Ch7 习题

1. $$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\}. \end{cases}$$

至多一个光滑解

Proof: ~~全~~ 设 $u_1$. $u_2$ 为原方程2个光滑解. $v = u_1 - u_2$ 要证 $v=0$

v 满足 $$\begin{cases} v_t - \Delta v = \cancel{f}\, 0 & \text{in } U_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ v = 0 & \text{on } U \times \{t=0\} \end{cases}$$

两边乘以v. 积分得.

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 - \int_U v\, \Delta v = 0$$

分部积分

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\int_U |\nabla v|^2\, dx \le 0$$

又: $\|v(0)\|_{L^2}^2 = 0$ 故 $\forall t \in (0,T]$. $\|v(t)\|_{L^2}^2 \le 0$

$$\Rightarrow \|v(t)\|_{L^2} = 0 \quad \underset{v \in C^\infty}{\Rightarrow} \quad v = 0 \quad \text{in } [0,T]$$

□

2. 设 u 是如下方程的光滑解.

$$\begin{cases} u_t - \Delta u = \cancel{f}\, 0 & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times [0,\infty) \\ u = g & \text{on } \partial U \times \{t=0\} \end{cases}$$

证明: $\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t}\|g\|_{L^2(U)}$

$\lambda_1 > 0$ 是 $-\Delta$ 的主特征值

Proof: 方程两边乘以u. 积分得

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\int_U |\nabla u|^2\, dx$$

又: $\lambda_1 = \inf\limits_{\substack{u \in H_0^1(U) \\ u \ne 0}} \dfrac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$

故 $\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 \le -\lambda_1 \|u(t)\|_{L^2}^2$.

由 Gronwall 不等式

$$\|u(t)\|_{L^2}^2 \le e^{-2\lambda_1 t}\|u(0)\|_{L^2}^2 = e^{-2\lambda_1 t}\|g\|_2^2$$

□

3. 设 $u$ 为
$$\begin{cases} \partial_t u + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$
的光滑解，其中 $L$ 为二阶椭圆算子

$v$ 为
$$\begin{cases} \partial_t v - L^* v = 0 & \text{in } U_T \\ v = 0 & \text{on } \partial U \times [0,T] \\ v = h & \text{on } U \times \{t=T\}. \end{cases}$$
的光滑解

求证：$\displaystyle\int_U g(x) v(x,0)\, dx = \int_U u(x,T) h(x)\, dx.$

证明：$\displaystyle\int_U u(x,T)h(x) - g(x)v(x,0)\, dx = \int_U u(x,T)v(x,T) - u(x,0)v(x,0)\, dx$

显见，这是关于 $t$ 分部积分出来的边界项

$$\text{上式} = \int_0^T \int_U (\partial_t u)\cdot v + (\partial_t v)\cdot u \, dx\, dt.$$

$\langle Lu, v\rangle = \langle u, L^* v\rangle$

$$= \int_0^T \int_U (\partial_t u)\cdot v + Lu\cdot v - u L^* v + \partial_t v \cdot u \, dx\, dt$$

$$= \int_0^T \int_U \underbrace{(\partial_t u + Lu)}_{=0} v + u\cdot \underbrace{(\partial_t v - L^* v)}_{=0}\, dx\, dt = 0$$

□

4. (Galerkin Method for Poisson).

$f \in L^2(U)$. $u_m = \sum_{k=1}^m d_m^k w_k$ solves $\displaystyle\int_U Du_m \cdot Dw_k \, dx = \int_U f\cdot w_k \, dx$, $1 \le k \le m$.

证明：$\{u_m\}$ 存在子列，在 $H_0^1(U)$ 中弱收敛于 $\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的弱解.

Pf. Step 1: $\{u_m\}$ 在 $H_0^1$ 中一致有界

$\displaystyle\int Du_m \cdot Dw_k \, dx = \int_U f\cdot w_k \, dx$. 两边乘以 $d_m^k$，对 $k$ 求和，

$\displaystyle\int \cancel{\ } \ \|Du_m\|_{L^2}^2 = \int_U f u_m \, dx$

$C\|u_m\|_{H_0^1}^2 \le \quad \le \|f\|_2 \|u_m\|_{H_0^1} \le \varepsilon \|u_m\|_{H_0^1}^2 + C(\varepsilon) \|f\|_{L^2}^2.$

$\varepsilon$ 充分小 $\Rightarrow \|u_m\|_{H_0^1} \le C\|f\|_{L^2}$ ✓

Step 2: $\exists$ 子列 $u_{m_k} \rightharpoonup u$ in $H_0^1(U)$,

$Du_{m_k} \rightharpoonup v$ in $L^2(U)$

$v \overset{\text{a.e.}}{=} Du$?

$\forall \varphi \in C_c^\infty$ $\quad \displaystyle\int Du_{m_k} \cdot D\varphi = \int f \varphi$, $\quad \displaystyle\int Du_{m_k} \cdot D\varphi \to \int v \cdot D\varphi$

$$\int Du_{m_k}\cdot D\varphi = -\int u_{m_k} \cdot D^2 \varphi \xrightarrow{k\to\infty} -\int u \cdot D^2\varphi = \int Du \cdot D\varphi$$

~~同理可得在 = $\int v D\varphi$~~

□

[7.5] 设

$$\begin{cases} \mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\ L^2(0,T;H_0^1(U)), \\ \mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U)). \end{cases}$$

证明:$\mathbf{u}' = \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))$.

**证明：**我们断言：

**Claim:** 对任意$\phi \in C_c^\infty(0,T), w \in H_0^1(U)$, 成立:

$$\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle = \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle,$$

其中$\langle \cdot, \cdot \rangle$代表$H^{-1}(U), H_0^1(U)$中元素之间的作用(pairing).

若Claim获证，那么在$H^{-1}(U)$中(即作为$H_0^1(U)$上的连续线性泛函)成立:

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{v}(t)\phi(t)dt.$$

再由时间弱导数定义知

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{u}'(t)\phi(t)dt.$$

这样就有$\mathbf{u}' = \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))$.

Claim的证明仍然由直接计算可得: 注意到$t \mapsto \pi(t)w \in L^2(0,T;H_0^1)$, 那么:

$$\begin{aligned}
\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle &= \int_0^T \langle \phi'(t)\mathbf{u}(t), w\rangle\, dt \\
&= \int_0^T \langle \mathbf{u}(t), \phi'(t)w\rangle\, dt \\
(\mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\ L^2(0,T;H_0^1(U))) &= \lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k(t), \phi'(t)w\rangle\, dt \\
&= \lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k(t)\phi'(t), w\rangle\, dt \\
&= \lim_{k\to\infty} \left\langle \int_0^T \mathbf{u}_k(t)\phi'(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty} \left\langle \int_0^T \mathbf{u}_k'(t)\phi(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty} \int_0^T \langle \mathbf{u}_k'(t)\phi(t), w\rangle\, dt \\
(\mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\ L^2(0,T;H^{-1}(U))) &= -\int_0^T \langle \mathbf{v}(t), \phi(t)w\rangle\, dt \\
&= \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle
\end{aligned}$$

6. 设 H是Hilbert空间, $u_k \rightharpoonup u$ in $L^2(0,T;H)$ , $\operatorname*{ess\,sup}_{0\le t\le T} \|u_k(t)\| \le C$ $\forall k\in \mathbb{Z}_+$

证明: $\operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\| \le C$

Proof: 先证: $\forall 0\le a\le b\le T$. $v\in H$, 有:

(hint). $\int_a^b (v, u_k(t))\,dt \le C\|v\|\cdot|b-a|$. (显然)

若 hint 成立, 则

$$\frac{1}{b-a}\int_a^b (v, u_k(t))\,dt \le C\|v\|$$

这样 由 Lebesgue 微分定理. $\xrightarrow{}$ t视[a.b]中点.

对 a.e. $t\in[0,T]$ 成立: $|\langle u_k^{(t)}, v\rangle| \le C\|v\|$

~~$k\to\infty$. 由弱收敛知. $|\langle u, v\rangle| \le C\|v\|$ a.e $t\in[0,T]$~~

~~$\Rightarrow \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\| \le C$.~~

6. H为Hilbert空间. $u_k \rightharpoonup u$ in $L^2(0,T;H)$. 且 $\operatorname*{ess\,sup}_{0\le t\le T} \|u_k(t)\| \le C$ $\forall k\in \mathbb{Z}_+$.

求证: $\operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\| \le C$

证明: 设 $f_{ab}(v) = \int_a^b (v, u(t))\,dt$.

显然 $f_{ab} \in L^2(0,T;H)$.

故 $\lim_{k\to\infty} f_{a,b}(u_k) = f_{a,b}(u) = \int_a^b (u(t), u(t))\,dt$.

而 $f_{a,b}(u_k) = \int_a^b (u_k(t), u(t))\,dt$

$$\le C\int_a^b \|u(t)\|_H\,dt$$

$$\le C\sqrt{b-a}\,\big\|\|u(t)\|_H\big\|_{L^2_t(a,b)}$$

$$\xRightarrow{k\to\infty} \int_a^b \|u(t)\|_H^2\,dt \le C^2(b-a) \quad \forall a,b\in[0,T]$$

$\Rightarrow \|u(t)\|_H \le C$ a.e. $t\in[0,T]$.

除以 b-a. 用Lebesgue 微分定理即可 □

令 $b\to a$

7. 设u是光滑解：$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U\times(0,\infty) \\ u = 0 & \text{on } \partial U\times[0,\infty) \\ u = g & \text{on } U\times\{t=0\} \end{cases}$

且函数c满足 $c \geq \gamma > 0$.

证明：$|u(x,t)| \leq Ce^{-\gamma t}$

Proof：设 $v = e^{\gamma t}u$.

则 $\partial_t v - \Delta v + cv = \gamma e^{\gamma t}u + e^{\gamma t}u_t - e^{\gamma t}\Delta u + ce^{\gamma t}u$

$$= \gamma v + (\underbrace{(\partial_t - \Delta + c)u}_{0})e^{\gamma t}$$

$$= \gamma v$$

$$\Rightarrow \begin{cases} \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$$

由弱极大值原理：

$\forall (x,t)\in U_T$.

$$|v(x,t)| = e^{\gamma t}|u(x,t)| \leq \sup_{\Gamma_T}|v(x,t)|$$

$$= \sup_{x\in U}|g(x)|.$$

$$\Rightarrow |u(x,t)| \leq e^{-\gamma t}\|g\|_{L^\infty}$$

8. 若u是7中方程的光滑解：$g\geq 0$，但c有界但不一定非负，证明 $u\geq 0$

Proof：令 $v = e^{-(\|c\|_{L^\infty}+1)t}u$.

$$\xRightarrow{\text{同7.}} \begin{cases} \partial_t v - \Delta v + (c + \|c\|_{L^\infty} + 1)v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$$

由弱极大值原理. $\min\limits_{\bar{U}_T} v \geq -\max\limits_{\Gamma_T} u^- = -\max\limits_{U} g^-$.

$$g\geq 0 \Rightarrow g^- \equiv 0 \Rightarrow \min_{\bar{U}_T} v \geq 0 \Rightarrow \min_{\bar{U}_T} u \geq 0. \qquad \square$$

9. 证明 7.1.3 中的 (54).
   7.2.3 中的 (59).

Proof:

(54) 是什么? $\forall u\in H^2(U)\cap H_0^1(U)$ 成立: $\beta\|u\|_{H^2}^2 \le (Lu,-\Delta u)+\gamma\|u\|_{L^2(U)}^2$

$(\exists\beta>0,\ \gamma\ge 0)$.

实际用上的只是对 Galerkin 逼近序列用此不等式.

在此,为了方便,我们设 $Lu=-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)$ 　在实际构造中 $\Delta u_m=0$ on $\partial U$.

$\begin{cases} u\in C^\infty, & u|_{\partial U}=0. \quad \Delta u|_{\partial U}=0.\end{cases}$

要证: $$-\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j(\Delta u)\,dx \ge \frac{\theta}{2}\int_U|D^2u|^2dx - C\int_U|u|^2dx$$

希望对 I 有何控制?

问题:

. 计算出 $u$ 的所有二阶导数 (along $\partial U$).

. 如何处理 $\frac{\partial u}{\partial x_n}$. (边界的法向导数)?

手段: 边界拉直 (on 某边界的邻域之内), 写成局部坐标.

. 直接计算. 先求 $\partial_{nn}u$. 再求 $\partial_\alpha\partial_n u$. 再求 $\partial_\alpha\partial_\beta u$.

. 希望的结果. $|I|\le \varepsilon\int_U|D^2u|^2+C_\varepsilon\int_U|u|^2dx$.

这样 $\xLeftarrow{\text{Young}}$ $|I|\lesssim\int_U|Du|^2dx$

因 I 本身是 $\partial U$ 的积分, 若转化为 U 中的积分只有靠"迹定理" $\xLeftarrow{\text{迹定理}}$ $|I|\lesssim\int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2\underset{\sim}{dS}$.

$\xLeftarrow{\text{局部坐标下,法向为}e_n}$ $|I|\lesssim\int_\Sigma(\partial_{x_n}u)^2\,d\mathcal{H}^{n-1}$ ← 于是这成为了我们的目标.

即: 设法用 $\partial_{x_n}u$. 等, 来给出 I 中各个导数 (尤其是二阶的) 的估计.
$\partial_{x_\alpha}\partial_{x_n}u$.

于是，我们现在要做的是，将∂U上的各阶导数用 $\partial_{x_n}u$、$\partial\partial_{x_n}^2 u$ 表示出来。
令中

Step 1：由单位分解（因∂U紧），可以假设 $a^{ij}$ 的支撑包含于某一点 $x_0\in\partial U$ 的邻域V内。

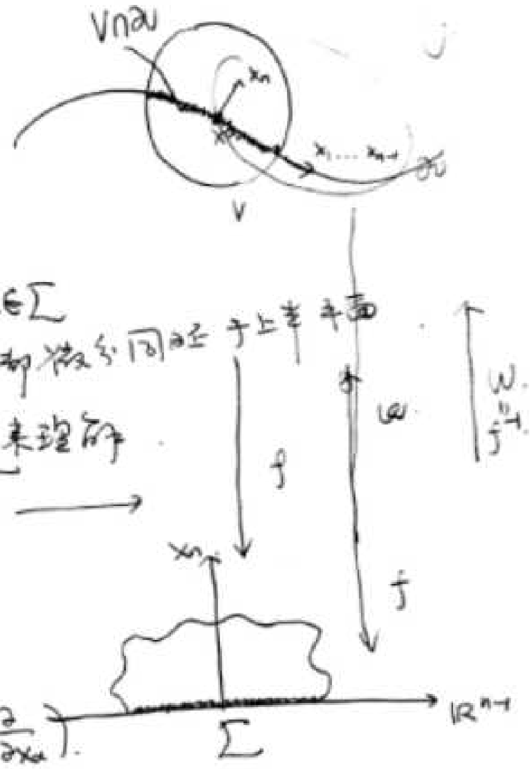
不妨假设：$x^0$ 是 ~~原点~~ 原点，
∂U在 $x^0$ 处的法向是 $x_n$ 轴 ($\vec{e}_n$)。

$\Sigma$ = V∩∂U 在 $x_n$ 轴的投影。

记 $x_n = w(x')$，$x' = (x_1, \dots, x_{n-1}) \in \Sigma$
$w \in C^2(\Sigma)$。

→ 这一段文字其实可以用 带边流形的定义来理解
如同 则存在 ~~坐标~~ 微分同胚 f：
w 即是 *该微分同胚之逆
~~坐标映射之逆~~

局部微分同胚于上半平面




$(\partial_\alpha = \frac{\partial}{\partial x_\alpha})$.

Step 2：计算 $\partial_\alpha\partial_\beta u$。 $(\partial_\alpha = \frac{\partial}{\partial x_\alpha})$.

(1°) 令 $v(x') = \partial_n u(x', w(x'))$ ⟶ 即 V∩∂U 任一点，用局部坐标可写成 $(x', w(x'))$

~~①~~ 求 $\partial_n^2 u$。
对上式求导 $(\alpha = 1, 2, \dots, n-1)$。

有：$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_n^2 u \cdot \partial_\alpha w$. $\quad\dots$ ①

又：$u|_{\partial U} = 0$。 故 $u(x', w(x')) = 0$ $\forall x' \in \Sigma$。

对 $x_\alpha$ 求导：$\partial_\alpha u + \partial_n u\, \partial_\alpha w = \partial_\alpha u + v \cdot \partial_\alpha w = 0$. on V∩∂U.

对 $x_\beta$ 求导. $\partial_\alpha\partial_\beta u + \partial_\alpha\partial_n u\, \partial_\beta w + \partial_\beta v\, \partial_\alpha w + v \cdot \partial_\alpha\partial_\beta w = 0$. $\quad\dots$ ②

$1 \le \alpha, \beta \le n-1$.
取 $\alpha = \beta$ 有：$\partial_\alpha^2 u + \partial_\alpha\partial_n u\, \partial_\alpha w + \partial_\alpha v\, \partial_\alpha w + v \cdot \partial_\alpha^2 w = 0$ $\quad\dots$ ③

对 $\alpha$ 从 $1\sim n-1$ 求和： ~~有~~

注意到 $\Delta u\big|_{\partial u}=0 \Rightarrow -\partial_n^2 u = \sum_{1\le\alpha\le n-1}\partial_\alpha^2 u$.

有：$-\partial_n^2 u + \sum_{1\le\alpha\le n-1} \partial_\alpha\partial_n u\cdot\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\,\partial_\alpha^2 w = 0$ ... ④

① 代入④，有：（重复希腊字母指标代表求和）.

$$-\partial_n^2 u + (\partial_\alpha v - \partial_n^2 u\,\partial_\alpha w)\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\,\partial_\alpha^2 w = 0$$

$$\Rightarrow \partial_n^2 u\left(1+\sum_\alpha(\partial_\alpha w)^2\right) = v\,\Delta_{n-1} w + 2\partial_\alpha v\,\partial_\alpha w$$

$$\Rightarrow \partial_n^2 u = \frac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}}\partial_\alpha v + \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}}v$$

这样，对 $\partial_n^2 u$，我们达到了目的，即用 $\partial_\alpha v$, $v$ (i.e. $\partial_\alpha\partial_n u$, $\partial_n u$) 表示.

方便起见，令 $\sigma_{nn}^\alpha = \dfrac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C^1(\Sigma)$

$$T_{nn} = \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C(\Sigma).$$

$$\Rightarrow \partial_n^2 u = \sigma_{nn}^\alpha\,\partial_\alpha v + T_{nn}\,v \qquad \cdots ⑤$$

上下指标表示求和

(2) 求 $\partial_\alpha\partial_n u$.

⑤代入① 即有：

$\alpha$ 换成 $\beta$，因这样与①中 $\alpha$ 重复.

$$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_\alpha w\left(\sigma_{nn}^\beta\,\partial_\beta v + T_{nn}\,v\right).$$

$$\Rightarrow \partial_\alpha\partial_n u = \partial_\alpha v - \partial_\alpha w\left(\sigma_{nn}^\beta\,\partial_\beta v + T_{nn}\,v\right).$$

$$= \partial_\beta v\left(\delta_\alpha^\beta - \sigma_{nn}^\beta\,\partial_\alpha w\right) - T_{nn}\,\partial_\alpha w\cdot v.$$

令 $T_{\alpha n} = T_{nn}\,\partial_\alpha w \in C(\Sigma)$ $\quad \sigma_{\alpha n}^\beta = \delta_\alpha^\beta - \sigma_{nn}^\beta\,\partial_\alpha w \in C^1(\Sigma)$

有：$\partial_\alpha\partial_n u = \sigma_{\alpha n}^\beta\,\partial_\beta v - T_{\alpha n}\,v$ ... ⑥

(3°) 求 $\partial_\beta\partial_\alpha u$.

①代入② 有:

$$\partial_{\alpha\beta} u + (\sigma_{\alpha n}^{\nu}\partial_\nu v + \tau_{\alpha n} v)\partial_\beta w + \partial_\beta v\,\partial_\alpha w + v\,\partial_\alpha\partial_\beta w = 0$$

$$\Rightarrow \partial_{\alpha\beta} u = -(\sigma_{\alpha n}^{\nu}\partial_\nu v + \tau_{\alpha n} v)\partial_\beta w - \partial_\beta v\,\partial_\alpha w - \partial_{\alpha\beta} w\cdot v = 0.$$

$$= \partial_\nu v\,(-\sigma_{\alpha n}^{\nu}\partial_\beta w - \delta_\beta^{\nu}\partial_\alpha w)$$

$$+ v\,(-\partial_{\alpha\beta} w - \tau_{\alpha n}\partial_\beta w)$$

令 $\sigma_{\alpha\beta}^{\nu} = -\sigma_{\alpha n}^{\nu}\partial_\beta w - \delta_\beta^{\nu}\partial_\alpha w$

$\tau_{\alpha\beta} = -\partial_{\alpha\beta} w - \tau_{\alpha n}\partial_\beta w$

$$\Rightarrow \partial_{\alpha\beta} u = \sigma_{\alpha\beta}^{\nu}\partial_\nu v + \tau_{\alpha\beta} v \qquad \sigma_{\alpha\beta}^{\nu}\in C^1(\Sigma),\ \tau_{\alpha\beta}\in C(\Sigma)$$

Step3: 完成估计:

如今, 可以用一些 $C^1(\Sigma)$ 之函 $g^\alpha$ 与 $C^0(\Sigma)$ 之函 $h$.

表示 I 如下:

$$I = \int_\Sigma v\,(g^\alpha\partial_\alpha v + hv)\,d\mathcal{H}^{n-1}\cdot\underbrace{|\det w|}_{=1}$$

$$|I| = \cancel{\left|\int_\Sigma \frac{v(g^\alpha\partial_\alpha v^2 + hv^2)}{2}\right|}$$

$$\left|\int_\Sigma \left(\tfrac{1}{2}g^\alpha\partial_\alpha(v^2) + hv^2\right)d\mathcal{H}^{n-1}\right|.$$

分部积分地

$$= \left|\int_\Sigma \left(h - \tfrac{1}{2}\partial_\alpha g^\alpha\right)v^2\,d\mathcal{H}^{n-1}\right|.$$ 迹定理

$$\lesssim \int_\Sigma v^2\,d\mathcal{H}^{n-1} \lesssim \int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2 dS \lesssim \int_U |\nabla u|^2\,dx$$

再由 CHS. T9 有 由 $\int_U |\nabla u|^2 dx \lesssim \|u\|_{L^2}\|D^2 u\|_{L^2} \lesssim \varepsilon\|D^2 u\|_{L^2}^2 + C(\varepsilon)\|u\|_{L^2}^2$ $\square$

10. 求证：如下方程至多一个光滑解，其中 $d$ 为常数

~~证明~~

$$\begin{cases} u_{tt} - du_t - u_{xx} = f & \text{in } (0,1)\times(0,T) \\ u = 0 & \text{on } (\{0\}\times[0,T]) \cup (\{1\}\times[0,T]) \\ u = g,\ u_t = h & \text{on } (0,1)\times\{t=0\} \end{cases}$$

证明：若有二个光滑解 $u_1, u_2$，令 $v = u_1 - u_2$

则：

$$\begin{cases} v_{tt} - dv_t - v_{xx} = 0 & \text{in } U\times(0,T) \\ v = 0 & \text{on } \partial U\times[0,T] \\ v = 0 \quad v_t = 0 & \text{on } U\times\{0\} \end{cases} \qquad U = (0,1).$$

令 $E(t) = \frac{1}{2}\left(\|\partial_t u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)}\right)$

$$E'(t) = \int_U u_t u_{tt} - du_t^2 + u_x u_{xt}\, dx$$

第3项分部积分

$$= \int_U u_t u_{tt} - du_t^2 - u_{xx} u_t \, dx$$

$$= \int_U u_t ( u_{tt} - du_t + u_{xx}) \, dx = 0$$

□

11. $\begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } (0,1)\times(0,T). \\ u = \partial_x u = 0 & \text{on } (\{0\}\times[0,T]) \cup (\{1\}\times[0,T]) \\ u = g.\ u_t = h. & \text{on } [0,1]\times\{t=0\} \end{cases}$

<u>存在至多一个光滑解</u>.

<u>Proof</u>:

$v = u_1 - u_2$.

$$\Rightarrow \partial_t^2 v + \partial_x^4 v = 0.$$

乘以 $v_t$ $\Rightarrow$ $\partial_t v \partial_t^2 v + \partial_t v \partial_x^4 v = 0.$

而: $\partial_t(\partial_t v)^2 = 2\partial_t^2 v \partial_t v.$

$\partial_t(\partial_x^2 v) = 2(\partial_x^2 v \cdot \partial_t \partial_x^2 v)$

所以上式 $\Rightarrow$ $\frac{1}{2}\partial_t\left(\|v_t\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2\right) = 0$

积分.

(分部2次.对$x$)

$\Rightarrow \|v_t\|^2 + \|v_{xx}\|^2 = \text{const.} = t=0$ 时的值 $= 0$

$\Rightarrow v = 0.$

□.

12: 设 $A$ 为实Banach空间$X$上的闭算子, 定义域为$D(A)$ ~~预解式$\rho(A)$~~ 若 $\lambda, \nu \in \rho(A)$

求证: (1) $R_\lambda - R_\nu = (\nu - \lambda) R_\lambda R_\nu$

(2) $R_\lambda R_\nu = R_\nu R_\lambda$.

证明: (1) 不妨 $\lambda \neq \nu$

$$R_\lambda - R_\nu = R_\lambda \cdot \underbrace{(\nu I - A) R_\nu}_{\text{Id}} - \underbrace{(\lambda I - A) R_\lambda}_{\text{Id}} R_\nu$$

$$= (\nu - \lambda) R_\lambda R_\nu$$

(2) 调换 $\lambda, \nu$. 有 $R_\nu R_\lambda (\lambda - \nu) = R_\nu - R_\lambda$.

$= -(R_\lambda - R_\nu)$

$= -(\nu - \lambda) R_\lambda R_\nu$

约掉 $\lambda - \nu$

$\Rightarrow R_\lambda R_\nu = R_\nu R_\lambda$

□

13. Justify the equality

$$A\int_0^\infty e^{-\lambda t}S(t)u\,dt=\int_0^\infty e^{-\lambda t}AS(t)u\,dt$$

used in (16) of §7.4.1. (Hint: Approximate the integral by a Riemann sum and recall $A$ is a closed operator.)

先证明：对 $\int_0^M$. $A$ 可与之换序：

$\forall M\in\mathbb{R}_+$. 对 $[0,M]$ 作 $k$ 等分：

$$[0,M]=\bigcup_{j=0}^{2^k-1}\left[\frac{j}{2^k}M,\ \frac{j+1}{2^k}M\right]$$

由 $I_k(t):=\sum_{j=1}^{k}e^{-\lambda t_j}S(t_j)u\,\chi_{t\in[t_{j-1},t_j]}\rightrightarrows e^{-\lambda t}S(t)u$ as $k\to\infty$

↑simple functions

及 $A$ 闭，知：

$$A\int_0^M e^{-\lambda t}S(t)u\,dt=A\left(\lim_{k\to\infty}\int_0^M I_k(t)\,dt\right)$$

$$\overset{A\text{闭}}{=}\lim_{k\to\infty}A\int_0^M I_k(t)\,dt$$

$$\overset{I_k\text{ simple}}{=}\lim_{k\to\infty}\int_0^M AI_k(t)\,dt$$

$$\underset{\text{MCT}}{\overset{A\text{闭}}{=}}\int_0^M\lim_{k\to\infty}AI_k(t)\,dt$$

$$=\int_0^M Ae^{-\lambda t}S(t)u\,dt\ \overset{A\text{ linear}}{=\!=}\ \int_0^M e^{-\lambda t}AS(t)u\,dt$$

又：$e^{-\lambda t}$ rapidly decays，
$\|S(t)\|\le 1$，
$A$ 闭

$$\Rightarrow\int_0^M e^{-\lambda t}S(t)u\,dt\ \underset{\exists}{\xrightarrow{M\to\infty}}\ \int_0^{+\infty}e^{-\lambda t}S(t)u\,dt.$$

$$\Rightarrow A\left(\int_0^M e^{-\lambda t}S(t)u\,dt\right)\xrightarrow{M\to\infty}A\left(\int_0^\infty e^{-\lambda t}S(t)u\,dt\right)$$

therefore "||" holds.

|| 刚刚证.

$$\int_0^M e^{-\lambda t}AS(t)u\,dt\xrightarrow{M\to\infty}\int_0^\infty e^{-\lambda t}AS(t)u\,dt.$$

||

$$\int_0^M e^{-\lambda t}S(t)Au\,dt\xrightarrow{M\to\infty}\int_0^\infty e^{-\lambda t}S(t)Au\,dt$$

□.

14.

设$\phi$是热方程的基本解，即 $\phi(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$

$\forall t>0$，令

$$[S(t)g](x)=\int_{\mathbb{R}^n}\phi(x-y,t)g(y)dy.\qquad x\in\mathbb{R}^n.$$

$g:\mathbb{R}^n\to\mathbb{R}$.

$S(0)g=g$.

则：$\{S(t)\}_{t\geq 0}$ 是$L^2(\mathbb{R}^n)$上的压缩半群

不是$L^\infty(\mathbb{R}^n)$上的压缩半群

· $S(t)$是$L^2$上的压缩半群

· $\|S(t)g\|_{L^2}=\|\phi * g\|_{L^2}$

$$\leq\|\phi\|_{L^1}\|g\|_{L^2}=\|g\|_{L^2}\quad\Rightarrow\|S(t)\|\leq 1.$$

· $S(t+s)g=\widehat{\phi(x-y,t+s)g}(\xi)$.

$\widehat{\phi}(\xi,t+s)\hat{g}(\xi)$.

$$=\hat{\phi}(\xi,t)\hat{\phi}(\xi,s)\hat{g}(\xi)=\widehat{S(t)S(s)g}$$

· $t\mapsto S(t)g$ 连续性：

$$\|S(t+h)g-S(t)g\|_{L^2}\leq\|S(h)g-g\|_{L^2}.$$

$$=\left\|\int_{\mathbb{R}^n}\phi(x-y,h)g(y)dy-g(x)\right\|_{L^2_x}.$$

$$=\left\|\int_{\mathbb{R}^n}\phi(x-y,h)(g(y)-g(x))dy\right\|_{L^2_x}.$$

$$=\left\|\int_{\mathbb{R}^n}\phi(y,h)(g(x-y)-g(x))dy\right\|_{L^2_x}$$

$$\leq\int_{\mathbb{R}^n}\phi(y,h)\,\|g(x-y)-g(x)\|_{L^2_x}\,dy.$$

$\downarrow_0$（平移连续性）.

$\xrightarrow{DCT}0$.

·$S(t)$ 不是 $L^\infty$ 上的压缩半群，因 在 $t=0$ 处不连续

令. ~~$g(x,y) = \chi_{\{x>0,\, y>0\}}$~~ $g(x) = \chi_{\{x>0\}}$. $g(0)=0$.

$$(S(t)g)(x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}}\, dy. \qquad (S(t)g)(0) = \frac{1}{2} \quad \forall t>0$$

$\Rightarrow \|S(t)g - g\|_{L^\infty} \geq \frac{1}{2}$

□

15. 设 $X$ 上有以 $A$ 为无穷小生成元的压缩半群 $\{S(t)\}_{t\geq 0}$

归纳地定义 $D(A^k) := \{u \in D(A^{k-1}) \mid A^{k-1}u \in D(A)\}$ $(k\geq 2)$

证明：若 $\exists k$. $u\in D(A^k)$, 则 $\forall t\geq 0$ $S(t)u \in D(A^k)$

Pf. $\forall j\in\{1,2,\dots,k-1\}$. 要证：$A^j S(t)u \in D(A)$

i.e. $\lim_{s\to 0^+} \frac{S(s)A^jS(t)u - A^jS(t)u}{s}$ $\exists$

$$\frac{S(s)A^jS(t)u - A^jS(t)u}{s}$$

$$= \frac{S(s)S(t)A^ju - S(t)A^ju}{s}$$

$$= \frac{S(t+s)(A^ju) - S(t)(A^ju)}{s}$$

归纳假设 $A^ju \in D(A)$

$=$ exists (以及 $S(0)=Id$).

□

16. 用T15证明：若 $u$ 是 $\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u=0 & \text{on } \partial U\times[0,T] \\ u=g & \text{on } U\times\{t=0\}\end{cases}$ 在 $X=L^2(U)$ 中的半群解

且 $g\in C_c^\infty(U)$, 则 $u(\cdot,t)\in C^\infty(U)$, $0\leq t\leq T$

Pf. 先证 $\{S(t)\}_{t\geq 0}$ 压缩，此为显见. 因 在方程两边同乘 $u$. 分部积分有

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2) + \|\nabla u\|^2 = 0. \quad \Rightarrow \frac{d}{dt}\|u\|_{L^2}^2 \leq 0$$

$\Rightarrow \|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2$

$\therefore \{S(t)\}$ 压缩

$\Rightarrow \|S(t)g\|_{L^2} \leq \|g\|_{L^2}$ $\forall g\in L^2$.

而 $g\in C_c^\infty(U) \subset H^{2k}(U)\cap H_0^{2k-1}(U) = D(\Delta^k)$

$k=1$ 是显然的. 再归纳即可.

由T15 知. $\forall t\geq 0$

$u(\cdot,t) = S(t)g \in H^{2k}(U)\cap H_0^{2k-1}(U)$. $\forall k\in\mathbb{Z}_+$

由Sobolev嵌入知 $u(\cdot,t)\in C^\infty(U)$.

□