

Ch 6 习题：本节假设  $L$ -算子有圆  $\cup \subset \mathbb{R}^n$  为有界开集， $\partial \cup \in C^\infty$ .

[6.1] 考虑带位势  $C$  的 Laplace 方程  $-\Delta u + cu = 0 \quad \dots (*)$   
和散度形式的方程  $-\operatorname{div}(a \nabla v) = 0, \quad a > 0.$

(1) 证明：若  $u$  为  $(*)$  的解， $w > 0$  也是  $(*)$  的解， $v = \frac{u}{w}$  是  $(**)$  的解  $(a = w^2)$

(2). 反之，若  $v$  是  $(**)$  的解， $u = va^{\frac{1}{2}}$  是  $(*)$  的解， $(a = w^2)$  且  $C$ .

证明： (1).  $-\Delta u + cu = 0, \quad -\Delta w + cw = 0$

$$v = \frac{u}{w}$$

$$\Rightarrow \partial_i v = \frac{\partial_i u \cdot w - u \cdot \partial_i w}{w^2} \Rightarrow \operatorname{div} v = \cancel{\partial_i \partial_i u w - u \partial_i \partial_i w} \cancel{w^2}$$

$$\begin{aligned} \operatorname{div}(a \nabla v) &= \sum_{i=1}^n \partial_i (a \partial_i v) \\ &= \sum_{i=1}^n \partial_i a \partial_i v + \sum_{i=1}^n a \partial_i \partial_i v \\ &= \sum_{i=1}^n \underbrace{\partial_i a}_{w^2} \frac{\partial_i u \cdot w - u \partial_i w}{w^2} + \sum_{i=1}^n a \frac{\partial_i(\partial_i u \cdot w - u \partial_i w) w^2 - 2w \partial_i w (\partial_i u \cdot w - u \partial_i w)}{w^4} \end{aligned}$$

$$a = w^2 \Rightarrow a \partial_i v = \partial_i u \cdot w - \partial_i w \cdot u$$

$$\begin{aligned} \operatorname{div}(a \nabla v) &= \sum_{i=1}^n a \partial_i (\partial_i u \cdot w - \partial_i w \cdot u) \\ &= \sum_{i=1}^n (\partial_i \partial_i u \cdot w + \partial_i u \partial_i w - \partial_i w \partial_i u) \quad \text{及 } \partial_i \partial_i w \cdot u. \end{aligned}$$

$$= \Delta u \cdot w \neq \Delta w \cdot u$$

$$= cuw - cwu = 0.$$

(2). 若  $-\operatorname{div}(a \nabla v) = 0$ .

$$\text{则 } \sum_{i=1}^n \partial_i (a \partial_i v) = 0 \Rightarrow \sum_{i=1}^n \partial_i a \cdot \partial_i v + a \sum_{i=1}^n \partial_i \partial_i v = 0$$

$$-\Delta u + cu = -\sum_{i=1}^n \partial_i (\partial_i (va^{\frac{1}{2}})) + cva^{\frac{1}{2}}.$$

$$v \in a^{\frac{1}{2}} V$$

$$= cva^{\frac{1}{2}} - \sum_{i=1}^n \left( \partial_i \left( \partial_i v \cdot a^{\frac{1}{2}} + \frac{1}{2} a^{\frac{1}{2}} \partial_i a \cdot v \right) \right)$$

$$\begin{aligned} &= cva^{\frac{1}{2}} - \sum_{i=1}^n \left( \partial_i \partial_i v \cdot a^{\frac{1}{2}} + \partial_i v \cdot \frac{1}{2} a^{\frac{1}{2}} \partial_i a + \frac{1}{2} a^{\frac{1}{2}} \partial_i a \partial_i v \right. \\ &\quad \left. + \frac{1}{4} (\partial_i \partial_i (a^{\frac{1}{2}})) v \right). \end{aligned}$$

$$= cva^{\frac{1}{2}} - a^{-\frac{1}{2}} \cdot \underbrace{\left( \sum_{i=1}^n \partial_i \partial_i v + a \partial_i \partial_i v \right)}_{-\frac{1}{2} \sum_{i=1}^n (\partial_i \partial_i a^{\frac{1}{2}}) v}.$$

$$= \cancel{cv} a^{\frac{1}{2}} - \cancel{a^{-\frac{1}{2}}} \cancel{v} \left( ca^{\frac{1}{2}} - \Delta \sqrt{a} \right)^0$$

$$\text{即 } c = \frac{\Delta \sqrt{a}}{\sqrt{a}}$$

□

$$[6.2]. \quad \text{设 } Lu = -\sum_{i,j=1}^n a^{ij} \partial_j (a^{ij} \partial_i u) + cu.$$

证明：存在常数  $\mu > 0$ . 使得  $c(x) \geq -\mu (x \in U)$  时有下述结论。由 Milgram 定理得证。

$$\text{证明: } B[u, v] = \sum_{i,j=1}^n \int_U a^{ij} \partial_i u \partial_j v + cuv \quad \forall u, v \in H_0^1(U)$$

$$\begin{aligned} \textcircled{1} \quad |B[u, v]| &\leq \|a^{ij}\|_{L^\infty} \sum_{i,j=1}^n \int_U |\partial_i u| |\partial_j v| + \|c\|_{L^\infty} \int_U |u| |v| dx \\ &\stackrel{\text{H\"older}}{\leq} C \left( \|Du\|_{L^2} \|Dv\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} \right) \\ &\leq C (\|u\|_{H_0^1} \|v\|_{H_0^1}) \end{aligned}$$

$$\textcircled{2} \quad |B[u, u]| = \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u + cu^2$$

一致有界性

$$\geq \theta \|Du\|_{L^2}^2 + \int c u^2.$$

~~由  $H_0^1 \hookrightarrow \text{Poincar\'e 不等式} \|u\|_{L^2} \leq C \|Du\|_{L^2}$  (for some  $C > 0$ )~~

$$= \theta \|Du\|_{L^2}^2 + (C + \mu) \|u\|_{L^2}^2 - (\mu + \varepsilon) \|u\|_{L^2}^2.$$

Poincar\'e 不等式: ~~由  $u \in H_0^1(U)$ . 且  $\exists C > 0$ .  $\|u\|_{L^2} \leq C \|Du\|_{L^2}$~~

$$\rightarrow (\theta - C^2 \mu) \|Du\|_{L^2}^2 + (C + \mu) \|u\|_{L^2}^2.$$

$$\theta \mu + \frac{1}{2} \varepsilon^2 \leq C^2 \mu - C^2 \varepsilon \Rightarrow \varepsilon_0. \Rightarrow \mu \leq \theta - (1 + \frac{1}{C^2}) \varepsilon_0$$

$$\rightarrow \theta \|Du\|_{L^2}^2 = C(C')^2 \|Du\|_{L^2}^2 - \frac{\theta \varepsilon_0}{C'}$$

$$= \theta \|Du\|_{L^2}^2 + \int (C + \mu + \varepsilon) u^2 - (\mu + \varepsilon) \int u^2$$

Poincar\'e: ~~由  $u \in H_0^1(U)$ . 则  $\exists C > 0$ .  $\|u\|_{L^2} \leq C \|Du\|_{L^2}$~~

$$\geq \theta \|Du\|_{L^2}^2 + \int (C + \mu + \varepsilon) u^2 - (\mu + \varepsilon) (C')^2 \|Du\|_{L^2}^2$$

$$\Rightarrow (\mu + \varepsilon)(C')^2 = \frac{\theta}{2} \quad (\varepsilon^2 \text{ 为 } 0) \quad \text{于是}$$

$$\text{上式} \geq \frac{\theta}{2} \|Du\|_{L^2}^2 = \frac{\theta}{4} \|Du\|_{L^2}^2 + \frac{\theta}{4} \|Du\|_{L^2}^2$$

$$\geq \frac{\theta}{4} \|Du\|_{L^2}^2 + \frac{\theta}{4} \frac{1}{C'} \|u\|_{L^2}^2$$

$$\geq \min\left\{\frac{\theta}{4}, \frac{\theta}{4C'}\right\} \|u\|_{H_0^1}^2$$

$$[6.3] \quad u \in H_0^2(U) \text{ 是如下边值问题 } \begin{cases} \Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \quad \text{的弱解, 若 } \int_U \Delta u \Delta v \, dx = \int_U f v \, dx \quad \forall v \in H_0^2(U)$$

今给定  $f \in L^2(U)$ , 证明该方程存在唯一弱解.

$$\text{证明: 全 B}[u, v] = \int_U \Delta u \Delta v \, dx$$

$$(1) |B[u, v]| = \int_U |\Delta u| \cdot |\Delta v| \, dx$$

$$\stackrel{\text{Hölder}}{\leq} C \|D^2 u\|_{L^2} \|D^2 v\|_{L^2}$$

$$\stackrel{u, v \in H_0^2(U)}{\leq} C' \|u\|_{H_0^2} \|v\|_{H_0^2}$$

由 Poincaré 不等式  $\|u\|_2 \leq \|D u\|_2$ . 由  $\|D^2 u\|_2 \neq 0$

$$(2) B[u, u] \stackrel{\text{设 } u \in C_c^\infty(U)}{=} \int_U \frac{\Delta u \Delta u}{|\Delta u| + |\Delta u|} \, dx$$

$$= \sum_{j, k=1}^n \int \partial_j^2 u \partial_k^2 u \, dx.$$

$$\stackrel{\text{分部积分}}{=} - \sum_{j=1}^n \int \partial_j u \cdot \partial_j^2 \partial_k u \, dx.$$

$$\stackrel{\text{再用积}}{=} \sum_{j, k=1}^n \int (\partial_j \partial_k u)^2 \, dx = \|D^2 u\|_{L^2}^2.$$

$$\stackrel{\text{Poincaré}}{\geq} c (\|D^2 u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ = c \|u\|_{H_0^2(U)}^2.$$

~~且  $u \in H_0^2(U)$ .~~ 由  $u \in C_c^\infty(U)$

$$\Rightarrow \|u_n\|_{H_0^2(U)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

对一切  $n \in \mathbb{N}$ ,  $\exists \{u_n\} \subset C_c^\infty(U)$  使  $\|u_n - u\|_{H_0^2(U)} \rightarrow 0$

$$\Rightarrow \|u_n\|_{H_0^2} \rightarrow \|u\|_{H_0^2}.$$

$$|\|\Delta u_n\|_{L^2}^2 - \|\Delta u\|_{L^2}^2| \leq \|\Delta(u_n - u)\|_{L^2}^2 \leq C \|D^2(u_n - u)\|_{L^2}^2 \rightarrow 0$$

$\therefore B[u, u] \geq c \|u\|_{H_0^2}^2$  对  $u \in H_0^2(U)$  成立.

即由(2), 由 Lax-Milgram 定理,  $\exists! u \in H_0^2(U)$

s.t.  $\forall v \in H_0^2(U)$ ,  $B[u, v] = (f, v)_2$ . given  $f \in L^2$ .

□

[6.4] 假设  $u \in H^1(U)$  是 Neumann 边值问题的弱解，是指：

设  $U$  连通

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

$\forall v \in H^1(U)$ , 成立:  $\int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx$ .

现设  $f \in L^2(U)$ . 证明：上述方程弱解存在  $\Leftrightarrow \int_U f \, dx = 0$ .

证明 :  $\Rightarrow$  令  $v = 1$

$$\Leftrightarrow \text{令 } B[u, v] = \int_U \nabla u \cdot \nabla v \, dx$$

$$H_0^1(U) = \{u \in H^1(U) \mid \int_U u \, dx = 0\}$$

Step 1:  $H_0^1(U)$  为 Hilbert 空间, 内积为  $B[\cdot, \cdot]$

实际上,  $\ell: H^1(U) \rightarrow \mathbb{R}$  作为  $H^1(U)$  上的连续线性泛函, 满足:

$$u \mapsto \int_U u \, dx$$

$$H_0^1(U) = \ell^{-1}(0)$$

而  $\{\}$  为  $\mathbb{R}$  闭  $\therefore \ell^{-1}(0)$  闭  $\Rightarrow H_0^1(U)$  为  $H^1(U)$  的闭子空间, 从而是 Hilbert 空间

$B[\cdot, \cdot]$  为内积? check: 双线性易见.

是因 (2):  $B[u, u] = 0 \Leftrightarrow u = 0$  in  $H^1$

实际上, 由  $U$  连通, 据 Poincaré 不等式:

$$\|u - \langle u \rangle_U\|_{L^2} \leq \|\nabla u\|_{L^2} = \sqrt{B[u, u]} = 0$$

$$\langle u \rangle_U = 0 \quad \Rightarrow \quad u = 0. \quad \checkmark$$

Step 2: 由 Riesz 表示定理,  $\forall f \in L^2(U), \int_U f = 0 \exists! u_f \in H_0^1(U)$ .

$$\text{s.t. } \forall v \in H_0^1(U), \int_U \nabla u_f \cdot \nabla v \, dx = B[u_f, v] = (f, v)$$

Step 3:  $\forall v \in H^1(U), v - \langle v \rangle_U \in H_0^1(U)$ . 由 Step 2 知.

给定  $f \in L^2$ .  $\exists! u_f \in H_0^1 \subset H^1$

$$\text{s.t. } (f, v - \langle v \rangle_U) = \int_U \nabla u_f \cdot \nabla (v - \langle v \rangle_U) \, dx = \int_U \nabla u_f \cdot \nabla v \, dx$$

$$\text{而 } \int_U f = 0 \quad \therefore \int f v = \int \nabla u_f \cdot \nabla v \, dx \quad \text{证毕!}$$

□

[6.5]

$$\text{设 } \begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

如何定义该方程的  $H^1(U)$  弱解?

若给定  $f \in L^2(U)$ , 如何证明解的存在唯一性?

证明: 全  $B[u, v] = \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} \operatorname{Tr} u \cdot \operatorname{Tr} v \, d\mathcal{H}^{n-1} \quad \forall u, v \in H^1(U)$

\* 为何如此? 若  $u, v \in C^\infty(U)$ , 则

$$\int -\Delta u \cdot v = \underset{\substack{\uparrow \\ \text{分部积分}}}{\int_U \nabla u \cdot \nabla v \, dx} - \int_{\partial U} v \cdot \nabla u \cdot \vec{\nu} \, d\mathcal{H}^{n-1}.$$

$$= \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \cdot \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}$$

$$-\frac{\partial u}{\partial \nu} = u \text{ on } \partial U$$

$$= \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} u \cdot v \, d\mathcal{H}^{n-1} \text{ 符合我们的式子.}$$

下面 check Lax-Milgram 定理的条件即可.

$$\textcircled{1} |B[u, v]| \leq C \|u\|_{H^1} \|v\|_{H^1} \text{ 显然 ( } \int_{\partial U} \text{ 项用迹定理即可).}$$

$$\textcircled{2} \text{ 下证: } B[u, u] = \int_U \nabla u \cdot \nabla u \, dx + \int_{\partial U} (\operatorname{Tr} u)^2 \, d\mathcal{H}^{n-1} \geq \beta \|u\|_{H^1(U)}^2.$$

若不然, by  $\forall n \in \mathbb{Z}_+$ ,  $\exists u_n \in H^1(U)$  with  $\|u_n\|_{H^1(U)} = 1$ .

$$\text{s.t. } n B[u_n, u_n] < \|u_n\|_{H^1(U)}^2 = 1.$$

$$\Rightarrow B[u_n, u_n] < \frac{1}{n}. \quad \rightarrow \text{由 } H^1(U) \text{ 的保证.}$$

由于  $\{u_n\} \subset H^1(U)$  一致有界. 由 Banach-Alaoglu 定理

exists  $u_n \rightharpoonup \text{some } u \in \overline{H^1(U)}$  in  $H^1(U)$ .

而  $H^1(U) \hookrightarrow L^2(U)$  故  $u_n \rightarrow u$  in  $L^2(U)$ .

这在本节是指  
 $u_n \rightarrow u$  in  $L^2$ .  
 $\nabla u_n \rightarrow \nabla u$  in  $L^2$ .

$$\text{但 } \|\nabla u_n\|_{L^2}^2 \leq B[u_n, u_n] \leq \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\|\nabla u\|_{L^2}^2 \leq \liminf_{k \rightarrow \infty} \|\nabla u_{n_k}\|_{L^2}^2 = 0.$$

$$\underline{u = \text{const.}} \Rightarrow \nabla u = 0 \text{ a.e.}$$

故现在有  $u_n \rightarrow u$  in  $H^1 \Rightarrow \|u\|_{H^1} = 1$ .

而  $\nabla u = 0$  表明  $u$  在  $U$  的每个连通分支中 const.

这是因为  
 $u=0$  on  $\partial U$  用  
 连通分支知

$u \in H_0^1(U)$  用

Poincaré 不等式

$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$

$$\text{但 } \|\operatorname{Tr} u\|_{L^2(\partial U)} \leq \|\operatorname{Tr}(u - u_n)\|_{L^2(\partial U)} + \|\operatorname{Tr} u_n\|_{L^2(\partial U)}$$

$$\leq \|\operatorname{Tr}\| \cdot \|u - u_n\|_{H^1} + \sqrt{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\leq \|\operatorname{Tr}\| \cdot \|u - u_n\|_{H^1} + \sqrt{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

① ② 通过 F. 由  
 Lax-Milgram 定理  
 即可得解.

□

[6.6]. 误U连通.  $\partial U = \Gamma_1 \cup \Gamma_2$ .  $\Gamma_i$  为不交闭集.

请解决如下问题  $\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases}$  的弱解，并讨论唯一性.

Pf. 猜形式：先设  $u, v \in C^\infty(U)$ .

$$-\Delta u = f \Rightarrow \int_U -\Delta u \cdot v \, dx = \int_U f v \, dx$$

左边分部积分可得.

$$\begin{aligned} \int_U f v \, dx &= - \int_{\partial U} \nabla u \cdot \vec{\nu} v \, dx + \int_U \nabla u \cdot \nabla v \, dx \\ &= \int_U \nabla u \cdot \nabla v \, dx - \underbrace{\int_U \frac{\partial u}{\partial \nu} v \, dx}. \end{aligned}$$

$\partial U = \Gamma_1 \cup \Gamma_2$ .  $\Gamma_1$  上:  $u = 0$  故分部积分时边界项消失

$$\Gamma_2 \perp, \frac{\partial u}{\partial \nu} = 0$$

$$\therefore \text{应这样} \quad \int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx \quad \forall u, v \in H^1(U).$$

唯一性与 [6.5] 类似. 跳过.

□.

[6.7].  $u \in H^1(\mathbb{R}^n)$  是且是  $-\Delta u + c(u) = f$  in  $\mathbb{R}^n$  的弱解.  $f \in L^2(\mathbb{R}^n)$ .

$c(u) \in L^2(\mathbb{R}^n)$ .  $c: \mathbb{R} \rightarrow \mathbb{R}$  smooth.  $c(0) = 0$ .  $c'(x) \geq 0$ . 证明:  $\|D^2 u\|_{L^2} \leq C \|f\|_{L^2}$ .

Pf.  $u$  为  $-\Delta u + c(u) = f$  弱解  $\Rightarrow \forall v \in H^1(\mathbb{R}^n)$ .  $\int_U \nabla u \cdot \nabla v + c(u) \cdot v \, dx = \int_U f v \, dx$

令  $v = -D_K^h D_K^h u$   $0 < h < 1$ . 则  $v \in H^1(\mathbb{R}^n)$  且满足

代入有..  $\int_U D_u \cdot (-D_K^h D_K^h u) \, dx - \int_U c(u) D_K^h D_K^h u \, dx = - \int f D_K^h D_K^h u \, dx$ .

由“ $D \leftrightarrow D_K^h$  可交换”与“差商形似分部积分”

$$\Rightarrow \int_U |D_K^h D_K^h u|^2 + D_K^h c(u) \cdot D_K^h u \, dx = - \int f D_K^h D_K^h u \, dx.$$

$$(2) = \frac{c(u(x+h\epsilon_k)) - c(u(x))}{h} D_K^h(u(x)) \stackrel{\text{中值}}{\exists \exists \in \mathbb{R}} c'(\xi) \left( \frac{u(x+h\epsilon_k) - u(x)}{h} \right)^2 \cdot D_K^h u(x)$$

$$\therefore \int_U |D_K^h D_K^h u|^2 \leq - \int f D_K^h D_K^h u \, dx = c'(\xi) |D_K^h u|^2 \geq 0.$$

$$\leq \left| \int f D_K^h D_K^h u \, dx \right|^2 \leq C \|f\|_{L^2}^2 + \varepsilon \|D_K^h D_K^h u\|_{L^2}^2$$

$$\text{取 } \varepsilon < \frac{1}{2} \text{ 即有 } \int_U |D_K^h D_K^h u|^2 \, dx \lesssim \|f\|_{L^2}^2$$

$$\Rightarrow D^2 u \in L^2. \|D^2 u\|_{L^2} \lesssim \|f\|_{L^2}$$

□

注: 此题解法有小问题, 应将函数限制为  $H^1(U)$  中全体在  $\Gamma_1$  上的迹为0的函数, 其范数应该设计为  $u$  的  $H_0^1$  范数 +  $\text{Tr}_{\Gamma_2} u$  的  $L^2$  范数。

[6.8]. 设  $u \in C^\infty(U)$  为  $L_u = -\sum_{i,j} a^{ij}(x) u_{x_i x_j} = 0$  的解.  $a^{ij}$  等数均有界.

$$\text{求证: } \| \nabla u \|_{L^\infty(U)} \leq C (\| \nabla u \|_{L^\infty(\partial U)} + \| u \|_{L^\infty(\partial U)})$$

证明: 令  $v = |\nabla u|^2 + \lambda u^2$ . 若能将入  $\lambda > 0$  选取合适, 使  $L_v \leq 0$ , 则么对

弱极大值原理即得.

直接计算:  $\partial_{x_i x_j} (u^2) = \partial_{x_i} (\partial_x u u_{x_j})$

$$= 2u_{x_i} u_{x_j} + 2u u_{x_i x_j}$$

$$\cdot |\nabla u|^2 = \sum_k 2u_{x_k}^2$$

$$\cdot \partial_{x_j} |\nabla u|^2 = \sum_k 2u_{x_k} u_{x_k x_j}$$

$$\cdot \partial_{x_i} \partial_{x_j} |\nabla u|^2 = 2 \sum_k (u_{x_k x_i} u_{x_k x_j} + u_{x_k} u_{x_k} u_{x_j x_i})$$

$$\Rightarrow L_v = -a^{ij} (|\nabla u|^2)_{x_i x_j} - \underbrace{\lambda a^{ij} (u^2)_{ij}}_{\text{上下指标表示 Einstein 和和.}} = -2 \sum_k a^{ij} (u_{x_k}^2)_{x_i x_j} - 2 \lambda a^{ij} u_{x_i} u_{x_j} - 2 \lambda u \underbrace{a^{ij} u_{x_i x_j}}_{\text{II.}}$$

$$= -2 \sum_{i,j} a^{ij} \left( \sum_k u_{x_k x_i} u_{x_k x_j} + \lambda u_{x_i} u_{x_j} \right) - 2 \sum_{k=1}^n u_{x_k} \sum_{i,j=1}^n a^{ij} u_{x_k x_i x_j}$$

$$= -2 \sum_{i,j} a^{ij} (\nabla u)_{x_i} \cdot (\nabla u)_{x_j} - 2 \lambda \sum_{i,j} a^{ij} u_{x_i} u_{x_j}$$

$$- 2 \sum_{k=1}^n u_{x_k} \cdot \left( \left( \sum_{i,j=1}^n a^{ij} u_{x_i x_j} \right)_{x_k} - \sum_{i,j=1}^n a^{ij} u_{x_k x_i x_j} \right).$$

$L$ -致椭圆

$$\geq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2.$$

$$- 2 \sum_{k=1}^n \sum_{i,j} u_{x_k} a^{ij}_{x_k} u_{x_i x_j}.$$

$$\leq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2 + C \underbrace{\left| \sum_{i,j,k} u_{x_i x_j} u_{x_k} \right|}_{\text{II.}}$$

$$\leq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2 + \frac{C}{\varepsilon} |\nabla^2 u|^2 + \frac{C\varepsilon}{2} |\nabla u|^2.$$

$$\text{取 } \varepsilon = \frac{C}{4\theta}. \text{ 上式 } L_v \leq (-2 \lambda \theta + \frac{C^2}{8\theta}) |\nabla u|^2 \stackrel{\lambda \rightarrow \infty}{\rightarrow} 0 \text{ 即可.}$$

[6.9] 设  $u$  是  $\begin{cases} L_u = -\sum_{i,j=1}^n a^{ij} \partial_{ij} u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$  的光滑解.

$f$  有界

固定  $x^0 \in \partial U$ . 定义  $w \in C^2$  为  $x^0$  处的闭去壁 (barrier). 是指

$$\begin{cases} Lw \geq 1 & \text{in } U \\ w(x^0) = 0 \\ w \geq 0 & \text{on } \partial U \end{cases}$$

证明:  $|\nabla u(x^0)| \leq C \left| \frac{\partial w}{\partial n}(x^0) \right|$

证明: 先对  $w$  用极值原理.  $\min_U w = \min_{\partial U} w = w(x^0)$ .

$$\text{令 } V_1 = u + w \|f\|_{L^\infty}$$

$$LV_1 \geq 0$$

再用弱极值原理知

$$\min_U V_1 = \min_{\partial U} V_1 = \|f\|_{L^\infty} w(x^0)$$

$$V_2 = u - w \|f\|_{L^\infty}$$

$$LV_2 \leq 0$$

$$\max_U V_2 = V_2(x^0).$$

据 Hopf 引理:  $0 \geq \frac{\partial V_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty} \frac{\partial w}{\partial \nu}(x^0)$ .

$$0 \leq \frac{\partial V_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty} \frac{\partial w}{\partial \nu}(x^0).$$

$$\text{而 } u = 0 \text{ on } \partial U \quad \text{且 } \nabla u / \nu \Rightarrow |\nabla u(x^0)| = \left| \frac{\partial u}{\partial \nu}(x^0) \right| \leq \|f\|_{L^\infty} \left| \frac{\partial w}{\partial \nu}(x^0) \right|$$

[6.10]  $U$  连通. 分别用能量法、极值原理

口

证明:  $\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$  的唯一光滑解为  $u = \text{const.}$

证明: (1) 能量法: 寻找能量泛函  $I[w] = \frac{1}{2} \int_U |\nabla w|^2 dx$  的极小化子.

而  $u = \text{const}$  小恰好使  $I$  极小. ( $I = 0$  了都)

$$I = 0 \Rightarrow \nabla w = 0 \Rightarrow w = \text{const}$$

∴ 只有常值解

(2) 极值原理法:

若  $u$  在  $U$  内部达极大值, 由  $U$  连通, 据强极值原理即可

若  $x^0 \in \partial U$  使得  $u(x^0) = \sup_{\bar{U}} u(x)$ .

且  $\forall x \in U, u(x^0) > u(x)$ .

则 Hopf 引理  $\Rightarrow \frac{\partial u}{\partial \nu}(x^0) > 0$ . 矛盾!

口

[6.11] 设  $u \in H^1(U)$  为  $-\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v = 0$  in  $U$  的有界弱解

$$\phi: \mathbb{R} \rightarrow \mathbb{R} \text{ 为 } C^\infty, w = \phi(u)$$

求证:  $\forall v \in H_0^1(U)$  且  $v \geq 0$ , 都有  $B[w, v] \leq 0$

$$\text{证明: } B[u, v] = \int_U \sum_{i,j} a^{ij} \partial_i u \partial_j v \, dx \quad u \in H^1(U), \\ v \in H_0^1(U)$$

由习题 5.17 知  $\phi(u) \in H^1(U)$ .

为了避免不能分部积分的尴尬, 我们先设  $v \geq 0$ ,  $v \in C_c^\infty(U)$ .  
Sobolev 函数

$$B[\phi(u), v] = \int_U \sum_{i,j} a^{ij} \partial_i (\phi(u)) \partial_j v \, dx$$

$$\begin{aligned} &= \int_U \sum_{i,j} \underbrace{\phi'(u) \cdot \partial_i u}_{\phi'(u) \partial_i u} \partial_j v \, dx \\ \text{注意到: } &\phi'(u) \partial_i u = \partial_j(\phi'(u)v) - \phi''(u) \partial_i u \partial_j v. \quad (\text{当 } u \in H^1(U) \text{ 时, Leibniz Rule 成立}) \\ &\Rightarrow \underbrace{\int_U \sum_{i,j} a^{ij} \partial_i u \partial_j (\phi'(u)v)}_{L-\text{有界}} \Big|_0^1 - \int_U \sum_{i,j} a^{ij} \underbrace{\phi''(u) \partial_i u \partial_j v}_{\text{注意原方程弱解意义}} \, dx \\ &\leq - \underbrace{\int_0^1 \phi''(u) |\nabla u|^2 \cdot v \, dx}_{\geq 0 \text{ (因为 } \phi \geq 0 \text{)}} \\ &\leq 0 \end{aligned}$$

□.

$$[6.12] \quad Lu = -\sum_{i,j=1}^n a^{ij} \partial_{ij} u + \sum_{i=1}^n b^i \partial_i u + cu.$$

设  $\exists v \in C^2(\bar{U}) \cap C(\bar{U})$  使  $\begin{cases} Lv \geq 0 \text{ in } U \\ v > 0 \text{ on } \partial U \end{cases}$

求证:  $\forall u \in C^2(\bar{U}) \cap C(\bar{U})$ , 只要  $\begin{cases} Lu \leq 0 \text{ in } U \\ u \leq 0 \text{ on } \partial U \end{cases}$ , 就有  $u \leq 0$  in  $U$ .

证明: 设  $u \in C^2(\bar{U}) \cap C(\bar{U})$ .  $Lu \leq 0$  in  $U$ .  $u \leq 0$  on  $\partial U$ .

令  $w = \frac{u}{v} \in C^2(\bar{U}) \cap C(\bar{U})$ .

如今, 我们希望构造一个有界子集  $M$ , s.t.  $Mw \leq 0$  in  $\{x \in \bar{U} | u > 0\} \subseteq U$ .

若能证此, 则进一步假设  $A = \{x \in \bar{U} | u > 0\} \neq \emptyset$ . 由弱极大值原理

$$0 < \sup_{\bar{A}} w = \sup_{\partial A} w = \frac{0}{v} = 0. \text{ 这不可能. } \text{ 若 } A = \emptyset \Rightarrow u \leq 0 \text{ in } U.$$

先求  $-a^{ij} \partial_{ij} w$ , 以便确定  $M$

$$\begin{aligned} -a^{ij} \partial_{ij} \left( \frac{u}{v} \right) &= -a^{ij} \partial_{ij} \left( \frac{\partial_i u \cdot v - \partial_i v \cdot u}{v^2} \right) = -a^{ij} \left( \partial_{ij} \left( \frac{\partial_i u}{v} \right) - \partial_{ij} \left( \frac{\partial_i v \cdot u}{v^2} \right) \right) \\ &= -a^{ij} \left( \frac{\partial_i \partial_j u \cdot v - \partial_i u \partial_j v}{v^2} - \frac{-2uv \partial_i v \partial_j v + v^2 \partial_i \partial_j v \cdot u + v^2 \partial_j v \cdot \partial_i u}{v^4} \right) \\ &= -\frac{a^{ij} \partial_i u \cdot v + a^{ij} \partial_j v \cdot u}{v^2} + \frac{a^{ij} \partial_i v \cdot \partial_j u - a^{ij} \partial_i u \cdot \partial_j v}{v^2} + a^{ij} \frac{2}{v} \cdot \frac{\cancel{\partial_i v - \partial_j u}}{\cancel{v^2}} \cdot \partial_j v \end{aligned}$$

对 i, j 求和, 上式第 2 项消失.

$$-\sum_{i,j} a^{ij} \partial_{ij} \left( \frac{u}{v} \right) = \frac{(Lu - b^i \partial_i u - cu)v + (-Lv + b^i \partial_i v + cv)u}{v^2} + a^{ij} \frac{2}{v} \partial_j v \partial_i w$$

上、下指标代表求和

$$= \frac{uLu}{v} - \frac{uLv}{v^2} - b^i \partial_i w + \frac{2}{v} a^{ij} \partial_j v \partial_i w.$$

$$\therefore \text{令 } Mw = \sum_{i,j} a^{ij} \partial_{ij} w + b^i \partial_i w \left( b^i - a^{ij} \partial_j v \cdot \frac{2}{v} \right)$$

$$= \frac{Lu}{v} - \frac{uLv}{v^2} \leq 0. \quad \text{on } \{x \in \bar{U} | u > 0\} \subseteq U.$$

$\uparrow_{\text{由 } c \text{ 确定. }} \quad \frac{Lu \leq 0}{v > 0} \quad \frac{u \leq 0}{Lv > 0}$

而  $M$  是一致有界的.

□

[6.13] (柯朗 极大极小原理)

设  $Lu = -\sum_{ij} \theta_j(a^{ij}) a_{ij} u$      $a^{ij} = a^{ji}$     对零边值问题. 设  $L$  有特征值

$$0 < \lambda_1 < \lambda_2 \leq \dots$$

求证:  $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$      $k \in \mathbb{Z}_+$ .

其中  $\sum_{k-1}$  是  $H_0^1(U)$  全体  $(k-1)$  维子空间

证明: 求证  $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$ .

$L^2(U)$  全体  $(k-1)$  维子空间

$\exists A = L^\dagger : L^2 \rightarrow H_0^1(U) \hookrightarrow L^2(U)$      $\forall A \in L^2(U) \rightarrow L^2(U)$  紧致子  
形式上通过  $f \mapsto u \mapsto u$ .

设  $\lambda_k$  对应特征向量  $w_k$ .     $\|w_k\|_2 = 1$ ,  $\langle w_i, w_j \rangle_{L^2} = \delta_{ij}$

$\forall L w_k = \lambda_k w_k \Rightarrow A w_k = \frac{1}{\lambda_k} w_k \therefore A$  的特征值  $\lambda_1^\dagger > \lambda_2^\dagger > \dots > 0$

由 Hilbert-Schmidt 定理.  $A$  关于  $\lambda_k^\dagger$  有特征向量  $e_k$ .     $\|e_k\|_{L^2(U)} = 1$

$\forall f \in L^2(U), f = \sum_i (f, e_i) e_i$      $\{e_k\}_{k \in \mathbb{Z}_+} \rightarrow L^2(U)$  标准正交基  
 $\Rightarrow B[u, u] = \langle Lu, u \rangle = \sum_{i=1}^{\infty} \lambda_i (u, e_i)^2$      $\forall u \in L^2, \|u\|_2 = 1$

①  $\forall S \in \sum_{k-1}, \exists u_k \in \text{Span}\{e_1, \dots, e_k\}$  s.t.  $u_k \perp S$  (by Hilbert-Schmidt thm).  
 $\Rightarrow \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u] \leq B[u_k, u_k] = \sum_{i=1}^k \lambda_i (u, e_i)^2 \leq \lambda_k$ .

②  $\exists S = \text{Span}\{e_1, \dots, e_{k-1}\} \quad \forall u \in S^\perp, \lambda_k = B[e_k, e_k] \leq \sum_{j \geq k} \lambda_j (u, e_j)^2$

①②  $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u] = B[u, u]$

由  $H_0^1(U) \subseteq L^2(U)$ ,  $\lambda_k \geq \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$ .

为证  $\leq$ , 取  $S = \text{Span}\left\{\frac{e_1}{\sqrt{\lambda_1}}, \dots, \frac{e_k}{\sqrt{\lambda_k}}\right\}$ . (由课本 6.5 节知,  $\left\{\frac{e_j}{\sqrt{\lambda_j}}\right\}_{j \in \mathbb{Z}_+} \rightarrow H_0^1(U)$  标准正交基)

从而  $\forall u \in S^\perp$  且  $\|u\|_2 = 1$     设  $u = \sum_{j \geq k} (a_j \sqrt{\lambda_j}) \frac{e_j}{\sqrt{\lambda_j}}$      $\forall j \in \mathbb{Z}_+$

$$\Rightarrow B[u, u] = \sum_{j \geq k} a_j^2 \lambda_j \geq \lambda_k \quad \text{证毕!}$$

□

14.  $\lambda_1$  是如下椭圆算子的特征值.

$$Lu = -\sum_{i,j} a^{ij} \partial_{ij} u + \sum_i b^i \partial_i u + cu.$$

由  $\lambda_1 = \sup_{\substack{u \in C^\infty(\bar{U}), \\ u > 0 \text{ in } U \\ u=0 \text{ on } \partial U}} \inf_{x \in U} \frac{Lu(x)}{u(x)}$

14题前半部分修正:

Pm. 令  $X = \{u \in C^\infty(\bar{U}) : u > 0 \text{ in } U, u|_{\partial U} = 0\}$ . 则由6.5节定理3, 存在  $w_1 \in X$  作为  $L$  关于  $\lambda_1$  的特征向量。注意，这个特征函数不仅仅是  $H_0^1(U)$  函数。事实上，由于Evans第六章习题假设了椭圆算子  $L$  的系数均为光滑函数，且区域  $U$  有界且具有光滑边界，所以可以不

(1) 断使用椭圆正则性定理，直接证得  $w_1 \in C^\infty(\bar{U})$ . 从而  $Lw_1 = \lambda_1 w_1$  在  $U$  中逐点成立。这样，就得到想要的不等式。

$$\inf_{x \in U} \frac{Lw_1}{w_1} = \lambda_1 \leq \sup_{u \in X} \inf_{x \in U} \frac{Lu}{u}.$$

$\lambda_1 \geq \sup_{u \in X} \inf_{x \in U} \frac{Lu}{u}$  的证明仍然同之前的答案。

(2)  $\forall u \in X. \quad \inf_{x \in U} \frac{Lu}{u} \leq \lambda_1.$   
 $\Leftrightarrow \inf_{x \in U} (Lu - \lambda_1 u) \leq 0.$

Consider.  $L^* w_1^* = \lambda_1 w_1^*$ .  $w_1^* > 0 \Rightarrow L^* \text{ 关于 } \lambda_1 \text{ 持征向量}.$

$$\Leftrightarrow (L^* w_1^*, u) = (\lambda_1 w_1^*, u).$$

$$\Leftrightarrow \cancel{(L^* w_1^*, u)} = (\lambda_1 w_1^*, u)$$

$$\Leftrightarrow \langle (Lu - \lambda_1 u, w_1^*) \rangle = 0$$

$$\Leftrightarrow \inf_x (Lu - \lambda_1 u) \leq 0$$

check:  $\lambda_1$  为  $L$  的特征值. ( $\lambda_1^* \neq \lambda_1^*$ )

$$\text{由: } \lambda_1^* (w_1^*, w_1)_{L^2} = \langle L^* w_1^*, w_1 \rangle_{L^2}$$

$$\Rightarrow \lambda_1^* = \lambda_1. \quad \begin{aligned} &= \langle w_1^*, L w_1 \rangle_{L^2} \\ &= \lambda_1 \langle w_1^*, w \rangle_{L^2}. \end{aligned}$$

□

[6.15]  $U(t) \subseteq \mathbb{R}^n$ ,  $\partial U(t)$  速度  $\vec{v}$ . vi. 魏特行值问题 ~~lambda~~

$\downarrow$  关于  $t \in C^\infty$ .

$$\begin{cases} -\Delta w = \lambda w & \text{in } U(t) \\ w = 0 & \text{on } \partial U(t) \end{cases}$$

$$\|w\|_{L^2(U(t))} = 1$$

$$\lambda = \lambda(t) \in C^\infty$$

$$w = w(x, t) \in C^\infty_x$$

利用  $\frac{d}{dt} \int_{U(t)} f dx = \int_{\partial U(t)} f \cdot (\vec{v} \cdot \vec{\nu}) dS + \int_{U(t)} \partial_t f dx$

去证明 Hadamard 变分公式

$$\dot{\lambda} = - \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS$$

证明:  $-\Delta w = \lambda w \Rightarrow - \int_{U(t)} w \cdot \Delta w = \lambda \int_{U(t)} w^2 = \lambda$   
 || 分部积分

$$\Rightarrow \lambda = \int_{U(t)} |\nabla w|^2 dx - \int_{\partial U(t)} \underline{w} \cdot \frac{\partial w}{\partial \vec{\nu}} dS = \int_{U(t)} |\nabla w|^2 dx$$

由  $w=0$  on  $\partial U(t)$  知.  $\nabla w \neq \frac{\partial w}{\partial \vec{\nu}}$

$$\therefore \dot{\lambda} = \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS + \int_{U(t)} \partial_t |\nabla w|^2 dx \quad \dots \textcircled{1}$$

① 第2项 =  $\int_{U(t)} 2 \nabla w \cdot \nabla (\partial_t w) dx.$

$$\stackrel{\text{分部积分}}{=} \int_{U(t)} 2w (-\Delta \partial_t w) dx \stackrel{-\Delta w = \lambda w}{=} \int_{U(t)} 2w \cdot \partial_t w dx$$

$$= \int_{U(t)} \cancel{+2\lambda w \cdot \partial_t w} dx + 2\dot{\lambda} \int_{U(t)} w^2 dx = 2\dot{\lambda}$$

$$= 2\dot{\lambda} + 2\lambda \int_{U(t)} \partial_t w^2 dx$$

$$= 2\dot{\lambda} + 2\lambda \left( \underbrace{\frac{d}{dt} \int_{U(t)} w^2 dx}_{|| \text{ 若 } \dot{\lambda} = 0} - \int_{\partial U(t)} \underline{w}^2 (\vec{v} \cdot \vec{\nu}) dS \right)$$

$$= 2\dot{\lambda}$$

于是  $\textcircled{1} \Rightarrow \dot{\lambda} = - \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS.$

□