这个文档比较模糊，清晰版可以在我的知乎主页上找到（知乎搜索用户名yx3x）。我的原稿放在家里没带在身边。

Ch7 习题

1. $\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$ 至多一个光滑解

Proof：设 u_1, u_2 为原方程2个光滑解. $v = u_1 - u_2$ 要证 $v = 0$

v 满足 $\begin{cases} v_t - \Delta v = f - f = 0 & \text{in } U_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ v = 0 & \text{on } U \times \{t=0\} \end{cases}$

两边乘 v，积分得

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 - \int_U v \Delta v = 0$$

分部积分

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\int_U |\nabla v|^2 dx \le 0$$

又：$\|v(0)\|_{L^2}^2 = 0$ 且 $\forall t \in (0,T]$，$\|v(t)\|_{L^2}^2 \le 0$

$$\Rightarrow \|v(t)\|_{L^2} = 0 \Rightarrow v = 0 \text{ in } [0,T]$$

$v \in C^\infty$

□

2. 设 u 是如下方程的光滑解.

$\begin{cases} u_t - \Delta u = f = 0 & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times [0,\infty) \\ u = g & \text{on } \partial U \times \{t=0\} \end{cases}$

证明：$\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t}\|g\|_{L^2(U)}$

$\lambda_1 > 0$ 是 $-\Delta$ 的主特征值

Proof：方程两边乘以 u，积分得

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\int_U |\nabla u|^2 dx$$

又：$\lambda_1 = \inf_{u \in H_0^1(U)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$

$u \neq 0$

故 $\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 \le -\lambda_1 \|u(t)\|_{L^2}^2$.

由 Gronwall 不等式

$$\|u(t)\|_{L^2}^2 \le e^{-2\lambda_1 t}\|u(0)\|_{L^2}^2 = e^{-2\lambda_1 t}\|g\|_{L^2}^2$$

□

3. 设 u 为 $\begin{cases} \partial_t u + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$ 的光滑解，其中 L 为二阶椭圆算子

v 为 $\begin{cases} \partial_t v - L^* v = 0 & \text{in } U_T \\ v = 0 & \text{on } \partial U \times [0,T] \\ v = h & \text{on } U \times \{t=T\}. \end{cases}$ 的光滑解

求证：$\int_U g(x) v(x,0)\, dx = \int_U u(x,T) h(x)\, dx.$

证明：$\int_U u(x,T) h(x) - g(x) v(x,0)\, dx = \int_U u(x,T) v(x,T) - u(x,0) v(x,0)\, dx$

显然，这是关于 t 分部积分出来的边界项

$$\text{上式} = \int_0^T \int_U (\partial_t u) \cdot v + (\partial_t v) \cdot u \, dx\, dt.$$

$\langle Lu, v\rangle = \langle u, L^* v\rangle$

$$= \int_0^T \int_U (\partial_t u) \cdot v + Lu \cdot v - u L^* v + \partial_t v \cdot u \, dx\, dt$$

$$= \int_0^T \int_U \underbrace{(\partial_t u + Lu)}_{=0} v + u \cdot \underbrace{(\partial_t v - L^* v)}_{=0} \, dx\, dt = 0$$

□

4. (Galerkin Method for Poisson).

$f \in L^2(U)$. $u_m = \sum_{k=1}^m d_m^k w_k$ solves $\int_U Du_m \cdot Dw_k \, dx = \int_U f \cdot w_k \, dx$, $1 \le k \le m$.

证明：$\{u_m\}$ 存在子列，在 $H_0^1(U)$ 中弱收敛于 $\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的弱解.

Pf. Step 1: $\{u_m\}$ 在 H_0^1 中一致有界

$\int Du_m \cdot Dw_k \, dx = \int_U f \cdot w_k \, dx$. 两边乘以 d_m^k，对 k 求和有：

$$\int \|Du_m\|_{L^2}^2 = \int_U f u_m \, dx$$

$C\|u_m\|_{H_0^1}^2 \le$

$$\le \|f\|_2 \|u_m\|_{H_0^1} \le \varepsilon \|u_m\|_{H_0^1}^2 + C(\varepsilon) \|f\|_{L^2}^2.$$

ε 充分小 $\Rightarrow \|u_m\|_{H_0^1} \le C\|f\|_{L^2}$ ✓

Step 2: $\exists$ 子列 $u_{m_k} \longrightarrow u$ in $H_0^1(U)$.

$Du_{m_k} \longrightarrow v$ in $L^2(U)$

$v \overset{a.e.}{=} Du$?

$\forall \varphi \in C_c^\infty$ $\int Du_{m_k} \cdot D\varphi = \int f \varphi$ $\to \int v \cdot D\varphi$

$$\| \quad -\int u_{m_k} \cdot D^2 \varphi$$

$\Downarrow k \to \infty$

$$-\int u \cdot D^2 \varphi = \int Du \cdot D\varphi$$

~~同理可得 左 $= \int v D\varphi$~~

□

[7.5] 设

$$\begin{cases} \mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\ L^2(0,T;H_0^1(U)), \\ \mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U)). \end{cases}$$

证明:$\mathbf{u}' = \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))$.

证明：我们断言：

Claim: 对任意$\phi \in C_c^\infty(0,T), w \in H_0^1(U)$, 成立:

$$\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle = \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle,$$

其中$\langle\cdot,\cdot\rangle$代表$H^{-1}(U), H_0^1(U)$中元素之间的作用(pairing).

若Claim获证，那么在$H^{-1}(U)$中(即作为$H_0^1(U)$上的连续线性泛函)成立:

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{v}(t)\phi(t)dt.$$

再由时间弱导数定义知

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{u}'(t)\phi(t)dt.$$

这样就有$\mathbf{u}' = \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))$.

Claim的证明仍然由直接计算可得: 注意到$t \mapsto \pi(t)w \in L^2(0,T;H_0^1)$, 那么:

$$\begin{aligned}
\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle &= \int_0^T \langle \phi'(t)\mathbf{u}(t), w\rangle\, dt \\
&= \int_0^T \langle \mathbf{u}(t), \phi'(t)w\rangle\, dt \\
(\mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\ L^2(0,T;H_0^1(U))) &= \lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k(t), \phi'(t)w\rangle\, dt \\
&= \lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k(t)\phi'(t), w\rangle\, dt \\
&= \lim_{k\to\infty} \left\langle \int_0^T \mathbf{u}_k(t)\phi'(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty} \left\langle \int_0^T \mathbf{u}_k'(t)\phi(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k'(t)\phi(t), w\rangle\, dt \\
(\mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))) &= -\int_0^T \langle \mathbf{v}(t), \phi(t)w\rangle\, dt \\
&= \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle
\end{aligned}$$

6. 设H是Hilbert空间，$u_k \rightharpoonup u$ in $L^2(0,T;H)$，$\operatorname*{ess\,sup}_{0\le t\le T}\|u_k(t)\|\le C$ $\forall k\in\mathbb{Z}_+$

证明：$\operatorname*{ess\,sup}_{0\le t\le T}\|u(t)\|\le C$

Proof：先证：$\forall 0\le a\le b\le T$，$v\in H$，有：

(hint). $\int_a^b (v, u_k(t))\,dt \le C\|v\|\cdot|b-a|$ （显然）

若hint成立，则

$$\frac{1}{b-a}\int_a^b (v, u_k(t))\,dt \le C\|v\|$$

这样由Lebesgue微分定理，

对a.e. $t\in[0,T]$成立：$|\langle u^{(t)}, v\rangle| \le C\|v\|$ （t是[a,b]中点）

$k\to\infty$，由弱收敛知，$|\langle u, v\rangle| \le C\|v\|$ a.e. $t\in[0,T]$

$\Rightarrow \operatorname*{ess\,sup}_{0\le t\le T}\|u(t)\|\le C$.

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6. H为Hilbert空间，$u_k \rightharpoonup u$ in $L^2(0,T;H)$，且 $\operatorname*{ess\,sup}_{0\le t\le T}\|u_k(t)\|\le C$ $\forall k\in\mathbb{Z}_+$.

求证：$\operatorname*{ess\,sup}_{0\le t\le T}\|u(t)\|\le C$

证明：设 $f_{a,b}(v)=\int_a^b (v, u(t))\,dt$.

显然 $f_{a,b}\in L^2(0,T;H)$.

故 $\lim_{k\to\infty} f_{a,b}(u_k) = f_{a,b}(u) = \int_a^b (u(t), u(t))\,dt$.

而 $f_{a,b}(u_k) = \int_a^b (u_k(t), u(t))\,dt$

$$\le C\int_a^b \|u(t)\|_H\,dt$$

$$\le C\sqrt{b-a}\,\Big\|\|u(t)\|_H\Big\|_{L^2_t(a,b)}$$

$$\overset{k\to\infty}{\Longrightarrow} \int_a^b \|u(t)\|_H^2\,dt \le C^2(b-a) \quad \forall a,b\in[0,T]$$

$\Rightarrow \|u(t)\|_H \le C$ a.e. $t\in[0,T]$.

除以b-a，用Lebesgue微分定理即可

令$b\to a$

□

7. 设 u 是光滑解：$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U\times(0,\infty) \\ u = 0 & \text{on } \partial U\times[0,\infty) \\ u = g & \text{on } U\times\{t=0\} \end{cases}$

且函数 c 满足 $c \geq \gamma > 0$.

证明：$|u(x,t)| \leq Ce^{-\gamma t}$.

Proof：设 $v = e^{\gamma t}u$.

则 $\partial_t v - \Delta v + cv = \gamma e^{\gamma t}u + e^{\gamma t}u_t - e^{\gamma t}\Delta u + ce^{\gamma t}u$

$= \gamma v + \underbrace{\left((\partial_t - \Delta + c)u\right)}_{0} e^{\gamma t}$

$= \gamma v$

$\Rightarrow \begin{cases} \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$

由弱极大值原理：

$\forall (x,t)\in U_T$.

$|v(x,t)| = e^{\gamma t}|u(x,t)| \leq \sup_{\Gamma_T} |v(x,t)|$

$= \sup_{x\in U} |g(x)|$.

$\Rightarrow |u(x,t)| \leq e^{-\gamma t}\|g\|_{L^\infty}$

8. 若 u 是下方程的光滑解：$g \geq 0$，ϕ c 有界但不一定非负，证明 $u \geq 0$

Proof：令 $v = e^{-(\|c\|_{L^\infty}+1)t}u$.

则 $\Rightarrow \begin{cases} \partial_t v - \Delta v + (c + \|c\|_{L^\infty} + 1)v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$

由弱极大值原理 $\min_{\bar{U}_T} v \geq -\max_{\Gamma_T} u^- = -\max_{U} g^-$.

$g \geq 0 \Rightarrow g^- \equiv 0 \rightarrow \min_{\bar{U}_T} v \geq 0 \Rightarrow \min_{\bar{U}_T} u \geq 0$.

□

9. theorem 7.1.3 中 in (5?).
7.2.3 中 in (5?).

Proof:
(5?) 是什么？ $\forall u \in H^2(U) \cap H_0^1(U)$. 欲证: $\beta\|u\|_{H^2}^2 \le (Lu, -\Delta u) + \gamma\|u\|_{L^2(U)}^2$.
$(\exists \beta>0, \gamma\ge 0)$.

实际用上 in 只是对 Galerkin 逼近序列 因此不妨.

在此为了方便，我们设 $Lu = -\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u)$ ← 在实际构造中 $\Delta u_m = 0$ on ∂U.

$\{u \in C^\infty,\ u|_{\partial U} = 0.\ \Delta u|_{\partial U} = 0$.

要证: $-\int_U \sum_{i,j} a^{ij}\partial_i u\, \partial_j(\Delta u)\,dx \ge \frac{\theta}{2}\int_U |D^2 u|^2 dx - C\int_U |u|^2 dx$

①出现三阶导，希望分步积分.

分步积分可得:

$$-\sum_{i,j}\int_U a^{ij}\partial_i u\,\partial_j(\Delta u) = -\sum_{i,j}\int_U a^{ij}\partial_i u\,\nabla\cdot(\nabla\partial_j u)\,dx$$
$$= \sum_{i,j}\int_U \nabla(a^{ij}\partial_i u)\cdot\nabla(\partial_j\nabla u)\,dx$$
$$-\sum_{i,j}\int_{\partial U} a^{ij}\partial_i u\,(\partial_j\nabla u)\cdot\vec{n}\,dS$$
$$= \sum_{i,j}\Big(\int_U a^{ij}\partial_{ik}u\,\partial_{jk}u\,dx + \int_U \partial_k a^{ij}\,\partial_i u\,\partial_{jk}u\,dx\Big)$$
$$- I.$$

$I = \sum_{i,j,k}\int_{\partial U} a^{ij}\partial_i u\,\partial_{jk}u\cdot\cos(\vec{n},e_k)\,dS$
$= \frac{\partial(\partial_j u)}{\partial\vec{n}}$, 记为

希望对 I 有何估计？

问题:
· 计算出 u 在所有二阶导数 (along ∂U).

· 如何处理 $\frac{\partial}{\partial n}$. (边界上的方向导数)?

手段: 边界拉直 (on 等边法) 、等式两边坐标.

· 直接计算. 先求 $\partial_{nn}u$. 再求 $\partial_\alpha\partial_n u$. 再求 $\partial_\alpha\partial_\beta u$.

· 希望的结果: $|I| \lesssim \varepsilon\int_U |D^2u|^2 + C_\varepsilon\int_U |u|^2 dx$.

运算 $\Leftarrow$ (Young) $|I| \lesssim \int_U |\nabla u|^2\,dx$

因 I 本身是 ∂U 的积分，转化为 U 中积分只有靠"迹定理" ← 迹定理 $\Leftarrow$ $|I| \lesssim \int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2 dS$.

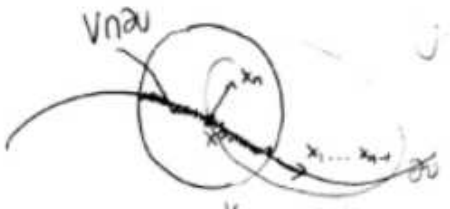
局部坐标下，法向为 e_n
$\Leftarrow$ $|I| \lesssim \int_\Sigma (\partial_n u)^2\, d\tilde{x}^{n-1}$ ← 于是变成为了我们的目标.

即: 设法用 $\partial_n u$. 等，来给出 I 中各个导数 (尤其是二阶导) 的估计.
$\partial_\alpha\partial_n u$.
$\partial_\alpha\partial_\beta u$.

于是，我们现在要做的是：将∂U上的导数用$\partial_{x_n} u$, $\partial_{x_n}^2 u$表示出来。

Step 1：由单位分解（因∂U紧），可以假设a^{ij}的支集包含于某一点$x_0 \in \partial U$的邻域V内。

不妨设：x^0是原点，∂U在x^0处的法向是x_n轴$(\vec{e}_n)$。



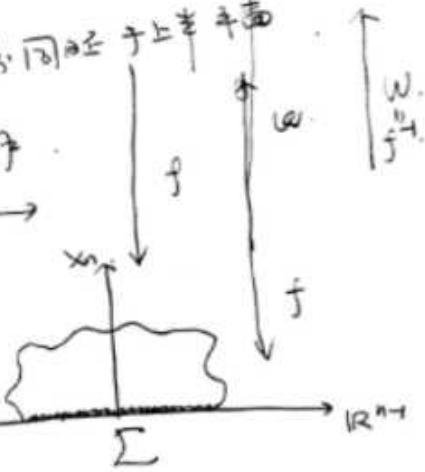
$\Sigma = V \cap \partial U$在$x_n$轴的投影。

记$x_n = w(x')$. $x' = (x_1, \dots, x_{n-1}) \in \Sigma$

$w \in C^2(\Sigma)$.

↓局部微分同胚于上半平面

↳ 这一步对某类可以用带边流形之定义来理解



如何在Σ附近微分同胚于f：

w即是"该微分同胚之逆"

Step 2：计算$\partial_\alpha \partial_\beta u$. $(\partial_\alpha = \frac{\partial}{\partial x_\alpha})$.

令$v(x') = \partial_n u(x', w(x'))$. ← 即$V \cap \partial U$任一点，用局部坐标可表成$(x', w(x'))$

(1°)

① 先求$\partial_n^2 u$.

对上式求导$(\alpha = 1, 2, \dots, n-1)$.

有：$\partial_\alpha v = \partial_\alpha \partial_n u + \partial_n^2 u \cdot \partial_\alpha w$. ... ①

又：$u|_{\partial U} = 0$. 故$u(x', w(x')) = 0$. $\forall x' \in \Sigma$.

对x_α求导：$\partial_\alpha u + \partial_n u \partial_\alpha w = \partial_\alpha u + v \cdot \partial_\alpha w = 0$.

对x_β求导：$\partial_\alpha \partial_\beta u + \partial_\beta \partial_n u \partial_\beta w + \partial_\beta v \partial_\alpha w + v \cdot \partial_\alpha \partial_\beta w = 0$. ... ②

$1 \le \alpha, \beta \le n-1$.

取$\alpha = \beta$有：$\partial_\alpha^2 u + \partial_\alpha \partial_n u \partial_\alpha w + \partial_\alpha v \partial_\alpha w + v \cdot \partial_\alpha^2 w = 0$ ... ③.

对 α 从 $1\sim n-1$ 求和：

注意到 $\Delta u\big|_{\partial u}=0 \Rightarrow -\partial_n^2 u = \sum_{1\le\alpha\le n-1}\partial_\alpha^2 u$.

有：$-\partial_n^2 u + \sum_{1\le\alpha\le n-1}\partial_\alpha\partial_n u\cdot\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\,\partial_\alpha^2 w = 0$ …④

①代入④，有：（重复指标代表求和）.

$$-\partial_n^2 u + (\partial_\alpha v - \partial_n^2 u\,\partial_\alpha w)\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\,\partial_\alpha^2 w = 0$$

$$\Rightarrow \partial_n^2 u\left(1+\sum_\alpha(\partial_\alpha w)^2\right) = v\Delta_{n-1}w + 2\partial_\alpha v\,\partial_\alpha w$$

$$\Rightarrow \partial_n^2 u = \frac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}}\partial_\alpha v + \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}}v$$

这样，对 $\partial_n^2 u$，我们达到了目的，即用 $\partial_\alpha v$，v（i.e. $\partial_\alpha\partial_n u$，$\partial_n u$）表示.

方便起见，令 $\sigma_{nn}^\alpha = \dfrac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C^1(\Sigma)$

$$\tau_{nn} = \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C(\Sigma).$$

$$\Rightarrow \partial_n^2 u = \sigma_{nn}^\alpha\partial_\alpha v + \tau_{nn}v \quad \boxed{\text{上下指标表示求和}}$$

…⑤

(2) 求 $\partial_\alpha\partial_n u$.

⑤代入①，即有：

α 换成 β，因避免与①中 α 重复.

$$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_\alpha w\left(\sigma_{nn}^\beta\partial_\beta v + \tau_{nn}v\right).$$

$$\Rightarrow \partial_\alpha\partial_n u = \partial_\alpha v - \partial_\alpha w\left(\sigma_{nn}^\beta\partial_\beta v + \tau_{nn}v\right).$$

$$= \partial_\beta v\left(\delta_\alpha^\beta - \sigma_{nn}^\beta\partial_\alpha w\right) - \tau_{nn}\partial_\alpha w\cdot v.$$

令 $\tau_{\alpha n} = \tau_{nn}\partial_\alpha v \in C(\Sigma)$　　$\sigma_{\alpha n}^\beta = \delta_\alpha^\beta - \sigma_{nn}^\beta\partial_\alpha w \in C^1(\Sigma)$

有：$\partial_\alpha\partial_n u = \sigma_{\alpha n}^\beta\partial_\beta v - \tau_{\alpha n}v$ …⑥

(3°) 求 $\partial_\beta\partial_\alpha u$.

回代(1)(2)有:

$$\partial_{\alpha\beta}u + (\sigma^{\nu}_{\alpha n}\partial_\gamma v + \tau_{\alpha n}v)\partial_\beta w + \partial_\beta v\,\partial_\alpha w + v\,\partial_\alpha\partial_\beta w = 0$$

$$\Rightarrow \partial_{\alpha\beta}u = -(\sigma^{\nu}_{\alpha n}\partial_\gamma v + \tau_{\alpha n}v)\partial_\beta w - \partial_\beta v\,\partial_\alpha w - \partial_{\alpha\beta}w\cdot v$$

$$= \partial_\gamma v\left(-\sigma^{\nu}_{\alpha n}\partial_\beta w - \vec{\sigma}^{\nu}_{\beta}\partial_\alpha w\right)$$

$$+ v\left(-\partial_{\alpha\beta}w - \tau_{\alpha n}\partial_\beta w\right)$$

$$\because \sigma^{\nu}_{\alpha\beta} = -\sigma^{\nu}_{\alpha n}\partial_\beta w - \vec{\sigma}^{\nu}_{\beta}\partial_\alpha w$$

$$\tau_{\alpha\beta} = -\partial_{\alpha\beta}w - \tau_{\alpha n}\partial_\beta w$$

$$\Rightarrow \partial_{\alpha\beta}u = \sigma^{\nu}_{\alpha\beta}\partial_\gamma v + \tau_{\alpha\beta}v \qquad \sigma^{\nu}_{\alpha\beta}\in C^1(\Sigma),\ \tau_{\alpha\beta}\in C(\Sigma)$$

Step3: 完成估计:

如今, 可以用一些 $C^1(\Sigma)$ 之类 g^α 与 $C^0(\Sigma)$ 之类 h.

表为 I 如下:

$$I = \int_\Sigma v\left(g^\alpha\partial_\alpha v + hv\right)d\mathcal{H}^{n-1}\cdot\underbrace{|\det w|}_{=1}$$

$$|I| = \left|\int_\Sigma \left(\tfrac{1}{2}g^\alpha\partial_\alpha(v^2) + hv^2\right)d\mathcal{H}^{n-1}\right|$$

分部积分 $= \left|\int_\Sigma \left(h - \tfrac{1}{2}\partial_\alpha g^\alpha\right)v^2\,d\mathcal{H}^{n-1}\right|$ 迹定理

$$\lesssim \int_\Sigma v^2\,d\mathcal{H}^{n-1} \lesssim \int\left|\frac{\partial u}{\partial n}\right|^2 dS \lesssim \int_U |\nabla u|^2\,dx.$$

再由 Chs. 可有 $\int_U|\nabla u|^2dx \le \|u\|_{L^2}\|D^2u\|_{L^2} \lesssim \varepsilon\|D^2u\|_{L^2}^2 + C(\varepsilon)\|u\|_{L^2}^2$ □

10. 求证：如下方程至多一个光滑解　$d>0$ ← 书上应该漏条件

$$\begin{cases} u_{tt} + du_t - u_{xx} = f & \text{in } (0,1)\times(0,T) \\ u = 0 & \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]) \\ u(0)=g \quad u_t(0)=h \end{cases}$$

证明：若有2个光滑解 ~~u_1~~ u_1, u_2，令 $v = u_1 - u_2$

则 v 满足

$$\begin{cases} v_{tt} + dv_t - v_{xx} = 0 & \text{in } (0,1)\times(0,T) \\ v = 0 & \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]) \\ v(0)=g \quad v_t(0)=h. \end{cases}$$

令 $E(t) = \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{1}{2}\|v_x\|_2^2$

则 $E'(t) = \int_0^1 v_t v_{tt} + v_x v_{xt}\,dx$

第2项分部积分.

$$= \int_0^1 v_t v_{tt} - v_{xx} v_t\,dx$$

$$= \int_0^1 (-dv_t) v_t\,dx$$

$$= -d\int_0^1 v_t^2 \le 0$$

又 $E(0)=0$，$\therefore E(t)\equiv 0 \Rightarrow v(t)=\text{const}$.

又 $v(0)=0$　$\therefore v\equiv 0$

□

11. $\begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } (0,1)\times(0,T) \\ u = \partial_x u = 0 & \text{on } (\{0\}\times[0,T]) \cup (\{1\}\times[0,T]) \\ u = g,\ u_t = h & \text{on } [0,1]\times\{t=0\} \end{cases}$

存在至多一个光滑解.

Proof:

$v = u_1 - u_2$.

$\Rightarrow \partial_t^2 v + \partial_x^4 v = 0$.

乘以 $v_t \Rightarrow \partial_t v \partial_t^2 v + \partial_t v \partial_x^4 v = 0$

而: $\partial_t (\partial_t v)^2 = 2\partial_t^2 v \partial_t v$.

$\partial_t (\partial_x^2 v) = 2(\partial_x^2 v \cdot \partial_t \partial_x^2 v)$

所以上式 $\Rightarrow$ 积分 $\frac{1}{2}\partial_t\left(\|v_t\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2\right) = 0$

(分部2次,对x)

$\Rightarrow \|v_t\|^2 + \|v_{xx}\|^2 = \text{const.} = t=0$ 时的值 $= 0$

$\Rightarrow v = 0$.

□.

12: 设 A 为实 Banach 空间 X 上的闭算子, 定义域为 D(A), 若 $\lambda, \nu \in \rho(A)$

求证: (1) $R_\lambda - R_\nu = (\nu - \lambda) R_\lambda R_\nu$

(2) $R_\lambda R_\nu = R_\nu R_\lambda$.

证明: (1) 不妨 $\lambda \neq \nu$

$R_\lambda - R_\nu = R_\lambda \cdot \underbrace{(\nu I - A) R_\nu}_{Id} - \underbrace{(\lambda I - A) R_\lambda R_\nu}_{Id}$

$= (\nu - \lambda) R_\lambda R_\nu$

(2) 调换 λ, ν. 有 $R_\nu R_\lambda (\lambda - \nu) = R_\nu - R_\lambda$.

$= -(R_\lambda - R_\nu)$

移项 $\lambda - \nu$ $= -(\nu - \lambda) R_\lambda R_\nu$

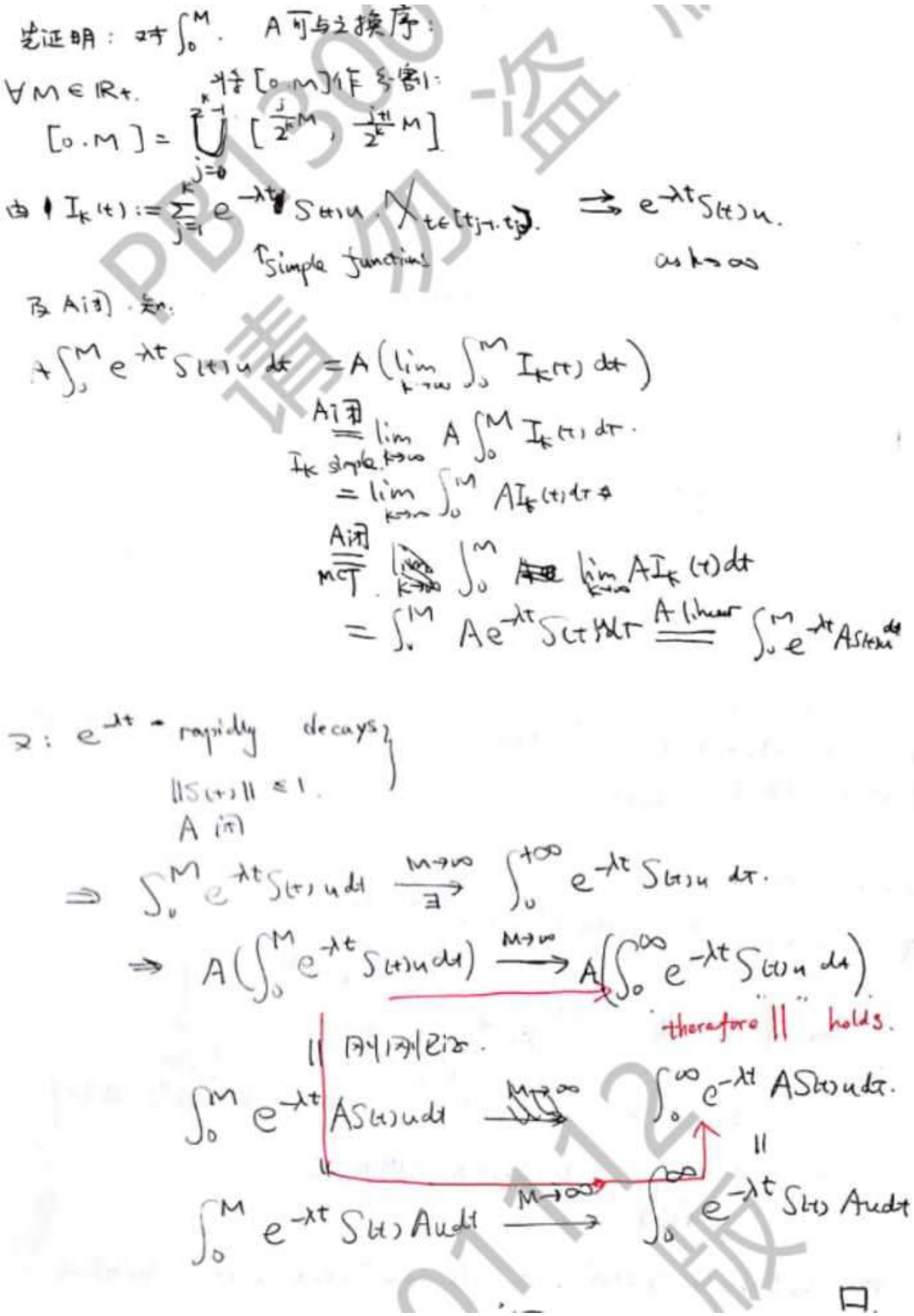
$\Rightarrow R_\lambda R_\nu = R_\nu R_\lambda$

□

13. Justify the equality

$$A\int_0^\infty e^{-\lambda t}S(t)u\,dt = \int_0^\infty e^{-\lambda t}AS(t)u\,dt$$

used in (16) of §7.4.1. (Hint: Approximate the integral by a Riemann sum and recall A is a closed operator.)



14.

设 Φ 是热方程的基本解，即 $\Phi(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$.

$\forall t>0$. 令

$$[S(t)g](x)=\int_{\mathbb{R}^n}\Phi(x-y,t)\,g(y)\,dy. \qquad x\in\mathbb{R}^n.$$

$g:\mathbb{R}^n\to\mathbb{R}$.

$S(0)g=g$.

则：$\{S(t)\}_{t\geq 0}$ 是 $L^2(\mathbb{R}^n)$ 上的压缩半群

不是 $L^\infty(\mathbb{R}^n)$ 上的压缩半群

· $S(t)$ 是 L^2 上的压缩半群

· $\|S(t)g\|_{L^2}=\|\Phi * g\|_{L^2}$

$\leq \|\Phi\|_{L^1}\|g\|_{L^2}=\|g\|_{L^2}\Rightarrow \|S(t)\|\leq 1.$

· ~~$t\mapsto S(t)$~~

$S(t+s)g=\Phi(x-y)\ \Phi(x-y,t+s)\ g(\xi)$.

$\hat{\Phi}(\xi,t+s)\,\hat{g}(\xi)$.

$=\hat{\Phi}(\xi,t)\hat{\Phi}(\xi,s)\hat{g}(\xi)=S(t)S(s)g$

· $t\mapsto S(t)g$ 连续性：

$\|S(t+h)g-S(t)g\|_{L^2}\leq\|S(h)g-g\|_{L^2}$.

$=\|\int_{\mathbb{R}^n}\Phi(x-y,h)g(y)\,dy-g(x)\|_{L^2_x}$.

$=\|\int_{\mathbb{R}^n}\Phi(x-y,h)\,(g(y)-g(x))\,dy\|_{L^2_x}$.

$=\|\int_{\mathbb{R}^n}\Phi(y,h)\,(g(x-y)-g(x))\,dy\|_{L^2_x}$

$\leq\int_{\mathbb{R}^n}\Phi(y,h)\cdot\|g(x-y)-g(x)\|_{L^2_x}\,dy$.

$\downarrow_0$ (平移连续性)

$\xrightarrow{DCT}0$.

· $S(t)$ 不是 L^∞ 上的压缩半群，因 在 $t=0$ 处不连续

令. $g(x)=\chi_{\{x>0\}}$ $\cancel{g(x,y)=\chi_{\{x>0,\ y>0\}}}$. $g(0)=0$.

$$(S(t)g)(x)=\int_{-\infty}^{x}\frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}}\,dy. \qquad (S(t)g)(0)=\frac{1}{2} \quad \forall t>0$$

$$\Rightarrow \|S(t)g-g\|_{L^\infty}\geq\frac{1}{2}$$

□

15. 设 X 上有以 A 为无穷小生成元的压缩半群 $\{S(t)\}_{t\geq0}$

归纳定义 $D(A^k):=\{u\in D(A^{k-1})\mid A^{k-1}u\in D(A)\}$ $(k\geq2)$

证明：若 $\exists k$. $u\in D(A^k)$. 则 $\forall t\geq0$ $S(t)u\in D(A^k)$

Pf: $\forall j\in\{1,2,\dots,k-1\}$. 要证：$A^jS(t)u\in D(A)$

i.e. $$\lim_{s\to0^+}\frac{S(s)A^jS(t)u-A^jS(t)u}{s}$$

$$\frac{S(s)A^jS(t)u-A^jS(t)u}{s}$$

$$=\frac{S(s)S(t)A^ju-S(t)A^ju}{s}$$

$$=\frac{S(t+s)(A^ju)-S(t)(A^ju)}{s}$$

归纳假设 $A^ju\in D(A)$

$=$ exists (以及 $S(0)=Id$).

□

16. 用T15证明：若 u 是 $\begin{cases}u_t-\Delta u=0 & \text{in } U_T\\ u=0 & \text{on } \partial U\times[0,T]\\ u=g & \text{on } U\times\{t=0\}\end{cases}$ 在 $X=L^2(U)$ 中的半群解

且 $g\in C_c^\infty(U)$，则 $u(\cdot,t)\in C^\infty(U)$ $0\leq t\leq T$

Pf. 先证 $\{S(t)\}_{t\geq0}$ 压缩，此为显见. 因 在方程两边同乘 u. 分部积分有

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2)+\|\nabla u\|^2=0. \Rightarrow \frac{d}{dt}\|u\|_{L^2}^2\leq0$$

$$\Rightarrow \|u(t)\|_{L^2}^2\leq\|u(0)\|_{L^2}^2$$

$$\Rightarrow \|S(t)g\|_{L^2}\leq\|g\|_{L^2}\ \forall g\in L^2.$$

$\therefore$ $\{S(t)\}$ 压缩

而 $g\in C_c^\infty(U)\subset H^{2k}(U)\cap H_0^{2k-1}(U)=D(\Delta^k)$ $k=1$ 是显见的. 之后归纳即可.

由T15知. $\forall t\geq0$

$$u(\cdot,t)=S(t)g\in H^{2k}(U)\cap H_0^{2k-1}(U).\quad \forall k\in\mathbb{Z}_+.$$

由Sobolev嵌入知 $u(\cdot,t)\in C^\infty(U)$.

□

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