1.(30 points) Consider two fields, $K = \mathbb{Q}(\sqrt{2}+\sqrt{6}), E = \mathbb{Q}(\sqrt{2}+i)$.

(1)Calculate $\dim_{\mathbb{Q}} K, \dim_{\mathbb{Q}} E, \dim_{\mathbb{Q}}(K \cap E)$.

(2)Calculate $\dim_{\mathbb{Q}}(KE)$ where KE is the smallest field containing K, E.

(3)Determine $\operatorname{Aut} K, \operatorname{Aut} E$. Prove or disprove $\operatorname{Aut} K \simeq \operatorname{Aut} E$. Prove or disprove $K \simeq E$.

2.(50 points) Consider Laurent polynomial ring over $\mathbb{C}$,

$$\mathbb{C}[x, x^{-1}] = \{\sum_{i \in I} a_i x^i | a_i \in \mathbb{C}, I \text{ is a finite sub set of } \mathbb{Z}.\}$$

(1)Prove or disprove: $\mathbb{C}[x, x^{-1}]$ is a PID.

(2)List all prime ideals of $\mathbb{C}[x, x^{-1}]$.

(3)As a $\mathbb{C}[x]-$module, prove or disprove: $\mathbb{C}[x, x^{-1}]$ is finitely generated. $\mathbb{C}[x, x^{-1}]$ is indecomposable.

(4)Prove or disprove: $\mathbb{Z}[x, x^{-1}]$ is a PID.

(5)Prove or disprove: $\mathbb{Z}[x, x^{-1}]/(x^2+3, 5)$ is finite ring. $\mathbb{Z}[x, x^{-1}]/(x^2+3, 5)$ is finite field.

3.(15 points) $a \in \mathbb{Z}$, $G = < \begin{bmatrix} -1 & 0 \\ a & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} >$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(1)Find all possible a such that G is Abelian.

(2)Find all possible a such that G is Finite.

(3)For finite G, determine $G \cap \mathrm{SL}_2(\mathbb{R})$.

4.(5 points)Given a matrix ring homomorphsim $\phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ such that $\phi(\lambda E_n) = \lambda E_m$ for all $\lambda \in \mathbb{R}$. Prove $n|m$.