1.(ring in this question is commutative)

R is a unital ring, $M_n(R)$ is a matrix ring over R.

(1)given a unital ring S, prove there is a ring isomorphism:

$$M_n(R) \times M_n(S) \cong M_n(R \times S)$$

(2)suppose R is a local ring ,of which m is the maximal ideal.given $x \in m$,prove: $1 + x$ is invertible in R

(3)prove:R is local $\iff$ $M_n(R)$ is local.

(4)suppose R is a local ring and S is a unital ring, prove:

$$M_n(R) \cong M_n(S) \iff R \cong S$$

2. $q = p^r$,p is prime. $\mathbb{F}_q$ is a finite field of order q.

(1)calculate the order of group $SL_n(\mathbb{F}_q)$.

(2)find a sylow-p subgroup of $SL_n(\mathbb{F}_q)$

(3)given a group G s.t. $|G| = p^m, m \in \mathbb{Z}^+$,and a homomorphism: $\rho : G \to GL_n.(\mathbb{F}_q)$

prove:there exists an invertible matrix P, $\forall X \in Im\rho$, PXP^{-1} is a upper triangular matrix.

3.for $N \in \mathbb{Z}^+, N \geq 2$,

$$F[N] = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1, a \equiv d \equiv 1, b \equiv c \equiv 0(mod\ N) \}$$

(1)prove that homomorphism $\rho : \mathbb{Z} \to \frac{\mathbb{Z}}{2\mathbb{Z}} : a \mapsto a + 2\mathbb{Z}$ induces a surjective ring homomorphism:

$$SL_2(\mathbb{Z}) \to SL_2(\frac{\mathbb{Z}}{2\mathbb{Z}})$$

(2)prove $F[2]$ is a normal subgroup of $SL_2(\mathbb{Z})$

(3)calculate:$[SL_2(\mathbb{Z}) : F[N]]$

4. (1) is $\mathbb{Z}[\sqrt{-2}]$ a UFD?

(2) show all integer x, y s.t. $x^2 = y^3 - 2$

5.consider abelian group homomorphism f, g:

$$N \xrightarrow{f} M \xrightarrow{g} L$$

where $Im\ g, Im\ f, Ker\ g, Ker\ f$ is finitely generated.prove:

(1) $Ker(g \circ f)$,$Im(g \circ f)$ are finitely generated abelian groups.

(2) suppose N, L are cyclic and $|N| = 27, |L| = 18, Ker\ g = Im\ f$,calculate invariant factors of M.

6.

(1)show that :$S_n = <(12), (12 \ldots n)>$

(2)is $f(x) = x^5 - 4x + 2$ reducible over $\mathbb{Q}$?

(3)calculate Galois group of $f(x)$