

Ch 6 习题：本节假设 L -算子有圆 $\cup \subset \mathbb{R}^n$ 为有界开集， $\partial \cup \in C^\infty$.

[6.1] 考虑带位势 C 的 Laplace 方程 $-\Delta u + cu = 0 \quad \dots (*)$
和散度形式的方程 $-\operatorname{div}(a \nabla v) = 0, \quad a > 0.$

(1) 证明：若 u 为 $(*)$ 的解， $w > 0$ 也是 $(*)$ 的解， $v = \frac{u}{w}$ 是 $(**)$ 的解 $(a = w^2)$

(2). 反之，若 v 是 $(**)$ 的解， $u = va^{\frac{1}{2}}$ 是 $(*)$ 的解， $(\forall \text{ 常数 } C)$.

证明： (1). $-\Delta u + cu = 0, \quad -\Delta w + cw = 0$

$$v = \frac{u}{w}$$

$$\Rightarrow \partial_i v = \frac{\partial_i u \cdot w - u \cdot \partial_i w}{w^2} \Rightarrow \operatorname{div} v = \cancel{\partial_i \partial_i u w - u \partial_i \partial_i w} \cancel{w^2}$$

$$\begin{aligned} -\operatorname{div}(a \nabla v) &= \sum_{i=1}^n \partial_i (a \partial_i v) \\ &= \sum_{i=1}^n \partial_i a \partial_i v + \sum_{i=1}^n a \cdot \partial_i \partial_i v \\ &= \sum_{i=1}^n \underbrace{\partial_i a \cdot \frac{\partial_i u \cdot w - u \partial_i w}{w^2}}_{w^2} + \sum_{i=1}^n a \cdot \frac{\partial_i(\partial_i u \cdot w - u \partial_i w)}{w^4} - 2w \partial_i w (\partial_i u \cdot w - u \partial_i w) \end{aligned}$$

$$a = w^2 \Rightarrow a \partial_i v = \partial_i u \cdot w - \partial_i w \cdot u$$

$$\begin{aligned} \operatorname{div}(a \nabla v) &= \sum_{i=1}^n a \partial_i (\partial_i u \cdot w - \partial_i w \cdot u) \\ &= \sum_{i=1}^n (\partial_i \partial_i u \cdot w + \partial_i u \partial_i w - \partial_i w \partial_i u - \cancel{\partial_i \partial_i w \cdot u}). \end{aligned}$$

$$= \Delta u \cdot w - \Delta w \cdot u$$

$$= cuw - cwu = 0.$$

(2). 若 $-\operatorname{div}(a \nabla v) = 0$.

$$\text{则 } \sum_{i=1}^n \partial_i (a \partial_i v) = 0 \Rightarrow \sum_{i=1}^n \partial_i a \cdot \partial_i v + a \sum_{i=1}^n \partial_i \partial_i v = 0$$

$$-\Delta u + cu = -\sum_{i=1}^n \partial_i (\partial_i (va^{\frac{1}{2}})) + cva^{\frac{1}{2}}.$$

$$v \in a^{\frac{1}{2}} V$$

$$= cva^{\frac{1}{2}} - \sum_{i=1}^n \left(\partial_i \left(\partial_i v \cdot a^{\frac{1}{2}} + \frac{1}{2} a^{\frac{1}{2}} \partial_i a \cdot v \right) \right)$$

$$\begin{aligned} &= cva^{\frac{1}{2}} - \sum_{i=1}^n \left(\partial_i \partial_i v \cdot a^{\frac{1}{2}} + \partial_i v \cdot \frac{1}{2} a^{\frac{1}{2}} \partial_i a + \frac{1}{2} a^{\frac{1}{2}} \partial_i a \partial_i v \right. \\ &\quad \left. + \cancel{\frac{1}{2} (\partial_i \partial_i (a^{\frac{1}{2}})) v} \right) \end{aligned}$$

$$= cva^{\frac{1}{2}} - a^{-\frac{1}{2}} \cdot \underbrace{\left(\sum_{i=1}^n \partial_i \partial_i v + a \partial_i \partial_i v \right)}_{-\frac{1}{2} \sum_{i=1}^n (\partial_i \partial_i a^{\frac{1}{2}}) v}.$$

$$= \cancel{cv} a^{\frac{1}{2}} - \cancel{a^{-\frac{1}{2}}} \underbrace{v (ca^{\frac{1}{2}} - \Delta \sqrt{a})}_0$$

$$\text{即 } c = \frac{\Delta \sqrt{a}}{\sqrt{a}}$$

□

$$[6.2]. \quad \text{设 } Lu = -\sum_{i,j=1}^n a^{ij} \partial_j (a^{ij} \partial_i u) + cu.$$

证明：存在常数 $\mu > 0$. 使得 $c(x) \geq -\mu (x \in U)$ 时有下述结论。由 Milgram 定理得证。

$$\text{证明: } B[u, v] = \sum_{i,j=1}^n \int_U a^{ij} \partial_i u \partial_j v + cuv \quad \forall u, v \in H_0^1(U)$$

$$\begin{aligned} \textcircled{1} \quad |B[u, v]| &\leq \|a^{ij}\|_{L^\infty} \sum_{i,j=1}^n \int_U |\partial_i u| |\partial_j v| + \|c\|_{L^\infty} \int_U |u| |v| dx \\ &\stackrel{\text{H\"older}}{\leq} C \left(\|Du\|_{L^2} \|Dv\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} \right) \\ &\leq C (\|u\|_{H_0^1} \|v\|_{H_0^1}) \end{aligned}$$

$$\textcircled{2} \quad |B[u, u]| = \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u + cu^2$$

一致有界性

$$\geq \theta \|Du\|_{L^2}^2 + \int c u^2.$$

~~由 $H_0^1 \hookrightarrow \text{Poincar\'e 不等式} \|u\|_{L^2} \leq C \|Du\|_{L^2}$ (for some $C > 0$)~~

$$= \theta \|Du\|_{L^2}^2 + (C + \mu) \|u\|_{L^2}^2 - (\mu + \varepsilon) \|u\|_{L^2}^2.$$

Poincar\'e 不等式: ~~由 $u \in H_0^1(U)$. 且 $\exists C > 0$. $\|u\|_{L^2} \leq C \|Du\|_{L^2}$~~

$$\rightarrow (\theta - C^2 \mu) \|Du\|_{L^2}^2 + (C + \mu) \|u\|_{L^2}^2.$$

$$\theta \mu + \frac{1}{2} \varepsilon^2 \leq C^2 \mu - C^2 \varepsilon \Rightarrow \varepsilon_0. \Rightarrow \mu \leq \theta - (1 + \frac{1}{C^2}) \varepsilon_0$$

$$\rightarrow \theta \|Du\|_{L^2}^2 = C(C')^2 \|Du\|_{L^2}^2 - \frac{\theta \varepsilon_0}{C'}$$

$$= \theta \|Du\|_{L^2}^2 + \int (C + \mu + \varepsilon) u^2 - (\mu + \varepsilon) \int u^2$$

Poincar\'e: ~~由 $u \in H_0^1(U)$. 则 $\exists C > 0$. $\|u\|_{L^2} \leq C \|Du\|_{L^2}$~~

$$\geq \theta \|Du\|_{L^2}^2 + \int (C + \mu + \varepsilon) u^2 - (\mu + \varepsilon) (C')^2 \|Du\|_{L^2}^2$$

$$\Rightarrow (\mu + \varepsilon)(C')^2 = \frac{\theta}{2} \quad (\varepsilon^2 \text{ 为 } 0) \quad \text{于是}$$

$$\text{上式} \geq \frac{\theta}{2} \|Du\|_{L^2}^2 = \frac{\theta}{4} \|Du\|_{L^2}^2 + \frac{\theta}{4} \|Du\|_{L^2}^2$$

$$\geq \frac{\theta}{4} \|Du\|_{L^2}^2 + \frac{\theta}{4} \frac{1}{C'} \|u\|_{L^2}^2$$

$$\geq \min\left\{\frac{\theta}{4}, \frac{\theta}{4C'}\right\} \|u\|_{H_0^1}^2$$

$$[6.3] \quad u \in H_0^2(U) \text{ 是如下边值问题 } \begin{cases} \Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \quad \text{的弱解, 若 } \int_U \Delta u \Delta v \, dx = \int_U f v \, dx \quad \forall v \in H_0^2(U)$$

今给定 $f \in L^2(U)$, 证明该方程存在唯一弱解.

$$\text{证明: 全 B}[u, v] = \int_U \Delta u \Delta v \, dx$$

$$(1) |B[u, v]| = \int_U |\Delta u| \cdot |\Delta v| \, dx$$

$$\stackrel{\text{Hölder}}{\leq} C \|D^2 u\|_{L^2} \|D^2 v\|_{L^2}$$

$$\stackrel{u, v \in H_0^2(U)}{\leq} C' \|u\|_{H_0^2} \|v\|_{H_0^2}$$

由 Poincaré 不等式 $\|u\|_2 \leq \|D u\|_2$. 由 $\|D^2 u\|_2 \neq 0$

$$(2) B[u, u] \stackrel{\text{设 } u \in C_c^\infty(U)}{=} \int_U \frac{\Delta u \Delta u}{|\Delta u| + |\Delta u|} \, dx$$

$$= \sum_{j, k=1}^n \int \partial_j^2 u \partial_k^2 u \, dx.$$

$$\stackrel{\text{分部积分}}{=} - \sum_{j=1}^n \int \partial_j u \cdot \partial_j^2 \partial_k u \, dx.$$

$$\stackrel{\text{再用积}}{=} \sum_{j, k=1}^n \int (\partial_j \partial_k u)^2 \, dx = \|D^2 u\|_{L^2}^2.$$

$$\stackrel{\text{Poincaré}}{\geq} c (\|D^2 u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ = c \|u\|_{H_0^2(U)}^2.$$

~~且 $u \in H_0^2(U)$.~~ 由 $u \in C_c^\infty(U)$

$$\Rightarrow \|u_n\|_{H_0^2(U)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

对一切 $n \in \mathbb{N}$, $\exists \{u_n\} \subset C_c^\infty(U)$ 使 $\|u_n - u\|_{H_0^2(U)} \rightarrow 0$

$$\Rightarrow \|u_n\|_{H_0^2} \rightarrow \|u\|_{H_0^2}.$$

$$|\|\Delta u_n\|_{L^2}^2 - \|\Delta u\|_{L^2}^2| \leq \|\Delta(u_n - u)\|_{L^2}^2 \leq C \|D^2(u_n - u)\|_{L^2}^2 \rightarrow 0$$

$\therefore B[u, u] \geq c \|u\|_{H_0^2}^2$ 对 $u \in H_0^2(U)$ 成立.

即由(2), 由 Lax-Milgram 定理, $\exists! u \in H_0^2(U)$

s.t. $\forall v \in H_0^2(U)$, $B[u, v] = (f, v)_2$. given $f \in L^2$. \square

[6.4] 假设 $u \in H^1(U)$ 是 Neumann 边值问题的弱解，是指：

设 U 连通

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

$\forall v \in H^1(U)$, 成立: $\int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx$.

现设 $f \in L^2(U)$. 证明：上述方程弱解存在 $\Leftrightarrow \int_U f \, dx = 0$.

证明 : \Rightarrow 令 $v = 1$

$$\Leftrightarrow \text{令 } B[u, v] = \int_U \nabla u \cdot \nabla v \, dx$$

$$H_0^1(U) = \{u \in H^1(U) \mid \int_U u \, dx = 0\}$$

Step 1: $H_0^1(U)$ 为 Hilbert 空间, 内积为 $B[\cdot, \cdot]$

实际上, $\ell: H^1(U) \rightarrow \mathbb{R}$ 作为 $H^1(U)$ 上的连续线性泛函, 满足:

$$u \mapsto \int_U u \, dx$$

$$H_0^1(U) = \ell^{-1}(0)$$

而 $\{\}$ 为 \mathbb{R} 闭 $\therefore \ell^{-1}(0)$ 闭 $\Rightarrow H_0^1(U)$ 为 $H^1(U)$ 的闭子空间, 从而是 Hilbert 空间

$B[\cdot, \cdot]$ 为内积? check: 双线性易见.

是因 (2): $B[u, u] = 0 \Leftrightarrow u = 0$ in H^1

实际上, 由 U 连通, 据 Poincaré 不等式:

$$\|u - \langle u \rangle_U\|_{L^2} \leq \|\nabla u\|_{L^2} = \sqrt{B[u, u]} = 0$$

$$\langle u \rangle_U = 0 \quad \Rightarrow \quad u = 0. \quad \checkmark$$

Step 2: 由 Riesz 表示定理, $\forall f \in L^2(U), \int_U f = 0 \exists! u_f \in H_0^1(U)$.

$$\text{s.t. } \forall v \in H_0^1(U), \int_U \nabla u_f \cdot \nabla v \, dx = B[u_f, v] = (f, v)$$

Step 3: $\forall v \in H^1(U), v - \langle v \rangle_U \in H_0^1(U)$. 由 Step 2 知.

给定 $f \in L^2$. $\exists! u_f \in H_0^1 \subset H^1$

$$\text{s.t. } (f, v - \langle v \rangle_U) = \int_U \nabla u_f \cdot \nabla (v - \langle v \rangle_U) \, dx = \int_U \nabla u_f \cdot \nabla v \, dx$$

$$\text{而 } \int_U f = 0 \quad \therefore \int f v = \int \nabla u_f \cdot \nabla v \, dx \quad \text{证毕!}$$

□

[6.5]

$$\text{设 } \begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

如何定义该方程的 $H^1(U)$ 弱解?

若给定 $f \in L^2(U)$, 如何证明解的存在唯一性?

证明: 全 $B[u, v] = \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} \operatorname{Tr} u \cdot \operatorname{Tr} v \, d\mathcal{H}^{n-1} \quad \forall u, v \in H^1(U)$

* 为何如此? 若 $u, v \in C^\infty(U)$, 则

$$\int -\Delta u \cdot v = \underset{\substack{\uparrow \\ \text{分部积分}}}{\int_U \nabla u \cdot \nabla v \, dx} - \int_{\partial U} v \cdot \nabla u \cdot \vec{\nu} \, d\mathcal{H}^{n-1}.$$

$$= \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \cdot \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}$$

$$-\frac{\partial u}{\partial \nu} = u \text{ on } \partial U$$

$$= \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} u \cdot v \, d\mathcal{H}^{n-1} \text{ 符合我们的式子.}$$

下面 check Lax-Milgram 定理的条件即可.

$$\textcircled{1} |B[u, v]| \leq C \|u\|_{H^1} \|v\|_{H^1} \text{ 显然 (} \int_{\partial U} \text{ 项用迹定理即可).}$$

$$\textcircled{2} \text{ 下证: } B[u, u] = \int_U \nabla u \cdot \nabla u \, dx + \int_{\partial U} (\operatorname{Tr} u)^2 \, d\mathcal{H}^{n-1} \geq \beta \|u\|_{H^1(U)}^2.$$

若不然, by $\forall n \in \mathbb{Z}_+$, $\exists u_n \in H^1(U)$ with $\|u_n\|_{H^1(U)} = 1$.

$$\text{s.t. } n B[u_n, u_n] < \|u_n\|_{H^1(U)}^2 = 1.$$

$$\Rightarrow B[u_n, u_n] < \frac{1}{n}. \quad \rightarrow \text{由 } H^1(U) \text{ 的保证.}$$

由于 $\{u_n\} \subset H^1(U)$ 一致有界. 由 Banach-Alaoglu 定理

exists $u_n \rightharpoonup \text{some } u \in \overline{H^1(U)}$ in $H^1(U)$.

而 $H^1(U) \hookrightarrow L^2(U)$ 故 $u_n \rightarrow u$ in $L^2(U)$.

这在本节上是指
 $u_n \rightarrow u$ in L^2 .
 $\nabla u_n \rightarrow \nabla u$ in L^2 .

$$\text{但 } \|\nabla u_n\|_{L^2}^2 \leq B[u_n, u_n] \leq \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\|\nabla u\|_{L^2}^2 \leq \liminf_{k \rightarrow \infty} \|\nabla u_{n_k}\|_{L^2}^2 = 0.$$

$$\underline{u = \text{const.}} \Rightarrow \nabla u = 0 \text{ a.e.}$$

故现在有 $u_n \rightarrow u$ in $H^1 \Rightarrow \|u\|_{H^1} = 1$.

而 $\nabla u = 0$ 表明 u 在 U 的每个连通分支中 const.

这是因为
 $u = 0$ on ∂U 用
 连通分支知

$u \in H_0^1(U)$ 用

Poincaré 不等式

$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} = 0$

$$\text{但 } \|\operatorname{Tr} u\|_{L^2(\partial U)} \leq \|\operatorname{Tr}(u - u_n)\|_{L^2(\partial U)} + \|\operatorname{Tr} u_n\|_{L^2(\partial U)}$$

$$\leq \|\operatorname{Tr}\| \cdot \|u - u_n\|_{H^1} + \sqrt{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\leq \|\operatorname{Tr}\| \cdot \|u - u_n\|_{H^1} + \sqrt{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

① ② 通过 F. 由
 Lax-Milgram 定理
 即可得解.

□

[6.6]. 误U连通. $\partial U = P_1 \cup P_2$. P_i 为不交闭集.

请解决如下问题 $\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } P_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } P_2 \end{cases}$ 的弱解, 并讨论唯一性.

Pf.: 猜形式: 先设 $u, v \in C^\infty(U)$.

$$-\Delta u = f \Rightarrow \int_U -\Delta u \cdot v \, dx = \int_U f v \, dx$$

左边分部积分可得.

$$\begin{aligned} \int_U f v \, dx &= - \int_{\partial U} \nabla u \cdot \vec{\nu} v \, dx + \int_U \nabla u \cdot \nabla v \, dx \\ &= \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dx. \end{aligned}$$

$\partial U = P_1 \cup P_2$. P_1 上: $u = 0$ 故分部积分时边界项消失

$$P_2 \perp, \frac{\partial u}{\partial \nu} = 0$$

$$\therefore \text{应这样} \int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx \quad \forall u, v \in H^1(U).$$

唯一性与 [6.5] 类似. 跳过.

□.

[6.7]. $u \in H^1(\mathbb{R}^n)$ 是. 且是 $-\Delta u + c(u) = f$ in \mathbb{R}^n 的弱解. $f \in L^2(\mathbb{R}^n)$.

$c(u) \in L^2(\mathbb{R}^n)$. $c: \mathbb{R} \rightarrow \mathbb{R}$ smooth. $c(0) = 0$. $c'(x) \geq 0$. 证明: $\|D^2 u\|_{L^2} \leq C \|f\|_{L^2}$.

Pf.: u 为 $-\Delta u + c(u) = f$ 弱解 $\Rightarrow \forall v \in H^1(\mathbb{R}^n)$. $\int_U \nabla u \cdot \nabla v + c(u) \cdot v \, dx = \int_U f v \, dx$

令 $v = -D_K^h D_K^h u$ $0 < h < 1$. $\exists \forall v \in H^1(\mathbb{R}^n)$ 且满足.

代入有.. $\int_U D_u \cdot (-D_K^h D_K^h u) \, dx - \int_U c(u) D_K^h D_K^h u \, dx = - \int f D_K^h D_K^h u \, dx$.

由“ $D \leftrightarrow D_K^h$ 可交换”与“差商形式的分部积分”

$$\Rightarrow \int_U |D_K^h D_K^h u|^2 + D_K^h c(u) \cdot D_K^h u \, dx = - \int f D_K^h D_K^h u \, dx.$$

$$(2) = \frac{c(u(x+h\epsilon)) - c(u(x))}{h} D_K^h(u(x)) \stackrel{\text{中值}}{\underset{\exists \beta \in \mathbb{R}}{=}} c'(\beta) \left(\frac{u(x+h\epsilon) - u(x)}{h} \right)^2 \cdot D_K^h(u(x))$$

$$\therefore \int_U |D_K^h D_K^h u|^2 \leq - \int f D_K^h D_K^h u \, dx = c'(\beta) |D_K^h u|^2 \geq 0.$$

$$\leq \left| \int f D_K^h D_K^h u \, dx \right|^2 \leq C \|f\|_{L^2}^2 + \varepsilon \|D_K^h D_K^h u\|_{L^2}^2$$

$$\text{取 } \varepsilon < \frac{1}{2} \text{ 即有 } \int_U |D_K^h D_K^h u|^2 \, dx \lesssim \|f\|_{L^2}^2$$

$$\Rightarrow D^2 u \in L^2. \|D^2 u\|_{L^2} \lesssim \|f\|_{L^2}$$

□

[6.8]. 设 $u \in C^\infty(U)$ 为 $L_u = -\sum_{i,j} a^{ij}(x) u_{x_i x_j} = 0$ 的解. a^{ij} 等数均有界.

$$\text{求证: } \| \nabla u \|_{L^\infty(U)} \leq C (\| \nabla u \|_{L^\infty(\partial U)} + \| u \|_{L^\infty(\partial U)})$$

证明: 令 $v = |\nabla u|^2 + \lambda u^2$. 若能将入 $\lambda > 0$ 选取合适, 使 $L_v \leq 0$, 则么对

弱极大值原理即得.

直接计算: $\partial_{x_i x_j} (u^2) = \partial_{x_i} (\partial_x u u_{x_j})$

$$= 2u_{x_i} u_{x_j} + 2u u_{x_i x_j}$$

$$\cdot |\nabla u|^2 = \sum_k 2u_{x_k}^2$$

$$\cdot \partial_{x_j} |\nabla u|^2 = \sum_k 2u_{x_k} u_{x_k x_j}$$

$$\cdot \partial_{x_i} \partial_{x_j} |\nabla u|^2 = 2 \sum_k (u_{x_k x_i} u_{x_k x_j} + u_{x_k} u_{x_k} u_{x_j x_i})$$

$$\Rightarrow L_v = -a^{ij} (|\nabla u|^2)_{x_i x_j} - \underbrace{\lambda a^{ij} (u^2)_{ij}}_{\text{上下指标表示 Einstein 和和.}} = -2 \sum_k a^{ij} (u_{x_k}^2)_{x_i x_j} - 2 \lambda a^{ij} u_{x_i} u_{x_j} - 2 \lambda u \underbrace{a^{ij} u_{x_i x_j}}_{\text{II.}}$$

$$= -2 \sum_{i,j} a^{ij} \left(\sum_k u_{x_k x_i} u_{x_k x_j} + \lambda u_{x_i} u_{x_j} \right) - 2 \sum_{k=1}^n u_{x_k} \sum_{i,j=1}^n a^{ij} u_{x_k x_i x_j}$$

$$= -2 \sum_{i,j} a^{ij} (\nabla u)_{x_i} \cdot (\nabla u)_{x_j} - 2 \lambda \sum_{i,j} a^{ij} u_{x_i} u_{x_j}$$

$$- 2 \sum_{k=1}^n u_{x_k} \cdot \left(\left(\sum_{i,j=1}^n a^{ij} u_{x_i x_j} \right)_{x_k} - \sum_{i,j=1}^n a^{ij} u_{x_k x_i x_j} \right).$$

L -致椭圆

$$\geq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2.$$

$$- 2 \sum_{k=1}^n \sum_{i,j} u_{x_k} a^{ij}_{x_k} u_{x_i x_j}.$$

$$\leq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2 + C \underbrace{\left| \sum_{i,j,k} u_{x_i x_j} u_{x_k} \right|}_{\text{II.}}$$

$$\leq -2 \theta \sum_{k=1}^n |\nabla u_{x_k}|^2 - 2 \lambda \theta |\nabla u|^2 + \frac{C}{\varepsilon} |\nabla^2 u|^2 + \frac{C\varepsilon}{2} |\nabla u|^2.$$

$$\text{取 } \varepsilon = \frac{C}{4\theta}. \text{ 上式 } L_v \leq (-2 \lambda \theta + \frac{C^2}{8\theta}) |\nabla u|^2 \stackrel{\lambda \rightarrow \infty}{\rightarrow} 0 \text{ 即可.}$$

[6.9] 设 u 是 $\begin{cases} L_u = -\sum_{i,j=1}^n a^{ij} \partial_{ij} u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的光滑解.

f 有界

固定 $x^0 \in \partial U$. 定 $w \in C^2$ 为 x^0 处的闭去壁 (barrier). 是指

$$\begin{cases} Lw \geq 1 & \text{in } U \\ w(x^0) = 0 \\ w \geq 0 & \text{on } \partial U \end{cases}$$

证明: $|\nabla u(x^0)| \leq C \left| \frac{\partial w}{\partial n}(x^0) \right|$

证明: 先对 w 用极值原理. $\min_U w = \min_{\partial U} w = w(x^0)$.

$$\text{令 } V_1 = u + w \|f\|_{L^\infty}$$

$$LV_1 \geq 0$$

再用弱极值原理知

$$\min_U V_1 = \min_{\partial U} V_1 = \|f\|_{L^\infty} w(x^0)$$

$$V_2 = u - w \|f\|_{L^\infty}$$

$$LV_2 \leq 0$$

$$\max_U V_2 = V_2(x^0).$$

据 Hopf 引理: $0 \geq \frac{\partial V_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty} \frac{\partial w}{\partial \nu}(x^0)$.

$$0 \leq \frac{\partial V_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty} \frac{\partial w}{\partial \nu}(x^0).$$

$$\text{而 } u = 0 \text{ on } \partial U \quad \text{且 } \nabla u / \nu \Rightarrow |\nabla u(x^0)| = \left| \frac{\partial u}{\partial \nu}(x^0) \right| \leq \|f\|_{L^\infty} \left| \frac{\partial w}{\partial \nu}(x^0) \right|$$

[6.10] U 连通. 分别用能量法、极值原理

口

证明: $\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$ 的唯一光滑解为 $u = \text{const.}$

证明: (1) 能量法: 寻找能量泛函 $I[w] = \frac{1}{2} \int_U |\nabla w|^2 dx$ 的极小化子.

而 $u = \text{const}$ 小恰好使 I 极小. ($I = 0$ 了都)

$$I = 0 \Rightarrow \nabla w = 0 \Rightarrow w = \text{const}$$

∴ 只有常值解

(2) 极值原理法:

若 u 在 U 内部达极值, 由 U 连通, 据强极值原理即可

若 $x^0 \in \partial U$ 使得 $u(x^0) = \sup_{\bar{U}} u(x)$.

且 $\forall x \in U, u(x^0) > u(x)$.

则 Hopf 引理 $\Rightarrow \frac{\partial u}{\partial \nu}(x^0) > 0$. 矛盾!

口.

[6.11] 设 $u \in H^1(U)$ 为 $-\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v = 0$ in U 的有界弱解

$$\phi: \mathbb{R} \rightarrow \mathbb{R} \text{ 为 } C^\infty, w = \phi(u)$$

求证: $\forall v \in H_0^1(U)$ 且 $v \geq 0$, 都有 $B[w, v] \leq 0$

$$\text{证明: } B[u, v] = \int_U \sum_{i,j} a^{ij} \partial_i u \partial_j v \, dx \quad u \in H^1(U), \\ v \in H_0^1(U)$$

由习题 5.17 知 $\phi(u) \in H^1(U)$.

为了避免不能分部积分的尴尬, 我们先设 $v \geq 0$, $v \in C_c^\infty(U)$.
Sobolev 函数

$$B[\phi(u), v] = \int_U \sum_{i,j} a^{ij} \partial_i (\phi(u)) \partial_j v \, dx$$

$$\begin{aligned} &= \int_U \sum_{i,j} \underbrace{\phi'(u) \cdot \partial_i u}_{\phi'(u) \partial_i u} \partial_j v \, dx \\ \text{注意到: } &\phi'(u) \partial_i u = \partial_j(\phi'(u)v) - \phi''(u) \partial_i u \partial_j v. \quad (\text{当 } u \in H^1(U) \text{ 时, Leibniz Rule 成立}) \\ &\Rightarrow \underbrace{\int_U \sum_{i,j} a^{ij} \partial_i u \partial_j (\phi'(u)v) \, dx}_{\substack{\text{L-积分有界} \\ 0}} - \int_U \sum_{i,j} a^{ij} \underbrace{\phi''(u) \partial_i u \partial_j v}_{\substack{\text{注意原方程弱解} \\ 0}} \, dx \\ &\leq - \underbrace{\int_0 \phi''(u) |\nabla u|^2 v \, dx}_{\substack{\geq 0 \\ (\text{因为 } v \geq 0)}} \leq 0 \end{aligned}$$

□.

$$[6.12] \quad Lu = -\sum_{i,j=1}^n a^{ij} \partial_{ij} u + \sum_{i=1}^n b^i \partial_i u + cu.$$

设 $\exists v \in C^2(\bar{U}) \cap C(\bar{U})$ 使 $\begin{cases} Lv \geq 0 \text{ in } U \\ v > 0 \text{ on } \partial U \end{cases}$

求证: $\forall u \in C^2(\bar{U}) \cap C(\bar{U})$. 只要 $\begin{cases} Lu \leq 0 \text{ in } U \\ u \leq 0 \text{ on } \partial U \end{cases}$, 就有 $u \leq 0$ in U .

证明: 设 $u \in C^2(\bar{U}) \cap C(\bar{U})$. $Lu \leq 0$ in U . $u \leq 0$ on ∂U .

令 $w = \frac{u}{v} \in C^2(\bar{U}) \cap C(\bar{U})$.

如今, 我们希望构造一个有界子集 M , s.t. $Mw \leq 0$ in $\{x \in \bar{U} | u > 0\} \subseteq U$.

若能证此, 则进一步假设 $A = \{x \in \bar{U} | u > 0\} \neq \emptyset$. 由弱极大值原理

$$0 < \sup_{\bar{A}} w = \sup_{\partial A} w = \frac{0}{v} = 0. \text{ 这不可能. } \text{ 若 } A = \emptyset \Rightarrow u \leq 0 \text{ in } U.$$

先求 $-a^{ij} \partial_{ij} w$, 以便确定 M

$$\begin{aligned} -a^{ij} \partial_{ij} \left(\frac{u}{v} \right) &= -a^{ij} \partial_{ij} \left(\frac{\partial_i u \cdot v - \partial_i v \cdot u}{v^2} \right) = -a^{ij} \left(\partial_{ij} \left(\frac{\partial_i u}{v} \right) - \partial_{ij} \left(\frac{\partial_i v \cdot u}{v^2} \right) \right) \\ &= -a^{ij} \left(\frac{\partial_i \partial_j u \cdot v - \partial_i u \partial_j v}{v^2} - \frac{-2uv \partial_i v \partial_j v + v^2 \partial_i \partial_j v \cdot u + v^2 \partial_j v \cdot \partial_i u}{v^4} \right) \\ &= -\frac{a^{ij} \partial_i u \cdot v + a^{ij} \partial_j v \cdot u}{v^2} + \frac{a^{ij} \partial_i v \cdot \partial_j u - a^{ij} \partial_i u \cdot \partial_j v}{v^2} + a^{ij} \frac{2}{v} \cdot \frac{\cancel{\partial_i v - \partial_j u}}{\cancel{v^2}} \cdot \partial_j v \end{aligned}$$

对 i, j 求和, 上式第 2 项消失.

$$-\sum_{i,j} a^{ij} \partial_{ij} \left(\frac{u}{v} \right) = \frac{(Lu - b^i \partial_i u - cu)v + (-Lv + b^i \partial_i v + cv)u}{v^2} + a^{ij} \frac{2}{v} \partial_j v \partial_i w$$

上、下指标代表求和

$$= \frac{uLu}{v} - \frac{uLv}{v^2} - b^i \partial_i w + \frac{2}{v} a^{ij} \partial_j v \partial_i w.$$

$$\therefore \text{令 } Mw = \sum_{i,j} a^{ij} \partial_{ij} w + b^i \partial_i w \left(b^i - a^{ij} \partial_j v \cdot \frac{2}{v} \right)$$

$$= \frac{Lu}{v} - \frac{uLv}{v^2} \leq 0. \quad \text{on } \{x \in \bar{U} | u > 0\} \subseteq U.$$

$\uparrow_{\text{由 } c \text{ 确定. }} \quad \frac{Lu \leq 0}{v > 0} \quad \frac{u \leq 0}{Lv > 0}$

而 M 是一致有界的.

□

[6.13] (柯朗 极大极小原理)

设 $Lu = -\sum_{ij} \theta_j(a^{ij}) a_{ij} u$ $a^{ij} = a^{ji}$ 对零边值问题. 设 L 有特征值

$$0 < \lambda_1 < \lambda_2 \leq \dots$$

求证: $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$ $k \in \mathbb{Z}_+$.

其中 \sum_{k-1} 是 $H_0^1(U)$ 全体 $(k-1)$ 维子空间

证明: 求证 $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$.

$L^2(U)$ 全体 $(k-1)$ 维子空间

$\exists A = L^\dagger : L^2 \rightarrow H_0^1(U) \hookrightarrow L^2(U)$ $\forall A \in L^2(U) \rightarrow L^2(U)$ 紧致子
形式上通过 $f \mapsto u \mapsto u$.

设 λ_k 对应特征函数 w_k . $\|w_k\|_2 = 1$, $\langle w_i, w_j \rangle_{L^2} = \delta_{ij}$

$\forall L w_k = \lambda_k w_k \Rightarrow A w_k = \frac{1}{\lambda_k} w_k \therefore A$ 的特征值 $\lambda_1^\dagger > \lambda_2^\dagger > \dots > 0$

由 Hilbert-Schmidt 定理. A 关于 λ_k^\dagger 有特征向量 e_k . $\|e_k\|_{L^2(U)} = 1$

$\forall f \in L^2(U)$. $f = \sum_i (f, e_i) e_i$ $\{e_k\}_{k \in \mathbb{Z}_+} \rightarrow L^2(U)$ 标准正交基
 $\Rightarrow B[u, u] = \langle Lu, u \rangle = \sum_{i=1}^{\infty} \lambda_i (u, e_i)^2$ $\forall u \in L^2$, $\|u\|_2 = 1$

① $\forall S \in \sum_{k-1}$. $\exists u_k \in \text{Span}\{e_1, \dots, e_k\}$ s.t. $u_k \perp S$ (by Hilbert-Schmidt thm).
 $\Rightarrow \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u] \leq B[u_k, u_k] = \sum_{i=1}^k \lambda_i (u, e_i)^2 \leq \lambda_k$.

② $\exists S = \text{Span}\{e_1, \dots, e_{k-1}\}$ $\forall u \in S^\perp$. $\lambda_k = B[e_k, e_k] \leq \sum_{j \geq k} \lambda_j (u, e_j)^2$

①② $\lambda_k = \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u] = B[u, u]$

由 $H_0^1(U) \subseteq L^2(U)$, $\lambda_k \geq \sup_{S \in \sum_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_2=1}} B[u, u]$.

为证 \leq . 取 $S = \text{Span}\left\{\frac{e_1}{\sqrt{\lambda_1}}, \dots, \frac{e_k}{\sqrt{\lambda_k}}\right\}$. (由课本 6.5 节知. $\left\{\frac{e_j}{\sqrt{\lambda_j}}\right\}_{j \in \mathbb{Z}_+} \rightarrow H_0^1(U)$ 标准正交基)

从而 $\forall u \in S^\perp$ 且 $\|u\|_2 = 1$ 设 $u = \sum_{j \geq k} (a_j \sqrt{\lambda_j}) \frac{e_j}{\sqrt{\lambda_j}}$ $\forall j \in \mathbb{Z}_+$

$$\Rightarrow B[u, u] = \sum_{j \geq k} a_j^2 \lambda_j \geq \lambda_k \quad \text{证毕!}$$

□

14. λ_1 是如下椭圆算子的特征值.

$$Lu = -\sum_{i,j} a^{ij} \partial_{ij} u + \sum_i b^i \partial_i u + cu.$$

$$\text{def } \lambda_1 = \sup_{\substack{u \in C^2(\bar{U}), \\ u > 0 \text{ in } U \\ u=0 \text{ on } \partial U}} \inf_{x \in U} \frac{Lu(x)}{u(x)}.$$

$$\text{Pmf: } X = \left\{ u \in C^2(\bar{U}) \mid \begin{array}{l} u > 0, \\ \text{in } U \\ u=0 \text{ on } \partial U \end{array} \right\}.$$

① 由 Thm 6.5.3. $\exists w_1 > 0$ $w_1 \in H^1(U)$. 为 L 关于 λ_1 的特征向量.

H.P. $\exists \{u_n\} \subset X$ $u_n \rightarrow w_1$ in H^1 .

$$\Rightarrow \sup_x \inf_u \frac{Lu}{u} \geq \inf_x \frac{Lu_n}{u_n} \underset{n \rightarrow \infty}{\uparrow} = \lambda_1.$$

② $\forall u \in X$. $\inf_{x \in U} \frac{Lu}{u} \leq \lambda_1$.

$$\Leftrightarrow \inf_{x \in U} (Lu - \lambda_1 u) \leq 0.$$

Consider. $L^* w_1^* = \lambda_1 w_1^*$. $w_1^* > 0$ 为 L^* 关于 λ_1 的特征向量.

$$\Leftrightarrow (L^* w_1^*, u) = (\lambda_1 w_1^*, u).$$

$$\Rightarrow \cancel{\langle L u, w_1^* \rangle} = \langle \lambda_1 u, w_1^* \rangle$$

$$\Leftrightarrow \langle (Lu - \lambda_1 u), w_1^* \rangle = 0$$

$$\Leftrightarrow \inf_x (Lu - \lambda_1 u) \leq 0$$

check: λ_1 为 L 的特征值. (w_1 为 λ_1^*)

$$\text{由: } \lambda_1^* (w_1^*, w_1)_{L^2} = \langle L^* w_1^*, w_1 \rangle_{L^2}$$

$$\Rightarrow \lambda_1^* = \lambda_1,$$

$$= \langle w_1^*, L w_1 \rangle_{L^2}$$

$$= \lambda_1 \langle w_1^*, u \rangle_{L^2}.$$

□

[6.15] $U(t) \subseteq \mathbb{R}^n$, $\partial U(t)$ 速度 \vec{v} . vi. 魏特行值问题 ~~lambda~~

\downarrow 关于 $t \in C^\infty$.

$$\begin{cases} -\Delta w = \lambda w & \text{in } U(t) \\ w = 0 & \text{on } \partial U(t) \end{cases}$$

$$\|w\|_{L^2(U(t))} = 1$$

$$\lambda = \lambda(t) \in C^\infty$$

$$w = w(x, t) \in C^\infty_x$$

利用 $\frac{d}{dt} \int_{U(t)} f dx = \int_{\partial U(t)} f \cdot (\vec{v} \cdot \vec{\nu}) dS + \int_{U(t)} \partial_t f dx$

去证明 Hadamard 变分公式

$$\dot{\lambda} = - \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS$$

证明: $-\Delta w = \lambda w \Rightarrow - \int_{U(t)} w \cdot \Delta w = \lambda \int_{U(t)} w^2 = \lambda$
 || 分部积分

$$\Rightarrow \lambda = \int_{U(t)} |\nabla w|^2 dx - \int_{\partial U(t)} \underline{w} \cdot \frac{\partial w}{\partial \vec{\nu}} dS = \int_{U(t)} |\nabla w|^2 dx$$

由 $w=0$ on $\partial U(t)$ 知. $\nabla w \neq \frac{\partial w}{\partial \vec{\nu}}$

$$\therefore \dot{\lambda} = \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS + \int_{U(t)} \partial_t |\nabla w|^2 dx \quad \dots \textcircled{1}$$

① 第2项 = $\int_{U(t)} 2 \nabla w \cdot \nabla (\partial_t w) dx.$

$$\stackrel{\text{分部积分}}{=} \int_{U(t)} 2w (-\Delta \partial_t w) dx \stackrel{-\Delta w = \lambda w}{=} \int_{U(t)} 2w \cdot \partial_t w dx$$

$$= \int_{U(t)} \cancel{+2\lambda w \cdot \partial_t w} dx + 2\dot{\lambda} \int_{U(t)} w^2 dx = 2\dot{\lambda}$$

$$= 2\dot{\lambda} + 2\lambda \int_{U(t)} \partial_t w^2 dx$$

$$= 2\dot{\lambda} + 2\lambda \left(\underbrace{\frac{d}{dt} \int_{U(t)} w^2 dx}_{|| \text{ 若 } \dot{\lambda} = 0} - \int_{\partial U(t)} \underline{w}^2 (\vec{v} \cdot \vec{\nu}) dS \right)$$

$$= 2\dot{\lambda}$$

于是 $\textcircled{1} \Rightarrow \dot{\lambda} = - \int_{\partial U(t)} \left| \frac{\partial w}{\partial \vec{\nu}} \right|^2 (\vec{v} \cdot \vec{\nu}) dS.$

□