Zero Surface Tension Limit of the Free-Boundary Problem in Incompressible Magnetohydrodynamics

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Abstract

We show that the solution of the free-boundary incompressible ideal magnetohydrodynamic (MHD) equations with surface tension converges to that of the free-boundary incompressible ideal MHD equations without surface tension given the Rayleigh-Taylor sign condition holds initially. This result is a continuation of the authors' previous works [16, 30, 15]. Our proof is based on the combination of the techniques developed in our previous works [16, 30, 15], Alinhac good unknowns, and a crucial anti-symmetric structure on the boundary.

Contents

1 Introduction

We consider the following 3D incompressible ideal MHD system which describes the motion of a conducting fluid with free surface boundary in an electro-magnetic field under the influence of surface tension

$$\begin{cases} D_t u - B\cdot\nabla B + \nabla P = 0, \quad P := p + \frac{1}{2}|B|^2 & \text{in } \mathcal{D}; \\ D_t B - B\cdot\nabla u = 0, & \text{in } \mathcal{D}; \\ \text{div } u = 0, \quad \text{div } B = 0, & \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

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with boundary conditions

$$\begin{cases} D_t|_{\partial\mathcal{D}} \in \mathcal{T}(\partial\mathcal{D}), \\ P = \sigma\mathcal{H} & \text{on } \partial\mathcal{D}, \\ B\cdot\hat{n} = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{1.2}$$

Here, $D_t := \partial_t + u\cdot\nabla$ denotes the material derivative, and $\mathcal{D} := \cup_{0\le t\le T}\{t\}\times\mathcal{D}_t$, where $\mathcal{D}_t \subseteq \mathbb{R}^3$ is a bounded domain occupied by the conducting fluid (plasma) whose boundary $\partial\mathcal{D}_t$ moves with the velocity of the fluid. Also, we denote by $v = (v_1, v_2, v_3)$ the fluid's velocity, $B = (B_1, B_2, B_3)$ the magnetic field, p the fluid's pressure, and $P := p + \frac{1}{2}|B|^2$ the total pressure. The quantity $\mathcal{H}$ is the mean curvature of the free surface $\partial\mathcal{D}_t$ and $\sigma \ge 0$ is a given constant, called surface tension coefficient. Finally, we use $\hat{n}$ to denote the exterior unit normal to $\partial\mathcal{D}_t$. The equations div $B = 0$ and $B\cdot\hat{n}|_{\partial\mathcal{D}} = 0$ are only the constraints for initial data that propagate to any $t > 0$ if initially hold (cf. [18]).

When $\sigma > 0$ (i.e., the MHD equations are under the influence of the surface tension), we proved in [15] that given a simply connected domain $\mathcal{D}_0 \subseteq \mathbb{R}^3$ and initial data u_0 and B_0 satisfying div $u_0 = 0$ and div $B_0 = 0$, $B_0\cdot\hat{n}|_{\partial\mathcal{D}_0} = 0$, there exist a set $\mathcal{D}$ and vector fields u and B that solve (1.1)-(1.2) with initial data

$$\mathcal{D}_0 = \{x : (0, x) \in \mathcal{D}\}, \quad (u, B) = (u_0, B_0), \quad \text{in } \{t = 0\}\times\Omega_0. \tag{1.3}$$

In addition, when $\sigma = 0$ (i.e., the surface tension is neglected) the initial-boundary value problem (1.1), (1.2), (1.3) is known to be ill-posed [19, 11] unless the Rayleigh-Taylor sign condition is imposed

$$-\nabla_n P \ge c_0 > 0, \quad \text{on } \Gamma, \tag{1.4}$$

for some $c_0 > 0$. In fact, *we only need to impose* (1.4) *on the initial data and then show that it propagates within the interval of existence.* The local well-posedness (LWP) of the initial-boundary value problem (1.1), (1.2), (1.3), (1.4) was obtained by the first author and Wang in [16].

The following terminologies will be used throughout the rest of this manuscript for the sake of clarity: We use "$\sigma > 0$ problem" to denote the initial-boundary value problem (1.1), (1.2), (1.3) with a positive surface tension coefficient σ. In addition, we use "$\sigma = 0$ problem" to denote the initial-boundary value problem (1.1), (1.2), (1.3) without the surface tension but with the physical condition (1.4). In this paper, we would like to prove the zero surface tension limit of (1.1)-(1.3), i.e., the solution to the "$\sigma > 0$ problem" converges to the solution to the "$\sigma = 0$ problem" as $\sigma \to 0_+$, provided the Rayleigh-Taylor sign condition (1.4) holds initially.

Before stating our result, we would like to review the related previous results. The free-boundary problem of MHD can be considered the simplified version of the plasma-vacuum free-interface model which is the basic theoretical model for plasma confinement and some astrophysical phenomena. For a detailed discussion, we refer to our previous work [15, Section 1.1.1]. In particular, the surface tension cannot be neglected when we use a free-boundary MHD system to model the liquid metal, film flow, etc [22, 31, 32]. Even if we consider the MHD flows in astrophysical plasmas, where the surface tension effect and magnetic diffusion are usually neglected, it is still useful to keep surface tension as a stabilization effect in numerical simulations of the magnetic Rayleigh-Taylor instability [37, 38].

In the absence of the magnetic field, i.e., $B = \mathbf{0}$, the free-boundary MHD system is reduced to the free-boundary incompressible Euler equations. The first breakthrough came into Wu [46, 47] for the LWP of 2D and 3D incompressible irrotational gravity water wave. In the case of nonzero vorticity, Christodoulou-Lindblad [5] first established the nonlinear a priori estimates and Lindblad [27] proved the LWP by Nash-Moser iteration. Later, Coutand-Shkoller [6] avoided the Nash-Moser iteration and extended the result to the case $\sigma > 0$. See also [48, 34, 35, 36, 8, 9] and references therein for the related works.

For the mathematical studies of free-boundary incompressible MHD system without surface tension, Hao-Luo [18, 20] proved the a priori estimates and the linearized LWP by generalizing [5, 26]. The first author and Wang [16] proved the LWP for the nonlinear problem. See also Lee [24, 25] for an alternative proof by using the vanishing viscosity-resistivity method, Sun-Wang-Zhang [39] for the incompressible MHD current-vortex sheets, Sun-Wang-Zhang [40] and the first author [12, 13] for the plasma-vacuum interface model, and the second and the third authors [29] for the minimal regularity estimates in a small fluid domain. For the compressible ideal MHD, we refer to [3, 41, 44, 33, 42, 28] and references therein.

However, when the surface tension is not neglected, much less has been developed and most of the previous results focus on the viscous or resistive MHD [4, 17, 45]. To the best of our knowledge, our previous works [30, 15] are the only available results on the free-boundary incompressible ideal MHD with surface tension. In addition, we also refer to Trakhinin-Wang [43] for the compressible case.

This paper is the continuation of our previous works [16, 30, 15]: we aim to prove the zero surface tension limit of the free-boundary problem in incompressible ideal MHD with surface tension. This can be achieved by establishing the uniform-in-σ

energy estimates. Our proof is based on the combination of the techniques in [16, 30, 15], Alinhac good unknowns, and the *newly-developed* anti-symmetric structure on the boundary.

1.1 Reformulation in Lagrangian coordinates

We reformulate the MHD equations in the Lagrangian coordinates and thus the free-surface domain becomes fixed. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Denoting coordinates on Ω by $y = (y_1, y_2, y_3)$, we define $\eta : [0, T] \times \Omega \to \mathcal{D}$ to be the flow map of the velocity v, i.e.,

$$\partial_t \eta(t, y) = u(t, \eta(t, y)), \quad \eta(0, y) = \eta_0(y), \tag{1.5}$$

where $\eta_0 : \Omega \to \mathcal{D}_0$ is a diffeomorphism. We introduce the Lagrangian velocity, magnetic field and pressure respectively by

$$v(t, y) = u(t, \eta(t, y)), \quad b(t, y) = B(t, \eta(t, y)), \quad q(t, y) = P(t, \eta(t, y)). \tag{1.6}$$

Let ∂ be the spatial derivative with respect to y variable. We introduce the cofactor matrix $A = [\partial\eta]^{-1}$. It's worth noting that since $\det(\partial\eta) = 1$, we have

$$A = \begin{pmatrix} \partial_2\eta \times \partial_3\eta \\ \partial_3\eta \times \partial_1\eta \\ \partial_1\eta \times \partial_2\eta \end{pmatrix} = \begin{pmatrix} \epsilon^{\alpha\lambda\tau}\partial_2\eta_\lambda\partial_3\eta_\tau \\ \epsilon^{\alpha\lambda\tau}\partial_3\eta_\lambda\partial_1\eta_\tau \\ \epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda\partial_2\eta_\tau \end{pmatrix}, \tag{1.7}$$

where $\epsilon^{\alpha\lambda\tau}$ is the fully antisymmetric symbol with $\epsilon^{123} = 1$. In light of (1.7), it is easy to see that A verifies the Piola's identity

$$\partial_\mu A^{\mu\alpha} = 0, \tag{1.8}$$

where the summation convention is used for repeated upper and lower indices. In above and throughout, all Greek indices range over $1, 2, 3$, and the Latin indices range over $1, 2$. For the sake of simplicity and clean notation, we consider the model case when

$$\Omega = \mathbb{T}^2 \times (0, 1) \text{ and } \eta_0 = \mathrm{Id}, \tag{1.9}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma$ and $\Gamma = \mathbb{T}^2 \times \{1\}$ is the top (moving) boundary, $\Gamma_0 = \mathbb{T}^2 \times \{0\}$ is the fixed bottom. This is known to be the reference domain. Using a partition of unity, e.g., [9, 29], a general domain can also be treated with the same tools we shall present. However, choosing Ω as above allows us to focus on the real issues of the problem without being distracted by the cumbersomeness of the partition of unity.

Remark. The use of Lagrangian coordinates loosens the restriction on the geometry of the free surface in comparison to treating the free interface as a graph. Specifically, with a large initial velocity, the free surface may fail to form a graph within a very short time interval. Nonetheless, using Lagrangian coordinates opens the possibility to go beyond that time.

The system (1.1)-(1.2) can be reformulated as:

$$\begin{cases} \partial_t v_\alpha - b_\beta A^{\mu\beta}\partial_\mu b_\alpha + A^\mu_\alpha \partial_\mu q = 0 & \text{in } [0, T] \times \Omega; \\ \partial_t b_\alpha - b_\beta A^{\mu\beta}\partial_\mu v_\alpha = 0 & \text{in } [0, T] \times \Omega; \\ A^{\mu\alpha}\partial_\mu v_\alpha = 0, \ \ A^{\mu\alpha}\partial_\mu b_\alpha = 0 & \text{in } [0, T] \times \Omega; \\ v \cdot N = b \cdot N = 0 & \text{on } \Gamma_0; \\ A^{\mu\alpha}N_\mu q + \sigma(\sqrt{g}\Delta_g \eta^\alpha) = 0 & \text{on } \Gamma; \\ A^{\mu\nu}b_\nu N_\mu = 0 & \text{on } \Gamma, \\ (\eta, v, b) = (\mathrm{Id}, v_0, b_0) & \text{on } \{t = 0\} \times \overline{\Omega}. \end{cases} \tag{1.10}$$

where N is the unit outer normal vector to $\partial\Omega$, particularly $N = (0, 0, -1)$ on Γ_0 and $N = (0, 0, 1)$ on Γ, and Δ_g is the Laplacian of the metric g_{ij} induced on $\partial\Omega(t)$ by the embedding η. Specifically, we have:

$$g_{ij} = \overline{\partial}_i \eta^\mu \overline{\partial}_j \eta_\mu, \ \ \Delta_g(\cdot) = \frac{1}{\sqrt{g}}\overline{\partial}_i(\sqrt{g}g^{ij}\overline{\partial}_j(\cdot)), \text{ where } g := \det(g_{ij}), \tag{1.11}$$

where $\overline{\partial} = (\overline{\partial}_1, \overline{\partial}_2)$ denotes the tangential derivative (with respect to Γ). In [30], the second and third authors proved the local a priori energy estimate of (1.12) for each fixed $\sigma > 0$. The local well-posedeness of this problem is established very recently in [15].

Furthermore, we are able to express the magnetic field b in terms of its initial data b_0 and η. Specifically, by the second equation of (1.10) and the divergence-free condition on b, we get $\partial_t(A^{\mu\alpha}b_\mu) = 0$ which implies $A^{\mu\alpha}b_\mu = b_0^\alpha$ and thus $b^\alpha = b_0^\mu \partial_\mu \eta^\alpha = (b_0 \cdot \partial)\eta^\alpha$. We refer to Gu-Wang [16, (1.13)-(1.15)] for the proof.

We formulate the free-boundary MHD equations with and without surface tension. For each $\sigma > 0$, let $(\eta^\sigma, v^\sigma, q^\sigma)$ verifies

$$\begin{cases} \partial_t \eta^\sigma = v^\sigma & \text{in } [0,T]\times\Omega; \\ \partial_t v^\sigma - (b_0\cdot\partial)^2\eta^\sigma + \nabla_{A(\eta^\sigma)} q^\sigma = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{A(\eta^\sigma)} v^\sigma := A(\eta^\sigma)^{\mu\nu}\partial_\mu v^\sigma_\nu = 0, & \text{in } [0,T]\times\Omega; \\ \operatorname{div} b_0 := \delta^{\mu\nu}\partial_\mu (b_0)_\nu = 0 & \text{in } \{0\}\times\Omega; \\ (v^\sigma)^3 = (b_0)^3 = 0 & \text{on } \Gamma_0; \\ A(\eta^\sigma)^{3\alpha} q^\sigma + \sigma(\sqrt{g^\sigma}\Delta_{g^\sigma}(\eta^\sigma)^\alpha) = 0 & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ (\eta^\sigma, v^\sigma) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}, \end{cases} \tag{1.12}$$

where $\nabla^\alpha_{A(\eta^\sigma)} := A(\eta^\sigma)^{\mu\alpha}\partial_\mu$ denotes the covariant derivative, and the induced metric $g_{ij} = g_{ij}(\eta^\sigma)$ is given in (1.11). The initial pressure q_0 is determined by v_0 and b_0 through the elliptic equation

$$-\Delta q_0 = (\partial v_0) : (\partial v_0) - (\partial b_0) : (\partial b_0), \quad \text{in } \Omega, \tag{1.13}$$

with boundary conditions

$$\begin{aligned} q_0 = 0, &\quad \text{on } \Gamma, \\ \frac{\partial q_0}{\partial N} = 0, &\quad \text{on } \Gamma_0. \end{aligned} \tag{1.14}$$

For the boundary condition on Γ, since $A = \mathbf{I}_3$ and $g_{ij} = \delta_{ij}$ at $t = 0$, the fifth equation of (1.12) becomes

$$q_0 + \sigma\delta^{ij}\overline{\partial}_i\overline{\partial}_j\eta^3(0) = 0,$$

and because $\overline{\partial}_i\overline{\partial}_j\eta^3(0) = 0$ on Γ, we have $q_0 = 0$ on Γ. The Neumann boundary condition follows from restricting the normal component of the momentum equation to Γ_0.

On the other hand, when $\sigma = 0$, let (ζ, w, r) verifies

$$\begin{cases} \partial_t \zeta = w & \text{in } [0,T]\times\Omega; \\ \partial_t w - (b_0\cdot\partial)^2\zeta + \nabla_{A(\zeta)} r = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{A(\zeta)} w = 0, & \text{in } [0,T]\times\Omega; \\ \operatorname{div} b_0 = 0 & \text{in } \{0\}\times\Omega; \\ w^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ r = 0 & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ (\zeta, w) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{1.15}$$

The initial pressure r_0 is determined by the elliptic equation

$$-\Delta r_0 = (\partial v_0) : (\partial v_0) - (\partial b_0) : (\partial b_0), \quad \text{in } \Omega, \tag{1.16}$$

with the boundary condition $r_0 = 0$ on $\Gamma \cup \Gamma_0$. This yields that $r_0 = q_0$. Moreover, in order for (1.15) to be well-posed, we assume the Rayleigh-Taylor sign condition

$$-\partial_3 r_0 \geq c_0 > 0, \quad \text{on } \Gamma \tag{1.17}$$

holds for some constant $c_0 > 0$.

1.2 The main result

This paper aims to show that if q_0 verifies the Rayleigh-Taylor sign condition, i.e.,

$$-\partial_3 q_0 \geq c_0 > 0, \quad \text{on } \Gamma, \tag{1.18}$$

then the solution of the "$\sigma > 0$ problem" (1.12) converges to the solution of the "$\sigma = 0$ problem" (1.15) as $\sigma \to 0$. Specifically, we prove

Theorem 1.1. Suppose the initial data (v_0, b_0, q_0) verifies:

1. v_0, b_0 are divergence-free vector fields with $v_0^3 = 0$ on Γ_0 and $b_0^3 = 0$ on $\Gamma \cup \Gamma_0$.
2. $v_0, b_0 \in H^5(\Omega)$; $\sqrt{\sigma} v_0$, $\sqrt{\sigma} b_0 \in H^{5.5}(\Omega)$; $\sigma \overline{\Delta} v_0^3, \sigma \overline{\Delta} b_0^3 \in H^{3.5}(\Gamma)$; $\sigma^{3/2} \overline{\Delta} v_0^3, \sigma^{3/2} \overline{\Delta} b_0^3 \in H^4(\Gamma)$, where $\overline{\Delta} := \delta^{ij} \overline{\partial}_i \overline{\partial}_j$.
3. The Rayleigh-Taylor sign condition
$$-\partial_3 q_0 \geq c_0 > 0, \quad \text{on } \Gamma. \tag{1.19}$$
4. The compatibility conditions up to the 4-th order, where the j-th order ($0 \leq j \leq 4$) condition reads
$$\begin{aligned} \partial_t^j q(0) &= \sigma \partial_t^j \mathcal{H}(0), \quad \text{on } \Gamma, \\ \partial_3 \partial_t^j q(0) &= 0, \quad \text{on } \Gamma_0. \end{aligned} \tag{1.20}$$

Then there exists some $T > 0$ independent of σ, such that the solutions to (1.12) and (1.15) exists, and as $\sigma \to 0$, we have

$$(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, q^\sigma) \xrightarrow{C^1_{t,y}([0,T]\times\Omega)} (w, (b_0 \cdot \partial)\zeta, r). \tag{1.21}$$

We will drop the superscript σ on $(\eta^\sigma, v^\sigma, q^\sigma)$ in the rest of this paper and simply denote them by (η, v, q) when no confusion can be raised.

Notation 1.2. (The Sobolev norms) Let $f = f(t, y)$ be a smooth function on $[0, T] \times \Omega$ and $g = g(t, y)$ be a smooth function on $[0, T] \times \Gamma$. We define $\|f\|_s := \|f(t, \cdot)\|_{H^s(\Omega)}$ and $|g|_s := |g(t, \cdot)|_{H^s(\Gamma)}$ throughout the rest of this paper.

Remark. The reason for us to require v_0 to have higher regularity on the boundary is that we need to express the initial data of q and its time derivatives in terms of v_0 and b_0. Specifically, since $q_t(0)$ verifies an elliptic equation with the boundary condition

$$\begin{aligned} q_t(0) &= \sigma \overline{\Delta} v_0^3 + \text{l.o.t}, \quad \text{on } \Gamma, \\ \partial_3 q_t(0) &= 0, \quad \text{on } \Gamma_0, \end{aligned}$$

given by the compatibility condition (1.20), then the standard elliptic estimate yields that the control of $\|q_t(0)\|_4$ and $\|\sqrt{\sigma} q_t(0)\|_{4.5}$ requires the bounds for $|\sigma \overline{\Delta} v_0^3|_{3.5}$ and $|\sigma^{3/2} \overline{\Delta} v_0^3|_4$, respectively. In addition, the bounds for $|\sigma \overline{\Delta} b_0^3|_{3.5}$ and $|\sigma^{3/2} \overline{\Delta} b_0^3|_4$ are also required when controlling $\|q_{tt}(0)\|_3$ and $\|\sqrt{\sigma} q_{tt}(0)\|_{3.5}$, respectively. The detailed arguments can be found in the Appendix A.

Remark. Our argument can be adapted to treat the general case when η_0 is replaced by a smooth map. However, our requirement on the initial data in Theorem 1.1 has to be modified slightly. We no longer have $q_0^\sigma = r_0$ since $q_0^\sigma = \sigma \mathcal{H}_0 \neq 0$ on Γ. Therefore, the initial data for the $\sigma > 0$ problem should be (v_0, b_0, q_0^σ), where q_0^σ verifies (1.18) for all $\sigma > 0$ and $q_0^\sigma \to r_0$ as $\sigma \to 0$.

Theorem 1.1 is a direct consequence of the following theorem where we establish an energy estimate of the $\sigma > 0$ problem that is uniform as $\sigma \to 0$ via the standard compactness argument, which is also proved in section 8.

Theorem 1.3. Let (v_0, b_0, q_0) be the initial data given in Theorem 1.1. Let

$$E(t) = E_1(t) + \sigma E_2(t), \tag{1.22}$$

with

$$E_1(t) := \|\eta(t)\|_5^2 + \sum_{k=0}^{5} \left(\left\|\partial_t^k v(t)\right\|_{5-k}^2 + \left\|\partial_t^k (b_0 \cdot \partial)\eta(t)\right\|_{5-k}^2 \right) + \left|\overline{\partial}^5 \eta(t) \cdot \hat{n}(t)\right|_0^2 \tag{1.23}$$

and

$$E_2(t) := \|\eta(t)\|_{5.5}^2 + \sum_{k=0}^{4} \left(\left\|\partial_t^k v(t)\right\|_{5.5-k}^2 + \left\|\partial_t^k (b_0 \cdot \partial)\eta(t)\right\|_{5.5-k}^2 \right) + \sum_{k=0}^{5} \left|\overline{\partial}^{6-k} \partial_t^k \eta(t) \cdot \hat{n}(t)\right|_0^2 + \left|\overline{\partial}^5 (b_0 \cdot \partial)\eta(t) \cdot \hat{n}(t)\right|_0^2, \tag{1.24}$$

where $\hat{n}$ is the Eulerian unit normal vector. Then there exists a constant $C = C_1 + C_2$, where C_1 depending on c_0, $\|v_0\|_5$, $\|b_0\|_5$, $|\sigma\overline{\Delta}v_0^3|_{3.5}$, $|\sigma\overline{\Delta}b_0^3|_{3.5}$, and C_2 depending on $\|\sqrt{\sigma}v_0\|_{5.5}$, $\|\sqrt{\sigma}b_0\|_{5.5}$, $|\sigma^{\frac{3}{2}}\overline{\Delta}v_0^3|_4$, $|\sigma^{\frac{3}{2}}\overline{\Delta}b_0^3|_4$ such that

$$E(t) \leq C, \quad \forall t \in [0, T] \tag{1.25}$$

holds uniformly for all σ. Furthermore, $C_2 \to 0$ as $\sigma \to 0$. Moreover, the higher regularity of v on the boundary can be recovered at the later time, i.e.,

$$|\sigma v^3(t)|_{5.5} + |\sigma^{\frac{3}{2}} v^3(t)|_6 + |\sigma b^3(t)|_{5.5} + |\sigma^{\frac{3}{2}} b^3(t)|_6 \leq P(E(t)), \quad \forall t \in (0, T]. \tag{1.26}$$

In other words, Theorem 1.3 states that given the initial data for the $\sigma = 0$ problem, we want to prove the a priori energy estimate for the $\sigma > 0$ problem, and such energy estimate has to be uniform in σ as σ tends to zero. That said, it is natural to come up with the energy (1.22), in which $E_1(t)$ corresponds to the $\sigma = 0$ problem and $E_2(t)$ corresponds to the $\sigma > 0$ problem. Also, $E(t)$ reduces to $E_1(t)$ when $\sigma = 0$.

1.3 Key new difficulties and comparison with the existing literature

1.3.1 Difference between Euler and MHD equations

The zero surface tension limit for the free-boundary (compressible) Euler equations is studied in [7], but the method developed there does not apply to the free-boundary MHD equations. Their treatment depends on the assumption that η is $1/2$-derivatives more regular in the interior than v when passing σ to 0. This can be done by assuming the vorticity is more regular initially[1]and this extra regularity can then be carried over to η through the Cauchy invariance. However, in MHD equations, the coupling between magnetic field and velocity denies the possibility of the higher regularity assumption on the vorticity and thus on η. This is due to that the Lorentz force (i.e., $B \cdot \nabla B$) in the momentum equation destroys the Cauchy invariance.

On the other hand, we mention here that the zero surface tension limit of 2D Euler equations was established by Ambrose-Masmoudi [2] but without vorticity. In this paper, we develop a unified framework of proving the zero surface tension limit for both incompressible Euler and MHD equations in 3D, without imposing the irrotational assumption[2]on v_0. That is, we can recover the zero surface tension limit of the 3D Euler equations by setting $b_0 = 0$.

1.3.2 Treating the higher-order interior and boundary terms

The aforementioned regularity issue prevents us from commuting $\overline{\partial}^5$ with the equation

$$\partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_A q = 0,$$

because we are unable to control the term generated when all derivatives land on A. Our method to overcome this issue is to invoke the so-called Alinhac good unknowns for the leading order terms involving v and q (i.e., (2.1)). The structure of these good unknowns ties to the covariant differentiation in the Eulerian coordinates, and thus the higher-order term will not show up. Nevertheless, the use of good unknowns alone cannot fully resolve the problem, as the surface tension introduces higher-order boundary terms in the energy estimate (i.e., the integrals on the RHS of (2.2)). None of these integrals can be controlled directly, but fortunately, we can overcome this issue by exploiting the structure of the equations as well as that of the good unknowns to generate cancellation schemes. We refer to Section 2.1.2 for the detailed explanations.

1.3.3 Application to other free-boundary fluid models

The method developed in this manuscript is applicable to study the zero surface limit problem for Euler equations without the extra regularity assumption on η. In particular, Theorem 1.3 yields the uniform-in-σ energy estimate for the free boundary incompressible Euler equations by setting $b_0 = 0$. Also, it is possible to adapt our method to study the zero surface tension limit in other complex fluid models, where no extra regularity assumption can be made on the flow map η.

[1]It is also well-known that in Euler equations, the extra regularity assumption on vorticity can be propagated to a later time.

[2]Physically, the Lorentz force twists the velocity field and thus introduces vorticity. In consequence, the irrotational assumption on the velocity becomes inadequate in MHD equations.

2 Strategy of the proof and some auxiliary results

2.1 Proof of Theorem 1.3: An overview

2.1.1 Necessity of time derivatives

The energy (1.22) consists of mixed space and time derivatives, and this is required as a result of the estimate of the pressure q. Specifically, we cannot equip the elliptic equation verified by q with Dirichlet boundary condition when $\sigma > 0$ (see, e.g., [5, 16]). Instead, we should impose the Neumann boundary condition, which contains the time derivative of v and thus forces us to analyze all the time derivatives of the variables v and $b = (b_0 \cdot \partial)\eta$.

2.1.2 Interior tangential estimates: Alinhac good unknowns and cancellation structures

The non-weighted full Sobolev norms are analyzed via div-curl analysis. The curl part can be controlled via the evolution equation of the Eulerian vorticity. The normal trace part in the div-curl decomposition should be reduced to the interior tangential estimates via Lemma 2.6. If the tangential derivatives contain at least one time derivatives, one may follow the ideas in our previous work [30, 15] to close the energy estimates and we refer to Section 6 for the proof.

However, when the tangential derivatives are purely spatial, we have to introduce the Alinhac good unknowns to proceed with the energy estimates because we cannot directly commute $\overline{\partial}^5$ with the covariant derivative ∇_A falling on q or v. Instead, we rewrite the term $\overline{\partial}^5(\nabla_A q)$ in terms of the covariant derivative of the Alinhac good unknowns plus a controllable "error" term. See Section 5 for the proof. Such remarkable observation was first due to Alinhac [1]. In the study of free-surface fluid, it is first (implicitly) applied to the Q-tensor energy introduced by Christodoulou-Lindblad [5]. Here, following [16, Section 4], the Alinhac good unknowns of v and q with respect to $\overline{\partial}^5$ are

$$\mathbf{V} := \overline{\partial}^5 v - \overline{\partial}^5\eta \cdot \nabla_A v, \quad \mathbf{Q} := \overline{\partial}^5 q - \overline{\partial}^5\eta \cdot \nabla_A q. \tag{2.1}$$

Under this setting, it suffices to analyze the evolution equation of the good unknowns to derive the $\overline{\partial}^5$-estimates.

Compared with the "$\sigma = 0$ problem", the boundary integral appearing in the tangential estimates, which reads $J := -\int_\Gamma A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha \,\mathrm{d}S$, becomes very difficult because the top order derivative of the pressure $\overline{\partial}^5 q$ no longer vanishes due to the presence of surface tension. It contributes to

$$-\int_\Gamma A^{3\alpha}\overline{\partial}^5 q \mathbf{V}_\alpha dS = -\int_\Gamma \overline{\partial}^5(A^{3\alpha}q)\mathbf{V}_\alpha \,\mathrm{d}S + \int_\Gamma q(\overline{\partial}^5 A^{3\alpha})\mathbf{V}_\alpha \,\mathrm{d}S + \cdots =: \mathrm{ST} + J_1 + \cdots, \tag{2.2}$$

and the remaining term reads

$$\mathrm{RT} := \int_\Gamma A^{3\alpha}(\overline{\partial}^5\eta \cdot \nabla_A q)\mathbf{V}_\alpha \,\mathrm{d}S.$$

Due to the presence of $\partial_3 q$, the term RT can be directly controlled by invoking the Rayleigh-Taylor sign condition and a standard cancellation structure of the Alinhac good unknowns which relies on the simple identity $\partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$ and also appears in the "$\sigma = 0$ problem". This part contributes to the non-weighted boundary energy $|\overline{\partial}^5\eta \cdot \hat{n}|_0^2$ that exactly controls the regularity of the second fundamental form of the free surface. The term ST contributes to the $\sqrt{\sigma}$-weighted energy $|\sqrt{\sigma}\overline{\partial}^6\eta \cdot \hat{n}|_0^2$ after plugging the surface tension equation $A^{3\alpha}q = -\sigma\sqrt{g}\Delta_g\eta^\alpha$ and integrating by parts, where the error terms can be either directly controlled or eliminated by using the structure of the good unknown $\mathbf{V}$ (e.g., (5.23)).

Finally, $\overline{\partial}^5 A^{3\alpha}$ in J_1 has the top order contribution $\overline{\partial}^6\eta \times \overline{\partial}\eta$ that cannot be directly controlled. To overcome this difficulty, we write $A^{3\alpha} = (\overline{\partial}_1\eta \times \overline{\partial}_2\eta)^\alpha$ and use it to observe a crucial symmetric structure on the boundary, while the 2D version of this symmetric structure was developed in [14] and plays an important role in the local well-posedness of free-surface incompressible elastodynamics.

We illustrate the observation of symmetric structure briefly. By plugging $A^{3\alpha} = (\overline{\partial}_1\eta \times \overline{\partial}_2\eta)^\alpha$ in J_1, the highest order terms are all of the form

$$J_{11} = \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta) \cdot \overline{\partial}^5 v \,\mathrm{d}S,$$

which cannot be controlled directly. To resolve this, we re-express J_{11} as

$$
\begin{aligned}
J_{11} =& \frac{d}{dt}\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S - \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S - \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}^5\eta\,\mathrm{d}S \\
&- \int_\Gamma \sigma\partial_t\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S \\
=& \frac{d}{dt}\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S + \underbrace{\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\overline{\partial}_1\eta\,\mathrm{d}S}_{\mathfrak{A}} + \int_\Gamma \sigma(\overline{\partial}^5 v\times\overline{\partial}_1(\mathcal{H}\overline{\partial}_2\eta))\cdot\overline{\partial}^5\eta\,\mathrm{d}S \\
&- \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}^5\eta\,\mathrm{d}S - \int_\Gamma \sigma\partial_t\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S
\end{aligned}
$$

Although we do not have control for $\mathfrak{A}$, by the antisymmetry of the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$, we have

$$
(\overline{\partial}^5 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\overline{\partial}_1\eta = -(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5 v,
$$

and thus $\mathfrak{A} = -J_{11}$. This implies

$$
J_{11} = \frac{1}{2}\frac{d}{dt}\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S + ...
$$

Nevertheless, the first term on the RHS does not have a positive sign. In consequence, we need to control it by the $\sqrt{\sigma}$-weighted energy $|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0^2$. To achieve this, we need to invoke the decomposition $\delta^{\alpha\beta} = \hat{n}^\alpha\hat{n}^\beta + \tau^\alpha\tau^\beta = \hat{n}^\alpha\hat{n}^\beta + g^{ij}\overline{\partial}_i\eta_\alpha\overline{\partial}_j\eta_\beta$, where τ denotes the unit Eulerian tangential vector, and use the vector identity $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ to obtain good cancellations of the error terms. We refer to section 5.2.3 for the details.

Remark. It appears that the aforementioned cancellation schemes are crucial when treating the higher-order terms ST and J_1. Nevertheless, neither of these terms would appear in the case when Alinhac good unknowns are not needed. For instance, the good unknowns are not employed when proving the local existence of the $\sigma > 0$ problem with fixed σ in [15] since we do not need to consider the energy $E_1(t)$ (defined in (1.23)). In addition to this, although the zero surface tension limit is proved in [7], no good unknowns are required owing to the extra regularity assumption on η.

2.1.3 Necessity of weighted energy and the control via surface tension

In the tangential estimates, especially in the boundary integrals, there are a lot of terms that have 5 derivatives weighted by the surface tension coefficients. Therefore, it is reasonable to include the weighted $H^{5.5}$-energy $\sigma E_2(t)$ to control these boundary terms via the trace lemma. To control the weighted higher-order energy $\sigma E_2(t)$, we again do the div-curl decomposition, while the $\sqrt{\sigma}$-weighted normal traces, i.e., the $\sqrt{\sigma}$-weighted Lagrangian normal projections, are no longer reduced by using Lemma 2.6. Instead, we notice that the boundary energies contributed by the surface tension in the *non-weighted tangential estimates* are exactly the $\sqrt{\sigma}$-weighted Eulerian normal projections with the same order as those $\sqrt{\sigma}$-weighted normal traces. Therefore, it remains to control the gap between the Eulerian normal $\hat{n}$ and the Lagrangian normal N, which is expected to be small due to the short time and $\hat{n} = N$ at $t = 0$. Hence, the energy estimates for $E(t) = E_1(t) + \sigma E_2(t)$ are closed. The detailed discussion can be found in Section 7.

2.2 The auxiliary results

In this subsection we record some well-known results that will be used frequently (and sometimes silently) throughout the rest of this manuscript.

2.2.1 Geometric identities

Lemma 2.1. Let $\hat{n}$ be the unit outer normal to $\eta(\Gamma)$ and $\mathcal{T}, \mathcal{N}$ be the tangential and normal bundle of $\eta(\Gamma)$ respectively. Denote $\Pi : \mathcal{T}|_{\eta(\Gamma)} \to \mathcal{N}$ to be the canonical normal projection. Denote $\overline{\partial}_A$ to be ∂_t or $\overline{\partial}_1, \overline{\partial}_2$. Then we have the identities

$$
\begin{aligned}
\hat{n} := n \circ \eta =& \frac{A^T N}{|A^T N|}, && (2.3)\\
|A^T N| =& |(A^{31}, A^{32}, A^{33})| = \sqrt{g}, && (2.4)\\
\Pi^\alpha_\lambda =& \hat{n}^\alpha \hat{n}_\lambda = \delta^\alpha_\lambda - g^{kl}\overline{\partial}_k \eta_\alpha \overline{\partial}_l \eta_\lambda, && (2.5)\\
\Pi^\alpha_\lambda =& \Pi^\alpha_\mu \Pi^\mu_\lambda, && (2.6)\\
-\Delta_g(\eta^\alpha|_\Gamma) =& \mathcal{H} \circ \eta \hat{n}^\alpha, && (2.7)\\
\sqrt{g}\Delta_g \eta^\alpha =& \sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}_i \overline{\partial}_j \eta^\lambda = \sqrt{g} g^{ij} \overline{\partial}_i \overline{\partial}_j \eta^\alpha - \sqrt{g} g^{ij} g^{kl} \overline{\partial}_k \eta^\alpha \overline{\partial}_l \eta^\mu \overline{\partial}_i \overline{\partial}_j \eta_\mu, && (2.8)\\
\overline{\partial}_A(\sqrt{g}\Delta_g \eta^\alpha) =& \overline{\partial}_i \Big(\sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}_A \overline{\partial}_j \eta^\lambda + \sqrt{g}(g^{ij}g^{kl} - g^{ik}g^{lj}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda \overline{\partial}_A \overline{\partial}_l \eta^\lambda \Big), && (2.9)\\
\overline{\partial}_A \hat{n}_\mu =& - g^{kl} \overline{\partial}_k \overline{\partial}_A \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu, && (2.10)\\
\partial_t(\sqrt{g} g^{ij}) =& \sqrt{g}(g^{ij}g^{kl} - 2g^{lj}g^{ik}) \overline{\partial}_k v^\lambda \overline{\partial}_l \eta_\lambda, && (2.11)\\
\overline{\partial}(\sqrt{g} g^{ij}) =& \sqrt{g}\left(\frac{1}{2} g^{ij} g^{kl} - g^{ik} g^{jl}\right) \underbrace{(\overline{\partial}\overline{\partial}_k \eta^\mu \overline{\partial}_l \eta_\mu + \overline{\partial}_k \eta^\mu \overline{\partial}\overline{\partial}_l \eta_\mu)}_{=\overline{\partial} g_{kl}}. && (2.12)
\end{aligned}
$$

Proof. See Disconzi-Kukavica [8, Lemma 2.5]. □

Notation 2.2. We shall use the notation $Q(\partial\eta)$ and $Q(\overline{\partial}\eta)$ to denote the rational functions of $\partial\eta$ and $\overline{\partial}\eta$, respectively.

This Q notation allows us to record error terms in a concise way. For example, for any tangential derivative $\overline{\partial}_A$, we have $\overline{\partial}_A Q(\overline{\partial}\eta) = \tilde{Q}^i_\alpha(\overline{\partial}\eta)\overline{\partial}_A \overline{\partial}_i \eta^\alpha$ where the term $\tilde{Q}^i_\alpha(\overline{\partial}\eta)$ is also a rational function of $\overline{\partial}\eta$. Also, recall that $g_{ij} = \overline{\partial}_i \eta_\mu \overline{\partial}_j \eta^\mu$ and $g = \det[g_{ij}]$ and $[g^{ij}] = [g_{ij}]^{-1}$. This means that g_{ij}, g and g^{ij} are rational functions of $\overline{\partial}\eta$ and so is Π.

The following lemma will be employed frequently in the rest of this paper.

Lemma 2.3 (**A priori assumptions**). Assume that $\|\eta\|_5, \|v\|_5, \|\sqrt{\sigma}\eta\|_{5.5}, \|\sqrt{\sigma}v\|_{5.5} \le N_0$ for some $N_0 \ge 1$. If $T \le \varepsilon/P(N_0)$ for some fixed polynomial P and η, v is defined on $[0, T]$, then the following inequality holds for $t \in [0, T]$:

$$\|A^{\mu\nu}(t) - \delta^{\mu\nu}\|_4 \lesssim \varepsilon, \quad \|A^{\mu\alpha}(t) A^\nu_\alpha(t) - \delta^{\mu\nu}\|_4 \lesssim \varepsilon, \tag{2.13}$$

$$\|\sqrt{\sigma}(A^{\mu\nu}(t) - \delta^{\mu\nu})\|_{4.5} \lesssim \varepsilon, \quad \|\sqrt{\sigma}(A^{\mu\alpha}(t) A^\nu_\alpha(t) - \delta^{\mu\nu})\|_{4.5} \lesssim \varepsilon, \tag{2.14}$$

$$|\sqrt{g} g^{ij} - \delta^{ij}|_{L^\infty} \le \varepsilon. \tag{2.15}$$

Proof. Since $A^{\mu\nu}(0) = A^{\mu\alpha}(0) A^\nu_\alpha(0) = \delta^{\mu\nu}$ and invoking the fundamental theorem of calculus, we have

$$\|A^{\mu\nu}(t) - \delta^{\mu\nu}\|_4 \lesssim \int_0^t \|\partial_t A^{\mu\nu}\|_4,$$

and

$$\|A^{\mu\alpha}(t) A^\nu_\alpha(t) - \delta^{\mu\nu}\|_4 = \int_0^t \|\partial_t (A^{\mu\alpha} A^\nu_\alpha)\|_4,$$

and so (2.13) follows because $t \le T \le \varepsilon/P(N_0)$. The inequalities are proved similarly. □

2.2.2 Sobolev inequalities

First, we list the Kato-Ponce inequality and its corollary which will be used in nonlinear product estimates.

Lemma 2.4 (**Kato-Ponce type inequalities**). Let $J = (I - \Delta)^{1/2}$, $s \ge 0$. Let f, g be smooth functions. Then the following estimates hold:

(1) $\forall s \ge 0$, we have

$$
\begin{aligned}
\|J^s(fg)\|_{L^2} &\lesssim \|f\|_{W^{s,p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|g\|_{W^{s,q_2}},\\
\|\partial^s(fg)\|_{L^2} &\lesssim \|f\|_{\dot{W}^{s,p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|g\|_{\dot{W}^{s,q_2}},
\end{aligned} \tag{2.16}
$$

with $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \le p_1, q_2 < \infty$;

(2) $\forall s \ge 1$, we have

$$\|J^s(fg) - (J^s f)g - f(J^s g)\|_{L^p} \lesssim \|f\|_{W^{1,p_1}}\|g\|_{W^{s-1,q_2}} + \|f\|_{W^{s-1,q_1}}\|g\|_{W^{1,q_2}} \tag{2.17}$$

for all the $1 < p < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$.

Proof. See Kato-Ponce [23]. □

The inequalities listed in the following corollary shall come in handy when estimating products in fractional Sobolev spaces. All of them are direct consequences of (2.16).

Corollary 2.5. Let f, g be given as above. We have

$$\|fg\|_{0.5} \lesssim \|f\|_{0.5}\|g\|_{1.5+\delta}, \tag{2.18}$$

$$\|fg\|_s \lesssim \|f\|_s\|g\|_{1.5+\delta} + \|f\|_{1.5+\delta}\|g\|_s, \quad s \ge 1.5. \tag{2.19}$$

Proof. (2.18) follows from setting $s = 0.5$, $p_1 = 2$, $p_2 = \infty$, and $q_1 = q_2 = 4$ in the first inequality of (2.16). (2.19) follows from setting $p_1 = 2$, $p_2 = \infty$, $q_1 = \infty$ and $q_2 = 2$ in the second inequality of (2.16). □

Lemma 2.6 (**Normal trace lemma**)**.** Let X be a smooth vector field. Then

$$\left|\overline{\partial}X \cdot N\right|_{-0.5} \lesssim \|\overline{\partial}X\|_0 + \|\text{div } X\|_0 \tag{2.20}$$

Proof. This can be proved by testing a $H^{0.5}(\Gamma)$ function and divergence theorem. See [16, Lemma 3.4]. □

2.2.3 Elliptic estimates

We illustrate the Hodge-type div-curl estimate, which will be adapted to study the full Sobolev norms of v and $(b_0 \cdot \partial)\eta$.

Lemma 2.7 (**The Hodge-type elliptic estimate**)**.** Let X be a smooth vector field and $s \ge 1$, then it holds that

$$\|X\|_s \lesssim \|X\|_0 + \|\text{div } X\|_{s-1} + \|\text{curl } X\|_{s-1} + |\overline{\partial}X \cdot N|_{s-1.5}. \tag{2.21}$$

Proof. This follows from the well-known identity $-\Delta X = \text{curl curl } X - \nabla \text{div } X$ and integrating by parts. □

Finally, the following H^1-elliptic estimates will be applied to control $\|\partial_t^3 q\|_1$. Its proof can be found in [21].

Lemma 2.8 (**Low regularity elliptic estimates**)**.** Assume $\mathfrak{B}^{\mu\nu}$ satisfies $\|\mathfrak{B}\|_{L^\infty} \le K$ and the ellipticity $\mathfrak{B}^{\mu\nu}(x)\xi_\mu\xi_\nu \ge \frac{1}{K}|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Assume W to be an H^1 solution to

$$\begin{cases} \partial_\nu(\mathfrak{B}^{\mu\nu}\partial_\mu W) = \text{div } \pi & \text{in } \Omega \\ \mathfrak{B}^{\mu\nu}\partial_\nu W N_\mu = h & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

where $\pi, \text{div } \pi \in L^2(\Omega)$ and $h \in H^{-0.5}(\partial\Omega)$ with the compatibility condition

$$\int_{\partial\Omega} (\pi \cdot N - h)dS = 0.$$

If $\|\mathfrak{B} - I\|_{L^\infty} \le \varepsilon_0$ which is a sufficently small constant depending on K, then we have:

$$\|W - \overline{W}\|_1 \lesssim \|\pi\|_0 + |h - \pi \cdot N|_{-0.5}, \text{ where } \overline{W} := \frac{1}{|\Omega|}\int_\Omega W dy, \tag{2.23}$$

3 Elliptic estimates for the pressure and its time derivatives

In this section we control the pressure q and its time derivatives. The estimates presented in this section are essentially the same as the ones in [15, Section 3]. Our conclusion is

Proposition 3.1. The total pressure q satisifies

$$\sum_{k=0}^{4} \|\partial_t^k q\|_{5-k}^2 + \sum_{k=0}^{3} \|\sqrt{\sigma}\partial_t^k q\|_{5.5-k}^2 \le P(E(t)). \tag{3.1}$$

3.1 Control of $\|q\|_5^2$ and $\sigma\|q\|_{5.5}^2$

Due to the presence of the surface tension, we need to consider the elliptic equation verified by q equipped with the Neumann boundary condition. We henceforth let $\Delta_A q := \text{div}_A(\nabla_A q) = A^{\mu\alpha}\partial_\mu(A^\nu_\alpha \partial_\nu q)$. Taking div_A in the second equation of (1.12) and invoking the identity $\text{div}_A v = \text{div}_A(b_0 \cdot \partial)\eta = 0$, we obtain

$$-\Delta_A q = -(\partial_t A^{\mu\alpha})\partial_\mu v_\alpha - ((b_0 \cdot \partial)A^{\mu\alpha})\partial_\mu (b_0\cdot\partial)\eta_\alpha + A^{\mu\alpha}(\partial_\mu b_0 \cdot \partial)(b_0\cdot\partial)\eta_\alpha,$$

and thus

$$-\Delta q = -\partial_\mu\left((\delta^{\mu\nu} - A^{\mu\alpha}A^\nu_\alpha)\partial_\nu q\right) - (\partial_t A^{\mu\alpha})\partial_\mu v_\alpha - ((b_0\cdot\partial)A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha + A^{\mu\alpha}(\partial_\mu b_0\cdot\partial)(b_0\cdot\partial)\eta_\alpha. \tag{3.2}$$

The Neumann boundary condition is derived by contracting $A^{\mu\alpha}N_\mu = A^{3\alpha}$ with the momentum equation of (1.12). This leads to

$$\frac{\partial q}{\partial N} = (\delta^{3\mu} - A^{3\alpha}A^\mu_\alpha)\partial_\mu q - A^{3\alpha}\partial_t v_\alpha + A^{3\alpha}(b_0\cdot\partial)^2\eta_\alpha, \quad \text{on } \Gamma. \tag{3.3}$$

Furthermore, since $A^{31} = A^{32} = 0$, $A^{33} = 1$, $v_3 = 0$, and $b_0^3 = 0$ on Γ_0, (3.3) implies that $\frac{\partial q}{\partial N} = 0$ on Γ_0. First, we estimate $\|q\|_5^2$. By the standard elliptic estimate, we have

$$\|q\|_5^2 \lesssim \|\text{RHS of (3.2)}\|_3^2 + |\text{RHS of (3.3)}|_{3.5}^2 + |q|_0^2. \tag{3.4}$$

Invoking (2.13) in Lemma 2.3, the standard product estimates imply

$$\|\text{RHS of (3.2)}\|_3^2 \lesssim \varepsilon\|q\|_5^2 + P(\|b_0\|_4, \|\eta\|_4, \|v\|_4, \|(b_0\cdot\partial)\eta\|_4). \tag{3.5}$$

and by the trace lemma,

$$|\text{RHS of (3.3)}|_{3.5}^2 \le \|\text{RHS of (3.3)}\|_4^2 \lesssim \varepsilon\|q\|_5^2 + P(\|b_0\|_4, \|\eta\|_5, \|(b_0\cdot\partial)\eta\|_5, \|\partial_t v\|_4). \tag{3.6}$$

Finally, since (2.3) implies $\hat{n} = A^{3\alpha}/\sqrt{g}$, the 6th equation in (1.12) becomes

$$q = -\sigma\Delta_g\eta\cdot\hat{n}, \quad \text{on } \Gamma, \tag{3.7}$$

and thus

$$\|q\|_0^2 \le P(\|\sqrt{\sigma}\eta\|_{5.5}, \|\eta\|_5). \tag{3.8}$$

Second, we estimate the σ-weighted term $\sigma\|q\|_{5.5}^2$. Again, the standard elliptic estimate implies

$$\sigma\|q\|_{5.5}^2 \lesssim \|\sqrt{\sigma}\cdot\text{RHS of (3.2)}\|_{3.5}^2 + |\sqrt{\sigma}\cdot\text{RHS of (3.3)}|_4^2 + |\sqrt{\sigma}q|_0^2. \tag{3.9}$$

However, we need to make sure that the top order terms on the RHS appear linearly so that they can be controlled in the σ-weighted norms. This is indeed the case thanks to the product estimate: Invoking (2.19) with $s = 3.5$, and both (2.13) and (2.14) in Lemma 2.3, we have

$$\|\sqrt{\sigma}\cdot\text{RHS of (3.2)}\|_{3.5}^2 \lesssim \varepsilon\|\sqrt{\sigma}q\|_{5.5}^2 + \varepsilon\|q\|_5^2 + P(\|b_0\|_{4.5}, \|\eta\|_{4.5}, \|v\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}). \tag{3.10}$$

Furthermore, invoking (2.19) with $s = 4.5$, then

$$\begin{aligned} |\sqrt{\sigma}\cdot\text{RHS of (3.3)}|_4^2 &\le \|\sqrt{\sigma}\cdot\text{RHS of (3.3)}\|_{4.5}^2 \\ &\lesssim \varepsilon\|\sqrt{\sigma}q\|_{5.5}^2 + \varepsilon\|q\|_5^2 + P(\|b_0\|_{4.5}, \|\eta\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\partial_t v\|_{3.5}, \|\sqrt{\sigma}\eta\|_{5.5}, \|\sqrt{\sigma}(b_0\cdot\partial)\eta\|_{5.5}, \|\sqrt{\sigma}\partial_t v\|_{4.5}). \end{aligned} \tag{3.11}$$

Summing these up, we have

$$\|q\|_5^2 + \|\sqrt{\sigma}q\|_{5.5}^2 \le P(E(t)). \tag{3.12}$$

3.2 Control of $\|\partial_t^k q\|_{5-k}^2$ and $\sigma\|\partial_t^k q\|_{5.5-k}^2$ for $k = 1, 2, 3$

In this subsection we derive the estimates for the time derivatives of q. To control $\|\partial_t q\|_4^2$ and $\sigma\|\partial_t q\|_{4.5}^2$, we consider the time differentiated (3.2) and (3.3), i.e.,

$$\begin{aligned}-\Delta q_t = & -\partial_\mu\left(\partial_t(\delta^{\mu\nu} - A^{\mu\alpha}A^\nu_\alpha)\partial_\nu q\right) - \partial_\mu\left((\delta^{\mu\nu} - A^{\mu\alpha}A^\nu_\alpha)\partial_\nu q_t\right) - (\partial_t^2 A^{\mu\alpha})\partial_\mu v_\alpha - (\partial_t A^{\mu\alpha})\partial_\mu\partial_t v_\alpha \\ & - ((b_0\cdot\partial)\partial_t A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha - ((b_0\cdot\partial)A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)v_\alpha \\ & + (\partial_t A^{\mu\alpha})(\partial_\mu b_0\cdot\partial)(b_0\cdot\partial)\eta_\alpha + A^{\mu\alpha}(\partial_\mu b_0\cdot\partial)(b_0\cdot\partial)v_\alpha,\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\frac{\partial q_t}{\partial N} = & (\delta^{3\mu} - A^{3\alpha}A^\mu_\alpha)\partial_\mu q_t + \partial_t(\delta^{3\mu} - A^{3\alpha}A^\mu_\alpha)\partial_\mu q - (\partial_t A^{3\alpha})\partial_t v_\alpha - A^{3\alpha}\partial_t^2 v_\alpha \\ & + (\partial_t A^{3\alpha})(b_0\cdot\partial)^2\eta_\alpha + A^{3\alpha}(b_0\cdot\partial)^2 v_\alpha, \quad \text{on } \Gamma.\end{aligned} \tag{3.14}$$

By the standard elliptic estimate, we have

$$\|q_t\|_4^2 \lesssim \|\text{RHS of (3.13)}\|_2^2 + |\text{RHS of (3.14)}|_{2.5}^2 + |q_t|_0^2, \tag{3.15}$$

$$\sigma\|q_t\|_{4.5}^2 \lesssim \left\|\sqrt{\sigma}\cdot\text{RHS of (3.13)}\right\|_{2.5}^2 + \left|\sqrt{\sigma}\cdot\text{RHS of (3.14)}\right|_3^2 + \left|\sqrt{\sigma}q_t\right|_0^2. \tag{3.16}$$

Here, the standard product estimates together with (2.13) imply

$$\|\text{RHS of (3.13)}\|_2^2 \lesssim \varepsilon\|q_t\|_4^2 + P(\|q\|_4, \|b_0\|_3, \|\eta\|_4, \|v\|_4, \|(b_0\cdot\partial)\eta\|_4, \|\partial_t v\|_3, \|\partial_t(b_0\cdot\partial)\eta\|_3), \tag{3.17}$$

$$\begin{aligned}|\text{RHS of (3.14)}|_{2.5}^2 &\lesssim \|\text{RHS of (3.14)}\|_3^2 \\ &\lesssim \varepsilon\|q_t\|_4^2 + P(\|q\|_4, \|b_0\|_3, \|\eta\|_4, \|v\|_4, \|(b_0\cdot\partial)\eta\|_4, \|\partial_t v\|_3, \|\partial_t(b_0\cdot\partial)\eta\|_4, \|\partial_t^2 v\|_3).\end{aligned} \tag{3.18}$$

Also, for the σ-weighted quantities, by invoking (2.13)-(2.14) as well as (2.19) with $s = 2.5$ and 3.5, respectively, we get

$$\begin{aligned}\|\sqrt{\sigma}\cdot\text{RHS of (3.13)}\|_{2.5}^2 \lesssim & \varepsilon\|\sqrt{\sigma}q_t\|_{4.5}^2 + \varepsilon\|q_t\|_4^2 \\ & + P(\|q\|_{4.5}, \|b_0\|_{3.5}, \|\eta\|_{4.5}, \|v\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\partial_t v\|_{3.5}, \|\partial_t(b_0\cdot\partial)\eta\|_{3.5}),\end{aligned} \tag{3.19}$$

$$\begin{aligned}&|\sqrt{\sigma}\cdot\text{RHS of (3.14)}|_3^2 \lesssim \|\sqrt{\sigma}\cdot\text{RHS of (3.14)}\|_{3.5}^2 \\ &\lesssim \varepsilon\|\sqrt{\sigma}q_t\|_{4.5}^2 + \varepsilon\|q_t\|_4^2 + P(\|q\|_{4.5}, \|b_0\|_{3.5}, \|\eta\|_{4.5}, \|v\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\partial_t v\|_{3.5}, \|\partial_t(b_0\cdot\partial)\eta\|_{4.5}, \|\sqrt{\sigma}\partial_t^2 v\|_{3.5}).\end{aligned} \tag{3.20}$$

Finally, to control $|q_t|_0$, we invoke

$$q_t = -\sigma\partial_t(\Delta_g\eta\cdot\hat{n}), \quad \text{on } \Gamma, \tag{3.21}$$

and (2.10) with $\overline{\partial}_A = \partial_t$ to obtain

$$\|q_t\|_0^2 \le P(\|\sigma\eta\|_{5.5}, \|\sigma v\|_{5.5}, \|\eta\|_5, \|v\|_5). \tag{3.22}$$

Thus,

$$\|q_t\|_4^2 + \sigma\|q_t\|_{4.5}^2 \lesssim P(E(t)). \tag{3.23}$$

Moreover, $\|\partial_t^k q\|_{5-k}^2$ and $\sigma\|\partial_t^k q\|_{5.5-k}^2$ for $k = 2, 3$ are estimated by adapting parallel arguments and so we omit the details.

3.3 Control of $\|\partial_t^4 q\|_1$: Low regularity elliptic estimate

We will also need the estimate of $\|\partial_t^4 q\|_1$ to control the full time derivatives in section 6.1. Due to the low regularity, we have to use Lemma 2.8. The proof is similar to our previous work [15, (3.47)-(3.53)]. First we should write the ∂_t^4-differentiated elliptic equation into the divergence form

$$\partial_\nu(A^{\nu\alpha}A^{\mu\alpha}\partial_t^4\partial_\mu q) = \partial_\nu\left(\left[A^{\nu\alpha}A^{\mu\alpha}, \partial_t^4\right]\partial_\mu q\right) + \partial_\nu\partial_t^4\left(A^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right), \tag{3.24}$$

with the boundary condition

$$A^{3\alpha}A^{\mu\alpha}\partial_\mu\partial_t^4 q = \left[A^{3\alpha}A^{\mu\alpha}, \partial_t^4\right]\partial_\mu q + \partial_t^4\left(A^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right), \text{ on } \Gamma. \tag{3.25}$$

Now if we set

$$\mathfrak{B}^{\nu\mu} := A^{\nu\alpha}A^{\mu\alpha}, \quad h := \text{RHS of (3.25)}$$

and

$$\pi^\nu := \left[A^{\nu\alpha}A^{\mu\alpha}, \partial_t^4\right]\partial_\mu q + \partial_t^4\left(A^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right)$$

then the elliptic system (3.24)-(3.25) is exactly of the form (2.22). The smallness of $\|\mathfrak{B} - \mathbf{I}_3\|_{L^\infty}$ follows from the a priori assumption (cf. Lemma 2.3). It is also straightforward to see that $\pi, \operatorname{div}\pi \in L^2$, i.e.,

$$\|\pi\|_0 + \|\operatorname{div}\pi\|_0 \lesssim P(E_1(t)). \tag{3.26}$$

and

$$h - \pi\cdot N = 0. \tag{3.27}$$

Then by Lemma 2.8 and using the estimates of $\partial_t^k q$ ($k \le 3$) we have

$$\left\|\partial_t^4 q - \overline{\partial_t^4 q}\right\|_1 \lesssim \|\pi\|_0 \lesssim P(E_1(t)). \tag{3.28}$$

Lastly, we need to control the H^1-norm of $\overline{\partial_t^4 q}$ by $\mathcal{P}$.

$$\begin{aligned}
\overline{\partial_t^4 q} &= \frac{1}{\operatorname{vol}(\Omega)}\int_\Omega \partial_t^4 q \,\mathrm{d}y = \frac{1}{\operatorname{vol}(\Omega)}\int_\Omega \partial_t^4 q\overline{\partial}_1 y_1 \,\mathrm{d}y = -\frac{1}{\operatorname{vol}(\Omega)}\int_\Omega y_1\overline{\partial}_1\partial_t^4 q \\
&\le C(\operatorname{vol}(\Omega))\|\overline{\partial}\partial_t^4 q\|_0\|y_1\|_0 = C(\operatorname{vol}(\Omega))\left\|\overline{\partial}(\partial_t^4 q - \overline{\partial_t^4 q})\right\|_0\|y_1\|_0 \le C(\operatorname{vol}(\Omega))\left\|\partial_t^4 q - \overline{\partial_t^4 q}\right\|_1.
\end{aligned} \tag{3.29}$$

This concludes the control of $\|\partial_t^4 q\|_1$, and we have

$$\|\partial_t^4 q\|_1 \lesssim P(E_1(t)). \tag{3.30}$$

4 Estimates for the non-weighted full Sobolev norms

We study the estimates for v, $(b_0\cdot\partial)\eta$, and their time derivatives in full Sobolev spaces. More precisely, we need to estimate

$$\|\partial_t^k v\|_{5-k}^2, \quad \|\partial_t^k (b_0\cdot\partial)\eta\|_{5-k}^2, \quad \text{for } k = 0,1,2,3,4.$$

We will adapt the Hodge-type elliptic div-curl estimate (2.21) to the quantities above. Specifically, we will replace X by $\partial_t^k v$ and $\partial_t^k(b_0\cdot\partial)\eta$, as well as their σ-weighted versions, in application.

4.1 Divergence estimates

First, we treat

$$\|\operatorname{div}\partial_t^k v\|_{4-k}^2, \quad \|\operatorname{div}\partial_t^k(b_0\cdot\partial)\eta\|_{4-k}^2, \ k = 0,\cdots,4.$$

We recall that $\operatorname{div}_A X := A^{\mu\alpha}\partial_\mu X_\alpha$. When $k = 0$, since $\operatorname{div}_A v = 0$ and $\operatorname{div}_A(b_0\cdot\partial)\eta = 0$, we have

$$\begin{aligned}
&\operatorname{div} v = \operatorname{div}_A v + (\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu v_\alpha = (\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu v_\alpha, \\
&\operatorname{div}(b_0\cdot\partial)\eta = (\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha.
\end{aligned} \tag{4.1}$$

So, by invoking (2.13) and (2.14) in Lemma 2.3, we have

$$\|\operatorname{div} v\|_4^2 + \|\operatorname{div}(b_0\cdot\partial)\eta\|_4^2 \lesssim \varepsilon(\|v\|_5^2 + \|(b_0\cdot\partial)\eta\|_5^2). \tag{4.2}$$

When $k=1$, we differentiate (4.1) with respect to time and get

$$\begin{aligned}
&\operatorname{div} v_t = \partial_t(\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu v_\alpha + (\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu\partial_t v_\alpha,\\
&\operatorname{div}(b_0\cdot\partial)\eta_t = \partial_t(\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha + (\delta^{\mu\alpha} - A^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\partial_t\eta_\alpha.
\end{aligned}\tag{4.3}$$

Thus, in light of the Sobolev product estimate (2.19), we have

$$\begin{aligned}
&\|\operatorname{div} v_t\|_3^2 + \|\operatorname{div}(b_0\cdot\partial)\eta_t\|_3^2 \lesssim \varepsilon(\|v_t\|_4^2 + \|(b_0\cdot\partial)\eta_t\|_4^2) + P(\|\eta\|_4, \|v\|_4)\\
\lesssim&\varepsilon(\|v_t\|_4^2 + \|(b_0\cdot\partial)\eta_t\|_4^2) + P(E_1(0)) + \int_0^T P(E_1(t))\,\mathrm{d}t.
\end{aligned}\tag{4.4}$$

The estimates for $k=2,3,4$ are similar and so we omit the details. In the end, we obtain

$$\sum_{k=0}^{4}\left(\|\operatorname{div}\partial_t^k v\|_{4-k}^2 + \|\operatorname{div}(b_0\cdot\partial)\partial_t^k\eta\|_{4-k}^2\right) \lesssim \varepsilon E_1(t) + P(E_1(0)) + \int_0^T P(E_1(t)).\tag{4.5}$$

4.2 Curl estimates

In this part we aim to control $\|\operatorname{curl}\partial_t^k v\|_{4-k}$ and $\|\operatorname{curl}\partial_t^k(b_0\cdot\partial)\eta\|_{4-k}$ for $0\le k\le 4$. Define $(\operatorname{curl}_A X)_\lambda := \epsilon_{\lambda\tau\alpha}A^{\mu\tau}\partial_\mu X^\alpha$. Similar to the estimates for the divergence, it suffices to study the estimates for $\operatorname{curl}_A\partial_t^k v$ and $\operatorname{curl}_A\partial_t^k(b_0\cdot\partial)\eta$ instead. Indeed, thank to (2.13)-(2.14), we have

$$\begin{aligned}
&\sum_{k=0}^{4}\left(\|\operatorname{curl}\partial_t^k v\|_{4-k}^2 + \|\operatorname{curl}(b_0\cdot\partial)\partial_t^k\eta\|_{4-k}^2\right)\\
\lesssim&\sum_{k=0}^{4}\left(\|\operatorname{curl}_A\partial_t^k v\|_{4-k}^2 + \|\operatorname{curl}_A(b_0\cdot\partial)\partial_t^k\eta\|_{4-k}^2\right) + \varepsilon E_1(t) + P(E_1(0)) + \int_0^T P(E_1(t)).
\end{aligned}\tag{4.6}$$

Applying curl_A to the second equation of (1.12), we get

$$\partial_t(\operatorname{curl}_A v) - (b_0\cdot\partial)(\operatorname{curl}_A((b_0\cdot\partial)\eta)) = \operatorname{curl}_{\partial_t A} v + [\operatorname{curl}_A, (b_0\cdot\partial)](b_0\cdot\partial)\eta := \mathcal{F},\tag{4.7}$$

where $(\operatorname{curl}_{\partial_t A} v)_\lambda := \epsilon_{\lambda\tau\alpha}\partial_t A^{\mu\tau}\partial_\mu X^\alpha$. Now, when $k=0$, by taking ∂^4 to (4.7), testing it with $\partial^4\operatorname{curl}_A v$, we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega|\partial^4\operatorname{curl}_A v|^2 - \int_\Omega(\partial^4\operatorname{curl}_A v)\left((b_0\cdot\partial)\partial^4(\operatorname{curl}_A((b_0\cdot\partial)\eta))\right)\\
=&\int_\Omega\left([\partial^4, (b_0\cdot\partial)]\operatorname{curl}_A(b_0\cdot\partial)\eta + \partial^4\mathcal{F}\right)(\partial^4\operatorname{curl}_A v).
\end{aligned}\tag{4.8}$$

Since $b_0^3=0$, we integrate $(b_0\cdot\partial)$ by parts in the second term on the LHS:

$$\begin{aligned}
&-\int_\Omega(\partial^4\operatorname{curl}_A v)\left((b_0\cdot\partial)\partial^4(\operatorname{curl}_A(b_0\cdot\partial)\eta)\right) = \int_\Omega(b_0\cdot\partial)(\partial^4\operatorname{curl}_A v)\partial^4(\operatorname{curl}_A((b_0\cdot\partial)\eta))\\
=&\int_\Omega\partial^4\operatorname{curl}_A((b_0\cdot\partial)v)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta) - \int_\Omega([\partial^4\operatorname{curl}_A, (b_0\cdot\partial)]v)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta).
\end{aligned}\tag{4.9}$$

Here, $\int_\Omega\partial^4\operatorname{curl}_A((b_0\cdot\partial)v)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta)$ contributes to

$$\frac{d}{dt}\frac{1}{2}\int_\Omega|\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta)|^2 - \int_\Omega\partial^4\operatorname{curl}_{\partial_t A}((b_0\cdot\partial)\eta)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta).\tag{4.10}$$

Therefore, (4.8) becomes

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left(\int_\Omega|\partial^4\operatorname{curl}_A v|^2 + \int_\Omega|\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta)|^2\right)\\
=&\int_\Omega\left([\partial^4, (b_0\cdot\partial)]\operatorname{curl}_A(b_0\cdot\partial)\eta + \partial^4\mathcal{F}\right)(\partial^4\operatorname{curl}_A v)\\
&+\int_\Omega([\partial^4\operatorname{curl}_A, (b_0\cdot\partial)]v)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta) + \int_\Omega\partial^4\operatorname{curl}_{\partial_t A}((b_0\cdot\partial)\eta)\partial^4\operatorname{curl}_A((b_0\cdot\partial)\eta).
\end{aligned}\tag{4.11}$$

Now, because

$$\|\mathcal{F}\|_4^2 \le \|\text{curl}_{\partial_t A} v\|_4^2 + \|[\text{curl}_A, (b_0\cdot\partial)](b_0\cdot\partial)\eta\|_4^2 \le P(\|b_0\|_5, \|\eta\|_5, \|v\|_5, \|(b_0\cdot\partial)\eta\|_5),$$

it is not hard to see that the terms on the RHS of (4.11) can be controlled by $P(E(t))$. Therefore,

$$\|\text{curl}_A v\|_4^2 + \|\text{curl}_A (b_0\cdot\partial)\eta\|_4^2 \lesssim \int_0^T P(E(t)). \tag{4.12}$$

The estimates for the cases when $k=1,2,3,4$ follow from a parallel argument: By taking $\partial^{4-k}\partial_t^k$ (or $\partial^{4.5-k}\partial_t^k$) to (4.7), testing it with $\partial^{4-k}\partial_t^k \text{curl}_A v$, the energy estimate then yields

$$\sum_{k=1}^{4}\left(\|\text{curl}_A \partial_t^k v\|_{4-k}^2 + \|\text{curl}_A (b_0\cdot\partial)\partial_t^k\eta\|_{4-k}^2\right) \lesssim \int_0^T P(E(t)). \tag{4.13}$$

Finally, in view of (4.6), we obtain

$$\sum_{k=0}^{4}\left(\|\text{curl}\, \partial_t^k v\|_{4-k}^2 + \|\text{curl}\, (b_0\cdot\partial)\partial_t^k\eta\|_{4-k}^2\right) \lesssim \varepsilon E_1(t) + P(E_1(0)) + \int_0^T P(E_1(t)). \tag{4.14}$$

4.3 Control of the boundary term

Finally, we need to treat

$$|\overline{\partial}\partial_t^k v^3|_{3.5-k}^2, \quad |\overline{\partial}\partial_t^k (b_0\cdot\partial)\eta^3|_{3.5-k}^2$$

which correspond to the last term in (2.21). For these non-weighted terms, it suffices to control

$$|\overline{\partial}^{5-k}\partial_t^k v^3|_{-0.5}^2, \quad |\overline{\partial}^{5-k}\partial_t^k (b_0\cdot\partial)\eta^3|_{-0.5}^2.$$

We invoke Lemma (2.6) to obtain

$$|\overline{\partial}^{5-k}\partial_t^k v^3|_{-0.5}^2 \lesssim \|\overline{\partial}^{5-k}\partial_t^k v\|_0^2 + \|\text{div}\, \partial_t^k v\|_{4-k}^2, \tag{4.15}$$

$$|\overline{\partial}^{5-k}\partial_t^k (b_0\cdot\partial)\eta^3|_{-0.5}^2 \lesssim \|\overline{\partial}^{5-k}\partial_t^k (b_0\cdot\partial)\eta\|_0^2 + \|\text{div}\, \partial_t^k (b_0\cdot\partial)\eta\|_{4-k}^2. \tag{4.16}$$

Here, the second term on the RHS of (4.15) and (4.16) has been treated in Section 4.1, while the first term corresponds to the tangential energy that will be studied in the upcoming sections.

5 Tangential estimate of spatial derivatives

By Lemma 2.6, we already reduce the control of boundary norms to the interior tangential estimate. In this section, we would like to do the $\overline{\partial}^5$-estimates for v and $b=(b_0\cdot\partial)\eta$, i.e., the control of tangential spatial derivatives.

5.1 Interior estimates: Alinhac good unknown

However, we cannot directly commute $\overline{\partial}^5$ with the covariant derivative ∇_A because the commutator contains $\overline{\partial}^5 A = \overline{\partial}^5\partial\eta\times\partial\eta$ whose L^2 norm cannot be controlled. The reason is that the essential highest order term in $\overline{\partial}^5(\nabla_A f)$, i.e., the standard derivatives of a covariant derivative, is actually the covariant derivative of Alinhac good unknown $\mathbf{f} := \overline{\partial}^5 f - \overline{\partial}^5\eta\cdot\nabla_A f$ instead of the term produced by simply commuting $\overline{\partial}^5$ with ∇_A. Specifically,

$$\begin{aligned}
\overline{\partial}^5(\nabla_A^\alpha f) &= \nabla_A^\alpha(\overline{\partial}^5 f) + (\overline{\partial}^5 A^{\mu\alpha})\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_A^\alpha(\overline{\partial}^5 f) - \overline{\partial}^4(A^{\mu\gamma}\overline{\partial}\partial_\beta\eta_\gamma A^{\beta\alpha})\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_A^\alpha(\overline{\partial}^5 f) - A^{\beta\alpha}\partial_\beta\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu f - ([\overline{\partial}^4, A^{\mu\gamma}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\gamma)\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \underbrace{\nabla_A^\alpha(\overline{\partial}^5 f - \overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu f)}_{=\nabla_A^\alpha \mathbf{f}} + \underbrace{\overline{\partial}^5\eta_\gamma\nabla_A^\alpha(\nabla_A^\gamma g) - ([\overline{\partial}^4, A^{\mu\gamma}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\gamma)\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f]}_{=:C^\alpha(f)},
\end{aligned}$$

We introduce the Alinhac good unknowns of v and q with respect to $\overline{\partial}^5$ by

$$\mathbf{V} := \overline{\partial}^5 v - \overline{\partial}^5 \eta \cdot \nabla_A v, \quad \mathbf{Q} := \overline{\partial}^5 q - \overline{\partial}^5 \eta \cdot \nabla_A q. \tag{5.1}$$

Then direct computation (e.g., see [16, Section 4.2.4]) shows that the good unknowns enjoy the following properties

$$\overline{\partial}^5 \underbrace{(\nabla_A \cdot v)}_{=0} = \nabla_A \cdot \mathbf{V} + C(v), \quad \overline{\partial}^5 (\nabla_A q) = \nabla_A \mathbf{Q} + C(q) \tag{5.2}$$

and

$$\|C(f)\|_0 \lesssim P(\|\eta\|_5)\|f\|_5. \tag{5.3}$$

Under this setting, we take $\overline{\partial}^5$ in the second equation of (1.12) and invoke (5.1) to get the evolution equation of the Alinhac good unknowns

$$\partial_t \mathbf{V} = -\nabla_A \mathbf{Q} + (b_0 \cdot \partial)(\overline{\partial}^5 (b_0 \cdot \partial)\eta) + \underbrace{\partial_t(\overline{\partial}^5 \eta \cdot \nabla_A v) - C(q) + [\overline{\partial}^5, (b_0 \cdot \partial)]((b_0 \cdot \partial)\eta)}_{=:\mathfrak{f}_0}. \tag{5.4}$$

Taking $L^2(\Omega)$ inner product with $\mathbf{V}$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2 \,\mathrm{d}y = -\int_\Omega \nabla_A \mathbf{Q} \cdot \mathbf{V} \,\mathrm{d}y + \int_\Omega \Big((b_0 \cdot \partial)(\overline{\partial}^5 (b_0 \cdot \partial)\eta)\Big) \cdot \mathbf{V} \,\mathrm{d}y + \int_\Omega \mathfrak{f}_0 \cdot \mathbf{V} \,\mathrm{d}y, \tag{5.5}$$

where the last term can be directly controlled

$$\int_\Omega \mathfrak{f}_0 \cdot \mathbf{V} \,\mathrm{d}y \lesssim P(\|\eta\|_5, \|v\|_5, \|\partial_t v\|_4, \|q\|_5, \|b_0\|_5, \|(b_0 \cdot \partial)\eta\|_5) \lesssim P(E_1(t)). \tag{5.6}$$

Then we integrate $(b_0 \cdot \partial)$ by parts in the second integral of (5.5) to produce the tangential energy of the magnetic field $(b_0 \cdot \partial)\eta$. Note that $b_0 \cdot N = 0$ on $\partial\Omega$ and $\operatorname{div} b_0 = 0$, no boundary term appears in this step.

$$\begin{aligned}
&\int_\Omega \Big((b_0 \cdot \partial)(\overline{\partial}^5 (b_0 \cdot \partial)\eta)\Big) \cdot \mathbf{V} \,\mathrm{d}y = -\int_\Omega (\overline{\partial}^5 (b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)\mathbf{V} \,\mathrm{d}y \\
=& -\int_\Omega (\overline{\partial}^5 (b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)(\overline{\partial}^5 \partial_t \eta) \,\mathrm{d}y + \int_\Omega (\overline{\partial}^5 (b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)(\overline{\partial}^5 \eta \cdot \nabla_A v) \,\mathrm{d}y \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\overline{\partial}^5 ((b_0 \cdot \partial)\eta)\right|^2 + \int_\Omega (\overline{\partial}^5 (b_0 \cdot \partial)\eta^\alpha)([\overline{\partial}^5, (b_0 \cdot \partial)]v_\alpha + (b_0 \cdot \partial)(\overline{\partial}^5 \eta \cdot \nabla_A v_\alpha)) \,\mathrm{d}y \\
\lesssim& -\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\overline{\partial}^5 ((b_0 \cdot \partial)\eta)\right|^2 + P(E_1(t)).
\end{aligned} \tag{5.7}$$

Next we analyze the first integral of (5.5). Integrate by parts, using Piola's identity $\partial_\mu A^{\mu\alpha} = 0$ and invoking (5.2), we get

$$\begin{aligned}
-\int_\Omega \nabla_A \mathbf{Q} \cdot \mathbf{V} \,\mathrm{d}y =& \int_\Omega \mathbf{Q}(\nabla_A \cdot \mathbf{V}) \,\mathrm{d}y \underbrace{- \int_\Gamma A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha \,\mathrm{d}S}_{=:J} - \int_{\Gamma_0} \underbrace{A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha}_{=0} \,\mathrm{d}S \\
=& -\int_\Omega \mathbf{Q}\, C(v) \,\mathrm{d}y + J \lesssim \|\mathbf{Q}\|_0 \|C(v)\|_0 + J \\
\lesssim& P(\|\eta\|_5, \|q\|_5, |v\|_5) + J,
\end{aligned} \tag{5.8}$$

where the boundary integral on Γ_0 vanishes due to $\eta|_{\Gamma_0} =$ Id and thus $A^{3\alpha}\mathbf{V}_\alpha = \overline{\partial}^5 v_3 = 0$. Therefore, it remains to analyze the boundary integral J.

5.2 Boundary estimates and cancellation structure

The boundary integral now reads

$$
\begin{aligned}
J &= -\int_\Gamma A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha \,\mathrm{d}S \\
&= -\int_\Gamma A^{3\alpha}\overline{\partial}^5 q\mathbf{V}_\alpha \,\mathrm{d}S \underbrace{+\int_\Gamma A^{3\alpha}(\overline{\partial}^5\eta\cdot\nabla_A q)\mathbf{V}_\alpha \,\mathrm{d}S}_{=:\,\mathrm{RT}} \\
&= \underbrace{-\int_\Gamma \overline{\partial}^5(A^{3\alpha}q)\mathbf{V}_\alpha \,\mathrm{d}S}_{=:\,\mathrm{ST}} + \int_\Gamma q(\overline{\partial}^5 A^{3\alpha})\mathbf{V}_\alpha \,\mathrm{d}S + \int_\Gamma \sum_{k=1}^{4}\binom{5}{k}\overline{\partial}^{5-k}A^{3\alpha}\overline{\partial}^k q\mathbf{V}_\alpha \,\mathrm{d}S + \mathrm{RT} \\
&=: \mathrm{ST} + J_1 + J_2 + \mathrm{RT}.
\end{aligned}
\tag{5.9}
$$

5.2.1 Non-weighted boundary energy: Rayleigh-Taylor sign condition

The term RT together with the Rayleigh-Taylor sign condition yields the non-weighted boundary energy. Recall that $\mathbf{V} = \overline{\partial}^5 v - \overline{\partial}^5\eta\cdot\nabla_A v$, then we have

$$
\begin{aligned}
\mathrm{RT} = &\int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\overline{\partial}^5 v_\alpha \,\mathrm{d}S \\
&-\int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha \,\mathrm{d}S \\
&+\sum_{i=1}^{2}\int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{i\beta}\overline{\partial}_i q(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
=:&\ \mathrm{RT}_1 + \mathrm{RT}_2 + \mathrm{RT}_3.
\end{aligned}
\tag{5.10}
$$

The term RT_1 gives the boundary energy term by writting $v_\alpha = \partial_t\eta_\alpha$.

$$
\begin{aligned}
\mathrm{RT}_1 =& \int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\partial_t\overline{\partial}^5\eta_\alpha \,\mathrm{d}S \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Gamma(-\partial_3 q)\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \,\mathrm{d}S + \int_\Gamma(\partial_t A^{3\alpha})A^{3\beta}\overline{\partial}^5\eta_\beta\partial_3 q\overline{\partial}^5\eta_\alpha \,\mathrm{d}S + \frac{1}{2}\int_\Gamma \partial_t\partial_3 q\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \,\mathrm{d}S \\
=:& -\frac{1}{2}\frac{d}{dt}\int_\Gamma(-\partial_3 q)\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \,\mathrm{d}S + \mathrm{RT}_{11} + \mathrm{RT}_{12}.
\end{aligned}
\tag{5.11}
$$

The term RT_{12} can be directly controlled by

$$
\mathrm{RT}_{12} \lesssim |\partial_t\partial_3 q|_{L^\infty}\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|_0^2 \lesssim P(E_1(t)).
\tag{5.12}
$$

The term RT_{11} is exactly canlled by RT_2 after plugging $\partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$

$$
\mathrm{RT}_{11} = -\int_\Gamma A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}A^{3\beta}\overline{\partial}^5\eta_\beta\partial_3 q\overline{\partial}^5\eta_\alpha \,\mathrm{d}S = -\,\mathrm{RT}_2.
\tag{5.13}
$$

Finally, invoking $q = -\sigma\sqrt{g}\Delta_g\eta\cdot\hat{n} = \sigma Q(\overline{\partial}\eta)\overline{\partial}^2\eta\cdot\hat{n}$, we can control RT_3 by the weighted energy and trace lemma

$$
\begin{aligned}
\mathrm{RT}_3 &= \sigma\int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{i\beta}\overline{\partial}_i(Q(\overline{\partial}\eta)\overline{\partial}^2\eta\cdot\hat{n})(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\lesssim P(|\partial\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty}|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \\
&\lesssim P(\|\eta\|_3)\|\eta\|_5\underbrace{\|\sqrt{\sigma}\eta\|_{5.5}(\|\sqrt{\sigma}v\|_{5.5} + \|\sqrt{\sigma}\eta\|_{5.5}\|v\|_3)}_{\le\sqrt{\sigma E_2}(\sqrt{\sigma E_2}+\sqrt{E_1}\sqrt{\sigma E_2})} \\
&\lesssim P(E_1(t))(\sigma E_2(t)).
\end{aligned}
\tag{5.14}
$$

Summarizing (5.10)-(5.14), we conclude the estimate of RT by

$$
\int_0^T \mathrm{RT}\,\mathrm{d}t \lesssim -\frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|_0^2 + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\tag{5.15}
$$

5.2.2 Control of the weighted boundary energy: surface tension

Now we analyze the term ST, where the surface tension gives the $\sqrt{\sigma}$-weighted top order boundary energy. Invoking $A^{3\alpha}q = -\sigma\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta^\beta$, we get

$$
\begin{aligned}
\mathrm{ST} = &\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+5\sigma\int_\Gamma \overline{\partial}(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}^4\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+\sigma\int_\Gamma \overline{\partial}^5(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+\sum_{k=2}^4\sigma\int_\Gamma\binom{5}{k}\overline{\partial}^k(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}^{5-k}\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
=:&\ \mathrm{ST}_1+\mathrm{ST}_2+\mathrm{ST}_3+\mathrm{ST}_4.
\end{aligned}
\tag{5.16}
$$

The term ST_4 can be directly controlled with the help of $\sqrt{\sigma}$-weighted energy

$$
\begin{aligned}
\mathrm{ST}_4 \lesssim\ &10|\overline{\partial}^2(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)|_{L^\infty}|\sqrt{\sigma}\overline{\partial}^3\overline{\partial}_{ij}^2\eta_\beta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
&+10|\overline{\partial}^3(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)|_{L^4}|\sqrt{\sigma}\overline{\partial}^2\overline{\partial}_{ij}^2\eta_\beta|_{L^4}(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
&+5|\sqrt{\sigma}\overline{\partial}^4(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)|_0|\overline{\partial}\overline{\partial}_{ij}^2\eta_\beta|_{L^\infty}(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
\lesssim\ &P(E_1(t))(\sigma E_2(t)).
\end{aligned}
\tag{5.17}
$$

In ST_1, we first integrate $\overline{\partial}_i$ by parts.

$$
\begin{aligned}
\mathrm{ST}_1 \overset{\overline{\partial}_i}{=} &-\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\overline{\partial}_j\eta_\beta\overline{\partial}_i(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&-\sigma\int_\Gamma \overline{\partial}_i(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}^5\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
=:&\ \mathrm{ST}_{11}+\mathrm{ST}_{12}.
\end{aligned}
\tag{5.18}
$$

In ST_{11}, we write $v_\alpha = \partial_t\eta_\alpha$ to produce the energy term

$$
\begin{aligned}
\mathrm{ST}_{11} =&-\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\overline{\partial}_j\eta_\beta\overline{\partial}_i(\overline{\partial}^5\partial_t\eta_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
=&-\frac{\sigma}{2}\frac{d}{dt}\int_\Gamma\left|\overline{\partial}^5\overline{\partial}\eta\cdot\hat{n}\right|^2\,\mathrm{d}S\\
&-\frac{\sigma}{2}\int_\Gamma(\sqrt{g}g^{ij}-\delta^{ij})\left(\overline{\partial}^5\overline{\partial}_i\eta\cdot\hat{n}\right)\left(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n}\right)\,\mathrm{d}S-\frac{\sigma}{2}\int_\Gamma\sqrt{g}\partial_t g^{ij}(\overline{\partial}^5\overline{\partial}_i\eta\cdot\hat{n})(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})\,\mathrm{d}S\\
&+\sigma\int_\Gamma g^{ij}\partial_t(\underbrace{\sqrt{g}\hat{n}^\alpha}_{=A^{3\alpha}})\overline{\partial}^5\overline{\partial}_i\eta_\alpha(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})\,\mathrm{d}S+\sigma\int_\Gamma\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\overline{\partial}_j\eta_\beta\overline{\partial}_i\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}S\\
&+\sigma\int_\Gamma\sqrt{g}g^{ij}\hat{n}^\alpha(\hat{n}^\beta\overline{\partial}^5\overline{\partial}_j\eta_\beta)(\overline{\partial}^5\eta\cdot\overline{\partial}_i(\nabla_A v_\alpha))\\
=:&-\frac{\sigma}{2}\frac{d}{dt}\int_\Gamma\left|\overline{\partial}^6\eta\cdot\hat{n}\right|^2\,\mathrm{d}S+\mathrm{ST}_{111}+\mathrm{ST}_{112}+\mathrm{ST}_{113}+\mathrm{ST}_{114}+\mathrm{ST}_{115}.
\end{aligned}
\tag{5.19}
$$

Invoking (2.15), the term ST_{111} can be absorbed by the weighted energy term

$$
\mathrm{ST}_{111}\leq\varepsilon\left|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2.
\tag{5.20}
$$

The terms ST_{112} and ST_{115} can be directly controlled

$$
\mathrm{ST}_{112}\lesssim|\sqrt{g}\partial_t g^{ij}|_{L^\infty}\left|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\lesssim P(E_1(t))(\sigma E_2(t)).
\tag{5.21}
$$

$$
\mathrm{ST}_{115}\lesssim|\sqrt{g}g^{ij}\overline{\partial}(\nabla_A v)|_{L^\infty}\left|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}\right|_0\|\sqrt{\sigma}\eta\|_{5.5}\lesssim P(E_1(t))(\sigma E_2(t)).
\tag{5.22}
$$

In ST_{113}, we use (2.3)-(2.4), i.e., $\sqrt{g}\hat{n}^\alpha = A^{3\alpha}$ and $\partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$ to produce cancellation with ST_{114}

$$\mathrm{ST}_{113} = -\sigma\int_\Gamma g^{ij}A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}\overline{\partial}^5\overline{\partial}_i\eta_\alpha(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})\,\mathrm{d}S = -\,\mathrm{ST}_{114}. \tag{5.23}$$

Summarizing (5.19)-(5.23), we conclude the estimate of ST_{11} by choosing $\varepsilon > 0$ sufficiently small

$$\int_0^T \mathrm{ST}_{11}\,\mathrm{d}t \lesssim -\frac{\sigma}{2}\left|\overline{\partial}^5\overline{\partial}\eta\cdot\hat{n}\right|_0^2\Big|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.24}$$

Next we control ST_{12}. First we have

$$\begin{aligned}\mathrm{ST}_{12} =& -\sigma\int_\Gamma \overline{\partial}_i(\sqrt{g}g^{ij}\hat{n}^\alpha)(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ &-\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ =:&\ \mathrm{ST}_{121} + \mathrm{ST}_{122}.\end{aligned} \tag{5.25}$$

The term ST_{121} can be directly controlled

$$\mathrm{ST}_{121} \lesssim |\overline{\partial}_i(\sqrt{g}g^{ij}\hat{n})|_{L^\infty}|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \lesssim P(E_1(t))(\sigma E_2(t)). \tag{5.26}$$

To control ST_{122}, we first integrate $\overline{\partial}_j$ by parts.

$$\begin{aligned}\mathrm{ST}_{122} =& -\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}_j\overline{\partial}^5 v_\alpha\,\mathrm{d}S\\ &+\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}_j\overline{\partial}^5\eta_\gamma(A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}S\\ &+\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}^5\eta_\gamma\overline{\partial}_j(A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}S\\ &+\sigma\int_\Gamma \overline{\partial}_j(\sqrt{g}g^{ij}\hat{n}^\alpha\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ =:&\ \mathrm{ST}_{1221} + \mathrm{ST}_{1222} + \mathrm{ST}_{1223} + \mathrm{ST}_{1224}.\end{aligned} \tag{5.27}$$

The term ST_{1223} can be directly controlled by the weighted energy

$$\mathrm{ST}_{1223} + \mathrm{ST}_{1224} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{5.28}$$

To control ST_{1221}, we write $v_\alpha = \partial_t\eta_\alpha$ and then integrate ∂_t by parts under the time integral. When ∂_t falls on $\sqrt{g}\hat{n}^\alpha = A^{3\alpha}$, the cancellation sturcture analogous to (5.23) is again produced.

$$\begin{aligned}\int_0^T \mathrm{ST}_{1221}\,\mathrm{d}t \overset{\partial_t}{=}& \sigma\int_0^T\int_\Gamma \partial_t(g^{ij}\overline{\partial}_i\hat{n}^\beta\overline{\partial}^5\eta_\beta)(\overline{\partial}_j\overline{\partial}^5\eta_\alpha\hat{n}^\alpha)\,\mathrm{d}S\\ &+\sigma\int_0^T\int_\Gamma \underbrace{(-A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha})}_{\partial_t(\sqrt{g}\hat{n}^\alpha)} g^{ij}\overline{\partial}_i\hat{n}^\beta\overline{\partial}^5\eta_\beta\overline{\partial}_j\overline{\partial}^5\eta_\alpha\,\mathrm{d}S\\ \lesssim& \int_0^T P(E_1(t))(\|\sqrt{\sigma}v\|_{5.5} + \|\sqrt{\sigma}\eta\|_{5.5})|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0\,\mathrm{d}t + (-\,\mathrm{ST}_{1222}).\end{aligned} \tag{5.29}$$

Therefore, ST_{122} is controlled by

$$\int_0^T \mathrm{ST}_{122}\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{5.30}$$

which together with (5.24) and (5.26) gives the control of ST_1

$$\int_0^T \mathrm{ST}_1\,\mathrm{d}t \lesssim -\frac{\sigma}{2}\int_\Gamma\left|\overline{\partial}^6\eta\cdot\hat{n}\right|^2\,\mathrm{d}S\Big|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.31}$$

It remains to control ST_2 and ST_3 in (5.16). From (5.18), we find that ST_2 has the same form as ST_{12}, so we omit the analysis of ST_2 and only list the result

$$\int_0^T \mathrm{ST}_2\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.32}$$

As for ST_3, we have

$$\begin{aligned}
\mathrm{ST}_3 =& \sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}^5\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+ \sigma\int_\Gamma \sqrt{g}g^{ij}(\overline{\partial}^5\hat{n}^\alpha)\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+ \sigma\int_\Gamma \overline{\partial}^5(\sqrt{g}g^{ij})\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+ \sum_{k=1}^4\sigma\int_\Gamma \overline{\partial}^k(\sqrt{g}g^{ij})\overline{\partial}^{5-k}(\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
=:& \mathrm{ST}_{31} + \mathrm{ST}_{32} + \mathrm{ST}_{33} + \mathrm{ST}_{34},
\end{aligned} \tag{5.33}$$

where ST_{34} can be directly controlled

$$\mathrm{ST}_{34} \lesssim P(|\eta|_{W^{3,\infty}})|\sqrt{\sigma}\eta|_5(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \lesssim P(E_1(t))(\sigma E_2(t)). \tag{5.34}$$

To control ST_{31} and ST_{32}, we need to invoke (2.10) to get

$$\overline{\partial}^5\hat{n}^\alpha = -\overline{\partial}^4\left(g^{kl}(\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\alpha\right) = -g^{kl}(\overline{\partial}^5\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\alpha - [\overline{\partial}^4, g^{kl}\overline{\partial}_l\eta^\alpha](\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n}),$$

and thus plug it into ST_{31} and ST_{32}:

$$\begin{aligned}
\mathrm{ST}_{31} =& -\sigma\int_\Gamma \sqrt{g}g^{ij}g^{kl}(\overline{\partial}^5\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\hat{n}_\alpha(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&-\sigma\int_\Gamma \sqrt{g}g^{ij}\left([\overline{\partial}^4, g^{kl}\overline{\partial}_l\eta^\alpha](\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n})\right)\overline{\partial}_i\overline{\partial}_j\eta_\beta\hat{n}_\alpha(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
\lesssim& P(E_1(t))|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
&+ P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
\lesssim& P(E_1(t))(\sigma E_2(t)),
\end{aligned} \tag{5.35}$$

and similarly

$$\begin{aligned}
\mathrm{ST}_{32} \lesssim& P(E_1(t))|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
&+ P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\
\lesssim& P(E_1(t))(\sigma E_2(t)),
\end{aligned} \tag{5.36}$$

For ST_{33}, we use the identity (2.12) to get

$$\begin{aligned}
\overline{\partial}^5(\sqrt{g}g^{ij}) =& \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)(\overline{\partial}^5\overline{\partial}_k\eta^\mu\overline{\partial}_l\eta_\mu + \overline{\partial}_k\eta^\mu\overline{\partial}^5\overline{\partial}_l\eta_\mu)\\
&+ \underbrace{\left[\overline{\partial}^4, \overline{\partial}_l\eta^\mu\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)\right]\overline{\partial}\overline{\partial}_k\eta_\mu + \left[\overline{\partial}^4, \overline{\partial}_k\eta^\mu\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)\right]\overline{\partial}\overline{\partial}_l\eta_\mu}_{R_{33}^{ij}},
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{ST}_{33} =& \sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)(\overline{\partial}^5\overline{\partial}_k\eta^\mu\overline{\partial}_l\eta_\mu + \overline{\partial}_k\eta^\mu\overline{\partial}^5\overline{\partial}_l\eta_\mu)\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&+ \sigma\int_\Gamma R_{33}^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
=:& \mathrm{ST}_{331} + \mathrm{ST}_{332}.
\end{aligned} \tag{5.37}$$

The term ST_{332} can be directly controlled

$$\mathrm{ST}_{332} \lesssim P(E_1(t))|\sqrt{\sigma}\eta|_5(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}). \tag{5.38}$$

In ST_{331}, we should first integrate the derivative $\overline{\partial}_k$ in $\overline{\partial}^5\overline{\partial}_k\eta^\mu$ (resp. $\overline{\partial}_l$ in $\overline{\partial}^5\overline{\partial}_l\eta_\mu$) by parts

$$\begin{aligned}
\mathrm{ST}_{331} =& -\sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&-\sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_k\eta^\mu\overline{\partial}^5\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_l(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&-\sigma\int_\Gamma \overline{\partial}_k\left(\sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&-\sigma\int_\Gamma \overline{\partial}_l\left(\sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_k\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
=:& \ \mathrm{ST}_{3311} + \mathrm{ST}_{3312} + \mathrm{ST}_{3313} + \mathrm{ST}_{3314},
\end{aligned} \tag{5.39}$$

where ST_{3313} and ST_{3314} can be directly controlled

$$\mathrm{ST}_{3313} + \mathrm{ST}_{3314} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{5.40}$$

For ST_{3311} and ST_{3312}, we need to write $v_\alpha = \partial_t\eta_\alpha$ and then integrate ∂_t by parts. For simplicity we only show the control of ST_{3311}.

$$\int_0^T \mathrm{ST}_{3311}\,\mathrm{d}t = -\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\partial_t\eta_\alpha\,\mathrm{d}S \tag{5.41}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}S \tag{5.42}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}^5\eta\cdot\overline{\partial}_k(\nabla_A v_\alpha)\,\mathrm{d}S \tag{5.43}$$

$$\overset{\partial_t}{=}\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5 v^\mu\overline{\partial}_l\eta_\mu\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}_k\overline{\partial}^5\eta_\alpha\hat{n}^\alpha)\,\mathrm{d}S \tag{5.44}$$

$$+\sigma\int_0^T\int_\Gamma \left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\partial_t(\sqrt{g}\hat{n}^\alpha)\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\eta_\alpha\,\mathrm{d}S \tag{5.45}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\partial_t\left(\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_l\eta_\mu\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu(\hat{n}^\alpha\overline{\partial}_k\overline{\partial}^5\eta_\alpha)\,\mathrm{d}S + (5.42) + (5.43). \tag{5.46}$$

Note that $\partial_t A(\sqrt{g}\hat{n}^\alpha) = \partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$, we know (5.45) + (5.42) = 0. Then the remaining quantities (5.43), (5.44) and (5.46) can all be directly controlled

$$(5.43) \lesssim \int_0^T |\sqrt{\sigma}\eta|_5^2 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{5.47}$$

$$(5.44) \lesssim \int_0^T |\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0|\sqrt{\sigma}v|_5 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{5.48}$$

$$(5.46) \lesssim \int_0^T |\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0|\sqrt{\sigma}\eta|_5 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.49}$$

Combining (5.37)-(5.49), we get the control of ST_{33}

$$\int_0^T \mathrm{ST}_{33}\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{5.50}$$

whic together with (5.34), (5.35) and (5.36) gives the control of ST_3

$$\int_0^T \mathrm{ST}_3\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.51}$$

Finally, (5.16), (5.17), (5.31), (5.32) and (5.51) gives the control of the boundary terms contributed by the surface tension as well as the $\sqrt{\sigma}$-weighted boundary energy

$$\int_0^T \mathrm{ST}\,\mathrm{d}t \lesssim -\frac{\sigma}{2}\int_\Gamma \left|\overline{\partial}^6\eta\cdot\hat{n}\right|^2 \mathrm{d}S\Big|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.52}$$

5.2.3 Control of the error terms

It remains to control J_1 and J_2 in (5.9). Note that $q = -\sigma\sqrt{g}\Delta_g\eta\cdot\hat{n} = \sigma Q(\overline{\partial}\eta)\overline{\partial}^2\eta\cdot\hat{n}$ on the boundary and $A^{3\alpha} = \overline{\partial}_1\eta\times\overline{\partial}_2\eta$. The term J_2 can be directly controlled

$$\begin{aligned}
J_2 &= 5\sigma\int_\Gamma \overline{\partial}^4 A^{3\alpha}\overline{\partial}(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&\quad + 10\sigma\int_\Gamma \overline{\partial}^3 A^{3\alpha}\overline{\partial}^2(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&\quad + 10\sigma\int_\Gamma \overline{\partial}^2 A^{3\alpha}\overline{\partial}^3(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&\quad + 5\sigma\int_\Gamma \overline{\partial} A^{3\alpha}\overline{\partial}^4(Q(\overline{\partial}\eta)(\overline{\partial}^2\eta\cdot\hat{n}))(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\
&\lesssim P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty}\left(|\sqrt{\sigma}\eta|_5 + |\sqrt{\sigma}\overline{\partial}^6\eta\cdot n|_0\right)\left(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}\right)\\
&\lesssim P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty}|\partial v|_{L^\infty}\left(\sqrt{E_1} + \sqrt{\sigma E_2}\right)\sqrt{\sigma E_2} \le P(E_1(t))(\sigma E_2(t)).
\end{aligned}\tag{5.53}$$

The control of J_1 needs more decliate computation. Recall that $A^{3\alpha} = (\overline{\partial}_1\eta\times\overline{\partial}_2\eta)^\alpha$, we have that

$$\begin{aligned}
J_1 &= \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\partial_t\eta\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}_1\eta\times\overline{\partial}^5\overline{\partial}_2\eta)\cdot\overline{\partial}^5\partial_t\eta\,\mathrm{d}S\\
&\quad - \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v)\,\mathrm{d}S - \int_\Gamma \sigma\mathcal{H}(\overline{\partial}_1\eta\times\overline{\partial}^5\overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v)\,\mathrm{d}S\\
&\quad + \sum_{k=1}^4\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^k\overline{\partial}_1\eta\times\overline{\partial}^{4-k}\overline{\partial}_2\eta)\cdot(\overline{\partial}^5 v - \overline{\partial}^5\eta\cdot\nabla_A v)\,\mathrm{d}S\\
&=: J_{11} + J_{12} + J_{13} + J_{14} + J_{15}.
\end{aligned}\tag{5.54}$$

Again, the term J_{15} is directly controlled by the weighted energy

$$J_{15} \le P(E_1(t))(\sigma E_2(t)). \tag{5.55}$$

Below we only show the control of J_{11} and J_{13}, and the control of J_{12} and J_{14} follows in the same way. For J_{11}, we integrate ∂_t by parts to get

$$\begin{aligned}
\int_0^T J_{11}\,\mathrm{d}t &= -\int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S - \int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\\
&\quad - \int_0^T\int_\Gamma \sigma\partial_t\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\Big|_0^T\\
&=: J_{111} + J_{112} + J_{113} + J_{114}.
\end{aligned}\tag{5.56}$$

Next we integrate $\overline{\partial}_1$ by parts in J_{111} and use the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ to get

$$
\begin{aligned}
J_{111} \overset{\overline{\partial}_1}{=} & \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v \times \overline{\partial}_2\eta)\cdot \overline{\partial}_1\overline{\partial}^5\eta \,\mathrm{d}S\ \mathrm{d}t \\
& + \int_0^T \int_\Gamma \sigma\overline{\partial}_1\mathcal{H}(\overline{\partial}^5 v \times \overline{\partial}_2\eta)\cdot \overline{\partial}^5\eta \,\mathrm{d}S\ \mathrm{d}t + \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v \times \overline{\partial}_1\overline{\partial}_2\eta)\cdot \overline{\partial}^5\eta \,\mathrm{d}S\ \mathrm{d}t \\
= & \underbrace{-\int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot \overline{\partial}^5 v \,\mathrm{d}S\ \mathrm{d}t}_{=-\int_0^T J_{11}\,\mathrm{d}t} \\
& + \int_0^T \int_\Gamma \sigma\overline{\partial}_1\mathcal{H}(\overline{\partial}^5 v \times \overline{\partial}_2\eta)\cdot \overline{\partial}^5\eta \,\mathrm{d}S\ \mathrm{d}t + \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v \times \overline{\partial}_1\overline{\partial}_2\eta)\cdot \overline{\partial}^5\eta \,\mathrm{d}S\ \mathrm{d}t \\
\lesssim & -\int_0^T J_{11}\,\mathrm{d}t + \int_0^T P(E_1(t))(\sigma E_2(t)).
\end{aligned} \tag{5.57}
$$

Therefore, we have

$$
\int_0^T J_{11}\,\mathrm{d}t \lesssim \frac{1}{2}(J_{112}+J_{113}+J_{114}) + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.58}
$$

Next we need to control J_{114} by $\mathcal{P}_0 + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t$. For that we need the following identity

$$
\delta^{\alpha\beta} = \hat{n}^\alpha\hat{n}^\beta + g^{ij}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta,
$$

which yields

$$
\begin{aligned}
J_{114} = & \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\alpha g^{ij}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
=: & J_{1141} + J_{1142}.
\end{aligned} \tag{5.59}
$$

In J_{1141} we integrate $\overline{\partial}_1$ by parts

$$
\begin{aligned}
J_{1141} \overset{\overline{\partial}_1}{=} & -\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\eta\times\overline{\partial}_2\eta)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_1\overline{\partial}^5\eta_\beta\,\mathrm{d}S - \int_\Gamma \sigma\overline{\partial}_1(\mathcal{H}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_2\eta)\overline{\partial}^5\eta\overline{\partial}^5\eta\,\mathrm{d}S \\
\lesssim & P(|\eta|_{W^{3,\infty}})\left(|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0^2\right) \\
\lesssim & P(E_1(T))(\sigma E_2(T))\int_0^T \|\sqrt{\sigma}\overline{\partial}^5 v(t)\|_0\,\mathrm{d}t,
\end{aligned} \tag{5.60}
$$

where we used $\overline{\partial}^5\eta|_{t=0} = \mathbf{0}$.

In J_{1142}, we notice that the integral vanishes if $i=2$ due to $(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}_2\eta = 0$. If $i=1$, then

$$
\begin{aligned}
J_{1142} = & \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}_1\eta\, g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S = -\int_\Gamma \sigma\mathcal{H}\underbrace{(\overline{\partial}_1\eta\times\overline{\partial}_2\eta)}_{=\sqrt{g}\hat{n}}\cdot\overline{\partial}^5\overline{\partial}_1\eta\, g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
\lesssim & |\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0 P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^2\eta|_{L^\infty}\,\mathrm{d}S \\
\lesssim & P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t
\end{aligned} \tag{5.61}
$$

The term J_{113} can be controlled in the same way as J_{114} so we omit the proof. Thus we already get

$$
J_{113} + J_{114} \lesssim P(E(T))\int_0^T P(E(t))\,\mathrm{d}t. \tag{5.62}
$$

It remains to analyze $\frac{1}{2}J_{112}$ which should be controlled together with J_{13}. Again we have

$$
\begin{aligned}
\frac{1}{2}J_{112} = & \frac{1}{2}\int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2 v)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S + \frac{1}{2}\int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2 v)_\alpha g^{ij}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
=: & J_{1121} + J_{1122},
\end{aligned} \tag{5.63}
$$

and the control of J_{1121} follows in the same way as (5.60) by integrating $\overline{\partial}_1$ by parts

$$J_{1121} \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{5.64}$$

For J_{1122} we need to do further decomposition

$$\begin{aligned}
J_{1122} =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}_i\eta \times \overline{\partial}_2 v)_\alpha g^{ij}\overline{\partial}^5\overline{\partial}_1\eta^\alpha \overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S\ \mathrm{d}t\\
=& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_i\eta \times \overline{\partial}_2 v)_\gamma \hat{n}^\gamma\hat{n}^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{ij}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S\ \mathrm{d}t\\
&-\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_i\eta \times \overline{\partial}_2 v)_\gamma g^{kl}\overline{\partial}_k\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{ij}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S\ \mathrm{d}t\\
=:&\ J_{11221} + J_{11222},
\end{aligned} \tag{5.65}$$

where J_{11221} is directly controlled by

$$J_{11221} \lesssim \int_0^T P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0\,\mathrm{d}t \lesssim \int_0^T P(E_1(t)) + \sigma E_2(t)\,\mathrm{d}t. \tag{5.66}$$

In J_{11222}, the integral vanishes if $i = k$, so we only need to investigate the cases $(i,k) = (1,2)$ and $(i,k) = (2,1)$, which contribute to

$$\begin{aligned}
J_{11222} =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1\eta \times \overline{\partial}_2 v)_\gamma g^{2l}\overline{\partial}_2\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S\ \mathrm{d}t\\
&-\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_2\eta \times \overline{\partial}_2 v)_\gamma g^{1l}\overline{\partial}_1\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{2j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S\ \mathrm{d}t\\
=& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1\eta \times \overline{\partial}_2 v)\cdot\overline{\partial}_2\eta\right)\left(g^{2l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta\right)\left(g^{1j}\overline{\partial}_j\eta\cdot\overline{\partial}^5\eta\right)\,\mathrm{d}S\ \mathrm{d}t\\
&+\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1\eta \times \overline{\partial}_2 v)\cdot\overline{\partial}_2\eta\right)\left(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta\right)\left(g^{2j}\overline{\partial}_j\eta\cdot\overline{\partial}^5\eta\right)\,\mathrm{d}S\ \mathrm{d}t.
\end{aligned} \tag{5.67}$$

Next we analyze J_{13}. First we do the following decomposition

$$\begin{aligned}
\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha &= \overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v_\alpha + \overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\underbrace{\overline{\partial}_j\eta_\gamma A^{\mu\gamma}}_{=\delta^\mu_j}\partial_\mu v_\alpha\\
&= \overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v_\alpha + \overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha,
\end{aligned}$$

and thus

$$\begin{aligned}
J_{13} =& \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v)\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v)\,\mathrm{d}S\\
=:&\ J_{131} + J_{132}.
\end{aligned} \tag{5.68}$$

The term J_{131} can be controlled similarly as J_{1141} in (5.60), i.e., integrating $\overline{\partial}_1$ by parts,

$$J_{131} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{5.69}$$

In J_{132}, we need do further decomposition

$$\begin{aligned}
J_{132} =& \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\gamma\hat{n}^\gamma\hat{n}^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\gamma g^{kl}\overline{\partial}_k\eta^\gamma\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S\\
=:&\ J_{1321} + J_{1322}.
\end{aligned} \tag{5.70}$$

The integral in J_{1322} vanishes if $k = 2$. When $k = 1$, we again use the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ and invoke $(\overline{\partial}_1\eta\times\overline{\partial}^2\eta) = \sqrt{g}\hat{n}$ to get

$$\begin{aligned}
J_{1322} =& \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\gamma g^{1l}\overline{\partial}_1\eta^\gamma\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S\\
=& -\int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\overline{\partial}_1\eta\right) g^{1l}\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S\\
\lesssim&\ |\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\mathcal{H}\ g^2\overline{\partial}\eta\ \overline{\partial}\eta\ \overline{\partial}v|_{L^\infty} \lesssim P(E_1(t))(\sigma E_2(t))
\end{aligned} \tag{5.71}$$

We recall $\hat{n}_\gamma = \sqrt{g}^{-1}(\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\gamma$ and use the vector identities $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ and $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ to get

$$
\begin{aligned}
(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta) =& -(\overline{\partial}^5\overline{\partial}_1\eta\times(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\cdot\overline{\partial}_2\eta\\
=& -(\overline{\partial}^5\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)\underbrace{(\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)}_{=g_{12}=-(\det g)g^{12}}+(\overline{\partial}^5\overline{\partial}_1\eta\cdot\overline{\partial}_1\eta)\underbrace{(\overline{\partial}_2\eta\cdot\overline{\partial}_2\eta)}_{=g_{22}=(\det g)g^{11}}.
\end{aligned}
$$

Plugging this into J_{1321} yields

$$
\begin{aligned}
J_{1321} =& \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)\sqrt{g}^{-1}\hat{n}^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S\\
=& \int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{ij}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_j v_\alpha\hat{n}^\alpha\sqrt{g})\,\mathrm{d}S\\
=& \int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{1i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_1 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S + \int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_2 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S\\
=:& J_{13211}+J_{13212}.
\end{aligned}
\tag{5.72}
$$

Integraing $\overline{\partial}_1$ by parts in J_{13211}, the highest order term is exactly the same as J_{13211} itself but with a minus sign. Therefore,

$$
\begin{aligned}
J_{13211} =& -\frac{1}{2}\int_\Gamma \sigma(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{1i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)\overline{\partial}_1(\mathcal{H}\overline{\partial}_1 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S\\
\lesssim& |\sqrt{\sigma}\overline{\partial}^5\eta|_0^2 P(|\overline{\partial}\eta|_{L^\infty})(|\overline{\partial}^2\eta\,\overline{\partial}v|_{L^\infty}) \lesssim P(E_1(t))(\sigma E_2(t)).
\end{aligned}
\tag{5.73}
$$

Now J_{13212} reads

$$
J_{13212} = -\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S,
\tag{5.74}
$$

which together with (5.67) yields that

$$
\begin{aligned}
J_{11222}+J_{13212} =& -\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
&-\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S,
\end{aligned}
\tag{5.75}
$$

and thus integrating $\overline{\partial}_1$ by parts in the first integral yields the cancellation with the second integral due to the symmetry

$$
\begin{aligned}
J_{11222}+J_{13212} =& \frac{1}{2}\int_\Gamma \sigma\mathcal{H}(\overline{\partial}_1(g^{1l}\overline{\partial}_l\eta)\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
&+\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_1(g^{2i}\overline{\partial}_i\eta)\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
&+\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)\overline{\partial}_1((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
\lesssim& |\sqrt{\sigma}\overline{\partial}^5\eta|_0^2 P(E_1(t)) \leq P(E_1(t))(\sigma E_2(t)).
\end{aligned}
\tag{5.76}
$$

Summarizing (5.54)-(5.56), (5.62)-(5.66), (5.68)-(5.76), we conclude the estimate of J_1 by

$$
\int_0^T J_1\,\mathrm{d}t \lesssim P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\tag{5.77}
$$

Finally, combining (5.9), (5.15), (5.52), (5.53) and (5.77), we conclude the $\overline{\partial}^5$-boundary estimate by

$$
\int_0^T J\,\mathrm{d}t \lesssim -\frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2 - \frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2 + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\tag{5.78}
$$

5.3 Finalizing the tangential estimate of spatial derivatives

Summarizing (5.5)-(5.8) and (5.78), we conclude the estimate of the Alinhac good unknowns by

$$\|\mathbf{V}\|_0^2 + \frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\bigg|_{t=T} \lesssim P(\|v_0\|_5) + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.79}$$

Finally, from the definition of the good unknowns (5.1) and $\overline{\partial}^5\eta|_{t=0} = \mathbf{0}$, we know

$$\|\overline{\partial}^5 v(T)\|_0^2 \lesssim \|\mathbf{V}(T)\|_0^2 + \|\overline{\partial}^5\eta(T)\|_0^2\|\nabla_A v(T)\|_{L^\infty}^2 \lesssim \|\mathbf{V}(T)\|_0^2 + P(E_1(T))\int_0^T \|\overline{\partial}^5 v(t)\|_0^2\,\mathrm{d}t,$$

and thus

$$\|\overline{\partial}^5 v\|_0^2 + \|\overline{\partial}^5(b_0\cdot\partial)\eta\|_0^2 + \frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\bigg|_{t=T} \lesssim P(\|v_0\|_5) + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{5.80}$$

6 Tangential estimates of time derivatives

Now we consider the tangential estimates involving time derivatives. Due to the appearance of time derivatives, we no longer need the Alinhac good unknowns thanks to the fact that A has the same spatial regularity as $\partial_t A$. Instead, we adapt similar techniques developed in our previous work [30] to reveal the subtle cancellation structure and $\sqrt{\sigma}$-weighted energy terms on the boundary.

6.1 Full time derivatives

First, we do the ∂_t^5-estimate which is the most difficult part in the tangential estimates of time derivatives. The reason is two-fold

- We do not have any estimate for the full-time derivatives of q,
- The full-time derivatives $\partial_t^5 v$ and $\partial_t^5(b_0\cdot\partial)\eta$ only have L^2 interior regularity and thus have no control on the boundary due to the failure of trace lemma.

From the second equation of (1.12), we have

$$\begin{aligned}
&\frac{1}{2}\|\partial_t^5 v\|_0^2 + \frac{1}{2}\|\partial_t^5(b_0\cdot\partial)\eta\|_0^2\bigg|_0^T \\
=&\underbrace{-\int_0^T\int_\Omega \partial_t^5(A^{\mu\alpha}\partial_\mu q)\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t}_{=:I} \\
&+\int_0^T\int_\Omega (b_0\cdot\partial)\partial_t^5((b_0\cdot\partial)\eta^\alpha)\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \partial_t^5((b_0\cdot\partial)v)\cdot\partial_t^5((b_0\cdot\partial)\eta)\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{6.1}$$

It is not difficult to see that the second integral cancels with the third one after integrating $(b_0\cdot\partial)$ by parts. Therefore it suffices to control I that reads

$$\begin{aligned}
I =& -\int_0^T\int_\Omega A^{\mu\alpha}\partial_t^5\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t - 5\int_0^T\int_\Omega \partial_t A^{\mu\alpha}\partial_t^4\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t \\
&-10\int_0^T\int_\Omega \partial_t^2 A^{\mu\alpha}\partial_t^3\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t - 10\int_0^T\int_\Omega \partial_t^3 A^{\mu\alpha}\partial_t^2\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t \\
&-5\int_0^T\int_\Omega \partial_t^4 A^{\mu\alpha}\partial_t\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega \partial_t^5 A^{\mu\alpha}\partial_\mu q\partial_t^5 v_\alpha\,\mathrm{d}y\,\mathrm{d}t \\
=:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{6.2}$$

where $I_2\sim I_5$ can all be directly controlled

$$I_2 + \cdots + I_5 \lesssim \int_0^T P(\|\partial_t^5 v\|_0, \|\partial_t^4 v, \partial_t^4 q\|_1, \|\partial_t^3 v, \partial_t^3 q\|_2, \|\partial_t^2 v, \partial_t^2 q\|_3, \|\partial_t v, \partial_t q, v, q, \eta\|_3)\,\mathrm{d}t \tag{6.3}$$

For I_1, we integrate ∂_μ by parts and then invoke the surface tension equation to get

$$\begin{aligned}
I_1 =& -\int_0^T\int_\Gamma A^{3\alpha}\partial_t^5 q\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega A^{\mu\alpha}\partial_t^5 q\partial_\mu\partial_t^5 v_\alpha \,\mathrm{d}y\,\mathrm{d}t \\
=& -\int_0^T\int_\Gamma \partial_t^5(A^{3\alpha}q)\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Gamma \partial_t^5 A^{3\alpha} q\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t \\
&+ \sum_{k=1}^4\binom{5}{k}\int_0^T\int_\Gamma \partial_t^k A^{3\alpha}\partial_t^{5-k}q\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Gamma A^{\mu\alpha}\partial_t^5 q\partial_\mu\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t \\
=:& I_{11}+I_{12}+I_{13}+I_{14}.
\end{aligned} \tag{6.4}$$

Apart from I_{11}, the most difficult term is I_{12} since $\partial_t^5 v$ cannot be controlled on the boundary. However, we can integrate ∂_μ by parts in I_6 to produce a cancellation. Invoking Piola's identity and the boundary conditions on Γ_0, we have

$$\begin{aligned}
I_6 \overset{\partial_\mu}{=}& -\int_0^T\int_\Gamma \partial_t^5 A^{3\alpha} q\partial_t^5 v_\alpha \,\mathrm{d}S\,\mathrm{d}t + \underbrace{\int_0^T\int_\Omega \partial_t^5 A^{\mu\alpha} q\partial_t^5\partial_\mu v_\alpha \,\mathrm{d}y\,\mathrm{d}t}_{=:I_{61}} \\
=& -I_{12}+I_{61}.
\end{aligned} \tag{6.5}$$

Next we control I_{61}. Recall that $A^{1\cdot}=\overline{\partial}_2\eta\times\partial_3\eta,\ A^{2\cdot}=\partial_3\eta\times\overline{\partial}_1\eta,\ A^{3\cdot}=\overline{\partial}_1\eta\times\overline{\partial}_2\eta$ which implies

$$\begin{aligned}
I_{61} =& \int_0^T\int_\Omega q\partial_t^5 A^{1\alpha}\partial_t^5\overline{\partial}_1 v_\alpha + q\partial_t^5 A^{2\alpha}\partial_t^5\overline{\partial}_2 v_\alpha + q\partial_t^5 A^{3\alpha}\partial_t^5\partial_3 v_\alpha \,\mathrm{d}y\,\mathrm{d}t \\
=& \int_0^T\int_\Omega q(\partial_t^4\overline{\partial}_2 v\times\partial_3\eta)\cdot\overline{\partial}_1\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega q(\overline{\partial}_2\eta\times\partial_t^4\partial_3 v)\cdot\overline{\partial}_1\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_0^T\int_\Omega q(\partial_t^4\partial_3 v\times\overline{\partial}_1\eta)\cdot\overline{\partial}_2\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega q(\partial_3\eta\times\partial_t^4\overline{\partial}_1 v)\cdot\overline{\partial}_2\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_0^T\int_\Omega q(\partial_t^4\overline{\partial}_1 v\times\overline{\partial}_2\eta)\cdot\partial_3\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega q(\overline{\partial}_1\eta\times\partial_t^4\overline{\partial}_2 v)\cdot\partial_3\partial_t^5 v\,\mathrm{d}y\,\mathrm{d}t \\
&+ \text{ lower order terms} =: I_{611}+\cdots+I_{616}+\text{ lower order terms}.
\end{aligned} \tag{6.6}$$

From the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$, we can divide these 6 terms into 3 pairs: $I_{611}+I_{614}$, $I_{612}+I_{615}$ and $I_{613}+I_{616}$.

$$I_{611}+I_{614} = \underbrace{\int_\Omega q(\partial_t^4\overline{\partial}_2 v\times\partial_3\eta)\cdot\overline{\partial}_1\partial_t^4 v\,\mathrm{d}y\Big|_0^T}_{I_{6a}} + \text{lower order terms}, \tag{6.7}$$

$$I_{612}+I_{615} = \underbrace{\int_\Omega q(\partial_t^4\partial_3 v\times\overline{\partial}_1\eta)\cdot\overline{\partial}_2\partial_t^4 v\,\mathrm{d}y\Big|_0^T}_{I_{6b}} + \text{lower order terms}, \tag{6.8}$$

$$I_{613}+I_{616} = \underbrace{\int_\Omega q(\partial_t^4\overline{\partial}_1 v\times\overline{\partial}_2\eta)\cdot\partial_3\partial_t^4 v\,\mathrm{d}y\Big|_0^T}_{I_{6c}} + \text{lower order terms}. \tag{6.9}$$

Now we shall control $I_{6a},\ I_{6b},\ I_{6c}$ by $\mathcal{P}_0+P(E_1(T))\int_0^T P(E_1(t))\,\mathrm{d}t$. For I_{6a}, we can equivalently write it to be

$$I_{6a} = \int_\Omega q\epsilon^{\alpha\beta\gamma}\partial_t^4\overline{\partial}_2 v_\alpha\partial_3\eta_\beta\overline{\partial}_1\partial_t^4 v_\gamma\,\mathrm{d}y.$$

When $\beta=1,2$, we know $\partial_3\eta_\beta|_{t=0}=0$, and thus

$$I_{6a}|_{\beta=1,2} \lesssim \|\partial_t^4 v\|_1^2\|q\|_{L^\infty}\int_0^T\|\partial_3\eta\|_{L^\infty}\,\mathrm{d}t \lesssim \mathcal{P}_0+P(E_1(T))\int_0^T P(E_1(t))\,\mathrm{d}t. \tag{6.10}$$

When $\beta = 3$, α, β can only be 1 or 2. Note that $\epsilon^{132} = -\epsilon^{231}$, we know $I_{6a}|_{\beta=3}$ only consists of two terms

$$I_{6a}|_{\beta=3} = \int_\Omega q\partial_3\eta_3\overline{\partial}_1\partial_t^4 v_1\overline{\partial}_2\partial_t^4 v_2\,\mathrm{d}y - \int_\Omega q\partial_3\eta_3\overline{\partial}_1\partial_t^4 v_2\overline{\partial}_2\partial_t^4 v_1\,\mathrm{d}y. \tag{6.11}$$

Then we integrate $\overline{\partial}_2$ by parts in the first term and $\overline{\partial}_1$ in the second term. We notice that the highest order terms cancel with each other

$$\begin{aligned}
I_{6a}|_{\beta=3} =& \int_\Omega q\partial_3\eta_3(\underbrace{\overline{\partial}_1\overline{\partial}_2\partial_t^4 v_1\partial_t^4 v_2 - \partial_t^4 v_2\overline{\partial}_1\overline{\partial}_2\partial_t^4 v_1}_{=0})\,\mathrm{d}y \\
&- \int_\Omega \overline{\partial}_2 q\overline{\partial}_1\partial_t^4 v_1\partial_t^4 v_2 - \overline{\partial}_1 q\partial_t^4 v_2\overline{\partial}_2\partial_t^4 v_1\,\mathrm{d}y + \text{ lower order terms} \\
\lesssim& \|q\|_3\|\partial_t^4 v\|_0\|\partial_t^4 v\|_1 \lesssim \varepsilon\|\partial_t^4 v\|_1^2 + \mathcal{P}_0 + \int_0^T P(E_1(t))\,\mathrm{d}t.
\end{aligned} \tag{6.12}$$

The estimates of I_{6b} and I_{6c} can proceed in the same way. We only show how to control I_{6b}. Again we expand the components of the cross product to get

$$I_{6b} = \int_\Omega q\epsilon^{\alpha\beta\gamma}\partial_t^4\partial_3 v_\alpha\overline{\partial}_1\eta_\beta\overline{\partial}_2\partial_t^4 v_\gamma\,\mathrm{d}y.$$

When $\beta = 2, 3$, $\overline{\partial}_1\eta_\beta|_{t=0} = 0$ and thus $I_{6b}|_{\beta=2,3} \lesssim \mathcal{P}_0 + P(E_1(T))\int_0^T P(E_1(t))\,\mathrm{d}t$. When $\beta = 1$, we write

$$I_{6b}|_{\beta=1} = \int_\Omega q\overline{\partial}_1\eta_1(\overline{\partial}\partial_t^4 v_2\partial_3\partial_t^4 v_3 - \overline{\partial}_2\partial_t^4 v_3\partial_3\partial_t^4 v_2)\,\mathrm{d}y.$$

Again we can integrate $\partial_3, \overline{\partial}_1$ by parts in each term respectively and produce similar cancellation as in (6.12). However, since ∂_3 is not tangential, we also have to control the boundary term as follows

$$\begin{aligned}
I_{6b1} :=& \int_\Gamma q\overline{\partial}_1\eta_1\partial_t^4 v_2\overline{\partial}_2\partial_t^4 v_3\,\mathrm{d}S = \int_\Gamma \sigma\mathcal{H}\overline{\partial}_1\eta_1\partial_t^4 v_2\overline{\partial}_2\partial_t^4 v_3\,\mathrm{d}S \\
\lesssim& \|\sqrt{\sigma}\partial_t^4 v\|_{1.5}|\overline{\partial}\eta|_{L^\infty}\int_0^T |\partial_t\mathcal{H}(t)|_{L^\infty}\,\mathrm{d}t \lesssim P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t,
\end{aligned} \tag{6.13}$$

where we used $\mathcal{H}|_{t=0} = 0$ due to the appearence of $\overline{\partial}^2\eta$ in $\mathcal{H}$. Therefore, we get

$$I_6 + I_{12} \lesssim \varepsilon\|\partial_t^4 v\|_1^2 + \mathcal{P}_0 + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t. \tag{6.14}$$

We then start to control I_{11} in (6.4) which is expected to produce the weighted boundary energy $\sigma|\overline{\partial}\partial_t^4 v \cdot \hat{n}|_0^2$. Below we use $\stackrel{L}{=}$ to denote equality modulo error terms that are effectively of lower order. Plugging the surface tension equation $A^{3\alpha}q = -\sigma\sqrt{g}\Delta_g\eta^\alpha$ and the identity (2.9), we get

$$\begin{aligned}
I_{11} \stackrel{L}{=}& -\sigma\int_0^T\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\partial_t^4\overline{\partial}_j v_\beta\overline{\partial}_i\partial_t^5 v_\alpha\,\mathrm{d}S\,\mathrm{d}t \\
&- \sigma\int_0^T\int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{ik}g^{jl})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta^\lambda\overline{\partial}_l\partial_t^4 v^\lambda\overline{\partial}_i\partial_t^5 v_\alpha\,\mathrm{d}S\,\mathrm{d}t \\
=:& I_{111} + I_{112}.
\end{aligned} \tag{6.15}$$

And the term I_{111} contributes to the weighted boundary energy

$$\begin{aligned}
I_{111} =& -\frac{\sigma}{2}\left|\overline{\partial}\partial_t^4 v\cdot\hat{n}\right|_0^2\Big|_0^T \underbrace{-\frac{\sigma}{2}\int_\Gamma(\sqrt{g}g^{ij} - \delta^{ij})\hat{n}^\alpha\hat{n}^\beta\partial_t^4\overline{\partial}_j v_\beta\overline{\partial}_i\partial_t^4 v_\alpha\,\mathrm{d}S}_{I_{1111}} \\
&\underbrace{+\sigma\int_0^T\int_\Gamma \partial_t(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\partial_t^4\overline{\partial}_i v_\alpha\partial_t^4\overline{\partial}v_\beta\,\mathrm{d}S\,\mathrm{d}t,}_{I_{1112}}
\end{aligned} \tag{6.16}$$

where I_{1111} is controlled similarly as (5.20) by invoking the a priori assumtpion (2.15)

$$I_{1111} \lesssim -\varepsilon \left| \sqrt{\sigma}\overline{\partial}\partial_t^4 v \cdot \hat{n} \right|_0^2, \tag{6.17}$$

and I_{1112} is directly controlled by the weighted energy

$$I_{1112} \lesssim \int_0^T |P(\overline{\partial}\eta)\overline{\partial}v|_{L^\infty} \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 \,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{6.18}$$

The control of I_{112} requires a remarkable identity, first introduced by Coutand-Shkoller [6, (12.10)-(12.11)] to rewrite I_{112}. Here we only list the result and we refer to our previous work [15, (4.33)-(4.37)] for the details.

$$I_{112} = \int_0^T \int_\Gamma \frac{\sigma}{\sqrt{g}} (\partial_t \det \mathbf{A}^1 + \det \mathbf{A}^2 + \det \mathbf{A}^3), \tag{6.19}$$

where

$$\mathbf{A}^1 = \begin{bmatrix} \overline{\partial}_1 \eta_\mu \partial_t^4 \overline{\partial}_1 v^\mu & \overline{\partial}_1 \eta_\mu \overline{\partial}_2 \partial_t^4 v^\mu \\ \overline{\partial}_2 \eta_\mu \partial_t^4 \overline{\partial}_1 v^\mu & \overline{\partial}_2 \eta_\mu \overline{\partial}_2 \partial_t^4 v^\mu \end{bmatrix},$$

and $\mathbf{A}^2_{ij} = \overline{\partial}_i v_\mu \overline{\partial}_j \partial_t^4 v^\mu$ and $\mathbf{A}^3_{ij} = \overline{\partial}_i \eta_\mu \overline{\partial}_j \partial_t^4 v^\mu$. Under this setting we have

$$I_{112} \lesssim \epsilon \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 + \mathcal{P}_0 + \int_0^T \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 P(E_1(t))\,\mathrm{d}t. \tag{6.20}$$

It now remains to control I_{13} in (6.4). First we recall that $A^{3\cdot} = \overline{\partial}_1\eta \times \overline{\partial}_2\eta$.

$$\begin{aligned} I_{13} \overset{L}{=} {} & 5\int_0^T \int_\Gamma \partial_t q(\partial_t^3 \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^5 v \,\mathrm{d}S\,\mathrm{d}t + 10\int_0^T \int_\Gamma \partial_t^2 q(\partial_t^2 \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^5 v \,\mathrm{d}S\,\mathrm{d}t \\ & + 10\int_0^T \int_\Gamma \partial_t^3 q(\partial_t \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^5 v \,\mathrm{d}S\,\mathrm{d}t + 5\int_0^T \int_\Gamma \partial_t^4 q(\overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^5 v \,\mathrm{d}S\,\mathrm{d}t \\ =: {} & I_{131} + \cdots + I_{134}. \end{aligned} \tag{6.21}$$

We only show the control of I_{131} and the rest three terms are easier. For I_{131}, we integrate ∂_t by parts to get

$$\begin{aligned} I_{131} \overset{L}{=} & -5\sigma \int_0^T \int_\Gamma \partial_t \mathcal{H}(\partial_t^4 \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^4 v \,\mathrm{d}S\,\mathrm{d}t + 5\sigma \int_\Gamma \partial_t \mathcal{H}(\partial_t^3 \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^4 v \,\mathrm{d}S \\ \overset{L}{=} & -5\sigma \int_0^T \int_\Gamma \partial_t \mathcal{H}(\partial_t^4 \overline{\partial} v \times \overline{\partial}\eta) \cdot \partial_t^4 v \,\mathrm{d}S\,\mathrm{d}t - 5\sigma \int_\Gamma \partial_t \mathcal{H}(\partial_t^3 v \times \overline{\partial}\eta) \cdot \overline{\partial}\partial_t^4 v \,\mathrm{d}S \\ \lesssim & \sqrt{\sigma} \int_0^T \|\partial_t^4 v\|_1 \| \sqrt{\sigma}\partial_t^4 v\|_{1.5} |\overline{\partial}\eta\, \partial_t \mathcal{H}|_{L^\infty} + 5\sigma \|\partial_t^4 v\|_{1.5} \|\partial_t^3 v\|_1 |\partial_t \mathcal{H}\, \overline{\partial}\eta|_{L^\infty} \\ \lesssim & \sqrt{\sigma} \int_0^T P(E_1(t))(\sqrt{\sigma\, E_2(t)})\,\mathrm{d}t + 5\sigma \|\partial_t^4 v\|_{1.5} \|\partial_t^3 v\|_1 |\partial_t \mathcal{H}\, \overline{\partial}\eta|_{L^\infty}. \end{aligned} \tag{6.22}$$

The last term should be controlled by using Young's inequality

$$\begin{aligned} & \sigma \|\partial_t^4 v\|_{1.5} \|\partial_t^3 v\|_1 |\partial_t \mathcal{H}\, \overline{\partial}\eta|_{L^\infty} \lesssim \sigma \left(\varepsilon \|\partial_t^4 v\|_{1.5}^2 + \|\partial_t^3 v\|_1^2 |\partial_t \mathcal{H}\, \overline{\partial}\eta|_{L^\infty}^2 \right) \\ \lesssim {} & \varepsilon \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 + \sigma \|\partial_t^3 v\|_1^2 |\partial_t \mathcal{H}\, \overline{\partial}\eta|_{L^\infty}^2 \\ \lesssim {} & \varepsilon \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 + \sigma \left(\mathcal{P}_0 + \int_0^T P(E_1(t))\,\mathrm{d}t \right). \end{aligned}$$

Summarizing (6.1)-(6.3) and (6.14)-(6.22), we conclude the ∂_t^5 estimates as follows

$$\begin{aligned} & \|\partial_t^5 v\|_0^2 + \|\partial_t^5 (b_0 \cdot \partial)\eta\|_0^2 + \frac{\sigma}{2} \left| \overline{\partial}\partial_t^4 v \cdot \hat{n} \right|_0^2 \\ \lesssim {} & \varepsilon \| \sqrt{\sigma}\partial_t^4 v\|_{1.5}^2 + \mathcal{P}_0 + P(E_1(T))(\sigma E_2(T)) \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \end{aligned} \tag{6.23}$$

Since we consider the zero surface tension limit, we can take $\sigma > 0$ sufficiently small to absorb the term $\sigma E_1(t)$ to LHS in the final step.

6.2 Mixed space-time derivatives

Replace ∂_t^5 in Section 6.1 by $\mathfrak{D}^4\partial_t$ with $\mathfrak{D} = \overline{\partial}$ or ∂_t, we can similarly get the $\mathfrak{D}^4\partial_t$ tangential estimates. The proof is similar and even easier since $\partial_t^k v$ $(1 \leq k \leq 4)$ has at least H^{5-k} regularity and may be controlled in $H^{4.5-k}$ norm on the boundary by using trace lemma. So we omit the proof and only list the result. For the related details one can refer to our previous work [15, Section 4.2].

$$
\begin{aligned}
&\sum_{k=1}^{4} \|\overline{\partial}^{5-k}\partial_t^k v\|_0^2 + \|\overline{\partial}^{5-k}\partial_t^k (b_0\cdot\partial)\eta\|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^{6-k}\partial_t^{k-1} v\cdot\hat{n}\right|_0^2 \\
&\lesssim \varepsilon(\sigma E_2(T)) + \mathcal{P}_0 + P(E_1(T))(\sigma E_2(T)) \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\end{aligned}
\tag{6.24}
$$

7 Control of weighted Sobolev norms

7.1 Weighted div-curl estimates

The estimate for $\|\sqrt{\sigma}v\|_{5.5}^2$ and $\|\sqrt{\sigma}(b_0\cdot\partial)\eta\|_{5.5}^2$ is done similarly as in Section 4.1 and 4.2. For the divergence, we directly get

$$
\sigma\|\text{div } v\|_{4.5}^2 + \sigma\|\text{div } (b_0\cdot\partial)\eta\|_{4.5}^2 \lesssim \varepsilon\sigma\left(\|v\|_{5.5}^2 + \|(b_0\cdot\partial)\eta\|_{5.5}^2\right), \tag{7.1}
$$

and similarly

$$
\sum_{k=0}^{4}\sigma\|\text{div } \partial_t^k v\|_{4.5-k}^2 + \sigma\|\text{div } \partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2 \lesssim \sum_{k=0}^{4}\varepsilon\sigma\left(\|\partial_t^k v\|_{5.5-k}^2 + \|\partial_t^k(b_0\cdot\partial)\eta\|_{5.5-k}^2\right) + P(E(0)) + \int_0^T P(E(t))\,\mathrm{d}t. \tag{7.2}
$$

The weighted curl estimates follow similarly as in section 4.2. By taking $\partial^{4.5}$ to (4.7), testing it with $\sigma\partial^{4.5}\text{curl}_A v$, and then integrating $(b_0\cdot\partial)$ by parts, we have

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\left(\int_\Omega |\sqrt{\sigma}\partial^{4.5}\text{curl}_A v|^2 + \int_\Omega |\sqrt{\sigma}\partial^{4.5}\text{curl}_A((b_0\cdot\partial)\eta)|^2\right) &= \int_\Omega \left(\sqrt{\sigma}[\partial^{4.5},(b_0\cdot\partial)]\text{curl}_A(b_0\cdot\partial)\eta + \sqrt{\sigma}\partial^{4.5}\mathcal{F}\right)(\sqrt{\sigma}\partial^{4.5}\text{curl}_A v) \\
&+ \int_\Omega (\sqrt{\sigma}[\partial^{4.5}\text{curl}_A,(b_0\cdot\partial)]v)(\sqrt{\sigma}\partial^{4.5}\text{curl}_A((b_0\cdot\partial)\eta)) + \int_\Omega \sqrt{\sigma}\partial^{4.5}\text{curl}_{\partial_t A}((b_0\cdot\partial)\eta)(\sqrt{\sigma}\partial^{4.5}\text{curl}_A((b_0\cdot\partial)\eta)).
\end{aligned}
\tag{7.3}
$$

Since

$$
\begin{aligned}
\|\sqrt{\sigma}\mathcal{F}\|_{4.5}^2 &\leq \|\sqrt{\sigma}\text{curl}_{\partial_t A} v\|_{4.5}^2 + \|\sqrt{\sigma}[\text{curl}_A,(b_0\cdot\partial)](b_0\cdot\partial)\eta\|_{4.5}^2 \\
&\leq P(\|b_0\|_{5.5},\|\eta\|_5,\|v\|_5,\|(b_0\cdot\partial)\eta\|_5,\|\sqrt{\sigma}\eta\|_{5.5},\|\sqrt{\sigma}v\|_{5.5},\|\sqrt{\sigma}(b_0\cdot\partial)\eta\|_{5.5}),
\end{aligned}
$$

then the terms on the RHS of (7.3) can be controlled by $P(E(t))$ as well. Therefore,

$$
\|\sqrt{\sigma}\text{curl}_A v\|_4^2 + \|\sqrt{\sigma}\text{curl}_A(b_0\cdot\partial)\eta\|_4^2 \lesssim \int_0^T P(E(t)), \tag{7.4}
$$

which yields

$$
\|\sqrt{\sigma}\text{curl } v\|_{4.5}^2 + \|\sqrt{\sigma}\text{curl } (b_0\cdot\partial)\eta\|_{4.5}^2 \lesssim \varepsilon\sigma\left(\|v\|_{5.5}^2 + \|(b_0\cdot\partial)\eta\|_{5.5}^2\right) + \int_0^T P(E(t)), \tag{7.5}
$$

and similarly

$$
\sum_{k=0}^{4}\sigma\|\text{curl } \partial_t^k v\|_{4.5-k}^2 + \sigma\|\text{curl } \partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2 \lesssim \sum_{k=0}^{4}\varepsilon\sigma\left(\|\partial_t^k v\|_{5.5-k}^2 + \|\partial_t^k(b_0\cdot\partial)\eta\|_{5.5-k}^2\right) + P(E(0)) + \int_0^T P(E(t))\,\mathrm{d}t. \tag{7.6}
$$

7.2 Control of the weighted boundary norms

We still need to control $\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k v\cdot N|_0$ and $\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_0$ for $0\le k\le 4$. For the boundary estimates of v, one can directly compare them with the energy terms contributed by surface tension. (cf. (5.80), (6.23)-(6.24))

$$\sqrt{\sigma}\left|(\overline{\partial}^{5-k}\partial_t^k v\cdot N)-\overline{\partial}^{5-k}\partial_t^k v\cdot\hat{n}\right|_0\lesssim\left|\sqrt{\sigma}\partial_t^k v\right|_{5-k}|\hat{n}-N|_{L^\infty}\lesssim\left\|\sqrt{\sigma}\partial_t^k v\right\|_{5.5-k}\int_0^T|\partial_t(\hat{n}-N)|_{L^\infty}^2\,\mathrm{d}t \tag{7.7}$$

As for $(b_0\cdot\partial)\eta$, when $k\ge 1$, we can directly control them by the norms of v

$$\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_0=\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^{k-1}(b_0\cdot\partial)v\cdot N|_0\lesssim\|b_0\|_{L^\infty}\|\sqrt{\sigma}\partial_t^{k-1}\partial v\|_{5.5-k}+\text{ lower order terms.} \tag{7.8}$$

When $k=0$, again we need to compare it with the projection onto the Eulerian normal direction

$$\sqrt{\sigma}\left|(\overline{\partial}^5((b_0\cdot\partial)\eta)\cdot N)-\overline{\partial}^5(b_0\cdot\partial)\eta\cdot\hat{n}\right|_0\lesssim\left|\sqrt{\sigma}(b_0\cdot\partial)\eta\right|_5|\hat{n}-N|_{L^\infty}\lesssim\left\|\sqrt{\sigma}(b_0\cdot\partial)\eta\right\|_{5.5}\int_0^T|\partial_t(\hat{n}-N)|_{L^\infty}^2\,\mathrm{d}t, \tag{7.9}$$

and thus it remains to control $\sqrt{\sigma}|\overline{\partial}^5(b_0\cdot\partial)\eta\cdot\hat{n}|_0$. In fact, this term naturally appears as a boundary energy term contributed by the surface tension in the $\overline{\partial}^4(b_0\cdot\partial)$ tangential estimate, which can be proceeded in the same way as $\overline{\partial}^4\partial_t$-estimate by just replacing ∂_t by $(b_0\cdot\partial)$. The reason for that is $(b_0\cdot\partial)\eta$ and η have the same spatial regularity, which is similar with the fact that $\partial_t\eta=v$ has the same spatial regularity as η. In other words, the tangential derivative $(b_0\cdot\partial)$ (note that $b_0\cdot N=0$ on $\partial\Omega$!) plays the same role as a time derivative if it falls on the flow map η. We just list the result of $\overline{\partial}^4(b_0\cdot\partial)$-estimate

$$\|\overline{\partial}^4(b_0\cdot\partial)v\|_0^2+\|\overline{\partial}^4(b_0\cdot\partial)^2\eta\|_0^2+\frac{\sigma}{2}\left|\overline{\partial}^5(b_0\cdot\partial)\eta\cdot\hat{n}\right|_0^2\Big|_{t=T}\lesssim\mathcal{P}_0+P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{7.10}$$

8 The zero surface tension limit

Now we conclude the energy estimates as follows. First, (4.5) and (4.14) give the div-curl control of the non-weighted Sobolev norms. Then by (4.15) and (4.16), the boundary normal traces are reduced to the interior tangential estimates which are established in (5.80) for the spatial derivatives and (6.23)-(6.24) for the space-time derivatives. In the control of the non-weighted Sobolev norms, the weighted energy $\sigma E_2(t)$ is needed to close the energy estimates. The $\sqrt{\sigma}$-weighted div-curl estimates are established in (7.2) and (7.6), while the boundary normal traces are no longer reduced to the interior tangential estimates via Lemma 2.6. Instead, we notice that, in the non-weighted tangential estimates, the surface tension contributes to $\sqrt{\sigma}$-weighted boundary energies which are exactly the *Eulerian* normal traces of the weighted variables. Therefore, it suffices to estimate the difference between the $\sqrt{\sigma}$-weighted Lagrangian normal traces and the $\sqrt{\sigma}$-weighted Eulerian normal traces, which is established in (7.7)-(7.9). Finally, we can get the following energy estimates

$$E(T)=E_1(T)+\sigma E_2(T)\lesssim\varepsilon E(T)+P(E(0))+P(E(T))\int_0^T P(E(t))\,\mathrm{d}t, \tag{8.1}$$

which together with Gronwall inequality implies that there exists some $T>0$, independent of σ, such that

$$\sup_{0\le t\le T}E(t)\le P(E(0))\le C. \tag{8.2}$$

Finally, we need to recover the higher boundary regularity of v by proving (1.26). Taking ∂_t in the surface tension equation $A^{3\alpha}q=-\sigma\sqrt{g}\Delta_g\eta^\alpha$ yields

$$\sigma\sqrt{g}g^{ij}(\overline{\partial}_{ij}^2v^\alpha-\Gamma_{ij}^k\overline{\partial}_k v^\alpha)=\sigma\partial_t(\sqrt{g}g^{ij})\overline{\partial}_{ij}^2\eta^\alpha-\sigma\partial_t(\sqrt{g}g^{ij}\Gamma_{ij}^k)\overline{\partial}_k\eta^\alpha-\partial_t(A^{3\alpha}q),\quad\text{ on }\Gamma. \tag{8.3}$$

One can mimic the proof of [8, Prop. 3.4] to verify $g\in\mathrm{BMO}(\Gamma)$ and thus the elliptic estimate in [10] can be applied to this equation. Invoking (3.23) and taking $\alpha=3$, we then have

$$\begin{aligned}|\sigma v^3(T)|_{5.5}&\lesssim|\text{RHS of (8.3)}|_{3.5}\lesssim|\partial_t(\sqrt{g}g^{ij})|_0|\sigma\overline{\partial}^2\eta^3|_{3.5}+P(E_1(T))\\&\lesssim P(E_1(T))+P(E_1(T))\int_0^T|\sigma\overline{\partial}^2v^3(t)|_{3.5}\,\mathrm{d}t,\end{aligned} \tag{8.4}$$

where we use the fact that $\overline{\partial}^2\eta|_{t=0} = \mathbf{0}$ and $\overline{\partial}\eta^3|_{t=0} = 0$. It is worth mentioning that, when estimating |RHS of (8.3)$|_{3.5}$, the top order term $|q_t|_{3.5} \lesssim \|q_t\|_4$ is controlled by considering the Neumann boundary condition of q_t, which then avoids circular arguments. Therefore, the standard Gronwall-type argument gives the control of $|\sigma v|_{5.5}$. As for $|\sigma^{\frac{3}{2}} v^3|_6$, one can multiply $\sqrt{\sigma}$ in (8.3) and again invoke the elliptic estimate to get

$$|\sigma^{\frac{3}{2}} v^3(T)|_6 \lesssim P(E_1(T))(\sigma E_2(T)) + P(E_1(T))(\sigma E_2(T)) \int_0^T |\sigma^{\frac{3}{2}}\overline{\partial}^2 v^3(t)|_4 \,\mathrm{d}t \tag{8.5}$$

which yields $|\sigma^{\frac{3}{2}} v^3(T)|_6 \leq P(E(T))$. The weighted bounds for the magnetic field $b^3 = (b_0 \cdot \partial)\eta^3$ follow in the same manner but just replacing the ∂_t-differentiated surface tension equation by the $(b_0 \cdot \partial)$-differentiated surface tension equation. Now, (1.26) follows from the Gronwall's inequality, and this concludes the proof of Theorem 1.3.

Now we prove the zero surface tension limit. Assume $(w, (b_0 \cdot \partial)\zeta, r)$ to be the solution of the incompressible MHD system without surface tension (1.15) and $(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, q^\sigma)$ to be the solution of (1.12) with $\sigma > 0$. Theorem 1.3 and Sobolev embedding implies that

$$\|v^\sigma\|^2_{C^1([0,T]\times\Omega)} + \|(b_0 \cdot \partial)\eta^\sigma\|^2_{C^1([0,T]\times\Omega)} + \|q^\sigma\|^2_{C^1([0,T]\times\Omega)} \lesssim C. \tag{8.6}$$

By Morrey's embedding, we can prove $v^\sigma, (b_0 \cdot \partial)\eta^\sigma, q^\sigma \in C^1_t H^4_y([0,T] \times \Omega) \hookrightarrow C^1_t C^{2,\frac{1}{2}}_y([0,T] \times \Omega)$, which implies the equi-continuity of $(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, q^\sigma)$ in $C^1([0,T] \times \Omega)$. By Arzelà-Ascoli lemma, we prove the uniform convergence (up to subsequence) of $(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, q^\sigma)$ as $\sigma \to 0_+$, and the limit is the solution $(w, (b_0 \cdot \partial)\zeta, r)$ to (1.15).

A Appendix: The initial data and compatibility conditions

We study the initial data under the general setting. Let η_0 be a smooth map but not necessarily the identity. In consequence, the initial interface $\Gamma(0)$ may not form a graph. The required regularity of the initial data is proved by solving the time-differentiated MHD system restricted at $\{t = 0\}$. Namely, given (η_0, v_0, b_0, q_0), and let $E(t)$ be defined as in (1.22). We show

$$E(0) \leq C, \tag{A.1}$$

where C depends on σ, η_0, v_0, b_0, and $\mathcal{H}_0$.

A.1 One time derivative

First, we control

$$\|\partial_t v(0)\|_4, \|\sqrt{\sigma}\partial_t v(0)\|_{4.5}, \|\partial_t b(0)\|_4, \|\sqrt{\sigma}\partial_t b(0)\|_{4.5}.$$

Since $\partial_t v(0) - (b_0 \cdot \partial)b_0 + \nabla_{A_0} q_0 = 0$ and $\partial_t b(0) = \partial_t (b_0 \cdot \partial)\eta(0) = (b_0 \cdot \partial)v_0$, we have

$$\|\partial_t v(0)\|_4 \lesssim \|b_0\|_4\|b_0\|_5 + \|\eta_0\|_5\|q_0\|_5, \quad \|\sqrt{\sigma}\partial_t v(0)\|_{4.5} \lesssim \|b_0\|_{4.5}\|\sqrt{\sigma}b_0\|_{5.5} + \|\eta_0\|_5\|\sqrt{\sigma}q_0\|_{5.5} + \|\sqrt{\sigma}\eta_0\|_{5.5}\|q_0\|_5, \tag{A.2}$$

and

$$\|\partial_t b(0)\|_4 \lesssim \|b_0\|_4\|v_0\|_5, \quad \|\sqrt{\sigma}\partial_t b(0)\|_{4.5} \lesssim \|b_0\|_{4.5}\|\sqrt{\sigma}v_0\|_{5.5}. \tag{A.3}$$

These imply that we need to bound $\|q_0\|_5$ and $\|\sqrt{\sigma}q_0\|_{5.5}$. Invoking the elliptic equation verified by q_0, i.e.,

$$\begin{cases} -\Delta_{A_0} q_0 = -(\partial_t A(0)^{\mu\alpha})\partial_\mu (v_0)_\alpha - ((b_0 \cdot \partial)A_0^{\mu\alpha})\partial_\mu (b_0)_\alpha + A_0^{\mu\alpha}(\partial_\mu b_0 \cdot \partial)(b_0)_\alpha, & \text{in } \Omega, \\ q_0 = \sigma\mathcal{H}_0, & \text{on } \Gamma(0), \\ \frac{\partial q_0}{\partial N} = 0, & \text{on } \Gamma_{\text{fix}}, \end{cases} \tag{A.4}$$

the elliptic estimate then yield

$$\|q_0\|_5 \leq C(\|\eta_0\|_4, \|v_0\|_4, \|b_0\|_4, \sigma|\mathcal{H}_0|_{4.5}), \quad \|\sqrt{\sigma}q_0\|_{5.5} \leq C(\|\eta_0\|_{4.5}, \|v_0\|_{4.5}, \|b_0\|_{4.5}, \sigma^{\frac{3}{2}}|\mathcal{H}_0|_5). \tag{A.5}$$

Remark. When $\eta_0 = \text{Id}$, we have $\mathcal{H}_0 = 0$, and so q_0 is determined by

$$\begin{cases} -\Delta q_0 = (\partial v_0) : (\partial v_0) + (\partial b_0) : (\partial b_0), & \text{in } \Omega, \\ q_0 = 0, & \text{on } \Gamma(0), \\ \frac{\partial q_0}{\partial N} = 0, & \text{on } \Gamma_{\text{fix}}, \end{cases} \tag{A.6}$$

which coincides with (1.13)-(1.14).

A.2 Two time derivatives

We adapt the same strategy to control

$$\|\partial_t^2 v(0)\|_3, \|\sqrt{\sigma}\partial_t^2 v(0)\|_{3.5}, \|\partial_t^2 b(0)\|_3, \|\sqrt{\sigma}\partial_t^2 b(0)\|_{3.5}.$$

Let $q_1 = \partial_t q(0)$. Since

$$\partial_t^2 v(0) - (b_0\cdot\partial)\partial_t b(0) + \nabla_{A_0} q_1 + \nabla_{\partial_t A(0)} q = 0,$$

and

$$\partial_t^2 b(0) = \partial_t^2 (b_0\cdot\partial)\eta(0) = (b_0\cdot\partial)\partial_t v(0),$$

then there hold

$$\|\partial_t^2 v(0)\|_3 \lesssim \|b_0\|_3\|\partial_t b(0)\|_4 + \|\eta_0\|_4\|q_1\|_4 + \|\eta_0\|_4\|v_0\|_4\|q\|_4, \tag{A.7}$$

$$\|\sqrt{\sigma}\partial_t v(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|\sqrt{\sigma}\partial_t b(0)\|_{4.5} + \|\eta_0\|_{4.5}\|\sqrt{\sigma} q_1\|_{4.5} + \sqrt{\sigma}\|\eta_0\|_{4.5}\|v_0\|_{4.5}\|q\|_{4.5}, \tag{A.8}$$

$$\|\partial_t^2 b(0)\|_3 \lesssim \|b_0\|_3\|\partial_t v(0)\|_4, \tag{A.9}$$

$$\|\sqrt{\sigma}\partial_t^2 b(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|\sqrt{\sigma}\partial_t v(0)\|_{4.5}. \tag{A.10}$$

It can be seen that we still need to control $\|q_1\|_4$ and $\|\sqrt{\sigma} q_1\|_{4.5}$. Invoking the elliptic equation satisfied by q_1,

$$\begin{aligned} -\Delta_{A_0} q_1 = &-(\partial_t^2 A(0)^{\mu\alpha})\partial_\mu (v_0)_\alpha - (\partial_t A(0)^{\mu\alpha})\partial_\mu(\partial_t v(0))_\alpha - ((b_0\cdot\partial)\partial_t A(0)^{\mu\alpha})\partial_\mu (b_0)_\alpha - ((b_0\cdot\partial)A_0^{\mu\alpha})\partial_\mu(\partial_t b(0))_\alpha \\ &+ \partial_t A(0)^{\mu\alpha}(\partial_\mu b_0\cdot\partial)(b_0)_\alpha + A_0^{\mu\alpha}(\partial_\mu b_0\cdot\partial)(\partial_t b(0))_\alpha, \end{aligned} \tag{A.11}$$

with the boundary conditions

$$\begin{aligned} q_1 &= \sigma\partial_t\mathcal{H}(0), \quad \text{on } \Gamma(0), \\ \frac{\partial q_1}{\partial N} &= 0, \quad \text{on } \Gamma_{\text{fix}}, \end{aligned}$$

we have

$$\|q_1\|_4 \le C(\|\eta_0\|_4, \|v_0\|_4, \|\partial_t v(0)\|_3, \|b_0\|_4, \|\partial_t b(0)\|_3, \|q_0\|_4, \sigma|\partial_t\mathcal{H}(0)|_{3.5}), \tag{A.12}$$

and

$$\|\sqrt{\sigma} q_1\|_{4.5} \le C(\|\eta_0\|_{4.5}, \|v_0\|_{4.5}, \|\partial_t v(0)\|_{3.5}, \|b_0\|_{4.5}, \|\partial_t b(0)\|_{3.5}, \|q_0\|_{4.5}, \sigma^{3/2}|\partial_t\mathcal{H}(0)|_4). \tag{A.13}$$

Since $\mathcal{H} = -\Delta_g \eta\cdot\hat{n}$, the control of $\sigma|\partial_t\mathcal{H}(0)|_{3.5}$ and $\sigma^{3/2}|\partial_t\mathcal{H}(0)|_4$ require that

$$\sigma|\Delta_{g_0} v_0\cdot\hat{n}_0|_{3.5}, \quad \sigma^{3/2}|\Delta_{g_0} v_0\cdot\hat{n}_0|_4 \tag{A.14}$$

to be bounded, respectively. Also, $\Delta_{g_0} v_0\cdot\hat{n}_0 = \overline{\Delta} v_0^3$ if $\eta_0 = \text{Id}$, which agrees with (2.) in Theorem 1.1.

A.3 Three and more time derivatives

The quantities

$$\|\partial_t^k v(0)\|_{5-k}, \|\sqrt{\sigma}\partial_t^k v(0)\|_{5.5-k}, \|\partial_t^k b(0)\|_{5-k}, \|\sqrt{\sigma}\partial_t^k b(0)\|_{5.5-k}, \quad k = 3, 4, 5,$$

are treated similarly by adapting the above arguments, and thus we shall only list the major differences. We need to control $\|q_2\|_3$ and $\|\sqrt{\sigma} q_2\|_{3.5}$ when $k = 3$, which require the bounds for $\sigma|\partial_t^2\mathcal{H}(0)|_{2.5}$ and $\sigma^{3/2}|\partial_t^2\mathcal{H}(0)|_3$, respectively. Analogous to (A.14), we require the boundness of

$$\sigma|\Delta_{g_0}(b_0\cdot\partial)b_0\cdot\hat{n}_0|_{2.5}, \quad \sigma^{3/2}|\Delta_{g_0}(b_0\cdot\partial)b_0\cdot\hat{n}_0|_3.$$

Also, since $(b_0\cdot\partial)$ is a tangential derivative on Γ, the above quantities reduce to

$$\sigma|\overline{\Delta} b_0^3|_{3.5}, \quad \sigma^{3/2}|\overline{\Delta} b_0^3|_4, \tag{A.15}$$

up to the leading order when $\eta_0 = \mathrm{Id}$. This is just (2.) in Theorem 1.1.

This procedure stops here since the next time derivative will require the boundness of

$$\sigma|\Delta_{g_0}(b_0\cdot\partial)\partial_t b(0)\cdot\hat{n}_0|_{1.5}, \quad \sigma^{3/2}|\Delta_{g_0}(b_0\cdot\partial)\partial_t b(0)\cdot\hat{n}_0|_2$$

at the top order. But thanks to the identity $b_0 = (b_0\cdot\partial)\eta_0$, the above quantities are turned into

$$\sigma|\Delta_{g_0}(b_0\cdot\partial)^2 v_0\cdot\hat{n}_0|_{1.5}, \sigma^{3/2}|\Delta_{g_0}(b_0\cdot\partial)^2 v_0\cdot\hat{n}_0|_2,$$

which are (A.14) at the leading order.

A.4 The compatibility condition

The above analysis yields that the initial data of the $\sigma > 0$ problem has to satisfy the compatibility conditions up to the fourth-order, where the j-th order ($0\leq j\leq 4$) compatibility conditions read

$$\begin{aligned} q_j = \sigma\partial_t^j\mathcal{H}(0), &\quad \text{on } \Gamma(0),\\ \partial_3 q_j = 0, &\quad \text{on } \Gamma_{\text{fix}}. \end{aligned} \tag{A.16}$$

Furthermore, this can be carried over to the $\sigma = 0$ problem, namely, the first line is replaced by $q_j = 0$ on $\Gamma(0)$.

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