Pures Appl., Vol. 128: 1-41, 2019.

[25] Guo, B., Zeng, L., Ni, G. *Decay rates for the Viscous Incompressible MHD with and without Surface Tension.* Comput. Math. Appl., 77(12), 3224-3249, 2019.

[26] Hao, C. *Remarks on the free boundary problem of compressible Euler equations in physical vacuum with general initial densities.* Discrete and Continuous Dynamical Systems (Series B), 20(9), 2885-2931, 2015.

[27] Hao, C. *On the motion of free interface in ideal incompressible MHD.* Arch. Ration. Mech. Anal., 224(2), 515-553, 2017.

[28] Hao, C., Luo, T. *A priori estimates for free boundary problem of incompressible inviscid magnetohydrodynamic flows.* Arch. Ration. Mech. Anal., 212(3), 805–847, 2014.

[29] Hao, C., Luo, T. *Ill-posedness of free boundary problem of the incompressible ideal MHD.* Comm. Math. Phys., 376(1), 259-286, 2020.

[30] Hao, C., Luo, T. *Well-posedness for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations.* arXiv: 1912.05908, preprint, 2019.

[31] Ifrim, M., Tataru, D. *The compressible Euler equations in a physical vacuum: a comprehensive Eulerian approach.* Preprint, arxiv: 2007.05668, 2020.

[32] Iguchi, T. *Well-posedness of the initial value problem for capillary-gravity waves.* Funkcial. Ekvac. 44(2), 219–241, 2001.

[33] Jang, J., Masmoudi, N. *Well-posedness of Compressible Euler Equations in a Physical Vacuum.* Comm. Pure Appl. Math., 68(1): 61-111, 2014.

[34] Kreiss, H.-O. *Initial boundary value problems for hyperbolic systems.* Comm. Pure Appl. Math., 23(3), 277-298, 1970.

[35] Lannes, D. *Well-posedness of the water-waves equations.* J. Amer. Math. Soc., 18(3), 605-654, 2005.

[36] Lee, D. *Initial value problem for the free boundary magnetohydrodynamics with zero magnetic boundary condition.* Comm. Math. Sci., 16(3): 589-615, 2018.

[37] Lee, D. *Uniform estimate of viscous free-boundary magnetohydrodynamics with zero vacuum magnetic field.* SIAM Journal of Mathematical Analysis, 49(4), 2710-2789, 2017.

[38] Lindblad, H. *Well-posedness for the linearized motion of an incompressible liquid with free surface boundary.* Comm. Pure Appl. Math., 56(2), 153-197, 2003.

[39] Lindblad, H. *Well-posedness for the motion of an incompressible liquid with free surface boundary.* Ann. Math., 162(1), 109-194, 2005.

[40] Lindblad, H. *Well-posedness for the linearized motion of a compressible liquid with free surface boundary.* Comm. Math. Phys., 236(2), 281-310, 2003.

[41] Lindblad, H. *Well-posedness for the motion of a compressible liquid with free surface boundary.* Comm. Math. Phys. 260(2), 319-392, 2005.

[42] Lindblad, H., Luo, C. *A priori estimates for the compressible Euler equations for a liquid with free surface boundary and the incompressible limit.* Comm. Pure Appl. Math., 71(7), 1273-1333, 2018.

[43] Luo, C. *On the Motion of a Compressible Gravity Water Wave with Vorticity.* Annals of PDE, 4(2), 2506-2576, 2018.

[44] Luo, C., Zhang, J. *A regularity result for the incompressible magnetohydrodynamics equations. with free surface boundary.* Nonlinearity,

33(4), 1499-1527, 2020.

[45] Luo, C., Zhang, J. *A priori estimates for the incompressible free-boundary magnetohydrodynamics equations with surface tension.* SIAM J. Math. Anal., 53(2), 2595-2630, 2021.

[46] Luo, C., Zhang, J. *Local well-posedness for the motion of a compressible gravity water wave with vorticity.* Preprint, 2020.

[47] Luo, T., Xin, Z., Zeng, H. *Well-posedness for the motion of physical vacuum of the three-dimensional compressible Euler equations with or without self-gravitation.* Arch. Ration. Mech. Anal., 213(3), 763-831, 2014.

[48] Masmoudi, N., Rousset, F. *Uniform Regularity and Vanishing Viscosity Limit for the Free Surface Navier-Stokes Equations.* Arch. Ration. Mech. Anal., 223(1), 301-417, 2017.

[49] Ming, M., Zhang, Z. *Well-posedness of the water-wave problem with surface tension.* J. Math. Pures Appliquées, 92(5), 429-455, 2009.

[50] Morando, A., Trakhinin, Y., Trebeschi, P. *Well-posedness of the linearized plasma-vacuum interface problem in ideal incompressible MHD.* Quarterly of Applied Mathematics, 72(3), 549-587, 2014.

[51] Ohno, M., Shirota, T. *On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics.* Arch. Ration. Mech. Anal., 144(3), 259-299, 1998.

[52] Ohno, M., Shizuta, Y., Yanagisawa, T. *The trace theorem on anisotropic Sobolev spaces.* Tohoku Math. J., 46(3), 393-401, 1994.

[53] Padula, M., Solonnikov, V. A. *On the free boundary problem of magnetohydrodynamics.* Zap. Nauchn. Semin. POMI, 385, 135–186 (2010). J. Math. Sci. (N.Y.) 178 (2011), no. 3, 313–344, 2010.

[54] Secchi, P. *Well-posedness for Mixed Problems for the Equations of Ideal Magneto-hydrodynamics.* Archiv. der. Math. 64(3), 237–245, 1995.

[55] Secchi, P. *Well-posedness of characteristic symmetric hyperbolic systems.* Arch. Ration. Mech. Anal., 134(2), 155-197, 1996.

[56] Secchi, P., Trakhinin, Y. *Well-posedness of the plasma–vacuum interface problem.* Nonlinearity, 27(1): 105-169, 2013.

[57] Shatah, J., Zeng, C. *Geometry and a priori estimates for free boundary problems of the Euler's equation.* Comm. Pure Appl. Math., 61(5), 698-744, 2008.

[58] Shatah, J., Zeng, C. *A priori estimates for fluid interface problems.* Comm. Pure Appl. Math., 61(6), 848-876, 2008.

[59] Shatah, J., Zeng, C.. *Local well-posedness for fluid interface problems.* Arch. Ration. Mech. Anal., 199(2), 653-705, 2011.

[60] Sun, Y., Wang, W., Zhang, Z. *Nonlinear Stability of the Current-Vortex Sheet to the Incompressible MHD Equations.* Comm. Pure Appl. Math., 71(2), 356-403, 2018.

[61] Sun, Y., Wang, W., Zhang, Z. *Well-posedness of the Plasma-Vacuum Interface Problem for Ideal Incompressible MHD.* Arch. Ration. Mech. Anal., 234(1), 81-113, 2019.

[62] Tao, T. *Nonlinear dispersive equations: Local and global analysis.* CBMS Reg. Conf. Ser. Math. 106. American Mathematical Society, Providence, RI, 2006.

[63] Trakhinin, Y., *The existence of current-vortex sheets in ideal compressible magnetohydrodynamics.* Arch. Ration. Mech. Anal., 191(2), 245-310, 2009.

[64] Trakhinin, Y. *Local existence for the free boundary problem for nonrelativistic and Relativistic compressible Euler equations with a vacuum boundary condition.* Comm. Pure Appl. Math., 62(11), 1551-1594, 2009.

[65] Trakhinin, Y. *On well-posedness of the plasma-vacuum interface problem: the case of non-elliptic interface symbol.* Comm. Pure Appl. Anal., 15 (4), 1371-1399, 2016.

[66] Trakhinin, Y., Wang, T. *Well-posedness of Free Boundary Problem in Non-relativistic and Relativistic Ideal Compressible Magnetohydrodynamics.* Arch. Ration. Mech. Anal., 239(2), 1131-1176, 2021.

[67] Trakhinin, Y., Wang, T. *Well-Posedness for the Free-Boundary Ideal Compressible Magnetohydrodynamic Equations with Surface Tension.* Math. Ann., to appear, 2021.

[68] Wang, Y., Xin, Z. *Vanishing viscosity and surface tension limits of incompressible viscous surface waves.* SIAM J. Math. Anal., 53(1), 574-648, 2021.

[69] Wang, Y., Xin, Z. *Global Well-posedness of Free Interface Problems for the incompressible Inviscid Resistive MHD.* arXiv: 2009.11636, preprint, 2020.

[70] Wu, S. *Well-posedness in Sobolev spaces of the full water wave problem in 2-D.* Invent. Math., 130(1), 39–72, 1997.

[71] Wu, S. *Well-posedness in Sobolev spaces of the full water wave problem in 3-D.* J. Amer. Math. Soc., 12(2), 445–495, 1999.

[72] Yanagisawa, T., Matsumura, A. *The fixed boundary value problems for the equations of ideal magneto-hydrodynamics with a perfectly conducting wall condition.* Comm. Math. Phys., 136(1), 119-140, 1991.

[73] Zhang. J. *A priori Estimates for the Free-Boundary Problem of Compressible Resistive MHD Equations and Incompressible Limit.* arxiv: 1911.04928, preprint, 2019.

[74] Zhang. J. *Local Well-posedness of the Free-Boundary Problem in Compressible Resistive Magnetohydrodynamics.* arxiv: 2012.13931, preprint, 2020.

[75] Zhang, P., Zhang, Z. *On the free boundary problem of three-dimensional incompressible Euler equations.* Comm. Pure Appl. Math., 61(7), 877-940, 2008.