Local Well-posedness of the Free-Boundary Incompressible Magnetohydrodynamics with Surface Tension

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Abstract

We prove the local well-posedness of the 3D free-boundary incompressible ideal magnetohydrodynamics (MHD) equations with surface tension, which describe the motion of a perfect conducting fluid in an electromagnetic field. We adapt the tangential smoothing method developed in [13] to generate an approximate problem with artificial viscosity indexed by $\kappa > 0$ whose solution converges to that of the MHD equations as $\kappa \to 0$. This paper is the continuation of the second and third authors' previous work [40] in which the a priori energy estimate for incompressible free-boundary MHD with surface tension is established. However, the existence is not a trivial consequence of the a priori estimate as it cannot be adapted directly to the approximate problem.

Contents

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1 Introduction

We consider the following 3D incompressible ideal MHD system which describes the motion of a conducting fluid with free surface boundary in an electro-magnetic field under the influence of surface tension

$$\begin{cases} (\partial_t + u\cdot\nabla)u - B\cdot\nabla B + \nabla P = 0, \quad P := p + \frac{1}{2}|B|^2 & \text{in } \mathcal{D}; \\ (\partial_t + u\cdot\nabla)B - B\cdot\nabla u = 0, & \text{in } \mathcal{D}; \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, & \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} (\partial_t + u\cdot\nabla)|_{\partial\mathcal{D}} \in \mathcal{T}(\partial\mathcal{D}), \\ P = \sigma\mathcal{H} & \text{on } \partial\mathcal{D}, \\ B\cdot n = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{1.2}$$

Here $\mathcal{D} := \cup_{0\leq t\leq T}\{t\}\times\mathcal{D}_t$ and $\mathcal{D}_t \subseteq \mathbb{R}^3$ is the bounded domain occupied by the conducting fluid (plasma) whose boundary $\partial\mathcal{D}_t$ moves with the velocity of the fluid. Here $u = (u_1, u_2, u_3)$ is the fluid velocity, $B = (B_1, B_2, B_3)$ is the magnetic field, p is the fluid pressure and $P := p + \frac{1}{2}|B|^2$ is the total pressure. The quantity $\mathcal{H}$ is the mean curvature of the free surface $\partial\mathcal{D}_t$, $\sigma > 0$ is a given constant, called surface tension coefficient and n denotes the exterior unit normal to $\partial\mathcal{D}_t$. Throughout the manuscript, we will use the notation $D_t := \partial_t + u\cdot\nabla$ to denote the material derivative.

The first boundary condition shows that the boundary of the plasma moves with the velocity of the fluid. It can be equivalently expressed as the velocity of $(\partial\mathcal{D}_t)$ is equal to $u\cdot n$. The second boundary condition shows that the motion of the plasma is under the influence of surface tension. Here we note that $\mathcal{H}$ is determined by the unknown moving domain and thus not known a priori. The third boundary condition implies that the plasma liquid is a perfect conductor. In other words, the induced electric field $E := u\times B$ satisfies $E\times n = \mathbf{0}$ on $\partial\mathcal{D}_t$. We also note that $\operatorname{div} B = 0$ and $B\cdot n|_{\partial\mathcal{D}_t} = 0$ are both required only for initial data and they automatically propagate to any positive time. Therefore, the system (1.1)-(1.2) is not over-determined.

Under the conditions above, we have the following conservation of physical energy [40, Section 1].

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}_t}|u|^2 + |B|^2\,dx + \sigma\int_{\partial\mathcal{D}(t)} dS\,(\partial\mathcal{D}(t))\right) = 0 \tag{1.3}$$

Given a simply connected domain $\mathcal{D}_0 \subseteq \mathbb{R}^3$ and initial data u_0 and B_0 satisfying $\operatorname{div} u_0 = 0$ and $\operatorname{div} B_0 = 0$, $B_0\cdot n|_{\partial\mathcal{D}_0} = 0$, we want to find a set $\mathcal{D}$ and vector fields u and B solving (1.1)-(1.2) with initial data

$$\mathcal{D}_0 = \{x : (0,x)\in\mathcal{D}\}, \quad (u,B) = (u_0, B_0), \quad \text{in } \{t=0\}\times\Omega_0. \tag{1.4}$$

Remark. When the surface tension is neglected, the classical Rayleigh-Taylor sign condition $-\nabla_n P \geq c_0 > 0$ is necessary for the well-posedness. Ebin [20] and Hao-Luo [29] constructed the counterexamples for Euler equations and MHD equations respectively to show that the free-boundary problems can be ill-posed when the Rayleigh-Taylor sign condition is violated.

1.1 History and Background

1.1.1 Physical background: Plasma-Vacuum model

The free-boundary problem (1.1)-(1.2) originates from the plasma-vacuum free-interface model, which is an important theoretic model both in laboratory and in astro-physical magnetohydrodynamics. The plasma is confined in a vacuum with another magnetic field $\hat{B}$, and there is a free interface $\Gamma(t)$, moving with the motion of plasma, between the plasma region $\Omega_+(t)$ and the vacuum region $\Omega_-(t)$. Such model requires that (1.1) holds in the plasma region $\Omega_+(t)$ and the pre-Maxwell system holds in vacuum $\Omega_-(t)$:

$$\operatorname{curl}\hat{B} = \mathbf{0}, \quad \operatorname{div}\hat{B} = 0. \tag{1.5}$$

On the interface $\Gamma(t)$, it is required that there is no jump in the *the normal component*:

$$B\cdot n = \hat{B}\cdot n = 0, \quad [P] := p + \frac{1}{2}|B|^2 - \frac{1}{2}|\hat{B}|^2 = \sigma\mathcal{H} \tag{1.6}$$

where n is the exterior unit normal to $\Gamma(t)$. Finally, the following boundary condition holds on the outside rigid wall of the vacuum region

$$\hat{B} \times n = J$$

where J is the given outer surface current density (as an external input of energy). Note that for ideal MHD, $B = n = \hat{B} \cdot n = 0$ and $\hat{B} \times n = J$ should also be a constraint on initial data which propagates instead of an imposed boundary condition. See more details in [21, Chapter 4, 6].

1.1.2 Review of previous results

In the absence of magnetic field, the system (1.1)-(1.2) is reduced to the free-boundary incompressible Euler equations. The study of free-surface incompressible Euler equations has blossomed in the past several decades. In the case of no surface tension ($\sigma = 0$), the first breakthrough is Wu [59, 60] in which she proved the local well-posedness (LWP) for the irrotational case without surface tension. Lannes [33] proved the LWP for water wave with bottom. See also [4, 41, 1, 2] for the LWP with or without surface tension and [3, 5] for the study of incompressible vortex sheets. In the case of nonzero vorticity, Christodoulou-Lindblad [10] first proved the a priori estimates and then Lindblad [37, 38] proved the LWP by using Nash-Moser iteration. Later Coutand-Shkoller [13, 14] proved the LWP by using tangential smoothing and the energy estimates without loss of regularity in the case of both $\sigma = 0$ and $\sigma > 0$. See also Zhang-Zhang [63] for the study of incompressible water wave. In the case of nonzero surface tension, we refer to [44, 13] for LWP, and [31, 16, 17] for low regularity estimates, and [8, 46, 47, 48] for the study of incompressible vortex sheets with surface tension.

However, the study of free-boundary MHD equations is far less developed as opposed to Euler equations. The strong coupling between the magnetic field and the motion of fluid destroys good properties of Euler equations such as the propagation of irrotational assumption. Most of the known results focus on the case of zero surface tension. When the surface tension is neglected, extra stabilization such as the Rayleigh-Taylor sign condition is required. Lee [35, 36] proved the LWP for viscous-resistive MHD and the vanishing viscosity-resistivity limit. For the free-boundary problem of ideal incompressible MHD under Rayleigh-Taylor sign condition, Hao-Luo [28] proved the a priori estimates and [30] proved the linearized LWP. Then the first author and Wang [25] proved the LWP. The second and the third authors [39] proved the minimal regularity $H^{\frac{5}{2}+\varepsilon}$ estimates for a small fluid domain. For the plasma-vacuum model under Rayleigh-Taylor sign condition, Hao [27] proved the a priori estimates when $J = \mathbf{0}$ and the first author [22, 23] proved the LWP for axi-symmetric case. We note that there is another non-collinearity condition[1] $|B \times \hat{B}| \geq c_0 > 0$ which gives extra 1/2-order regularity of the free interface than Rayleigh-Taylor sign condition for the plasma-vacuum model. Under this condition, Morando-Trebeschi-Trakhinin [42] proved the LWP for linearized plasma-vacuum system and Sun-Wang-Zhang [50] proved the nonlinear LWP. Coulombo-Morando-Secchi-Trebeschi [11] proved the a priori estimates for 3D incompressible current-vortex sheets and Sun-Wang-Zhang [49] proved the LWP. So far, the energy estimates and well-posedness of the plasma-vacuum model in general cases under Rayleigh-Taylor sign condition are still open problems.

In the case of nonzero surface tension, there are very few results for the free-boundary MHD system and most previous works focus on the resistive or viscous MHD. To the best of our knowledge, The second and the third authors' previous work [40] which proved the $H^{7/2}$ a priori estimates is the only avaliable result for incompressible ideal MHD with surface tension. We also refer to Chen-Ding [7] for inviscid limit, Wang-Xin [58] for GWP of incompressible resistive MHD around a transversal uniform magnetic field, and Padula-Solonnikov [43], Guo-Zeng-Ni [26] for incompressible viscous-resistive MHD.

Finally, for compressible MHD, we refer to Secchi-Trakhinin [45] for the LWP of plasma-vacuum model under non-collinearity condition, and Chen-Wang [6], Trakhinin [53] and Wang-Yu [57] for compressible current-vortex sheets in 3D and 2D. Very recently, Trakhinin-Wang proved the LWP of free-boundary compressible ideal MHD under Rayleigh-Taylor sign condition [55] or with surface tension [56]. All these results are proved by Nash-Moser iteration and thus there is no energy estimate without regularity loss. The third author proved the LWP [62] and the incompressible limit [61] of compressible resistive MHD under Rayleigh-Taylor sign condition with energy estimates of no regularity loss. Finding suitable energy estimates without regularity loss for compressible ideal MHD with or without surface tension is also a widely open

[1] Such condition comes from the study of stability of current-vortex sheet which is a two-fluid (plasma-plasma) model in free-boundary MHD.

problem. The plasma-vacuum model in compressible MHD under Rayleigh-Taylor sign condition is also unsolved. See Trakhinin [54] for detailed discussion.

In the presenting manuscript, we prove the local well-posedness with energy estimates of no regularity loss for the free-boundary problem in incompressible ideal MHD with surface tension. Our result is a necessary step to study the plasma-vacuum model under the influence of surface tension, which is an original theoretical model in the study of confined plasma in both laboratory and astro-physical MHD.

1.2 Reformulation in Lagrangian coordinates

We reformulate the MHD equations in Lagrangian coordinates and thus the free-surface domain becomes fixed. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Denoting coordinates on Ω by $y = (y_1, y_2, y_3)$, we define $\eta : [0,T] \times \Omega \to \mathcal{D}$ to be the flow map of the velocity u, i.e.,

$$\partial_t \eta(t,y) = u(t, \eta(t,y)), \quad \eta(0,y) = y. \tag{1.7}$$

We introduce the Lagrangian velocity, magnetic field and pressure respectively by

$$v(t,y) = u(t,\eta(t,y)), \quad b(t,y) = B(t,\eta(t,y)), \quad q(t,y) = P(t,\eta(t,y)). \tag{1.8}$$

Let ∂ be the spatial derivative with respect to y variable. We introduce the cofactor matrix $a = [\partial\eta]^{-1}$ and $J := \det[\partial\eta]$, which is well-defined since $\eta(t,\cdot)$ is almost the identity map when t is sufficiently small. It's worth noting that a verifies the Piola's identity and $J = 1$ in the incompressible case, i.e.,

$$\partial_\mu (J a^{\mu\alpha}) = 0 \text{ and } J = 1. \tag{1.9}$$

Here, the Einstein summation convention is used for repeated upper and lower indices. In above and throughout, all Greek indices range over 1, 2, 3, and the Latin indices range over 1, 2.

Under this setting, the system (1.1)-(1.2) can be reformulated as:

$$\begin{cases} \partial_t v_\alpha - b_\beta a^{\mu\beta}\partial_\mu b_\alpha + a^\mu_\alpha \partial_\mu q = 0 & \text{in } [0,T]\times\Omega; \\ \partial_t b_\alpha - b_\beta a^{\mu\beta}\partial_\mu v_\alpha = 0 & \text{in } [0,T]\times\Omega; \\ a^{\mu\alpha}\partial_\mu v_\alpha = 0, \ \ a^{\mu\alpha}\partial_\mu b_\alpha = 0 & \text{in } [0,T]\times\Omega; \\ v\cdot N = b\cdot N = 0 & \text{on } \Gamma_0; \\ a^{\mu\alpha}N_\mu q + \sigma(\sqrt{g}\Delta_g \eta^\alpha) = 0 & \text{on } \Gamma; \\ a^{\mu\nu} b_\nu N_\mu = 0 & \text{on } \Gamma, \end{cases} \tag{1.10}$$

where N is the unit outer normal vector to $\partial\Omega$, a^T is the transpose of a, $|\cdot|$ is the Euclidean norm and Δ_g is the Laplacian of the metric g_{ij} induced on $\partial\Omega(t)$ by the embedding η. Specifically, we have:

$$g_{ij} = \overline{\partial}_i \eta^\mu \overline{\partial}_j \eta_\mu, \ \Delta_g(\cdot) = \frac{1}{\sqrt{g}}\overline{\partial}_i(\sqrt{g} g^{ij}\overline{\partial}_j(\cdot)), \text{ where } g := \det(g_{ij}). \tag{1.11}$$

For the sake of simplicity and clean notation, here we consider the model case[2] when

$$\Omega = \mathbb{T}^2 \times (0,1), \tag{1.12}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma$ and $\Gamma = \mathbb{T}^2 \times \{1\}$ is the top (moving) boundary, $\Gamma_0 = \mathbb{T}^2 \times \{0\}$ is the fixed bottom. Using a partition of unity, e.g., [17], a general domain can also be treated with the same tools we shall present. However, choosing Ω as above allows us to focus on the real issues of the problem without being distracted by the cumbersomeness of the partition of unity. Let N stands for the outward unit normal of $\partial\Omega$. In particular, we have $N = (0,0,-1)$ on Γ_0 and $N = (0,0,1)$ on Γ.

[2] $\mathbb{T}^2 \times (0,1)$ is called the reference domain, which allows us to work in one coordinate patch. See Coutand-Shkoller [13] for more detailed discussion.

By the second equation of (1.10) and the divergence-free condition on b, we get $\partial_t(a^{\mu\alpha}b_\mu) = 0$ which implies $a^{\mu\alpha}b_\mu = b_0^\alpha$ and thus $b^\alpha = b_0^\mu \partial_\mu \eta^\alpha = (b_0 \cdot \partial)\eta^\alpha$. See Gu-Wang [25, (1.13)-(1.15)] for the proof. Therefore, the system (1.10) can be equivalently written as the following system of (η, v, q)

$$
\begin{cases}
\partial_t \eta = v & \text{in } [0,T]\times\Omega;\\
\partial_t v - (b_0\cdot\partial)^2\eta + \nabla_a q = 0 & \text{in } [0,T]\times\Omega;\\
\text{div}\,_a v = 0, & \text{in } [0,T]\times\Omega;\\
\text{div}\, b_0 = 0 & \text{in } [0,T]\times\Omega;\\
v^3 = b_0^3 = 0 & \text{on } \Gamma_0;\\
a^{3\alpha} q + \sigma(\sqrt{g}\Delta_g \eta^\alpha) = 0 & \text{on } \Gamma;\\
b_0^3 = 0 & \text{on } \Gamma,\\
(\eta, v) = (\text{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}.
\end{cases}
\tag{1.13}
$$

Here $\nabla_a^\alpha := a^{\mu\alpha}\partial_\mu$ denotes the covariant derivative and div $_a$ denotes the Eulerian divergence.

Remark. The initial data of q is determined by v_0 and b_0. Acutually q_0 satisfies an elliptic equation

$$-\Delta q_0 = (\partial v_0)(\partial v_0) - (\partial b_0)(\partial b_0),$$

which can be solved with Neumann boundary condition.

1.3 Main result

We prove the local well-posedness of (1.13) in the presenting manuscript. We denote $\|f\|_s := \|f(t,\cdot)\|_{H^s(\Omega)}$ for any function $f(t,y)$ on $[0,T]\times\Omega$ and $|f|_s := |f(t,\cdot)|_{H^s(\Gamma)}$ for any function $f(t,y)$ on $[0,T]\times\Gamma$. Let Π be the canonical normal project defined on the tangent bundle of the moving interface. Our main result is:

Theorem 1.1. Let $v_0 \in H^{4.5}(\Omega)\cap H^5(\Gamma)$ and $b_0 \in H^{4.5}(\Omega)$ be divergence-free vector fields with $(b_0\cdot N)|_\Gamma = 0$. Then there exists some $T > 0$, only depending on σ, v_0, b_0, such that the system (1.13) with initial data (v_0, b_0, q_0) has a unique strong solution (η, v, q) with the energy estimates

$$\sup_{0\le t\le T} E(t) \le C, \tag{1.14}$$

where C is a constant depends on $\|v_0\|_{4.5}, \|b_0\|_{4.5}$, and

$$
\begin{aligned}
E(t) := {}& \|\eta\|_{4.5}^2 + \|v\|_{4.5}^2 + \|\partial_t v\|_{3.5}^2 + \left\|\partial_t^2 v\right\|_{2.5}^2 + \left\|\partial_t^3 v\right\|_{1.5}^2 + \left\|\partial_t^4 v\right\|_0^2\\
&+ \|(b_0\cdot\partial)\eta\|_{4.5}^2 + \|\partial_t(b_0\cdot\partial)\eta\|_{3.5}^2 + \left\|\partial_t^2(b_0\cdot\partial)\eta\right\|_{2.5}^2 + \left\|\partial_t^3(b_0\cdot\partial)\eta\right\|_{1.5}^2 + \left\|\partial_t^4(b_0\cdot\partial)\eta\right\|_0^2\\
&+ \left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v\right)\right|_0^2 + \left|\overline{\partial}(\Pi\overline{\partial}^3 v)\right|_0^2 + \left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)\right|_0^2.
\end{aligned}
\tag{1.15}
$$

Moreover, the $H^5(\Gamma)$-regularity of v on the free-surface can also be recovered, in the sense that there exists some $0 < T_1 < T$, depending only on σ, v_0, b_0, such that

$$\sup_{0\le t\le T_1} |\eta(t)|_5^2 + |v(t)|_5^2 \le C. \tag{1.16}$$

□

Remark (**Smoothing effect of** $b_0\cdot\partial$**).** It can be seen that in (1.15) v and $(b_0\cdot\partial)\eta$ are of the same interior regularity (i.e., $H^{4.5}(\Omega)$). This suggests that $(b_0\cdot\partial)$ and ∂_t behaves the same when falling on the flow map η. This observation turns out to be very important when studying the energy of the approximate equations (1.18) defined below.

1.4 Strategy of the proof

1.4.1 Necessity of the tangential smoothing

In [40], the second and third authors proved the a priori estimates of (1.13). However, it is often highly nontrivial to prove the local well-posedness for a free-boundary problem of inviscid fluid, especially when equipped with the Young-Laplace boundary condition, by a simple iteration scheme and fixed-point argument for the linearized equations. The reason is that the linearization breaks the subtle cancellation structure on the free surface and thus causes the loss of tangential derivatives of the flow map η, which also occurs for incompressible Euler equations with surface tension.

In their remarkable work [13], Coutand and Shkoller introduced an approximate system in the Lagrangian coordinates by smoothing the nonlinear coefficients in the tangential direction. This can be adapted to study the MHD equations and the tangential smoothing preserves the essential transport-type structure of the original equations. Specifically, we define Λ_κ to be the standard mollifier with parameter $\kappa > 0$ on $\mathbb{R}^2$ as in (2.18). Let $\tilde{\eta} := \Lambda_\kappa^2 \eta$ and $\tilde{a} = [\partial\tilde{\eta}]^{-1}$. Then we set nonlinear κ-approximation problem by replacing a with $\tilde{a}$. However, such construction is not applicable to MHD because we also need to control $\|[\Lambda_\kappa^2, (b_0 \cdot \partial)]\eta\|_{4.5}$ in which there is a normal derivative $b_0^3\partial_3$ that is not compatible with the tangential mollification. Motivated by Gu-Wang [25], we first mollify the flow map on the boundary, then extend it into the interior by the harmonic extension, i.e.,

$$\begin{cases} -\Delta\tilde{\eta} = -\Delta\eta & \text{in } \Omega, \\ \tilde{\eta} = \Lambda_\kappa^2\eta & \text{on } \Gamma. \end{cases} \tag{1.17}$$

Define $\tilde{a} := [\partial\tilde{\eta}]^{-1}$, $\tilde{J} := \det[\partial\tilde{\eta}]$ and $\tilde{A} := \tilde{J}\tilde{a}$, then we have the Piola's identity $\partial_\mu\tilde{A}^{\mu\alpha} = 0$. The nonlinear approximate system is defined to be

$$\begin{cases} \partial_t\eta = v & \text{in } [0,T]\times\Omega; \\ \partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\tilde{A}} q = 0 & \text{in } [0,T]\times\Omega; \\ \mathrm{div}_{\tilde{a}} v = 0, & \text{in } [0,T]\times\Omega; \\ \mathrm{div}\, b_0 = 0 & \text{in } \{t=0\}\times\Omega; \\ v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \tilde{A}^{3\alpha} q = -\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha + \kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ (\eta, v) = (\mathrm{Id}, v_0) & \text{in } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{1.18}$$

In this paper, we will (i). derive the uniform-in-κ a priori estimates of the system (1.18), and then (ii). solve the nonlinear κ-approximation system (1.18).

1.4.2 Necessity of the artificial viscosity

There is an artificial viscosity term $\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha$ in the smoothed surface tension equation on the boundary. This was first introduced by Coutand-Shkoller in [13] where the authors mentioned that the artificial viscosity term appears to be necessary in order to prove the existence of an inviscid fluid with non-trivial vorticity and surface tension. This term also appears in the subsequent work that studies the free-surface fluid with surface tension, e.g., Cheng-Coutand-Shkoller [8] for the vortex sheets, Coutand-Hole-Shkoller [12] for the compressible Euler, and very recently Trakhinin-Wang [56] for the compressible MHD.

Remark. Very recently, the first author and Lei [24] proved the LWP of incompressible elastodynamics with surface tension by proving the inviscid limit of visco-elastodynamics system in standard Sobolev spaces. We also note that the inviscid limit of free-boundary MHD was recently proved by Chen-Ding [7] in co-normal Sobolev spaces. However, analogous inviscid limit in standard Sobolev space is not applicable to MHD due to the existence of MHD boundary layers.

An essential reason for introducing such artificial viscosity term is that the presence of surface tension forces us to control all of the time derivatives. In particular, the pressure q satisfies an elliptic equation and it appears that one can only get control of it by considering the Neumann boundary condition instead of Dirichlet boundary condition due to the presence of surface tension. The Neumann boundary condition contains the time derivative of v, and thus we have to include the time derivatives in our energy.

However, the full time derivatives of v and $(b_0\cdot\partial)\eta$ only has $L^2(\Omega)$ regularity and we cannot get estimates of the full time derivatives of q due to the low spatial regularity. Therefore, we do not have any control for the terms containing full time derivatives on the boundary due to the failure of Sobolev trace lemma. For the original system, one can use the subtle cancellation structure developed in [16, 40] to resolve this difficulty. But such cancellation structure no longer holds for the nonlinear κ-approximate problem due to the presence of tangential smoothing. Therefore, introducing the artificial viscosity term could produce κ-weighted higher order terms on the boundary, which enables us to finish the energy control.

Remark. The Young-Laplace boundary condition only gives us the information in the Eulerian normal direction. Therefore, the artificial viscosity can only be imposed in the smoothed Eulerian normal direction $(\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha)$ instead of all the components, otherwise the system would be over-determined.

1.4.3 Difference from the case without surface tension

The first author and Wang [25] proved the LWP of incompressible MHD without surface tension, in which the pressure q can be controlled by the elliptic equation with Dirichlet (zero) boundary condition and thus one can avoid the estimates of all time derivatives which turn out to be very complicated in the presenting manuscript. This tells an essential difference from the case without the surface tension.

On the other hand, as mentioned in [13, 16, 40], surface tension has a stronger stabilization effect than the Rayleigh-Taylor sign condition in the case without surface tension. In fact, the presence of surface tension allows us to control the boundary norms of the normal component of v and $(b_0\cdot\partial)\eta$ by comparing with the corresponding Eulerian normal projections instead of using normal trace lemma (cf. Lemma 2.4) to reduce to interior tangential estimates. We refer Section 3.3 for details. This property allows us to gain extra 1/2 derivatives in the interior, and there is no need to introduce the Alinhac good unknowns and correction terms as in [25].

1.4.4 Illustration on the energy functional

Let Π be the canonical normal project defined on the tangent bundle of the moving interface and $\tilde{n}$ be the (Eulerian) unit normal (We refer Lemma 2.1 for the precise definition). The energy functional of the nonlinear approximate problem (1.18) is defined to be

$$E_\kappa(T)=E_\kappa^{(1)}(T)+E_\kappa^{(2)}(T)+E_\kappa^{(3)}(T),$$

where

$$\begin{aligned}
E_\kappa^{(1)}(T):=&\|\eta(\kappa)\|_{4.5}^2+\|v(\kappa)\|_{4.5}^2+\|\partial_t v(\kappa)\|_{3.5}^2+\left\|\partial_t^2 v(\kappa)\right\|_{2.5}^2+\left\|\partial_t^3 v(\kappa)\right\|_{1.5}^2+\left\|\partial_t^4 v(\kappa)\right\|_0^2\\
&+\|(b_0\cdot\partial)\eta(\kappa)\|_{4.5}^2+\|\partial_t(b_0\cdot\partial)\eta(\kappa)\|_{3.5}^2+\left\|\partial_t^2(b_0\cdot\partial)\eta(\kappa)\right\|_{2.5}^2+\left\|\partial_t^3(b_0\cdot\partial)\eta(\kappa)\right\|_{1.5}^2+\left\|\partial_t^4(b_0\cdot\partial)\eta(\kappa)\right\|_0^2\\
&+\left|\overline{\partial}\left(\Pi\partial_t^3 v(\kappa)\right)\right|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v(\kappa)\right)\right|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v(\kappa)\right)\right|_0^2+\left|\overline{\partial}(\Pi\overline{\partial}^3 v(\kappa))\right|_0^2+\left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta(\kappa))\right|_0^2,\\
E_\kappa^{(2)}(T):=&\frac{1}{\sigma}\int_0^T\Big(\left|\sqrt{\kappa}\partial_t^4 v(\kappa)\cdot\tilde{n}(\kappa)\right|_1^2+\left|\sqrt{\kappa}\partial_t^3 v(\kappa)\cdot\tilde{n}(\kappa)\right|_2^2+\left|\sqrt{\kappa}\partial_t^2 v(\kappa)\cdot\tilde{n}(\kappa)\right|_3^2\\
&+\left|\sqrt{\kappa}\partial_t v(\kappa)\cdot\tilde{n}(\kappa)\right|_4^2+\left|\sqrt{\kappa}(b_0\cdot\partial)v(\kappa)\cdot\tilde{n}(\kappa)\right|_4^2\Big)\,dt,\\
E_\kappa^{(3)}(T):=&\int_0^T\Big(\left\|\sqrt{\kappa}\partial_t^4 v(\kappa)\right\|_{1.5}^2+\left\|\sqrt{\kappa}\partial_t^4(b_0\cdot\partial)\eta\right\|_{1.5}^2+\left\|\sqrt{\kappa}\partial_t^3 v(\kappa)\right\|_{2.5}^2+\left\|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)\eta\right\|_{2.5}^2\Big)\\
&+\left\|\sqrt{\kappa}\partial_t^2 v(\kappa)\right\|_{3.5}^2+\left\|\sqrt{\kappa}\partial_t^2(b_0\cdot\partial)\eta\right\|_{3.5}^2+\left\|\sqrt{\kappa}\partial_t v(\kappa)\right\|_{4.5}^2+\left\|\sqrt{\kappa}\partial_t(b_0\cdot\partial)\eta\right\|_{4.5}^2\Big)\,dt.
\end{aligned}$$

The energy constructed above looks much more complicated than (1.15), but it is in fact quite natural. First, $E_\kappa^{(1)}$ constitutes the non-weighted energies which are needed in order to close the a priori estimate for the MHD equations without the artificial viscosity (cf. Luo-Zhang [40]). Then $E_\kappa^{(2)}$ consists of the κ-weighted higher order energy terms produced by the artificial viscosity when dealing with the tangential estimates.

Besides, extra error terms are generated when all the derivatives fall on the smoothed Eulerian normal $\tilde{n}$ in the construction of $E_\kappa^{(2)}$. Since $E_\kappa^{(2)}$ only gives us higher order control of the normal component instead of all components. Most of the top order error terms should be treated by moving them to the interior with the help of Sobolev trace lemma, and we use $E_\kappa^{(3)}$ to record all of them. Nevertheless, due to the coupling structure between the velocity and the magnetic field, the terms in $E_\kappa^{(3)}$ must be controlled together via the Hodge-type div-curl estimate and thus we have to include the associated magnetic terms in $E_\kappa^{(3)}$ as well.

When closing the energy estimates of $E_\kappa^{(2)}$ and $E_\kappa^{(3)}$, and $\overline{\partial}^3\partial_t$-tangential estimates, one needs the control of $\sqrt{\kappa}$-weighted $H^5(\Gamma)$-norms of η, v and $(b_0\cdot\partial)\eta$ recorded in Lemma 3.5. These $\sqrt{\kappa}$-weighted bounds can be established by considering $\overline{\partial}^4, \overline{\partial}^4\partial_t, \overline{\partial}^4(b_0\cdot\partial)$-differentiated smoothed Young-Laplace boundary condition. See also Coutand-Shkoller [13, Lemma 12.6].

Remark. In the proof of Lemma 3.5, the self-adjointness of Λ_κ is used to keep the struture and close the energy estimates. This is the reason that we need to mollify η twice in (1.17).

1.4.5 Difference between Euler equations and MHD with surface tension

As mentioned in [39, 40], the irrotational assumption for Euler equations no longer holds for MHD system, which makes it impossible to get a higher regularity of the flow map η than that of the velocity v. Without such property, one cannot control the $\overline{\partial}^4$-estimates as Coutand-Shkoller did in [13] for incompressible Euler equations. Besides, the $|\sqrt{\kappa}\eta|_6$ regularity for 3D incompressible Euler equations cannot be achieved either. But this does not affect the proof for MHD system unless one wants to get a $H^6(\Gamma)$-postpriori estimates for the flow map η.

1.4.6 Penalization method to solve the linearized problem

Finally, it remains to solve the nonlinear approximation problem. With the help of tangential smoothing, it is not difficult for us to finish the iteration from the linearized approximate problem to the nonlinear one. But it is still difficult to solve the linearized approximate problem by the fixed-point argument even if one can get the a priori estimates without the loss of regularity. The reason is that we do not have any suitable equation for q and thus the structure of the linearized system is no longer preserved in the verification of fixed-point argument. Motivated by [13], we use the penalization method to solve the linearized system. We introduce a penalized pressure defined by $q_\lambda := -\lambda^{-1}\mathrm{div}_{\mathring{A}} w_\lambda$ and prove the existence of L^2-weak solution to the penalized problem by Galerkin's method. Then we take the weak limit by passing $\lambda\to 0$ to get the weak solution of the linearized approximate problem. Finally, one can prove the weak solution is strong by H^1-estimates together with the inverse theorem of div-curl decomposition (cf. Lemma 2.6 (2)).

Remark. The penalization method is not needed in the compressible case because the free-boundary compressible MHD is a first-order symmetric hyperbolic system with characteristic boundary conditions and the corresponding linearized problem can be solved by the duality argument in Lax-Phillips [34]. We refer to Trakhinin-Wang [55, 56] for details.

Remark. We cannot directly prove the weak solution of the penalized problem is strong as in [13] because the presence of magnetic field makes the divergence part out of control. That is why we first take the weak limit and then verify the H^1-estimates for the linearized system.

Remark. In the a priori estimates and iteration process of the linearized approximate problem, the energy control is much simpler than the uniform-in-κ estimates of the nonlinear approximate problem (3.2) because we no longer require the energy is κ-independent. Therefore, one can use the elliptic estimates for equations with merely BMO-coefficients proved by Dong-Kim [19] (see also Disconzi-Kukavica [16, Proposition 3.4].) to get the boundary control. See Section 8 for details.

1.5 Organization of the paper

The presenting manuscript is organized as follows. In Section 2 we record the lemmas that are repeatedly used in the proof. Then we introduce the nonlinear κ-approximation problem and do the div-curl-boundary estimates in Section 3. The non-weighted energy $E_\kappa^{(1)}$ and $\sqrt{\kappa}$-weighted boundary norms are treated in Section

4 and $\sqrt{\kappa}$-weighted interior norms are treated in Section 5. Then the uniform-in-κ estimates for the nonlinear κ-approximate problem are closed in Section 6. In Section 7 we solve the linearized approximate system by penalization method. In Section 8 we use Picard iteration to solve the nonlinear κ-approximate problem. Finally, the local well-posedness and energy estimates of the original system are established in Section 9.

The following notations will be frequently used in the rest of this manuscript.
List of Notations:

- $\Omega := \mathbb{T}^2 \times (0,1)$. $\Gamma := \mathbb{T}^2 \times \{1\}$ is the free boundary and $\Gamma_0 := \mathbb{T}^2 \times \{0\}$ is the fixed bottom.
- $\|\cdot\|_s$: We denote $\|f\|_s := \|f(t,\cdot)\|_{H^s(\Omega)}$ for any function $f(t,y)$ on $[0,T]\times\Omega$.
- $|\cdot|_s$: We denote $|f|_s := |f(t,\cdot)|_{H^s(\Gamma)}$ for any function $f(t,y)$ on $[0,T]\times\Gamma$.
- $\|\cdot\|_{\dot{H}^s}, |\cdot|_{\dot{H}^s}$: Homogeneous Sobolev norm, replacing H^s above by $\dot{H}^s$.
- $P(\cdot)$: A generic non-decreasing continuous function in its arguments;
- $\overline{\partial}, \overline{\Delta}$: $\overline{\partial} = \partial_1, \partial_2$ denotes the tangential derivative and $\overline{\Delta} := \partial_1^2 + \partial_2^2$ denotes the tangential Laplacian.
- $\nabla_a^\alpha f := a^{\mu\alpha}\partial_\mu f$, $\text{div}_a \mathbf{f} := a^{\mu\alpha}\partial_\mu \mathbf{f}_\alpha$ and $(\text{curl}_a \mathbf{f})_\lambda := \epsilon_{\lambda\tau\alpha} a^{\mu\tau}\partial_\mu \mathbf{f}^\alpha$, where $\epsilon_{\lambda\tau\alpha}$ is the sign of the 3-permutation $(\lambda\tau\alpha) \in S_3$.

□

2 Preliminary lemmas

2.1 Geometric identities

The following geometric identities will be used repeatedly (and silently) throughout this manuscript.

Lemma 2.1. Let $\hat{n}$ the unit outer normal to $\eta(\Gamma)$ and $\mathcal{T}, \mathcal{N}$ be the tangential and normal bundle of $\eta(\Gamma)$ respectively. Denote $\Pi : \mathcal{T}|_{\eta(\Gamma)} \to \mathcal{N}$ to be the canonical normal projection. Denote $\overline{\partial}_A$ be ∂_t or $\overline{\partial}_1, \overline{\partial}_2$. Then we have the identities

$$\hat{n} := n \circ \eta = \frac{a^T N}{|a^T N|}, \tag{2.1}$$

$$|a^T N| = |(a^{31}, a^{32}, a^{33})| = \sqrt{g}, \tag{2.2}$$

$$\Pi^\alpha_\lambda = \hat{n}^\alpha \hat{n}_\lambda = \delta^\alpha_\lambda - g^{kl}\overline{\partial}_k \eta_\alpha \overline{\partial}\eta_\lambda, \tag{2.3}$$

$$\Pi^\alpha_\lambda = \Pi^\alpha_\mu \Pi^\mu_\lambda, \tag{2.4}$$

$$-\Delta_g(\eta^\alpha|_\Gamma) = \mathcal{H} \circ \eta \hat{n}^\alpha, \tag{2.5}$$

$$\sqrt{g}\Delta_g \eta^\alpha = \sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}_i \overline{\partial}_j \eta^\lambda = \sqrt{g} g^{ij} \overline{\partial}_i \overline{\partial}_j \eta^\alpha - \sqrt{g} g^{ij} g^{kl} \overline{\partial}_k \eta^\alpha \overline{\partial}_l \eta^\mu \overline{\partial}_i \overline{\partial}_j \eta_\mu, \tag{2.6}$$

$$\overline{\partial}_A(\sqrt{g}\Delta_g \eta^\alpha) = \overline{\partial}_i \Big(\sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}_A \overline{\partial}_j \eta^\lambda + \sqrt{g}(g^{ij} g^{kl} - g^{ik} g^{lj}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda \overline{\partial}_A \overline{\partial}_l \eta^\lambda \Big), \tag{2.7}$$

$$\overline{\partial}_A \hat{n}_\mu = - g^{kl} \overline{\partial}_k \overline{\partial}_A \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu, \tag{2.8}$$

$$\partial_t(\sqrt{g} g^{ij}) = \sqrt{g}(g^{ij} g^{kl} - 2 g^{lj} g^{ik}) \overline{\partial}_k v^\lambda \overline{\partial}_l \eta_\lambda. \tag{2.9}$$

Proof. See Lemma 2.5 in Disconzi-Kukavica [16]. □

Remark. Recall that $g_{ij} = \overline{\partial}_i \eta_\mu \overline{\partial}_j \eta^\mu$ and $g = \det[g_{ij}]$ and $[g^{ij}] = [g_{ij}]^{-1}$. This means that g_{ij}, g and g^{ij} are rational functions of $\overline{\partial}\eta$ and so is Π.

Notation 2.2. We shall use the notation $Q(\partial\eta)$ and $Q(\overline{\partial}\eta)$ to denote the rational functions of $\partial\eta$ and $\overline{\partial}\eta$, respectively. This Q notation allows us to record error terms in a concise way and so it will be used frequently throughout the rest of this paper. For example, for any tangential derivative $\overline{\partial}_A$, we have $\overline{\partial}_A Q(\overline{\partial}\eta) = \tilde{Q}^i_\alpha(\overline{\partial}\eta)\overline{\partial}_A \overline{\partial}_i \eta^\alpha$ where the term $\tilde{Q}^i_\alpha(\overline{\partial}\eta)$ is also a rational function of $\overline{\partial}\eta$. For more details of such notation, we refer readers to Section 11 in Coutand-Shkoller [13] and Remark 2.4 in Disconzi-Kukavica [16].

2.2 Sobolev inequalities

First we list the Kato-Ponce estimates which will be used in div-curl estimates.

Lemma 2.3 (**Kato-Ponce type inequalities**). Let $J = (I - \Delta)^{1/2}$, $s \geq 0$. Then the following estimates hold:

(1) $\forall s \geq 0$, we have

$$\begin{aligned} \|J^s(fg)\|_{L^2} &\lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{W^{s,q_2}}, \\ \|\partial^s(fg)\|_{L^2} &\lesssim \|f\|_{\dot{W}^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{\dot{W}^{s,q_2}}, \end{aligned} \tag{2.10}$$

with $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \leq p_1, q_2 < \infty$;

(2) $\forall s \geq 1$, we have

$$\|J^s(fg) - (J^s f)g - f(J^s g)\|_{L^p} \lesssim \|f\|_{W^{1,p_1}}\|g\|_{W^{s-1,q_2}} + \|f\|_{W^{s-1,q_1}}\|g\|_{W^{1,q_2}} \tag{2.11}$$

for all the $1 < p < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$.

Proof. See Kato-Ponce [32]. □

Lemma 2.4 (**Normal trace theorem**). It holds that for a vector field X

$$\left|\overline{\partial} X \cdot N\right|_{-0.5} \lesssim \|\overline{\partial} X\|_0 + \|\text{div } X\|_0 \tag{2.12}$$

Proof. This can be proved by testing a $H^{0.5}(\Gamma)$ function and divergence theorem. See [25, Lemma 3.4]. □

Lemma 2.5 (**Trace lemma for harmonic function**). Suppose that $s \geq 0.5$ and u solves the boundary-valued problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma \end{cases}$$

where $g \in H^s(\Gamma)$. Then it holds that

$$|g|_s \lesssim \|u\|_{s+0.5} \lesssim |g|_s$$

Proof. The LHS follows from the standard Sobolev trace lemma, while the RHS is the property of Poisson integral, which can be found in [52, Proposition 5.1.7]. □

2.3 Elliptic estimates

First we illustrate the div-curl elliptic estimate.

Lemma 2.6 (**Hodge-type decomposition and the inverse theorem**).

(1) Let X be a smooth vector field and $s \geq 1$, then it holds that

$$\|X\|_s \lesssim \|X\|_0 + \|\text{curl } X\|_{s-1} + \|\text{div } X\|_{s-1} + |\overline{\partial} X \cdot N|_{s-1.5}. \tag{2.13}$$

(2) Let $\Omega \subseteq \mathbb{R}^3$ be a bounded H^{k+1}-domain with $k > 1.5$. Given $\mathbf{F}, G \in H^{l-1}(\Omega)$ with $\text{div } \mathbf{F} = 0$. Consider the equations

$$\text{curl } X = \mathbf{F}, \quad \text{div } X = G \quad \text{in } \Omega. \tag{2.14}$$

If $\mathbf{F}$ satisfies $\int_\gamma \mathbf{F} \cdot N\, dS = 0$ for each connected component γ of $\partial\Omega$ and $h \in H^{l-0.5}(\partial\Omega)$ satisfies $\int_{\partial\Omega} h\, dS = \int_\Omega G\, dy$, then $\forall 1 \leq l \leq k$, there exists a solution $X \in H^l(\Omega)$ to (2.14) with boundary condition $X \cdot N|_{\partial\Omega} = h$ such that

$$\|X\|_{H^l(\Omega)} \leq C(|\partial\Omega|_{H^{k+0.5}})\left(\|\mathbf{F}\|_{H^{l-1}(\Omega)} + \|G\|_{H^{l-1}(\Omega)} + |h|_{H^{l-0.5}(\partial\Omega)}\right). \tag{2.15}$$

Such solution is unique if Ω is the disjoint union of simply connected open sets.

Proof. (1) This follows from the well-known identity $-\Delta X = \text{curl curl } X - \nabla \text{div } X$ and integrating by parts. (2) This is the main result of Cheng-Shkoller [9]. □

Next, the following H^1-elliptic estimates which will be applied to control $\|\partial_t^3 q\|_1$.

Lemma 2.7 (**Low regularity elliptic estimates**). Assume $\mathfrak{B}^{\mu\nu}$ satisfies $\|\mathfrak{B}\|_{L^\infty} \le K$ and the ellipticity $\mathfrak{B}^{\mu\nu}(x)\xi_\mu\xi_\nu \ge \frac{1}{K}|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Assume W to be an H^1 solution to

$$\begin{cases} \partial_\nu(\mathfrak{B}^{\mu\nu}\partial_\mu W) = \operatorname{div}\pi & \text{in } \Omega \\ \mathfrak{B}^{\mu\nu}\partial_\nu W N_\mu = h & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where $\pi, \operatorname{div}\pi \in L^2(\Omega)$ and $h \in H^{-0.5}(\partial\Omega)$ with the compatibility condition

$$\int_{\partial\Omega} (\pi \cdot N - h)dS = 0.$$

If $\|\mathfrak{B} - I\|_{L^\infty} \le \varepsilon_0$ which is a sufficently small constant depending on K, then we have:

$$\|W - \overline{W}\|_1 \lesssim \|\pi\|_0 + |h - \pi \cdot N|_{-0.5}, \text{ where } \overline{W} := \frac{1}{|\Omega|}\int_\Omega W dy, \tag{2.17}$$

Proof. See [31, Lemma 3.2]. □

2.4 Properties of tangential mollification

Let $\zeta = \zeta(y_1, y_2) \in C_c^\infty(\mathbb{R}^2)$ be a standard cut-off function such that $\operatorname{Spt}\zeta = \overline{B(0,1)} \subseteq \mathbb{R}^2$, $0 \le \zeta \le 1$ and $\int_{\mathbb{R}^2}\zeta = 1$. The corresponding dilation is

$$\zeta_\kappa(y_1, y_2) = \frac{1}{\kappa^2}\zeta\left(\frac{y_1}{\kappa}, \frac{y_2}{\kappa}\right), \quad \kappa > 0.$$

Now we define

$$\Lambda_\kappa f(y_1, y_2, y_3) := \int_{\mathbb{R}^2} \zeta_\kappa(y_1 - z_1, y_2 - z_2) f(z_1, z_2)\, dz_1\, dz_2. \tag{2.18}$$

The following lemma records the basic properties of tangential smoothing.

Lemma 2.8 (**Regularity and Commutator estimates**). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. For $\kappa > 0$, we have: (1) The following regularity estimates:

$$\|\Lambda_\kappa f\|_s \lesssim \|f\|_s, \quad \forall s \ge 0; \tag{2.19}$$

$$|\Lambda_\kappa f|_s \lesssim |f|_s, \quad \forall s \ge -0.5; \tag{2.20}$$

$$|\overline{\partial}\Lambda_\kappa f|_0 \lesssim \kappa^{-s}|f|_{1-s}, \quad \forall s \in [0,1]; \tag{2.21}$$

$$|f - \Lambda_\kappa f|_{L^\infty} \lesssim \sqrt{\kappa}|\overline{\partial} f|_{0.5} \tag{2.22}$$

$$|f - \Lambda_\kappa f|_{L^p} \lesssim \kappa|\overline{\partial} f|_{L^p}, \tag{2.23}$$

$$|f - \Lambda_\kappa f|_{L^2} \lesssim \sqrt{\kappa}|\overline{\partial}^{\frac{1}{2}} f|_0. \tag{2.24}$$

(2) Commutator estimates: Define the commutator $[\Lambda_\kappa, f]g := \Lambda_\kappa(fg) - f\Lambda_\kappa(g)$. Then it satisfies

$$|[\Lambda_\kappa, f]g|_0 \lesssim |f|_{L^\infty}|g|_0, \tag{2.25}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim |f|_{W^{1,\infty}}|g|_0, \tag{2.26}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_{0.5} \lesssim |f|_{W^{1,\infty}}|g|_{0.5}. \tag{2.27}$$

Proof. We refer [13, 25, 62] for the proof except for (2.24). The inequality (2.24) can be proved by integrating $\overline{\partial}^{\frac{1}{2}}$ by parts and then using Minkowski inequality

$$\begin{aligned} |f - \Lambda_\kappa f|_0 =& \left|\int_{\mathbb{R}^2\cap B(0,\kappa)} \zeta_\kappa(z)\,(f(y-z) - f(y))\, dz\right|_{L^2_y} \\ =&\kappa\left|\int_{\mathbb{R}^2\cap B(0,\kappa)} \overline{\partial}^{\frac{1}{2}}\zeta_\kappa(z)\overline{\partial}^{\frac{1}{2}} f(y - \theta z) dz\right|_{L^2_y} \\ \lesssim&\kappa\left|\overline{\partial}^{\frac{1}{2}} f\right|_0 \left|\overline{\partial}^{\frac{1}{2}}\zeta_\kappa\right|_{L^1(\mathbb{R}^2\cap B(0,\kappa))} \lesssim \kappa^2\left|\overline{\partial}^{\frac{1}{2}} f\right|_0 \left|\overline{\partial}^{\frac{1}{2}}\zeta_\kappa\right|_{L^2}. \end{aligned}$$

Then by interpolation, we have

$$\left|\overline{\partial}^{\frac{1}{2}}\zeta_\kappa\right|_{L^2} \lesssim |\zeta_\kappa|_{L^2}^{\frac{1}{2}}\left|\overline{\partial}\zeta_\kappa\right|_{L^2}^{\frac{1}{2}} \lesssim \left(\frac{1}{\kappa}|\zeta|_{L^2}\right)^{\frac{1}{2}}\left(\frac{1}{\kappa^2}|\overline{\partial}\zeta|_{L^2}\right)^{\frac{1}{2}} \lesssim \kappa^{-\frac{3}{2}},$$

and thus

$$|f-\Lambda_\kappa f|_0 \lesssim \sqrt{\kappa}\left|\overline{\partial}^{\frac{1}{2}} f\right|_0.$$

□

3 The nonlinear approximate system

For $\kappa > 0$, we denote Λ_κ to be the standard mollifier on $\mathbb{R}^2$ as defined as (2.18). Define $\tilde{\eta} := \Lambda_\kappa \eta$ to be the smoothed version of η solved by the following elliptic system

$$\begin{cases} -\Delta\tilde{\eta} = -\Delta\eta, & \text{in } \Omega, \\ \tilde{\eta} = \Lambda_\kappa^2\eta & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and $\tilde{a} := [\partial\tilde{\eta}]^{-1}$, $\tilde{J} := \det[\partial\tilde{\eta}]$, $\tilde{A} := \tilde{J}\tilde{a}$ and $\tilde{n} = n \circ \tilde{\eta}$. Now we introduce the nonlinear κ-approximation system of (1.13).

$$\begin{cases} \partial_t\eta = v & \text{in } [0,T]\times\Omega; \\ \partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\tilde{A}} q = 0 & \text{in } [0,T]\times\Omega; \\ \mathrm{div}_{\tilde{a}} v = 0, & \text{in } [0,T]\times\Omega; \\ \mathrm{div}\, b_0 = 0 & \text{in } \{t=0\}\times\Omega; \\ v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \tilde{A}^{3\alpha} q = -\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha + \kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ (\eta, v) = (\mathrm{Id}, v_0) & \text{in } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{3.2}$$

Here $\overline{\Delta} := \overline{\partial}_1^2 + \overline{\partial}_2^2$ is the tangential Laplacian. The re-formulated boundary condition on Γ is used here since we find that it is more convenient to apply when studying (3.2). We remark here that in absence of $\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha$ the boundary condition is just a reformulation of

$$\tilde{A}^{3\alpha} q = -\sigma\sqrt{g}\Delta_g\eta^\alpha. \tag{3.3}$$

Invoking (2.1) and the identity $\tilde{J}|\tilde{a}^T N| = \sqrt{\tilde{g}}$, where $\tilde{g} = g(\tilde{\eta})$, we have

$$\tilde{A}^{3\alpha}/\sqrt{\tilde{g}} = \tilde{J}\tilde{a}^{\mu\alpha}N_\mu/\tilde{J}|\tilde{a}^T N| = \tilde{n}^\alpha, \tag{3.4}$$

and so (3.3) becomes

$$q\tilde{n}^\alpha = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}\Delta_g\eta^\alpha.$$

Also, because $\tilde{n}\cdot\tilde{n} = 1$ (Euclidean dot product), we obtain

$$q\tilde{n}^\alpha = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha.$$

In view of (3.4), this is equivalent to

$$\tilde{A}^{3\alpha} q = -\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha$$

By adding the artificial viscosity term $\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha$ on the RHS, the boundary condition of (3.2) is then achieved:

$$\tilde{A}^{3\alpha}q=-\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha+\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha. \tag{3.5}$$

In addition, since $\tilde{A}^{3\alpha}\tilde{n}_\alpha=\sqrt{\tilde{g}}$, (3.5) can be written as

$$\sqrt{\tilde{g}}q=-\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})+\kappa(1-\overline{\Delta})(v\cdot\tilde{n}). \tag{3.6}$$

Despite being equivalent to each other, (3.5) and (3.6) will be adapted to different scenarios. In fact, (3.5) will be used in Section 4 for the tangential energy estimate, whereas we find (3.6) more convenient when dealing with the boundary estimate in Section 3.3.

Let's state the main theorem. Our goal is to derive the uniform-in-κ a priori estimates for the nonlinear approximation system (3.2).

Proposition 3.1. Given the divergence-free vector fields $v_0\in H^{4.5}(\Omega)\cap H^5(\Gamma)$ and $b_0\in H^{4.5}(\Omega)$ satisfying $b_0^3=0$, there exists some $T_1>0$ independent of $\kappa>0$, such that the solution $(\eta(\kappa),v(\kappa),q(\kappa))$ to (3.2) satisfies the following uniform-in-κ estimates

$$\sup_{0\le t\le T_1}E_\kappa(t)\le C, \tag{3.7}$$

where C is a constant depends on $\|v_0\|_{4.5},\|b_0\|_{4.5}$, provided the following a priori assumption hold for all $t\in[0,T_1]$

$$\|\tilde{J}(t)-1\|_{3.5}+\|\mathrm{Id}-\tilde{A}(t)\|_{3.5}+\|\mathrm{Id}-\tilde{A}^T\tilde{A}\|_{3.5}\le\varepsilon. \tag{3.8}$$

Here the energy functional E_κ of (3.2) is defined to be

$$E_\kappa=E_\kappa^{(1)}+E_\kappa^{(2)}+E_\kappa^{(3)}, \tag{3.9}$$

where

$$\begin{aligned}
E_\kappa^{(1)}(T):=&\|\eta(\kappa)\|_{4.5}^2+\|v(\kappa)\|_{4.5}^2+\|\partial_t v(\kappa)\|_{3.5}^2+\left\|\partial_t^2 v(\kappa)\right\|_{2.5}^2+\left\|\partial_t^3 v(\kappa)\right\|_{1.5}^2+\left\|\partial_t^4 v(\kappa)\right\|_0^2\\
&+\|(b_0\cdot\partial)\eta(\kappa)\|_{4.5}^2+\|\partial_t(b_0\cdot\partial)\eta(\kappa)\|_{3.5}^2+\left\|\partial_t^2(b_0\cdot\partial)\eta(\kappa)\right\|_{2.5}^2+\left\|\partial_t^3(b_0\cdot\partial)\eta(\kappa)\right\|_{1.5}^2+\left\|\partial_t^4(b_0\cdot\partial)\eta(\kappa)\right\|_0^2\\
&+\left|\overline{\partial}\left(\Pi\partial_t^3 v(\kappa)\right)\right|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v(\kappa)\right)\right|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v(\kappa)\right)\right|_0^2+\left|\overline{\partial}(\Pi\overline{\partial}^3 v(\kappa))\right|_0^2+\left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta(\kappa))\right|_0^2,\\
E_\kappa^{(2)}(T):=&\frac{1}{\sigma}\int_0^T\Big(\left|\sqrt{\kappa}\partial_t^4 v(\kappa)\cdot\tilde{n}(\kappa)\right|_1^2+\left|\sqrt{\kappa}\partial_t^3 v(\kappa)\cdot\tilde{n}(\kappa)\right|_2^2+\left|\sqrt{\kappa}\partial_t^2 v(\kappa)\cdot\tilde{n}(\kappa)\right|_3^2\\
&+\left|\sqrt{\kappa}\partial_t v(\kappa)\cdot\tilde{n}(\kappa)\right|_4^2+\left|\sqrt{\kappa}(b_0\cdot\partial)v(\kappa)\cdot\tilde{n}(\kappa)\right|_4^2\Big)\,dt,\\
E_\kappa^{(3)}(T):=&\int_0^T\Big(\left\|\sqrt{\kappa}\partial_t^4 v(\kappa)\right\|_{1.5}^2+\left\|\sqrt{\kappa}\partial_t^4(b_0\cdot\partial)\eta\right\|_{1.5}^2+\left\|\sqrt{\kappa}\partial_t^3 v(\kappa)\right\|_{2.5}^2+\left\|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)\eta\right\|_{2.5}^2\Big)\\
&+\left\|\sqrt{\kappa}\partial_t^2 v(\kappa)\right\|_{3.5}^2+\left\|\sqrt{\kappa}\partial_t^2(b_0\cdot\partial)\eta\right\|_{3.5}^2+\left\|\sqrt{\kappa}\partial_t v(\kappa)\right\|_{4.5}^2+\left\|\sqrt{\kappa}\partial_t(b_0\cdot\partial)\eta\right\|_{4.5}^2\Big)\,dt.
\end{aligned}$$

The proof of this theorem is organized as follows: The rest of this section is devoted to the estimate of the full Sobolev of the pressure q, and the velocity field v and the magnetic field $(b_0\cdot\partial)\eta$ as well as their time derivatives. In Section 4 we study the tangential energy estimate of v and $(b_0\cdot\partial)\eta$, which ties to the control of the boundary Sobolev norms of the time derivatives of v and $(b_0\cdot\partial)\eta$ that arose from the div-curl estimate. The terms in the weighted boundary top order energy $E_\kappa^{(2)}$ are created during this process owing to the artificial viscosity. Lastly, we investigate the weighted top order energy functional $E_\kappa^{(3)}$ in Section 5. In fact, we need this energy to control the error terms generated by the artificial viscosity on the boundary when all derivatives land on the Eulerian normal $\tilde{n}$.

Let $T\le T_\kappa$, where $[0,T_\kappa]$ is the interval of existence for the solution of the κ-problem for some fixed κ. The key step for showing (3.7) is to prove

$$\sup_{0\le t\le T}E_\kappa(t)\le\mathcal{P}_0+C(\varepsilon)\sup_{0\le t\le T}E_\kappa(t)+(\sup_{0\le t\le T}\mathcal{P})\int_0^T\mathcal{P}, \tag{3.10}$$

holds true independent of κ, where
$$\mathcal{P} = P(E_\kappa(t)),$$
and
$$\mathcal{P}_0 = P(E_\kappa(0), \|q(0)\|_{4.5}, \|q_t(0)\|_{3.5}, \|q_{tt}(0)\|_{2.5}),$$
with P denoting a non-decreasing continuous function in its arguments, and $C(\varepsilon)$ is a constant that is proportional to ε (and thus $C(\varepsilon) \ll 1$ whenever $\varepsilon \ll 1$). For the simplicity of notations, we will omit the κ in $(\eta(\kappa), v(\kappa), q(\kappa))$ in the rest of this section. Also, we may assume that $\sup_{0\le t\le T} E_\kappa(t) = E(T)$, and this allows us to drop $\sup_{0\le t\le T}$ in (3.10). In other words, we only need to show
$$E_\kappa(T) \le \mathcal{P}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{3.11}$$

Before going to the proof, we need the following preliminary estimates for $\tilde{\eta}$ and its derivatives.

Lemma 3.2. We have
$$\|\tilde{\eta}\|_{4.5} \lesssim \|\eta\|_{4.5} \tag{3.12}$$
$$\|(b_0\cdot\partial)\tilde{\eta}\|_{4.5} \lesssim P(\|b_0\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\eta\|_{4.5}). \tag{3.13}$$

Proof. (3.12) follows from standard elliptic estimates and property of mollification. To prove (3.13), we take $(b_0\cdot\partial)$ in (3.1)
$$\begin{cases} -\Delta((b_0\cdot\partial)\tilde{\eta}) = -\Delta((b_0\cdot\partial)\eta) - [(b_0\cdot\partial),\Delta]\,\eta + [(b_0\cdot\partial),\Delta]\tilde{\eta} & \text{in } \Omega, \\ \tilde{\eta} = \Lambda_\kappa^2\eta & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

and standard elliptic estimates yields that
$$\begin{aligned} \|(b_0\cdot\partial)\tilde{\eta}\|_{4.5} \lesssim& \|-\Delta((b_0\cdot\partial)\eta) - [(b_0\cdot\partial),\Delta]\,\eta + [(b_0\cdot\partial),\Delta]\tilde{\eta}\|_{2.5} \\ &+ \left|\Lambda_\kappa^2((b_0\cdot\partial)\eta)\right|_4 + \left|\left[(b_0\cdot\partial),\Lambda_\kappa^2\right]\eta\right|_4 \\ \lesssim& \|(b_0\cdot\partial)\eta\|_{4.5} + \|b_0|_{4.5}\|\eta\|_{4.5} \\ &+ \sum_{l=1}^{2}\left|\left[\Lambda_\kappa^2, b_0^l\right]\overline{\partial}_l\eta\right|_{1/2} + \left|\left[\Lambda_\kappa^2, b_0^l\right]\overline{\partial}_l\overline{\partial}^{7/2}\eta\right|_{1/2} + \left|\left[\overline{\partial}^{7/2}, [\Lambda_\kappa^2, b_0^l]\overline{\partial}_l\right]\eta\right|_{1/2} \\ \lesssim& P(\|b_0\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\eta\|_{4.5}), \end{aligned} \tag{3.15}$$
where commutator estimates in Lemma 2.8 is also used. □

The next lemma concerns some auxiliary results which come in handy when studying Proposition 3.1.

Lemma 3.3. Assume that $\|\eta\|_{4.5}, \|v\|_{4.5} \le N_0$, where $N_0 \ge 1$. If $T \le \varepsilon/P(N_0)$ for some fixed polynomial P and η, v is defined on $[0,T]$, then the following inequality holds for $t \in [0,T]$:
$$\|\tilde{a}^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} \lesssim \varepsilon, \quad \|a^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} \lesssim \varepsilon, \tag{3.16}$$
$$\|\tilde{A}^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} \lesssim \varepsilon, \tag{3.17}$$
$$\forall 0 \le s \le 1.5, \ \ |\overline{\partial}^s(\tilde{n} - N)|_{L^\infty(\Gamma)} \lesssim \varepsilon, \quad |\overline{\partial}^s(\hat{n} - N)|_{L^\infty(\Gamma)} \lesssim \varepsilon, \tag{3.18}$$
$$|\tilde{n} - N|_3 \lesssim \varepsilon, \quad |\hat{n} - N|_3 \lesssim \varepsilon, \tag{3.19}$$
$$|\delta^{ij} - \sqrt{g}g^{ij}|_3 \le \varepsilon, \tag{3.20}$$
$$|\overline{\partial}\eta\cdot n|_3 \le \varepsilon, \quad |\overline{\partial}^2\eta|_2 \le \varepsilon. \tag{3.21}$$

Proof. Since
$$A^{1\alpha} = \epsilon^{\alpha\lambda\tau}\partial_2\eta_\lambda\partial_3\eta_\tau, \quad A^{2\alpha} = -\epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda\partial_3\eta_\tau, \quad A^{3\alpha} = \epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda\partial_2\eta_\tau, \tag{3.22}$$

where $\epsilon^{\alpha\lambda\tau}$ is the fully antisymmnetric symbol with $\epsilon^{123} = 1$, we have $|\tilde{A} - \delta| \le \int_0^t |\partial_t Q(\partial\tilde{\eta})| \le \int_0^t |Q(\partial\tilde{\eta})\partial\tilde{v}|$, where Q is defined in Notation 2.2. Then (3.17) follows from (2.19) and the Kato-Ponce inequality (Lemma 2.3). Both inequalities in (3.16) are proved similarly.

In addition, for (3.18), it suffices to prove the first inequality. Since $\tilde{n}|_{t=0} = N$ and $\tilde{n} = Q(\overline{\partial}\tilde{\eta})$, for each fixed $0 \le s \le 1.5$, there holds

$$|\overline{\partial}^s(\tilde{n} - N)|_{L^\infty(\Gamma)} \le \int_0^t |\overline{\partial}^s\partial_t\tilde{n}|_{L^\infty(\Gamma)} = \int_0^t |\overline{\partial}^s(Q(\overline{\partial}\tilde{\eta})\overline{\partial}\tilde{v})|_{L^\infty(\Gamma)} \lesssim \int_0^t |Q(\overline{\partial}\tilde{\eta})\overline{\partial}\tilde{v}|_3,$$

by the Sobolev embedding. Now, the trace lemma and the Kato-Ponce inequality yield $\int_0^t |Q(\overline{\partial}\tilde{\eta})\overline{\partial}\tilde{v}|_3 \lesssim \int_0^t \|Q(\overline{\partial}\tilde{\eta})\overline{\partial}\tilde{v}\|_{3.5} \le \int_0^t P(N_0)$ and so (3.18) follows.

Moreover, we have

$$|\tilde{n} - N|_3 \le \int_0^t |Q(\overline{\partial}\tilde{\eta})\partial\tilde{v}|_3 \lesssim \int_0^t \|Q(\overline{\partial}\tilde{\eta})\overline{\partial}\tilde{v}\|_{3.5},$$

which verifies (3.19).

In addition, owing to the fact that $(\delta^{ij} - \sqrt{g}g^{ij})|_{t=0} = 0$ and the identity (2.9), there holds

$$|\delta^{ij} - \sqrt{g}g^{ij}|_3 \le \int_0^t |\partial_t(\sqrt{g}g^{ij})|_3 = \int_0^T \|Q(\partial\eta)\overline{\partial}v\|_{3.5},$$

which yields (3.20). Finally, a similar proof yields (3.21) since $\overline{\partial}\eta \cdot \tilde{n}|_{t=0} = \overline{\partial}\eta^3|_{t=0} = 0$ and $\overline{\partial}^2\eta|_{t=0} = 0$. □

Remark. The inequalities in Lemma 3.3 can in fact be view as an extended list of the a priori assumptions. Moreover, (3.8) is in fact a direct consequence of (3.16) and (3.17).

We also need the following corollary of Lemma 2.8 that "extends" (2.22) and (2.24) to the interior of Ω when applied to η and its time derivatives.

Lemma 3.4. Let $k = 0, \cdots, 4$. Then

$$\|\partial\partial_t^k(\tilde{\eta} - \eta)\|_0 \lesssim \|\sqrt{\kappa}\partial_t^k\eta\|_{1.5}. \tag{3.23}$$

Further, for $\ell = 0, 1, 2$, there holds

$$\|\partial\partial_t^\ell(\tilde{\eta} - \eta)\|_{L^\infty} \lesssim \|\sqrt{\kappa}\partial_t^\ell\eta\|_{3.5}. \tag{3.24}$$

Proof. The definition of $\tilde{\eta}$ in (3.1) implies that $\tilde{\eta} - \eta$ together its time derivatives is a harmonic function in Ω. So we invoke Lemma 2.5 to get

$$\|\partial\partial_t^k(\tilde{\eta} - \eta)\|_0 \lesssim \|\partial_t^k(\tilde{\eta} - \eta)\|_1 \lesssim |\partial_t^k(\Lambda_\kappa\eta - \eta)|_{0.5}$$

where $|\partial_t^k(\Lambda_\kappa\eta - \eta)|_{0.5} \lesssim |\sqrt{\kappa}\partial_t^k\eta|_1$ in light of (2.24). This, together with the trace lemma give (3.23). Moreover, (3.24) follow from (3.23) and the Sobolev embedding. □

Remark. It is possible to prove an improved estimate for (3.24), i.e.,

$$\|\partial\partial_t^\ell(\tilde{\eta} - \eta)\|_{L^\infty} \lesssim \|\sqrt{\kappa}\partial_t^\ell\eta\|_{2.5}. \tag{3.25}$$

This can be done by adapting the following Schauder estimate for div-curl systems: Let X be a smooth vector field on $\overline{\Omega}$. For fixed $0 < \delta < \frac{1}{2}$, we have

$$\|\partial X\|_{C^{0,\delta}(\Omega)} \lesssim \|\mathrm{div}\, X\|_{C^{0,\delta}(\Omega)} + \|\mathrm{curl}\, X\|_{C^{0,\delta}(\Omega)} + \|X\|_{C^{0.5,\delta}(\partial\Omega)} + \|X\|_2. \tag{3.26}$$

This inequality in fact reduces to the one in Lemma 8.2 of [18] in the absence of the boundary term. Thus, in view of (3.1), we have

$$\|\partial\partial_t^\ell(\tilde{\eta} - \eta)\|_{L^\infty} \le \|\partial\partial_t^\ell(\tilde{\eta} - \eta)\|_{C^{0,\delta}(\Omega)} \lesssim |\partial_t^\ell(\Lambda_\kappa\eta - \eta)|_{C^{0.5,\delta}(\Gamma)} + \|\partial_t^\ell(\tilde{\eta} - \eta)\|_2,$$

where the last term on the RHS is $\lesssim \|\sqrt{\kappa}\partial_t^\ell\eta\|_{2.5}$. In addition to this, (2.22) and the Sobolev embedding suggest that $|\partial_t^\ell(\Lambda_\kappa\eta-\eta)|_{C^{0.5,\delta}(\Gamma)}\lesssim|\sqrt{\kappa}\partial_t^\ell\eta|_{2+\delta}$, and so (3.25) follows after using the trace lemma.

Nevertheless, we mention here that (3.25) will not be applied in the rest of this manuscript. Despite not being sharp, (3.24) turns out to be sufficient.

Finally, we state the following two lemmas that concern the boundary elliptic estimates of $\sqrt{\kappa}\tilde{\eta}$ and $\kappa(b_0\cdot\partial)\tilde{\eta}$. These lemmas will be adapted to control the boundary error terms generated when derivatives land on the Eulerian normal $\tilde{n}$.

Lemma 3.5. Let $\mathcal{M}_0=P(\|v_0\|_{4.5},\sqrt{\kappa}\|v_0\|_{8.5},\sqrt{\kappa}\|b_0\|_{8.5})$. Then

$$|\sqrt{\kappa}\eta|_5^2\leq\mathcal{M}_0+C(\varepsilon)E_\kappa(T)+\mathcal{P}\int_0^T\mathcal{P}, \tag{3.27}$$

$$\int_0^T|\sqrt{\kappa}v|_5^2\leq\mathcal{M}_0+C(\varepsilon)E_\kappa(T)+\mathcal{P}\int_0^T\mathcal{P}, \tag{3.28}$$

$$\int_0^T|\sqrt{\kappa}(b_0\cdot\partial)\eta|_5^2\leq\mathcal{M}_0+C(\varepsilon)E_\kappa(T)+\mathcal{P}\int_0^T\mathcal{P}. \tag{3.29}$$

Discussion of the proof: The inequality (3.27) is Lemma 12.6 in [13]. The key step of the proof for (3.27) is to consider the $\overline{\partial}^4$-differentiated modified boundary condition (3.6), and then test it with $\Pi\overline{\partial}^4\eta$. During this process we need the control of $\|q\|_{4.5}^2$ (which is given by (3.30)) and $\|\eta\|_{4.5}^2$. Also, we mention here that the following highest order term will be generated during the testing process

$$\kappa\int_0^T\int_\Gamma(v^j\overline{\partial}^5\overline{\partial}_j\tilde{\eta}\cdot\tilde{n})(\overline{\partial}^5\eta\cdot\tilde{n}),$$

which cannot be controlled directly. Instead, we need to commute one tangential smooth operator Λ_κ from $\tilde{\eta}$ to η and hence create a positive term after pulling $\overline{\partial}_j$ out. In fact, this is the *only* place that this operation is required.

The proof for (3.28) and (3.29) follows from the same idea by studying the $\overline{\partial}^4\partial_t$ and $\overline{\partial}^4(b_0\cdot\partial)$-differentiated (3.6) tested with $\Pi\overline{\partial}^4v$ and $\Pi\overline{\partial}^4(b_0\cdot\partial)\eta$, respectively. In the former case we need the control of $\|v\|_{4.5}^2$ and $\|q_t\|_{3.5}^2$ (which is given by (3.30)), and in the latter case we need the control of $\|(b_0\cdot\partial)\eta\|_{4.5}^2$ and $\|(b_0\cdot\partial)q\|_{3.5}^2$, where $\|(b_0\cdot\partial)q\|_{3.5}^2\lesssim\|b_0\|_{3.5}^2\|q\|_{4.5}^2$ in light of the Kato-Ponce inequality (2.10).

3.1 Elliptic estimates of pressure

We prove the following proposition in this section.

Proposition 3.6. The pressure q in (3.2) and its time derivatives satisfy the following estimates

$$\|q\|_{4.5}+\|\partial_tq\|_{3.5}+\|\partial_t^2q\|_{2.5}+\|\partial_t^3q\|_1\lesssim\mathcal{P}. \tag{3.30}$$

First, we give control of the pressure q. Taking $\mathrm{div}_{\tilde{A}}$ in the second equation of (3.2), we get the following elliptic system for q

$$\begin{aligned}-\Delta_{\tilde{A}}q&=\left[\mathrm{div}_{\tilde{A}},\partial_t\right]v+\left[\mathrm{div}_{\tilde{A}},(b_0\cdot\partial)\right](b_0\cdot\partial)\eta+(b_0\cdot\partial)\mathrm{div}_{\tilde{A}}\left((b_0\cdot\partial)\eta\right)\\&=-\partial_t\tilde{A}^{\mu\alpha}\partial_\mu v_\alpha+\partial_\beta((b_0\cdot\partial)\tilde{\eta}_\nu)\partial_\nu\tilde{a}^{\mu\nu}\tilde{A}^{\beta\alpha}\partial_\mu(b_0\cdot\partial)\eta_\alpha\\&\quad+(b_0\cdot\partial)\underbrace{\mathrm{div}_{\,a}\left((b_0\cdot\partial)\eta\right)}_{=\mathrm{div}\,b_0=0}+(b_0\cdot\partial)\left((\tilde{A}^{\mu\alpha}-a^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha\right),\end{aligned}$$

and thus

$$\begin{aligned}-\Delta q=&-\partial_\nu\left((\delta^{\mu\nu}-\tilde{A}^{\mu\alpha}\tilde{A}^{\nu\alpha})\partial_\mu q\right)-\partial_t\tilde{A}^{\mu\alpha}\partial_\mu v_\alpha+\partial_\beta((b_0\cdot\partial)\tilde{\eta}_\nu)\partial_\nu\tilde{a}^{\mu\nu}\tilde{A}^{\beta\alpha}\partial_\mu(b_0\cdot\partial)\eta_\alpha\\&+(b_0\cdot\partial)\left((\tilde{A}^{\mu\alpha}-a^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha\right).\end{aligned} \tag{3.31}$$

We impose Neumann boundary condition to (3.31) by contracting $\tilde{A}^{\mu\alpha}N_\mu=\tilde{A}^{3\alpha}$ with the second equation of (3.2)

$$\frac{\partial q}{\partial N}=(\delta^{\mu3}-\tilde{A}^{\mu\alpha}\tilde{A}^{3\alpha})\partial_\mu q-\tilde{A}^{3\alpha}\partial_tv_\alpha+\tilde{A}^{3\alpha}(b_0\cdot\partial)^2\eta_\alpha. \tag{3.32}$$

By standard elliptic estimates, we have

$$\|q\|_{4.5} \lesssim \|\text{RHS of (3.31)}\|_{2.5} + |\text{RHS of (3.32)}|_3 + |q|_0.$$

Here, $|q|_0$ can be directly bounded by invoking the boundary condition of q, i.e.,

$$q = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\Delta_g\eta\cdot\tilde{n}) + \kappa\frac{1}{\sqrt{\tilde{g}}}(1-\overline{\Delta})(v\cdot\tilde{n}), \tag{3.33}$$

and thus

$$|q|_0 \lesssim \mathcal{P}. \tag{3.34}$$

Invoking the a priori assumption (3.8), we have

$$\begin{aligned}\|\text{RHS of (3.31)}\|_{2.5} \lesssim&\varepsilon\|q\|_{4.5} + \|\partial\tilde{\eta}\|_{2.5}\left(\|\partial v\|_{2.5}^2 + \|\partial(b_0\cdot\partial)\eta\|_{2.5}\|\partial(b_0\cdot\partial)\tilde{\eta}\|_{2.5}\right)\\ &+\|\tilde{A}-a\|_{3.5}\|b_0\|_{2.5}\|(b_0\cdot\partial)\eta\|_{3.5} + \|\tilde{A}-a\|_{2.5}\|b_0\|_{2.5}\|(b_0\cdot\partial)\eta\|_{4.5}\\ \lesssim&\varepsilon\|q\|_{4.5} + P(\|b_0\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\eta\|_{3.5}, \|v\|_{3.5}) + \kappa\|\overline{\partial}a\|_{2.5}\|b_0\|_{2.5}\|(b_0\cdot\partial)\eta\|_{4.5}\\ \lesssim&\varepsilon\|q\|_{4.5} + \mathcal{P},\end{aligned} \tag{3.35}$$

and

$$|\text{RHS of (3.32)}|_3 \lesssim \varepsilon\|q\|_{4.5} + |\overline{\partial}\tilde{\eta}|_{2.5}^2\left(\|\partial_t v\|_{3.5} + \|b_0\|_{3.5}\|(b_0\cdot\partial)\eta\|_{4.5}\right) \lesssim \varepsilon\|q\|_{4.5} + \mathcal{P}. \tag{3.36}$$

Summing up (3.34)-(3.36) and choosing $\varepsilon > 0$ sufficiently small, we get the estimates of q

$$\|q\|_{4.5} \lesssim \mathcal{P}. \tag{3.37}$$

Next we take ∂_t in (3.31)-(3.32) to get the equations of $\partial_t q$

$$\begin{aligned}-\Delta\partial_t q =& -\partial_\nu\left((\delta^{\mu\nu} - \tilde{A}^{\mu\alpha}\tilde{A}^{\nu\alpha})\partial_\mu\partial_t q\right) - \partial_\nu\left((\delta^{\mu\nu} - \partial_t(\tilde{A}^{\mu\alpha}\tilde{A}^{\nu\alpha}))\partial_\mu q\right)\\ &-\partial_t^2\tilde{A}^{\mu\alpha}\partial_\mu v_\alpha - \partial_t\tilde{A}^{\mu\alpha}\partial_t\partial_\mu v_\alpha + \partial_t(\partial((b_0\cdot\partial)\tilde{\eta})\cdot\partial\tilde{a}\cdot\tilde{A}\cdot\partial((b_0\cdot\partial)\eta))\\ &+(b_0\cdot\partial)\left((\partial_t\tilde{A}-\partial_t a)\partial((b_0\cdot\partial)\eta) + (\tilde{A}-a)\partial((b_0\cdot\partial)v)\right),\end{aligned} \tag{3.38}$$

with Neumann boundary condition

$$\frac{\partial\partial_t q}{\partial N} = (\delta^{\mu 3} - \tilde{A}^{\mu\alpha}\tilde{A}^{3\alpha})\partial_\mu\partial_t q - \partial_t(\tilde{A}^{\mu\alpha}\tilde{A}^{3\alpha})\partial_\mu Q - \tilde{A}^{3\alpha}(\partial_t^2 v_\alpha - (b_0\cdot\partial)^2 v_\alpha) - \partial_t\tilde{A}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha \tag{3.39}$$

Similarly we have

$$\|\partial_t q\|_{3.5} \lesssim \|\text{RHS of (3.38)}\|_{1.5} + |\text{RHS of (3.39)}|_2 + |\partial_t q|_0.$$

The control of the first two terms follows similarly as $\|q\|_{4.5}$:

$$\|\text{RHS of (3.38)}\|_{1.5} + |\text{RHS of (3.39)}|_2 \lesssim \mathcal{P}. \tag{3.40}$$

As for the boundary term, we take ∂_t in the surface tension equation to get

$$\partial_t q = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\Delta_g v\cdot\tilde{n}) + \kappa\frac{1}{\sqrt{\tilde{g}}}(1-\overline{\Delta})(\partial_t v\cdot\tilde{n}) + \text{lower-order temrs}$$

and thus

$$\|\partial_t q\|_0 \lesssim \mathcal{P}. \tag{3.41}$$

Summing up (3.40)-(3.41) and choosing $\varepsilon > 0$ to be sufficiently small, we get

$$\|\partial_t q\|_{3.5} \lesssim \mathcal{P}. \tag{3.42}$$

Time differentiating (3.38)-(3.39) and (3.33) again, we can silimarly get the estimates of $\|\partial_t^2 q\|_{2.5}$:

$$\|\partial_t^2 q\|_{2.5} \lesssim \mathcal{P}. \tag{3.43}$$

The treatment is similar to what has been done before and so we omit the details.

However, we cannot use the similar method to control $\|\partial_t^3 q\|_1$ because the standard elliptic estimates requires at least H^2-regularity. Instead, we invoke Lemma 2.7 which allows us to perform the low regularity H^1-estimate for ∂_t^3-differentiated elliptic system (3.31)-(3.32). To use this Lemma, we need to first rewrite the elliptic equations into the divergence form. Recall that the elliptic equation (3.31) is derived by taking smoothed Eulerian divergence $\mathrm{div}_{\tilde{A}}$. This, together with Piola's identity $\partial_\nu \tilde{A}^{\nu\alpha} = 0$ give that

$$-\partial_\nu(\tilde{A}^{\nu\alpha}\tilde{A}^{\mu\alpha}\partial_\mu q) = \partial_\nu\left(\tilde{A}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right),$$

with the boundary condition

$$\tilde{A}^{3\alpha}\tilde{A}^{\mu\alpha}\partial_\mu q = \tilde{A}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha.$$

Taking ∂_t^3 derivatives, we get

$$\partial_\nu(\tilde{A}^{\nu\alpha}\tilde{A}^{\mu\alpha}\partial_t^3\partial_\mu q) = \partial_\nu\left(\left[\tilde{A}^{\nu\alpha}\tilde{A}^{\mu\alpha}, \partial_t^3\right]\partial_\mu q\right) + \partial_\nu\partial_t^3\left(\tilde{A}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right), \tag{3.44}$$

with the boundary condition

$$\tilde{A}^{3\alpha}\tilde{A}^{\mu\alpha}\partial_\mu\partial_t^3 q = \left[\tilde{A}^{3\alpha}\tilde{A}^{\mu\alpha}, \partial_t^3\right]\partial_\mu q + \partial_t^3\left(\tilde{A}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right). \tag{3.45}$$

Now if we set

$$\mathfrak{B}^{\nu\mu} := \tilde{A}^{\nu\alpha}\tilde{A}^{\mu\alpha}, \quad h := \text{RHS of (3.45)}$$

and

$$\pi^\nu := \left[\tilde{A}^{\nu\alpha}\tilde{A}^{\mu\alpha}, \partial_t^3\right]\partial_\mu q + \partial_t^3\left(\tilde{A}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right)$$

then the elliptic system (3.44)-(3.45) is exactly of the form (2.16). The a priori assumption (3.8) shows that $\|\mathfrak{B} - \mathrm{Id}\,\|_{L^\infty}$ is sufficiently small. Now it is straightforward to see that $\pi, \mathrm{div}\,\pi \in L^2$, i.e.,

$$\|\pi\|_0 + \|\mathrm{div}\,\pi\|_0 \lesssim \mathcal{P}. \tag{3.46}$$

Also, since

$$h - \pi\cdot N = 0, \tag{3.47}$$

then by Lemma 2.7 and invoking (3.37), (3.42), (3.43), we have

$$\left\|\partial_t^3 q - \overline{\partial_t^3 q}\right\|_1 \lesssim \|\pi\|_0 \lesssim \mathcal{P}. \tag{3.48}$$

Lastly, we need to control the H^1-norm of $\overline{\partial_t^3 q}$ by $\mathcal{P}$.

$$\begin{aligned}\overline{\partial_t^3 q} &= \frac{1}{\mathrm{vol}\,(\Omega)}\int_\Omega \partial_t^3 q\,dy = \frac{1}{\mathrm{vol}\,(\Omega)}\int_\Omega \partial_t^3 q\overline{\partial}_1 y_1\,dy = -\frac{1}{\mathrm{vol}\,(\Omega)}\int_\Omega y_1\overline{\partial}_1\partial_t^3 q \\ &\le C(\mathrm{vol}\,(\Omega))\|\overline{\partial}\partial_t^3 q\|_0\|y_1\|_0 = C(\mathrm{vol}\,(\Omega))\left\|\overline{\partial}(\partial_t^3 q - \overline{\partial_t^3 q})\right\|_0\|y_1\|_0 \le C(\mathrm{Vol}(\Omega))\left\|\partial_t^3 q - \overline{\partial_t^3 q}\right\|_1.\end{aligned} \tag{3.49}$$

This concludes the control of $\|\partial_t^3 q\|_1$, and we have

$$\|\partial_t^3 q\|_1 \lesssim \mathcal{P}. \tag{3.50}$$

3.2 The div-curl estimates

Invoking Lemma 2.6, we have the following inequalities for $0 \le k \le 3$

$$\|v\|_{4.5}^2 \lesssim \|v\|_0^2 + \|\mathrm{div}\,v\|_{3.5}^2 + \|\mathrm{curl}\,v\|_{3.5}^2 + |\overline{\partial}v^3|_3^2, \tag{3.51}$$

$$\|(b_0\cdot\partial)\eta\|_{4.5}^2 \lesssim \|(b_0\cdot\partial)\eta\|_0^2 + \|\mathrm{div}\,(b_0\cdot\partial)\eta\|_{3.5}^2 + \|\mathrm{curl}\,(b_0\cdot\partial)\eta\|_{3.5}^2 + |\overline{\partial}(b_0\cdot\partial)\eta^3|_3^2, \tag{3.52}$$

$$\|\partial_t^k v\|_{4.5-k}^2 \lesssim \|\partial_t^k v\|_0^2 + \|\mathrm{div}\,\partial_t^k v\|_{3.5-k}^2 + \|\mathrm{curl}\,\partial_t^k v\|_{3.5-k}^2 + |\overline{\partial}\partial_t^k v^3|_{3-k}^2, \tag{3.53}$$

$$\|\partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2 \lesssim \|\partial_t^k(b_0\cdot\partial)\eta\|_0^2 + \|\mathrm{div}\,\partial_t^k(b_0\cdot\partial)\eta\|_{3.5-k}^2 + \|\mathrm{curl}\,\partial_t^k(b_0\cdot\partial)\eta\|_{3.5-k}^2 + |\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta^3|_{3-k}^2. \tag{3.54}$$

We note that the L^2-norms in (3.51) and (3.52) are controlled by energy conservation law. We will omit the control of L^2-norms appearing in the div-curl estimates in the rest of this manuscript.

Divergence estimates

For the velocity vector field, one has

$$\operatorname{div} v = \underbrace{\operatorname{div}_{\tilde{a}} v}_{=0} + (\delta^{\mu\alpha} - \tilde{a}^{\mu\alpha})\partial_\mu v_\alpha = \operatorname{div}_{\mathrm{Id}-\tilde{a}} v, \tag{3.55}$$

and thus

$$\|\operatorname{div} v\|_{3.5} \leq \|\operatorname{div}_{\tilde{a}} v\|_{3.5} + \|(\delta^{\mu\alpha} - \tilde{a}^{\mu\alpha})\partial_\mu v_\alpha\|_{3.5} \leq 0 + \varepsilon\|v\|_{4.5}. \tag{3.56}$$

Time differentiating (3.55), one has

$$\begin{aligned}\|\operatorname{div} \partial_t v\|_{2.5} \lesssim & \|\operatorname{div}_{\mathrm{Id}-\tilde{a}} \partial_t v\|_{2.5} + \|\operatorname{div}_{\partial_t \tilde{a}} v\|_{2.5} \\ \lesssim & \varepsilon\|\partial_t^2 v\|_{2.5} + \|\partial_t \tilde{a}\|_{2.5}\|v\|_{3.5} \lesssim \varepsilon\|\partial_t^2 v\|_{2.5} + \|\partial\tilde{\eta}\|_{2.5}\|v\|_{3.5}^2 \\ \lesssim & \varepsilon\|\partial_t^2 v\|_{2.5} + P(\|v_0\|_{3.5}) + \|\eta\|_{3.5}\int_0^T P(\|v\|_{4.5}),\end{aligned} \tag{3.57}$$

where in the last step we write $\|v\|_{3.5}$ in terms of initial data plus time integral and use Young's inequality.

Repeatedly time differentiating (3.55), we can similarly derive the divergence estimates of $\partial_t^k v$

$$\|\operatorname{div} \partial_t^2 v\|_{1.5} + \|\operatorname{div} \partial_t^3 v\|_{0.5} \lesssim \varepsilon(\|\partial_t^2 v\|_{2.5} + \|\partial_t^3 v\|_{1.5}) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{3.58}$$

As for $(b_0 \cdot \partial)\eta$, one no longer has $\operatorname{div}_{\tilde{a}}((b_0 \cdot \partial)\eta) = 0$ due to the tangential mollification. Instead, one can compute the evolution equation. Invoking $\operatorname{div}_{\tilde{a}} v = 0$ and $\partial_t \eta = v$, we have

$$\partial_t(\operatorname{div}_{\tilde{a}}((b_0 \cdot \partial)\eta)) = [\operatorname{div}_{\tilde{a}}, (b_0 \cdot \partial)]v + \operatorname{div}_{\partial_t \tilde{a}}(b_0 \cdot \partial)\eta. \tag{3.59}$$

The commutator $[\operatorname{div}_{\tilde{a}}, (b_0 \cdot \partial)]v$ only contains first order derivative of v and $(b_0 \cdot \partial)\eta$. One has

$$\begin{aligned}[\operatorname{div}_{\tilde{a}}, (b_0 \cdot \partial)]v = & \tilde{a}^{\mu\alpha}\partial_\mu b_0^\nu \partial_\nu v_\alpha - b_0^\nu \partial_\nu \tilde{a}^{\mu\alpha}\partial_\mu v_\alpha \\ = & \tilde{a}^{\mu\alpha}\partial_\mu b_0^\nu \partial_\nu v_\alpha + b_0^\nu \partial_\nu \tilde{a}^{\mu\gamma}\partial_\beta \partial_\nu \tilde{a}^{\beta\alpha}\partial_\mu v_\alpha \\ = & \tilde{a}^{\mu\alpha}\partial_\mu b_0^\nu \partial_\nu v_\alpha + \partial_\beta((b_0 \cdot \partial)\tilde{\eta}_\gamma)\partial_\nu \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}\partial_\mu v_\alpha - \partial_\beta b_0^\nu \underbrace{\partial_\nu \tilde{\eta}_\gamma \tilde{a}^{\mu\gamma}}_{\delta_{\nu\mu}} \tilde{a}^{\beta\alpha}\partial_\mu v_\alpha \\ = & \partial_\beta((b_0 \cdot \partial)\tilde{\eta}_\gamma)\partial_\nu \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}\partial_\mu v_\alpha.\end{aligned}$$

Therefore, taking $\partial^{3.5}$ in (3.59) and doing L^2 estimates, we get the divergence control of the magnetic field

$$\begin{aligned}\|\operatorname{div}_{\tilde{a}}(b_0 \cdot \partial)\eta\|_{3.5}^2 \leq & \|\operatorname{div} b_0\|_{3.5}^2 + \int_0^T \left(\partial^{3.5}\operatorname{div}_{\tilde{a}}((b_0 \cdot \partial)\eta)\right) \cdot \left([\operatorname{div}_{\tilde{a}}, (b_0 \cdot \partial)]v + \operatorname{div}_{\partial_t \tilde{a}}(b_0 \cdot \partial)\eta\right) \\ \lesssim & \int_0^T P(\|b_0\|_{4.5}, \|(b_0 \cdot \partial)\eta\|_{4.5}, \|v\|_{4.5}, \|\eta\|_{4.5})\, dt,\end{aligned} \tag{3.60}$$

and thus

$$\|\operatorname{div}\,(b_0 \cdot \partial)\eta\|_{3.5-k}^2 \lesssim \varepsilon^2\|(b_0 \cdot \partial)\eta\|_{4.5}^2 + \int_0^T \mathcal{P}\, dt. \tag{3.61}$$

Similarly, one can take $\partial^{3.5-k}\partial_t^k$ for $1 \leq k \leq 3$ in (3.59), then compute the L^2 estimates to get

$$\|\operatorname{div}_{\tilde{a}} \partial_t^k (b_0 \cdot \partial)\eta\|_{3.5-k}^2 \lesssim \varepsilon^2\|\partial_t^k (b_0 \cdot \partial)\eta\|_{4.5-k}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}\, dt. \tag{3.62}$$

Curl estimates

The curl estimates can be derived by the evolution equation of $\text{curl}_{\tilde{a}} v$. Taking $\text{curl}_{\tilde{a}}$ in the second equation of (3.2), we get

$$\partial_t(\text{curl}_{\tilde{A}} v) - (b_0 \cdot \partial)\text{curl}_{\tilde{A}}((b_0 \cdot \partial)\eta) = \text{curl}_{\partial_t \tilde{A}} v + [\text{curl}_{\tilde{A}}, (b_0 \cdot \partial)](b_0 \cdot \partial)\eta. \tag{3.63}$$

Then we take $\partial^{3.5}$, compute L^2 estimates and integrate $(b_0 \cdot \partial)$ by parts (recall that $b_0 \cdot N|_{\partial\Omega} = 0$ and $\text{div}\, b_0 = 0$) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{3.5}\text{curl}_{\tilde{A}} v|^2 + |\partial^{3.5}\text{curl}_{\tilde{A}}(b_0 \cdot \partial)\eta|^2\, dy \\
=&\int_\Omega \left(\left[\partial^{3.5}(b_0 \cdot \partial)\right]\text{curl}_{\tilde{A}}(b_0 \cdot \partial)\eta + \partial^{3.5}\left(\text{curl}_{\partial_t \tilde{A}} v + [\text{curl}_{\tilde{A}}, (b_0 \cdot \partial)](b_0 \cdot \partial)\eta\right)\right)(\partial^{3.5}\text{curl}_{\tilde{A}} v)\, dy \\
&+ \int_\Omega \partial^{3.5}(\text{curl}_{\tilde{A}}(b_0 \cdot \partial)\eta) \cdot \left(\left[\partial^{3.5}\text{curl}_{\tilde{A}}, (b_0 \cdot \partial)\right] v + \partial^{3.5}(\text{curl}_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta)\right)\, dy \\
\lesssim& P(\|b_0\|_{4.5}, \|(b_0 \cdot \partial)\eta\|_{4.5}, \|v\|_{4.5}, \|\tilde{A}\|_{3.5}, \|(b_0 \cdot \partial)\tilde{\eta}\|_{4.5}) \lesssim \mathcal{P},
\end{aligned} \tag{3.64}$$

and thus by the a priori assumption (3.8), we have

$$\|\text{curl}\, v\|_{3.5}^2 + \|\text{curl}\,(b_0 \cdot \partial)\eta\|_{3.5}^2 \lesssim \varepsilon^2(\|v\|_{4.5}^2 + \|(b_0 \cdot \partial)\eta\|_{4.5}^2) + \int_0^T \mathcal{P}\, dt. \tag{3.65}$$

Similarly, replacing $\partial^{3.5}$ by $\partial^{3.5-k}\partial_t^k$ for $1 \le k \le 3$, we can similarly get the following curl estimates

$$\|\text{curl}_{\tilde{A}}\partial_t^k(b_0 \cdot \partial)\eta\|_{3.5-k}^2 \lesssim \varepsilon^2\|\partial_t^k(b_0 \cdot \partial)\eta\|_{4.5-k}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}\, dt. \tag{3.66}$$

3.3 Boundary estimates

We need to control the boundary term $|\overline{\partial}\partial_t^k v \cdot N|_{3-k}$ and $|\overline{\partial}\partial_t^k(b_0 \cdot \partial)\eta \cdot N|_{3-k}$. In the case of no zero surface tension, one can use the normal trace theorem to reduce $|\overline{\partial}X \cdot N|_{s-1.5}$ to the interior tangential estimates $\|\overline{\partial}^s X\|_0$. But the interior tangential estimates, especially in the full spatial derivative case, are out of control due to the appearance of surface tension.

3.3.1 Control of $|\overline{\partial}\partial_t^k v \cdot N|_{3-k}$

Theorem 3.7. For $k = 0, 1, 2, 3$, one has

$$|\overline{\partial}\partial_t^k v^3|_{3-k}^2 \lesssim \|\overline{\partial}(\Pi\overline{\partial}^{3-k}\partial_t^k v)\|_0^2 + \mathcal{P}\int_0^T \mathcal{P}. \tag{3.67}$$

First we study the case when $k = 3$. Let us consider the projection of $\partial_t^3 v$ to the Eulerian normal direction, i.e., $(\Pi\partial_t^3 v)^3$ instead of Lagrangian normal direction. The reason is twofold.

1. Recall that (2.6) in Lemma 2.1 gives that

   $$\sqrt{g}g^{ij}\Delta_g \eta^\alpha = \sigma\sqrt{g}g^{ij}\Pi_\lambda^\alpha\overline{\partial}_{ij}^2\eta^\lambda.$$

   So if we test ∂_t^4-differentiated version of (2.6) with $\partial_t^4 v$ and integrate by parts, then the term $|\overline{\partial}(\Pi\partial_t^3 v)|_0^2$ is produced as part of energy term,i.e.,

   $$\int_\Gamma \sigma\sqrt{g}g^{ij}\Pi_\lambda^\alpha\partial_t^4\overline{\partial}_{ij}^2\eta^\lambda \cdot \partial_t^4 v_\alpha = -\frac{1}{2}\frac{d}{dt}\int_\Gamma \left|\overline{\partial}(\Pi\partial_t^3 v)\right|^2\, dS + \cdots \tag{3.68}$$

2. The difference between X^3 and $(\Pi X)^3$ is expected to be small within a short period of time.

We will make the above assertions precise. For any vector field X, the following identity holds:

$$X^3 = \delta^3_\lambda X^\lambda = (\delta^3_\lambda - g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda)X^\lambda + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda \\ = \Pi^3_\lambda X^\lambda + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda = (\Pi X)^3 + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda. \tag{3.69}$$

Using $\overline{\partial}\eta^3 = \int_0^T \overline{\partial}v^3\,dt$ (this is true since $\overline{\partial}\eta^3 = 0$ initially), we can control the difference between $(\Pi X)^3$ and X^3 as

$$\begin{aligned}\left|\overline{\partial}\left((\Pi X)^3 - X^3\right)\right|_0^2 \lesssim& \left|g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda\overline{\partial}X^\lambda\right|_0^2 + \left|\overline{\partial}(g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda)X^\lambda\right|_0^2 \\ \lesssim& P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}X|_0^2\int_0^T\left|\overline{\partial}v^3\right|_{1.5}^2\,dt + |X|_{L^4}^2|\overline{\partial}(g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda)|_{L^4}^2 \\ \lesssim& \|X\|_{1.5}^2P(|\overline{\partial}\eta|_{L^\infty})\int_0^T\mathcal{P}.\end{aligned} \tag{3.70}$$

Let $X = \partial_t^3 v$. Since $\|\partial_t^3 v\|_{1.5}^2$ is included in the energy $E_\kappa^{(1)}$, then (3.70) implies

$$\left|\overline{\partial}\left((\Pi\partial_t^3 v)^3 - \partial_t^3 v^3\right)\right|_0^2 \lesssim \mathcal{P}\int_0^T\mathcal{P}, \tag{3.71}$$

and thus

$$\left|\overline{\partial}\partial_t^3 v^3\right|_0^2 \lesssim \left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \mathcal{P}\int_0^T\mathcal{P}. \tag{3.72}$$

Finally, (3.67) follows from a parallel argument by assigning $X = \overline{\partial}\partial_t^2 v, \overline{\partial}^2\partial_t v, \overline{\partial}^3 v$, respectively.

3.3.2 Control of $|\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_{3-k}$

First, when $k \geq 1$, the control of $|\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_{3-k}$ requires to that of $|\overline{\partial}\partial_t^l v\cdot N|_{3-l}$ (modulo lower order terms generated when derivatives land on b_0) for $l = 0, 1, 2$, which has been done in the previous subsection.

Thus it suffices to study the control of $|(b_0\cdot\partial)\eta^3|_4$. In Luo-Zhang [40], the boundary condition forms an elliptic equation $-\sigma\sqrt{g}\Delta_g\eta^\alpha = a^{3\alpha}q$ and thus one can take $(b_0\cdot\partial)$ and then use elliptic estimates. However, the boundary condition now takes the form (3.6) in the smoothed approximate equations and it appears that there is no appropriate boundary H^2-control for $\kappa(b_0\cdot\partial)\overline{\Delta}(v\cdot\tilde{n})$. Specifically, it does not seem to be possible to control $|\kappa(b_0\cdot\partial)\overline{\Delta}(v\cdot\tilde{n})|_2$ by $\mathcal{P}_0 + C(\varepsilon)E_\kappa(T) + \int_0^T\mathcal{P}$ due to the lack of time integrals.

Our strategy here is to adapt the inequality (3.70) with $X = \overline{\partial}^3(b_0\cdot\partial)\eta$. In particular, we have

$$\begin{aligned}\left|\overline{\partial}\left((\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)^3 - \overline{\partial}^3(b_0\cdot\partial)\eta^3\right)\right|_0^2 \lesssim& \|\overline{\partial}^3(b_0\cdot\partial)\eta\|_{1.5}^2P(|\overline{\partial}\eta|_{L^\infty})\int_0^T\mathcal{P} \\ \lesssim& \mathcal{P}\int_0^T\mathcal{P},\end{aligned} \tag{3.73}$$

where the last inequality holds since $\|(b_0\cdot\partial)\eta\|_{4.5}^2$ is included in $E_\kappa^{(1)}$. Therefore,

$$\left|\overline{\partial}^4(b_0\cdot\partial)\eta^3\right|_0^2 \lesssim \left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)\right|_0^2 + \mathcal{P}\int_0^T\mathcal{P}. \tag{3.74}$$

4 Tangential energy estimates

The purpose of this section is to investigate the a priori energy estimate for the tangentially differentiated approximate κ-problem (3.2). In particular, we will study the energy estimate for

$$\partial_t^4, \overline{\partial}\partial_t^3, \overline{\partial}^2\partial_t^2, \overline{\partial}^3\partial_t, \overline{\partial}^3(b_0\cdot\partial)$$

differentiated κ-problem, respectively.

4.1 Control of full time derivatives

Now we compute the L^2-estimate of $\partial_t^4 v$ and $\partial_t^4(b_0\cdot\partial)\eta$. This turns out to be the most difficult case compare to the cases with at least one tangential spatial derivatives that will be treated in Section 4.2. This is due to the fact that $\partial_t^4 v$ can only be controlled in $L^2(\Omega)$ and so one has to control some higher order interior terms instead. These interior terms will be treated by adapting the geometric cancellation scheme introduced in [16] together with an error term which can be controlled by terms in $E_\kappa^{(3)}(t)$.

For the sake of simplicity and clean arguments, we shall focus on treating the leading order terms. We henceforth adopt:

Notation 4.1. We use $\stackrel{L}{=}$ to denote equality modulo error terms that are effectively of lower order. For instance, $X \stackrel{L}{=} Y$ means that $X = Y + \mathcal{R}$, where $\mathcal{R}$ consists of lower order terms with respect to Y.

Invoking (3.2) and integrating $(b_0\cdot\partial)$ by parts, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega |\partial_t^4 v|^2 + \left|\partial_t^4(b_0\cdot\partial)\eta\right|^2\,dy\\
=&\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^5 v^\alpha\,dy\,dt + \int_0^T\int_\Omega \partial_t^4(b_0\cdot\partial)\eta_\alpha\partial_t^4(b_0\cdot\partial)v^\alpha\,dy\,dt\\
=&\int_0^T\int_\Omega \partial_t^4 v_\alpha\partial_t^4(b_0\cdot\partial)^2\eta_\alpha\,dy\,dt - \int_0^T\int_\Omega \partial_t^4 v_\alpha\partial_t^4(\tilde{A}^{\mu\alpha}\partial_\mu q)\,dy\,dt\\
&+\int_0^T\int_\Omega \partial_t^4(b_0\cdot\partial)\eta_\alpha\partial_t^4(b_0\cdot\partial)v^\alpha\,dy\,dt\\
=&-\int_0^T\int_\Omega \partial_t^4(b_0\cdot\partial)v_\alpha\partial_t^4(b_0\cdot\partial)\eta_\alpha\,dy\,dt - \int_0^T\int_\Omega \partial_t^4 v_\alpha\partial_t^4(\tilde{A}^{\mu\alpha}\partial_\mu q)\,dy\,dt\\
&+\int_0^T\int_\Omega \partial_t^4(b_0\cdot\partial)\eta_\alpha\partial_t^4(b_0\cdot\partial)v^\alpha\,dy\,dt\\
=&-\int_0^T\int_\Omega \partial_t^4 v_\alpha\partial_t^4(\tilde{A}^{\mu\alpha}\partial_\mu q)\,dy\,dt =: I.
\end{aligned}\tag{4.1}$$

Then we integrate ∂_μ by parts, I becomes

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t^4\partial_\mu v_\alpha\partial_t^4(\tilde{A}^{\mu\alpha}q) + \underbrace{\int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^4(\tilde{A}^{3\alpha}q)}_{I_0}\\
=&\int_0^T\int_\Omega \tilde{A}^{\mu\alpha}\partial_t^4\partial_\mu v_\alpha\partial_t^4 q + \underbrace{\int_0^T\int_\Omega \partial_t^4\partial_\mu v_\alpha[\partial_t^4,\tilde{A}^{\mu\alpha}]q}_{I_1} + I_0\\
=&\int_0^T\int_\Omega \underbrace{\partial_t^4\mathrm{div}_{\tilde{A}}v}_{=0}\,\partial_t^4 q - \underbrace{\int_0^T\int_\Omega [\partial_t^4,\tilde{A}^{\mu\alpha}]\partial_\mu v_\alpha\partial_t^4 q}_{L} + I_1 + I_0.
\end{aligned}\tag{4.2}$$

I_I yields a top order interior term when all 4 time derivatives land on $\tilde{A}^{\mu\alpha}$, i.e.,

$$I_{11} = \int_0^T\int_\Omega \partial_t^4\partial_\mu v_\alpha(\partial_t^4\tilde{A}^{\mu\alpha})q.\tag{4.3}$$

If $\tilde{A}^{\mu\alpha}$ were $A^{\mu\alpha}$ then this term could be controlled by adapting the cancellation scheme developed in [16]. This motivate us to consider

$$\int_0^T\int_\Omega \partial_t^4\partial_\mu v_\alpha(\partial_t^4 A^{\mu\alpha})q + \int_0^T\int_\Omega \partial_t^4\partial_\mu v_\alpha\Big(\partial_t^4(\tilde{A}^{\mu\alpha}-A^{\mu\alpha})\Big)q = I_{111}+I_{112}.\tag{4.4}$$

Invoking (3.22), we have $\partial_t^4 \tilde{A} = \sum_{i+j=3} b_{ij}(\partial_t^i \partial\tilde{\eta})(\partial\partial_t^j \tilde{v})$ and $\partial_t^4 A = \sum_{i+j=3} b_{ij}(\partial_t^i \partial\eta)\partial\partial_t^j v$, where we denoted $A^{\mu\alpha}$ by A and $\tilde{A}^{\mu\alpha}$ by $\tilde{A}$ by a slight abuse of notations. These imply that

$$\partial_t^4(\tilde{A}-A) = \sum_{i+j=3} b_{ij}\Big(\partial_t^i\partial\tilde{\eta}\Big(\partial\partial_t^j(\tilde{v}-v)\Big) + \partial_t^i(\partial\tilde{\eta}-\partial\eta)\partial\partial_t^j v\Big),$$

and so $\|\partial_t^4(\tilde{A}-A)\|_0$ consists the sum of $\|i_\ell\|_0$, $\ell = 1,\cdots,8$, where

$$i_1 = (\partial\partial_t^2\tilde{v})\partial(\tilde{v}-v), \quad i_2 = (\partial\partial_t\tilde{v})\partial\partial_t(\tilde{v}-v), \quad i_3 = (\partial\tilde{v})\partial\partial_t^2(\tilde{v}-v), \quad i_4 = (\partial\tilde{\eta})\partial\partial_t^3(\tilde{v}-v),$$
$$i_5 = \partial\partial_t^2(\tilde{v}-v)\partial v, \quad i_6 = \partial\partial_t(\tilde{v}-v)\partial\partial_t v, \quad i_7 = \partial(\tilde{v}-v)\partial\partial_t^2 v, \quad i_8 = \partial(\tilde{\eta}-\eta)\partial\partial_t^3 v.$$

The L^2-norm of these quantities can be controlled by invoking Lemma 3.4. Specifically,

$$\begin{aligned}
\|i_1\|_0 \le& \|\partial(\tilde{v}-v)\|_{L^\infty}\|\partial\partial_t^2\tilde{v}\|_0 \lesssim \sqrt{\kappa}\|v\|_{3.5}\|\partial_t^2 v\|_1,\\
\|i_2\|_0 \le& \|\partial\partial_t(\tilde{v}-v)\|_{L^\infty}\|\partial\partial_t\tilde{v}\|_0 \lesssim \sqrt{\kappa}\|\partial_t v\|_{3.5}\|\partial_t v\|_1,\\
\|i_3\|_0 \le& \|\partial\partial_t^2(\tilde{v}-v)\|_0\|\partial\tilde{v}\|_{L^\infty} \lesssim \sqrt{\kappa}\|\partial_t^2 v\|_{1.5}\|v\|_3,\\
\|i_4\|_0 \le& \|\partial\tilde{\eta}\|_{L^\infty}\|\partial\partial_t^3(\tilde{v}-v)\|_0 \lesssim \sqrt{\kappa}\|\eta\|_3\|\partial_t^3 v\|_{1.5},
\end{aligned}$$

and

$$\begin{aligned}
\|i_5\|_0 \le& \|\partial\partial_t^2(\tilde{v}-v)\|_0\|\partial v\|_{L^\infty} \lesssim \sqrt{\kappa}\|\partial_t^2 v\|_{1.5}\|v\|_3,\\
\|i_6\|_0 \le& \|\partial\partial_t(\tilde{v}-v)\|_{L^\infty}\|\partial\partial_t v\|_0 \lesssim \sqrt{\kappa}\|\partial_t v\|_{3.5}\|\partial_t v\|_1,\\
\|i_7\|_0 \le& \|\partial(\tilde{v}-v)\|_{L^\infty}\|\partial\partial_t^2 v\|_0 \lesssim \sqrt{\kappa}\|v\|_{3.5}\|\partial_t^2 v\|_1,\\
\|i_8\|_0 \le& \|\partial(\tilde{\eta}-\eta)\|_{L^\infty}\|\partial\partial_t^3 v\|_{L^2} \lesssim \sqrt{\kappa}\|\eta\|_{3.5}\|\partial_t^3 v\|_1.
\end{aligned}$$

Summing these up, we obtain

$$I_{112} \le \int_0^T \|\partial_t^4\partial v\|_0\|\partial_t^4(\tilde{A}^{\mu\alpha}-A^{\mu\alpha})\|_0\|q\|_{L^\infty} \le \frac{\varepsilon}{2}\int_0^T \|\sqrt{\kappa}\partial_t^4\partial v\|_0^2 + \frac{1}{2\varepsilon}\int_0^T \mathcal{P}, \tag{4.5}$$

where the first term on the RHS contributes to $\varepsilon\mathcal{P}$, and we bound $\|q\|_{L^\infty}$ by $\|q\|_2 \le \mathcal{P}$ through (3.30).

We next control I_{111}. The argument is largely similar to that used in Section 3.1.3 of [16] which replies on exploiting the geometric structure in order to create cancellation among the leading order terms. Invoking (3.22) and then expanding the index μ in I_{111}, we have

$$\begin{aligned}
I_{111} =& \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\lambda\partial_3\eta_\tau\overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_\tau\overline{\partial}_2\partial_t^4 v_\alpha\\
&+ \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_3\partial_t^3 v_\tau\overline{\partial}_2\eta_\lambda\overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\tau\overline{\partial}_2\eta_\lambda\partial_3\partial_t^4 v_\alpha\\
&+ \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\tau\overline{\partial}_1\eta_\lambda\partial_3\partial_t^4 v_\alpha - \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_3\partial_t^3 v_\tau\overline{\partial}_1\eta_\lambda\overline{\partial}_2\partial_t^4 v_\alpha + I_{low}\\
=:& I_{1111} + I_{1112} + \cdots + I_{1116} + I_{low},
\end{aligned}\tag{4.6}$$

where I_{low} consists terms of the form $\int_0^T\int_\Omega q\partial\partial_t^2 v\partial v\partial\partial_t^3 v$. This term can be treated by integrating ∂_t by parts,

$$\int_0^T\int_\Omega q\partial\partial_t^2 v\partial v\partial\partial_t^4 v = \int_\Omega q\partial\partial_t^2 v\partial v\partial\partial_t^3 v\Big|_0^T - \int_0^T\int_\Omega \partial_t(q\partial\partial_t^2 v\partial v)\partial\partial_t^3 v,$$

where the second term is controlled by $\int_0^T \mathcal{P}$, whereas

$$\left|\int_\Omega q\partial\partial_t^2 v\partial v\partial\partial_t^3 v\Big|_0^T\right| \lesssim \mathcal{P}_0 + \varepsilon\|\partial_t^3 v\|_1^2 + \|q\|_{L^\infty}^2\|\partial v\|_{L^\infty}^2\|\partial\partial_t^2 v\|_0^2 \le \mathcal{P}_0 + \varepsilon\|\partial_t^3 v\|_1^2 + \int_0^T \mathcal{P}.$$

To control the leading terms in (4.6), we consider $I_{1111}+I_{1112}$, $I_{1113}+I_{1114}$, and $I_{1115}+I_{1116}$. For $I_{1111}+I_{1112}$, integrating ∂_t by parts in I_{1112}, we have

$$
\begin{aligned}
I_{1111}+I_{1112} \le\ & \underbrace{\int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^4 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\partial_t^3 v_\alpha}_{=0} \\
& - \int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\partial_t^3 v_\alpha\Big|_0^T + I'_{low},
\end{aligned}
\tag{4.7}
$$

where I'_{low} consists terms of the form $\int_0^T\int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_t(q\partial\eta)(\partial\partial_t^3 v)^2$ which can be controlled by $\int_0^T\mathcal{P}$. Next we treat the first term on the RHS of (4.7). It suffices to consider $-\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\partial_t^3 v_\alpha\Big|_{t=T} := \mathcal{T}$ as

$$
\int_\Omega q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\partial_t^3 v_\alpha\Big|_{t=0} \le \mathcal{P}_0.
$$

We shall drop $\Big|_{t=T}$ in $\mathcal{T}$ for the sake of clean notations. Expanding in τ, we find

$$
\mathcal{T} = -\int_\Omega q\epsilon^{\alpha\lambda i}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_i \overline{\partial}_2\partial_t^3 v_\alpha - \int_\Omega q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_3 \overline{\partial}_2\partial_t^3 v_\alpha. \tag{4.8}
$$

Since $\partial_3\eta_i|_{t=0}=0$, we can write $\partial_3\eta_i = \int_0^T\partial_3 v_i$, and so

$$
-\int_\Omega q\epsilon^{\alpha\lambda i}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_i \overline{\partial}_2\partial_t^3 v_\alpha \le \mathcal{P}\int_0^T\mathcal{P}. \tag{4.9}
$$

In addition to this, we have $\partial_3\eta_3 = 1+\int_0^T\partial_3 v_3$, and so

$$
-\int_\Omega q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_3 \overline{\partial}_2\partial_t^3 v_\alpha \le -\int_\Omega q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda \overline{\partial}_2\partial_t^3 v_\alpha + \mathcal{P}\int_0^T\mathcal{P}. \tag{4.10}
$$

To treat the first term on the RHS, we expand $\epsilon^{\alpha\lambda 3}$ and get

$$
-\int_\Omega q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda \overline{\partial}_2\partial_t^3 v_\alpha = -\int_\Omega q(\overline{\partial}_1\partial_t^3 v_2\overline{\partial}_2\partial_t^3 v_1 - \overline{\partial}_1\partial_t^3 v_1\overline{\partial}_2\partial_t^3 v_2). \tag{4.11}
$$

Integrating by parts $\overline{\partial}_2$ in the first term and $\overline{\partial}_1$ in the second term, we have

$$
\begin{aligned}
& -\int_\Omega q(\overline{\partial}_1\partial_t^3 v_2\overline{\partial}_2\partial_t^3 v_1 - \overline{\partial}_1\partial_t^3 v_1\overline{\partial}_2\partial_t^3 v_2) \\
=\ & \underbrace{\int_\Omega q\overline{\partial}_1\overline{\partial}_2\partial_t^3 v_2\partial_t^3 v_1 - \int_\Omega q\partial_t^3 v_1\overline{\partial}_1\overline{\partial}_2\partial_t^3 v_2}_{=0} + \int_\Omega \overline{\partial}_2 q\overline{\partial}_1\partial_t^3 v_2\partial_t^3 v_1 - \int_\Omega \overline{\partial}_1 q\partial_t^3 v_1\overline{\partial}_1\partial_t^3 v_2.
\end{aligned}
$$

Here,

$$
\Big|\int_\Omega \overline{\partial}_2 q\overline{\partial}_1\partial_t^3 v_2\partial_t^3 v_1 - \int_\Omega \overline{\partial}_1 q\partial_t^3 v_1\overline{\partial}_1\partial_t^3 v_2\Big| \lesssim \epsilon\|\partial_t^3 v\|_1^2 + \|\partial_t^3 v\|_0^2\|\partial q\|_{L^\infty}^2 \le \varepsilon\|\partial_t^3 v\|_1^2 + \mathcal{P}_0 + \int_0^T\mathcal{P}.
$$

Therefore,

$$
I_{1111}+I_{1112} \le \varepsilon E(T) + \mathcal{P}_0 + \mathcal{P}\int_0^T\mathcal{P}. \tag{4.12}
$$

On the other hand, $I_{1113}+I_{1114}$ and $I_{1115}+I_{1116}$ are treated similarly with only one exception. Previously, we integrated $\overline{\partial}_1$ and $\overline{\partial}_2$ by parts in (4.11) and so there is no boundary terms. However, when controlling

$I_{1113} + I_{1114}$, we need to integrate $\overline{\partial}_1$ and ∂_3 by parts when treating (4.11), and thus the following boundary term will appear:

$$\int_\Gamma q\partial_t^3 v_1 \overline{\partial}_1 \partial_t^3 v_3. \tag{4.13}$$

To control this term, we invoke the identity

$$\overline{\partial}_1\partial_t^3 v^3 = \Pi^3_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda + g^{kl}\overline{\partial}_k \eta^3 \overline{\partial}_l \eta_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda = \Pi^3_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda + g^{kl}\left(\int_0^T \overline{\partial}_k v^3\right)\overline{\partial}_l \eta_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda, \tag{4.14}$$

and thus (4.13) becomes

$$\begin{aligned}
&\int_\Gamma q\partial_t^3 v_1 \Pi^3_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda + \int_\Gamma q\partial_t^3 v_1 g^{kl}(\int_0^T \overline{\partial}_k v^3)\overline{\partial}_l \eta_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda \\
\lesssim &\varepsilon |\Pi \overline{\partial} \partial_t^3 v|_0^2 + |q|^2_{L^\infty} |\partial_t^3 v|_0^2 + \left| q\partial_t^3 v_1 g^{kl}(\int_0^T \overline{\partial}_k v^3)\overline{\partial}_l \eta_\lambda \right|_{0.5} |\overline{\partial}\partial_t^3 v^\lambda|_{-0.5} \\
\lesssim &\varepsilon |\Pi \overline{\partial} \partial_t^3 v|_0^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}.
\end{aligned}$$

The extra term generated when analyzing $I_{1115} + I_{1116}$ is of the same type integral and thus can be treated by the same method. Therefore,

$$I_{111} \le \varepsilon E(T) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.15}$$

Next we study

$$I_1 - I_{11} = 4\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t^3 \tilde{A}^{\mu\alpha}\partial_t q + 6\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t^2 \tilde{A}^{\mu\alpha}\partial_t^2 q + 4\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t \tilde{A}^{\mu\alpha}\partial_t^3 q := I_{12} + I_{13} + I_{14}. \tag{4.16}$$

For I_{12}, we integrating ∂_t by parts and obtain

$$4\int_\Omega \partial_t^3 \partial_\mu v_\alpha \partial_t^3 \tilde{A}^{\mu\alpha}\partial_t q - 4\int_0^T \int_\Omega \partial_t^3 \partial_\mu v_\alpha \partial_t (\partial_t^3 \tilde{A}^{\mu\alpha}\partial_t q).$$

Here, the second term is $\le \int_0^T \mathcal{P}$, and since

$$\partial_t^3 \tilde{A} = Q(\partial\tilde{\eta})\partial\partial_t^2 \tilde{v} + \text{lower order terms}$$

then the first term is bounded by

$$\varepsilon \|\partial_t^3 v\|_1^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}.$$

I_{13} is treated by adapting a similar method and so we omit the details. However, we can't integrate ∂_t by parts in order to control I_{14} as we do not have a bound for $\partial_t^4 q$. We integrate ∂_μ by parts instead.

$$I_{14} = 4\int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{A}^{3\alpha}\partial_t^3 q - 4\int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_\mu (\partial_t \tilde{A}^{3\alpha}\partial_t^3 q).$$

There is no problem to control the second integral by $\int_0^T \mathcal{P}$. For the first integral, invoking the boundary condition (3.6), we obtain

$$-4\sigma\int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{A}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}\Delta_g \eta\cdot\tilde{n}\Big) + 4\int_0^T \int_\Gamma \kappa\partial_t^4 v_\alpha \partial_t \tilde{A}^{3\alpha}\partial_t^3\Big(\frac{1}{\sqrt{\tilde{g}}}(1-\overline{\Delta})(v\cdot\tilde{n})\Big) = I_{141} + I_{142}. \tag{4.17}$$

Invoking (2.6), I_{141} becomes

$$\begin{aligned}I_{141}=&-4\sigma\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{A}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\mathring{g}}}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\Big)\\&-4\sigma\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{A}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\mathring{g}}}g^{ij}g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}_i\overline{\partial}_j\eta_\mu\overline{\partial}_k\eta\cdot\tilde{n}\Big).\end{aligned}$$

It suffices for us to consider the first integral only since the second integral is of the same type. Integrating by parts $\overline{\partial}_j$ first and then ∂_t, the first integral becomes

$$-4\sigma\int_0^T\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{A}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\mathring{g}}}g^{ij}\overline{\partial}_j\partial_t^3 v\cdot\tilde{n}\Big)-4\sigma\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{A}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\mathring{g}}}g^{ij}\overline{\partial}_j\partial_t^2 v\cdot\tilde{n}\Big)+\mathcal{R}.$$

Since $\|\partial_t^3 v\|_{3.5}$ is part of $E_\kappa^{(1)}(t)$, the trace lemma implies that the first integral is bounded straightforwardly by $\int_0^T\mathcal{P}$. Moreover, for the second integral, we have

$$\begin{aligned}&4\sigma\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{A}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\mathring{g}}}g^{ij}\overline{\partial}_j\partial_t^2 v\cdot\tilde{n}\Big)\\ \lesssim&\varepsilon|\partial_t^3 v|_1^2+P(\|\partial\eta\|_{L^\infty},\|\partial v\|_{L^\infty})|\partial_t^2 v|_1^2\le\varepsilon\|\partial_t^3 v\|_{1.5}^2+\mathcal{P}_0+\int_0^T\mathcal{P},\end{aligned}\tag{4.18}$$

where we used the trace lemma in the last inequality. In addition,

$$I_{142}\overset{L}{=}-4\int_0^T\int_\Gamma(\sqrt{\kappa}\partial_t^4 v_\alpha)\partial_t\tilde{A}^{3\alpha}\Big(\frac{1}{\sqrt{\mathring{g}}}\overline{\Delta}(\sqrt{\kappa}\partial_t^3 v\cdot\tilde{n})\Big).$$

Integrating $\overline{\partial}$ by parts,then

$$\begin{aligned}I_{142}\overset{L}{=}&4\int_0^T\int_\Gamma(\sqrt{\kappa}\overline{\partial}\partial_t^4 v_\alpha)\partial_t\tilde{A}^{3\alpha}\frac{1}{\sqrt{\mathring{g}}}(\sqrt{\kappa}\overline{\partial}\partial_t^3 v\cdot\tilde{n})\\ \lesssim&4\int_0^T\varepsilon|\sqrt{\kappa}\overline{\partial}\partial_t^4 v|_0^2+P(\|\partial\eta\|_{L^\infty},\|\partial v\|_{L^\infty})|\sqrt{\kappa}\overline{\partial}\partial_t^3 v|_0^2\,dt\\ \lesssim&\varepsilon\int_0^T\|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2+\mathcal{P}_0+\int_0^T\mathcal{P}.\end{aligned}$$

Now, we start to analyze the boundary integral I_0 in (4.2). This is essentially identical to the case of the incompressible Euler equations, which has been treated in [13], Sect.12. Indeed, as what appears in the previous paper [40] concerning the a priori estimate, we found that the magnetic field plays no role in the estimate of I_0. But we shall provide the control of the top order terms for the sake of the completeness of our proof.

By plugging the boundary condition

$$\tilde{A}^{3\alpha}q=-\sigma\sqrt{g}(\Delta_g\eta\cdot\tilde{n})\tilde{n}^\alpha+\kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha$$

in I_0 we obtain

$$\frac{1}{\sigma}I_0=\int_0^T\int_\Gamma\partial_t^4 v_\alpha\partial_t^4(\sqrt{g}\Delta_g\eta\cdot\tilde{n}\tilde{n}^\alpha)\,dS\,dt-\frac{\kappa}{\sigma}\int_0^T\int_\Gamma\partial_t^4 v_\alpha\partial_t^4[(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha]\,dS\,dt,\tag{4.19}$$

where, after integrating one tangential derivative by parts, the second term becomes

$$-\frac{\kappa}{\sigma}\sum_{\ell=0,1}\Big(\int_0^T\int_\Gamma\overline{\partial}^\ell\partial_t^4 v_\alpha\partial_t^4[\overline{\partial}^\ell(v\cdot\tilde{n})\tilde{n}^\alpha]\,dS\,dt+\int_0^T\int_\Gamma\partial_t^4 v_\alpha\partial_t^4[\overline{\partial}^\ell(v\cdot\tilde{n})\overline{\partial}^\ell\tilde{n}^\alpha]\,dS\,dt\Big).\tag{4.20}$$

The first term on the RHS contributes to the positive energy term (after moving to the LHS)

$$\frac{\kappa}{\sigma}\int_0^T \int_\Gamma \left|\partial_t^4 v\cdot\tilde{n}\right|_1^2\, dS\, dt$$

together with errors terms. The most difficult error term is

$$\kappa\int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\partial_t^4\overline{\partial}\tilde{n})\, dS\, dt, \tag{4.21}$$

where the other errors are either with the same type of integrand or are effectively of lower order by one derivative with the case above. Since $\overline{\partial}\tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}^2\tilde{\eta}\cdot\tilde{n}$, we have

$$\begin{aligned}
&\frac{\kappa}{\sigma}\int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\partial_t^4\overline{\partial}\tilde{n})\, dS\, dt\\
\overset{L}{=}&\frac{\kappa}{\sigma}\int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\overline{\partial}^2\partial_t^3\tilde{v}\cdot\tilde{n})\, dS\, dt\\
\leq&\int_0^T P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})|\sqrt{\kappa}\overline{\partial}\partial_t^4 v|_0|\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v\cdot\tilde{n}|_0\\
\lesssim&\int_0^T |\sqrt{\kappa}\overline{\partial}\partial_t^4 v|_0^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)}) + \left(\int_0^T |\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v\cdot\tilde{n}|_0^2\right)^2\\
\lesssim&\int_0^T \|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2 + \left(\int_0^T \|\sqrt{\kappa}\partial_t^3 v\|_{2.5}^2\right)^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})\\
\leq& E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)}).
\end{aligned}$$

Here, the last term can be controlled appropriately because

$$\begin{aligned}
|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)} &\lesssim \|\eta\|_3 \leq \|\eta_0\|_3 + \int_0^T \|v\|_3,\\
|v|_{L^\infty(\Gamma)} &\lesssim \|v\|_2 \leq \|v_0\|_2 + \int_0^T \|v_t\|_2,
\end{aligned}$$

and so $\sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)}) \leq \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}$. In addition, the second term on the RHS of (4.20) can be treated by the same argument.

Next we analyze the first term on the RHS of (4.19). Since $\hat{n}\cdot\hat{n} = 1$, invoking (2.5) in Lemma 2.1 and we obtain

$$\Delta_g\eta\cdot\hat{n}\hat{n}^\alpha = -\mathcal{H}\circ\eta\hat{n}^\alpha = \Delta_g\eta^\alpha, \tag{4.22}$$

and so we are able to rewrite

$$\begin{aligned}
\sqrt{g}\Delta_g\eta\cdot\tilde{n}\tilde{n}^\alpha &= \sqrt{g}\Delta_g\eta\cdot\hat{n}\hat{n}^\alpha + \sqrt{g}\Delta_g\eta\cdot\tilde{n}(\tilde{n}^\alpha-\hat{n}^\alpha) + \sqrt{g}\Delta_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha\\
&= \sqrt{g}\Delta_g\eta^\alpha + \sqrt{g}\Delta_g\eta\cdot\tilde{n}(\tilde{n}^\alpha-\hat{n}^\alpha) + \sqrt{g}\Delta_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha.
\end{aligned} \tag{4.23}$$

In light of this, the first term on the RHS of (4.19) becomes

$$\begin{aligned}
&\int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^4(\sqrt{g}\Delta_g\eta^\alpha)\, dS\, dt + \int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^4[\sqrt{g}\Delta_g\eta\cdot\tilde{n}(\tilde{n}^\alpha-\hat{n}^\alpha)]\, dS\, dt\\
&+\int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^4[\sqrt{g}\Delta_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha]\, dS\, dt.
\end{aligned} \tag{4.24}$$

We shall study the main term $I_{00} = \int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^4(\sqrt{g}\Delta_g\eta^\alpha)\, dS\, dt$. The error terms involving $\tilde{n}-\hat{n}$ are treated using (2.23) and they are identical to the Euler case. We refer [13] for the details. Invoking (2.6)-(2.7), we have

$$\begin{aligned}
I_{00} =& \int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^3\overline{\partial}_i\left(\sqrt{g}g^{ij}\Pi_\lambda^\alpha\overline{\partial}_j v^\lambda\right) dS\, dt\\
&+\int_0^T \int_\Gamma \partial_t^4 v_\alpha\partial_t^3\overline{\partial}_i\left(\sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\overline{\partial}_l v^\lambda\right).
\end{aligned} \tag{4.25}$$

Integrating $\overline{\partial}_i$ by parts and expanding the parenthesis, we get

$$
\begin{aligned}
(4.25) = &-\int_0^T\int_\Gamma \sqrt{g}g^{ij}\Pi^\alpha_\lambda\partial_t^3\overline{\partial}_jv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\\
&-\int_0^T\int_\Gamma \sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\partial_t^3\overline{\partial}_lv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\\
&-3\int_0^T\int_\Gamma \partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\lambda)\partial_t^2\overline{\partial}_jv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\,dS\,dt\\
&-3\int_0^T\int_\Gamma \partial_t\left(\sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\right)\partial_t^2\overline{\partial}_lv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\\
&-3\int_0^T\int_\Gamma \partial_t^2(\sqrt{g}g^{ij}\Pi^\alpha_\lambda)\partial_t\overline{\partial}_jv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\,dS\,dt\\
&-3\int_0^T\int_\Gamma \partial_t^2\left(\sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\right)\partial_t\overline{\partial}_lv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\\
&-\int_0^T\int_\Gamma \partial_t^3(\sqrt{g}g^{ij}\Pi^\alpha_\lambda)\overline{\partial}_jv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\,dS\,dt\\
&-\int_0^T\int_\Gamma \partial_t^3\left(\sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\right)\overline{\partial}_lv^\lambda\partial_t^4\overline{\partial}_iv_\alpha\\
=:&I_{01}+\cdots+I_{08}.
\end{aligned}
\tag{4.26}
$$

The main terms are I_{01} and I_{02} which produces the term $|\overline{\partial}(\Pi\partial_t^3v)|_0^2$ as a part of energy functional, and the others can be controlled by estimating $I_{03}+I_{04}, I_{05}+I_{06}, I_{07}+I_{08}$ and integrating ∂_t by parts. In I_{01}, we integrate ∂_t by parts and use (2.4) in Lemma 2.1 to get

$$
\begin{aligned}
I_{01}=&-\frac{1}{2}\int_\Gamma\sqrt{g}g^{ij}\Pi^\alpha_\lambda\partial_t^3\overline{\partial}_jv^\lambda\partial_t^3\overline{\partial}_iv_\alpha\Big|_0^T+\frac{1}{2}\int_0^T\int_\Gamma\partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\lambda)\partial_t^3\overline{\partial}_jv^\lambda\partial_t^3\overline{\partial}_iv_\alpha\,dS\,dt\\
=&\frac{1}{2}\int_\Gamma\sqrt{g}g^{ij}\overline{\partial}_i(\Pi^\alpha_\mu\partial_t^3v_\alpha)\overline{\partial}_j(\Pi^\mu_\lambda\partial_t^3v^\lambda)+\int_\Gamma\sqrt{g}g^{ij}\overline{\partial}\Pi^\alpha_\mu\partial_t^3v_\alpha\overline{\partial}_j(\Pi^\mu_\lambda\partial_t^3v^\lambda)\\
&-\frac{1}{2}\int_\Gamma\overline{\partial}_i\Pi^\alpha_\mu\overline{\partial}_j\Pi^\mu_\lambda\partial_t^3v_\alpha\partial_t^3v^\lambda+\frac{1}{2}\int_0^T\int_\Gamma\partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\lambda)\partial_t^3\overline{\partial}_jv^\lambda\partial_t^3\overline{\partial}_iv_\alpha\,dS\,dt+I_{01}|_{t=0}\\
=:&I_{011}+I_{012}+I_{013}+I_{014}+I_{01}|_{t=0}.
\end{aligned}
\tag{4.27}
$$

The term I_{011} produces the energy term

$$
\begin{aligned}
I_{011}=&-\frac{1}{2}\int_\Gamma\left|\overline{\partial}(\Pi\partial_t^3v)\right|^2\,dS-\frac{1}{2}\int_\Gamma(\sqrt{g}g^{ij}-\delta^{ij})\overline{\partial}_i(\Pi^\alpha_\mu\partial_t^3v_\alpha)\overline{\partial}_j(\Pi^\mu_\lambda\partial_t^3v^\lambda)\,dS\\
\lesssim&-\frac{1}{2}\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2+\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2\left|\sqrt{g}g^{ij}-\delta^{ij}\right|_{1.5}\\
\lesssim&-\frac{1}{2}\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2+\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2\int_0^TP(\|\partial_t\overline{\partial}\eta\|_2,\|\overline{\partial}\eta\|_2)\,dt.
\end{aligned}
\tag{4.28}
$$

The terms $I_{012}, I_{013}, I_{014}$ can all be directly controlled. Because $\overline{\partial}^2\eta|_{t=0}=0$, then

$$
\begin{aligned}
I_{012}\lesssim&\left|\sqrt{g}g^{-1}\right|_{L^\infty}|\overline{\partial}\Pi|_{L^\infty}|\partial_t^3v|_0|\overline{\partial}(\Pi\partial_t^3v)|_0\\
\lesssim&P(|\overline{\partial}\eta|_{L^\infty},|\overline{\partial}^2\eta|_{L^\infty})\|\partial_t^3v\|_{0.5}|\overline{\partial}(\Pi\partial_t^3v)|_0\\
\lesssim&\varepsilon\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2+P(\|\eta\|_4)\|\partial_t^3v\|_0\|\partial_t^3v\|_1\\
\lesssim&\varepsilon\left(\left|\overline{\partial}(\Pi\partial_t^3v)\right|_0^2+\|\partial_t^3v\|_{1.5}^2\right)+\mathcal{P}_0+\int_0^TP(\|\eta\|_4,\|v\|_4,\|\partial_t^4v\|_0)\,dt,
\end{aligned}
\tag{4.29}
$$

and

$$
\begin{aligned}
I_{013} \lesssim & |\overline{\partial}\Pi|_{L^4}^2 |\partial_t^3 v|_{L^4}^2 \lesssim P(\|\overline{\partial}\eta\|_2)\|\overline{\partial}^2\eta\|_2\|\partial_t^3 v\|_0\|\partial_t^3 v\|_1 \\
\lesssim & \varepsilon\|\partial_t^3 v\|_{1.5}^2 + P(\|\overline{\partial}\eta\|_2)\|\partial_t^3 v\|_0 \int_0^T \|\overline{\partial}^2 v\|_2\, dt,
\end{aligned}
\tag{4.30}
$$

and

$$
I_{014} \lesssim \int_0^T \left|\partial_t^3\overline{\partial}v\right|_0^2 \left|\partial_t(\sqrt{g}g^{ij}\Pi)\right|_{L^\infty} \lesssim \int_0^T P(\|\partial_t^3 v\|_{1.5}, \|v\|_3, \|\eta\|_3)\, dt. \tag{4.31}
$$

Combining (4.27) with (4.28)-(4.31), we get the estimates of I_{01} as follows

$$
I_{01} \lesssim \varepsilon\left(\left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \|\partial_t^3 v\|_{1.5}^2\right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}\, dt. \tag{4.32}
$$

Next we control $I_{02} := -\int_0^T \int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\partial_t^3\overline{\partial}_l v^\lambda\partial_t^4\overline{\partial}_i v_\alpha$. We expand the summation on l, i and find that:

- When $l = i$, this integral is zero thanks to the symmetry.
- When $l = 1, i = 2$, the integrand becomes $\sqrt{g}^{-1}(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda)\partial_t^3\overline{\partial}_1 v^\lambda\partial_t^4\overline{\partial}_2 v^\alpha$.
- When $l = 2, i = 1$, the integrand becomes $-\sqrt{g}^{-1}(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda)\partial_t^3\overline{\partial}_2 v^\lambda\partial_t^4\overline{\partial}_1 v^\alpha$.

Here, we use g^{-1} to denote $\det[g^{-1}] = g^{11}g^{22} - g^{12}g^{21}$. Therefore, we have

$$
\begin{aligned}
I_{02} =& -\int_0^T \int_\Gamma \frac{1}{\sqrt{g}}\left(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda\right)\left(\partial_t^3\overline{\partial}_1 v^\lambda\partial_t^4\overline{\partial}_2 v^\alpha + \partial_t^3\overline{\partial}_2 v^\lambda\partial_t^4\overline{\partial}_1 v^\alpha\right)\, dS\ dt \\
=& \int_0^T \int_\Gamma \frac{1}{\sqrt{g}}\frac{d}{dt}\left(\det\underbrace{\begin{bmatrix}\overline{\partial}_1\eta_\mu\partial_t^3\overline{\partial}_1 v^\mu & \overline{\partial}_1\eta_\mu\partial_t^3\overline{\partial}_2 v^\mu \\ \overline{\partial}_2\eta_\mu\partial_t^3\overline{\partial}_1 v^\mu & \overline{\partial}_2\eta_\mu\partial_t^3\overline{\partial}_2 v^\mu\end{bmatrix}}_{=:\mathbf{A}}\right) + \text{ lower order terms} \\
\overset{\partial_t}{=}& \int_\Gamma \frac{1}{\sqrt{g}}\det\mathbf{A}\Big|_0^T - \int_0^T \int_\Gamma \partial_t\left(\frac{1}{\sqrt{g}}\right)\det\mathbf{A}
\end{aligned}
\tag{4.33}
$$

The first term in the last line of (4.33) can be expanded into two terms

$$
\int_\Gamma \frac{1}{\sqrt{g}}\det\mathbf{A} = \int_\Gamma \frac{1}{\sqrt{g}}\left(\overline{\partial}_1\eta_\mu\overline{\partial}_2\eta_\lambda\overline{\partial}_1\partial_t^3 v^\mu\overline{\partial}_2\partial_t^3 v^\lambda - \overline{\partial}_1\eta_\mu\overline{\partial}_2\eta_\lambda\overline{\partial}_2\partial_t^3 v^\mu\overline{\partial}_1\partial_t^3 v^\lambda\right). \tag{4.34}
$$

It can be seen that the top order terms cancel with each other if one integrates $\overline{\partial}_1$ by parts in the first term and $\overline{\partial}_2$ by parts in the second. The remaining terms are all of the form $-\int_\Gamma Q_{\mu\lambda}(\overline{\partial}\eta, \overline{\partial}^2\eta)\partial_t^3 v^\mu\overline{\partial}\partial_t^3 v^\lambda$, which can be controlled as

$$
\begin{aligned}
& -\int_\Gamma Q_{\mu\lambda}(\overline{\partial}\eta, \overline{\partial}^2\eta)\partial_t^3 v^\mu\overline{\partial}\partial_t^3 v^\lambda \\
\lesssim & P(|\overline{\partial}^2\eta|_{L^\infty}, |\overline{\partial}\eta|_{L^\infty})|\partial_t^3 v|_0|\overline{\partial}\partial_t^3 v|_0 \\
\lesssim & \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \frac{1}{4\varepsilon}\|\partial_t^3 v\|_{0.5}\int_0^T P(\|\overline{\partial}^2 v\|_2)\, dt.
\end{aligned}
\tag{4.35}
$$

The second term of (4.33) can be directly controlled, i.e.,

$$
\int_0^T \int_\Gamma \partial_t\left(\frac{1}{\sqrt{g}}\right)\det\mathbf{A} \lesssim |\partial_t\overline{\partial}\eta|_{L^\infty}|\overline{\partial}\eta|_{L^\infty}^2|\overline{\partial}\partial_t^3 v|_0^2\, dt \lesssim \int_0^T \mathcal{P}. \tag{4.36}
$$

Therefore, we get the estimates of I_{02}:

$$
I_{02} \lesssim \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}\, dt, \tag{4.37}
$$

Next we control the remaining terms in I_0, i.e., $I_{03},\cdots,I_{08}$. The strategy here is to study $I_{03}+I_{04}, I_{05}+I_{06}, I_{07}+I_{08}$, where

$$
\begin{aligned}
I_{03}+I_{04} =& -3\int_0^T\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^4\overline{\partial}v\,dS\,dt \\
\overset{\partial_t}{=}& 3\int_0^T\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v + 3\int_0^T\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^3\overline{\partial}v\partial_t^3\overline{\partial}v + 3\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v\Big|_0^T \\
=& 3\int_0^T\int_\Gamma\left(\underbrace{Q(\overline{\partial}\eta)\overline{\partial}v}_{\partial_t(Q(\overline{\partial}\eta))}\overline{\partial}v + Q(\overline{\partial}\eta)\overline{\partial}\partial_t v\right)\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v \\
&+3\int_0^T\int_\Gamma Q(\overline{\partial}\eta)\overline{\partial}v\partial_t^3\overline{\partial}v\partial_t^3\overline{\partial}v + \int_\Gamma Q(\overline{\partial}\eta)\overline{\partial}v\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v\,dS\Big|_0^T \\
\lesssim& \mathcal{P}_0 + \int_0^T \mathcal{P} + |Q(\overline{\partial}\eta)\overline{\partial}v|_{L^\infty}|\partial_t^2 v|_1|\partial_t^3 v|_1 \\
\lesssim& \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}.
\end{aligned} \tag{4.38}
$$

Similarly, by plugging $\partial_t^3(Q(\overline{\partial}\eta)) = Q(\overline{\partial}\eta)(\overline{\partial}\partial_t v\overline{\partial}v\overline{\partial}v + \overline{\partial}\partial_t v\overline{\partial}v + \overline{\partial}\partial_t^2 v)$ into $I_{05}+I_{06}$, we get

$$
\begin{aligned}
I_{05}+I_{06} =& \int_0^T\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t\overline{\partial}v\partial_t^4\overline{\partial}v\,dS\,dt \\
\overset{\partial_t}{=}& -\int_0^T\int_\Gamma \partial_t^3(Q(\overline{\partial}\eta))\partial_t\overline{\partial}v\partial_t^3\overline{\partial}v\,dS\,dt - \int_0^T\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v + \int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t\overline{\partial}v\partial_t^3\overline{\partial}v\Big|_0^T \\
\lesssim& \mathcal{P}_0 + \int_0^T \mathcal{P} + |\overline{\partial}v|_{L^\infty}^2|\partial_t^3\overline{\partial}v|_0|\overline{\partial}\partial_t v|_0 \\
\lesssim& \mathcal{P}_0 + \int_0^T \mathcal{P} + \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \|\partial_t v\|_{1.5}^4 + \|v\|_3^8 \\
\lesssim& \mathcal{P}_0 + \int_0^T \mathcal{P} + \varepsilon\|\partial_t^3 v\|_{1.5}^2.
\end{aligned} \tag{4.39}
$$

Following the same way as above, we can control $I_{07}+I_{08}$ by $\mathcal{P}_0+\int_0^T\mathcal{P}+\varepsilon\|\partial_t^3 v\|_{1.5}^2$ so we omit the details. Combining this with (**??**)-(4.26), (4.32), (4.37)-(4.39), we get the estimates of I_0 by

$$
I_0 + \left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2 \lesssim \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.40}
$$

Now the only term left to control in (4.2) is L. Expanding $[\partial_t^4, \tilde{A}^{\mu\alpha}]$, we have

$$
\begin{aligned}
L =& \int_0^T\int_\Omega \partial_t^4\tilde{A}^{\mu\alpha}\partial_\mu v_\alpha\partial_t^4 q\,dy\,dt + 4\int_0^T\int_\Omega \partial_t^3\tilde{A}^{\mu\alpha}\partial_t\partial_\mu v_\alpha\partial_t^4 q\,dy\,dt \\
&+6\int_0^T\int_\Omega \partial_t^2\tilde{A}^{\mu\alpha}\partial_t^2\partial_\mu v_\alpha\partial_t^4 q\,dy\,dt + 4\int_0^T\int_\Omega \partial_t\tilde{A}^{\mu\alpha}\partial_t^3\partial_\mu v_\alpha\partial_t^4 q\,dy\,dt \\
=:& L_{21}+L_{22}+L_{23}+L_{24}.
\end{aligned} \tag{4.41}
$$

Despite having the right amount of derivatives, there is no direct control of $\|\partial_t^4 q\|_0$ and so we have to make some extra efforts to control $L_{21},\cdots,L_{24}$.

The hardest term to treat here is L_{21}. By plugging the relation $\partial_t^4\tilde{A}^{\mu\alpha} = -\tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\tilde{A}^{\beta\alpha}+$ lower order terms to L_{21}, we get

$$
L_{21} \overset{L}{=} \int_0^T\int_\Omega \tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\tilde{A}^{\beta\alpha}\partial_\mu v_\alpha\partial_t^4 q. \tag{4.42}
$$

Since

$$\tilde{A}^{\beta\alpha}\partial_t^4 q = \partial_t^4(\tilde{A}^{\beta\alpha}q) - (\partial_t^4\tilde{A}^{\beta\alpha})q - 4(\partial_t^3\tilde{A}^{\beta\alpha})\partial_t q - 6(\partial_t^2\tilde{A}^{\beta\alpha})\partial_t^2 q - 4(\partial_t\tilde{A}^{\beta\alpha})\partial_t^3 q,$$

and thus one can write the RHS of (4.42) as

$$\begin{aligned}&\int_0^T\int_\Omega \tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\tilde{A}^{\beta\alpha}q) - 4\int_0^T\int_\Omega \tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^3\tilde{A}^{\beta\alpha}\partial_t q\\ &- 6\int_0^T\int_\Omega \tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^2\tilde{A}^{\beta\alpha}\partial_t^2 q - 4\int_0^T\int_\Omega \tilde{a}^{\mu\nu}\partial_\beta\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t\tilde{A}^{\beta\alpha}\partial_t^3 q\\ &=: L_{211} + L_{212} + L_{213} + L_{214}.\end{aligned}$$

It is not hard to see that $L_{212}, L_{213}, L_{214}$ can all be controlled directly by $\int_0^T \mathcal{P}$ thanks to (3.30). To treat L_{211}, we integrate ∂_β by parts and get

$$\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\tilde{A}^{3\alpha}q) - \int_0^T\int_\Omega \partial_t^3\tilde{v}_\nu\partial_\beta\Big(\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\partial_t^4(\tilde{A}^{\beta\alpha}q)\Big) = L_{2111} + L_{2112}.$$

Since $L_{2112} \stackrel{L}{=} -\int_0^T\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\partial_t^4\partial_\beta(\tilde{A}^{\beta\alpha}q)$, we integrate ∂_t by parts in the last term and get

$$-\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\partial_t^3\partial_\beta(\tilde{A}^{\beta\alpha}q)\Big|_0^T + \int_0^T\int_\Omega \partial_t\Big(\partial_t^3\tilde{v}_\nu\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\Big)\partial_t^3\partial_\beta(\tilde{A}^{\beta\alpha}q) = L_{21121} + L_{21122}. \tag{4.43}$$

Now, since $\partial_\beta\tilde{A}^{\beta\alpha} = 0$, we can write

$$\partial_t^3\partial_\beta(\tilde{A}^{\beta\alpha}q) = -\partial_t^4 v^\alpha + \partial_t^3(b_0\cdot\partial)^2\eta^\alpha. \tag{4.44}$$

In light of this, we have

$$L_{21122} \le \int_0^T \mathcal{P}. \tag{4.45}$$

Also,

$$\begin{aligned}L_{21121} =& -\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{a}^{\mu\nu}\partial_\mu v_\alpha(-\partial_t^4 v^\alpha + \partial_t^3(b_0\cdot\partial)^2\eta^\alpha)\Big|_0^T\\ \lesssim& \mathcal{P}_0 + \|\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\|_{L^\infty}\|\partial_t^3 v\|_0(\|\partial_t^4 v\|_0 + \|\partial_t^3(b_0\cdot\partial)\eta\|_1)\\ \lesssim& \mathcal{P}_0 + \varepsilon(\|\partial_t^3 q\|_1^2 + \|\partial_t^3(b_0\cdot\partial)\eta\|_1) + \|\partial_t^3 v\|_0^4 + \|\tilde{a}^{\mu\nu}\partial_\mu v_\alpha\|_2^8\\ \lesssim& \mathcal{P}_0 + \varepsilon(\|\partial_t^3 q\|_1^2 + \|\partial_t^3(b_0\cdot\partial)\eta\|_1) + \mathcal{P}\int_0^T \mathcal{P}.\end{aligned}$$

Moreover, by plugging the boundary condition (3.6) to L_{2111} we obtain

$$-\sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}\Delta_g\eta\cdot\tilde{n}\tilde{n}^\alpha) + \kappa\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4\Big((1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha\Big) = L_{21111} + L_{21112}.$$

Invoking (2.6), we have

$$\begin{aligned}L_{21111} =& -\sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\tilde{n}^\alpha)\\ &+ \sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}_i\overline{\partial}_j\eta_\mu\overline{\partial}_k\eta\cdot\tilde{n}\tilde{n}^\alpha).\end{aligned}$$

It suffices to control the first term only since the second term has a highest order contribution with the same type of integrand. Also,

$$\begin{aligned}-\sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\tilde{n}^\alpha) \stackrel{L}{=}& -\sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^3 v\cdot\tilde{n}\tilde{n}^\alpha\\ &- \sigma\int_0^T\int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}(\partial_t^4\tilde{n}^\alpha).\end{aligned}$$

Now, since

$$\partial_t^4 \tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}\partial_t^3 v \cdot \tilde{n} + \text{lower-order terms,} \tag{4.46}$$

and so we have, after using the Sobolev embedding and trace lemma, that

$$\sigma \int_0^T \int_\Gamma \left| \tilde{a}^{\mu\nu}\partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \sqrt{g} g^{ij} \overline{\partial}_i \overline{\partial}_j \eta \cdot \tilde{n}(\partial_t^4 \tilde{n}^\alpha) \right| \leq \int_0^T \mathcal{P}. \tag{4.47}$$

In addition, by integrating $\overline{\partial}_i$ by parts and then using the trace lemma, we have

$$\sigma \int_0^T \int_\Gamma \left| \tilde{a}^{\mu\nu}\partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \sqrt{g} g^{ij} \overline{\partial}_i \overline{\partial}_j \partial_t^3 v \cdot \tilde{n}\tilde{n}^\alpha \right| \leq \int_0^T \mathcal{P}. \tag{4.48}$$

Moreover, we still need to control L_{21112}. In light of (4.46), we only need to study the case when all four time derivatives land on $\overline{\Delta} v$, i.e.,

$$-\kappa \int_0^T \int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \overline{\Delta}(\partial_t^4 v \cdot \tilde{n})\tilde{n}^\alpha.$$

Integrating $\overline{\partial}$ by parts, this term has the contributes to

$$\kappa \int_0^T \int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3 \overline{\partial}\tilde{v}_\nu \partial_\mu v_\alpha \overline{\partial}\partial_t^4 v \cdot \tilde{n}\tilde{n}^\alpha,$$

up to terms with the same type integrand, whose analysis (and bound) is identical. To control the main term, one has

$$\begin{aligned}
&\kappa \int_0^T \int_\Gamma \tilde{a}^{\mu\nu}\partial_t^3 \overline{\partial}\tilde{v}_\nu \partial_\mu v_\alpha \overline{\partial}\partial_t^4 v \cdot \tilde{n}\tilde{n}^\alpha = \sqrt{\kappa} \int_0^T \int_\Gamma Q(\overline{\partial}\tilde{\eta}, \partial v)\overline{\partial}\partial_t^3 v \sqrt{\kappa}\overline{\partial}\partial_t^4 v \\
\leq& \sqrt{\kappa} \int_0^T Q(\|\overline{\partial}\tilde{\eta}\|_{L^\infty}, \|\partial v\|_{L^\infty})\|\partial_t^3 v\|_{1.5}\| \sqrt{\kappa}\partial_t^4 v\|_{1.5} \\
\leq& \frac{1}{2}\Big(\sqrt{\kappa} \int_0^T Q(\|\overline{\partial}\tilde{\eta}\|_{L^\infty}, \|\partial v\|_{L^\infty})\|\partial_t^3 v\|_{1.5}^2 + \int_0^T \| \sqrt{\kappa}\partial_t^4 v\|_{1.5}^2 \Big) \\
\leq& \sqrt{\kappa} E_\kappa^{(3)} + \int_0^T \mathcal{P}.
\end{aligned}$$

Finally, combining (4.1) with the computations above, we finally get the control of full time derivatives

$$\left\|\partial_t^4 v\right\|_0^2 + \left\|\partial_t^4 (b_0\cdot\partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.49}$$

4.2 Control of mixed space-time tangential derivatives

To finish the control of $E_\kappa(T)$, it remains to study the tangential energies generated by the $\overline{\partial}\partial_t^3$, $\overline{\partial}^2\partial_t^2$, $\overline{\partial}^3\partial_t$ and $\overline{\partial}^3(b_0\cdot\partial)$-differentiated κ-problem. Generally speaking, the energy estimate becomes much simpler when the tangential spatial derivative(s) $\overline{\partial}$ is taken into account. This is due to that we can in fact avoid the higher order terms in the interior, i.e., terms associated to I_{11} in (4.3). This can be done by having all top orders terms on the boundary, and those terms can be controlled thanks to the extra 0.5 interior regularity.

The $\overline{\partial}\partial_t^3$-tangential energy: Similar to (4.1), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega \left|\overline{\partial}\partial_t^3 v\right|_0^2 + \left|\overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta\right|^2 \, dy \\
=& \underbrace{-\int_0^T \int_\Omega \overline{\partial}\partial_t^3(\tilde{A}^{\mu\alpha}\partial_\mu q)\overline{\partial}\partial_t^3 v_\alpha \, dy\, dt}_{I^*} \\
&+ \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)^2\eta_\alpha \overline{\partial}\partial_t^3 v_\alpha \, dy\, dt + \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta\alpha\overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \, dy\, dt.
\end{aligned} \tag{4.50}$$

By integrating $(b_0 \cdot \partial)$ by parts in the second term, we can get the cancellation with the third term

$$\begin{aligned}
&\int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)^2\eta_\alpha \overline{\partial}\partial_t^3 v_\alpha \, dy\, dt + \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \, dy\, dt \\
=& -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta_\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \, dy\, dt + \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta_\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \, dy\, dt \\
&+ \int_0^T \int_\Omega \left[\overline{\partial}, (b_0\cdot\partial)\right] \partial_t^3 (b_0\cdot\partial)\eta^\alpha \cdot \overline{\partial}\partial_t^3 v_\alpha - \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta^\alpha \cdot \left[(b_0\cdot\partial), \overline{\partial}\right] \partial_t^3 v_\alpha \, dy\, dt \\
\lesssim& \int_0^T P(\|b_0\|_3, \|\partial_t^3 v\|_1, \|\partial_t^2 v\|_2)\, dt
\end{aligned} \tag{4.51}$$

The main term I^* is treated a bit differently compare to I in (4.2). Specifically, one commutes $\tilde{A}^{\mu\alpha}$ with $\overline{\partial}\partial_t^3$ first and then integrate by parts. This allows us to avoid the appearance of the higher order interior terms.

$$\begin{aligned}
I^* &= -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \tilde{A}^{\mu\alpha} \overline{\partial}\partial_t^3 \partial_\mu q \, dy\, dt \underbrace{- \int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \, \left[\overline{\partial}\partial_t^3, \tilde{A}^{\mu\alpha}\right] \partial_\mu q \, dy\, dt}_{L_1^*} \\
&\overset{\partial_\mu}{=} \int_0^T \int_\Omega \tilde{A}^{\mu\alpha} \overline{\partial}\partial_t^3 \partial_\mu v_\alpha \overline{\partial}\partial_t^3 q \, dy\, dt \underbrace{- \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \tilde{A}^{3\alpha} \overline{\partial}\partial_t^3 q \, dS\, dt}_{I_B^*} + L_1^* \\
&= \int_0^T \int_\Omega \underbrace{\overline{\partial}\partial_t^3 (\mathrm{div}_{\tilde{A}} v)}_{=0} \overline{\partial}\partial_t^3 q \, dy\, dt + \underbrace{\int_0^T \int_\Omega \left[\tilde{A}^{\mu\alpha}, \overline{\partial}\partial_t^3\right] \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 q \, dy\, dt}_{L_2^*} + I_B^* + L_1^*.
\end{aligned} \tag{4.52}$$

Here, L_1^* and L_2^* can be directly controlled. For simplicity we only list the computation of the highest order terms

$$\begin{aligned}
L_1^* &= -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \, \left[\overline{\partial}\partial_t^3, \tilde{A}^{\mu\alpha}\right] \partial_\mu q \, dy\, dt \\
&\overset{L}{=} -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \, \overline{\partial}\partial_t^3 \tilde{A}^{\mu\alpha} \partial_\mu q \, dy\, dt \lesssim \int_0^T \mathcal{P}\, dt.
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
L_2^* &= \int_0^T \int_\Omega \left[\tilde{A}^{\mu\alpha}, \overline{\partial}\partial_t^3\right] \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 q \, dy\, dt \\
&\overset{L}{=} \int_0^T \int_\Omega \overline{\partial}\partial_t^3 \tilde{A}^{\mu\alpha} \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 q \, dy\, dt \lesssim \int_0^T \mathcal{P}\, dt.
\end{aligned} \tag{4.54}$$

Next we analyze the boundary integral I_B^*.

$$\begin{aligned}
I_B^* = &-\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^3 (\tilde{A}^{3\alpha} q)\, dS\, dt \\
&+ \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^3 \tilde{A}^{3\alpha} q \, dS\, dt + \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t^3 \tilde{A}^{3\alpha} \overline{\partial} q \, dS\, dt \\
&+ 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^2 \tilde{A}^{3\alpha} \partial_t q \, dS\, dt + 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t^2 \tilde{A}^{3\alpha} \overline{\partial}\partial_t q \, dS\, dt \\
&+ 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t \tilde{A}^{3\alpha} \partial_t^2 q \, dS\, dt + 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t \tilde{A}^{3\alpha} \overline{\partial}\partial_t^2 q \, dS\, dt \\
&+ \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial} \tilde{A}^{3\alpha} \partial_t^3 q \, dS\, dt \\
=:& J_0 + J_1 + \cdots + J_7.
\end{aligned} \tag{4.55}$$

Since we have $H^{1.5}(\Omega)$ regularity for $\partial_t^3 v$ and $H^1(\Omega)$ regularity for $\partial_t^3 q$, the top order terms contributed by J_1 to J_7 can all be directly controlled by the trace lemma. In the end, we have

$$J_1 + \cdots + J_7 \lesssim \int_0^T \mathcal{P}. \tag{4.56}$$

By plugging the boundary condition

$$\tilde{A}^{3\alpha} q = -\sigma\sqrt{g}(\Delta_g \eta \cdot \tilde{n})\tilde{n}^\alpha + \kappa(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha$$

in J_0, we obtain

$$\frac{1}{\sigma}J_0 = \int_0^T\int_\Gamma \overline{\partial}\partial_t^3(\sqrt{g}\Delta_g\eta\cdot\tilde{n}\tilde{n}^\alpha)\overline{\partial}\partial_t^3 v_\alpha \, dS\, dt - \frac{\kappa}{\sigma}\int_0^T\int_\Gamma \overline{\partial}\partial_t^3[(1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha]\overline{\partial}\partial_t^3 v_\alpha\, dS\, dt \tag{4.57}$$

For the second term, after integrating one $\overline{\partial}$ by parts, it contributes to the positive energy term (after moving to the LHS)

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma |\partial_t^3 v\cdot\tilde{n}|_2^2\, dS\, dt, \tag{4.58}$$

and some error terms. Here, the most difficult error term reads

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\partial_t^3\overline{\partial}^2\tilde{n})\, dS\, dt \tag{4.59}$$

which can be treated as follows:

$$\begin{aligned}
&\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\partial_t^3\overline{\partial}^2\tilde{n})\, dS\, dt \overset{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\partial_t^2\tilde{v}\cdot\tilde{n})\, dS\, dt \\
\le & \int_0^T P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)}, |v|_{L^\infty(\Gamma)})|\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v|_0 |\sqrt{\kappa}\overline{\partial}^3\partial_t^2 v\cdot\tilde{n}|_0 \\
\lesssim & \int_0^T |\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v|_0^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)}, |v|_{L^\infty(\Gamma)}) + \left(\int_0^T |\sqrt{\kappa}\overline{\partial}^3\partial_t^2 v\cdot\tilde{n}|_0^2\right)^2 \\
\lesssim & \int_0^T \|\sqrt{\kappa}\partial_t^3 v\|_{2.5}^2 + \left(\int_0^T \|\sqrt{\kappa}\partial_t^2 v\|_{3.5}^2\right)^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)}, |v|_{L^\infty(\Gamma)}) \\
\le & E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}.
\end{aligned}$$

The first term in (4.57) is treated analogous to the first term in (4.19). The main term we need to study in this case reads

$$\begin{aligned}
&\int_0^T\int_\Gamma (\overline{\partial}\partial_t^3(\sqrt{g}\Delta_g\eta^\alpha))(\overline{\partial}\partial_t^3 v)\, dS\, dt \\
=& \int_0^T\int_\Gamma \overline{\partial}\partial_t^2\overline{\partial}_i\left(\sqrt{g}g^{ij}\Pi^\alpha_\lambda\overline{\partial}_j v^\lambda + \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta^\lambda\overline{\partial}_l v_\lambda\right)\overline{\partial}\partial_t^3 v^\lambda\, dS\, dt
\end{aligned}$$

Integrating $\overline{\partial}_i$ by parts, we get

$$\begin{aligned}
J_{00} \overset{\overline{\partial}_i}{=} & -\int_0^T\int_\Gamma \sqrt{g}g^{ij}\Pi^\alpha_\lambda\overline{\partial}\partial_t^2\overline{\partial}_j v^\lambda\overline{\partial}\partial_t^3\overline{\partial}_i v_\alpha\, dS\, dt \\
& -\int_0^T\int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta^\lambda\overline{\partial}\overline{\partial}_l\partial_t^2 v_\lambda\partial_t^3\overline{\partial}_i\overline{\partial} v_\alpha\, dS\, dt + \mathcal{R}_0 \\
=: & J_{01} + J_{02} + \mathcal{R}_0,
\end{aligned} \tag{4.60}$$

where $\mathcal{R}_0$ consists terms that can be treated in the same way as in $I_{03}, \cdots, I_{08}$ in (4.26).

In J_{01}, we can integrate ∂_t by parts and mimic the proof of (4.27) to get

$$J_{01} + \left|\overline{\partial}^2(\Pi\partial_t^2 v)\right|_0^2 \lesssim \varepsilon\left(\left|\overline{\partial}(\Pi\overline{\partial}\partial_t^2 v)\right|_0^2 + \|\partial_t^2 v\|_{2.5}^2\right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}\, dt \tag{4.61}$$

J_{02} can also be controlled similarly as I_{02}. We find that the integrand is zero if $l = i$. So it suffices to compute the case $(l, i) = (1, 2)$ and $(2, 1)$. Similarly we get

$$J_{02} = \int_\Gamma \frac{1}{\sqrt{g}} \det\begin{bmatrix} \overline{\partial}_1\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu & \overline{\partial}_1\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu \\ \overline{\partial}_2\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu & \overline{\partial}_2\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu \end{bmatrix} dS\Big|_0^T + \int_0^T \mathcal{P} + \mathcal{R}. \tag{4.62}$$

The main term can be computed as follows

$$\begin{aligned}
&\int_\Gamma \frac{1}{\sqrt{g}} \det\begin{bmatrix} \overline{\partial}_1\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu & \overline{\partial}_1\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu \\ \overline{\partial}_2\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu & \overline{\partial}_2\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu \end{bmatrix} dS \\
=&\int_\Gamma \frac{1}{\sqrt{g}}\left(\overline{\partial}_1\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu\overline{\partial}_2\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu - \overline{\partial}_1\eta_\mu\overline{\partial}_2\partial_t^2\overline{\partial}v^\mu\overline{\partial}_2\eta_\mu\overline{\partial}_1\partial_t^2\overline{\partial}v^\mu\right) \\
\overset{\overline{\partial}_1,\overline{\partial}_2}{=}&\int_\Gamma Q^i_{\mu\lambda}(\overline{\partial}\eta, \overline{\partial}^2\eta)\overline{\partial}\partial_t^2 v^\mu\overline{\partial}_i\overline{\partial}\partial_t^2 v^\lambda\, dS \\
\lesssim& P\left(|\overline{\partial}\eta|_{L^\infty}, |\overline{\partial}^2\eta|_{L^\infty}\right)|\overline{\partial}\partial_t^2 v|_0\overline{\partial}^2\partial_t^2 v|_0 \\
\lesssim&\varepsilon\|\partial_t^2 v\|_{2.5}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P},
\end{aligned} \tag{4.63}$$

and thus we get the control of J_{02}

$$J_{02} \lesssim \varepsilon\left(\|\partial_t^2 v\|_{2.5}^2\right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.64}$$

Combining (4.50)-(4.61) and (4.64), we get the $\overline{\partial}\partial_t^3$-tangential estimates as follows

$$\left\|\overline{\partial}\partial_t^3 v\right\|_0^2 + \left\|\overline{\partial}\partial_t^3(b_0\cdot\partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \varepsilon\|\partial_t^2 v\|_{2.5}^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.65}$$

The $\overline{\partial}^2\partial_t^2$, $\overline{\partial}^3\partial_t$ and $\overline{\partial}^3(b_0\cdot\partial)$-tangential energies: The control of the other tangential energies that involving at least one $\overline{\partial}$ is follows from the arguments above by replacing $\overline{\partial}\partial_t^3$ to the corresponding derivatives. Hence, we shall omit the details and only illustrate the major difference.

First, we mention that the derivatives $\overline{\partial}^3\partial_t$ and $\overline{\partial}^3(b_0\cdot\partial)$ behaves the same since both v and $(b_0\cdot\partial)\eta$ are of the same interior regularity.

Second, one needs to pay attention to the terms that analogous to the error term generated by (4.57) during the construction of the energy term. In particular, we need to study the top order error term analogous to (4.59). Setting $\mathfrak{D} = \partial_t, \overline{\partial}$ or $(b_0\cdot\partial)$, and so $\overline{\partial}^2\partial_t^2, \overline{\partial}^3\partial_t, \overline{\partial}^3(b_0\cdot\partial)$ can be denoted systematically by $\overline{\partial}^2\mathfrak{D}^2$. Now we consider

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^3\mathfrak{D}^2 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\mathfrak{D}^2\tilde{n})\, dS\, dt. \tag{4.66}$$

When $\mathfrak{D}^2 = \partial_t^2$ then (4.66) is treated similar to (4.59). This is due to that

$$\overline{\partial}^3\partial_t^2\tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}^4\partial_t\tilde{v}\cdot\tilde{n} + \text{lower-order terms},$$

and $\int_0^T|\partial_t^4 v|_{1.5}^2$ is included in $E_\kappa^{(3)}$. In the end, we obtain

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^3\partial_t^2 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\partial_t^2\tilde{n})\, dS\, dt \le E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}.$$

On the other hand, when $\mathfrak{D}^2 = \overline{\partial}\partial_t, \overline{\partial}(b_0\cdot\partial)$, then using the fact that $\mathfrak{D}\tilde{n} = Q(\overline{\partial}\tilde{\eta})\mathfrak{D}\overline{\partial}\tilde{\eta}\cdot\tilde{n}$, we have

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^4\partial_t v\cdot\tilde{n})(v\cdot\overline{\partial}^4\partial_t\tilde{n})\,dS\,dt \stackrel{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^4\partial_t v\cdot\tilde{n})(v\cdot\overline{\partial}^5 v\cdot\tilde{n})\,dS\,dt, \tag{4.67}$$

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^4(b_0\cdot\partial)v\cdot\tilde{n})(v\cdot\overline{\partial}^4(b_0\cdot\partial)\tilde{n})\,dS\,dt \stackrel{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}^4(b_0\cdot\partial)v\cdot\tilde{n})(v\cdot\overline{\partial}^5(b_0\cdot\partial)\tilde{\eta}\cdot\tilde{n})\,dS\,dt. \tag{4.68}$$

The terms on the RHS requires $\int_0^T|\sqrt{\kappa}v|_5^2$ and $\int_0^T|\sqrt{\kappa}(b_0\cdot\partial)\eta|_5^2$, respectively, to control. However, owing to (3.27) and (3.29), both of them can be controlled by $\mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T\mathcal{P}$. Hence,

$$\left\|\overline{\partial}^2\partial_t^2 v\right\|_0^2 + \left\|\overline{\partial}^2\partial_t^2(b_0\cdot\partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \varepsilon E_\kappa(T) + \mathcal{P}_0 + \mathcal{P}\int_0^T\mathcal{P}, \tag{4.69}$$

$$\left\|\overline{\partial}^3\partial_t v\right\|_0^2 + \left\|\overline{\partial}^3\partial_t(b_0\cdot\partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^3 v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \varepsilon E_\kappa(T) + \mathcal{M}_0 + \mathcal{P}\int_0^T\mathcal{P}, \tag{4.70}$$

$$\left\|\overline{\partial}^3(b_0\cdot\partial)v\right\|_0^2 + \left\|\overline{\partial}^3(b_0\cdot\partial)^2\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \varepsilon E_\kappa(T) + \mathcal{M}_0 + \mathcal{P}\int_0^T\mathcal{P}. \tag{4.71}$$

5 Estimates for the higher order weighted interior norms

It remains to control $E_\kappa^{(3)}(T)$ in order to complete the proof of Proposition 3.1.

5.1 Full time derivatives

We shall first study the first two terms, i.e.,

$$\int_0^T\left(\left\|\sqrt{\kappa}\partial_t^4 v\right\|_{1.5}^2 + \left\|\sqrt{\kappa}\partial_t^4(b_0\cdot\partial)\eta\right\|_{1.5}^2\right)dt = K_1 + K_2.$$

These terms appear to be the most difficult ones to control. In particular, they yield error terms that contribute to the top order and can only be controlled in $L^2([0,T])$. In other words, we cannot use the time integral to create terms that can be controlled by $\mathcal{P}\int_0^T\mathcal{P}$.

The goal is to show:

$$K_1 + K_2 \le \mathcal{P}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T\mathcal{P}. \tag{5.1}$$

The control of K_1, K_2 relies on the div-curl estimate and so the H^1-norms of $\partial_t^4 v$ and $\partial_t^4(b_0\cdot\partial)\eta$ have to be studied together owing to the strong coupling structure of the MHD equations. In particular,

$$K_1 \le \int_0^T\left(\left\|\sqrt{\kappa}\mathrm{div}\,\partial_t^4 v\right\|_{0.5} + \left\|\sqrt{\kappa}\mathrm{curl}\,\partial_t^4 v\right\|_{0.5} + \left|\sqrt{\kappa}\partial_t^4 v^3\right|_1\right)dt =: K_{11} + K_{12} + K_{13}, \tag{5.2}$$

$$K_2 \le \int_0^T\left(\left\|\sqrt{\kappa}\mathrm{div}\,\partial_t^4(b_0\cdot\partial)\eta\right\|_{0.5} + \left\|\sqrt{\kappa}\mathrm{curl}\,\partial_t^4(b_0\cdot\partial)\eta\right\|_{0.5} + \left|\sqrt{\kappa}\partial_t^4(b_0\cdot\partial)\eta^3\right|_1\right)dt =: K_{21} + K_{22} + K_{23}. \tag{5.3}$$

Bound for K_{13} and K_{23}: For K_{13}, there holds

$$K_{13} \le \underbrace{\int_0^T|\sqrt{\kappa}\partial_t^4 v\cdot\tilde{n}|_1}_{\le E_\kappa^{(2)}} + \int_0^T|\sqrt{\kappa}\partial_t^4 v\cdot(N-\tilde{n})|_1,$$

and for the error term, we have

$$\int_0^T|\sqrt{\kappa}\partial_t^4 v\cdot(N-\tilde{n})|_1 \lesssim \int_0^T\|\sqrt{\kappa}\partial_t^4 v\|_{1.5}\cdot|N-\tilde{n}|_{1+} \lesssim \varepsilon\mathcal{P},$$

where (3.19) is used in the last inequality. To control K_{23}, since $\partial_t\eta = v$ we have $\int_0^T \left|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)v^3\right|_1$ and so it suffices to control $\int_0^T \left|\sqrt{\kappa}\partial_t^3 v^3\right|_2$. This term can then be treated similar to K_{13}.

Bound for K_{11} and K_{21}: First we state the following application of the Kato-Ponce inequality which shall be used frequently. Let $f \in H^{0.5}(\Omega)$ and g be a smooth function. Then

$$\|fg\|_{0.5} \lesssim \|f\|_{0.5}\|g\|_{1.5+}. \tag{5.4}$$

For K_{11}, we have

$$K_{11} \le \int_0^T (\|\sqrt{\kappa}\text{div}_{\tilde{a}}\partial_t^4 v\|_{0.5}^2 + \|\sqrt{\kappa}\text{div}_{a-\tilde{a}}\partial_t^4 v\|_{0.5}^2). \tag{5.5}$$

Since $\|a-\tilde{a}\|_{1.5+}^2 \le \kappa P(\|\eta\|_{3.5})$ thanks to (3.23), the error term can be controlled as

$$\int_0^T \|\sqrt{\kappa}\text{div}_{a-\tilde{a}}\partial_t^4 v\|_{0.5}^2 \lesssim \int_0^T \|a-\tilde{a}\|_{1.5+}^2\|\sqrt{\kappa}\partial\partial_t^4 v\|_{0.5}^2, \tag{5.6}$$

which can be controlled by the RHS of (5.1) when κ is small. For the first term, since $\text{div}_{\tilde{a}} v = 0$ we have

$$\int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{a}}\partial_t^4 v\|_{0.5}^2 = \int_0^T \|\sqrt{\kappa}[\partial_t^4, \tilde{a}]\partial v\|_{0.5}^2 \stackrel{L}{=} \int_0^T \|\sqrt{\kappa}\partial_t\tilde{a}\partial\partial_t^3 v\|_{0.5}^2 + \|\sqrt{\kappa}\partial_t^4\tilde{a}\partial v\|_{0.5}^2. \tag{5.7}$$

It is not hard to see that that $\int_0^T \|\sqrt{\kappa}\partial_t\tilde{a}\partial\partial_t^3 v\|_{0.5}^2 \le \int_0^T \mathcal{P}$ as $\partial_t^3 v \in H^{1.5}(\Omega)$ a priori. In addition, since $\partial_t\tilde{a}^{\mu\alpha} = Q(\partial\tilde{\eta})$, we obtain

$$\int_0^T \|\sqrt{\kappa}\partial_t^4\tilde{a}\partial v\|_{0.5}^2 \stackrel{L}{=} \int_0^T \|\sqrt{\kappa}Q(\partial\tilde{\eta})\partial\partial_t^3 v\partial v\|_{0.5}^2 \le \int_0^T \mathcal{P}, \tag{5.8}$$

The control of K_{21} is a bit more involved. We cannot commute ∂_t^4 to (3.59) as this would yield $\text{div}_{\partial_t^5\tilde{a}}(b_0\cdot\partial)\eta$ on the RHS which is out of control. However, by writing $\text{div}\,\partial_t^4(b_0\cdot\partial)\eta = \text{div}\,\partial_t^3(b_0\cdot\partial)v$ and then we have

$$\int_0^T \|\sqrt{\kappa}\text{div}\,\partial_t^3(b_0\cdot\partial)v\|_{0.5} \le \int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{a}}\partial_t^3(b_0\cdot\partial)v\|_{0.5} + \int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{a}-a}\partial_t^3(b_0\cdot\partial)v\|_{0.5}. \tag{5.9}$$

The second term on the RHS is again easy to control similar to (5.6). For the first term, because

$$\begin{aligned}
\text{div}_{\tilde{a}}\partial_t^3(b_0\cdot\partial)v =& \partial_t^3\text{div}_{\tilde{a}}((b_0\cdot\partial)v) - [\partial_t^3, \text{div}_{\tilde{a}}](b_0\cdot\partial)v \\
=& \partial_t^3([\text{div}_{\tilde{a}}, (b_0\cdot\partial)]v) - [\partial_t^3, \text{div}_{\tilde{a}}](b_0\cdot\partial)v \\
=& \sum_{0\le i\le 3}(\partial_t^i\tilde{a}^{\mu\alpha})(\partial_\mu b_0^\nu)(\partial_t^{3-i}\partial_\nu v) - \sum_{1\le j\le 3}(\partial_t^j\tilde{a}^{\mu\alpha})\partial_\mu((b_0\cdot\partial)\partial_t^{3-j}v),
\end{aligned}$$

then it can be seen, after counting the derivatives that both

$$\sum_{0\le i\le 3}\int_0^T \|\sqrt{\kappa}(\partial_t^i\tilde{a}^{\mu\alpha})(\partial_\mu b_0^\nu)(\partial_t^{3-i}\partial_\nu v)\|_{0.5}^2, \quad \sum_{1\le j\le 3}\int_0^T \|\sqrt{\kappa}(\partial_t^j\tilde{a}^{\mu\alpha})\partial_\mu((b_0\cdot\partial)\partial_t^{3-j}v)\|_{0.5}^2$$

can be controlled by $\int_0^T \mathcal{P}$ owing to the fact that $\partial_t^k v \in H^{4.5-k}(\Omega)$, $k = 2, 3$.

Bound for K_{12} and K_{22}: We would like to state the following strategy that will come in handy when dealing with the leading order terms in K_{12} and K_{22}. Let X be the term such that $\int_0^T \|\sqrt{\kappa}X\|_{0.5}^2$ is part of $E_\kappa^{(3)}$ and Y be a lower order term such that $\|Y\|_{1.5+}^2$ is controlled by $E_\kappa^{(1)}$. Then

$$\begin{aligned}
\int_0^T\int_0^t \|\sqrt{\kappa}XY\|_{0.5}^2\,dt \le& T\int_0^T \|\sqrt{\kappa}XY\|_{0.5}^2 \lesssim T\sup_t\|Y\|_{1.5+}^2\int_0^T \|\sqrt{\kappa}X\|_{0.5}^2 \\
\le& \frac{\varepsilon}{2}\Big(\int_0^T \|\sqrt{\kappa}X\|_{0.5}^2\Big)^2 + \frac{T^2}{2\varepsilon}\sup_t\|Y\|_{1.5+}^4,
\end{aligned} \tag{5.10}$$

which is bounded by the RHS of (3.11) if T is sufficiently small.

K_{12} and K_{22} will be considered together via studying the evolution equation verified by curl $\partial_t^4 v$ and curl $\partial_t^4(b_0\cdot\partial)\eta$. But this evolution equation cannot be derived by taking ∂_t^4 to (3.63) as this would yield curl $_{\partial_t^5 \tilde{A}} v$ in the source which is out of control. Instead, we commute $\partial_t^4\text{curl}_{\tilde{a}}$ to the equation $\partial_t v + (b_0\cdot\partial)^2\eta = \nabla_{\tilde{a}} q$ and get

$$\partial_t^4\text{curl}_{\tilde{a}}\partial_t v + \partial_t^4\text{curl}_{\tilde{a}}((b_0\cdot\partial)^2\eta) = 0.$$

This yields the following evolution equation by commuting three time derivatives through $\text{curl}_{\tilde{a}}$ in the first term on the LHS:

$$\partial_t\text{curl}_{\tilde{a}}\partial_t^4 v + \text{curl}_{\tilde{a}}((b_0\cdot\partial)^2\partial_t^4\eta) = -\partial_t([\partial_t^3, \text{curl}_{\tilde{a}}]v_t) - [\partial_t^4, \text{curl}_{\tilde{a}}](b_0\cdot\partial)^2\eta := f, \tag{5.11}$$

and, after expansion, the source term f becomes:

$$f = \partial_t\Big(\sum_{1\le j\le 3}\epsilon_{\alpha\beta\gamma}(\partial_t^j\tilde{a}^{\mu\beta})\partial_\mu\partial_t^{4-j}v^\gamma\Big) + \sum_{1\le j\le 4}\epsilon_{\alpha\beta\gamma}(\partial_t^j\tilde{a}^{\mu\beta})\partial_\mu(b_0\cdot\partial)^2\partial_t^{4-j}\eta^\gamma. \tag{5.12}$$

By multiplying $\kappa\overline{\partial}(\text{curl}\,_{\tilde{a}}\partial_t^4 v)$ to the evolution equation (5.11) and then integrating in space, we have

$$\int_\Omega \kappa(\text{curl}_{\tilde{a}}\partial_t^4 v)\overline{\partial}(\text{curl}_{\tilde{a}}\partial_t^4 v) + \int_\Omega \kappa\Big(\text{curl}_{\tilde{a}}((b_0\cdot\partial)^2\partial_t^4\eta\Big)\overline{\partial}(\text{curl}_{\tilde{a}}\partial_t^4 v) = \int_\Omega \kappa f\overline{\partial}(\text{curl}_{\tilde{a}}\partial_t^4 v), \tag{5.13}$$

where the first term contributes to $\frac{1}{2}\frac{d}{dt}\|\text{curl}_{\tilde{a}}\partial_t^4 v\|_{0.5}^2$ after integrating $\overline{\partial}^{\frac{1}{2}}$ by parts.

Next, if we integrate $(b_0\cdot\partial)$ by parts in $\int_\Omega \kappa\Big(\text{curl}_{\tilde{a}}((b_0\cdot\partial)^2\partial_t^4\eta\Big)\overline{\partial}(\text{curl}_{\tilde{a}}\partial_t^4 v)$ and then integrate $\overline{\partial}^{\frac{1}{2}}$ by parts, we obtain $\frac{1}{2}\frac{d}{dt}\|\text{curl}_{\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}^2$ up to terms involving commutators (which will be recorded below). In particular, the following energy inequality is achieved:

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}^2 + \frac{1}{2}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}^2\\
\lesssim&\mathcal{P}_0 + \int_0^T \|\sqrt{\kappa}f\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}\,dt + \int_0^T \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)](b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}\,dt\\
&+\int_0^T \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)]\partial_t^4 v\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}(b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\,dt\\
&+\int_0^T \|\sqrt{\kappa}\text{curl}\,_{\partial_t\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}(b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\,dt.
\end{aligned} \tag{5.14}$$

Hence, by integrating in time one more time, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}^2 + \frac{1}{2}\int_0^T \|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}^2\\
\lesssim&\int_0^T \mathcal{P}_0 + \int_0^T\int_0^t \|\sqrt{\kappa}f\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}\,dt\\
&+\int_0^T\int_0^t \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)](b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}\partial_t^4 v\|_{0.5}\,dt\\
&+\int_0^T\int_0^t \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)]\partial_t^4 v\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}(b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\,dt\\
&+\int_0^T\int_0^t \|\sqrt{\kappa}\text{curl}\,_{\partial_t\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}\|\sqrt{\kappa}\text{curl}\,_{\tilde{a}}(b_0\cdot\partial)\partial_t^4\eta\|_{0.5}\,dt,
\end{aligned} \tag{5.15}$$

where we have dropped one dt for the sake of concise notations. This suggests that we should control

$$\int_0^T\int_0^t \|\sqrt{\kappa}f\|_{0.5}^2\,dt,\quad \int_0^T\int_0^t \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)](b_0\cdot\partial)\partial_t^4\eta\|_{0.5}^2\,dt,$$

$$\int_0^T\int_0^t \|\sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0\cdot\partial)]\partial_t^4 v\|_{0.5}^2\,dt,\quad \int_0^T\int_0^t \|\sqrt{\kappa}\text{curl}\,_{\partial_t\tilde{a}}\partial_t^4(b_0\cdot\partial)\eta\|_{0.5}^2\,dt.$$

For the second term, we have

$$\int_0^T \int_0^t \| \sqrt{\kappa}[\text{curl}\,_{\tilde{a}}, (b_0 \cdot \partial)](b_0 \cdot \partial)\partial_t^4 \eta\|_{0.5}^2 \, dt \lesssim \int_0^T \int_0^t \|\tilde{a}(\partial b_0)(\partial(b_0 \cdot \partial)\partial_t^4 \eta\|_{0.5}^2 \, dt,$$

which can be controlled by the RHS of (5.1) by adapting (5.10). The third and forth term are treated analogously. For $\int_0^T \int_0^t \| \sqrt{\kappa} f\|_{0.5}^2 \, dt$, invoking (5.12), we need to consider

$$i = \sum_{1 \le j \le 3} \int_0^T \int_0^t \| \sqrt{\kappa}\partial_t(\epsilon_{\alpha\beta\gamma}(\partial_t^j \tilde{a}^{\mu\beta})\partial_\mu \partial_t^{4-j} v^\gamma)\|_{0.5}^2 \, dt, \tag{5.16}$$

$$ii = \sum_{1 \le j \le 4} \int_0^T \int_0^t \| \sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t^j \tilde{a}^{\mu\beta})\partial_\mu (b_0 \cdot \partial)^2 \partial_t^{4-j} \eta^\gamma\|_{0.5}^2 \, dt. \tag{5.17}$$

Here, $i \stackrel{L}{=} \int_0^T \int_0^t \| \sqrt{\kappa}(\partial_t \tilde{a})(\partial\partial_t^4 v)\|_{0.5}^2$, which controlled appropriately by adapting (5.10). Moreover,

$$ii \stackrel{L}{=} \int_0^T \int_0^t \| \sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t \tilde{a}^{\mu\beta})\partial_\mu (b_0 \cdot \partial)^2 \partial_t^3 \eta^\gamma\|_{0.5}^2 \lesssim \int_0^T \int_0^t \| \sqrt{\kappa}(\partial_t \tilde{a})\partial\partial_t^3[(b_0 \cdot \partial)^2 \eta]\|_{0.5}^2 \, dt \le \int_0^T \mathcal{P}, \tag{5.18}$$

since $\int_0^T \| \sqrt{\kappa}\partial_t^3 (b_0 \cdot \partial)\eta\|_{2.5}^2$ is included in $E_\kappa^{(3)}$, and this concludes the control of $K_1 + K_2$.

Remark. There is an alternative way to control the last integral in (5.18). We may use the equation to replace $(b_0 \cdot \partial)^2 \eta$ by $\partial_t v + \nabla_{\tilde{a}} q$, and this allow us to control this integral without using $\int_0^T \| \sqrt{\kappa}\partial_t^3 (b_0 \cdot \partial)\eta\|_{2.5}^2$. In fact, one can show

$$\int_0^T \int_0^t \| \sqrt{\kappa}\partial_t^3 q\|_{2.5}^2 \, dt \le \mathcal{P}$$

by employing the elliptic estimate we used in Section 3.1 (similar to the control of (5.23)), and so

$$\int_0^T \int_0^t \| \sqrt{\kappa}(\partial_t \tilde{a})\partial\partial_t^3[(b_0 \cdot \partial)^2 \eta]\|_{0.5}^2 \, dt \le \int_0^T \int_0^t \| \sqrt{\kappa}(\partial_t \tilde{a})\partial\partial_t^4 v\|_{0.5}^2 + \| \sqrt{\kappa}(\partial_t \tilde{a})\partial\partial_t^3 \nabla_{\tilde{a}} q\|_{0.5}^2 \, dt \le \int_0^T \mathcal{P},$$

because $\int_0^T \| \sqrt{\kappa}\partial_t^4 v\|_{1.5}^2$ is part of $E_\kappa^{(3)}$.

5.2 Mixed space-time derivatives

The treatment for the remaining terms of $E_\kappa^{(3)}$ is parallel and so we shall only sketch the details. We shall consider

$$\int_0^T \left(\left\| \sqrt{\kappa}\partial_t^k v\right\|_{5.5-k}^2 + \left\| \sqrt{\kappa}\partial_t^k (b_0 \cdot \partial)\eta\right\|_{5.5-k}^2\right) dt, \quad k = 1, 2, 3.$$

First, the boundary terms contributed by the time derivative(s) of $(b_0 \cdot \partial)\eta$, i.e., terms analogous to K_{23}, reads

$$\int_0^T | \sqrt{\kappa}\partial_t^k (b_0 \cdot \partial)\eta|_{5-k}, \quad k = 1, 2, 3.$$

Generally speaking, for each fixed k, the control of the above term requires that of $\int_0^T | \sqrt{\kappa}\partial_t^{k-1} v|_{6-k}$, and this process stops when $k = 1$. In particular, for each fixed $k = 2, 3$, we write $\int_0^T | \sqrt{\kappa}\partial_t^k (b_0 \cdot \partial)\eta|_{5-k}^2$ as $\int_0^T |\partial_t^{k-1}(b_0 \cdot \partial)v|_{5-k}^2$, which can then be controlled together with $\int_0^T |\partial_t^i v|_{5-i}^2$ with $i = 1, 2$. On the other hand, when $k = 1$, the control of $\int_0^T | \sqrt{\kappa}\partial_t (b_0 \cdot \partial)\eta|_4$ requires that of

$$\int_0^T | \sqrt{\kappa}(b_0 \cdot \partial)v|_4^2 \lesssim P(\|b_0\|_{4.5}) \int_0^T | \sqrt{\kappa} v|_5^2,$$

where, in view of (3.27), we have $\int_0^T | \sqrt{\kappa} v|_5^2 \le \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}$.

Second, the control of the analogous terms of ii (defined in (5.17)) for $k = 1, 2, 3$ requires a similar analysis as above. For each fixed k, we need to investigate

$$ii' = \sum_{1\le j\le k} \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t^j \tilde{a}^{\mu\beta})\partial_\mu(b_0\cdot\partial)^2\partial_t^{k-j}\eta^\gamma\|_{4.5-k}^2\,dt. \tag{5.19}$$

Again, it suffices to consider the most difficult term contributed by setting $j = 1$, i.e.,

$$ii' = \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t \tilde{a}^{\mu\beta})\partial_\mu(b_0\cdot\partial)^2\partial_t^{k-1}\eta^\gamma\|_{4.5-k}^2\,dt \tag{5.20}$$

$$\lesssim \int_0^T \int_0^t P(\|v\|_{4.5}, \|b_0\|_{4.5}, \|\eta\|_{4.5})\|\sqrt{\kappa}\partial^3\partial_t^{k-1}\eta\|_{4.5-k}^2\,dt. \tag{5.21}$$

In (5.21), it can be seen that when $k = 2, 3$, $\int_0^T \int_0^t \|\sqrt{\kappa}\partial^3\partial_t^{k-1}\eta\|_{4.5-k}\,dt$ is bounded by $\int_0^T \int_0^t \|\sqrt{\kappa}\partial^3 v\|_{2.5}^2\,dt$ and $\int_0^T \int_0^t \|\sqrt{\kappa}\partial^3\partial_t v\|_{1.5}^2\,dt$, respectively. Moreover, when $k = 1$, we need to consider (5.20) instead. The strategy here is to replace $(b_0\cdot\partial)^2\eta$ by $\partial_t v + \nabla_{\tilde{a}} q$, and so

$$ii' = \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t \tilde{a}^{\mu\beta})\partial_\mu\partial_t v^\gamma\|_{3.5}\,dt + \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t \tilde{a}^{\mu\beta})\partial_\mu\nabla_{\tilde{a}}^\gamma q\|_{3.5}\,dt, \tag{5.22}$$

where the first term is bounded by the RHS of (5.1) owing to (5.10). For the second term, since $v \in H^{4.5}(\Omega)$, so it suffices to consider the case when all derivatives land on $\nabla_{\tilde{a}} q$, whose control requires that of

$$\int_0^T \int_0^t \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2\,dt \tag{5.23}$$

after adapting (5.10). Actually, we are able to prove a slightly stronger bound by removing one time integral, i.e., we want to bound $\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2$. By the div-curl estimate, one has

$$\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2 \lesssim \int_0^T \left(\|\sqrt{\kappa}\text{div}\,\nabla_{\tilde{a}} q\|_{3.5}^2 + \|\sqrt{\kappa}\text{curl}\,\nabla_{\tilde{a}} q\|_{3.5}^2 + |\sqrt{\kappa}N\cdot\nabla_{\tilde{a}} q|_3^2 + |\sqrt{\kappa}q|_0^2\right).$$

Here,

$$\int_0^T \|\sqrt{\kappa}\text{div}\,\nabla_{\tilde{a}} q\|_{3.5}^2 \lesssim \int_0^T \|\sqrt{\kappa}\Delta_{\tilde{a}} q\|_{3.5}^2 + \int_0^T \|\sqrt{\kappa}\text{div}_{\,\tilde{a}-\delta}\nabla_{\tilde{a}} q\|_{3.5}^2, \tag{5.24}$$

and by invoking (3.16), (3.30), (5.10), and since $v \in H^{4.5}(\Omega)$ a priori, we have

$$\int_0^T \|\sqrt{\kappa}\text{div}_{\,\tilde{a}-\delta}\nabla_{\tilde{a}} q\|_{3.5}^2 \lesssim \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2 + \int_0^T \mathcal{P}.$$

Similarly, because $\text{curl}_{\,\tilde{a}}\nabla_{\tilde{a}} q = 0$, we have

$$\int_0^T \|\sqrt{\kappa}\text{curl}\,\nabla_{\tilde{a}} q\|_{3.5}^2 \lesssim \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2 + \int_0^T \mathcal{P}.$$

Moreover, invoking (3.19), (5.10) and the trace lemma, then

$$\begin{aligned}\int_0^T |\sqrt{\kappa}N\cdot\nabla_{\tilde{a}} q|_3^2 &\lesssim \int_0^T |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{a}} q|_3^2 + \int_0^T |\sqrt{\kappa}(N-\tilde{n})\cdot\nabla_{\tilde{a}} q|_3^2\\ &\lesssim \int_0^T |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{a}} q|_3^2 + \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2 + \int_0^T \mathcal{P}.\end{aligned}$$

As a consequence, (5.24) becomes

$$\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{a}} q\|_{4.5}^2 \lesssim \int_0^T \left(\|\sqrt{\kappa}\Delta_{\tilde{a}} q\|_{3.5}^2 + |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{a}} q|_3^2 + |\sqrt{\kappa}q|_0^2\right). \tag{5.25}$$

To control the RHS, we recall that q verifies

$$- \Delta_{\tilde{a}} q = -\partial_t \tilde{a}^{\mu\alpha} \partial_\mu v_\alpha + \partial_\beta((b_0 \cdot \partial)\tilde{\eta}_\nu)\partial_\nu \tilde{a}^{\mu\nu} \tilde{a}^{\beta\alpha} \partial_\mu (b_0 \cdot \partial)\eta_\alpha \tag{5.26}$$

with the Dirichlet and Neumann boundary conditions

$$\sqrt{\tilde{g}} q = -\sigma \sqrt{g}(\Delta_g \eta \cdot \tilde{n}) + \kappa(1 - \overline{\Delta})(v \cdot \tilde{n}), \tag{5.27}$$

$$\tilde{n} \cdot \nabla_{\tilde{a}} q = -\partial_t v \cdot \tilde{n} + (b_0 \cdot \partial)^2 \eta \cdot \tilde{n}. \tag{5.28}$$

Now,

$$\int_0^T \| \sqrt{\kappa} \Delta_{\tilde{a}} q\|_{3.5}^2 \le \int_0^T \kappa \|\partial_t \tilde{a} \partial v\|_{3.5}^2 + \int_0^T \kappa \|\partial((b_0 \cdot \partial)\eta)(\partial_t a)(a\partial(b_0 \cdot \partial)\eta)\|_{3.5}^2, \tag{5.29}$$

and because $v, (b_0 \cdot \partial)\eta \in H^{4.5}(\Omega)$ a priori, the RHS is bounded by $\int_0^T \mathcal{P}$. Also, it is not hard to see, via the trace lemma and the Dirichlet boundary condition, that

$$\int_0^T | \sqrt{\kappa} q|_0^2 \le \int_0^T \mathcal{P}.$$

Next, we control $\int_0^T | \sqrt{\kappa} \tilde{n} \cdot \nabla_{\tilde{a}} q|_3^2$. In view of the Neumann boundary condition (5.28), it contributes to

$$\int_0^T \kappa |\partial_t v \cdot \tilde{n}|_3^2, \quad \int_0^T | \sqrt{\kappa}(b_0 \cdot \partial)^2 \eta \cdot \tilde{n}|_3^2.$$

For the first term, since $\partial_t v \in H^{3.5}(\Omega)$ and $\eta \in H^{4.5}(\Omega)$ a priori, as well as $\overline{\partial}\tilde{n} = Q(\overline{\partial}\eta)\overline{\partial}^2 \eta$, we have, after employing the trace theorem, that

$$\int_0^T \kappa |\partial_t v \cdot \tilde{n}|_3^2 \le \int_0^T \mathcal{P}.$$

Also, for the second term,

$$\int_0^T | \sqrt{\kappa}(b_0 \cdot \partial)^2 \eta \cdot \tilde{n}|_3^2 \stackrel{L}{=} \int_0^T | \sqrt{\kappa}(b_0 \cdot \partial)^2 \partial^3 \eta \cdot \tilde{n}|_0^2 \le \int_0^T P(\|b_0\|_{4.5}, \|\eta\|_{4.5}) | \sqrt{\kappa} \overline{\partial}^5 \eta|_0^2,$$

which can be controlled by $\mathcal{M}_0 + C(\varepsilon) E_\kappa(T) + \mathcal{P} \int_0^T \mathcal{P}$ owing to (3.27).

In summary, we have

$$E_\kappa^{(3)} \le \mathcal{M}_0 + C(\varepsilon) E_\kappa(T) + \mathcal{P} \int_0^T \mathcal{P}. \tag{5.30}$$

6 Closing the nonlinear energy estimate

In this section we conclude the proof of Proposition 3.1.

6.1 Regularity of initial data

Our first task is to remove the extra regularity assumptions on the initial data. These additional regularities are introduced in $\mathcal{M}_0$ (defined in Lemma 3.5). In addition to this, one has to control $\|q(0)\|_{4.5}, \|q_t(0)\|_{3.5}, \|q_{tt}(0)\|_{2.5}$ in terms of v_0 and b_0 by the elliptic estimate, and extra regularity on v_0 and b_0 shall appear due to the viscosity term.

Note that q_0 verifies the elliptic equation

$$\begin{cases} -\Delta q_0 = (\partial v_0)(\partial v_0) - (\partial b_0)(\partial b_0) & \text{in } \Omega \\ q_0 = \kappa(1 - \overline{\Delta}) v^3 & \text{on } \Gamma \\ \frac{\partial q_0}{\partial N} = 0 & \text{on } \Gamma_0 \end{cases} \tag{6.1}$$

by standard elliptic estimates, we get

$$\|q_0\|_{4.5} \lesssim \|\partial v_0\|_{2.5}^2 + \|\partial b_0\|_{2.5}^2 + \kappa\|v_0^3\|_{4.5} + \kappa|v_0^3|_6.$$

Moreover, note that the energy functional contains time derivatives of v and $(b_0\cdot\partial)\eta$, so we need to express their initial data in terms of v_0 and b_0 as well. We invoke $\partial_t v(0) - (b_0 \cdot \partial)b_0 = -\partial q_0$ to get

$$\|\partial_t v(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|b_0\|_{4.5} + \|q_0\|_{4.5},$$

and

$$\|\partial_t (b_0 \cdot \partial)\eta(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|v_0\|_{4.5}.$$

Similarly, we consider the ∂_t-differentiated elliptic equation of q to get

$$\|\partial_t q(0)\|_{3.5} \lesssim P(\|v_0\|_{4.5}, \|b_0\|_{4.5})(|v_0^3|_5 + \kappa|\partial_t v(0)|_5),$$

and further

$$\|\partial_t^2 q(0)\|_{2.5} + \|\partial_t^3 q(0)\|_1 \lesssim P(\|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5)(1 + \kappa|\overline{\Delta}\partial_t^2 v(0)|_2).$$

By Sobolev trace lemma, we need to bound $\kappa\|\partial_t^2 v(0)\|_{4.5}$ which requires the control of $\kappa(\|v_0\|_{6.5} + \|b_0\|_{5.5} + \|\partial_t q(0)\|_{5.5})$. We replace 3.5 by 5.5 in the estimates of $\partial_t q(0)$, and thus we need to control

$$\kappa^2|\partial_t v(0)|_7 \lesssim \kappa^2(\|b_0\|_{7.5}\|b_0\|_{8.5} + \|q_0\|_{8.5}).$$

Finally, replacing 4.5 by 8.5 in the estimates of q_0, we need to control

$$\kappa^2(\|v_0\|_{7.5}^2 + \|b_0\|_{7.5}^2) + \kappa^3(|v_0|_8 + |v_0|_{10}).$$

In view of the above analysis and the definition of $\mathcal{M}_0$, we need to control κ-weighted norms of $\|v_0\|_{8.5}, \|b_0\|_{8.5}$ and $|v_0|_{10}$. However, our given initial data is $v_0 \in H^{4.5}(\Omega) \cap H^5(\Gamma)$ and $b_0 \in H^{4.5}$ and so we have to remove the additional regularity assumptions on the initial data. This can be done by adapting a similar argument in Section 12 of Coutand-Shkoller [13]. We define Ω_κ to be the regularized version of Ω tangentially mollified by $\zeta_{\exp^{-\kappa}}$ and define E_{Ω_κ} to be the extension operator from Ω to Ω_κ. Next we set

$$\mathbf{v}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(v_0),\ \mathbf{b}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(b_0),\ \mathbf{q}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(q_0).$$

Therefore, integrating by parts repeatedly to transfer derivatives to the mollifier $\zeta_{\exp^{-\kappa}}$, we get

$$\|\kappa\mathbf{v}_0\|_{8.5} + \|\kappa\mathbf{b}_0\|_{8.5} + \|\kappa\mathbf{q}_0\|_{8.5} + |\kappa\mathbf{v}_0|_{10} \lesssim \|v_0\|_{4.5} + \|b_0\|_{4.5} + \|q_0\|_{4.5} \le C, \tag{6.2}$$

where C is the constant depends on $\|v_0\|_{4.5}, \|b_0\|_{4.5}$ which appears in (3.7).

6.2 Nonlinear a priori estimates

Now we summarize the a priori estimates of the nonlinear κ-approximation system (3.2).

1. (3.30) gives the elliptic estimates of q and its time derivatives.
2. (3.56)-(3.58) and (3.61), (3.62) give the divergence estimate and (3.65)-(3.66) give the curl estimate.
3. (3.67) and (3.74) control the boundary part of $v, (b_0 \cdot \partial)\eta$ and its time derivative.
4. (4.49), (4.65), (4.69)-(4.71) provide control of the mixed tangential derivatives of v and $(b_0 \cdot \partial)\eta$ and the Eulerian normal projections of v. Note that these estimate depends on $E_\kappa^{(3)}$ on the RHS.
5. Finally, (5.30) provides the estimate for $E_\kappa^{(3)}$.

Thus, by combining these estimates and then invoking (6.2), we obtain a Gronwall-type inequality:

$$E_\kappa(T) - E_\kappa(0) \lesssim C(\varepsilon)E_\kappa(T) + C(\|v_0\|_{4.5}, \|b_0\|_{4.5}) + P(E_\kappa(T))\int_0^T E_\kappa(t)\,dt. \tag{6.3}$$

We pick $\varepsilon > 0$ suitably small such that the ε-terms can be absorbed to LHS. Therefore, by the nonlinear Gronwall inequality in Chapter 2 of Tao [51], we know there exists some time $T > 0$ independent of κ, such that

$$\sup_{0\le t\le T} E_\kappa(t) \le C. \tag{6.4}$$

This concludes the proof for Proposition 3.1.

7 Existence and uniqueness for the linearized approximate system

Since we have obtained an uniform-in-κ a priori energy estimate for the approximate κ-problem (3.2), our next goal is to construct a solution for this system for each fixed $\kappa > 0$.

Assumption 7.1. *We shall assume that $\kappa > 0$ is fixed throughout the rest of this manuscript.*

Let $T > 0$. We define

$$\mathbf{X} = \{u \in L^\infty(0,T; H^{4.5}(\Omega)) : \sup_{[0,T]} \|u\|_{4.5} \leq 2\|v_0\|_{4.5} + 1\}, \tag{7.1}$$

which is a closed subset of the space $L^\infty(0,T; H^{4.5}(\Omega))$.

In order to solve the approximate κ-problem (3.2) for each fixed $\kappa > 0$, we study the following linearized problem whose fixed-point shall provide the desired solutions. Fix an arbitrary function $\mathring{\eta} = \mathring{\eta}(t,y)$ whose time derivative $\mathring{\eta}_t \in \mathbf{X}$, we denote by $\mathring{a}$, $\mathring{g}$, $\mathring{J}$ and $\mathring{A}$ the associated quantities in Lagrangian coordinates and $\mathring{\tilde{\eta}} := \Lambda_\kappa \mathring{\eta}$, $\mathring{\tilde{a}} := [\partial\mathring{\tilde{\eta}}]^{-1}$, $\mathring{\tilde{J}} := \det[\partial\mathring{\tilde{\eta}}]$, $\mathring{\tilde{A}} := \mathring{\tilde{J}}\mathring{\tilde{a}}$ and $\mathring{\tilde{n}}$ to be the associated smoothed quantities.

We aim to construct η and v that solve

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T]\times\Omega; \\ \partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\mathring{\tilde{A}}} q = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{\mathring{\tilde{A}}} v = 0, \quad \operatorname{div} b_0 = 0 & \text{in } [0,T]\times\Omega; \\ v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \mathring{\tilde{A}}^{3\alpha} q = -\sigma\sqrt{\mathring{g}}(\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\tilde{n}^\alpha + \kappa(1-\overline{\Delta})(v\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha & \text{on } \Gamma; \\ (\eta, v) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{7.2}$$

The rest of this section is devoted to show the existence of η, v by first establishing the existence of the weak solution and then boosting up their regularity. The construction of the solution for the nonlinear κ-problem will be postponed until the next section.

We will adapt the method developed in Coutand-Shkoller[13] to study the weak solution for (7.2). Also, due to technical reasons, it appears that it is more convenient for us to first construct the weak solution of (7.2) in $L^2(0,T; H^{-1}(\Omega))$ and then prove that this solution in fact has $L^2(0,T; H^1(\Omega))$ regularity.

7.1 The penalized problem

The goal of this subsection is to study the penalized version (of the divergence-free condition on the velocity) of the linearized κ-problem (7.2). In particular, for $0 < \lambda \ll 1$, let w_λ, ξ_λ be the solutions for (7.2) with

$$\operatorname{div}_{\mathring{\tilde{A}}} w_\lambda = -\lambda q_\lambda \tag{7.3}$$

where q_λ is defined to be the penalized pressure. In this case, (7.2) becomes

$$\begin{cases} \partial_t \xi_\lambda = w_\lambda & \text{in } [0,T]\times\Omega; \\ \partial_t w_\lambda - (b_0\cdot\partial)^2\xi_\lambda + \nabla_{\mathring{\tilde{A}}} q_\lambda = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{\mathring{\tilde{A}}} w_\lambda = -\lambda q_\lambda, \quad \operatorname{div} b_0 = 0 & \text{in } [0,T]\times\Omega; \\ w_\lambda^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \mathring{\tilde{A}}^{3\alpha} q_\lambda = -\sigma\sqrt{\mathring{g}}(\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha + \kappa(1-\overline{\Delta})(v\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha & \text{on } \Gamma; \\ (\xi_\lambda, w_\lambda) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{7.4}$$

Since each penalized problem is indexed by λ (recall κ is fixed), we shall denote them by "λ-problem" throughout the rest of this section.

7.1.1 Weak solution for the λ-problem

First of all, for each fixed λ, we will solve the λ-problem by the Galerkin approximation and obtain a weak solution. By introducing a basis $(e_k)_{k=1}^\infty$ of $L^2(\Omega)\cap H^1(\Omega)$, and considering the approximation

$$\partial_t\xi_m(t,y) = w_m(t,y), \tag{7.5}$$

$$w_m(t,y) = \sum_{k=1}^m z_k(t)e_k(y), \quad m\geq 2, \quad t\in[0,T], \tag{7.6}$$

one can form a system of ODE by multiplying a test vector field ϕ, whose component $\phi_\alpha\in\text{span}(e_1,\cdots,e_m)$ to the λ-problem. Specifically, we have

$$\int_\Omega (w_m^\alpha)_t\phi_\alpha - \int_\Omega [(b_0\cdot\partial)^2\xi_m^\alpha]\phi_\alpha + \int_\Omega [\mathring{A}^{\mu\alpha}\partial_\mu q_m]\phi_\alpha = 0. \tag{7.7}$$

We recall that $(b_0\cdot\partial)|_\Gamma$ is tangential to Γ. Owing to this and the boundary condition of q_m, we obtain, after integration by parts, that

$$\int_\Omega (w_m^\alpha)_t\phi_\alpha + \int_\Omega [(b_0\cdot\partial)\xi_m^\alpha][(b_0\cdot\partial)\phi_\alpha] + \kappa\sum_{l=0,1}\int_\Gamma \overline{\partial}^l(w_m\cdot\mathring{n})\overline{\partial}^l(\phi\cdot\mathring{n})$$
$$- \int_\Omega q_m[\mathring{A}^{\mu\alpha}\partial_\mu\phi_\alpha] = \sigma\int_\Gamma(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{n})(\phi\cdot\mathring{n}), \tag{7.8}$$

$$w_m(0) = (v_0)_m, \quad \xi_m(0) = \text{Id}, \tag{7.9}$$

where $(v_0)_m$ is the projection of v_0 onto $\text{span}(e_1,\cdots,e_m)$.

Let $\phi_\alpha = e_k$, $k=1,2,\cdots,m$. Then (7.8)-(7.9), and (7.3) yield a system of ODE, and the standard ODE theory gives the the existence and uniqueness of ξ_m and w_m in $[0,T_\lambda]$ for some $T_\lambda>0$. *We mention here that it is important to introduce the penalized pressure* (7.3) *since* (7.8) *would not form a system of ODE otherwise*.

Setting $\phi = w_m$, and since $\sigma|\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}^\alpha|_0\leq\mathcal{N}_0$, where $\mathcal{N}_0$ denotes a generic polynomial function such that

$$\mathcal{N}_0 = P(\|\eta_0\|_{4.5}, \|v_0\|_{4.5}, \|b_0\|_{4.5}),$$

then (7.8) gives us

$$\|w_m\|_0^2 + \|(b_0\cdot\partial)\xi_m\|_0^2 + \lambda\int_0^t\|q_m\|_0^2 + \kappa\int_0^t|w_m\cdot\mathring{n}|_1^2 \leq \mathcal{N}_0, \quad t\in[0,T_\lambda] \tag{7.10}$$

Also, because the RHS of (7.10) is independent of λ, we must have that the solution (ξ_m, w_m) is defined on $[0,T]$ (possibly after setting T smaller). In addition, there is a subsequence, which is still denoted with the index m, satisfying

$$(b_0\cdot\partial)\xi_m \rightharpoonup (b_0\cdot\partial)\xi_\lambda, \quad w_m\rightharpoonup w_\lambda, \quad q_m\rightharpoonup q_\lambda, \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{7.11}$$

$$w_m\cdot\mathring{n}\rightharpoonup w_\lambda\cdot\mathring{n}, \quad \text{in } L^2(0,T;H^1(\Gamma)), \tag{7.12}$$

where w_λ, $(b_0\cdot\partial)\xi_\lambda$, and q_λ verify the estimate

$$\|w_\lambda\|_0^2 + \|(b_0\cdot\partial)\xi_\lambda\|_0^2 + \lambda\int_0^t\|q_\lambda\|_0^2 + \kappa\int_0^t|w_\lambda\cdot\mathring{n}|_1^2 \leq \mathcal{P}_0, \quad t\in[0,T]. \tag{7.13}$$

Now, let Y be a Banach space. We denote its dual by Y', and, for $\Psi\in H^s(\Omega)' = H^{-s}(\Omega)$ and $\Phi\in H^s(\Omega)$, the pairing between Ψ and Φ is denoted by $\langle\Psi,\Phi\rangle_s$. It follows from the ODE (7.8) defining w_m, that $\partial_t w_\lambda\in L^2(0,T;H^{-\frac12+})$, where $H^{-\frac12+} := H^{-\frac12+\delta}$ for some $0<\delta\ll1$, and $(b_0\cdot\partial)^2\xi_\lambda\in L^2(0,T;H^{-\frac12+})$ as well. Now, for $\phi\in L^2(0,T;H^{\frac12-})$, we have

$$\int_0^T\langle\partial_t w_\lambda^\alpha,\phi_\alpha\rangle_{\frac12-} + \int_0^T\langle(b_0\cdot\partial)^2\xi_\lambda,\phi_\alpha\rangle_{\frac12-}$$
$$+\kappa\sum_{l=0,1}\int_0^T\int_\Gamma\overline{\partial}^l(w_\lambda\cdot\mathring{n})\overline{\partial}^l(\phi\cdot\mathring{n}) - \int_0^T\langle q_\lambda,\mathring{A}^{\mu\alpha}\partial_\mu\phi_\alpha\rangle_{\frac12+} \tag{7.14}$$
$$=\sigma\int_0^T\int_\Gamma(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{n})(\phi\cdot\mathring{n}).$$

In light of (7.14), we can see that $\partial_t w_\lambda \in L^2(0,T;H^{-\frac{1}{2}+})$, and $q_\lambda \in L^2(0,T;H^{\frac{1}{2}+})$, and the regularity of q_λ implies $\nabla_{\mathring{A}} q_\lambda \in L^2(0,T;H^{-\frac{1}{2}-})$. Therefore, we have that

$$\partial_t w_\lambda - (b_0\cdot\partial)^2\xi_\lambda + \nabla_{\mathring{A}} q_\lambda = 0 \tag{7.15}$$

holds in $L^2(0,T;H^{-\frac{1}{2}-}(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$. In addition, by commuting $\operatorname{curl}_{\mathring{A}}$ through (7.15) we get the following evolution equation verified by $\operatorname{curl}_{\mathring{A}} w_\lambda$ and $\operatorname{curl}_{\mathring{A}}(b_0\cdot\partial)\xi_\lambda$:

$$\partial_t(\operatorname{curl}_{\mathring{A}} w_\lambda) - (b_0\cdot\partial)\operatorname{curl}_{\mathring{A}}((b_0\cdot\partial)\xi_\lambda) = [\operatorname{curl}_{\mathring{A}}, (b_0\cdot\partial)]((b_0\cdot\partial)\xi_\lambda) + \operatorname{curl}_{\partial_t\mathring{A}} w_\lambda. \tag{7.16}$$

7.1.2 The limit as $\lambda \to 0$

By plugging (7.3) into $\lambda\int_0^t \|q_\lambda\|_0^2$, the estimate (7.13) implies the following:

$$\int_0^t \left(\|w_\lambda\|_0^2 + \|(b_0\cdot\partial)\xi_\lambda\|_0^2 + \frac{1}{\lambda}\|\operatorname{div}_{\mathring{A}} w_\lambda\|_0^2 + \kappa|w_\lambda\cdot\mathring{\tilde{n}}|_1^2\right)dt \le \mathcal{N}_0, \quad t\in[0,T] \tag{7.17}$$

Thanks to this, the sequences $\{w_\lambda\}$ and $\{(b_0\cdot\partial)\xi_\lambda\}$ admits converging subsequences (still denoted with index λ) such that

$$w_\lambda \rightharpoonup v, \quad (b_0\cdot\partial)\xi_\lambda \rightharpoonup (b_0\cdot\partial)\eta, \quad \operatorname{div}_{\mathring{A}} w_\lambda \rightharpoonup \operatorname{div}_{\mathring{A}} v, \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{7.18}$$

$$w_\lambda\cdot\mathring{\tilde{n}} \rightharpoonup v\cdot\mathring{\tilde{n}} \quad \text{in } L^2(0,T;H^1(\Gamma)). \tag{7.19}$$

Moreover, in view of (7.13), we must have that

$$\operatorname{div}_{\mathring{A}} v = 0, \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{7.20}$$

Also, this implies the evolution equation verified by $\operatorname{div}(b_0\cdot\partial)\eta$, i.e.,

$$\partial_t \operatorname{div}_{\mathring{A}}((b_0\cdot\partial)\eta) = [\operatorname{div}_{\mathring{A}}, (b_0\cdot\partial)]v + (\partial_t\mathring{\tilde{A}}^{\mu\alpha})\partial_\mu((b_0\cdot\partial)\eta_\alpha). \tag{7.21}$$

Our next goal is to show that η, v is a weak solution for (7.2) and we also need to get a bound for $\int_0^t \|v_t\|^2_{H^{-\frac{1}{2}+}(\Omega)}$ for $t\in[0,T]$. This quantity in fact ties to the $L^2(0,T;H^{\frac{1}{2}+})$ regularity of the pressure function q (to be defined later in this section). The main argument here is an adaption of what's in the Section 8 of [13]. First, we consider a vector field $f\in L^2(0,T;H^{\frac{1}{2}-})$. Define φ be the solution for the elliptic problem

$$\mathring{\tilde{A}}^\nu_\alpha \partial_\nu(\mathring{\tilde{A}}^{\mu\alpha}\partial_\mu\varphi) = \operatorname{div}_{\mathring{A}} f, \quad \text{in } \Omega, \tag{7.22}$$

$$\varphi = 0, \quad \text{on } \partial\Omega, \tag{7.23}$$

and let g, h be the vector fields such that $g = \nabla_{\mathring{A}}\varphi$ and $h = f - g$. Here, it is clear that $g, h \in L^2(0,T;H^{\frac{1}{2}-})$ and $\operatorname{div}_{\mathring{A}} h = 0$. Now, (7.14) yields, after replacing ϕ by h, that h verifies the following variational equation

$$\begin{aligned}
&\int_0^T \langle\partial_t w_\lambda, h\rangle_{\frac{1}{2}-} + \int_0^T \langle(b_0\cdot\partial)^2\xi_\lambda, h\rangle_{\frac{1}{2}-} + \kappa\sum_{l=0,1}\int_0^T\int_\Gamma \overline{\partial}^l(w_\lambda\cdot\mathring{\tilde{n}})\overline{\partial}^l(h\cdot\mathring{\tilde{n}}) \\
=&\sigma\int_0^T\int_\Gamma(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})(h\cdot\mathring{\tilde{n}}).
\end{aligned} \tag{7.24}$$

On the other hand, since $\operatorname{div}_{\mathring{A}} v = 0$, we have $\mathring{A}^{\mu\alpha}\partial_\mu\partial_t v_\alpha = -(\partial_t\mathring{\tilde{A}}^{\mu\alpha})\partial_\mu v_\alpha$. This identity and (7.23) yield

$$\langle v_t, g\rangle_{\frac{1}{2}-} = \langle v_t, \nabla_{\mathring{A}}\varphi\rangle_{\frac{1}{2}-} = \int_\Omega (\partial_t\mathring{\tilde{A}}^{\mu\alpha})\partial_\mu v_\alpha\varphi.$$

In light of this and (7.24), we obtain

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_0^T \langle\partial_t w_\lambda, f\rangle_{\frac{1}{2}-} + \int_0^T \langle(b_0\cdot\partial)^2\xi_\lambda^\alpha, f\rangle_{\frac{1}{2}-} \\
=&\int_0^T\int_\Omega(\partial_t\mathring{\tilde{A}}^{\mu\alpha})\partial_\mu v_\alpha\varphi - \kappa\sum_{l=0,1}\int_0^T\int_\Gamma\overline{\partial}^l(v\cdot\mathring{\tilde{n}})\overline{\partial}^l(h\cdot\mathring{\tilde{n}}) + \sigma\int_0^T\int_\Gamma(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})(h\cdot\mathring{\tilde{n}}),
\end{aligned} \tag{7.25}$$

and so

$$\lim_{\lambda\to 0}\int_0^T \|w_{\lambda t}\|^2_{H^{-\frac{1}{2}+}} + \|(b_0\cdot\partial)^2\xi_\lambda\|^2_{H^{-\frac{1}{2}+}} \le \mathcal{N}_0. \tag{7.26}$$

As a consequence, we have $w_{\lambda t} \rightharpoonup v_t$ and $(b_0\cdot\partial)^2\xi_\lambda \rightharpoonup (b_0\cdot\partial)^2\eta$ in $L^2(0,T;H^{-\frac{1}{2}+})$. The former ensures that $v\in C^0(0,T;L^2)$ and so the initial data of $w_\lambda(0)$ and $v(0)$ agrees and equals to v_0.

Moreover, by employing the Lagrange multiplier lemma (i.e., Lemma 7.4 in [13])[3], there exists $q\in L^2(0,T;H^{\frac{1}{2}+})$, in terms of the pressure function, such that

$$\begin{aligned}&\int_0^T \langle\partial_t v,\phi\rangle_{\frac{1}{2}-} + \int_0^T \langle(b_0\cdot\partial)^2\eta,\phi\rangle_{\frac{1}{2}-} - \int_0^T\langle q,\mathring{A}^{\mu\alpha}\partial_\mu\phi_\alpha\rangle_{\frac{1}{2}+} + \kappa\sum_{l=0,1}\int_0^T\int_\Gamma \overline{\partial}^l(v\cdot\mathring{n})\overline{\partial}^l(\phi\cdot\mathring{n})\\ =&\sigma\int_0^T\int_\Gamma(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{n})(\phi\cdot\mathring{n}).\end{aligned} \tag{7.27}$$

holds for any test function $\phi\in L^2(0,T;H^{\frac{1}{2}-})$. This yields that (η,v,q) verifies

$$\partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\mathring{A}} q = 0, \quad \text{and } \operatorname{div}_{\mathring{A}} v = 0, \quad \text{in } L^2(0,T;H^{-1}),$$

and so we've shown that η, v is indeed a weak solution for (7.2). Furthermore, (7.26) implies, after employing the Lagrange multiplier lemma in [13], that

$$\int_0^T \|q\|^2_{H^{\frac{1}{2}+}} \le \mathcal{P}_0. \tag{7.28}$$

Remark. The $\frac{1}{2}+$ interior regularity of q is required here as this controls the $H^{0+}(\Gamma)$-norm of q on the boundary. We refer Section 7.2.2 for the details.

Finally, we consider the difference between (7.27) with v and v', respectively, i.e.,

$$\begin{aligned}&\int_0^T\langle\partial_t(v-v'),\phi\rangle_{\frac{1}{2}-} + \int_0^T\langle(b_0\cdot\partial)^2(\eta-\eta'),\phi\rangle_{\frac{1}{2}+}\\ +&\kappa\sum_{l=0,1}\int_0^T\int_\Gamma\overline{\partial}^l((v-v')\cdot\mathring{n})\overline{\partial}^l(\phi\cdot\mathring{n}) - \int_0^T\langle(q-q'),\mathring{\tilde{A}}^{\mu\alpha}\partial_\mu\phi\rangle_{\frac{1}{2}+} = 0.\end{aligned} \tag{7.29}$$

where (v',η') is assumed to be another solution with the initial data. The uniqueness of the weak solution follows from setting $\phi = v-v'$.

7.2 H^1 Regularity estimates of v, $(b_0\cdot\partial)\eta$ and q

We shall show that v, $(b_0\cdot\partial)\eta$ and q are in fact $L^2(0,T;H^1(\Omega))$. Let

$$e(t) := \int_0^t \|\eta\|_1^2 + \|v\|_1^2 + \|(b_0\cdot\partial)\eta\|_1^2\,dt, \quad t\in[0,T]. \tag{7.30}$$

Our goal is to show

$$e(T)\le P(\mathcal{N}_0). \tag{7.31}$$

It suffices to consider $\int_0^T\|v\|_1^2$ and $\int_0^T\|(b_0\cdot\partial)\eta\|_1^2$ only since

$$\int_0^T\|\eta\|_1^2 \le \int_0^T\left(\|\eta_0\|_1^2 + \int_0^t\|v\|_1^2\,dt\right)dt.$$

[3] We in fact need a small modification here. Since we need our $q\in H^{\frac{1}{2}+}$, we need to consider the linear functional $\langle\operatorname{div}_{\mathring{A}}\phi,p\rangle_{\frac{1}{2}+}$ defined on $X(t)$, where $X(t)=\{\phi\in H^{\frac{1}{2}-}(\Omega):\operatorname{div}_{\mathring{A}}\phi\in H^{-\frac{1}{2}-}(\Omega))\}$.

Thanks to Lemma 2.6(2), it suffices for us to control

$$\int_0^T \|\text{div}\, v\|_0^2, \quad \int_0^T \|\text{curl}\, v\|_0^2, \quad \int_0^T |v^3|_{0.5}^2,$$

as well as

$$\int_0^T \|\text{div}\,(b_0\cdot\partial)\eta\|_0^2, \quad \int_0^T \|\text{curl}\,(b_0\cdot\partial)\eta\|_0^2, \quad \int_0^T |(b_0\cdot\partial)\eta^3|_{0.5}^2,$$

in order to control $\int_0^T \|v\|_1^2$ and $\int_0^T \|(b_0\cdot\partial)\eta\|_1^2$.

7.2.1 Control of the divergence and curl

The estimates we need here are essentially the same as those in Section 3.2 but without considering the time differentiated quantities. Firstly, since (3.17) in Lemma 3.3 remains true with $\tilde{A}$ replaced by $\mathring{\tilde{A}}$, then

$$\int_0^T \|\text{div}\, v\|_0^2 \le \int_0^T \|(\mathring{\tilde{A}}^{\mu\alpha}-\delta^{\mu\alpha})\partial_\mu v_\alpha\|_0^2 \le \varepsilon \int_0^T \|\partial v\|_0^2 \le \varepsilon e(T). \tag{7.32}$$

Secondly, because $\text{div}_{\mathring{\tilde{A}}}(b_0\cdot\partial)\eta$ verifies the evolution equation

$$\partial_t \text{div}_{\mathring{\tilde{A}}}((b_0\cdot\partial)\eta) = [\text{div}_{\mathring{\tilde{A}}}, (b_0\cdot\partial)]v + (\partial_t \mathring{\tilde{A}}^{\mu\alpha})\partial_\mu((b_0\cdot\partial)\eta_\alpha). \tag{7.33}$$

So, one needs to bound

$$\int_0^T \int_0^t \|\text{RHS of (7.33)}\|_0^2\, dt$$

in order to control $\int_0^T \|\text{div}_{\mathring{\tilde{A}}}((b_0\cdot\partial)\eta)\|_0^2$. We have

$$\int_0^T \int_0^t \|\partial_t \mathring{\tilde{A}}^{\mu\alpha}\partial_\mu((b_0\cdot\partial)\eta)\|_0^2\, dt \le \int_0^T \int_0^t \|\partial_t \mathring{\tilde{A}}\|_{L^\infty}^2 \|\partial((b_0\cdot\partial)\eta)\|_0^2\, dt \tag{7.34}$$

$$\le \int_0^T \int_0^t \mathcal{N}_0 \|\partial((b_0\cdot\partial)\eta)\|_0^2\, dt \le T\mathcal{N}_0 e(T). \tag{7.35}$$

Moreover, by writing $[\text{div}_{\mathring{\tilde{A}}}, (b_0\cdot\partial)]v = \mathring{A}^{\mu\alpha}((\partial_\mu b_0)\cdot\partial)v_\alpha - ((b_0\cdot\partial)\mathring{A}^{\mu\alpha})\partial_\mu v_\alpha$, one gets

$$\int_0^T \int_0^t \|[\text{div}_{\mathring{\tilde{A}}}(b_0\cdot\partial)]v\|_0^2 \le \int_0^T \int_0^t \mathcal{N}_0\|\partial v\|_0^2\, dt \le T\mathcal{N}_0 e(T). \tag{7.36}$$

Thus,

$$\int_0^T \|\text{div}_{\mathring{\tilde{A}}}((b_0\cdot\partial)\eta)\|_0^2 \le T\mathcal{N}_0 e(T). \tag{7.37}$$

In addition, since

$$\|\text{div}\,(b_0\cdot\partial)\eta\|_0^2 \le \|\text{div}_{\mathring{\tilde{A}}}(b_0\cdot\partial)\eta\|_0^2 + \|\mathring{A}-\delta\|_{L^\infty}^2 \|\partial(b_0\cdot\partial)\eta\|_0^2,$$

invoking (3.17), we conclude that

$$\int_0^T \|\text{div}\,(b_0\cdot\partial)\eta\|_0^2 \le \varepsilon e(T) + T\mathcal{N}_0 e(T). \tag{7.38}$$

Thirdly, the evolution equation satisfied by $\text{curl}_{\mathring{\tilde{A}}} v$ and $\text{curl}_{\mathring{\tilde{A}}}(b_0\cdot\partial)\eta$ reads

$$\partial_t(\text{curl}_{\mathring{\tilde{A}}} v)_\alpha - (b_0\cdot\partial)\text{curl}_{\mathring{\tilde{A}}}((b_0\cdot\partial)\eta)_\alpha = [\text{curl}_{\mathring{\tilde{A}}}, (b_0\cdot\partial)]((b_0\cdot\partial)\eta)_\alpha + \text{curl}\,{}_{\partial_t \mathring{\tilde{A}}} v_\alpha, \tag{7.39}$$

and this yields the following $L^2([0,T];L^2(\Omega))$-energy identity after testing with $\mathrm{curl}_{\mathring{A}} v$ and integrating in space and time:

$$\|\mathrm{curl}_{\mathring{A}} v\|_0^2 + \|\mathrm{curl}_{\mathring{A}} (b_0\cdot\partial)\eta\|_0^2 \lesssim \int_0^t \|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)\eta\|_0^2 + \int_0^t \|\mathrm{curl}\,_{\partial_t\mathring{A}} v\|_0^2 + \int_0^t \|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)v\|_0^2 + \int_0^t \|\mathrm{curl}\,_{\partial_t\mathring{A}} (b_0\cdot\partial)\eta\|_0^2.$$

Integrating in time one more time, we achieve

$$\int_0^T \left(\|\mathrm{curl}_{\mathring{A}} v\|_0^2 + \|\mathrm{curl}_{\mathring{A}} (b_0\cdot\partial)\eta\|_0^2\right) \lesssim \int_0^T \int_0^t \left(\|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)\eta\|_0^2 + \|\mathrm{curl}\,_{\partial_t\mathring{A}} v\|_0^2\right) dt + \int_0^T \int_0^t \left(\|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)v\|_0^2 + \|\mathrm{curl}\,_{\partial_t\mathring{A}} (b_0\cdot\partial)\eta\|_0^2\right) dt.$$

It suffices to control the first two terms on the RHS since the third and fourth term can then be controlled by an analogous method with the same bound.

For the first term on the RHS, since one can express

$$[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)\eta_\alpha = \epsilon_{\alpha\beta\gamma}((b_0\cdot\partial)\mathring{A}^{\nu\beta})\partial_\nu\eta^\gamma - \epsilon_{\alpha\beta\gamma}\mathring{A}^{\nu\beta}(\partial_\nu b_0\cdot\partial)\eta^\gamma$$

and so

$$\int_0^T \int_0^t \|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)\eta_\alpha\|_0 \,dt \lesssim T\mathcal{N}_0 e(T). \tag{7.40}$$

Similarly,

$$\|[(b_0\cdot\partial), \mathrm{curl}_{\mathring{A}}](b_0\cdot\partial)v_\alpha\|_0 \le T\mathcal{N}_0 e(T). \tag{7.41}$$

In addition, for the second term, writing $\mathrm{curl}\,_{\partial_t\mathring{A}} v = \epsilon_{\alpha\beta\gamma}(\partial_t\mathring{A}^{\nu\beta})\partial_\nu v^\gamma$, one obtains

$$\int_0^T \int_0^t \|\mathrm{curl}\,_{\partial_t\mathring{A}} v\|_0 \,dt \le T\mathcal{N}_0 e(T). \tag{7.42}$$

Summing these up, we obtain

$$\int_0^T \int_0^t \left(\|\mathrm{curl}_{\mathring{A}} v\|_0^2 + \|\mathrm{curl}_{\mathring{A}} (b_0\cdot\partial)\eta\|_0^2\right) dt \le T\mathcal{N}_0 e(T). \tag{7.43}$$

7.2.2 Control of the boundary terms

First we state some supplementary results which will come in handy when treating the boundary estimates. The following inequality is a direct consequence of (2.10). Let $f \in H^{0.5}(\partial\Omega)$ and g be a smooth function. Then

$$|fg|_{0.5} \lesssim |f|_{0.5}|g|_{1+}. \tag{7.44}$$

Also, we remark here that (3.18), (3.19) remain true by replacing $\tilde{n}$ by $\mathring{\tilde{n}}$.

Control of $\int_0^T |v^3|_{0.5}^2$: It suffices to control $\int_0^T |v\cdot\mathring{\tilde{n}}|_{0.5}^2$ since

$$\int_0^T |v^3|_{0.5}^2 \le \int_0^T |v\cdot\mathring{\tilde{n}}|_{0.5}^2 + \int_0^T |v\cdot(\mathring{\tilde{n}} - N)|_{0.5}^2, \tag{7.45}$$

where, after invoking (7.44) and the trace lemma, we have

$$\int_0^T |v\cdot(\mathring{\tilde{n}} - N)|_{0.5}^2 \le \int_0^T |v|_{0.5}^2 |\mathring{\tilde{n}} - N|_{1+}^2 \lesssim \varepsilon e(T). \tag{7.46}$$

Moreover, the control of $\int_0^T |v\cdot\mathring{\tilde{n}}|_{0.5}^2$ is a direct consequence of (7.17) as $\lambda \to 0$, i.e.,

$$\int_0^T |v\cdot\mathring{\tilde{n}}|_{0.5}^2 \lesssim \frac{1}{\kappa}\int_0^T |v\cdot\mathring{\tilde{n}}|_1^2 \le \frac{\mathcal{N}_0}{\kappa}. \tag{7.47}$$

Control of $\int_0^T |(b_0\cdot\partial)\eta^3|_{0.5}^2$: Similar to the control $\int_0^T |v^3|_{0.5}^2$, it suffices to bound $\int_0^T |(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_{0.5}^2$ only.
Since $(b_0\cdot\partial)|_\Gamma = b_0\cdot\overline{\partial}$ and $\overline{\partial}(\eta\cdot\mathring{\tilde{n}})|_{t=0} = \overline{\partial}\eta^3|_{t=0} = 0$, we have

$$(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}}) = \int_0^T \partial_t(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})\,dt. \tag{7.48}$$

Hence,

$$\int_0^T |(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_{0.5}^2 \le \int_0^T \left|\int_0^t \partial_t(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})\right|_{0.5}^2 dt \lesssim \int_0^T\int_0^t |\partial_t(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_{0.5}^2\,dt,$$

by Jensen's inequality. Here, the term on the second line is equal to

$$\int_0^T\int_0^t |(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})|_{0.5}^2\,dt + \int_0^T\int_0^t |(b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}})|_{0.5}^2 = I + II.$$

Since $\partial_t\mathring{\tilde{n}} = Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}}\cdot\mathring{\tilde{n}}$, invoking (7.44) and the trace lemma, we have $II \le T\mathcal{N}_0 e(T)$.
Next, invoking (7.44), we have

$$I \lesssim \|b_0\|_{0.5}\int_0^T\int_0^t |v\cdot\mathring{\tilde{n}}|_2^2\,dt.$$

By employing the boundary condition we obtain the following elliptic equation verified by $v\cdot\mathring{\tilde{n}}$ on Γ:

$$\overline{\Delta}(v\cdot\mathring{\tilde{n}}) = \frac{1}{\kappa}\Big((v\cdot\mathring{\tilde{n}}) + \sqrt{\mathring{\tilde{g}}}q + \sigma\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}}\Big). \tag{7.49}$$

By the virtual of the elliptic estimate, we have

$$\int_0^T\int_0^t |v\cdot\mathring{\tilde{n}}|_{2+}^2\,dt \le \kappa^{-1}\int_0^T\int_0^t \Big(|v\cdot\mathring{\tilde{n}}|_{0+}^2 + |\sqrt{\mathring{\tilde{g}}}q|_{0+}^2 + \sigma|\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}^3|_{0+}^2\Big)\,dt. \tag{7.50}$$

It is clear that the third term can be controlled by $T\mathcal{N}_0$, and first term is bounded by $T\mathcal{N}_0 e(T)$ via the trace lemma. Therefore,

$$\int_0^T\int_0^t |v\cdot\mathring{\tilde{n}}|_{2+}^2 \le \frac{1}{\kappa}(T\mathcal{N}_0 e(T) + \int_0^T\int_0^t \mathcal{N}_0|q|_{0+}^2\,dt). \tag{7.51}$$

Here, in light of (7.28), we have $\int_0^t |q|_{0+}^2 \le \mathcal{N}_0$ as a consequence of the trace theorem.
In summary, we have

$$e(T) \le \kappa^{-1}\mathcal{N}_0 + \varepsilon e(T) + T\mathcal{N}_0 e(T), \tag{7.52}$$

and this implies (7.31) if T is chosen sufficiently small, say $T = \frac{\varepsilon}{\mathcal{N}_0}$.

7.2.3 The strong solution for the linearized equations

Since $v, (b_0\cdot\partial)\eta \in L^2(0,T;H^1(\Omega))$ and so $v_t, (b_0\cdot\partial)^2\eta \in L^2(0,T;L^2(\Omega))$, we can now proceed as what has been done in Section 7 of [13] to bound q in $L^2(0,T;H^1(\Omega))$. Alternatively, one may also adapt Lemma 2.7 to achieve the same objective.

Therefore, we have obtained a strong solution for the linearized κ-problem (7.2). This allows us to further boost the regularity of the linearized solution to $H^{4.5}(\Omega)$ via classical methods in the upcoming section. Then we are able to achieve a solution for the nonlinear κ-problem by approximating it by a sequence of linearized solutions.

8 Existence for the nonlinear approximate κ-problem

We aim to construct a solution to the nonlinear κ-problem for *each fixed* $\kappa > 0$. Let $(\eta_0, v_0, q_0) = (\mathrm{Id}, 0, 0)$. For each $n \geq 0$, Let $(\eta_{(m+1)}, v_{(m+1)}, q_{(m+1)})$ be the solution for (7.2) with initial data (Id, v_0, q_0), where the (linearized) coefficients are determined by $(\eta_{(m)}, v_{(m)}, q_{(m)})$. The goal is to prove that the sequence $\{(\eta_{(m)}, v_{(m)})\}_{m\geq 0}$ strongly converges and the limit verifies the nonlinear approximate κ-problem. This can be done by standard Picard iteration. We will first establish the $H^{4.5}$-energy estimate for $(\eta_{(m)}, v_{(m)})$, and then this estimate can be carried over to the difference between two successive systems (7.2) which yields the convergence of $(\eta_{(m)}, v_{(m)})$ as $m \to \infty$.

8.1 A priori estimate of the linearized approximate problem

Let $m \geq 0$ be fixed and assume the solutions $(\eta_{(l)}, v_{(l)}, q_{(l)})$ are known for all $l \leq m$. For the sake of clean notations, we will denote $(\eta_{(m+1)}, v_{(m+1)}, q_{(m+1)})$ by (η, v) and $(\eta_{(m)}, v_{(m)}, q_{(m)})$ by $(\mathring{\eta}, \mathring{v}, \mathring{q})$ if no confusion is raised.

Proposition 8.1. For each fixed $\kappa > 0$, there exists some $T_\kappa > 0$ such that the solution (η, v) for (7.2) satisfies

$$\sup_{0\leq t\leq T_\kappa} \mathcal{E}(t) \leq C, \tag{8.1}$$

where C is a constant depends on $\|v_0\|_{4.5}, \|b_0\|_{4.5}$, provided that

$$\|\mathring{J}(t) - 1\|_{3.5} + \|\operatorname{Id} - \mathring{A}(t)\|_{3.5} + \|\operatorname{Id} - \mathring{A}^T \mathring{A}\|_{3.5} \leq \varepsilon. \tag{8.2}$$

holds for all $t \in [0, T_\kappa]$. Here the energy functional $\mathcal{E}$ of (7.2) is defined to be

$$\mathcal{E}(t) = \mathcal{E}^{(1)}(t) + \mathcal{E}^{(2)}(t), \tag{8.3}$$

where

$$\begin{aligned}
\mathcal{E}^{(1)}(t) &:= \|\eta\|_{4.5}^2 + \|v\|_{4.5}^2 + \|\partial_t v\|_{3.5}^2 + \left\|\partial_t^2 v\right\|_{2.5}^2 + \left\|\partial_t^3 v\right\|_{1.5}^2 + \left\|\partial_t^4 v\right\|_0^2 \\
&\quad + \|(b_0\cdot\partial)\eta\|_{4.5}^2 + \|\partial_t(b_0\cdot\partial)\eta\|_{3.5}^2 + \left\|\partial_t^2(b_0\cdot\partial)\eta\right\|_{2.5}^2 + \left\|\partial_t^3(b_0\cdot\partial)\eta\right\|_{1.5}^2 + \left\|\partial_t^4(b_0\cdot\partial)\eta\right\|_0^2 \\
\mathcal{E}^{(2)}(t) &:= \frac{\kappa}{\sigma}\int_0^T \left|\partial_t^4 v\cdot\mathring{\tilde{n}}\right|_1^2 dt + \kappa\Big(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4(b_0\cdot\partial)\eta\|_{1.5}^2\Big).
\end{aligned}$$

It can be seen that $\mathcal{E}(t)$ constructed above is significantly simpler than $E_\kappa(t)$ given in (3.9). In particular, *no* boundary terms appear in $\mathcal{E}^{(1)}(t)$ since $-\sigma\sqrt{\mathring{g}}(\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha$ is a fixed term in the linearized equations. In addition to this, we only need to perform the tangential energy estimate consists four time derivatives. Since κ is fixed, the boundary terms that involve at least two spatial derivatives can be controlled by study the elliptic equation generated by the boundary condition (i.e., (8.13)). Also, the following observation shall be used frequently throughout the rest of this section.;

Removing extra (tangential) spatial derivatives: In light of (2.21), we are able to absorb additional tangential spatial derivatives when necessary. This will allows us to greatly simplify most of the estimates on the boundary.

Thanks to the Gronwall's inequality, (8.1) is a direct consequence of

$$\sup_{0\leq t\leq T_\kappa} \mathcal{E}(t) \lesssim_{\kappa^{-1}} C(\|v_0\|_{4.5}, \|b_0\|_{4.5}) + C(\varepsilon)\sup_{0\leq t\leq T_\kappa}\mathcal{E}(t) + (\sup_{0\leq t\leq T_\kappa}\mathcal{P})\int_0^T \mathcal{P}, \tag{8.4}$$

where $\mathcal{P} = P(\mathcal{E}(t), \|\mathring{v}\|_{4.5}, \|(b_0\cdot\partial)\mathring{\eta}\|_{4.5})$ (after a slight abuse of notations). Also, we will drop the subscript κ and denote $T_\kappa = T$ for the sake of clean notations. Similar to (3.11) we shall assume that $\sup_{0\leq t\leq T}\mathcal{E}(t) = \mathcal{E}(T)$, and this allows us to drop sup $\sup_{0\leq t\leq T_\kappa}$ in (8.4). In other words, we only need to show

$$\mathcal{E}(T) \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\mathcal{E}(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{8.5}$$

where $\mathcal{P}_0 = \mathcal{P}(\mathcal{E}(0), \|q(0)\|_{4.5}, \|q_t(0)\|_{3.5}, \|q_{tt}(0)\|_{2.5})$. We remark here that (8.5) does not have to be uniform in κ, and so the RHS may depend on $\frac{1}{\kappa}$. This fact allows us to greatly simplify some of the boundary estimates (See Section 8.1.2). Also, it suffices to put $\mathcal{P}_0$ on the RHS of (8.5) since (6.2) allows us to control $\mathcal{P}_0$ by C.

8.1.1 Interior estimates

We control

$$\|\partial_t^k v\|_{4.5-k}^2, \quad \|\partial_t^k (b_0\cdot\partial)\eta\|_{4.5-k}^2, \quad k=0,1,2,3, \tag{8.6}$$

by applying the div-curl estimate:

$$\|\partial_t^k v\|_{4.5-k}^2 \lesssim \|\partial_t^k \text{div}\, v\|_{3.5-k}^2 + \|\partial_t^k \text{curl}\, v\|_{3.5-k}^2 + |\partial_t^k v^3|_{4-k}, \tag{8.7}$$

$$\|\partial_t^k (b_0\cdot\partial)\eta\|_{4.5-k}^2 \lesssim \|\partial_t^k \text{div}\,(b_0\cdot\partial)\eta\|_{3.5-k}^2 + \|\partial_t^k \text{curl}\,(b_0\cdot\partial)\eta\|_{3.5-k}^2 + |\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}. \tag{8.8}$$

Actually, the estimates for the divergence and curl of v and $(b_0\cdot\partial)\eta$, together with their time derivatives are identical to those in Section 3.2, and so we shall not repeat the proofs.

We also need the estimates for the interior Sobolev norms of the pressure q, which is identical to (3.30) in Sections 3.1. Furthermore, the estimate for the top order interior term in $\mathcal{E}^{(2)}$, i.e.,

$$\kappa\Big(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4 (b_0\cdot\partial)\eta\|_{1.5}^2\Big) \le \mathcal{P}_0 + C(\varepsilon)\mathcal{E}(T) + \mathcal{P}\int_0^T \mathcal{P} \tag{8.9}$$

is identical to what has been done in Section 5.

8.1.2 Boundary estimates

This subsection is devoted to control the boundary terms $|\partial_t^k v^3|_{4-k}$ and $|\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}$ for $k=0,1,2,3$. Our goal is to show

Lemma 8.2. For $k=0,1,2,3$, we have

$$|\partial_t^k v^3|_{4-k}^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\mathcal{E}(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{8.10}$$

$$|\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\mathcal{E}(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{8.11}$$

Note that we no longer require the energy bound to be κ-independent. Hence, *we are able to use* (2.21) *to absorb extra tangential spatial derivatives on the smoothed variables, i.e., variables with˜on top.* We can absorb at most two tangential spatial derivatives since $\tilde{\cdot} = \Lambda_\kappa^2\cdot$. Recall that the boundary condition in the linearized equations reads

$$\sqrt{\mathring{\tilde{g}}}q = -\sigma\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}} + \kappa(1-\overline{\Delta})(v\cdot\mathring{\tilde{n}}). \tag{8.12}$$

This can be converted to an elliptic equation satisfied by $v\cdot\mathring{\tilde{n}}$, i.e.,

$$\overline{\Delta}(v\cdot\mathring{\tilde{n}}) = v\cdot\mathring{\tilde{n}} - \kappa^{-1}\left(\sqrt{\mathring{\tilde{g}}}q + \sigma\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}}\right). \tag{8.13}$$

Now, invoking the standard elliptic estimate and (2.6), we get

$$\begin{aligned} |v\cdot\mathring{\tilde{n}}|_4^2 \lesssim& |v\cdot\mathring{\tilde{n}}|_2^2 + \kappa^{-1}\left(\left|\sqrt{\mathring{\tilde{g}}}q\right|_2^2 + \sigma P(|\overline{\partial}\mathring{\eta}|_{L^\infty}, |\overline{\partial}^2\mathring{\eta}|_{L^\infty})|\mathring{\eta}|_4^2\right) \\ \lesssim_{\kappa^{-1}}& \mathcal{P}_0 + \int_0^T \mathcal{P}, \end{aligned} \tag{8.14}$$

where the used the trace lemma and (3.30) in the second inequality.

For the magnetic field, since $(b_0\cdot\partial) = b_0^j\overline{\partial}_j$ on Γ and hence $(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_{t=0} = 0$. Thus,

$$(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}}) = \int_0^T \partial_t\Big((b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})\Big) = \int_0^T (b_0\cdot\partial)(v\cdot\mathring{\tilde{n}}) + \int_0^T (b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}}). \tag{8.15}$$

Since $\partial_t \mathring{\tilde{n}} = -\mathring{\tilde{g}}^{kl}\overline{\partial}_k \mathring{\tilde{v}} \cdot \mathring{\tilde{n}}\overline{\partial}_l \mathring{\tilde{\eta}} = Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}} \cdot \mathring{\tilde{n}}$, and invoking (2.21) and the Jensen's inequality, we have

$$\left|\int_0^T (b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}})\right|_4^2 \lesssim T\int_0^T |(b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}})|_4^2 \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P} \tag{8.16}$$

Here, we need (2.21) in order to control the leading order term generated when $\overline{\partial}^4(b_0\cdot\partial)$ fall on $\overline{\partial}\mathring{\tilde{v}}$ (which is part of $\partial_t\mathring{\tilde{n}}$), i.e.,

$$\int_0^T Q(|\mathring{\tilde{\eta}}|_{L^\infty}, |\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|(b_0\cdot\partial)\overline{\partial}\mathring{\tilde{v}}|_4^2 \le \int_0^T |b_0|_4^2 Q(|\mathring{\tilde{\eta}}|_{L^\infty}, |\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|\overline{\partial}^2\mathring{\tilde{v}}|_4^2 \lesssim_{\kappa^{-1}} \int_0^T |b_0|_4^2 Q(|\mathring{\tilde{\eta}}|_{L^\infty}, |\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|\mathring{v}|_4^2.$$

In addition,

$$\left|\int_0^T (b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})\right|_4^2 \lesssim T\int_0^T |(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})|_4^2, \tag{8.17}$$

and the RHS can be controlled by studying the elliptic equation satisfied by $(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})$. Taking $(b_0\cdot\partial)$ on (8.13) and we get

$$\overline{\Delta}(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}}) = [\overline{\Delta},(b_0\cdot\partial)](v\cdot\mathring{\tilde{n}}) + (b_0\cdot\partial)(v\cdot\mathring{\tilde{n}}) - \kappa^{-1}\left((b_0\cdot\partial)(\sqrt{\mathring{\tilde{g}}}q) + \sigma(b_0\cdot\partial)(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\right), \tag{8.18}$$

then the elliptic estimate implies

$$\int_0^T |(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})|_4^2 \lesssim_{\kappa^{-1}} P(|b_0|_4)\int_0^T \left(|v\cdot\mathring{\tilde{n}}|_4^2 + \left|\sqrt{\mathring{\tilde{g}}}q\right|_3^2 + \sigma P(|\mathring{\eta}|_{H^4})|(b_0\cdot\partial)\mathring{\eta}|_4^2\right) \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P}. \tag{8.19}$$

Thus,

$$|(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_4^2 \lesssim_\kappa \int_0^T \mathcal{P}. \tag{8.20}$$

We can obtain the bounds for $|v^3|_4^2$ and $|(b_0\cdot\partial)\eta^3|_4^2$ from (8.14) and (8.20), respectively. Indeed, we have

$$|v^3|_4^2 \le |v\cdot\mathring{\tilde{n}}|_4^2 + |v\cdot(N-\mathring{\tilde{n}})|_4^2, \tag{8.21}$$

$$|(b_0\cdot\partial)\eta^3|_4^2 \le |(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_4^2 + |(b_0\cdot\partial)(\eta\cdot(N-\mathring{\tilde{n}}))|_4^2. \tag{8.22}$$

Since

$$N-\mathring{\tilde{n}} = -\int_0^T \partial_t\mathring{\tilde{n}} = \int_0^T Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}}\cdot\mathring{\tilde{n}},$$

invoking the proof for (3.19) and (2.21), we have

$$|N-\mathring{\tilde{n}}|_5 \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P}. \tag{8.23}$$

Therefore,

$$|v\cdot(N-\mathring{\tilde{n}})|_4^2 + |(b_0\cdot\partial)(\eta\cdot(N-\mathring{\tilde{n}}))|_4^2 \lesssim_{\kappa^{-1}} \mathcal{P}\int_0^T \mathcal{P}. \tag{8.24}$$

Now, we can take time derivative ∂_t in (8.13) to get the elliptic equation of $\partial_t(v\cdot\mathring{\tilde{n}})$ on the boundary, i.e.,

$$\overline{\Delta}\partial_t(v\cdot\mathring{\tilde{n}}) = \partial_t(v\cdot\mathring{\tilde{n}}) - \kappa^{-1}\left(\partial_t(\sqrt{\mathring{\tilde{g}}}q) + \sigma\partial_t(\sqrt{\mathring{g}}\Delta_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\right). \tag{8.25}$$

Then standard elliptic estimate gives

$$\begin{aligned}|\partial_t(v\cdot\mathring{\tilde{n}})|_3^2 \lesssim& |\partial_t(v\cdot\mathring{\tilde{n}})|_1^2 + \kappa^{-1}\left(\left|\partial_t\left(\mathring{\tilde{A}}^{33}q\right)\right|_1^2 + \sigma P(|\overline{\partial}\mathring{\eta}|_{L^\infty}, |\overline{\partial}^2\mathring{\eta}|_{L^\infty}, |\overline{\partial}\mathring{v}|_{L^\infty})|\mathring{v}|_3^2\right)\\ \lesssim_{\kappa^{-1}}& \mathcal{P}_0 + \int_0^T \mathcal{P}.\end{aligned} \tag{8.26}$$

This estimate implies the estimate for $|\partial_t v^3|_3^2$ by writing $|\partial_t v^3|_3^2 \le |\partial_t(v\cdot\mathring{\tilde{n}})|_3^2 + |\partial_t(v\cdot(N-\mathring{\tilde{n}}))|_3^2$ and then adapting the arguments from (8.21)-(8.24). Moreover, in light of the estimate for $|v^3|_4^2$, we have

$$|\partial_t(b_0\cdot\partial)\eta^3|_3 = |(b_0\cdot\partial)v^3|_3^2 \le P(|b_0|_3)|v^3|_4^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{8.27}$$

Similarly, by taking two time derivatives to (8.13), we can control $|\partial_t^2(v\cdot\mathring{\tilde{n}})|_2$ by the standard elliptic estimate, i.e.,

$$|\partial_t^2(v\cdot\mathring{\tilde{n}})|_2^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + \int_0^T \mathcal{P}, \tag{8.28}$$

and this yields

$$|\partial_t^2 v^3|_2^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{8.29}$$

In addition to this, $|\partial_t^2(b_0\cdot\partial)\eta^3|_2^2$ reduces to $|\partial_t v^3|_3^2$, whose bound is given above. Also, in the case when there are three time derivatives, $|\partial_t^3(b_0\cdot\partial)\eta^3|_1^2$ reduces to $|\partial_t^2 v^3|_2^2$, which is just (8.29).

Finally, $|\partial_t^3 v^3|_1^2$ can be controlled with the help $\mathcal{E}^{(2)}$. We can make use of the κ-weighted higher order terms to directly control the time integrated terms on the boundary. Specifically, by writing $|\partial_t^3 v^3|_1 \le \mathcal{P}_0 + \int_0^T |\partial_t^4 v^3|_1$, we have

$$|\partial_t^3 v^3|_1^2 \lesssim \mathcal{P}_0 + \left(\int_0^T |\partial_t^4 v^3|_1\right)^2 \lesssim \mathcal{P}_0 + T\int_0^T |\partial_t^4 v^3|_1^2, \tag{8.30}$$

where

$$T\int_0^T |\partial_t^4 v^3|_1^2 \le T\int_0^T |\partial_t^4 v\cdot\mathring{\tilde{n}}|_1^2 + T\int_0^T |\partial_t^4 v\cdot(N-\mathring{\tilde{n}})|_1^2. \tag{8.31}$$

Here, the second term on the RHS is $\lesssim_{\kappa^{-1}} TC(\varepsilon)\int_0^T \|\partial_t^4 v\|_{1.5}^2$ whereas the first term is $\lesssim_{\kappa^{-1}} T\int_0^T |\partial_t^4 v\cdot\mathring{\tilde{n}}|_1^2$. Therefore, by choosing T sufficiently small, we have

$$|\partial_t^3 v^3|_1^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\mathcal{E}(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{8.32}$$

8.1.3 Tangential estimate with four time derivatives

We still need to control

$$\|\partial_t^4 v\|_0^2, \quad \|\partial_t^4(b_0\cdot\partial)\eta\|_0^2$$

in order to finish the control of $\mathcal{E}$. In fact, we only need to control $\|\partial_t^4 v\|_0^2$ since $\|\partial_t^4(b_0\cdot\partial)\eta\|_0^2$ reduces to $\|\partial_t^3 v\|_1^2$ which has been done previously.

Now we compute the L^2-estimate of $\partial_t^4 v$ and $\partial_t^4(b_0\cdot\partial)\eta$. Invoking (7.2) and integrating $(b_0\cdot\partial)$ by parts, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega |\partial_t^4 v|^2 + \left|\partial_t^4(b_0\cdot\partial)\eta\right|^2 \,dy\,dt = -\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4(\mathring{\tilde{A}}^{\mu\alpha}\partial_\mu q)\,dy\,dt \\
&= -\int_0^T\int_\Omega \partial_t^4 v_\alpha \mathring{\tilde{A}}^{\mu\alpha}\partial_t^4\partial_\mu q\,dy\,dt \underbrace{-\int_0^T\int_\Omega \partial_t^4 v_\alpha \left[\partial_t^4, \mathring{\tilde{A}}^{\mu\alpha}\right]\partial_\mu q\,dy\,dt}_{\mathring{I}_1} \\
&= \int_0^T\int_\Omega \mathring{\tilde{A}}^{\mu\alpha}\partial_t^4\partial_\mu v_\alpha \partial_t^4 q\,dy\,dt \underbrace{-\int_0^T\int_\Gamma \partial_t^4 v_\alpha \mathring{\tilde{A}}^{3\alpha}\partial_t^4 q\,dS\,dt}_{\mathring{I}_B} + \mathring{I}_1 \\
&= \int_0^T\int_\Omega \underbrace{\partial_t^4(\mathrm{div}_{\mathring{\tilde{A}}} v)}_{=0}\partial_t^4 q\,dy\,dt + \underbrace{\int_0^T\int_\Omega \left[\mathring{\tilde{A}}^{\mu\alpha}, \partial_t^4\right]\partial_\mu v_\alpha\, \partial_t^4 q\,dy\,dt}_{\mathring{I}_2} + \mathring{I}_B + \mathring{I}_1.
\end{aligned} \tag{8.33}$$

Here, $\mathring{I}_1$ and $\mathring{I}_2$ can be straightforwardly controlled by $\int_0^T \mathcal{P}$. We start to analyze the boundary integral I_B.

$$
\begin{aligned}
\mathring{I}_B =& -\int_0^T \int_\Gamma \partial_t^4 v_\alpha \mathring{\tilde{A}}^{3\alpha} \partial_t^4 q = -\int_0^T \int_\Gamma \sqrt{\mathring{\bar{g}}} (\partial_t^4 v \cdot \mathring{\tilde{n}})(\partial_t^4 q) \\
=& \sigma \int_0^T \int_\Gamma \sqrt{\mathring{\bar{g}}} (\partial_t^4 v \cdot \mathring{\tilde{n}}) \partial_t^4 (\sqrt{\mathring{\bar{g}}} \mathring{\bar{g}}^{-\frac{1}{2}} \Delta_{\mathring{\bar{g}}} \mathring{\tilde{\eta}} \cdot \mathring{\tilde{n}}) \\
&-\kappa \int_0^T \int_\Gamma \sqrt{\mathring{\bar{g}}} (\partial_t^4 v \cdot \mathring{\tilde{n}}) \partial_t^4 \left(\mathring{\bar{g}}^{-\frac{1}{2}} (1-\overline{\Delta})(v \cdot \mathring{\tilde{n}}) \right) := \mathring{I}_{B1} + \mathring{I}_{B2}.
\end{aligned} \tag{8.34}
$$

Invoking the identity (2.6), we have

$$
\begin{aligned}
\mathring{I}_{B1} =& \sigma \int_0^T \int_\Gamma \sqrt{\mathring{\bar{g}}} (\partial_t^4 v \cdot \mathring{\tilde{n}}) \partial_t^4 (\sqrt{\mathring{\bar{g}}} \mathring{\bar{g}}^{-\frac{1}{2}} \mathring{\bar{g}}^{ij} \overline{\partial}_i \overline{\partial}_j \mathring{\tilde{\eta}} \cdot \mathring{\tilde{n}}) - \sigma \int_0^T \int_\Gamma \sqrt{\mathring{\bar{g}}} (\partial_t^4 v \cdot \mathring{\tilde{n}}) \partial_t^4 (\sqrt{\mathring{\bar{g}}} \mathring{\bar{g}}^{-\frac{1}{2}} \mathring{\bar{g}}^{ij} \mathring{\bar{g}}^{kl} \overline{\partial}_l \mathring{\tilde{\eta}}^\mu \overline{\partial}_i \overline{\partial}_j \mathring{\tilde{\eta}}_\mu \overline{\partial}_k \mathring{\tilde{\eta}} \cdot \mathring{\tilde{n}}) \\
=& \mathring{I}_{B11} + \mathring{I}_{B12}.
\end{aligned} \tag{8.35}
$$

Since

$$
\mathring{I}_{B11} \stackrel{L}{=} \sigma \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}}) (\sqrt{\mathring{\bar{g}}} \mathring{\bar{g}}^{ij} \overline{\partial}_i \overline{\partial}_j \partial_t^3 \mathring{\tilde{v}} \cdot \mathring{\tilde{n}})
$$

we integrate $\overline{\partial}_i$ by parts and get

$$
\begin{aligned}
\mathring{I}_{B11} \stackrel{L}{=} & -\sigma \int_0^T \int_\Gamma (\overline{\partial}_i \partial_t^4 v \cdot \mathring{\tilde{n}}) (\sqrt{\mathring{\bar{g}}} \mathring{\bar{g}}^{ij} \overline{\partial}_j \partial_t^3 \mathring{\tilde{v}} \cdot \mathring{\tilde{n}}) \\
\lesssim_{\kappa^{-1}} & \varepsilon \int_0^T \| \sqrt{\kappa} \partial_t^4 v \|_{1.5}^2 + \int_0^T \mathcal{P} \le \varepsilon \mathcal{E}(T) + \int_0^T \mathcal{P},
\end{aligned}
$$

and $\mathring{I}_{B12}$ can be treated in the same fashion.

Next we study $\mathring{I}_{B2}$. We have

$$
\mathring{I}_{B2} \stackrel{L}{=} \kappa \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}}) \overline{\Delta} (\partial_t^4 v \cdot \mathring{\tilde{n}}) + \kappa \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}}) \overline{\Delta} (v \cdot \partial_t^4 \mathring{\tilde{n}}) := \mathring{I}_{B21} + \mathring{I}_{B22}, \tag{8.36}
$$

where $\mathring{I}_{B21}$ contributes to the positive energy term $\int_0^T |\partial_t^4 v \cdot \mathring{\tilde{n}}|_1^2$ after integrating $\overline{\partial}$ by parts and moving the resulting term to the LHS. In addition, since $\partial_t^4 \mathring{\tilde{n}} = Q(\overline{\partial} \mathring{\tilde{\eta}}) \overline{\partial} \partial_t^3 \mathring{\tilde{v}} \cdot \mathring{\tilde{n}} +$ lower-order terms,

$$
\begin{aligned}
\mathring{I}_{B22} \stackrel{L}{=} & -\kappa \int_0^T \int_\Gamma (\overline{\partial} \partial_t^4 v \cdot \mathring{\tilde{n}}) (v \cdot Q(\overline{\partial} \mathring{\tilde{\eta}}) \overline{\partial}^2 \partial_t^3 \mathring{\tilde{v}} \cdot \mathring{\tilde{n}}) \\
\lesssim & \varepsilon \int_0^T \| \partial_t^4 v \|_{1.5}^2 + \int_0^T |v|_{L^\infty}^2 Q(|\overline{\partial} \mathring{\tilde{\eta}}|_{L^\infty}) |\overline{\partial}^2 \partial_t^3 \mathring{\tilde{v}}|_0^2 \\
\lesssim_{\kappa^{-1}} & \varepsilon \mathcal{E}(T) + \int_0^T |v|_{L^\infty}^2 Q(|\overline{\partial} \mathring{\tilde{\eta}}|_{L^\infty}) |\overline{\partial} \partial_t^3 \mathring{v}|_0^2 \le \varepsilon \mathcal{E}(T) + \int_0^T \mathcal{P},
\end{aligned} \tag{8.37}
$$

where we used (2.21) in the second to the last inequality to control $|\overline{\partial}^2 \partial_t^3 \mathring{\tilde{v}}|_0^2$ by $\kappa^{-1} |\overline{\partial} \partial_t^3 \mathring{v}|_0^2$. This concludes the proof of Proposition 8.1.

8.2 Picard iteration

Now we prove that the sequence $\{(\eta_{(m)}, v_{(m)}, q_{(m)})\}_{m \in \mathbb{N}^*}$ has a strongly convergent subsequence. We define $[f]_{(m)} := f_{(m+1)} - f_{(m)}$ for any function f and then $([\eta]_{(m)}, [v]_{(m)}, [q]_{(m)})$ satisfies the following system

$$
\begin{cases}
\partial_t [\eta]_{(m)} = [v]_{(m)} & \text{in } \Omega, \\
\partial_t [v]_{(m)} - (b_0 \cdot \partial)^2 [\eta]_{(m)} + \nabla_{\tilde{A}_{(m)}} [q]_{(m)} = -\nabla_{[\tilde{A}]_{(m-1)}} q_{(m)} & \text{in } \Omega, \\
\text{div}\,_{\tilde{A}_{(m)}} [v]_{(m)} = -\text{div}\,_{[\tilde{A}]_{(m-1)}} v_{(m)} & \text{in } \Omega, \\
[q]_{(m)} = \kappa (1 - \overline{\Delta}) ([v]_{(m)} \cdot \tilde{n}_{(m)}) + h_{(m)} & \text{on } \Gamma, \\
([\eta]_{(m)}, [v]_{(m)})|_{t=0} = (\mathbf{0}, \mathbf{0}).
\end{cases} \tag{8.38}
$$

where

$$
\begin{aligned}
h_{(m)} =&\kappa(1-\overline{\Delta})(v_{(m)}\cdot[\tilde{n}]_{(m-1)})\\
&-\sigma\Big(\sqrt{\tilde{g}_{(m)}}g^{ij}_{(m)}\Pi^{\lambda}_{(m)\alpha}\overline{\partial}_i\overline{\partial}_j\eta_{(m)\lambda}\tilde{n}^{\alpha}_{(m)}-\sqrt{\tilde{g}_{(m-1)}}g^{ij}_{(m-1)}\Pi^{\lambda}_{(m-1)\alpha}\overline{\partial}_i\overline{\partial}_j\eta_{(m-1)\lambda}\tilde{n}^{\alpha}_{(m-1)}\Big)
\end{aligned}
$$

We also define the energy functional of $([\eta]_{(m)},[v]_{(m)},[q]_{(m)})$ to be

$$
[\mathcal{E}]_{(m)}:=[\mathcal{E}]^{(1)}_{(m)}+[\mathcal{E}]^{(2)}_{(m)}, \tag{8.39}
$$

where

$$
\begin{aligned}
[\mathcal{E}]^{(1)}_{(m)}(T):=&\left\|[\eta]_{(m)}\right\|^2_{3.5}+\left\|[v]_{(m)}\right\|^2_{3.5}+\left\|\partial_t[v]_{(m)}\right\|^2_{2.5}+\left\|\partial_t^2[v]_{(m)}\right\|^2_{1.5}+\left\|\partial_t^3[v]_{(m)}\right\|^2_{0}\\
&+\left\|(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_{3.5}+\left\|\partial_t(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_{2.5}+\left\|\partial_t^2(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_{1.5}+\left\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_{0}\\
[\mathcal{E}]^{(2)}_{(m)}(T):=&\frac{\kappa}{\sigma}\int_0^T\left|\partial_t^3[v]_{(m)}\cdot\tilde{n}_{(m)}\right|^2_1\,dt+\kappa\Big(\int_0^T\|\partial_t^3[v]_{(m)}\|^2_{1.5}+\int_0^T\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|^2_{1.5}\Big).
\end{aligned}\tag{8.40}
$$

8.2.1 The div-curl estimates

For $k=0,1,2$

$$
\|\partial_t^k[v]_{(m)}\|_{3.5-k}\lesssim\|\partial_t^k[v]_{(m)}\|_0+\|\text{div}\,\partial_t^k[v]_{(m)}\|_{2.5-k}+\|\text{curl}\,\partial_t^k[v]_{(m)}\|_{2.5-k}+|\overline{\partial}\partial_t^k[v]_{(m)}\cdot N|_{2-k}, \tag{8.41}
$$

$$
\begin{aligned}
\|\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{3.5-k}\lesssim&\|\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_0+\|\text{div}\,\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{2.5-k}+\|\text{curl}\,\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{2.5-k}\\
&+|\overline{\partial}\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\cdot N|_{2-k}.
\end{aligned}\tag{8.42}
$$

Again, each part in the div-curl estimates should follow in the same way as in Section 3.2 so we omit the proof. Similarly, to control the interior terms in $[\mathcal{E}]^{(2)}_{(m)}$, we also need a similar div-curl decomposition for the κ-weighted terms and follow the method in Section 5. Then we have

$$
\begin{aligned}
&\kappa\Big(\int_0^T\|\partial_t^3[v]_{(m)}\|^2_{1.5}+\int_0^T\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|^2_{1.5}\Big)\\
\lesssim&\mathcal{P}_0+\varepsilon[\mathcal{E}]_{(m)}(T)+P([\mathcal{E}]_{(m)}(T),\mathcal{E}_{(m),(m-1)}(T))\int_0^T P([\mathcal{E}]_{(m),(m-1)}(t),\mathcal{E}_{(m),(m-1)}(t))\,dt.
\end{aligned}\tag{8.43}
$$

8.2.2 Elliptic estimates of pressure

Similarly as in Section 3.1, one can derive the elliptic equation verified by $[q]_{(m)}$ and its time derivatives with Neumann boundary conditions. The only difference is that we need to control the contribution of $(\nabla_{[\tilde{A}]_{(m-1)}}q_{(m)})$ and its time derivatives, but this is straightforward. For example, we need to control $\|\text{div}\,_{\tilde{A}_{(m)}}(\nabla_{[\tilde{A}]_{(m-1)}}q_{(m)})\|_{1.5}$ in the estimate of $\|[q]_{(m)}\|_{3.5}$.

$$
\|\text{div}\,_{\tilde{A}_{(m)}}(\nabla_{[\tilde{A}]_{(m-1)}}q_{(m)})\|_{1.5}\lesssim P(\|[\tilde{A}]_{(m-1)}\|_{2.5},\|q_{(m)}\|_{3.5},\|\tilde{A}_{(m)}\|_{2.5}),
$$

and the boundary contribution

$$
|\tilde{A}_{(m)}N\cdot\nabla_{[\tilde{A}]_{(m-1)}}q_{(m)}|_2\lesssim P(\|[\tilde{A}]_{(m-1)}\|_{2.5},\|q_{(m)}\|_{3.5},\|\tilde{A}_{(m)}\|_{2.5}).
$$

8.2.3 Boundary estimates

The boundary estimates also follow in the same way as Section 8.1.2 because the energy is not required to be indepedent of κ. We can derive an elliptic equation on Γ, analogous with (8.13)

$$
\kappa\overline{\Delta}([v]_{(m)}\cdot\tilde{n}_{(m)})=\kappa([v]_{(m)}\cdot\tilde{n}_{(m)})+h_{(m)}-[q]_{(m)}. \tag{8.44}
$$

Then using the boundary elliptic estimates, we get

$$
\begin{aligned}
|[v]_{(m)}\cdot\tilde n_{(m)}|_3 \lesssim_{\kappa^{-1}} &|[v]_{(m)}\cdot\tilde n_{(m)}|_1 + |h_{(m)}|_1 + \|[q]_{(m)}\|_{1.5}\\
\lesssim&|[v]_{(m)}\cdot\tilde n_{(m)}|_1 + \|[q]_{(m)}\|_{1.5} + |v_{(m)}|_3 P(|\overline{\partial}[\tilde\eta]_{(m-1)}, \overline{\partial}^2[\eta]_{(m-1)}|_1, |\overline{\partial}\eta_{(m-1)}|_2)\\
\lesssim&\mathcal{P}_0 + P(\mathcal{E}_{(m),(m-1)}(T))\int_0^T P([\mathcal{E}]_{(m),(m-1)}(t), \mathcal{E}_{(m),(m-1)}(t))\,dt.
\end{aligned}
\tag{8.45}
$$

As for the magnetic field, we use the fact that $(b_0\cdot\partial) = b_0^j\overline{\partial}_j$ on Γ to get

$$
(b_0\cdot\partial)[\eta]_{(m)}\cdot\tilde n_{(m)} = 0 + \int_0^T (b_0\cdot\partial)[v]_{(m)}\cdot\tilde n_{(m)} + (b_0\cdot\partial)[\eta]_{(m)}\cdot\partial_t\tilde n_{(m)}.
$$

Similarly as in Section 8.1.2, one can directly control the $H^3(\Gamma)$-norm of the second term. Then the first term can be controlled by using elliptic estimates in $(b_0\cdot\partial)$-differentiated elliptic equation (8.44). We omit the detailed proof because there is no essential difference from the argument in Section 8.1.2.

$$
|(b_0\cdot\partial)[\eta]_{(m)}\cdot\tilde n_{(m)}|_3 \lesssim_{\kappa^{-1}} \int_0^T P([\mathcal{E}]_{(m)}(t), \mathcal{E}_{(m)}(t))\,dt. \tag{8.46}
$$

Taking one time derivative, we can similarly control the boundary norm of $\partial_t[v]_{(m)}$ and $\partial_t(b_0\cdot\partial)[\eta]_{(m)}$. We skip the details.

$$
\left|\partial_t[v]_{(m)}\cdot\tilde n_{(m)}, \partial_t(b_0\cdot\partial)[\eta]_{(m)}\right|_2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + P(\mathcal{E}_{(m),(m-1)}(T))\int_0^T P([\mathcal{E}]_{(m),(m-1)}(t), \mathcal{E}_{(m),(m-1)}(t))\,dt. \tag{8.47}
$$

For the $H^1(\Gamma)$-norm of $\partial_t^2[v]_{(m)}$ and $\partial_t^2(b_0\cdot\partial)[\eta]_{(m)}$, one can use the κ-weighted interior terms in $[\mathcal{E}]_{(m)}^{(2)}$ and Sobolev trace lemma to get the control

$$
\begin{aligned}
&\left|\partial_t^2[v]_{(m)}^3, \partial_t^2(b_0\cdot\partial)[\eta]_{(m)}^3\right|_1 \lesssim \left\|\partial_t^2[v]_{(m)}, \partial_t^2(b_0\cdot\partial)[\eta]_{(m)}\right\|_{1.5}\\
\lesssim&\mathcal{P}_0 + \int_0^T \left\|\partial_t^3[v]_{(m)}\cdot\tilde n_{(m)}, \partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\right\|_{1.5}\,dt\\
\lesssim&\mathcal{P}_0 + \sqrt{\frac{T}{\kappa}}\left\|\sqrt{\kappa}\partial_t^3[v]_{(m)},\ \sqrt{\kappa}\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\right\|_{L_t^2H_y^{1.5}}\\
\lesssim_{\kappa^{-0.5}}&\mathcal{P}_0 + \sqrt{T}P([\mathcal{E}]_{(m)}^{(2)}(T)).
\end{aligned}
\tag{8.48}
$$

Finally, we need to control the difference between $X\cdot N$ and $X\cdot\tilde n_{(m)}$, which should be done in the same way as (8.21)-(8.23), so we do not repeat the calculations. For $k=0,1$, we have for $X=[v]_{(m)}, (b_0\cdot\partial)[\eta]_{(m)}$

$$
|\partial_t^k X^3 - \partial_t^k(X\cdot\tilde n_{(m)})|_{3-k} \lesssim_{\kappa^{-1}} \int_0^T P([\mathcal{E}]_{(m)}, \mathcal{E}_{(m),(m-1)}(t))\,dt. \tag{8.49}
$$

Combining (8.45)-(8.49), we get the boundary estimates as

$$
\sum_{k=0}^2 \left|\partial_t^k([v]_{(m)}^3, (b_0\cdot\partial)[\eta]_{(m)}^3)\right|_{3-k} \lesssim_{\kappa^{-1}} \mathcal{P}_0 + P(\mathcal{E}_{(m),(m-1)}(T))\int_0^T P([\mathcal{E}]_{(m),(m-1)}(t), \mathcal{E}_{(m),(m-1)}(t))\,dt. \tag{8.50}
$$

8.2.4 Estimates of full time derivatives

Now it remains to control the L^2-norm of full time derivatives. By replacing ∂_t^4 in Section 8.1.3 by ∂_t^3, we can do analogous computation to control $\|\partial_t^3[v]_{(m)}\|_0$ and $\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|_0$. The κ-weighted boundary terms in $[\mathcal{E}]_{(m)}^{(2)}$ are produced in the analogues of (8.34). The only difference is that we should control the extra contribution (under time integral) of $\nabla_{[\tilde A]_{(m-1)}}q_{(m)}$ in the interior and the σ-coefficient part in the term $h_{(m)}$ on the boundary. These quantities can all be directly controlled

$$
\|\partial_t^3\nabla_{[\tilde A]_{(m-1)}}q_{(m)}\|_0 \lesssim P\left(\|[v]_{(m-1)}, \partial_t[v]_{(m-1)}, \partial_t^2[v]_{(m-1)}\|_2, \|\partial_t^3 q_{(m)}\|_1, \|\partial_t^2 q_{(m)}, \partial_t q_{(m)}, q_{(m)}\|_2\right).
$$

$$|\partial_t^3 h_{(m),\sigma}|_0 \lesssim P\left(|\partial_t^2 v_{(m),(m-1)}|_2, |\overline{\partial}\eta_{(m),(m-1)}, \overline{\partial}v_{(m),(m-1)}, \overline{\partial}\partial_t v_{(m),(m-1)}|_{L^\infty}\right).$$

Therefore, one can get

$$\begin{aligned}
&\|\partial_t^3[v]_{(m)}\|_0^2 + \|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|_0^2 + \frac{\kappa}{\sigma}\int_0^T \left|\partial_t^3[v]_{(m)}\cdot\tilde{n}_{(m)}\right|_1^2\,dt \\
\lesssim&\mathcal{P}_0 + \int_0^T P([\mathcal{E}]_{(m)}(t), \mathcal{E}_{(m),(m-1)}(t))\,dt
\end{aligned} \tag{8.51}$$

8.3 Well-posedness of the nonlinear approximate problem

We conclude this section with the following proposition.

Proposition 8.3 (**Local well-posedness of the nonlinear κ-approximation problem**). For each fixed $\kappa > 0$, there exists $T'_\kappa > 0$ such that the nonlinear κ-approxiamtion problem (3.2) has a unique strong solution $(\eta(\kappa), v(\kappa), q(\kappa))$ in $[0, T'_\kappa]$ that satisfies

$$\sup_{0\le t\le T'_\kappa} \mathcal{E}'(t) \le C \tag{8.52}$$

where $\mathcal{E}'(t) = \mathcal{E}^{(1)'}(t) + \mathcal{E}^{(2)'}(t)$

$$\begin{aligned}
\mathcal{E}^{(1)'}(t) :=& \|\eta\|_{4.5}^2 + \|v\|_{4.5}^2 + \|\partial_t v\|_{3.5}^2 + \left\|\partial_t^2 v\right\|_{2.5}^2 + \left\|\partial_t^3 v\right\|_{1.5}^2 + \left\|\partial_t^4 v\right\|_0^2 \\
&+ \|(b_0\cdot\partial)\eta\|_{4.5}^2 + \|\partial_t(b_0\cdot\partial)\eta\|_{3.5}^2 + \left\|\partial_t^2(b_0\cdot\partial)\eta\right\|_{2.5}^2 + \left\|\partial_t^3(b_0\cdot\partial)\eta\right\|_{1.5}^2 + \left\|\partial_t^4(b_0\cdot\partial)\eta\right\|_0^2 \\
\mathcal{E}^{(2)'}(t) :=& \frac{\kappa}{\sigma}\int_0^T \left|\partial_t^4 v\cdot\tilde{n}\right|_1^2\,dt + \kappa\left(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4(b_0\cdot\partial)\eta\|_{1.5}^2\right).
\end{aligned} \tag{8.53}$$

Proof. Summarizing (8.41)-(8.43), (8.50)-(8.51), we can get the following inequality

$$[\mathcal{E}]_{(m)}(T) \lesssim_\kappa^{-1} \mathcal{P}_0 + \varepsilon[\mathcal{E}](T) + TP([\mathcal{E}]_{(m)}(T)) + P([\mathcal{E}]_{(m)}(T), \mathcal{E}_{(m),(m-1)}(T))\int_0^T P([\mathcal{E}]_{(m),(m-1)}(t), \mathcal{E}_{(m),(m-1)}(t)).$$

By Gronwall-type inequality in Tao [51] and the conclusion of Proposition 8.1, there exists some $T'_\kappa > 0$, such that $\forall t \in [0, T'_\kappa]$

$$[\mathcal{E}]_{(m)}(t) \le \frac{1}{4}[\mathcal{E}]_{(m-1)}(t),$$

which implies $[\mathcal{E}]_{(m)}(t) \le 4^{-m}\mathcal{P}_0$. Let $m \to \infty$, we know the sequence $\{(\eta_{(m)}, v_{(m)}, q_{(m)})\}$ must strongly converge. The strong limit is denoted by $(\eta(\kappa), v(\kappa), q(\kappa))$ which exactly solves the nonlinear κ-approximation problem (3.2). By taking $m \to \infty$ in the energy of linearized equation (7.2), one can also get the energy estimates. □

9 Local well-posedness

9.1 Uniqueness and well-posedness

Combining the conclusions of Proposition 3.1 and Propostion 8.3 and letting $\kappa \to 0_+$, we actually prove that there exists some time $T' > 0$ (only depends on the initial data), such that the original system (1.13) has solution (η, v, q) satisfying the energy estimates

$$\sup_{0\le t\le T} E(t) \le C,$$

where $C = C(\|v_0\|_{4.5}, \|b_0\|_{4.5})$, and the energy functional E is defined to be

$$\begin{aligned}
E(t) :=& \|\eta\|_{4.5}^2 + \|v\|_{4.5}^2 + \|\partial_t v\|_{3.5}^2 + \left\|\partial_t^2 v\right\|_{2.5}^2 + \left\|\partial_t^3 v\right\|_{1.5}^2 + \left\|\partial_t^4 v\right\|_0^2 \\
&+ \|(b_0\cdot\partial)\eta\|_{4.5}^2 + \|\partial_t(b_0\cdot\partial)\eta\|_{3.5}^2 + \left\|\partial_t^2(b_0\cdot\partial)\eta\right\|_{2.5}^2 + \left\|\partial_t^3(b_0\cdot\partial)\eta\right\|_{1.5}^2 + \left\|\partial_t^4(b_0\cdot\partial)\eta\right\|_0^2 \\
&+ \left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v\right)\right|_0^2 + \left|\overline{\partial}(\Pi\overline{\partial}^3 v)\right|_0^2 + \left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)\right|_0^2,
\end{aligned} \tag{9.1}$$

To establish the local well-posedness, it remains to prove the uniqueness. Let $\{(\eta_{(m)}, v_{(m)}, q_{(m)})\}_{m=1,2}$ be two solutions of (1.13) satisfying the energy estimates. Then we define

$$[\eta] := \eta_{(1)} - \eta_{(2)},\ [v] := v_{(1)} - v_{(2)},\ [q] := q_{(1)} - q_{(2)},\ [a] := a_{(1)} - a_{(2)}.$$

Then $([\eta], [v], [q])$ satisfies the following system

$$\begin{cases} \partial_t[\eta] = [v] & \text{in } [0,T]\times\Omega; \\ \partial_t[v] - (b_0\cdot\partial)^2[\eta] + \nabla_{a_{(1)}}[q] = -\nabla_{[a]}q_{(2)} & \text{in } [0,T]\times\Omega; \\ \text{div}_{\,a_{(1)}}[v] = -\text{div}_{\,[a]}v_{(2)}, & \text{in } [0,T]\times\Omega; \\ \text{div } b_0 = 0 & \text{in } [0,T]\times\Omega; \\ [v^3] = b_0^3 = 0 & \text{on } \Gamma_0; \\ [q]n_{(1)} = -\sigma g_{(1)}^{ij}\Pi_{(1)}\overline{\partial}{}^2_{ij}[\eta] - \sigma\sqrt{g_{(1)}}\Delta_{[g]}\eta_{(2)} & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ ([\eta],[v]) = (\mathbf{0},\mathbf{0}) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{9.2}$$

Define

$$\begin{aligned} [E](t) := {}& \|[\eta]\|_{3.5}^2 + \|[v]\|_{3.5}^2 + \|\partial_t[v]\|_{2.5}^2 + \left\|\partial_t^2[v]\right\|_{1.5}^2 + \left\|\partial_t^3[v]\right\|_0^2 \\ & + \|(b_0\cdot\partial)[\eta]\|_{3.5}^2 + \|\partial_t(b_0\cdot\partial)[\eta]\|_{2.5}^2 + \left\|\partial_t^2(b_0\cdot\partial)[\eta]\right\|_{1.5}^2 + \left\|\partial_t^3(b_0\cdot\partial)[\eta]\right\|_0^2 \\ & + \left|\overline{\partial}\left(\Pi_{(1)}\partial_t^2[v]\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi_{(1)}\overline{\partial}\partial_t[v]\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi_{(1)}\overline{\partial}^2 v\right)\right|_0^2 + \left|\overline{\partial}(\Pi_{(1)}\overline{\partial}^2(b_0\cdot\partial)\eta)\right|_0^2. \end{aligned} \tag{9.3}$$

Then we can mimic the proof in Section 3 to get the energy estimates of $[E]$

$$[E](T) \lesssim P([E](T), E(T))\int_0^T P([E](t), E(t))\,dt,$$

which together with Gronwall-type inequality yields

$$\exists T\in[0,T'],\quad [E](t) = 0\ \ \forall t\in[0,T]$$

which establishes the local well-posedness of (1.13) in $[0,T]$.

9.2 Regularity of initial data and free surface

Finally, we need to prove that the norms of time derivatives can be controlled by $\|v_0\|_{4.5}, \|b_0\|_{4.5}$ and $|v_0|_5$. This part is exactly the same as in Section 6.1 or [40, Section 7.1]. Finally, the boundary condition of (1.13) gives us an elliptic equation of η on Γ

$$-\sigma\sqrt{g}\Delta_g\eta^\alpha = a^{3\alpha}q.$$

By using elliptic estimates in Dong-Kim [19] (see also [16, Proposition 3.4]), one has

$$|\eta|_5 \lesssim |a^{3\alpha}q|_3 \lesssim |(\overline{\partial}\eta\times\overline{\partial}\eta)q|_3 \lesssim P(\|\eta\|_{4.5})\|q\|_{3.5}.$$

Similarly, taking a time derivative gives us the elliptic equation of v^α

$$\begin{aligned} \sqrt{g}g^{ij}\overline{\partial}{}^2_{ij}v^\alpha = {}& \sqrt{g}g^{ij}\Gamma_{ij}^k\overline{\partial}_k v^\alpha - \partial_t(\sqrt{g}g^{ij})\overline{\partial}{}^2_{ij}\eta^\alpha - \partial_t(\sqrt{g}g^{ij}\Gamma_{ij}^k)\overline{\partial}_k\eta^\alpha \\ & - \sigma^{-1}(\partial_t a^{3\alpha}q + a^{3\alpha}\partial_t q), \end{aligned}$$

and thus by the similar argument in [40, Section 5.1] we get

$$|v(t)|_5 \lesssim P(E(t))\quad \text{in } [0,T].$$

This concludes the proof of Theorem 1.1.

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