

Evans Chapter 5 习题 $(U \subset \mathbb{R}^n, \partial U \in C^\infty)$ ($B(x, r)$ 表示闭球)

[5.1] 设 $k \in \mathbb{Z}_+$, $0 < r \leq 1$. 证明: $C^{k,r}(\bar{U})$ 是 Banach 空间.

证明: Step 1: 马上证 $\|\cdot\|_{C^{k,r}(\bar{U})}$ 是范数. $\|u\|_{C^{k,r}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,r}(\bar{U})}$

① 正定性.

$\|u\|_{C^{k,r}(\bar{U})} \geq 0$ 为显见.

若 $\|u\|_{C^{k,r}(\bar{U})} = 0$. 则 $\|D^\alpha u\|_{C(\bar{U})} = 0 \quad \forall |\alpha| \leq k$.

$\Rightarrow \|u\|_{C(\bar{U})} = 0 \Rightarrow u=0 \text{ in } \bar{U}$.

② 齐次性. $\|\lambda u\|_{C^{k,r}(\bar{U})} = |\lambda| \cdot \|u\|_{C^{k,r}(\bar{U})} \quad \forall \lambda \in \mathbb{C}$ 显见

③ 三角不等式. 设 $u, v \in C^{k,r}(\bar{U})$

$$\|u+v\|_{C^{k,r}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha(u+v)\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha(u+v)]_{C^{0,r}(\bar{U})}$$

$$\begin{aligned} \|\cdot\|_{C(\bar{U})} &\text{是范数} \\ &\leq \sum_{|\alpha| \leq k} (\|D^\alpha u\|_{C(\bar{U})} + \|D^\alpha v\|_{C(\bar{U})}) + \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x)+v(x)-u(y)-v(y)|}{|x-y|^r} \end{aligned}$$

$$\leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| \leq k} \|D^\alpha v\|_{C(\bar{U})} + \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x)-u(y)|}{|x-y|^r}$$

$$+ \sum_{|\alpha|=k} \sup_{\substack{x \neq y \\ x, y \in U}} \frac{|v(x)-v(y)|}{|x-y|^r}$$

$$= \|u\|_{C^{k,r}(\bar{U})} + \|v\|_{C^{k,r}(\bar{U})}.$$

Step 1 证毕!

Step 2: $(C^{k,r}(\bar{U}), \|\cdot\|_{C^{k,r}(\bar{U})})$ Banach.

设 $\{u_n\}$ 为 $C^{k,r}(\bar{U})$ 中的 Cauchy 序列. 由 $\|u_n - u_m\|_{C(\bar{U})} \sum_{|\alpha| \leq k} \|D^\alpha u_n - D^\alpha u_m\|_{C(\bar{U})} \rightarrow 0$

由 $C(\bar{U})$ 完整

$$\sum_{|\alpha|=k} [D^\alpha u_n - D^\alpha u_m]_{C^{0,r}(\bar{U})} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

$$\text{特别地. } \sup_{|\alpha| \leq k} \sup_{x \in \bar{U}} |D^\alpha(u_n - u_m)(x)| \rightarrow 0$$

由 $(C^k(\bar{U}), \|\cdot\|_{C^k(\bar{U})})$ Banach 完整. $\exists u \in C^k(\bar{U}). u_n \rightarrow u \text{ in } C^k(\bar{U})$.

下面先证 $[D^\alpha u_n - D^\alpha u]_{C^{0,r}(\bar{U})} \rightarrow 0 \quad \text{as } n \rightarrow \infty$

$$\text{上式} = \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|D^\alpha u_n(x) - D^\alpha u(x) - (D^\alpha u_n(y) - D^\alpha u(y))|}{|x-y|^r}$$

这两步应该调换一

下顺序, 先证明 u

在 $C^{k,r}$ 里面, 再证

明收敛性

于是, 只须证 $u \in C^{k,r}(\bar{U})$, 这只需要 $|\alpha|=k$, $[D^\alpha u]_{C^{0,r}(\bar{U})} < \infty$

事实上 $\forall x, y \in \bar{U}, x \neq y$, $|D^\alpha u(x) - D^\alpha u(y)| \leq \limsup_{n \rightarrow \infty} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y)|}{|x-y|^r}$

$$\leq \limsup_{n \rightarrow \infty} [D^\alpha u_n]_{C^{0,r}(\bar{U})} < \infty \quad (\text{由上式有界}) \quad \square$$

[5.2] $0 < \beta < r \leq 1$ 时, 证明:

$$\|u\|_{C^{0,r}(\bar{U})} \leq \|u\|_{C(\bar{U})}^{\frac{1-r}{1-\beta}} \|u\|_{C^{0,1}(\bar{U})}^{\frac{1-\beta}{1-r}}$$

证明: $\|u\|_{C^{0,r}(\bar{U})} = \|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^r}$

$$\leq \|u\|_{C(\bar{U})}^{\frac{1+r}{1-\beta}} \|u\|_{C(\bar{U})}^{\frac{r-\beta}{1-\beta}} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{(x-y)^{\frac{\beta(1-r)}{1-\beta}}} \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{1+\frac{r-\beta}{1-\beta}}}$$

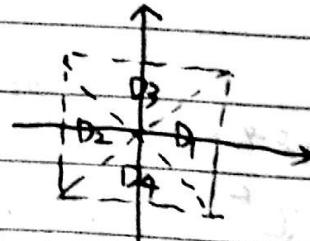
离散 Hölder

$$\leq (\|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{\frac{1-r}{1-\beta}}})^{\frac{1-r}{1-\beta}} \cdot (\|u\|_{C(\bar{U})} + \sup_{\substack{x \neq y \\ x, y \in \bar{U}}} \frac{|u(x) - u(y)|}{|x-y|^{\frac{r-\beta}{1-\beta}}})^{\frac{r-\beta}{1-\beta}}$$

$$= \|u\|_{C^{0,\beta}(\bar{U})} \|u\|_{C^{0,1}(\bar{U})}^{\frac{r-\beta}{1-\beta}}$$

[5.3] $U = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ □

令 $u(x) = \begin{cases} 1-x_1 & x_1 > 0, |x_2| < x_1 \rightarrow D_1 \\ 1+x_1 & x_1 < 0, |x_2| \leq -x_1 \rightarrow D_2 \\ 1-x_2 & x_2 > 0, |x_1| < x_2 \rightarrow D_3 \\ 1+x_2 & x_2 < 0, |x_1| \leq -x_2 \rightarrow D_4 \end{cases}$



问题: $p \in [1, +\infty]$, 且 $u \in W^{k,p}(U)$.

证明: $u \in L^p(U)$ 为显见. $\forall 1 \leq p \leq +\infty$, 下面先求 u 的弱导数.

Claim: $(-1, 0)$ in D_1

$$\nabla u = \begin{cases} (1, 0) & \text{in } D_2 \\ (0, -1) & \text{in } D_3 \\ (0, 1) & \text{in } D_4 \end{cases} \quad \text{是 } u \text{ 的一阶弱导数 } D_u$$

check: $\forall \varphi \in C_c^\infty(U)$.

$$\int_U \nabla u \cdot \varphi = \sum_{i=1}^4 \int_{D_i} \nabla u \cdot \varphi \underset{\substack{\nabla \in L^p \\ \varphi \in L^{\frac{p}{p-1}}}}{=} \int_{D_1} (-1, 0) \varphi + \int_{D_2} (1, 0) \varphi + \int_{D_3} (0, -1) \varphi + \int_{D_4} (0, 1) \varphi$$

$$= \sum_{i=1}^4 \int_{D_i} \nabla u \cdot \varphi \underset{\substack{\text{弱导数} \\ \text{分部积分}}}{=} \sum_{i=1}^4 - \int_{D_i} u \nabla \varphi + \int_{\partial D_i} u \varphi n_i \underset{\substack{\text{在 } U \text{ 内 } \varphi = 0 \\ \left. \varphi \right|_{\partial U} = 0}}{=}$$

$= - \int_U u \cdot \nabla \varphi \, dx$. 从而 V 的确是 u 的一个弱导数.

$v \in L^p \wedge 1 \leq p \leq +\infty \Leftrightarrow v \in W^{1,p}(\bar{U}) \quad 1 \leq p < +\infty$

□

[5.4]. 设 $n=1$, $u \in W^{1,p}(0,1)$, $1 \leq p < +\infty$

(1) 证明 u a.e. 等于一个绝对连续函数 $u^* \in L^p(0,1)$.

$$(2) \text{ 若 } p < +\infty \text{ 时, } |u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p dt \right)^{\frac{1}{p}}$$

证明 (1)

Lemma (周民强角早题指南 P256). 设 $f \in L^p[a,b]$, $\forall \varphi \in C_c^1(a,b)$, 若有 $\int_a^b f(x) \varphi'(x) = 0$ 则 $f(x) = c$. a.e.

Proof: 设 g 是任一紧支于 (a,b) 的连续函数.

$$h \text{ 是 } \dots \dots \dots \int_a^b h(x) dx = 1.$$

$$\text{令 } \varphi(x) = \int_a^x g(t) dt - \int_a^x h(t) dt \cdot \int_a^b g(t) dt, \quad x \in [a,b].$$

则 $\varphi \in C_c^1(a,b)$.

$$\varphi'(x) = g(x) - h(x) \int_a^b g(t) dt, \quad \forall x \in [a,b]$$

$$\text{从而 } 0 = \int f(x) \varphi'(x) dx$$

$$= \int_a^b f(x) \left(g(x) - \int_a^b g(t) dt \cdot h(x) \right) dx$$

$$= \int_a^b f(x) g(x) - \int_a^b f(x) h(x) dx \cdot \int_a^b g(x) dx = \int_a^b \left(f(x) - \int_a^b f(t) h(t) dt \right) g(x) dx$$

于是: $f(x) - \int_a^b f(t) h(t) dt = 0 \quad \text{a.e.} \Rightarrow f(x) = c \quad \text{a.e.}$

注: 这用到了 $f \in L^p(\mathbb{R}^d)$ 若 $\forall \varphi \in C_c^1(a,b)$, $\int f \varphi = 0$ 则 $f = 0$ a.e.

该命题可由反证法得出: 假设 $m(E) > 0$, $f(x) > 0$ in E.

则 存在紧支连续函数 $\{\varphi_k\}$, $\|\varphi_k - \chi_E\|_1 \rightarrow 0$

$$\left\{ |\varphi_k| \leq 1, \varphi_k \rightarrow \chi_E \text{ a.e. in } E. \right.$$

由 $|f \varphi| \leq |f| \quad \forall x \in E$

DCT

$$\therefore 0 < \int_E f(x) dx = \int_{\mathbb{R}^d} f(x) \chi_E(x) dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f(x) \varphi_k(x) dx = 0. \quad \#$$

引理证毕.

回到原题. 令 $u^* = \int_0^x u'(t) dt$, 其中 u' 为 u 的弱导数.

则 u^* 绝对连续. 下证 $u = u^* + \text{const}$ a.e.

$\forall \varphi \in C_c^\infty(0,1)$, $\exists u \varphi \in C_c^1(0,1)$.

$$\int_0^1 (u^* - u) \varphi' dx = \int_0^1 \int_0^x u' dt \cdot \varphi' dx - \int_0^1 u \varphi' dx$$

$$= \int_0^1 \int_0^t \varphi'(x) dx \cdot u'(t) dt + \int_0^1 u' \varphi(x) dx$$

$$= \int_0^1 (\underbrace{\varphi(1)}_0 - \varphi(t)) u'(t) dt + \int_0^1 u(x) \varphi(x) dx$$

$$= 0$$

由上得证.

(2). 由 u a.e.= $\frac{1}{p}$ 弱延拓出 $\frac{1}{p}$ 元

$$|u(x) - u(y)| = \left| \int_0^1 \chi_{\{x \leq t \leq y\}} u'(t) dt \right|$$

a.e. $x, y \in [0,1]$
 $\exists \varepsilon > x \leq y$

$$\leq |x-y|^{\frac{1}{p'}} \left(\int_x^y |u'|^p dt \right)^{\frac{1}{p}} \frac{1}{p} + \frac{1}{p'} = 1$$

Hölder. \square

5. U 有解, U, V 互. $V \subset U$. 证明: $\exists \zeta \in C^\infty(U)$, s.t. $\zeta \equiv 1$ on V

{
= 0 near ∂U

证明: 取开集 W . $V \subset W \subset \bar{W} \subset U$

$$\text{令 } \zeta(x) = (\chi_W * \eta_\varepsilon)(x) \quad (\varepsilon < \frac{1}{2} \min \{ \text{dist}(\partial V, \partial W), \text{dist}(\partial W, \partial U) \})$$

$$\text{在 } V \text{ 上, } \zeta(x) = \int_{\mathbb{R}^n} \chi_W(y) \eta_\varepsilon(x-y) dy$$

$$= \int_{B(0, \varepsilon)} \eta_\varepsilon(y) \cdot \chi_W(x-y) dy.$$

$$x \in V \text{ 时, } \forall y \in B(0, \varepsilon) \quad |x-y| \leq |x| + |\gamma| \leq |x| + \frac{1}{2} \text{dist}(\partial V, \partial W)$$

$$\Rightarrow x-y \in W \Rightarrow \chi_W(x-y) = 1$$

$$\Rightarrow \zeta(x) = 1 \quad \forall x \in V$$

$$\text{同理, 令 } \zeta_\varepsilon = \{x \in U \mid \text{dist}(x, \partial U) < \frac{\varepsilon}{3}\}$$

$$\text{要证, } \zeta_\varepsilon = 0 \text{ in } V_\varepsilon. \Rightarrow \zeta = 0 \text{ near } \partial U.$$

$\zeta \in C^\infty$ 由 mollifier 性质易得

\square

[5.6] U 有界. $\{V_i\}_i^N$ 是 \mathbb{R}^n 中的开集. $U \subset \bigcup_{i=1}^N V_i$. 证明: 存在 C^∞ 函数 $\{\zeta_i\}$,

$$\text{s.t. } \begin{cases} 0 \leq \zeta_i \leq 1 \\ \text{Spt } \zeta_i \subset V_i \quad (1 \leq i \leq N) \\ \sum_{i=1}^N \zeta_i = 1 \quad \text{in } U \end{cases}$$

证明: 对 $\bar{U} \subset \bigcup_{i=1}^N V_i$.

对每个 V_i , 由 [5.5] 知 存在 C^∞ 函数 η_i ($1 \leq i \leq N$) s.t. $0 \leq \eta_i \leq 1$

$\forall x \in \bar{U}$, 存在以 x 为中心的闭球 $B(x) \subseteq V_i$ (for some i). $\text{Spt } \eta_i \subseteq V_i$

因 \bar{U} 紧 $\bar{U} \subseteq \bigcup_{x \in \bar{U}} \overset{\circ}{B}(x)$. 故存在有限覆盖 $\bigcup_{j=1}^m B(x_j)$. $\sum_{j=1}^m \eta_j(x) = 1$

对任何 $i \in \{1, 2, \dots, N\}$, 令 $U_i = \bigcup_{j: \overset{\circ}{B}(x_j) \subseteq V_i} B(x_j)$

$$\text{则 } \bar{U} \subseteq \bigcup_{i=1}^N U_i$$

由上一题, $\exists \varphi_i \in C^\infty$, $0 \leq \varphi_i \leq 1$, $\varphi_i = 1$ in U_i

$$\text{Spt } \varphi_i \subseteq V_i$$

$$\text{令 } \eta_1 = \varphi_1, \eta_2 = \varphi_2(1-\varphi_1), \dots, \eta_N = \varphi_N(1-\varphi_1) \cdots (1-\varphi_{N-1})$$

则 $\text{Spt } \eta_i \subset V_i$.

$$\eta_1 + \dots + \eta_N = 1 - \prod_{i=1}^{N-1} (1-\varphi_i). \text{ 因 } \forall x \in \bar{U}, \text{ 总有一个 } \varphi_i \text{ 是 } 1. \text{ 故 } \eta_1 + \dots + \eta_N = 1$$

□

[5.7] U 有界, 且存在 C^∞ 向量场 $\vec{\alpha}$, 使 $\vec{\alpha} \cdot \vec{v} \geq 1$ along ∂U (\vec{v} 为 ∂U 外单位外法向).

($\leq p < \infty$. 请对 $\int_{\partial U} |\vec{\alpha} \cdot \vec{v}|^p |u|^p ds$ 用 Gauss-Green 公式 证明: $\forall u \in C^1(\bar{U})$).

$$\int_{\partial U} |u|^p ds \leq C \int_U |\nabla u|^p + |u|^p dx$$

$$\text{证明: } \int_{\partial U} |u|^p ds \leq \int_{\partial U} (|u|^p \vec{\alpha}) \cdot \vec{v} ds \stackrel{\text{Gauss-Green}}{=} \int_U \operatorname{div}(|u|^p \vec{\alpha}) dx.$$

$$= \sum_{i=1}^n \int_U \partial_i (|u|^p \alpha_i) dx \quad (\vec{\alpha} = (\alpha_1, \dots, \alpha_n))$$

$$= \sum_{i=1}^n \int_U \partial_i |u|^p \alpha_i dx + \sum_{i=1}^n \int_U |u|^p \partial_i \alpha_i dx$$

$$\stackrel{\vec{\alpha} \in C^\infty}{\leq} C \sum_{i=1}^n \int_U \partial_i |u|^p + C \int_U |u|^p dx$$

$$\leq C \sum_{i=1}^n \int_U p |u|^{p-1} |\partial_i u| + C \int_U |u|^p dx$$

$$\leq C \left(\int_U |u|^p + \int_U (|u|^{p-1} |\nabla u|)^p dx \right)$$

$$\stackrel{\text{Young 不等式}}{\leq} \int_U (|u|^p + |\nabla u|^p) dx$$

□

[6.8] U 有界, $\partial U \in C^1$. 证明: $T: L^p(U) \rightarrow L^p(\partial U)$ 为有界线性算子, 且 $Tu = u|_{\partial U}$.

$\forall u \in C(\bar{U}) \cap L^p(\bar{U})$

证明: 反设存在这样的 T , 令 $u_m = \max\{0, 1 - \text{dist}(x, \partial U)\}$

$$\text{由 } Tu_m = 1 \text{ on } \partial U \quad \|u_m\|_{L^p(\partial U)} = \left(\int_U 1 \, dH^{n-1} \right)^{\frac{1}{p}} = H^{n-1}(\partial U) > 0$$

∂U 的 $n-1$ 维 Hausdorff 测度

但 $\|u_m\|_{L^p(U)} \rightarrow 0$ as $m \rightarrow \infty$.

$$\text{check: } \int |u_m|^p \, dx \xrightarrow[u_m \rightarrow 0 \text{ as } m \rightarrow \infty \text{ in } U]{} 0.$$

DCT

$$\text{则 } \|T\| = \sup \geq \limsup_{m \rightarrow \infty} \frac{\|Tu_m\|_{L^p(\partial U)}}{\|u_m\|_{L^p(U)}} = \limsup_{m \rightarrow \infty} \frac{H^{n-1}(\partial U)}{0} = +\infty$$

$\Rightarrow T$ 有界矛盾!

□

[5.9] 分部积分定理: $\|Du\|_2 \leq C\|u\|_2^{\frac{1}{2}} \|D^2u\|_2^{\frac{1}{2}}$. $\forall u \in C_c^\infty(U)$.

(1) U 有界, $\partial U \in C^\infty$ 证明 上述不等式对 $u \in H_0^1(U) \cap H^2(U)$ 成立.

证明: $\forall u \in C_c^\infty(U)$ 令

$$\begin{aligned} \|Du\|_2^2 &= \int_U |Du|^2 \, dx = \sum_{i=1}^n \int_U (\partial_i u)^2 \, dx \\ &\stackrel{\text{分部积分}}{=} - \sum_{i=1}^n \int_U u \cdot \partial_i \partial_i u \, dx \\ &= - \int_U u \cdot \Delta u \, dx \leq \int_U |u| \cdot |\Delta u| \, dx \\ &\leq C \int_U |u| \cdot |D^2u| \, dx \\ &\leq C \|u\|_2 \|D^2u\|_2. \end{aligned}$$

(2) $\forall u \in H_0^1(U) \cap H^2(U)$

存在 $\exists \{v_n\} \subset C_c^\infty(U)$ $v_n \rightarrow u$ in $H_0^1(U)$

$\{w_n\} \subset C^\infty(U)$: $w_n \rightarrow u$ in $H^2(U)$

$$\text{则 } \int_U Dv_k \cdot Dw_k = \sum_{i=1}^n \int_U \partial_i v_k \cdot \partial_i w_k \, dx$$

$$\stackrel{\text{分部积分}}{=} - \sum_{i=1}^n \int_U v_k \partial_i \partial_i w_k \, dx$$

$$= - \int_U v_k \Delta w_k \, dx \leq C \int_U |v_k| |D^2 w_k| \, dx \stackrel{\text{嵌入定理}}{\leq} C \|v_k\|_2 \|D^2 w_k\|_2$$

$\leq C \|v_k\|_2 \|D^2 w_k\|_2. \quad (*)$

$$\text{再び: } \|V_k\|_2 \rightarrow \|u\|_2.$$

$$\cancel{\|D^2 w_k\|_2} \rightarrow \|D^2 u\|_2.$$

$$\text{よし } (\|V_k\|_2 \cdot \|D^2 w_k\|_2) \rightarrow \|u\|_2 \|D^2 u\|_2.$$

$$\text{而 } \int D_u \cdot D_u - \int D_{V_k} \cdot D_{w_k}$$

$$= \int D_u \cdot (D_u - D_{w_k}) dx + \int D_{w_k} \cdot (D_u - D_{V_k}) dx$$

$$\leq \|D_u\|_2 \|D_u - D_{w_k}\|_2 + \|D_{w_k}\|_2 \|D_u - D_{V_k}\|_2.$$

$\rightarrow 0$ as $k \rightarrow \infty$ - なぜなら ($\{D_{w_k}\} \subset L^2$ だから).

したがって (*) 両辺 $k \rightarrow \infty$ で等しい \square

$$[5. 10]. (1) \forall u \in C_c^\infty(U), 2 \leq p < \infty \quad \|Du\|_p \leq C \|u\|_p^{1/2} \|D^2 u\|_p^{1/2}.$$

$$(2) \forall u \in C_c^\infty(U), 1 \leq p < \infty \quad \|Du\|_{2p} \leq C \|u\|_\infty^{1/2} \|D^2 u\|_p^{1/2}.$$

$$\text{証明: (1) } \|Du\|_p^p = \int_U |Du|^p dx$$

$$= \int_U |Du|^{p-2} (Du)^2 dx = \sum_{i=1}^n \int_U \partial_i u (\partial_i u |Du|^{p-2}) dx$$

$$\stackrel{\text{Hölder}}{=} - \sum_{i=1}^n \int_U u \partial_i (\partial_i u |Du|^{p-2}) dx.$$

$u \in C_c^\infty(U), u|_{\partial U} = 0$

$$= - \underbrace{\int_U u \cdot \Delta u \cdot |Du|^{p-2} dx}_{I_1} - \underbrace{\sum_{i=1}^n \int_U u \partial_i u \cdot \cancel{\partial_i} \partial_i |Du|^{p-2} dx}_{I_2}$$

$$I_1 = - \int_U u \cdot \Delta u \cdot |Du|^{p-2} dx$$

$$\leq C \int_U |u| \cdot |\Delta u| \cdot |Du|^{p-2} dx. \stackrel{\text{Hölder}}{=} C \|u\|_p \|D^2 u\|_p \left(\int_U |Du|^{p-2 \cdot \frac{p}{p-2}} dx \right)^{\frac{p-2}{p}}$$

$$= C \|u\|_p \|D^2 u\|_p \cdot \|Du\|_p^{p-2}.$$

$$I_2 = - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \partial_i |Du|^{p-2} dx$$

$$= - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \partial_i \left(\sum_{j=1}^n (\partial_j u)^2 \right)^{\frac{p-2}{2}} dx$$

$$= - \sum_{i=1}^n \int_U u \cdot \partial_i u \cdot \left(\sum_{j=1}^n (\partial_j u)^2 \right)^{\frac{p-4}{2}} \cdot \sum_{j=1}^n \partial_i \partial_j u \cdot \partial_j u |Du|^{p-4} dx$$

$$= -(p-2) \sum_{i=1}^n \int_U u \cdot |Du|^{p-4} \sum_{j=1}^n \partial_i u (\partial_i \partial_j u) \partial_j u dx$$

$$\leq C \int_U |u| \cdot |Du|^{p-4} \cdot (\partial_1 u, \dots, \partial_n u) \cdot \begin{pmatrix} \partial_1 u & \dots & \partial_n u \\ \vdots & \ddots & \vdots \\ \partial_n u & \dots & \partial_1 u \end{pmatrix} dx$$

$$\leq C \int_U |u| \cdot |Du|^{p-4} \cdot |Du| \cdot |D^2 u| \cdot |Du| dx$$

$$= C \int_U |u| \cdot |Du|^{p-2} \cdot |D^2 u| dx \stackrel{\text{由H\"older, F\o I}}{\leq} C \|u\|_p \|Du\|_p^{p-2} \|D^2 u\|_p$$

$$\therefore P \int |Du|^p dx \leq C \int_U |u| \cdot |Du|^{p-2} |D^2 u| dx \\ \leq C \|u\|_p \|Du\|_p^{p-2} \|D^2 u\|_p$$

两边开 $\frac{2}{p}$: 次方根号.

$$(2) \quad \boxed{\text{由(1)有}} \|Du\|_p^{2p} \leq C \int_U |u| \cdot |Du|^{2p-2} |D^2 u| dx \\ \leq C \|u\|_\infty \int_U |Du|^{2p-2} |D^2 u| dx \\ \stackrel{\text{H\"older}}{\leq} C \|u\|_\infty \|Du\|_{2p}^{2p-2} \|D^2 u\|_p.$$

开平方根号.

□

[5.11] U 连通, $u \in W^{1,p}(U)$, $Du = 0$ a.e. in U $\Rightarrow u = \text{const}$ a.e. in U.

证明: 此题不能用 Poincar\'e 不等式 $\|u - (u)_U\|_p \leq C \|Du\|_p$. 因为该是
结论不适用于非 Poincar\'e 不等式

令 $u^\varepsilon = \eta^\varepsilon * u$. $\forall \varepsilon \in U$.

由 ε 充分小时. $Du^\varepsilon = \eta^\varepsilon * Du = (Du)^\varepsilon$ in V.

$u^\varepsilon \in C^\infty(V) \Rightarrow \exists$ 常数 C_ε s.t. $u^\varepsilon = C_\varepsilon$ in V.

而 $\|u^\varepsilon\|_p = \|u^\varepsilon * u\|_p \leq \|\eta^\varepsilon\|_1 \|u\|_p = \|u\|_p < \infty$ uniformly on ε .

$\Rightarrow \{C_\varepsilon\}_{\varepsilon>0}$ 有界 故有收敛子列 $C_{\varepsilon_i} \rightarrow C \in \mathbb{R}$.

由 $C \in L^p$, 由控制收敛定理易有

$\|u^\varepsilon - C\|_p \rightarrow 0$ as $i \rightarrow \infty$

又 $\|u^\varepsilon - u\|_p \rightarrow 0$ as $i \rightarrow \infty$ i.e. $u = C$ a.e. in any $V \subset \subset U$

故 $u = C$ a.e. in U

□

[5.12] 举例说明. 若 $\|D^h u\|_{L^1(V)} \leq C$ $\forall 0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$. 则 u 不解
推出 $u \in W^{1,1}(V)$.

证明: 令 $U = (-2017, 2017)^n$, $V = (-0.001, 1.001)^n$

$$V = (0, 1)^n.$$

$$u(x) = \begin{cases} 1 & 0 < x_1 < \frac{1}{2} \\ 0 & \text{否则.} \end{cases}$$

$$u(x) \in L^\infty(V)$$

$$\|D^h u\|_{L^1(V)} = \int_V |D^h u| dx \leq \left(\sqrt{1 + \frac{1}{h^2}} \right) \int_{\frac{1}{2}-h}^{\frac{1}{2}+h} \int_0^1 \cdots \int_0^1 \left| \frac{1}{h} \right| dx_m \cdots dx_1 = 1$$

但 $u \notin W^{1,1}(V)$. 否则 $\partial_{x_1} u$ 为 u 的 x_1 方向弱偏导. 且 $\partial_{x_1} u \in L^1(V)$.

$\forall \phi \in C_c^\infty(V) \int_V \partial_{x_1} u \phi dx = 0$ (~~因 $\partial_{x_1} u = 0$ a.e. (因 u 只取 0, 1 值)~~)

$$-\int_V u \cdot \partial_{x_1} \phi = -\int_{V \cap \{x_1 > \frac{1}{2}\}} \partial_{x_1} \phi dx$$

这不可能. □

[5.13] 设 $\bar{U} \subset \mathbb{R}^n$ 开, $u \in W^{1,\infty}(\bar{U})$ 但 u 不是 \bar{U} 上的 Lipschitz 连续函数.

证明: 令 $U = \bar{B}(0, 1) - \{(x, y) \in \bar{B}(0, 1) \mid x \geq 0, y \geq 0\}$.

$$u(x) = \text{sgn}(y) \cdot (\max\{0, x_0\})^2 \cdot \max\{\text{sgn} y, 0\}$$

则 $u(x)$ 在 U 中可微.

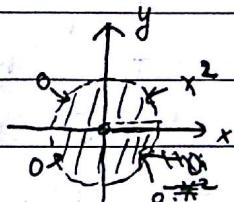
$$\partial_{x_1} u(x) = 2 \max\{\text{sgn} y, 0\} \max\{0, x\}.$$

$$\partial_{x_2} u(x) = 0.$$

$$\rightarrow u \in W^{1,\infty}(U).$$

但 u 不是 Lip. 因 $\forall \varepsilon > 0 \quad u(\frac{1}{2}, \varepsilon) - u(\frac{1}{2}, -\varepsilon) = \frac{1}{2}$

$$\Rightarrow \text{Lip}(u) \geq \frac{\frac{1}{2}}{2\varepsilon} = \frac{1}{4\varepsilon} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0^+.$$



□

[5.14] 证明. $\cup = B(0,1)$ のとき $u = \log \log (1 + \frac{1}{|x|}) \in W^{1,n}(\cup)$

$$\text{証明: } \partial_i u(x) = \frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{1}{1 + \frac{1}{|x|}} \left(-\frac{1}{|x|^2} \cdot \operatorname{sgn} x_i \right).$$

$$= -\frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{x_i}{|x|^3} \cdot \frac{1}{1 + \frac{1}{|x|}}$$

$$= -\frac{1}{\log(1 + \frac{1}{|x|})} \cdot \frac{x_i}{|x|^2} \cdot \frac{1}{|x| + 1}$$

$$\rightarrow |\partial_i u| \leq C \cdot \frac{1}{|x|} \cdot \frac{1}{\log(1 + \frac{1}{|x|})}$$

$$\int_{B(0,1)} |\partial_i u|^n dx \leq C \int_0^1 \left(\frac{1}{\log(1 + \frac{1}{p})} \right)^n \cdot \frac{1}{p^n} \cdot p^{n-1} dp.$$

↑ ここで $p = |x|$

$$\begin{aligned} z &= \log(1 + \frac{1}{p}) \\ &\leq C \int_1^\infty \frac{1}{\log 2} \cdot \frac{1}{z^n} dz < \infty \end{aligned}$$

$$\int_{B(0,1)} |u|^p dx = \int_0^1 \left| \log \log \left(1 + \frac{1}{p} \right) \right|^p p^{n-1} dp$$

$$= \int_{\frac{1}{e-1}}^1 \left| \log \log \left(1 + \frac{1}{p} \right) \right|^p p^{n-1} dp \quad I_1$$

$$+ \int_{\frac{1}{e-1}}^1 \left(\log \log \left(1 + \frac{1}{p} \right) \right)^p p^{n-1} dp \quad I_2$$

$$I_2 \leq \int_{\frac{1}{e-1}}^1 \left(\log \left(1 + \frac{1}{p} \right) \right)^p p^{n-1} dp$$

$$\leq \int_{\frac{1}{e-1}}^1 (\log 2)^p p^{n-1} dp < \infty$$

$$I_1 = - \int_0^{\frac{1}{e-1}} \log \log \left(1 + \frac{1}{p} \right)^p p^{n-1} dp$$

$$\leftarrow - \int_0^1 \left(\log \frac{p}{p+1} \right)^p p^{n-1} dp$$

$$= - \int_0^1 \log \left(1 - \frac{1}{p+1} \right)^p p^{n-1} dp$$

$$= (-1)^{n+1} \int_0^1 \log\left(1 + \frac{1}{p}\right)^n p^{n-1} dp.$$

$$\leq C \int_0^1 \frac{1}{p} dp$$

$$I_1 \cdot \mathbb{E} \stackrel{z=\frac{1}{p}}{=} \int_{e^{-1}}^{\infty} \log \log (1+z)^n \cdot \frac{dz}{z^{n+1}}$$

$$\leq C \int_{e^{-1}}^{\infty} \frac{dz}{z^{n+\frac{1}{2}}} < \infty \quad \text{thus, } u \in W^{1,n}(U)$$

□

[5.15] Fix $\alpha > 0$. $U = B(0, 1)$. 证明: 存在常数 $C(n, \alpha)$, 使

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx. \quad \text{其中 } \exists n. \{x \in U \mid u(x) = 0\} \geq \alpha. \quad u \in H^1(U)$$

$$\underline{\text{证明}}: \int_U u^2 dx \stackrel{\hat{z} < u = \frac{1}{|U|} \int_U u dx}{=} \int_{U-A} (u - \langle u \rangle + \langle u \rangle)^2 dx. \quad \text{其中 } A = \{x \mid u(x) = 0\}$$

$$= \int_{U-A} (u - \langle u \rangle)^2 dx + 2 \underbrace{\int_{U-A} (u - \langle u \rangle) dx \cdot \langle u \rangle}_{\text{Poincaré 不等式: } \|u - \langle u \rangle\|_p \leq C \|Du\|_p} + \int_{U-A} \langle u \rangle^2 dx$$

$$\leq C \|Du\|_p^2 + \int_{U-A} \langle u \rangle^2 dx$$

$$= C \|Du\|_p^2 + |\langle u \rangle^2| \cdot |U-A|$$

$$= C \|Du\|_p^2 + \frac{1}{|U|^2} \left(\int_{U-A} |u| dx \right)^2 \cdot |U-A|.$$

Hölder

$$\leq C \|Du\|_p^2 + \frac{1}{|U|^2} \cdot \left(\int_{U-A} |u|^2 dx \right) \cdot |U-A|^2.$$

$$\text{全 } 1 - C_0 = \frac{|U-A|^2}{|U|^2}$$

$$= C \|Du\|_p^2 + (1 - C_0) \|u\|_p^2$$

$$\Rightarrow \exists C' > 0. \quad \int_U u^2 dx \leq C' \int_U |Du|^2 dx$$

□

$$[5.16] \text{ 证明: } \forall n \geq 3, \exists \text{ const. } C. \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C \int_{\mathbb{R}^n} |Du|^2 dx, \forall u \in H^1(\mathbb{R}^n)$$

证明: 先设 $u \in C_c^\infty(\mathbb{R}^n)$ 且 $F(x) = \frac{x}{|x|^{n-2}}$.

$$\text{由 } \int_{\mathbb{R}^n} u^2 \operatorname{div} F dx = - \int_{\mathbb{R}^n} D(u^2) \cdot F(x) dx.$$

$$= -2 \int_{\mathbb{R}^n} u D(u) \cdot F(x) dx$$

$$= -2 \int_{\mathbb{R}^n} Du \cdot (u F) dx$$

$$\Rightarrow \left| \int_{\mathbb{R}^n} u^2 \operatorname{div} F dx \right| = 2 \left| \int_{\mathbb{R}^n} Du \cdot u F dx \right|$$

$$\leq 2 \|Du\|_2 \|u F\|_{L^2}$$

$$\text{由 } \operatorname{div} F(x) = \frac{n-2}{|x|^2}, |F(x)|^2 = \frac{1}{|x|^{n-2}} \text{ 代入有:}$$

$$\frac{(n-2)^2}{4} \left(\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \right)^2 \leq \int_{\mathbb{R}^n} |Du|^2 dx \cdot \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx$$

$$\Rightarrow \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |Du|^2 dx.$$

对一般的 $u \in H^1(\mathbb{R}^n)$. 由于 $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n) \iff U = \mathbb{R}^d$

故 $\exists u_k \in C_c^\infty(\mathbb{R}^n)$, $u_k \rightarrow u$ in $H^1(\mathbb{R}^n)$.

$$\text{从而 } \int_{\mathbb{R}^n} |Du_k|^2 dx \rightarrow \int_{\mathbb{R}^n} |Du|^2 dx$$

$$\frac{(n-2)}{4} \int_{\mathbb{R}^n} \frac{u_k^2}{|x|^2} dx.$$

$$\Rightarrow \frac{u_k}{|x|} \in L^2(\mathbb{R}^n). \text{ 由 } u_k \rightarrow u \text{ in } L^2$$

故存在子列 $u_{k_j} \rightarrow u$ a.e.

$$\Rightarrow \left(\frac{u_{k_j}}{|x|} \right)^2 \rightarrow \left(\frac{u}{|x|} \right)^2 \text{ a.e.}$$

$$\Rightarrow \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \liminf_{j \rightarrow \infty} \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u_{k_j}^2}{|x|^2} dx \leq \limsup_{j \rightarrow \infty} \int_{\mathbb{R}^n} |Du_{k_j}|^2 dx$$

Fatou's Lemma

$$= \int_{\mathbb{R}^n} |Du|^2 dx$$

[5.17] (待证) $F: \mathbb{R} \rightarrow \mathbb{R}$ 是 C' 的, 且 F' 有界 $u \in W^{1,p}(U)$, $1 \leq p \leq \infty$.

证明: (1) 若 $\int_U |F''(u)|^p dx < \infty$, 则 $V := F(u) \in W^{1,p}(U)$, $\partial_i V = F'(u) \partial_i u$.
 (2) 若 $\int_U |F''(u)|^p dx = \infty$, 但 $F(0) = 0$ 则 (1) 结论也对.

Rmk: 证明过程中会体现. (2) 中 $F(0) = 0$ 是必须的.

Proof: $\Leftrightarrow |F(u) - F(0)| \leq \|F'\|_{L^\infty} |u| \in L^p$ (中值定理).
 故 $F(u) - F(0) \in L^p(U)$

若 $\int_U |F''(u)|^p dx < \infty$, 则 $F(u) \in L^p(U) \Rightarrow F(u) \in L^p(U)$.

若 $\int_U |F''(u)|^p dx = \infty$, 则 $F(0) = 0 \Leftrightarrow F(u) \in L^p(U)$.

$$F(u) - F(0) \in L^p(U)$$

从而 $F'(u) \partial_i u \in L^p(U)$ 显见, 因 $F' \in L^\infty(U)$, $\partial_i u \in L^p(U)$.

下面证明 $\partial_i F(u) = F'(u) \partial_i u$, $i = 1, 2, \dots, n$.

令 $\forall \varepsilon \in U$, $u^\varepsilon = \eta_\varepsilon * u$. 使 $u^\varepsilon \in C_c^\infty(U)$.

$\forall \phi \in C_c^\infty(U)$ 且 $\text{Supp } \phi \subset V \subset U$, $u^\varepsilon = \eta_\varepsilon * u$.

要证: $\int_V F(u) \partial_i \phi dx = - \int_V F'(u) \partial_i u \cdot \phi dx$.

$$\text{左} = \int_V F(u) \partial_i \phi dx \stackrel{\text{①}}{=} \lim_{\varepsilon \rightarrow 0^+} \int_V F(u^\varepsilon) \partial_i \phi dx.$$

$$\stackrel{\text{分部积分}}{=} - \lim_{\varepsilon \rightarrow 0^+} \int_V F'(u^\varepsilon) \partial_i u^\varepsilon \cdot \phi dx \quad (\text{设 } \varepsilon > 0).$$

因 $u^\varepsilon \in C_c^\infty(U_\varepsilon)$
链式法则可得

$$\stackrel{\text{②}}{=} - \int_V F'(u) \partial_i u \cdot \phi dx = - \int_V F'(u) \partial_i u \cdot \phi dx$$

$$\text{check ①: } \int_V |F(u) - F(u^\varepsilon)| \cdot |\partial_i \phi| dx \leq \int_V |u - u^\varepsilon| \cdot |\partial_i \phi| dx \cdot \|F'\|_{L^\infty}$$

$$\leq \|F'\|_{L^\infty(U)} \|u - u^\varepsilon\|_{L^p(V)} \|\partial_i \phi\|_{L^p(V)}$$

$\rightarrow 0$ as $\varepsilon \rightarrow 0^+$ (因 $u^\varepsilon \rightarrow u$ in $L^p(V)$)

$$\text{②: } \left| \int_V F(u) \partial_i u \phi dx - \int_V F(u^\varepsilon) \partial_i u^\varepsilon \cdot \phi dx \right|$$

$$\leq \int_V |F(u) - F(u^\varepsilon)| |\partial_i u| |\phi| dx + \int_V |F'(u^\varepsilon)| \cdot |\partial_i u - \partial_i u^\varepsilon| |\phi| dx$$

A

B

$$x \cdot B = \int_V |F(u^\varepsilon)| \cdot |\partial_i u - \partial_i u^\varepsilon| \cdot |\phi| dx$$

$$\leq \|F\|_{L^\infty(\Omega)} \cdot \|\partial_i u - \partial_i u^\varepsilon\|_{L^p(V)} \cdot \|\phi\|_{L^p(V)}.$$

$\rightarrow 0$ as $\varepsilon \rightarrow 0^+$ (因 $u^\varepsilon \rightarrow u$ in $W^{1,p}$).

$$A = \int_V |F'(u) - F'(u^\varepsilon)| \cdot |\partial_i u| \cdot |\phi| dx$$

由 $u^\varepsilon \rightarrow u$ a.e. in V (光滑性质).

F' 连续 $\Rightarrow F'(u) \rightarrow F'(u^\varepsilon)$ a.e. in V

$$\text{又: } |F'(u) - F'(u^\varepsilon)| \cdot |\partial_i u| \cdot |\phi| \leq 2\|F'\|_{L^\infty} |\partial_i u| \cdot |\phi| \in L^1 \text{ (由 Hölder 即得)}$$

故由控制收敛定理. $A \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

证毕! \square

[5.18] $1 \leq p \leq \infty$. U 有界.

1) 证明: 若 $u \in W^{1,p}(U)$, 则 $|u| \in W^{1,p}(U)$.

2) 若 $u \in W^{1,p}(U)$, 则 $u^+, u^- \in W^{1,p}(U)$.

$$Du^+ = \begin{cases} Du & L^n\text{-a.e. on } \{u > 0\} \\ 0 & L^n\text{-a.e. on } \{u \leq 0\} \end{cases} \quad Du^- = \begin{cases} 0 & L^n\text{-a.e. on } \{u \geq 0\} \\ -Du & L^n\text{-a.e. on } \{u < 0\} \end{cases}$$

3). $u \in W^{1,p}(U)$. 则 $Du = 0$ a.e. on $\{u = 0\}$

Proof: 只用证(2). (2) \Rightarrow (1) 显见

若(2)对, 则 $Du = Du^+ - Du^- = 0$ on $\{u = 0\}$ L^n -a.e.

下证(2). 令 $F_\varepsilon(r) = (\sqrt{r^2 + \varepsilon^2} - \varepsilon) \chi_{\{r \geq 0\}} \in C^1(\mathbb{R})$

且 $F'_\varepsilon(r) \in L^\infty(\mathbb{R})$. (Fix $\varepsilon > 0$).

$$\text{由 17 题 } \int_U F_\varepsilon(u) \partial_i \phi dx = - \int_U F'_\varepsilon(u) \partial_i u \cdot \phi dx \dots (*)$$

注意到. $u^+ = \lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u)$.

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u)$$

(*) 左边令 $\varepsilon \rightarrow 0$. 极限 $\Rightarrow \int_U F(u) \partial_i \phi dx = \int_U u^+ \partial_i \phi dx$

这由控制收敛即得. (*) 右边同理 $\rightarrow - \int \partial_i u \cdot \chi_{\{u > 0\}} \phi dx$

$\partial_i u^+ = \partial_i u \chi_{\{u>0\}}$ 同理 u^- 有类似结果, 因 $u^\pm \in L^2$. □

[5.19] 设 $u \in H^1(U)$ 按书上 Hint 证明 $D_u = 0$ a.e. in $\{u=0\}$

证: 取 ϕ 是 C_c^∞ , 有界, 不减函数. ϕ' 有界. $\phi(z) = z$ $|z| \leq 1$.

$$\text{令 } u^\varepsilon(x) = \varepsilon \phi(\frac{u}{\varepsilon})$$

① claim $u^\varepsilon \rightarrow 0$ in $L^2(U)$.

$$\forall \varphi \in C_c^\infty(U). \int_U u^\varepsilon \varphi \, dx = \varepsilon \int_U \phi(\frac{u}{\varepsilon}) \varphi \, dx$$

$$\leq \varepsilon \cdot \|\phi\|_{L^\infty} \|\varphi\|_{L^1} \xrightarrow{\varepsilon \rightarrow 0^+} 0$$

$$\therefore \forall \varphi \in C_c^\infty(U). \langle \varphi, u^\varepsilon \rangle \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+$$

$\times C_c^\infty(U) \stackrel{\text{dense}}{\subset}$

$$\therefore \|u^\varepsilon\|_{L^2}^2 = \varepsilon^2 \int_U |\phi(\frac{u}{\varepsilon})|^2 \leq \|\phi\|_{L^\infty}^2 \|u\|_{L^2}^2$$

因 ϕ' 有界 $\forall x \in \mathbb{R}, |\phi'(x)| \leq \|\phi'\|_{L^\infty} |x|$.

$$\phi(0) = 0$$

$$\therefore \|u^\varepsilon\|_{L^2}^2 \leq \varepsilon^2 \int_U \frac{u^2}{\varepsilon^2} \|\phi'\|_{L^\infty}^2 dx = \|u\|_{L^2}^2 < \infty$$

$$\therefore \begin{cases} \|u^\varepsilon\|_{L^2} \rightarrow 0 \\ \forall \varphi \in C_c^\infty(U) \subset (L^2(U))^* = L^2(U), \langle \varphi, u^\varepsilon \rangle \rightarrow 0 \end{cases} \Rightarrow u^\varepsilon \rightarrow 0 \text{ in } L^2(U)$$

$$\text{dense}$$

② $\partial_i u^\varepsilon \rightarrow 0$ in $L^2(U)$.

$$\|\partial_i u^\varepsilon\|_{L^2}^2 = \int |\partial_i u^\varepsilon|^2 = \int |\phi'(\frac{u}{\varepsilon}) \cdot \partial_i u|^2 dx \leq \|\phi'\|_{L^\infty}^2 \|\partial_i u\|_{L^2}^2 < \infty$$

$\forall \varphi \in C_c^\infty(U)$,

$$\langle \partial_i u^\varepsilon, \varphi \rangle = \int \partial_i u^\varepsilon \cdot \varphi = - \int u^\varepsilon \cdot \partial_i \varphi \rightarrow 0 \quad (\text{因 } u^\varepsilon \rightarrow 0 \text{ in } L^2)$$

$\therefore \partial_i u^\varepsilon \rightarrow 0$ in $L^2(U)$.

$$\text{如今 } \int D_u^\varepsilon \cdot D_u dx = \sum_{i=1}^n \int \partial_i u^\varepsilon \partial_i u dx.$$

$$\rightarrow 0 \quad (\text{因 } \partial_i u \in L^2, \partial_i u^\varepsilon \rightarrow 0 \text{ in } L^2)$$

$$2: \int D_u^\varepsilon \cdot D_u = \sum_{i=1}^n \int \partial_i u^\varepsilon \partial_i u dx$$

$$= \sum_{i=1}^n \int \partial_i u \cdot \phi'(\frac{u}{\varepsilon}) \partial_i u dx$$

$$= \int |Du|^2 \phi'(\frac{u}{\varepsilon}) dx$$

由 P 在 $\{u=0\}$ 上, 令 $\varepsilon \rightarrow 0^+$ 有 $D_u=0$ a.e. on $\{u=0\}$

D

Pmk: 19 不帶 U 有界. 因 $\phi'(0)=0$, 17(2) 生效.

[5.20] 若 $u \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$. 則 $u \in L^\infty(\mathbb{R}^d)$. 且 $\|u\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^s(\mathbb{R}^d)}$.

證明: $u \in H^s(\mathbb{R}^d)$ 則 $u \in L^2(\mathbb{R}^d)$ ($s > \frac{d}{2}$)

$$|u(x)| \leq \lim_{N \rightarrow \infty} \int_{|x| \leq N} |\hat{u}(\xi)| e^{2\pi i x \cdot \xi} d\xi.$$

$$\leq \lim_{N \rightarrow \infty} \int_{|x| \leq N} |\hat{u}(\xi)| \langle \xi \rangle^s \frac{1}{\langle \xi \rangle^s} d\xi.$$

Claim: $S(\mathbb{R}^d)$ 在 $H^s(\mathbb{R}^d)$ 中稠密

若 ~~不是~~ claim 成立, 則 我們只用對 $u \in S(\mathbb{R}^d)$ 証明即可 (再延拓)

$$u \in S(\mathbb{R}^d) \quad |u(x)| = |(\hat{u}(\xi))_N^V|$$

$$\geq \left| \int_{\mathbb{R}^d} |\hat{u}(\xi)| e^{2\pi i x \cdot \xi} d\xi \right|$$

$$\leq \int_{\mathbb{R}^d} |\hat{u}(\xi)| d\xi$$

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}} = \int_{\mathbb{R}^d} |\hat{u}(\xi)| \langle \xi \rangle^s \frac{1}{\langle \xi \rangle^s} d\xi$$

$$\leq \left\| \frac{1}{|\xi|^s} \right\|_{L^2} \left\| |\xi|^s \hat{u} \right\|_{L^2}$$

$$= C_{s,d} \|u\|_{H^s(\mathbb{R}^d)}$$

再证 claim: 因 $S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, 故 $\exists v_k \in S(\mathbb{R}^d)$

$$v_k \rightarrow |\xi|^s \hat{u} \text{ in } L^2(\mathbb{R}^d)$$

$$\text{令 } u_k = (|\xi|^{-s} v_k)^{\vee} \text{ 这里合理的, 因 } v_k \cdot |\xi|^{-s} \in S(\mathbb{R}^d)$$

$$\text{故 } \|u_k - u\|_{H^s} = \left\| (\hat{u}_k - \hat{u}) |\xi|^s \right\|_{L^2}$$

$$= \left\| (|\xi|^{-s} v_k - \hat{u}) |\xi|^s \right\|_{L^2}$$

$$= \|v_k - \hat{u} |\xi|^s\|_{L^2} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

21题表明 $s > d/2$ 时, H^s 是一个 Banach 代数, 也就是对乘积封闭;

一般地, 对 $s \geq 0$, 结论是 $\|uv\|_{H^s} \leq C(\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s})$, 证明需要用到调和分析里面的 Littlewood-Paley 分解, 在此略去。借此(称作 Moser 不等式), 再由 20 题便可得到 21 题结论。

[5.21] 若 $u, v \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, 则 $uv \in H^s(\mathbb{R}^d)$.

且 $\|uv\|_{H^s(\mathbb{R}^d)} \leq C_{s,d} \|u\|_{H^s} \|v\|_{H^s}$. 右 $s > \frac{d}{2}$ 时 H^s 是代数.

证明:

$$\|uv\|_{H^s(\mathbb{R}^d)} = \|\hat{u}\hat{v}|\xi|^s\|_{L^2}.$$

$$= \|(\hat{u} * \hat{v}) |\xi|^s\|_{L^2}$$

$$= \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) |\xi|^s |\eta|^s d\eta \right\|_{L^2}$$

$$\text{这儿, } |\xi|^s = |1 + |\xi|^s|. \quad (\text{与 20 题那个差不多}).$$

$$\leq (1 + |\xi|^s)(1 + |\eta|^s).$$

$$= |\xi| =$$

① $|\xi| < \frac{|\eta|}{2}$ or $|\xi| > \frac{|\eta|}{2}$ 时,

$$|\xi|^s = 1 + |\xi|^s \cdot \begin{cases} \leq 1 + \frac{|\eta|^s}{2^s} = \frac{1}{2^s} (1 + |\eta|^s) \leq (1 + |\xi - \eta|^s)(1 + |\eta|^s) \\ \text{若 } |\xi| > |\eta| \text{ 则 } |\frac{\xi}{\eta}| > 2 \end{cases}$$

$$\Rightarrow 3\xi^2 - 8\xi\eta + 4\eta^2 \geq 0$$

$$\Rightarrow 1 + |\xi| \leq 2|\xi - \eta|$$

$$\Rightarrow |\xi|^s \leq C(|\xi - \eta|^s + |\eta|^s).$$

$$② \frac{|\eta|}{2} < |\xi| \leq 2|\eta| \text{ 时.}$$

$$1 + |\xi|^s \leq 2^s (1 + |\eta|^s) (1 + |\xi - \eta|^s).$$

$$\text{故 } \langle \xi \rangle^s \leq C_s (\langle \xi - \eta \rangle^s + \langle \eta \rangle^s).$$

代入有:

$$\begin{aligned} \|uv\|_{H^s} &\leq \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) \langle \xi - \eta \rangle^s d\eta \right\|_{L_\xi^2} \\ &\quad + \left\| \int \hat{u}(\xi - \eta) \hat{v}(\eta) \langle \eta \rangle^s d\eta \right\|_{L_\xi^2} \end{aligned}$$

$$= \left\| \hat{u} \cdot \langle \xi \rangle^s \right\|_{L^2} + \left\| \hat{u} * (\langle \cdot \rangle^s \hat{v}) \right\|_{L^2}.$$

$$\|f * g\|_{L^2} \leq \|f\|_L^1 \|g\|_{L^2}.$$

$$\leq \|\hat{u} \langle \xi \rangle^s\|_{L^2} \|\hat{v}\|_{L^1} + \|\hat{v} \langle \xi \rangle^s\|_{L^2} \|\hat{u}\|_{L^1}$$

$$= \|u\|_{H^s} \|\hat{v} \langle \xi \rangle^s \langle \xi \rangle^{-s}\|_{L^1} + \|v\|_{H^s} \|\hat{u} \langle \xi \rangle^s \langle \xi \rangle^{-s}\|_{L^1},$$

$$\leq \|u\|_{H^s} \|\hat{v} \langle \xi \rangle^s\|_{L^2} \|\langle \xi \rangle^{-s}\|_{L^2}$$

$$+ \|v\|_{H^s} \|\hat{u} \langle \xi \rangle^s\|_{L^2} \|\langle \xi \rangle^{-s}\|_{L^2}$$

$$\lesssim_s \|u\|_{H^s} \|v\|_{H^s}$$

□

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