

Evans Chapter 5 习题  $(U \subset \mathbb{R}^n, \partial U \in C^\infty)$  ( $B(x, r)$  表示闭球)

[5.1] 设  $k \in \mathbb{Z}_+$ ,  $0 < r \leq 1$ . 证明:  $C^{k,r}(\bar{U})$  是 Banach 空间.

证明: Step 1: 马上证  $\|\cdot\|_{C^{k,r}(\bar{U})}$  是范数.  $\|u\|_{C^{k,r}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,r}(\bar{U})}$

① 正定性:

$$\|u\|_{C^{k,r}(\bar{U})} \geq 0 \text{ 为显见.}$$

$$\text{若 } \|u\|_{C^{k,r}(\bar{U})} = 0. \text{ 则 } \|D^\alpha u\|_{C(\bar{U})} = 0 \quad \forall |\alpha| \leq k.$$

$$\Rightarrow \|u\|_{C(\bar{U})} = 0 \Rightarrow u = 0 \text{ in } \bar{U}.$$

$$② \text{齐次性. } \|\lambda u\|_{C^{k,r}(\bar{U})} = |\lambda| \|u\|_{C^{k,r}(\bar{U})} \quad \forall \lambda \in \mathbb{C} \text{ 显见}$$

③ 三角不等式. 设  $u, v \in C^{k,r}(\bar{U})$

$$\|u+v\|_{C^{k,r}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha(u+v)\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha(u+v)]_{C^{0,r}(\bar{U})}$$

$$\begin{aligned} \|\cdot\|_{C(\bar{U})} &\text{ 是范数} \\ &\leq \sum_{|\alpha| \leq k} (\|D^\alpha u\|_{C(\bar{U})} + \|D^\alpha v\|_{C(\bar{U})}) + \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x)+v(x)-u(y)-v(y)|}{|x-y|^r}. \end{aligned}$$

$$\leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| \leq k} \|D^\alpha v\|_{C(\bar{U})} + \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x)-u(y)|}{|x-y|^r}$$

$$+ \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in U}} \frac{|v(x)-v(y)|}{|x-y|^r}$$

$$= \|u\|_{C^{k,r}(\bar{U})} + \|v\|_{C^{k,r}(\bar{U})}.$$

Step 1 证毕!

Step 2:  $(C^{k,r}(\bar{U}), \|\cdot\|_{C^{k,r}(\bar{U})})$  Banach.

设  $\{u_n\}$  为  $C^{k,r}(\bar{U})$  中的 Cauchy 序列. 由  $\|u_n - u_m\|_{C(\bar{U})} \sum_{|\alpha| \leq k} \|D^\alpha u_n - D^\alpha u_m\|_{C(\bar{U})} \rightarrow 0$

由  $C(\bar{U})$  完

$$\sum_{|\alpha|=k} [D^\alpha u_n - D^\alpha u_m]_{C^{0,r}(\bar{U})} \rightarrow 0$$

$$\text{特别地. } \sup_{|\alpha| \leq k} \sup_{x \in \bar{U}} |D^\alpha(u_n - u_m)(x)| \rightarrow 0$$

as  $n, m \rightarrow \infty$

由  $(C^k(\bar{U}), \|\cdot\|_{C^k(\bar{U})})$  Banach 完.  $\exists u \in C^k(\bar{U}). u_n \rightarrow u \text{ in } C^k(\bar{U})$ .

下面先证  $[D^\alpha u_n - D^\alpha u]_{C^{0,r}(\bar{U})} \rightarrow 0$  as  $n \rightarrow \infty$

$$\text{设 } \delta = \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|D^\alpha u_n(x) - D^\alpha u(x) - (D^\alpha u_n(y) - D^\alpha u(y))|}{|x-y|^r}$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty \quad (\text{因 } D^\alpha u_n \rightarrow D^\alpha u).$$

于是, 只须证  $u \in C^{k,r}(\bar{U})$ , 这只需求  $\|u\|_{C^{k,r}(\bar{U})} = k, [D^\alpha u]_{C^{0,r}(\bar{U})} < \infty$

事实上  $\forall x, y \in \bar{U}, x \neq y, |D^\alpha u(x) - D^\alpha u(y)| \leq \limsup_{n \rightarrow \infty} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y)|}{|x-y|^r}$

$$\leq \limsup_{n \rightarrow \infty} [D^\alpha u_n]_{C^{0,r}(\bar{U})} < \infty \quad (\text{由上面已证})$$

□

[5.2]  $0 < \beta < r \leq 1$  时, 证明:

$$\|u\|_{C^{0,r}(\bar{U})} \leq \|u\|_{C(\bar{U})}^{\frac{1-r}{1-\beta}} \|u\|_{C^{0,1}(\bar{U})}^{\frac{1-\beta}{1-r}}$$

证明:  $\|u\|_{C^{0,r}(\bar{U})} = \|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^r}$

$$\leq \|u\|_{C(\bar{U})}^{\frac{1+r}{1-\beta}} \|u\|_{C(\bar{U})}^{\frac{r-\beta}{1-\beta}} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{(x-y)^{\frac{\beta(1-r)}{1-\beta}}} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{1+\frac{r-\beta}{1-\beta}}}$$

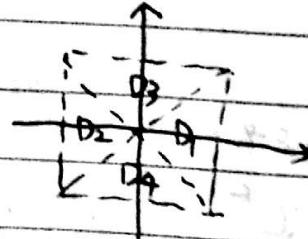
离散 Hölder

$$\leq (\|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{\frac{1-r}{1-\beta}}})^{\frac{1-r}{1-\beta}} \cdot (\|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{\frac{r-\beta}{1-\beta}}})^{\frac{r-\beta}{1-\beta}}$$

$$= \|u\|_{C^{0,\beta}(\bar{U})} \|u\|_{C^{0,1}(\bar{U})}^{\frac{r-\beta}{1-\beta}}$$

[5.3]  $U = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$  □

令  $u(x) = \begin{cases} 1-x_1 & x_1 > 0, |x_2| < x_1 \rightarrow D_1 \\ 1+x_1 & x_1 < 0, |x_2| \leq -x_1 \rightarrow D_2 \\ 1-x_2 & x_2 > 0, |x_1| < x_2 \rightarrow D_3 \\ 1+x_2 & x_2 < 0, |x_1| \leq -x_2 \rightarrow D_4 \end{cases}$



问题:  $p \in [1, +\infty]$ , 且  $u \in W^{k,p}(U)$ .

证明:  $u \in L^p(U)$  为显见.  $\forall 1 \leq p \leq +\infty$ , 下面先求  $u$  的弱导数.

Claim:  $(-1, 0)$  in  $D_1$

$$\nabla u = \begin{cases} (1, 0) & \text{in } D_2 \\ (0, -1) & \text{in } D_3 \\ (0, 1) & \text{in } D_4 \end{cases} \quad \text{是 } u \text{ 的一阶弱导数 } D_u$$

check:  $\forall \varphi \in C_c^\infty(U)$ .

$$\int_U \nabla u \cdot \varphi = \sum_{i=1}^4 \int_{D_i} \nabla u \cdot \varphi \underset{\substack{\nabla \in L^p \\ \varphi \in L^{\frac{p}{p-1}}}}{=} \int_{D_1} (-1, 0) \varphi + \int_{D_2} (1, 0) \varphi + \int_{D_3} (0, -1) \varphi + \int_{D_4} (0, 1) \varphi$$

$$= \sum_{i=1}^4 \int_{D_i} \nabla u \cdot \varphi \underset{\substack{\text{弱导数} \\ \text{分部积分}}}{=} \sum_{i=1}^4 - \int_{D_i} u \nabla \varphi + \int_{\partial D_i} u \varphi n_i \underset{\substack{\text{在 } U \text{ 内零} \\ \varphi|_{\partial U} = 0}}{=}$$

$= - \int_U u \cdot \nabla \varphi \, dx$ . 从而  $V$  的确是  $u$  的一个弱导数.

$v \in L^p \wedge 1 \leq p \leq +\infty \Leftrightarrow v \in W^{1,p}(\bar{U}) \quad 1 \leq p < +\infty$

□

[5.4]. 设  $n=1$ ,  $u \in W^{1,p}(0,1)$ ,  $1 \leq p < +\infty$

(1) 证明  $u$  a.e. 等于一个绝对连续函数  $u^* \in L^p(0,1)$ .

$$(2) \text{ 设 } 1 < p < +\infty \text{ 时, } |u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left( \int_0^1 |u'|^p dt \right)^{\frac{1}{p}}$$

证明 (1)

Lemma (周民强角早题指南 P256). 设  $f \in L^p[a,b]$ ,  $\forall \varphi \in C_c^1(a,b)$ , 若有  $\int_a^b f(x) \varphi'(x) = 0$  则  $f(x) = c$ . a.e.

Proof: 设  $g$  是任一紧支于  $(a,b)$  的连续函数.

$$h \text{ 是 } \dots \dots \dots \int_a^b h(x) dx = 1.$$

$$\text{令 } \varphi(x) = \int_a^x g(t) dt - \int_a^x h(t) dt \cdot \int_a^b g(t) dt, \quad x \in [a,b].$$

则  $\varphi \in C_c^1(a,b)$ .

$$\varphi'(x) = g(x) - h(x) \int_a^b g(t) dt, \quad \forall x \in [a,b]$$

$$\text{从而 } 0 = \int f(x) \varphi'(x) dx$$

$$= \int_a^b f(x) \left( g(x) - \int_a^b g(t) dt \cdot h(x) \right) dx$$

$$= \int_a^b f(x) g(x) - \int_a^b f(x) h(x) dx \cdot \int_a^b g(x) dx = \int_a^b \left( f(x) - \int_a^b f(t) h(t) dt \right) g(x) dx$$

于是:  $f(x) - \int_a^b f(t) h(t) dt = 0 \quad \text{a.e.} \Rightarrow f(x) = c \quad \text{a.e.}$

注: 这用到了  $f \in L^p(\mathbb{R}^d)$  若  $\forall \varphi \in C_c^1(a,b)$ ,  $\int f \varphi = 0$  则  $f = 0$  a.e.

该命题可由反证法得出: 假设  $m(E) > 0$ ,  $f(x) > 0$  in E.

则 存在紧支连续函数  $\{\varphi_k\}$ ,  $\|\varphi_k - \chi_E\|_1 \rightarrow 0$

$$\left\{ |\varphi_k| \leq 1, \varphi_k \rightarrow \chi_E \text{ a.e. in } E. \right.$$

由  $|f \varphi| \leq |f| \quad \forall x \in E$

DCT

$$\therefore 0 < \int_E f(x) dx = \int_{\mathbb{R}^d} f(x) \chi_E(x) dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f(x) \varphi_k(x) dx = 0. \quad \#$$

引理证毕.

回到原题. 令  $u^* = \int_0^x u'(t) dt$ , 其中  $u'$  为  $u$  的弱导数.

则  $u^*$  绝对连续. 下证  $u = u^* + \text{const}$  a.e.

$\forall \varphi \in C_c^\infty(0,1)$ ,  $\exists u \varphi \in C_c^1(0,1)$ .

$$\int_0^1 (u^* - u) \varphi' dx = \int_0^1 \int_0^x u' dt \cdot \varphi' dx - \int_0^1 u \varphi' dx$$

$$= \int_0^1 \int_0^t \varphi'(x) dx \cdot u'(t) dt + \int_0^1 u' \varphi(x) dx$$

$$= \int_0^1 (\underbrace{\varphi(1)}_0 - \varphi(t)) u'(t) dt + \int_0^1 u(x) \varphi(x) dx$$

$$= 0$$

由上得证.

(2). 由  $u$  a.e.= $\frac{1}{p}$  弱延拓出  $\frac{1}{p}$  元

$$|u(x) - u(y)| = \left| \int_0^1 \chi_{\{x \leq t \leq y\}} u'(t) dt \right|$$

a.e.  $x, y \in [0,1]$   
 $\exists \varepsilon > x \leq y$

$$\leq |x-y|^{\frac{1}{p'}} \left( \int_x^y |u'|^p dt \right)^{\frac{1}{p}} \frac{1}{p} + \frac{1}{p'} = 1$$

Hölder.  $\square$

5.  $U$  有解,  $U, V$  互.  $V \subset U$ . 证明:  $\exists \zeta \in C^\infty(U)$ , s.t.  $\zeta \equiv 1$  on  $V$

{  
= 0 near  $\partial U$

证明: 取开集  $W$ .  $V \subset W \subset \bar{W} \subset U$

$$\text{令 } \zeta(x) = (\chi_W * \eta_\varepsilon)(x) \quad (\varepsilon < \frac{1}{2} \min \{ \text{dist}(\partial V, \partial W), \text{dist}(\partial W, \partial U) \})$$

$$\text{在 } V \text{ 上, } \zeta(x) = \int_{\mathbb{R}^n} \chi_W(y) \eta_\varepsilon(x-y) dy$$

$$= \int_{B(0, \varepsilon)} \eta_\varepsilon(y) \cdot \chi_W(x-y) dy.$$

$$x \in V \text{ 时, } \forall y \in B(0, \varepsilon) \quad |x-y| \leq |x| + |\gamma| \leq |x| + \frac{1}{2} \text{dist}(\partial V, \partial W)$$

$$\Rightarrow x-y \in W \Rightarrow \chi_W(x-y) = 1$$

$$\Rightarrow \zeta(x) = 1 \quad \forall x \in V$$

$$\text{同理, 令 } \zeta_\varepsilon = \{x \in U \mid \text{dist}(x, \partial U) < \frac{\varepsilon}{3}\}$$

$$\text{要证, } \zeta_\varepsilon = 0 \text{ in } V_\varepsilon. \Rightarrow \zeta = 0 \text{ near } \partial U.$$

$\zeta \in C^\infty$  由 mollifier 性质易得

$\square$

[5.6]  $U$  有界.  $\{V_i\}_i^N$  是  $\mathbb{R}^n$  中的开集.  $U \subset \bigcup_{i=1}^N V_i$ . 证明: 存在  $C^\infty$  函数  $\{\zeta_i\}$ ,

$$\text{s.t. } \begin{cases} 0 \leq \zeta_i \leq 1 \\ \text{Spt } \zeta_i \subset V_i \quad (1 \leq i \leq N) \\ \sum_{i=1}^N \zeta_i = 1 \quad \text{in } U \end{cases}$$

证明: 对  $\bar{U} \subset \bigcup_{i=1}^N V_i$ .

对每个  $V_i$ , 由 [5.5] 知 存在  $C^\infty$  函数  $\eta_i$  ( $1 \leq i \leq N$ ) s.t.  $0 \leq \eta_i \leq 1$

$\forall x \in \bar{U}$ , 存在以  $x$  为中心的闭球  $B(x) \subseteq V_i$  (for some  $i$ ).  $\text{Spt } \eta_i \subseteq V_i$

因  $\bar{U}$  紧  $\bar{U} \subseteq \bigcup_{x \in \bar{U}} \overset{\circ}{B}(x)$ . 故存在有限覆盖  $\bigcup_{j=1}^m B(x_j)$ .  $\sum_{j=1}^m \eta_j(x) = 1$

对任何  $i \in \{1, 2, \dots, N\}$ , 令  $U_i = \bigcup_{j: \overset{\circ}{B}(x_j) \subseteq V_i} B(x_j)$

$$\text{则 } \bar{U} \subseteq \bigcup_{i=1}^N U_i$$

由上一题,  $\exists \varphi_i \in C^\infty$ ,  $0 \leq \varphi_i \leq 1$ ,  $\varphi_i = 1$  in  $U_i$

$$\text{Spt } \varphi_i \subseteq V_i$$

$$\text{令 } \eta_1 = \varphi_1, \eta_2 = \varphi_2(1-\varphi_1), \dots, \eta_N = \varphi_N(1-\varphi_1) \cdots (1-\varphi_{N-1})$$

则  $\text{Spt } \eta_i \subset V_i$ .

$$\eta_1 + \dots + \eta_N = 1 - \prod_{i=1}^{N-1} (1-\varphi_i). \text{ 因 } \forall x \in \bar{U}, \text{ 总有一个 } \varphi_i \text{ 是 } 1. \text{ 故 } \eta_1 + \dots + \eta_N = 1$$

□

[5.7]  $U$  有界, 且存在  $C^\infty$  向量场  $\vec{\alpha}$ , 使  $\vec{\alpha} \cdot \vec{v} \geq 1$  along  $\partial U$  ( $\vec{v}$  为  $\partial U$  外单位外法向).

( $\leq p < \infty$ . 请对  $\int_{\partial U} |\vec{\alpha} \cdot \vec{v}|^p |u|^p ds$  用 Gauss-Green 公式 证明:  $\forall u \in C^1(\bar{U})$ ).

$$\int_{\partial U} |u|^p ds \leq C \int_U |\nabla u|^p + |u|^p dx$$

$$\text{证明: } \int_{\partial U} |u|^p ds \leq \int_{\partial U} (|u|^p \vec{\alpha}) \cdot \vec{v} ds \stackrel{\text{Gauss-Green}}{=} \int_U \operatorname{div}(|u|^p \vec{\alpha}) dx.$$

$$= \sum_{i=1}^n \int_U \partial_i (|u|^p \alpha_i) dx \quad (\vec{\alpha} = (\alpha_1, \dots, \alpha_n))$$

$$= \sum_{i=1}^n \int_U \partial_i |u|^p \alpha_i dx + \sum_{i=1}^n \int_U |u|^p \partial_i \alpha_i dx$$

$$\stackrel{\vec{\alpha} \in C^\infty}{\leq} C \sum_{i=1}^n \int_U \partial_i |u|^p + C \int_U |u|^p dx$$

$$\leq C \sum_{i=1}^n \int_U p |u|^{p-1} |\partial_i u| + C \int_U |u|^p dx$$

$$\leq C \left( \int_U |u|^p + \int_U (|u|^{p-1} |\nabla u|)^p dx \right)$$

$$\stackrel{\text{Young 不等式}}{\leq} \int_U (|u|^p + |\nabla u|^p) dx$$

□

[6.8]  $U$  有界,  $\partial U \in C^1$ . 证明:  $T: L^p(U) \rightarrow L^p(\partial U)$  为有界线性算子, 且  $Tu = u|_{\partial U}$ .

$\forall u \in C(\bar{U}) \cap L^p(\bar{U})$

证明: 反设存在这样的  $T$ , 令  $u_m = \max\{0, 1 - \text{dist}(x, \partial U)\}$

$$\text{由 } Tu_m = 1 \text{ on } \partial U \quad \|u_m\|_{L^p(\partial U)} = \left( \int_U 1 \, dH^{n-1} \right)^{\frac{1}{p}} = H^{n-1}(\partial U) > 0$$

$\partial U$  的  $n-1$  维 Hausdorff 测度

但  $\|u_m\|_{L^p(U)} \rightarrow 0$  as  $m \rightarrow \infty$ .

$$\text{check: } \int |u_m|^p \, dx \xrightarrow[u_m \rightarrow 0 \text{ as } m \rightarrow \infty \text{ in } U]{} 0.$$

DCT

$$\text{则 } \|T\| = \sup \geq \limsup_{m \rightarrow \infty} \frac{\|Tu_m\|_{L^p(\partial U)}}{\|u_m\|_{L^p(U)}} = \limsup_{m \rightarrow \infty} \frac{H^{n-1}(\partial U)}{0} = +\infty$$

$\Rightarrow T$  有界矛盾!

□

[5.9] 分部积分定理:  $\|Du\|_2 \leq C\|u\|_2^{\frac{1}{2}} \|D^2u\|_2^{\frac{1}{2}}$ .  $\forall u \in C_c^\infty(U)$ .

(1)  $U$  有界,  $\partial U \in C^\infty$  证明 上述不等式对  $u \in H_0^1(U) \cap H^2(U)$  成立.

证明:  $\forall u \in C_c^\infty(U)$  令

$$\begin{aligned} \|Du\|_2^2 &= \int_U |Du|^2 \, dx = \sum_{i=1}^n \int_U (\partial_i u)^2 \, dx \\ &\stackrel{\text{分部积分}}{=} - \sum_{i=1}^n \int_U u \cdot \partial_i \partial_i u \, dx \\ &= - \int_U u \cdot \Delta u \, dx \leq \int_U |u| \cdot |\Delta u| \, dx \\ &\leq C \int_U |u| \cdot |D^2u| \, dx \\ &\leq C \|u\|_2 \|D^2u\|_2. \end{aligned}$$

(2)  $\forall u \in H_0^1(U) \cap H^2(U)$

存在  $\exists \{v_n\} \subset C_c^\infty(U)$   $v_n \rightarrow u$  in  $H_0^1(U)$

$\{w_n\} \subset C^\infty(U)$ :  $w_n \rightarrow u$  in  $H^2(U)$

$$\text{则 } \int_U Dv_k \cdot Dw_k = \sum_{i=1}^n \int_U \partial_i v_k \cdot \partial_i w_k \, dx$$

$$\stackrel{\text{分部积分}}{=} - \sum_{i=1}^n \int_U v_k \partial_i \partial_i w_k \, dx$$

$$= - \int_U v_k \Delta w_k \, dx \leq C \int_U |v_k| |D^2 w_k| \, dx \stackrel{\text{嵌入定理}}{\leq} C \|v_k\|_2 \|D^2 w_k\|_2$$

$\leq C \|v_k\|_2 \|D^2 w_k\|_2. \quad (*)$

$$\text{再び: } \|V_k\|_2 \rightarrow \|u\|_2.$$

$$\cancel{\|D^2 w_k\|_2} \rightarrow \|D^2 u\|_2.$$

$$\text{よし } (\|V_k\|_2 \cdot \|D^2 w_k\|_2) \rightarrow \|u\|_2 \|D^2 u\|_2.$$

$$\text{而 } \int D_u \cdot D_u - \int D_{V_k} \cdot D_{w_k}$$

$$= \int D_u \cdot (D_u - D_{w_k}) dx + \int D_{w_k} \cdot (D_u - D_{V_k}) dx$$

$$\leq \|D_u\|_2 \|D_u - D_{w_k}\|_2 + \|D_{w_k}\|_2 \|D_u - D_{V_k}\|_2.$$

$\rightarrow 0$  as  $k \rightarrow \infty$  - なぜなら ( $\{D_{w_k}\} \subset L^2$  だから).

したがって (\*) 両辺  $k \rightarrow \infty$  で等しい  $\square$

$$[5. 10]. (1) \forall u \in C_c^\infty(U), 2 \leq p < \infty \quad \|Du\|_p \leq C \|u\|_p^{1/2} \|D^2 u\|_p^{1/2}.$$

$$(2) \forall u \in C_c^\infty(U), 1 \leq p < \infty \quad \|Du\|_{2p} \leq C \|u\|_\infty^{1/2} \|D^2 u\|_p^{1/2}.$$

$$\text{証明: (1) } \|Du\|_p^p = \int_U |Du|^p dx$$

$$= \int_U |Du|^{p-2} (Du)^2 dx = \sum_{i=1}^n \int_U \partial_i u (\partial_i u |Du|^{p-2}) dx$$

$$\stackrel{\text{Hölder}}{=} - \sum_{i=1}^n \int_U u \partial_i (\partial_i u |Du|^{p-2}) dx.$$

$u \in C_c^\infty(U), u|_{\partial U} = 0$

$$= - \underbrace{\int_U u \cdot \Delta u \cdot |Du|^{p-2} dx}_{I_1} - \underbrace{\sum_{i=1}^n \int_U u \partial_i u \cdot \cancel{\partial_i} \partial_i |Du|^{p-2} dx}_{I_2}$$

$$I_1 = - \int_U u \cdot \Delta u \cdot |Du|^{p-2} dx$$

$$\leq C \int_U |u| \cdot |\Delta u| \cdot |Du|^{p-2} dx. \stackrel{\text{Hölder}}{=} C \|u\|_p \|D^2 u\|_p \left( \int_U |Du|^{p-2 \cdot \frac{p}{p-2}} dx \right)^{\frac{p-2}{p}}$$

$$= C \|u\|_p \|D^2 u\|_p \cdot \|Du\|_p^{p-2}.$$

$$I_2 = - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \partial_i |Du|^{p-2} dx$$

$$= - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \partial_i \left( \sum_{j=1}^n (\partial_j u)^2 \right)^{\frac{p-2}{2}} dx$$

$$= - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \left( \sum_{j=1}^n (\partial_j u)^2 \right)^{\frac{p-4}{2}} \cdot \sum_{j=1}^n \partial_i \partial_j u \cdot \partial_j u |Du|^{p-4} dx$$

$$= -(p-2) \sum_{i=1}^n \int_U u \cdot |Du|^{p-4} \sum_{j=1}^n \partial_i u (\partial_i \partial_j u) \partial_j u dx$$

$$\leq C \int_U |u| \cdot |Du|^{p-4} \cdot (\partial_1 u, \dots, \partial_n u) \cdot \begin{pmatrix} \partial_1 u & \dots & \partial_n u \\ \vdots & \ddots & \vdots \\ \partial_n u & \dots & \partial_1 u \end{pmatrix} dx$$

$$\leq C \int_U |u| \cdot |Du|^{p-4} \cdot |Du| \cdot |D^2 u| \cdot |Du| dx$$

$$= C \int_U |u| \cdot |Du|^{p-2} \cdot |D^2 u| dx \stackrel{\text{由H\"older, F\o I}}{\leq} C \|u\|_p \|Du\|_p^{p-2} \|D^2 u\|_p$$

$$\therefore \int |Du|^p dx \leq C \int_U |u| \cdot |Du|^{p-2} |D^2 u| dx \\ \leq C \|u\|_p \|Du\|_p^{p-2} \|D^2 u\|_p$$

两边开 $\frac{2}{p}$ : 次方根号.

$$(2) \quad \boxed{\text{由(1)有}} \|Du\|_p^{2p} \leq C \int_U |u| \cdot |Du|^{2p-2} |D^2 u| dx \\ \leq C \|u\|_\infty \int_U |Du|^{2p-2} |D^2 u| dx \\ \stackrel{\text{H\"older}}{\leq} C \|u\|_\infty \|Du\|_{2p}^{2p-2} \|D^2 u\|_p.$$

开平方根号.

□

[5.11] U 连通,  $u \in W^{1,p}(U)$ ,  $Du = 0$  a.e. in U  $\Rightarrow u = \text{const}$  a.e. in U.

证明: 此题不能用 Poincar\'e 不等式  $\|u - (u)_U\|_p \leq C \|Du\|_p$ . 因为该是  
结论不适用于非 Poincar\'e 不等式

令  $u^\varepsilon = \eta^\varepsilon * u$ .  $\forall \varepsilon \in U$ .

由  $\varepsilon$  充分小时.  $Du^\varepsilon = \eta^\varepsilon * Du = (Du)^\varepsilon$  in V.

$u^\varepsilon \in C^\infty(V) \Rightarrow \exists$  常数  $C_\varepsilon$  s.t.  $u^\varepsilon = C_\varepsilon$  in V.

而  $\|u^\varepsilon\|_p = \|u^\varepsilon * u\|_p \leq \|\eta^\varepsilon\|_1 \|u\|_p = \|u\|_p < \infty$  uniformly on  $\varepsilon$ .

$\Rightarrow \{C_\varepsilon\}_{\varepsilon>0}$  有界 故有收敛子列  $C_{\varepsilon_i} \rightarrow C \in \mathbb{R}$ .

由  $C \in L^p$ , 由控制收敛定理易有

$\|u^\varepsilon - C\|_p \rightarrow 0$  as  $i \rightarrow \infty$

又  $\|u^\varepsilon - u\|_p \rightarrow 0$  as  $i \rightarrow \infty$  i.e.  $u = C$  a.e. in any  $V \subset \subset U$

故  $u = C$  a.e. in U

□

[5.12] 举例说明. 若  $\|D^h u\|_{L^1(V)} \leq C$   $\forall 0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$ . 则  $u$  不解  
推出  $u \in W^{1,1}(V)$ .

证明: 令  $U = (-2017, 2017)^n$   $U = (-0.001, 1.001)^n$

$$V = (0, 1)^n.$$

$$u(x) = \begin{cases} 1 & 0 < x_1 < \frac{1}{2} \\ 0 & \text{否则.} \end{cases}$$

$$u(x) \in L^\infty(V)$$

$$\|D^h u\|_{L^1(V)} = \int_V |D^h u| dx \leq \left( \sqrt{1 + \frac{1}{h^2}} \right) \int_{\frac{1}{2}-h}^{\frac{1}{2}} \int_0^1 \cdots \int_0^1 \left| \frac{1}{h} \right| dx_m \cdots dx_1 = 1$$

但  $u \notin W^{1,1}(V)$ . 否则  $\partial_{x_1} u$  为  $u$  的  $x_1$  方向弱偏导. 且  $\partial_{x_1} u \in L^1(V)$ .

$\forall \phi \in C_c^\infty(V) \int_V \partial_{x_1} u \phi dx = 0$  (~~因  $\partial_{x_1} u = 0$  a.e. (因  $u$  只取 0, 1 值)~~)

$$-\int_V u \cdot \partial_{x_1} \phi = -\int_{V \cap \{x_1 > \frac{1}{2}\}} \partial_{x_1} \phi dx$$

这不可能. □

[5.13] 设  $\bar{U} \subset \mathbb{R}^n$  开,  $u \in W^{1,\infty}(\bar{U})$  但  $u$  不是  $\bar{U}$  上的 Lipschitz 连续函数.

证明: 令  $U = \bar{B}(0, 1) - \{(x, y) \in \bar{B}(0, 1) \mid x \geq 0, y \geq 0\}$ .

$$u(x) = \text{sgn}(y) \cdot (\max\{0, x_0\})^2 \cdot \max\{\text{sgn} y, 0\}$$

则  $u(x)$  在  $U$  中可微.

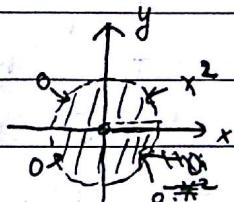
$$\partial_{x_1} u(x) = 2 \max\{\text{sgn} y, 0\} \max\{0, x\}.$$

$$\partial_{x_2} u(x) = 0.$$

$$\rightarrow u \in W^{1,\infty}(U).$$

但  $u$  不是 Lip. 因  $\forall \varepsilon > 0 \quad u(\frac{1}{2}, \varepsilon) - u(\frac{1}{2}, -\varepsilon) = \frac{1}{2}$

$$\Rightarrow \text{Lip}(u) \geq \frac{\frac{1}{2}}{2\varepsilon} = \frac{1}{4\varepsilon} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0^+.$$



□

[5.14] 证明.  $\cup = B(0,1)$  のとき  $u = \log \log (1 + \frac{1}{|x|}) \in W^{1,n}(\cup)$

$$\text{証明: } \partial_i u(x) = \frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{1}{1 + \frac{1}{|x|}} \left( -\frac{1}{|x|^2} \cdot \operatorname{sgn} x_i \right).$$

$$= -\frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{x_i}{|x|^3} \cdot \frac{1}{1 + \frac{1}{|x|}}$$

$$= -\frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{x_i}{|x|^2} \cdot \frac{1}{|x| + 1}$$

$$\rightarrow |\partial u| \leq C \cdot \frac{1}{|x|} \cdot \frac{1}{\log(1 + \frac{1}{|x|})}$$

$$\int_{B(0,1)} |\partial u|^n dx \leq C \int_0^1 \left( \frac{1}{\log(1 + \frac{1}{p})} \right)^n \cdot \frac{1}{p^n} \cdot p^{n-1} dp.$$

↑ ここで  $p = |x|$

$$\begin{aligned} z &= \log(1 + \frac{1}{p}) \\ &\leq C \int_1^\infty \frac{1}{\log 2} \cdot \frac{1}{z^n} dz < \infty \end{aligned}$$

$$\int_{B(0,1)} |u|^p dx = \int_0^1 \left| \log \log \left( 1 + \frac{1}{p} \right) \right|^p p^{n-1} dp$$

$$= \int_{\frac{1}{e-1}}^1 \left| \log \log \left( 1 + \frac{1}{p} \right) \right|^p p^{n-1} dp \quad I_1$$

$$+ \int_{\frac{1}{e-1}}^1 \left( \log \log \left( 1 + \frac{1}{p} \right) \right)^p p^{n-1} dp \quad I_2$$

$$I_2 \leq \int_{\frac{1}{e-1}}^1 \left( \log \left( 1 + \frac{1}{p} \right) \right)^p p^{n-1} dp$$

$$\leq \int_{\frac{1}{e-1}}^1 (\log 2)^p p^{n-1} dp < \infty$$

$$I_1 = - \int_0^{\frac{1}{e-1}} \log \log \left( 1 + \frac{1}{p} \right)^p p^{n-1} dp$$

$$\leftarrow - \int_0^1 \left( \log \frac{p}{p+1} \right)^p p^{n-1} dp$$

$$= - \int_0^1 \log \left( 1 - \frac{1}{p+1} \right)^p p^{n-1} dp$$

$$= (-1)^{n+1} \int_0^1 \log\left(1 + \frac{1}{p}\right)^n p^{n-1} dp.$$

$$\leq C \int_0^1 \frac{1}{p} dp$$

$$I_1 \cdot \mathbb{E} \stackrel{z=\frac{1}{p}}{=} \int_{e^{-1}}^{\infty} \log \log (1+z)^n \cdot \frac{dz}{z^{n+1}}$$

$$\leq C \int_{e^{-1}}^{\infty} \frac{dz}{z^{n+\frac{1}{2}}} < \infty \quad \text{thus, } u \in W^{1,n}(U)$$

□

[5.15] Fix  $\alpha > 0$ .  $U = \mathring{B}(0, 1)$ . 证明: 存在常数  $C(n, \alpha)$ , 使

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx. \quad \text{其中 } \exists n. \{x \in U \mid u(x) = 0\} \geq \alpha. \quad u \in H^1(U)$$

$$\underline{\text{证明: }} \int_U u^2 dx \stackrel{\mathbb{E} \langle u \rangle = \frac{1}{|U|} \int_U u dx}{=} \int_{U-A} (u - \langle u \rangle + \langle u \rangle)^2 dx. \quad \text{其中 } A = \{x \mid u(x) = 0\}$$

$$= \int_{U-A} (u - \langle u \rangle)^2 dx + 2 \underbrace{\int_{U-A} (u - \langle u \rangle) dx \cdot \langle u \rangle}_{\text{Poincaré 不等式: } \|u - \langle u \rangle\|_p \leq C \|Du\|_p} + \int_{U-A} \langle u \rangle^2 dx$$

Poincaré 不等式:  $\|u - \langle u \rangle\|_p \leq C \|Du\|_p$

$$\leq C \|Du\|_{L^2}^2 + \int_{U-A} \langle u \rangle^2$$

$$= C \|Du\|_{L^2}^2 + |\langle u \rangle^2| \cdot |U-A|$$

$$= C \|Du\|_{L^2}^2 + \frac{1}{|U|^2} \left( \int_{U-A} |u| dx \right)^2 \cdot |U-A|$$

Hölder

$$\leq C \|Du\|_{L^2}^2 + \frac{1}{|U|^2} \cdot \left( \int_{U-A} |u|^2 dx \right) \cdot |U-A|^2$$

$$\text{全 } 1 - C_0 = \frac{|U-A|^2}{|U|^2}$$

$$= C \|Du\|_{L^2}^2 + (1 - C_0) \|u\|_{L^2}^2$$

$$\Rightarrow \exists C' > 0. \quad \int_U u^2 dx \leq C' \int_U |Du|^2 dx$$

□

$$[5.16] \text{ 证明: } \forall n \geq 3, \exists \text{ const. } C. \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C \int_{\mathbb{R}^n} |Du|^2 dx, \forall u \in H^1(\mathbb{R}^n)$$

证明: 先设  $u \in C_c^\infty(\mathbb{R}^n)$  且  $F(x) = \frac{x}{|x|^{n-2}}$ .

$$\text{由 } \int_{\mathbb{R}^n} u^2 \operatorname{div} F dx = - \int_{\mathbb{R}^n} D(u^2) \cdot F(x) dx.$$

$$= -2 \int_{\mathbb{R}^n} u D(u) \cdot F(x) dx$$

$$= -2 \int_{\mathbb{R}^n} Du \cdot (u F) dx$$

$$\Rightarrow \left| \int_{\mathbb{R}^n} u^2 \operatorname{div} F dx \right| = 2 \left| \int_{\mathbb{R}^n} Du \cdot u F dx \right|$$

$$\leq 2 \|Du\|_2 \|u F\|_{L^2}$$

$$\text{由 } \operatorname{div} F(x) = \frac{n-2}{|x|^2}, |F(x)|^2 = \frac{1}{|x|^{n-2}} \text{ 代入有:}$$

$$\frac{(n-2)^2}{4} \left( \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \right)^2 \leq \int_{\mathbb{R}^n} |Du|^2 dx \cdot \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx$$

$$\Rightarrow \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |Du|^2 dx.$$

对一般的  $u \in H^1(\mathbb{R}^n)$ . 由于  $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n) \iff U = \mathbb{R}^d$

故  $\exists u_k \in C_c^\infty(\mathbb{R}^n)$ ,  $u_k \rightarrow u$  in  $H^1(\mathbb{R}^n)$ .

$$\text{从而 } \int_{\mathbb{R}^n} |Du_k|^2 dx \rightarrow \int_{\mathbb{R}^n} |Du|^2 dx$$

$$\frac{(n-2)}{4} \int_{\mathbb{R}^n} \frac{u_k^2}{|x|^2} dx.$$

$$\Rightarrow \frac{u_k}{|x|} \in L^2(\mathbb{R}^n). \text{ 由 } u_k \rightarrow u \text{ in } L^2$$

故存在子列  $u_{k_j} \rightarrow u$  a.e.

$$\Rightarrow \left( \frac{u_{k_j}}{|x|} \right)^2 \rightarrow \left( \frac{u}{|x|} \right)^2 \text{ a.e.}$$

$$\Rightarrow \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \liminf_{j \rightarrow \infty} \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u_{k_j}^2}{|x|^2} dx \leq \limsup_{j \rightarrow \infty} \int_{\mathbb{R}^n} |Du_{k_j}|^2 dx$$

Fatou's Lemma

$$= \int_{\mathbb{R}^n} |Du|^2 dx$$

[5.17] (待证)  $F: \mathbb{R} \rightarrow \mathbb{R}$  是  $C'$  的, 且  $F'$  有界  $u \in W^{1,p}(U)$

$1 \leq p \leq \infty$ .

证明: (1) 若  $\int_U |F''(u)|^p dx < \infty$ , 则  $V := F(u) \in W^{1,p}(U)$ ,  $\partial_i V = F'(u) \partial_i u$ .

(2) 若  $\int_U |F''(u)|^p dx = +\infty$ , 但  $F(0) = 0$  则 (1) 结论也对.

Rmk: 证明过程中会体现. (2) 中  $F(0) = 0$  是必须的.

Proof:  $\Leftrightarrow |F(u) - F(0)| \leq \|F'\|_{L^\infty} |u| \in L^p$  (中值定理).  
若  $F(u) - F(0) \in L^p(U)$

若  $\int_U |F''(u)|^p dx < \infty$ , 则  $F(u) \in L^p(U) \Rightarrow F(u) \in L^p(U)$ .

若  $\int_U |F''(u)|^p dx = +\infty$ , 则  $F(0) = 0 \Leftrightarrow F(u) \in L^p(U)$ .

$$F(u) - F(0) \in L^p(U)$$

从而  $F'(u) \partial_i u \in L^p(U)$  显见, 因  $F' \in L^\infty(U)$ ,  $\partial_i u \in L^p(U)$ .

下面证明  $\partial_i F(u) = F'(u) \partial_i u$   $i = 1, 2, \dots, n$ .

令  $\forall \varepsilon \in U$ ,  $u^\varepsilon = \eta_\varepsilon * u$ . 使  $u^\varepsilon \in C_c^\infty(U)$ .

$\forall \phi \in C_c^\infty(U)$  且  $\text{Supp } \phi \subset V \subset U$ .  $u^\varepsilon = \eta_\varepsilon * u$ .

要证:  $\int_V F(u) \partial_i \phi dx = - \int_V F'(u) \partial_i u \cdot \phi dx$ .

$$\text{左} = \int_V F(u) \partial_i \phi dx \stackrel{\text{①}}{=} \lim_{\varepsilon \rightarrow 0^+} \int_V F(u^\varepsilon) \partial_i \phi dx.$$

$$\stackrel{\text{分部积分}}{=} - \lim_{\varepsilon \rightarrow 0^+} \int_V F'(u^\varepsilon) \partial_i u^\varepsilon \cdot \phi dx \quad (\text{设 } \varepsilon \text{ 为 } 0).$$

因  $u^\varepsilon \in C_c^\infty(U_\varepsilon)$   
链式法则可得

$$\stackrel{\text{②}}{=} - \int_V F'(u) \partial_i u \cdot \phi dx = - \int_V F'(u) \partial_i u \cdot \phi dx$$

$$\text{check ①: } \int_V |F(u) - F(u^\varepsilon)| \cdot |\partial_i \phi| dx \leq \int_V |u - u^\varepsilon| \cdot |\partial_i \phi| dx \cdot \|F'\|_{L^\infty}$$

$$\leq \|F'\|_{L^\infty(U)} \|u - u^\varepsilon\|_{L^p(V)} \|\partial_i \phi\|_{L^p(V)}$$

$\rightarrow 0$  as  $\varepsilon \rightarrow 0^+$  (因  $u^\varepsilon \rightarrow u$  in  $L^p(V)$ )

$$\text{②} \left| \int_V F(u) \partial_i u \phi dx - \int_V F(u^\varepsilon) \partial_i u^\varepsilon \cdot \phi dx \right|$$

$$\leq \int_V |F(u) - F(u^\varepsilon)| |\partial_i u| |\phi| dx + \int_V |F'(u^\varepsilon)| \cdot |\partial_i u - \partial_i u^\varepsilon| |\phi| dx$$

A

B

$$x \cdot B = \int_V |F(u^\varepsilon)| \cdot |\partial_i u - \partial_i u^\varepsilon| \cdot |\phi| dx$$

$$\leq \|F\|_{L^\infty(\Omega)} \cdot \|\partial_i u - \partial_i u^\varepsilon\|_{L^p(V)} \cdot \|\phi\|_{L^p(V)}.$$

$\rightarrow 0$  as  $\varepsilon \rightarrow 0^+$  (因  $u^\varepsilon \rightarrow u$  in  $W^{1,p}$ ).

$$A = \int_V |F'(u) - F'(u^\varepsilon)| \cdot |\partial_i u| \cdot |\phi| dx$$

由  $u^\varepsilon \rightarrow u$  a.e. in  $V$  (光滑性质).

$F'$  连续  $\Rightarrow F'(u) \rightarrow F'(u^\varepsilon)$  a.e. in  $V$

$$\text{又: } |F'(u) - F'(u^\varepsilon)| \cdot |\partial_i u| \cdot |\phi| \leq 2\|F'\|_{L^\infty} |\partial_i u| \cdot |\phi| \in L^1 \text{ (由 Hölder 即得)}$$

故由控制收敛定理.  $A \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ .

证毕!  $\square$

[5.18]  $1 \leq p \leq \infty$ .  $U$  有界.

1) 证明: 若  $u \in W^{1,p}(U)$ , 则  $|u| \in W^{1,p}(U)$ .

2) 若  $u \in W^{1,p}(U)$ , 则  $u^+, u^- \in W^{1,p}(U)$ .

$$Du^+ = \begin{cases} Du & L^n\text{-a.e. on } \{u > 0\} \\ 0 & L^n\text{-a.e. on } \{u \leq 0\} \end{cases} \quad Du^- = \begin{cases} 0 & L^n\text{-a.e. on } \{u \geq 0\} \\ -Du & L^n\text{-a.e. on } \{u < 0\} \end{cases}$$

3).  $u \in W^{1,p}(U)$ . 则  $Du = 0$  a.e. on  $\{u = 0\}$

Proof: 只用证(2). (2)  $\Rightarrow$  (1) 显见

若(2)对, 则  $Du = Du^+ - Du^- = 0$  on  $\{u = 0\}$   $L^n$ -a.e.

下证(2). 令  $F_\varepsilon(r) = (\sqrt{r^2 + \varepsilon^2} - \varepsilon) \chi_{\{r \geq 0\}} \in C^1(\mathbb{R})$

且  $F'_\varepsilon(r) \in L^\infty(\mathbb{R})$ . (Fix  $\varepsilon > 0$ ).

$$\text{由 17 题 } \int_U F_\varepsilon(u) \partial_i \phi dx = - \int_U F'_\varepsilon(u) \partial_i u \cdot \phi dx \dots (*)$$

注意到.  $u^+ = \lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u)$ .

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u)$$

(\*) 左边令  $\varepsilon \rightarrow 0$ . 极限  $\Rightarrow \int_U F(u) \partial_i \phi dx = \int_U u^+ \partial_i \phi dx$

这由控制收敛即得. (\*) 右边同理  $\rightarrow - \int \partial_i u \cdot \chi_{\{u > 0\}} \phi dx$

$\partial_i u^+ = \partial_i u \chi_{\{u>0\}}$  同理  $u^-$  有类似结果, 因  $u^\pm \in L^2$ . □

[5.19] 设  $u \in H^1(U)$  按书上 Hint 证明  $D_u = 0$  a.e. in  $\{u=0\}$

证: 取  $\phi$  是  $C_c^\infty$ , 有界, 不减函数.  $\phi'$  有界.  $\phi(z) = z$   $|z| \leq 1$ .

$$\text{令 } u^\varepsilon(x) = \varepsilon \phi(\frac{u}{\varepsilon})$$

① claim  $u^\varepsilon \rightarrow 0$  in  $L^2(U)$ .

$$\forall \varphi \in C_c^\infty(U). \int_U u^\varepsilon \varphi \, dx = \varepsilon \int_U \phi(\frac{u}{\varepsilon}) \varphi \, dx$$

$$\leq \varepsilon \cdot \|\phi\|_{L^\infty} \|\varphi\|_{L^1} \xrightarrow{\varepsilon \rightarrow 0^+} 0$$

$$\therefore \forall \varphi \in C_c^\infty(U). \langle \varphi, u^\varepsilon \rangle \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+$$

$\times C_c^\infty(U) \stackrel{\text{dense}}{\subset}$

$$\therefore \|u^\varepsilon\|_{L^2}^2 = \varepsilon^2 \int_U |\phi(\frac{u}{\varepsilon})|^2 \leq \|\phi\|_{L^\infty}^2 \|u\|_{L^2}^2$$

因  $\phi'$  有界  $\forall x \in \mathbb{R}, |\phi'(x)| \leq \|\phi'\|_{L^\infty} |x|$ .

$$\phi(0) = 0$$

$$\therefore \|u^\varepsilon\|_{L^2}^2 \leq \varepsilon^2 \int_U \frac{u^2}{\varepsilon^2} \|\phi'\|_{L^\infty}^2 dx = \|u\|_{L^2}^2 < \infty$$

$$\therefore \begin{cases} \|u^\varepsilon\|_{L^2} - \text{致有界} \\ \forall \varphi \in C_c^\infty(U) \subset (L^2(U))^* = L^2(U), \langle \varphi, u^\varepsilon \rangle \rightarrow 0 \end{cases} \Rightarrow u^\varepsilon \rightarrow 0 \text{ in } L^2(U)$$

$$\text{dense}$$

②  $\partial_i u^\varepsilon \rightarrow 0$  in  $L^2(U)$ .

$$\|\partial_i u^\varepsilon\|_{L^2}^2 = \int |\partial_i u^\varepsilon|^2 = \int |\phi'(\frac{u}{\varepsilon}) \cdot \partial_i u|^2 dx \leq \|\phi'\|_{L^\infty}^2 \|\partial_i u\|_{L^2}^2 < \infty$$

$\forall \varphi \in C_c^\infty(U)$ ,

$$\langle \partial_i u^\varepsilon, \varphi \rangle = \int \partial_i u^\varepsilon \cdot \varphi = - \int u^\varepsilon \cdot \partial_i \varphi \rightarrow 0 \quad (\text{因 } u^\varepsilon \rightarrow 0 \text{ in } L^2)$$

$\therefore \partial_i u^\varepsilon \rightarrow 0$  in  $L^2(U)$ .

$$\text{如今 } \int D_u^\varepsilon \cdot D_u dx = \sum_{i=1}^n \int \partial_i u^\varepsilon \partial_i u dx.$$

$$\rightarrow 0 \quad (\text{因 } \partial_i u \in L^2, \partial_i u^\varepsilon \rightarrow 0 \text{ in } L^2)$$

$$2: \int D_u^\varepsilon \cdot D_u = \sum_{i=1}^n \int \partial_i u^\varepsilon \partial_i u dx$$

$$= \sum_{i=1}^n \int \partial_i u \cdot \phi'(\frac{u}{\varepsilon}) \partial_i u dx$$

$$= \int |Du|^2 \phi'(\frac{u}{\varepsilon}) dx$$

由 P 在  $\{u=0\}$  上, 令  $\varepsilon \rightarrow 0^+$  有  $D_u=0$  a.e. on  $\{u=0\}$

D

Pmk: 19 不帶  $U$  有界. 因  $\phi'(0)=0$ , 17(2) 生效.

[5.20]. 若  $u \in H^s(\mathbb{R}^d)$ ,  $s > \frac{d}{2}$ . 則  $u \in L^\infty(\mathbb{R}^d)$ . 且  $\|u\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^s(\mathbb{R}^d)}$ .

證明:  $u \in H^s(\mathbb{R}^d)$  則  $u \in L^2(\mathbb{R}^d)$  ( $s > \frac{d}{2}$ )

$$|u(x)| \leq \lim_{N \rightarrow \infty} \int_{|x| \leq N} |\hat{u}(\xi)| e^{2\pi i x \cdot \xi} d\xi.$$

$$\leq \lim_{N \rightarrow \infty} \int_{|x| \leq N} |\hat{u}(\xi)| \langle \xi \rangle^s \frac{1}{\langle \xi \rangle^s} d\xi.$$

Claim:  $S(\mathbb{R}^d)$  在  $H^s(\mathbb{R}^d)$  中稠密

若 ~~不是~~ claim 成立, 則 我們只用對  $u \in S(\mathbb{R}^d)$  証明即可 (再延拓)

$$u \in S(\mathbb{R}^d) \quad |u(x)| = |(\hat{u}(\xi))_N^V|$$

$$\geq \left| \int_{\mathbb{R}^d} |\hat{u}(\xi)| e^{2\pi i x \cdot \xi} d\xi \right|$$

$$\leq \int_{\mathbb{R}^d} |\hat{u}(\xi)| d\xi$$

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}} = \int_{\mathbb{R}^d} |\hat{u}(\xi)| \langle \xi \rangle^s \frac{1}{\langle \xi \rangle^s} d\xi$$

$$\leq \left\| \frac{1}{|\xi|^s} \right\|_{L^2} \left\| \langle \xi \rangle^s \hat{u} \right\|_{L^2}.$$

$$= C_{s,d} \|u\|_{H^s(\mathbb{R}^d)}.$$

再证 claim: 因  $S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ , 故  $\exists v_k \in S(\mathbb{R}^d)$

$$v_k \rightarrow \langle \xi \rangle^s \hat{u} \text{ in } L^2(\mathbb{R}^d).$$

$$\text{令 } u_k = (\langle \xi \rangle^{-s} v_k)^{\vee} \text{ 这里合理的, 因 } v_k \langle \xi \rangle^{-s} \in S(\mathbb{R}^d)$$

$$\text{故. } \|u_k - u\|_{H^s} = \|(\hat{u}_k - \hat{u}) \langle \xi \rangle^s\|_{L^2}.$$

$$= \|(\langle \xi \rangle^{-s} v_k - \hat{u}) \langle \xi \rangle^s\|_{L^2}.$$

$$= \|v_k - \hat{u} \langle \xi \rangle^s\|_{L^2} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

□

[5.21] 若  $u, v \in H^s(\mathbb{R}^d)$ ,  $s > \frac{d}{2}$ , 则  $uv \in H^s(\mathbb{R}^d)$ .

且  $\|uv\|_{H^s(\mathbb{R}^d)} \leq C_{s,d} \|u\|_{H^s} \|v\|_{H^s}$ . 右  $s > \frac{d}{2}$  时  $H^s$  是代数.

证明:

$$\|uv\|_{H^s(\mathbb{R}^d)} = \|\hat{u}\hat{v}\langle \xi \rangle^s\|_{L^2}. \quad \text{这个其实也应该先对 Schwartz 函数证明, 我偷个懒。}$$

$$= \|(\hat{u} * \hat{v}) \langle \xi \rangle^s\|_{L^2}.$$

$$= \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) \langle \xi \rangle^s d\eta \right\|_{L^2(\xi)}^2$$

$$\text{这儿, } \langle \xi \rangle^s = 1 + |\xi|^s. \quad (\text{与20题那个类似})$$

$$\leq (1 + |\xi|^s) (1 + |\eta|^s).$$

$$= \langle \xi \rangle =$$

①  $|\xi| < \frac{|\eta|}{2}$  or  $|\xi| > \frac{3}{2}|\eta|$  时,

$$\langle \xi \rangle^s = 1 + |\xi|^s. \quad \left\{ \begin{array}{l} \leq 1 + \frac{|\eta|^s}{2^s} = \frac{1}{2^s} (1 + |\eta|^s) \leq (1 + |\xi - \eta|^s) (1 + |\eta|^s) \\ \text{若 } |\xi| > 2|\eta| \text{ 则 } |\frac{\xi}{\eta}| > 2 \end{array} \right.$$

$$\Rightarrow 3\xi^2 - 8\xi\eta + 4\eta^2 \geq 0$$

$$\Rightarrow 1 + |\xi| \leq 2|\xi - \eta|$$

$$\Rightarrow \langle \xi \rangle^s \leq C (\langle \xi - \eta \rangle^s + \langle \eta \rangle^s).$$

$$② \frac{|\eta|}{2} < |\xi| \leq 2|\eta| \text{ 时.}$$

$$1 + |\xi|^s \leq 2^s (1 + |\eta|^s) (1 + |\xi - \eta|^s).$$

$$\text{故 } \langle \xi \rangle^s \leq C_s (\langle \xi - \eta \rangle^s + \langle \eta \rangle^s).$$

代入有:

$$\begin{aligned} \|uv\|_{H^s} &\leq \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) \langle \xi - \eta \rangle^s d\eta \right\|_{L_\xi^2} \\ &\quad + \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) \langle \eta \rangle^s d\eta \right\|_{L_\xi^2} \end{aligned}$$

$$= \left\| \hat{u} \cdot \langle \xi \rangle^s \right\|_{L^2} + \left\| \hat{u} * (\langle \cdot \rangle^s \hat{v}) \right\|_{L^2}.$$

$$\|f * g\|_{L^2} \leq \|f\|_L^1 \|g\|_{L^2}.$$

$$\leq \|\hat{u} \langle \xi \rangle^s\|_{L^2} \|\hat{v}\|_{L^1} + \|\hat{v} \langle \xi \rangle^s\|_{L^2} \|\hat{u}\|_{L^1}$$

$$= \|u\|_{H^s} \|\hat{v} \langle \xi \rangle^s \langle \xi \rangle^{-s}\|_{L^1} + \|v\|_{H^s} \|\hat{u} \langle \xi \rangle^s \langle \xi \rangle^{-s}\|_{L^1},$$

$$\leq \|u\|_{H^s} \|\hat{v} \langle \xi \rangle^s\|_{L^2} \|\langle \xi \rangle^{-s}\|_{L^2}$$

$$+ \|v\|_{H^s} \|\hat{u} \langle \xi \rangle^s\|_{L^2} \|\langle \xi \rangle^{-s}\|_{L^2}$$

$$\lesssim_s \|u\|_{H^s} \|v\|_{H^s}$$

□

证: