Local Well-posedness for the Motion of a Compressible Gravity Water Wave with Vorticity

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Abstract

In this paper we prove the local well-posedness for the 3D compressible Euler equations describing the motion of a liquid in an unbounded initial domain with moving physical vacuum boundary. The liquid is under the influence of gravity but without surface tension, and it is not assumed to be irrotational. We apply the tangential smoothing method introduced in Coutand-Shkoller [10, 11] to construct the approximation system with energy estimates uniform in the smooth parameter. It should be emphasized that, when doing the a priori estimates, we need neither the higher order wave equation of the pressure and delicate elliptic estimates, nor the higher regularity assumption on the initial vorticity. Instead, motivated by Gu-Wang [24] we generalize the Alinhac good unknown method to the estimates of full spatial derivatives. This technique is widely used in the study of free-boundary problems of incompressible fluids but seldomly for compressible fluids before.

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1 Introduction

In this paper we study the motion of a compressible gravity water wave in $\mathbb{R}^3$ described by the compressible Euler equations:

$$\begin{cases} D_t u := (\partial_t + u\cdot\nabla)u = -\frac{1}{\rho}\nabla p - ge_3, & \text{in } \mathcal{D} \\ D_t\rho + \rho \text{div}\, u = 0 & \text{in } \mathcal{D} \\ p = p(\rho) & \text{in } \mathcal{D} \end{cases} \tag{1.1}$$

where $\mathcal{D} = \bigcup_{0\le t\le T}\{t\}\times\mathcal{D}_t$ with $\mathcal{D}_t := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 \le \Sigma(t,x_1,x_2)\}$ representing the unbounded domain occupied by the fluid at each fixed time t, whose boundary $\partial\mathcal{D}_t = \{(x_1,x_2,x_3) : x_3 = \Sigma(t,x_1,x_2)\}$ moves with the velocity of the fluid. In (1.1), u, ρ, p represent the fluid velocity, density and pressure, respectively, and $g > 0$ is the gravity constant. The third equation of (1.1) is known to be the equation of states which satisfies

$$p'(\rho) > 0, \quad \text{for } \rho \ge \bar\rho_0, \tag{1.2}$$

where $\bar\rho_0 := \rho|_{\partial\mathcal{D}}$ is a positive constant (we set $\bar\rho_0 = 1$ for simplicity), which is in the case of a barotropic liquid. The equation of states is required in order to close the system of compressible Euler equations.

The initial and boundary conditions of the system (1.1) are

$$\mathcal{D}_0 = \{x : (0,x)\in\mathcal{D}\}, \quad \text{and } u = u_0, \rho = \rho_0 \quad \text{on } \{0\}\times\mathcal{D}_0, \tag{1.3}$$

$$D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}) \quad \text{and } p|_{\partial\mathcal{D}} = 0, \tag{1.4}$$

where $T(\partial\mathcal{D})$ stands for the tangent bundle of $\partial\mathcal{D}$. We introduce the new variable $h = h(\rho) := \int_1^\rho p'(\lambda)\lambda^{-1}\, d\lambda$, which is known to be the enthalpy of the fluid. It can be seen that $h'(\rho) > 0$ and $h|_{\partial\mathcal{D}} = 0$ thanks to (1.2). Since ρ can then be thought as a function of h, we define $e(h) := \log\rho(h)$. Under these new variables, (1.1) and (1.3)-(1.4) becomes

$$\begin{cases} D_t u = -\nabla h - ge_3, & \text{in } \mathcal{D}, \\ \text{div}\, u = -D_t e(h), & \text{in } \mathcal{D}, \\ \mathcal{D}_0 = \{x : (0,x)\in\mathcal{D}\}, & \\ u = u_0, h = h_0 & \text{on } \{0\}\times\mathcal{D}_0, \\ D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}) \quad \text{and } h = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{1.5}$$

The system (1.5) looks exactly like the incompressible Euler equations, where h takes the position of p but $\text{div}\, u$ is no longer 0 but determined as a function of ρ (and hence h). In addition, in Ebin [17], the free-boundary problem (1.5) is known to be ill-posed unless the physical sign condition (also known as the Taylor sign condition)

$$-\nabla_{\mathcal{N}} h \ge c_0 > 0, \quad \text{on } \partial\mathcal{D}_t. \tag{1.6}$$

holds. Here, $\mathcal{N}$ is the outward unit normal of $\partial\mathcal{D}_t$ and $\nabla_{\mathcal{N}} := \mathcal{N}\cdot\nabla$. The condition (1.6) is a natural physical condition which says that the enthalpy and hence the pressure and density is larger in the interior than on the boundary. We remark here that (1.6) can be derived by the strong maximum principle if the water wave is assumed to be irrotational [60, 61, 44], and the existence of the positive constant c_0 is a consequence of the presence of the gravity. Otherwise, we have merely that $-\nabla_{\mathcal{N}} h > 0$, which is insufficient to close the a priori energy estimate for (1.5).

We would like to impose the following natural conditions on $e(h)$: For each fixed $k \ge 1$, there exists a constant $C > 0$ such that

$$C^{-1} \le |e^{(k)}(h)| \le C. \tag{1.7}$$

In fact, (1.7) holds true if the equation of states is given by

$$p(\rho) = \gamma^{-1}(\rho^\gamma - 1), \quad \gamma \ge 1. \tag{1.8}$$

In particular, when $\gamma = 1$, a directly computation yields that $e(h) = h$.

Finally, in order for the initial boundary value problem (1.5)-(1.6) to be solvable the initial data has to satisfy certain compatibility conditions at the boundary. Since $h|_{\partial\mathcal{D}} = 0$ and $D_t \in T(\partial\mathcal{D})$, the second equation of (1.5) implies that $\operatorname{div} u|_{\partial\mathcal{D}} = 0$. We must therefore have $h_0|_{\partial\mathcal{D}_0} = 0$ and $\operatorname{div} u_0|_{\partial\mathcal{D}_0} = 0$, which is the zero-th compatibility condition. In general, for each $k \geq 0$, the k-th order compatibility condition reads

$$D_t^k h|_{\{0\}\times\partial\mathcal{D}_0} = 0, \quad k = 0, 1, \cdots, k. \tag{1.9}$$

In [44] Sect. 7, we have proved that for each fixed $k \geq 0$, there exists initial data verifying the compatibility condition up to order k such that the initial energy norm is bounded.

The study of the motion of a fluid has a long history in mathematics, and the study of the free-boundary problems has blossomed over the past two decades or so. However, much of this activity has focus on incompressible fluid models, i.e., the velocity vector field satisfies $\operatorname{div} u = 0$ and the density ρ is fixed to be a constant. Also, the pressure p is not determined by the equation of states. Rather, it is a Lagrange multiplier enforcing the divergence free constraint. It is worth mentioning here that when the fluid domain is unbounded and the velocity u_0 is irrotational (i.e., $\operatorname{curl} u_0 = 0$, a condition that preserved by the evolution), this problem is called the (incompressible) water wave problem, which has received a great deal of attention. The local well-posedness (LWP) for the free-boundary incompressible Euler equations in either bounded or unbounded domains have been studied in [1, 5, 6, 7, 10, 11, 15, 27, 37, 38, 40, 43, 48, 49, 50, 51, 52, 57, 59, 60, 61, 64, 65]. In addition, the long time well-posedness for the water wave problem with small initial data is available in [2, 4, 13, 18, 19, 26, 28, 30, 31, 32, 33, 47, 58, 62, 63, 66], and there are recent results concern the life-span for the water wave problem with vorticity [20, 29, 53].

On the other hand, much less is known for the free-boundary compressible Euler equations, especially for the ones modeling a liquid, as opposed to a gas whose density can be zero on the moving boundary. The LWP for the free-boundary compressible gas model was obtained in [9, 12, 25, 34, 35, 45], whereas for suitable initial data (e.g., data satisfying the compatibility condition), the LWP for the free-boundary compressible liquid model with *a bounded* fluid domain is available in [8, 14, 16, 22, 39, 41, 42]. Also, the problem is known to be the compressible water wave problem if the fluid (liquid) domain is unbounded, and little is known for this case. The only existence result is due to Trakhinin [56], who proved the LWP for the compressible gravity water wave with vorticity using the Nash-Moser iteration [1]. Recently, Luo [44] established the a priori energy estimates for the compressible gravity water wave with vorticity and proved the incompressible limit by adapting the the approach used in Lindblad-Luo [42] to an unbounded domain.

The goal of this paper is to prove the LWP for the motion of a compressible gravity water wave using the classical approach. The main idea is to approximate the nonlinear compressible water wave problem in the Lagrangian coordinates using a sequence of "tangentially smoothed" problems, whose solutions converge to that of the original problem when the smoothing coefficient goes to 0. This in the incompressible free-boundary Euler equations goes back to Coutand-Shkoller [10]. Also, for its application in the compressible free-boundary Euler equations modeling a liquid in a bounded domain, Coutand-Hole-Shkoller [8] obtained the LWP for the case with surface tension and Ginsberg-Lindblad-Luo [22] obtained the LWP for the self-gravitating liquid. However, here we use a different set of approximate problems by adapting what appears in [22] which yields a simpler construction of the sequence of approximate solutions. This will be discussed in Sect. 1.3.

1.1 The Lagrangian coordinates

We introduce the Lagrangian coordinates, under which the moving boundary becomes fixed. Let $\Omega := \mathbb{R}^2 \times (-\infty, 0)$ to be the lower half space of $\mathbb{R}^3$. Denoting coordinates on Ω by $y = (y_1, y_2, y_3)$, we define $\eta : [0, T] \times \Omega \to \mathcal{D}$ to be the flow map of u, i.e.,

$$\partial_t \eta(t, y) = u(t, \eta(t, y)), \quad \eta(0, y) = y. \tag{1.10}$$

It is not hard to see that in the (t, y) coordinates $D_t = \partial_t$ and the boundary $\Gamma := \partial\Omega$ becomes fixed (i.e., $\Gamma = \mathbb{R}^2$). We introduce the Lagrangian velocity by $v(t, y) := u(t, \eta(t, y))$, and denote the Lagrangian enthalpy $h(t, y) := h(t, \eta(t, y))$ by a slight abuse of notations.

[1] In fact, Trakhinin studied a compressible gas in an unbounded domain and he claimed that his approach holds true also for the liquid case.

Let $\partial = \partial_y$ be the spatial derivative in the Lagrangian coordinates. We introduce the matrix $a = (\partial\eta)^{-1}$, specifically $a^{\mu\alpha} = a^{\mu}_{\alpha} := \frac{\partial y^\mu}{\partial \eta^\alpha}$. This is well-defined since $\eta(t,\cdot)$ is almost an identity map whenever t is sufficiently small. In terms of v, h and a, (1.5)-(1.6) becomes

$$\begin{cases} \partial_t v^\alpha = -\nabla_a^\alpha h - ge_3, & \text{in } [0,T]\times\Omega, \\ \operatorname{div}_a v = -\partial_t e(h), & \text{in } [0,T]\times\Omega, \\ \eta = \mathrm{Id}, v = v_0, h = h_0 & \text{on } \{0\}\times\Omega, \\ \partial_t|_{[0,T]\times\Gamma} \in T([0,T]\times\Gamma) & \\ h = 0 & \text{on } \Gamma. \end{cases} \tag{1.11}$$

Here, $\nabla_a^\alpha = a^{\mu\alpha}\partial_\mu$ and $\operatorname{div}_a v = \nabla_a \cdot v = a^{\mu\alpha}\partial_\mu v_\alpha$, where the summation convention is used for repeated upper and lower indices, and in above and throughout, we adopt the convention that the Greek indices range over $1,2,3$, and the Latin indices range over 1 and 2. In addition, since $\eta(0,\cdot) = \mathrm{Id}$, we have $a(0,\cdot) = I$, where I is the identity matrix, and u_0 and v_0 agree. Furthermore, let $J := \det(\partial\eta)$. Then J satisfies

$$\partial_t J = Ja^{\mu\alpha}\partial_\mu v_\alpha. \tag{1.12}$$

Finally, we assume the physical sign condition holds initially

$$-\partial_3 h_0 \geq c_0 > 0 \tag{1.13}$$

and it can shown that (1.13) propagates to a later time.

1.2 The main result

The goal of this paper is to prove the LWP of the compressible gravity water wave system in the Lagrangian coordinates. Specifically, we want to construct a solution to (1.11) with localized initial data (v_0, h_0), i.e., $|v_0(y)| \to 0$ and $|h_0(y)| \to 0$ as $|y| \to \infty$ that satisfies the compatibility condition (1.9) up to order 5 as well as (1.13). The localized data is required so that the initial L^2-based higher order energy functional is bounded and the existence of such data can be found in [44], Sect. 7. Also, we remark here that (1.13) remains hold thanks to the presence of the gravity ([44], Sect. 7).

Definition 1.1. We define the higher order energy functional

$$\mathcal{E}(t) = \|\partial\eta\|^2_{L^\infty(\Omega)} + \|\partial^2\eta\|^2_{H^2(\Omega)} + \sum_{k=0}^{4}(\|\partial_t^{4-k}v\|^2_{H^k(\Omega)} + \|\partial_t^{4-k}h\|^2_{H^k(\Omega)}) + |a^{3\alpha}\overline{\partial}^4\eta_\alpha|^2_{L^2(\Gamma)}, \tag{1.14}$$

where $\overline{\partial} = (\partial_1, \partial_2)$ is the tangential Lagrangian spatial derivative.

Remark. The term $\|\partial\eta\|^2_{L^\infty(\Omega)} + \|\partial^2\eta\|^2_{H^2(\Omega)}$ can be replaced by $\|\eta\|^2_{H^4(\Omega)}$ if Ω is bounded. However, we have to be more careful in the case of an unbounded Ω since neither η nor $\partial\eta$ are in $L^2(\Omega)$, which is due to that $\eta_0 = \mathrm{Id}$. However, the requirement of $\|\eta\|^2_{H^4(\Omega)}$ can in fact be weakened to $\|\partial\eta\|^2_{L^\infty(\Omega)} + \|\partial^2\eta\|^2_{H^2(\Omega)}$. This is because that the energy estimate requires the control of $\|a\|_{L^\infty(\Omega)}$ and $\|\partial a\|_{H^2(\Omega)}$, which can be controlled by $\|\partial\eta\|_{L^\infty(\Omega)}$ and $\|\partial^2\eta\|_{H^2(\Omega)}$, respectively.

Theorem 1.1. Suppose that the initial data $v_0, h_0 \in H^4(\Omega)$ satisfies the compatibility condition (1.9) up to order 5 and the physical sign condition (1.13). Then there exists a $T_0 > 0$ and a unique solution (η, v, h) to (1.11) on the time interval $[0, T_0]$ which satisfies

$$\sup_{t\in[0,T_0]} \mathcal{E}(t) \leq P(\|v_0\|_{H^4(\Omega)}, \|h_0\|_{H^4(\Omega)}), \tag{1.15}$$

where P is a generic polynomial.

1.3 Strategy of the proof

The strategy of proving Theorem (1.1) contains three parts:

1. The a priori energy estimates in certain functional spaces.
2. A suitable approximate problem which is asymptotically consistent with the a priori estimate.
3. Construction of solutions to the approximate problem.

These steps are highly nontrivial in the case of a compressible water wave thanks to the nontrivial divergence of the velocity field and the unbounded fluid domain. The rest of this section is devoted to the elaboration of these steps. Also, we assume $e(h) = h$ in the rest of this section for the sake of simple exposition. But general $e(h)$ will be studied in the paper.

Construction for the approximate problem:

We approximate (1.11) by considering the "tangentially-smoothed" equations, and the smoothing operator is defined as follows. Let $\zeta = \zeta(y_1, y_2) \in C_c^\infty(\mathbb{R}^2)$ be the standard cut-off function such that $\text{Spt}\,\zeta = \overline{B(0,1)} \subseteq \mathbb{R}^2$, $0 \le \zeta \le 1$ and $\int_{\mathbb{R}^2} \zeta = 1$. The corresponding dilation is

$$\zeta_\kappa(y_1, y_2) = \frac{1}{\kappa^2}\zeta\left(\frac{y_1}{\kappa}, \frac{y_2}{\kappa}\right), \quad \kappa > 0,$$

and we define the smoothing operator as

$$\Lambda_\kappa f(y_1, y_2, y_3) := \int_{\mathbb{R}^2} \zeta_\kappa(y_1 - z_1, y_2 - z_2) f(z_1, z_2, z_3)\, dz_1\, dz_2. \tag{1.16}$$

Let $\tilde{a} = \tilde{a}(\tilde{\eta})$ be the smoothed version of a with $\tilde{\eta} := \Lambda_\kappa^2 \eta$. We define the approximate system by replacing a by $\tilde{a}$ in (1.11), i.e.,

$$\begin{cases} \partial_t \eta = v + \psi & \text{in } \Omega, \\ \partial_t v^\alpha = -\nabla_{\tilde{a}}^\alpha h - g e_3 & \text{in } \Omega, \\ \text{div}_{\tilde{a}} v = -\partial_t h & \text{in } \Omega, \\ h = 0 & \text{on } \Gamma, \\ (\eta, v, h)|_{\{t=0\}} = (\text{Id}, v_0, h_0), \end{cases} \tag{1.17}$$

where $\psi = \psi(\eta, v)$ is a correction term which solves the half-space Laplace equation

$$\begin{cases} \Delta \psi = 0 & \text{in } \Omega, \\ \psi = \overline{\Delta}^{-1}\mathbb{P}\left(\overline{\Delta}\eta_\beta \tilde{a}^{i\beta}\overline{\partial}_i \Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2 \eta_\beta \tilde{a}^{i\beta}\overline{\partial}_i v\right) & \text{on } \Gamma, \end{cases} \tag{1.18}$$

where $\overline{\Delta} := \partial_1^2 + \partial_2^2$ is the tangential Laplacian operator and $\overline{\Delta}^{-1} f := (|\xi|^{-2}\hat{f})^\vee$ is the inverse of $\overline{\Delta}$ on $\mathbb{R}^2$. The index β ranges from 1 to 3 and i ranges from 1 to 2, as stated after (1.11). The notation $\mathbb{P}f := P_{\ge 1} f$ denotes the standard Littlewood-Paley projection in $\mathbb{R}^2$ which removes the low-frequency part, i.e.,

$$P_{\ge 1} f := ((1 - \chi(\xi))\hat{f}(\xi))^\vee,$$

where $0 \le \chi(\xi) \le 1$ is a $C_c^\infty(\mathbb{R}^d)$ cut-off function which is supported in $\{|\xi| \le 2\}$ and equals to 1 in $\{|\xi| \le 1\}$. The Littlewood-Paley projection $\mathbb{P}$ is necessary when we apply the elliptic estimates to control ψ:

$$\|\psi\|_{H^{3.5}(\partial\Omega)} = \|\overline{\Delta}^{-1}\mathbb{P}f\|_{H^{3.5}(\partial\Omega)} \lesssim \|f\|_{H^{1.5}(\partial\Omega)},$$

otherwise the low-frequency part of $\overline{\Delta}^{-1} f$ loses control. Also, we remark here that the correction term $\psi \to 0$ as $\kappa \to 0$.

In [22], the compressible Euler equations are approximated by a "fully smoothed system", in the sense that all variables are replaced by their smoothed version. It can be seen that we smoothed the nonlinear coefficients (i.e., replacing $a^{\mu\alpha}$ by $\tilde{a}^{\mu\alpha}$) only in (1.17). The advantage of doing this has two folds:

- The existence of the approximate system (1.17) can be obtained by passing to the limit as $n \to \infty$ in a sequence of approximate solutions $(\eta^{(n)}, v^{(n)}, h^{(n)})$ which are constructed by solving a linearized version of (1.17) (see (1.25)).

- We do not need to construct the initial data for each linear approximate system as what was done in [22].

On the other hand, the appearance of the correction term ψ (which was first introduced by Gu and Wang in [24]) is crucial in order to eliminate the higher order terms on the boundary when performing the tangential energy estimate, which shall be explained in the following paragraph.

Discussion on the tangential energy estimate

The crucial part of the a priori energy estimates for the approximate system (1.17) is the estimate for the tangential part of the energy. In particular, the top order tangential energy with full spatial derivatives reads

$$\underbrace{\|\overline{\partial}^4 v\|^2_{L^2(\Omega)} + \|\overline{\partial}^4 h\|^2_{L^2(\Omega)}}_{=\mathcal{E}_{TI}} + \underbrace{|\tilde{a}^{3\alpha}\overline{\partial}^4 \Lambda_\kappa \eta_\alpha|^2_{L^2(\Gamma)}}_{=\mathcal{E}_{TB}}. \tag{1.19}$$

The control of $\frac{d}{dt}\mathcal{E}_{TI}$ requires the control of $\overline{\partial}^4\tilde{a}$ in $L^2(\Omega)$. In [22], this is treated by adding κ-weighted higher order terms to the energy, which corresponds to the fifth order full spatial energy of the wave equation verified by h. In this paper, however, we adapt the Alinhac's good unknowns for v and h, i.e., $\mathbf{V} := \overline{\partial}^2\overline{\Delta} v - \overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} v$, and $\mathbf{H} := \overline{\partial}^2\overline{\Delta} h - \overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} h$. to avoid the extra higher order terms when commuting $\nabla_{\tilde{a}}$ with $\overline{\partial}^4$. See (3.76)-(3.77) for details. Such a remarkable observation is due to Alinhac [3]. Now $\mathbf{V}$ and $\mathbf{H}$ satisfy

$$\partial_t \mathbf{V} = -\nabla_{\tilde{a}}\mathbf{H} + \text{error}, \quad \nabla_{\tilde{a}}\cdot\mathbf{V} = \overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\tilde{a}} v) + \text{error}, \quad \text{in } \Omega \tag{1.20}$$

$$\mathbf{H} = -\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta \tilde{a}^{3\beta}\partial_3 h \quad \text{on } \Gamma, \tag{1.21}$$

multiplying $\mathbf{V}$ through (1.20), integrating over Ω and integrating $\nabla_{\tilde{a}}$ in $-\int_\Omega \nabla_{\tilde{a}}\mathbf{H}\cdot\mathbf{V}$ by parts yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2 = \int_\Omega \mathbf{H}\overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\tilde{a}} v)\,dy + \int_\Gamma \partial_3 h\tilde{a}^{3\beta}\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta \mathbf{V}_\alpha\,dS + \text{error}. \tag{1.22}$$

Here, $\|\mathbf{V}\|^2_{L^2(\Omega)}$ bounds $\|\overline{\partial}^4 v\|^2_{L^2(\Omega)}$, and the first term on the RHS is equal to $-\frac{d}{dt}\frac{1}{2}\|\overline{\partial}^2\overline{\Delta} h\|^2_{L^2(\Omega)}$ modulo error, where $\|\overline{\partial}^2\overline{\Delta} h\|_{L^2(\Omega)} \approx \|\overline{\partial}^4 h\|_{L^2(\Omega)}$. In addition, the second term on the RHS is equal to

$$\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta(\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha - \overline{\partial}^2\overline{\Delta}\psi - \overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} v_\alpha)\,dS \tag{1.23}$$

by plugging the definition of $\mathbf{V}$ and invoking the first equation of (1.17). Now, after "moving" one Λ_κ from $\tilde{\eta}_\beta$ to η_α, we have

$$\begin{aligned}
&\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha \\
=&\frac{1}{2}\frac{d}{dt}\int_\Gamma \partial_3 h|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|^2\,dS \\
&+\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\gamma\tilde{a}^{i\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS - \int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma\tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha\,dS + \text{error}.
\end{aligned} \tag{1.24}$$

The higher order terms on the second line are cancelled out for the original problem (i.e., $\kappa = 0$) but we do not have the cancellation when $\kappa > 0$. However, both of them can indeed be cancelled by the term

$$\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\psi\,dS$$

in (1.23) up to lower order terms plus the low-frequency term

$$\overline{\partial}^2 \left((\mathrm{Id} - \mathbb{P}) \left(\overline{\Delta} \eta_\beta \tilde{a}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 v - \overline{\Delta} \Lambda_\kappa^2 \eta_\beta \tilde{a}^{i\beta} \overline{\partial}_i v \right) \right),$$

which can be controlled by using Bernstein's inequality (2.3) in Lemma 2.3. Hence, the issue above is resolved. See Section 3.6.3 for details.

Finally, we mention here that the use of Alinhac good unknowns has been widely adapted to the control of spatial derivatives in the study of incompressible fluids, see e.g., [24, 46, 59]. However, to the best of our knowledge, there appears to be no previous application of the Alinhac good unknowns to the control of spatial derivatives in the study of compressible Euler equations. Nevertheless, we mention here that Ginsberg-Lindblad [21] have adapted these good unknowns study the LWP for the free-boundary relativistic Euler equations in a bounded space-time domain.

Discussion on the existence of the approximate system

The approximate system (1.17) can be solved by an iteration of the approximate solutions. Specifically, let $(\eta^{(0)}, v^{(0)}, h^{(0)}) = (\eta^{(1)}, v^{(1)}, h^{(1)}) = (\mathrm{Id}, 0, 0)$ (i.e., the trivial solution). For each $n \geq 1$, we define $(\eta^{(n+1)}, v^{(n+1)}, h^{(n+1)})$ be the solution of the linear system of equations

$$\begin{cases} \partial_t \eta^{(n+1)} = v^{(n+1)} + \psi^{(n)} & \text{in } \Omega, \\ \partial_t v^{(n+1)} = -\nabla_{\tilde{a}^{(n)}} h^{(n+1)} - g e_3 & \text{in } \Omega, \\ \mathrm{div}_{\tilde{a}^{(n)}} v^{(n+1)} = -\partial_t h^{(n+1)} & \text{in } \Omega, \\ h^{(n+1)} = 0 & \text{on } \Gamma, \\ (\eta^{(n+1)}, v^{(n+1)}, h^{(n+1)})|_{t=0} = (\mathrm{Id}, v_0, h_0), \end{cases} \tag{1.25}$$

Here, $a^{(n)} := [\partial \eta^{(n)}]^{-1}$, $\tilde{a}^{(n)} := \Lambda_\kappa^2 a^{(n)}$ and the correction term $\psi^{(n)}$ is determined by (1.18) with $\eta = \eta^{(n)}, v = v^{(n)}, \tilde{a} = \tilde{a}^{(n)}$. The existence of $(\eta^{(n+1)}, v^{(n+1)}, h^{(n+1)})$ follows from showing that the map $\Xi : \mathbb{X} \to \mathbb{X}$ (defined below) has a fixed point, where the Banach space $\mathbb{X}$ define as

$$\mathbb{X} = \left\{ (\xi, w, \pi) : (w, \xi)|_{t=0} = (v_0, \mathrm{Id}), \sup_{t \in [0,T]} \left(\|w, \partial_t \pi, \nabla_{\tilde{a}^{(n)}} \pi\|_{Z^4} + \|\partial_t \xi\|_{Z^3} + \|\partial^2 \xi\|_{Z^2} + \|\partial \xi\|_{L^\infty} \right) \leq M \right\}. \tag{1.26}$$

Here, Z^k denotes the mixed space-time L^2-Sobolev norm of order $\leq k$.The map Ξ is given by

$$\Xi(\xi, w, \pi) = (\eta^{(n+1)}, v^{(n+1)}, h^{(n+1)})$$

where we define $\eta^{(n+1)}$, $v^{(n+1)}$ and $h^{(n+1)}$, respectively, by

$$\partial_t \eta^{(n+1)} = w + \psi^{(n)}, \eta^{(n+1)}(0) = \mathrm{Id}, \tag{1.27}$$

$$\partial_t v^{(n+1)} = -\nabla_{\tilde{a}^{(n)}} \pi - g e_3, v^{(n+1)}(0) = v_0, \tag{1.28}$$

and

$$\begin{cases} \partial_t^2 h^{(n+1)} - \Delta_{\tilde{a}^{(n)}} h^{(n+1)} = \partial_t \tilde{a}^{\nu\alpha}_{(n)} \partial_\nu v_\alpha^{(n+1)} & \text{in } \Omega, \\ h^{(n+1)} = 0 & \text{on } \Gamma, \\ (h^{(n+1)}, \partial_t h^{(n+1)})|_{t=0} = (h_0, h_1). \end{cases} \tag{1.29}$$

The estimates for $\eta^{(n+1)}$ and $v^{(n+1)}$ are straightforward since they verify transport equations. However, the estimate for $\|\nabla_{\tilde{a}^{(n)}} h^{(n+1)}\|_{Z^4}$ requires that of $\|\nabla_{\tilde{a}^{(n)}} h^{(n+1)}\|_{H^4}$ which cannot be done directly by commuting $\overline{\partial}^4$ through the wave equation, since there is no hope to control the corresponding source term consists $\partial_t \tilde{a}^{\nu\alpha}_{(n)} (\overline{\partial}^4 \partial_\nu v_\alpha^{(n+1)})$ in L^2.

The key observation here is that $\overline{\partial}^3 \partial_t \partial_\nu v_\alpha^{(n+1)}$ can in fact be controlled thanks to (1.28) and the finiteness of $\|\nabla_{\tilde{a}^{(n)}} \pi\|_{Z^4}$, and so there is no problem to control the wave energies by commuting $\mathcal{D}^3 \partial_t$ (where $\mathcal{D} = \overline{\partial}$

or ∂_t) through the wave equation (1.29). Now, the remaining $\|\nabla_{\tilde{a}^{(n)}} h^{(n+1)}\|_{H^4}$ can be treated using the elliptic estimate derived in [22]

$$\|\nabla_{\tilde{a}} f\|_{H^r} \leq C(\|\partial\tilde{\eta}\|_{L^\infty}, \|\partial^2\tilde{\eta}\|_{H^{r-2}})\Big(\|\Delta_{\tilde{a}} f\|_{H^{r-1}} + \|\overline{\partial}\partial\tilde{\eta}\|_{H^{r-1}}\|f\|_{H^r(\Omega)}\Big). \tag{1.30}$$

Plugging $f = h$, $\tilde{a} = \tilde{a}^{(n)}$ and $\tilde{\eta} = \tilde{\eta}^{(n)}$, we have that the control of $\|\nabla_{\tilde{a}^{(n)}} h^{(n+1)}\|_{H^4}$ requires that of $\|\Delta_{\tilde{a}^{(n)}} h^{(n+1)}\|_{H^3}$ up to the highest order. But this term is under control since (1.29) suggests that

$$\|\Delta_{\tilde{a}^{(n)}} h^{(n+1)}\|_{H^3} \leq \|\partial_t^2 h^{(n+1)}\|_{H^3} + \|\partial_t \tilde{a}^{\nu\alpha}_{(n)} \partial_\nu v_\alpha^{(n+1)}\|_{H^3},$$

where the second term is of lower order and the first term can be controlled by invoking the wave energy with 2 time derivatives.

The following list of notations will be adopted for the rest of this paper.

List of Notations:

- $\Omega := \mathbb{R}^2 \times (-\infty, 0)$ and $\Gamma := \mathbb{R}^2 \times \{0\}$.
- $\|\cdot\|_s$: We denote $\|f\|_s := \|f(t,\cdot)\|_{H^s(\Omega)}$ for any function $f(t,y)$ on $[0,T] \times \Omega$.
- $|\cdot|_s$: We denote $|f|_s := |f(t,\cdot)|_{H^s(\Gamma)}$ for any function $f(t,y)$ on $[0,T] \times \Gamma$.
- $\|\cdot\|_{\dot{H}^s}, |\cdot|_{\dot{H}^s}$: Homogeneous Sobolev norm, replacing H^s above by $\dot{H}^s$.
- $P(\cdots)$: A generic polynomial in its arguments;
- $\mathcal{P}_0$: $\mathcal{P}_0 = P(\|v_0\|_4, \|h_0\|_4)$;
- $[T, f]g := T(fg) - T(f)g$, and $[T, f, g] := T(fg) - T(f)g - fT(g)$, where T denotes a differential operator or the mollifier and f, g are arbitrary functions.
- $\overline{\partial}, \overline{\Delta}$: $\overline{\partial} = \partial_1, \partial_2$ denotes the tangential derivative and $\overline{\Delta} := \partial_1^2 + \partial_2^2$ denotes the tangential Laplacian.
- $\nabla_a^\alpha f := a^{\mu\alpha}\partial_\mu f$, div ${}_a\mathbf{f} := a^{\mu\alpha}\partial_\mu \mathbf{f}_\alpha$ and $(\text{curl}\,{}_a\mathbf{f})_\lambda := \epsilon_{\lambda\mu\alpha} a^{\nu\mu}\partial_\nu \mathbf{f}^\alpha$, where $\epsilon_{\lambda\mu\alpha}$ is the sign of the 3-permutation $(\lambda\mu\alpha) \in S_3$.

□

2 Preliminary lemmas

We need the following Lemmas in this manuscript.

2.1 Sobolev inequalities

Lemma 2.1. **(Kato-Ponce type inequalities)** Let $J = (I - \Delta)^{1/2}$, $s \geq 0$. Then the following estimates hold:

(1) $\forall s \geq 0$, we have

$$\begin{aligned} \|J^s(fg)\|_{L^2} &\lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{W^{s,q_2}}, \\ \|\partial^s(fg)\|_{L^2} &\lesssim \|f\|_{\dot{W}^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{\dot{W}^{s,q_2}}, \end{aligned} \tag{2.1}$$

with $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \leq p_1, q_2 < \infty$;

(2) $\forall s \geq 1$, we have

$$\|J^s(fg) - (J^s f)g - f(J^s g)\|_{L^p} \lesssim \|f\|_{W^{1,p_1}}\|g\|_{W^{s-1,q_2}} + \|f\|_{W^{s-1,q_1}}\|g\|_{W^{1,q_2}} \tag{2.2}$$

for all the $1 < p < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$.

Proof. See Kato-Ponce [36]. □

Lemma 2.2. Suppose that $s \geq 0.5$ and u solves the boundary-valued problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma \end{cases}$$

where $g \in H^s(\Gamma)$. Then it holds that

$$|g|_s \lesssim \|u\|_{s+0.5} \lesssim |g|_s$$

Proof. The LHS follows from the standard Sobolev trace lemma, while the RHS is the property of Poisson integral, which can be found in Proposition 5.1.7 in M. Taylor's book [55]. □

Lemma 2.3. (Bernstein-type inequalities) Let $0 \leq \chi(\xi) \leq 1$ be a $C_c^\infty(\mathbb{R}^d)$ cut-off function which is supported in $\{|\xi| \leq 2\}$ and equals to 1 in $\{|\xi| \leq 1\}$. Define the Littlewood-Paley projection $P_{\leq N}$ in $\mathbb{R}^d$ with respect to χ by

$$P_{\leq N} f := \left(\chi(\xi/N)\hat{f}(\xi)\right)^\vee, \ \ P_{\geq N} f := \left((1-\chi(\xi/N))\hat{f}(\xi)\right)^\vee, \ \ P_N f := \left((\chi(\xi/N) - \chi(2\xi/N))\hat{f}(\xi)\right)^\vee.$$

Then the following inequalities hold

$$\|P_{\leq N} f\|_{\dot{H}_x^s(\mathbb{R}^d)} \lesssim_{s,d} N^s \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall s \geq 0; \tag{2.3}$$

$$\|P_{\geq N} f\|_{\dot{H}_x^s(\mathbb{R}^d)} \lesssim_{s,d} \|f\|_{\dot{H}_x^s(\mathbb{R}^d)}, \quad \forall s \in \mathbb{R}. \tag{2.4}$$

Analogous results also hold for $H_x^s(\mathbb{R}^d)$.

Proof. For the first inequality, we apply Plancherel's identity to get

$$\|P_{\leq N} f\|_{\dot{H}_x^s(\mathbb{R}^d)} = \||\partial|^s P_{\leq N} f\|_{L^2(\mathbb{R}^d)} = \||\xi|^s \chi(\xi/N)\hat{f}(\xi)\|_{L^2(\mathbb{R}^d)} \lesssim N^s \cdot 1 \cdot \|f\|_{L^2(\mathbb{R}^d)}.$$

Note that $s \geq 0$ is used in the last inequality. For the second inequality, we just replace $\chi(\xi/N)$ above by $1-\chi(\xi/N)$ and notice that $0 \leq 1-\chi(\xi/N) \leq 1$ to get

$$\|P_{\geq N} f\|_{\dot{H}_x^s(\mathbb{R}^d)} \leq \||\xi|^s \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{\dot{H}_x^s(\mathbb{R}^d)}.$$

Analogous results hold for $H^s(\mathbb{R}^d)$ by replacing $|\partial|$ and $|\xi|$ with $\langle\partial\rangle$ and $\langle\xi\rangle$ respectively. One can see Appendix A in Tao [54] for more Bernstein-type inequalities. □

2.2 Properties of tangential smoothing operator

As stated in the introduction, we are going to use the tangential smoothing to construct the approximate solutions. Here we list the definition and basic properties which are repeatedly used in this paper. Let $\zeta = \zeta(y_1, y_2) \in C_c^\infty(\mathbb{R}^2)$ be a standard cut-off function such that $\operatorname{Spt}\zeta = \overline{B(0,1)} \subseteq \mathbb{R}^2$, $\ 0 \leq \zeta \leq 1$ and $\int_{\mathbb{R}^2} \zeta = 1$. The corresponding dilation is

$$\zeta_\kappa(y_1, y_2) = \frac{1}{\kappa^2}\zeta\left(\frac{y_1}{\kappa}, \frac{y_2}{\kappa}\right), \ \ \kappa > 0.$$

Now we define

$$\Lambda_\kappa f(y_1, y_2, y_3) := \int_{\mathbb{R}^2} \zeta_\kappa(y_1 - z_1, y_2 - z_2) f(z_1, z_2, z_3)\, dz_1\, dz_2. \tag{2.5}$$

The following lemma records the basic properties of tangential smoothing.

Lemma 2.4. (Regularity and Commutator estimates) For $\kappa > 0$, we have

(1) The following regularity estimates:

$$\|\Lambda_\kappa f\|_s \lesssim \|f\|_s, \quad \forall s \geq 0; \tag{2.6}$$
$$|\Lambda_\kappa f|_s \lesssim |f|_s, \quad \forall s \geq -0.5; \tag{2.7}$$
$$|\overline{\partial}\Lambda_\kappa f|_0 \lesssim \kappa^{-s}|f|_{1-s}, \quad \forall s \in [0,1]; \tag{2.8}$$
$$|f - \Lambda_\kappa f|_{L^\infty} \lesssim \sqrt{\kappa}|f|_{1.5}. \tag{2.9}$$

(2) Commutator estimates: Define the commutator $[\Lambda_\kappa, f]g := \Lambda_\kappa(fg) - f\Lambda_\kappa(g)$. Then it satisfies

$$|[\Lambda_\kappa, f]g|_0 \lesssim |f|_{L^\infty}|g|_0, \tag{2.10}$$
$$|[\Lambda_\kappa, f]\overline{\partial}g|_0 \lesssim |f|_{W^{1,\infty}}|g|_0, \tag{2.11}$$
$$|[\Lambda_\kappa, f]\overline{\partial}g|_{0.5} \lesssim |f|_{W^{1,\infty}}|g|_{0.5}. \tag{2.12}$$

Proof. (1): The estimates (2.7) and (2.8) follows directly from the definition (2.5) and the basic properties of convolution. (2.9) is derived by using Sobolev embedding and Hölder's inequality:

$$\begin{aligned} |f - \Lambda_\kappa f| &= \left|\int_{\mathbb{R}^2 \cap B(0,\kappa)} \zeta_\kappa(z)(f(y-z) - f(y))\,dz\right| \\ &\lesssim |\zeta_\kappa|_{L^{4/3}}|\kappa\overline{\partial}f|_{L^4} \\ &\lesssim \sqrt{\kappa}|\zeta|_{L^{4/3}}|f|_{1.5}. \end{aligned}$$

(2): The first three estimates can be found in Lemma 5.1 in Coutand-Shkoller [11]. To prove the fourth one, we note that

$$\overline{\partial}([\Lambda_\kappa, f]g) = \Lambda_\kappa(\overline{\partial}f\overline{\partial}g) + \Lambda_\kappa(f\overline{\partial}^2 g) - \overline{\partial}f\Lambda_\kappa\overline{\partial}g - f\Lambda_\kappa\overline{\partial}^2 g = [\Lambda_\kappa, \overline{\partial}f]\overline{\partial}g + [\Lambda_\kappa, f]\overline{\partial}^2 g.$$

From (2.10) and (2.11) we know

$$|\overline{\partial}[\Lambda_\kappa, f]g|_0 \lesssim |\overline{\partial}f|_{L^\infty}|\overline{\partial}g|_0 + |f|_{W^{1,\infty}}|\overline{\partial}g|_0 \lesssim |f|_{W^{1,\infty}}|g|_1. \tag{2.13}$$

Therefore (2.12) follows from the interpolation of (2.11) and (2.13). □

2.3 Elliptic estimates

Lemma 2.5. (Hodge-type decomposition) Let X be a smooth vector field and $s \geq 1$, then it holds that

$$\|X\|_s \lesssim \|X\|_0 + \|\text{curl } X\|_{s-1} + \|\text{div } X\|_{s-1} + |X \cdot N|_{s-0.5}. \tag{2.14}$$

Proof. This follows from the well-known identity $-\Delta X = \text{curl curl } X - \nabla\text{div } X$. □

Lemma 2.6. (Interior elliptic estimate) The following elliptic estimate holds for $r \geq 2$.

$$\|\nabla_{\tilde{a}} f\|_{H^r} \leq C(\|\partial\tilde{\eta}\|_{L^\infty}, \|\partial^2\tilde{\eta}\|_{r-2})\Big(\|\Delta_{\tilde{a}} f\|_{r-1} + \|\overline{\partial}\partial\tilde{\eta}\|_{r-1}\|f\|_r\Big). \tag{2.15}$$

Proof. See Ginsberg-Lindblad-Luo [22] Proposition 5.3. The original version of this estimate is

$$\|\nabla_{\tilde{a}} f\|_{H^r} \leq C(\|\tilde{\eta}\|_r)\Big(\|\Delta_{\tilde{a}} f\|_{r-1} + \|\overline{\partial}\tilde{\eta}\|_r\|f\|_r\Big). \tag{2.16}$$

However, the remark after (1.14) suggests that the dependence of $\|\tilde{\eta}\|_r$ in the original proof can be weakened to $\|\partial\tilde{\eta}\|_{L^\infty}, \|\partial^2\tilde{\eta}\|_{r-2}$. Similarly, we can replace $\|\overline{\partial}\tilde{\eta}\|_r$ by $\|\overline{\partial}\partial\tilde{\eta}\|_{r-1}$. □

3 The Approximate system and uniform a priori estimates

In this section we are going to introduce the approximation of the water wave problem and derive its uniform a priori estimates.

3.1 The approximate system

For $\kappa > 0$, we consider the following approximate system

$$\begin{cases} \partial_t \eta = v + \psi & \text{in } \Omega, \\ \partial_t v = -\nabla_{\tilde{a}} h - g e_3 & \text{in } \Omega, \\ \text{div}_{\tilde{a}} v = -e'(h) \partial_t h & \text{in } \Omega, \\ h = 0 & \text{on } \Gamma, \\ (\eta, v, h)|_{\{t=0\}} = (\text{Id}, v_0, h_0). \end{cases} \tag{3.1}$$

Here $\tilde{a} := (\partial \tilde{\eta})^{-1}$ where $\tilde{\eta}$ is the smoothed version of the flow map η defined by $\tilde{\eta} := \Lambda_\kappa^2 \eta$. The term $\psi = \psi(\eta, v)$ is a correction term which solves the half-space Laplacian equation

$$\begin{cases} \Delta \psi = 0 & \text{in } \Omega, \\ \psi = \overline{\Delta}^{-1} \mathbb{P} \left(\overline{\Delta} \eta_\beta \tilde{a}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 v - \overline{\Delta} \Lambda_\kappa^2 \eta_\beta \tilde{a}^{i\beta} \overline{\partial}_i v \right) & \text{on } \Gamma, \end{cases} \tag{3.2}$$

where $\mathbb{P} f := P_{\geq 1} f$ denotes the standard Littlewood-Paley projection in $\mathbb{R}^2$ defined is Lemma 2.3, which removes the low-frequency part. $\overline{\Delta} := \partial_1^2 + \partial_2^2$ denotes the tangential Laplacian operator and $\overline{\Delta}^{-1} f := (|\xi|^{-2} \hat{f})^\vee$ is the inverse of $\overline{\Delta}$ on $\mathbb{R}^2$.

Remark:

1. The correction term $\psi \to 0$ as $\kappa \to 0$. We introduce such a term to eliminate the higher order boundary terms which appears in the tangential estimates of v. These higher order boundary terms are zero when $\kappa = 0$ but are out of control when $\kappa > 0$.

2. The Littlewood-Paley projection is necessary here because we will repeatedly use

$$|\overline{\Delta}^{-1} \mathbb{P} f|_s \lesssim |\mathbb{P} f|_{H^{s-2}} \approx |\mathbb{P} f|_{\dot{H}^{s-2}} \lesssim |f|_{\dot{H}^{s-2}},$$

which can be proved by using Bernstein inequality (2.4). Without $\mathbb{P}$ the low-frequency part loses control when taking $\overline{\Delta}^{-1}$.

Fix any $\kappa > 0$, we will prove in Section 4 that there exists a $T_\kappa > 0$ depending on the initial data and $\kappa > 0$ such that there is a unique solution $(v(\kappa), h(\kappa), \eta(\kappa))$ to (3.1) in $[0, T_\kappa]$. For simplicity we omit the κ and only write v, h, η in this manuscript. The remaining context in this section is to derive the uniform-in-κ a priori estimates for the solutions to (3.1). **This guarantees that we are able to obtain the solution of the original problem in some fixed time interval by passing $\kappa \to 0$.**

Define the energy functional

$$\mathcal{E}_\kappa := \|\partial \eta\|_{L^\infty}^2 + \|\partial^2 \eta\|_2^2 + \sum_{k=0}^{4} (\|\partial_t^{4-k} v\|_k^2 + \|\partial_t^{4-k} h\|_k^2) + |\tilde{a}^{3\alpha} \overline{\partial}^4 \Lambda_\kappa \eta_\alpha|_0^2. \tag{3.3}$$

The rest of this section is devoted to prove:

Proposition 3.1. Let $\mathcal{E}_\kappa$ be defined as above. Then there exists a time $T > 0$ independent of κ such that

$$\sup_{0 \leq t \leq T} \mathcal{E}_\kappa(t) \leq P(\|v_0\|_4, \|h_0\|_4). \tag{3.4}$$

Proposition 3.1 is a direct consequence of the following proposition:

Proposition 3.2. Let $\mathcal{E}_\kappa$ be defined as above. Then it holds that

$$\mathcal{E}_\kappa(t) \leq P(\|v_0\|_4, \|h_0\|_4) + \int_0^t P(\mathcal{E}_\kappa(\tau)) \, d\tau, \quad \forall t \in [0, T] \tag{3.5}$$

provided the following a priori assumptions hold

$$-\partial_3 h(t) \geq \frac{c_0}{2} \quad \text{on } \Gamma, \tag{3.6}$$

$$\|\tilde{J}(t) - 1\|_3 \leq \epsilon \quad \text{in } \Omega, \tag{3.7}$$

$$\|\mathrm{Id} - \tilde{a}(t)\|_3 \leq \epsilon \quad \text{in } \Omega, \tag{3.8}$$

where $\tilde{J} := \det(\partial\tilde{\eta})$ and we use $\epsilon > 0$ to denote the sufficiently small number which appears here and the ϵ-Young inequality.

Remark. It suffices to show that (3.5) holds true when $t = T$. Also, (3.5) can in fact be reduced to

$$\mathcal{E}_\kappa(T) \leq \mathcal{E}_\kappa(0) + \int_0^T P(\mathcal{E}_\kappa(\tau))\,d\tau. \tag{3.9}$$

In [44] we are able to prove that there exists initial data satisfying the compatibility condition (1.9) up to order 5 such that $\mathcal{E}_\kappa(0) \leq P(\|v_0\|_4, \|h_0\|_4)$ holds. For notation simplicity we define $\mathcal{P}_0 := P(\|v_0\|_4, \|h_0\|_4)$.

3.2 Estimates for the flow map and correction term

First we bound the flow map and the correction term together with their smoothed version by the quantities in $\mathcal{E}_\kappa$. The following estimates will be repeatedly use in this section.

Lemma 3.3. Let (v, h, η) be the solution to (3.1). Then we have

$$\|\partial\tilde{\eta}\|_{L^\infty} \lesssim \|\partial\eta\|_{L^\infty}, \tag{3.10}$$

$$\|\partial^2\tilde{\eta}\|_2 \lesssim \|\partial^2\eta\|_2, \tag{3.11}$$

$$\|\psi\|_4 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_3), \tag{3.12}$$

$$\|\partial_t\psi\|_4 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3), \tag{3.13}$$

$$\|\partial_t^2\psi\|_3 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2), \tag{3.14}$$

$$\|\partial_t^3\psi\|_2 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1). \tag{3.15}$$

and

$$\|\partial_t\tilde{\eta}\|_4 \lesssim \|\partial_t\eta\|_4 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4), \tag{3.16}$$

$$\|\partial_t^2\tilde{\eta}\|_3 \lesssim \|\partial_t^2\eta\|_3 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3), \tag{3.17}$$

$$\|\partial_t^3\tilde{\eta}\|_2 \lesssim \|\partial_t^3\eta\|_2 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2), \tag{3.18}$$

$$\|\partial_t^4\tilde{\eta}\|_1 \lesssim \|\partial_t^4\eta\|_1 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1). \tag{3.19}$$

Proof. First, (3.10) and (3.11) follow from (2.6), i.e., $\|\partial\tilde{\eta}\|_{L^\infty} = \|\Lambda_\kappa^2\partial\eta\|_{L^\infty} \lesssim \|\partial\eta\|_{L^\infty}$, $\|\partial^2\tilde{\eta}\|_2 = \|\Lambda_\kappa^2\partial^2\eta\|_2 \lesssim \|\partial^2\eta\|_2$. To bound $\partial_t^k\tilde{\eta}$, it suffices to bound the same norm of $\partial_t^k\eta$ and then apply (2.6) again. From the first equation of (3.1), one has $\partial_t^{k+1}\eta = \partial_t^k v + \partial_t^k\psi$, so the estimates (3.16)-(3.19) automatically holds once we prove (3.12)-(3.15).

Commuting time derivatives through (3.2), we get the equations for $\partial_t^k\psi$ ($k = 0, 1, 2, 3, 4$):

$$\begin{cases} \Delta\partial_t^k\psi = 0 & \text{in } \Omega, \\ \partial_t^k\psi = \overline{\Delta}^{-1}\mathbb{P}\partial_t^k\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right) & \text{on } \Gamma. \end{cases} \tag{3.20}$$

By the standard elliptic estimates, Sobolev trace lemma and Bernstein inequality (2.4) in Lemma 2.3, we can get

$$\begin{aligned} \|\psi\|_4 &\lesssim \left|\overline{\Delta}^{-1}\mathbb{P}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{3.5} \\ &\lesssim \left|\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right|_{1.5} \\ &\lesssim \|\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\|_2 \\ &\lesssim \|\partial^2\eta\|_2\|\tilde{a}\|_{L^\infty}\|v\|_3 \leq P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_3). \end{aligned} \tag{3.21}$$

Also, when $k = 1, 2, 3$, one has

$$\begin{aligned}\|\partial_t\psi\|_4 &\lesssim \left|\overline{\Delta}^{-1}\partial_t\mathbb{P}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{3.5} \\ &\lesssim \left|\partial_t\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{1.5} \\ &\lesssim \|\partial_t(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v)\|_2 \\ &\lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|\partial_t v\|_3),\end{aligned} \tag{3.22}$$

$$\begin{aligned}\|\partial_t^2\psi\|_3 &\lesssim \left|\overline{\Delta}^{-1}\partial_t^2\mathbb{P}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{2.5} \\ &\lesssim \left|\partial_t^2\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{0.5} \\ &\lesssim \|\partial_t^2(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v)\|_1 \\ &\lesssim P(\|\partial^2\eta\|_2, \|v\|_4, \|\partial\eta\|_{L^\infty}, \|\partial_t v\|_3, \|\partial_t^2 v\|_2),\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\|\partial_t^3\psi\|_2 &\lesssim \left|\overline{\Delta}^{-1}\partial_t^3\mathbb{P}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{1.5} \\ &\lesssim \left|\ \mathbb{P}\partial_t^3\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{-0.5}\end{aligned} \tag{3.24}$$

where in the last step we apply the Bernstein's inequality (2.4).

Combining with $\partial_t^{k+1}\eta = \partial_t^k v + \partial_t^k\psi$, (3.16), (3.17) and (3.18) directly follows from (3.22) and (3.23), respectively. When $k = 3$, one has to be cautious because the leading order term in (3.24) is of the form $(\partial_t^3\overline{\Delta}\eta)\tilde{a}\overline{\partial}v$ and $\overline{\Delta}\eta\tilde{a}(\partial_t^3\overline{\partial}v)$ which can only be bounded in $L^2(\Omega)$ by the quantites in $\mathcal{E}_\kappa$ and thus loses control on the boundary. To control these terms on the boundary, we have to use the fact that $\dot{H}^{0.5}(\mathbb{R}^2) = (\dot{H}^{0.5}(\mathbb{R}^2))^*$.

First we separate them from other lower order terms which has $L^2(\Gamma)$ control.

$$\begin{aligned}&\mathbb{P}\partial_t^3\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right) \\ =\ &\mathbb{P}\underbrace{\left(\partial_t^3\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \partial_t^3\overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v + \overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\partial_t^3\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\partial_t^3\overline{\partial}_i v\right)}_{\text{leading order terms=:X}} + \mathbb{P}Y.\end{aligned} \tag{3.25}$$

The control of Y is straightforward by using Sobolev trace lemma and (3.16), (3.17),

$$\begin{aligned}|\mathbb{P}Y|_{-0.5} \le |\mathbb{P}Y|_0 &\lesssim \|Y\|_{0.5} \\ &\lesssim P(\|\partial_t^2\eta\|_{2.5}, \|\partial_t\eta\|_{3.5}, \|\partial_t^2\tilde{a}\|_{1.5}, \|\partial_t^2 v\|_{1.5}, \|\partial_t v\|_{2.5}) \\ &\lesssim P(\|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2).\end{aligned} \tag{3.26}$$

As for the $|\mathbb{P}X|_{-0.5}$ term, we first use the Bernstein inequality (2.4) to get $|\mathbb{P}X|_{-0.5} \approx |\mathbb{P}X|_{\dot{H}^{-0.5}} \lesssim |X|_{\dot{H}^{-0.5}}$. Then the duality between $\dot{H}^{-0.5}$ and $\dot{H}^{0.5}$ yields that for any test function $\phi \in \dot{H}^{0.5}(\mathbb{R}^2)$ with $|\phi|_{\dot{H}^{0.5}} \le 1$, one has

$$\begin{aligned}\langle\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\partial_t^3\overline{\partial}_i\Lambda_\kappa^2 v, \phi\rangle &= \langle\partial_t^3\overline{\partial}_i\Lambda_\kappa^2 v, \overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\phi\rangle \\ &= \langle\overline{\partial}_i^{0.5}\partial_t^3\Lambda_\kappa^2 v, \overline{\partial}_i^{0.5}(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\phi)\rangle \\ &\lesssim |\partial_t^3\Lambda_\kappa^2 v|_{\dot{H}^{0.5}}|\overline{\Delta}\eta\tilde{a}\phi|_{\dot{H}^{0.5}} \\ &\lesssim \|\partial_t^3 v\|_1(|\phi|_{\dot{H}^{0.5}}|\overline{\Delta}\eta\tilde{a}|_{L^\infty} + |\overline{\Delta}\eta\tilde{a}|_{\dot{W}^{0.5,4}}|\phi|_{L^4}) \\ &\lesssim \|\partial_t^3 v\|_1(\|\partial^2\eta\|_2\|a\|_{L^\infty})|\phi|_{\dot{H}^{0.5}}.\end{aligned} \tag{3.27}$$

Here we integrate 1/2-order tangential derivative on Γ by part in the second step, and then apply trace lemma to control $|\partial_t^3\Lambda_\kappa^2 v|_{\dot{H}^{0.5}}$ and Kato-Ponce product estimate (2.1) to bound $|\overline{\Delta}\eta\tilde{a}\phi|_{\dot{H}^{0.5}}$. Taking supremum over all $\phi \in H^{0.5}(\mathbb{R}^2)$ with $|\phi|_{\dot{H}^{0.5}} \le 1$, we have by the definition of $\dot{H}^{0.5}$-norm that

$$|\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\partial_t^3\overline{\partial}_i\Lambda_\kappa^2 v|_{\dot{H}^{-0.5}} \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|\partial_t^3 v\|_1). \tag{3.28}$$

Similarly as above, we have

$$|\overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\partial_t^3\overline{\partial}_i v|_{\dot{H}^{-0.5}} \lesssim P(\|\partial^2\eta\|_2, \|\partial_t^3 v\|_1), \tag{3.29}$$

$$|\partial_t^3\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \partial_t^3\overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v|_{\dot{H}^{-0.5}} \lesssim P(\|\partial_t^3\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_3). \tag{3.30}$$

Combining (3.24)-(3.30) and the bound (3.18) for $\partial_t^3\eta$, we get

$$\|\partial_t^3\psi\|_2 \lesssim P(\|\partial^2\eta\|_2, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1),$$

which is exactly (3.15). Hence, (3.19) directly follows from (3.15) and $\partial_t^4\eta = \partial_t^3(v + \psi)$.

□

3.3 Estimates for the enthalpy h: Wave equation

In this section we are going to control to Sobolev norm $\|\partial_t^{4-k}h\|_k$, i.e., the $L^2(\Omega)$-norm of $\partial_t^{4-k}\partial^k h$. We introduce the notation $\mathfrak{D}$ to denote either ∂_t or $\overline{\partial}$ for the simplicity of notations. First we take the Eulerian divergence (i.e.,$\text{div}_{\tilde{a}}$) in the second equation of system (3.1) and use the third equation of (3.1) to get a wave equation of h:

$$\tilde{J}e'(h)\partial_t^2 h - \partial_\nu(E^{\nu\mu}\partial_\mu h) = \underbrace{\tilde{J}\partial_t\tilde{a}^{\nu\alpha}\partial_\nu v_\alpha}_{:=F} - \tilde{J}e''(h)(\partial_t h)^2, \tag{3.31}$$

where $E^{\nu\mu} = \tilde{J}\tilde{a}^{\nu\alpha}\tilde{a}^\mu_\alpha$. Note that the matrix E is symmetric and positive-definite thanks to (3.7).

3.3.1 L^2-estimate of h

To bound $\|h\|_s$ we have to control $\|h\|_0$ and $\|\partial^s h\|_0$ separately since the Poincaré's inequality is invalid in Ω. The control of $\|h\|_0$ is straightforward, i.e.,

$$\|h(T)\|_0 \le \|h_0\|_0 + \int_0^T \|\partial_t h(t)\|_0\,dt. \tag{3.32}$$

In other words, the control of L^2-norm of h is reduced to the control of $\|h_t\|_0$ which will be done in the next step by using the wave equation.

3.3.2 Estimates for ∂h

We multiply (3.31) by $\partial_t h$ and then integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{J}e'(h)|\partial_t h|^2\,dy - \int_\Omega \partial_t h\partial_\nu(E^{\nu\mu}\partial_\mu h)\,dy = \int_\Omega F\partial_t h\,dy + \int_\Omega \partial_t\tilde{J}e'(h)|\partial_t h|^2 \tag{3.33}$$

First, $\|\partial_t\tilde{J}\|_{L^\infty}$ can be easily bounded, because by the definition of determinant, $\tilde{J}$ is the sum of $(\partial\tilde{\eta})(\partial\tilde{\eta})(\partial\tilde{\eta})$ and thus $\partial_t\tilde{J} \approx \partial(v+\psi)(\partial\tilde{\eta})(\partial\tilde{\eta})$. So we have

$$\|\partial_t\tilde{J}\|_{L^\infty} \lesssim \|\partial\eta\|_{L^\infty}^2\|\partial v\|_2\|\partial\psi\|_2. \tag{3.34}$$

Similarly, since $E = \tilde{J}\tilde{a}\cdot\tilde{a}$, we have the L^∞ bound for $\partial_t E$:

$$\|\partial_t E\|_{L^\infty} \lesssim \|J\|_{L^\infty}\|a\|_{L^\infty}\|\partial_t a\|_{L^\infty} + \|\partial_t\tilde{J}\|_{L^\infty}\|\partial\eta\|_{L^\infty}^2 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial v\|_2, \|\partial\psi\|_2). \tag{3.35}$$

Integrating ∂_μ by parts, we can get

$$\begin{aligned}-\int_\Omega \partial_t h\partial_\nu(E^{\nu\mu}\partial_\mu h)\,dy &= \int_\Omega E^{\nu\mu}\partial_t\partial_\nu h\partial_\mu h\,dy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega E^{\nu\mu}\partial_\nu h\partial_\mu h\,dy - \frac{1}{2}\int_\Omega \partial_t E^{\nu\mu}\partial_\nu h\partial_\mu h\,dy.\end{aligned} \tag{3.36}$$

Therefore, we get after integrating in time that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \tilde{J}e'(h)|\partial_t h|^2 + |\partial h|^2 \lesssim \frac{1}{2}\int_\Omega \tilde{J}e'(h)|\partial_t h|^2 + E^{\nu\mu}\partial_\nu h\partial_\mu h\,dy \\
&= \frac{1}{2}\int_\Omega e'(h)|\partial_t h|^2 + E^{\nu\mu}\partial_\nu h\partial_\mu h\,dy\Big|_{t=0} - \int_0^T \frac{1}{2}\int_\Omega \partial_t E^{\nu\mu}\partial_\nu h\partial_\mu h\,dy\,dt \\
&\quad + \int_0^T\int_\Omega F\partial_t h\,dy\,dt + \int_0^T\int_\Omega \partial_t\tilde{J}e'(h)(\partial_t h)^2 \qquad (3.37)\\
&\lesssim \mathcal{P}_0 + \int_0^T \|\partial_t E\|_{L^\infty}\|\partial h\|_0^2 + \|\partial_t h\|_0\|F\|_{L^\infty}\,dt \\
&\lesssim \mathcal{P}_0 + \int_0^T P(\|\partial_t h\|_0, \|\partial h\|_0, \|\partial\eta\|_{L^\infty}, \|\partial v\|_2)\,dt.
\end{aligned}
$$

3.3.3 Estimates for $\mathfrak{D}^3\partial h$

Let $\mathfrak{D}^3 = \overline{\partial}^3, \overline{\partial}^2\partial_t, \overline{\partial}\partial_t^2$ and ∂_t^3, i.e., all the 3rd-order tangential derivatives. (recall that $D_t \in \mathcal{T}(\partial\mathcal{D})$, so ∂_t is also in the tangential space of Γ.) Applying $\mathfrak{D}^3$ to (3.31), we get

$$
\tilde{J}e'(h)\partial_t^2\mathfrak{D}^3 h - \partial_\nu(E^{\nu\mu}\mathfrak{D}^3\partial_\mu h) = \mathfrak{D}^3 F \underbrace{-[\mathfrak{D}^3, \tilde{J}e'(h)]\partial_t^2 h + \mathfrak{D}^3(\tilde{J}e''(h)(\partial_t h)^2)}_{F_3} + \partial_\nu([\mathfrak{D}^3, E^{\nu\mu}]\partial_\mu h). \tag{3.38}
$$

Multiplying (3.38) by $\partial_t\mathfrak{D}^3 h$, then integrating ∂_ν by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{J}e'(h)|\mathfrak{D}^3\partial_t h|^2 + E^{\mu\nu}\partial_\nu\mathfrak{D}^3 h\partial_\mu\mathfrak{D}^3 h\,dy && (3.39)\\
=\ &\frac{1}{2}\int_\Omega \partial_t E^{\nu\mu}\mathfrak{D}^3\partial_\nu h\mathfrak{D}^3\partial_\mu h\,dy && (3.40)\\
&+\int_\Omega F_3\partial_t\mathfrak{D}^3 h\,dy && (3.41)\\
&+\int_\Omega \mathfrak{D}^3 F\partial_t\mathfrak{D}^3 h\,dy && (3.42)\\
&+\int_\Omega \partial_\nu([\mathfrak{D}^3, E^{\nu\mu}]\partial_\mu h)\partial_t\mathfrak{D}^3 h\,dy. && (3.43)
\end{aligned}
$$

(3.40) can be directly bounded by the energy:

$$
(3.40) \lesssim \|\partial_t E\|_{L^\infty}\|\mathfrak{D}^3\partial h\|_0^2 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial v\|_2, \|\partial\psi\|_2, \|\mathfrak{D}^3\partial h\|_0) \tag{3.44}
$$

To estimate (3.41), it suffices to bound $\|F_3\|_0$. The precise form of F_3 is

$$
F_3 = \sum_{m=2}^{5}\sum e^{(m)}(h)(\partial_t^{i_1}\mathfrak{D}^{j_1}h)\cdots(\partial_t^{i_m}\mathfrak{D}^{j_m}h),
$$

where the second sum is taken over the set $\{i_1+\cdots+i_m = 2,\ j_1+\cdots+j_m = 3,\ 1 \le i_m + j_m \le 4\}$. Invoking the condition imposed on $e(h)$ (i.e., (1.7)), one has

$$
\sum_{\mathfrak{D}^3}\|F_3\|_0 \lesssim P(\|\partial_t^4 h\|_0, \|\partial_t^3 h\|_1, \|\partial_t^2 h\|_2, \|\partial_t h\|_3, \|\mathfrak{D}h\|_2). \tag{3.45}
$$

As for (3.42), one has $\mathfrak{D}^3 F = \mathfrak{D}^3(\tilde{J}\tilde{a}^{\nu\alpha}\tilde{a}^\mu_\alpha)$.

- When $\mathfrak{D}^3 = \overline{\partial}^3$, then $\|\mathfrak{D}^3(\tilde{J}\tilde{a}^{\nu\alpha}\tilde{a}^\mu_\alpha)\|_0 \lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2)$.

- When $\mathfrak{D}^3$ contains at least one time derivative, then

$$\begin{aligned}\|\mathfrak{D}^3 F\|_0 &= \|\mathfrak{D}^2\partial_t(\tilde{J}\tilde{a}^{\nu\alpha}\tilde{a}^{\mu}_{\alpha})\|_0\\ &\lesssim \|\partial_t J\|_2\|a\|_{L^\infty}^2 + \|J\|_2\|\partial_t a\|_2\|a\|_{L^\infty}\\ &\lesssim P(\|\partial v\|_2, \|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}).\end{aligned}$$

Therefore,

$$(3.42) \lesssim P(\|\partial^2\eta\|_2, \|\partial v\|_2)\|\partial_t\mathfrak{D}^3 h\|_0 \tag{3.46}$$

Finally, one has to be cautious when controlling (3.43): The leading order term in $\partial_\nu([\mathfrak{D}^3, E^{\nu\mu}]\partial_\mu h)$ is $\partial\mathfrak{D}^3 E$. If $\mathfrak{D}^3 = \overline{\partial}^3$, then this term loses control in L^2. To avoid this problem, one can integrate ∂_ν by parts, and then integrate ∂_t by parts in the time integral of (3.43) to replace the ∂_ν falling on E by ∂_t. This is because J and $\partial_t J$ (also for a and $\partial_t a$) have the same spatial regularity. If $\mathfrak{D}^3$ contains at least one time derivative, then the L^2-norm of $\partial\mathfrak{D}^3 E$ can be controlled directly thanks to the same reason above.

- $\mathfrak{D}^3$ contains at least one time derivative, i.e., $\mathfrak{D}^3 = \mathfrak{D}^2\partial_t$. Then

$$\begin{aligned}&\int_\Omega \partial_\nu([\mathfrak{D}^2\partial_t, E^{\nu\mu}]\partial_\mu h)\partial_t\mathfrak{D}^3 h\,dy\\ &\lesssim \|[\mathfrak{D}^2\partial_t, E^{\nu\mu}]\partial_\mu h\|_1\|\partial_t\mathfrak{D}^3 h\|_0\\ &\lesssim (\|\mathfrak{D}^2\partial_t E\|_1\|h\|_3 + \|\mathfrak{D}^2 E\|_{L^\infty}\|\partial_t h\|_2 + \|\mathfrak{D}\partial_t E\|_{L^\infty}\|\mathfrak{D}h\|_2 + \|\mathfrak{D}E\|_{L^\infty}\|\mathfrak{D}\partial_t h\|_1 + \|\partial_t E\|_{L^\infty}\|\mathfrak{D}^2 h\|_1)\|\partial_t\mathfrak{D}^3 h\|_0.\end{aligned}$$

So

$$\sum_{\mathfrak{D}^3\setminus\{\overline{\partial}^3\}}(3.43) \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|h\|_3, \|\partial_t h\|_2, \|\partial_t^2 h\|_1)\|\partial_t\mathfrak{D}^3 h\|_0. \tag{3.47}$$

- When $\mathfrak{D}^3 = \overline{\partial}^3$, we consider the time integral of (3.43). We first integrate ∂_ν by parts, then integrate ∂_t by parts to get the following equality

$$\begin{aligned}&\int_0^T\int_\Omega \partial_\nu([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_t\overline{\partial}^3 h\,dy\,dt\\ &= -\int_0^T\int_\Omega([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\partial_t\overline{\partial}^3 h\,dy\,dt + \int_0^T\int_\Gamma([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)N_\nu\underbrace{\partial_t\overline{\partial}^3 h}_{=0}\,dS\,dt\\ &= \int_0^T\int_\Omega \partial_t([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\overline{\partial}^3 h\,dy\,dt - \int_\Omega([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\overline{\partial}^3 h\,dy\Big|_{t=0}^{t=T}.\end{aligned}$$

The leading order term in the first integral is $\overline{\partial}^3\partial_t E$ which has L^2- control, so one can bound this directly by using Hölder's inequality

$$\int_0^T\int_\Omega \partial_t([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\overline{\partial}^3 h\,dy\,dt \lesssim \int_0^T P(\|\partial^2\eta\|_2, \|\partial v\|_3, \|\partial h\|_2, \|\partial_t h\|_3)\|\overline{\partial}^3 h\|_1\,dt. \tag{3.48}$$

As for the second integral, we can use Hölder's inequality first, then use ϵ-Young's inequality and Jensen's inequality

$$
\begin{aligned}
&-\int_\Omega ([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\overline{\partial}^3 h\, dy\Big|_{t=T}\\
&\lesssim \|\overline{\partial}^3 E(T)\|_0\|\partial h(T)\|_{L^\infty}\|\overline{\partial}^3 h(T)\|_1 + \|\overline{\partial}^2 E(T)\|_1\|\partial\overline{\partial} h(T)\|_1\|\overline{\partial}^3 h(T)\|_1 + \|\overline{\partial} E(T)\|_{L^\infty}\|\partial\overline{\partial}^2 h(T)\|_0\|\overline{\partial}^3 h(T)\|_1\\
&\lesssim \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \frac{1}{8\epsilon}(\|\overline{\partial} E(T)\|_2^4 + \|\overline{\partial} E(T)\|_{L^\infty}^4 + \|h(T)\|_3^4)\\
&\lesssim \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \frac{1}{8\epsilon}\left(\|\overline{\partial} E(0)\|_2^4 + \|\overline{\partial} E(0)\|_{L^\infty}^4 + \|h(0)\|_3^4 + \int_0^T \|\partial_t\overline{\partial} E(t)\|_2^4 + \|\partial_t h(t)\|_3^4\, dt\right)\\
&\lesssim \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \mathcal{P}_0 + \int_0^T P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2, \|v\|_4, \|\partial_t h\|_3)\, dt.
\end{aligned}
\tag{3.49}
$$

The above estimates along with

$$
\int_\Omega ([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_\nu\overline{\partial}^3 h\, dy\Big|_{t=0} \lesssim \mathcal{P}_0
$$

give the bound for the time integral of (3.43):

$$
\int_0^T\int_\Omega \partial_\nu([\overline{\partial}^3, E^{\nu\mu}]\partial_\mu h)\partial_t\overline{\partial}^3 h\, dy\, dt \lesssim \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \mathcal{P}_0 + \int_0^T P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|\partial_t h\|_3)\, dt.
\tag{3.50}
$$

Now, summing up (3.44), (3.45), (3.46), (3.47), (3.48), (3.50) and then plugging it into (3.39), we get the tangential derivative estimates of h:

$$
\sum_{\mathfrak{D}^3}\int_\Omega \tilde{J}e'(h)|\mathfrak{D}^3\partial_t h|^2 + |\partial\mathfrak{D}^3 h|^2\, dy\Big|_{t=0}^{t=T} \lesssim \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \mathcal{P}_0 + \int_0^T P(\mathcal{E}_\kappa(t))\, dt.
\tag{3.51}
$$

Note that we have used E is symmetric and positive-definite. Choosing $\epsilon > 0$ sufficiently small, the term $\epsilon\|\overline{\partial}^3 h(T)\|_1^2$ can absorbed by the LHS of (3.51).

3.3.4 Estimates for the full Sobolev norm

Up to now, we have control all the tangential space-time derivative of ∂h. Therefore it suffices to control ≥ 2 normal derivatives of h. Actually this follows directly from the wave equation (3.31)

$$
e'(h)\partial_t^2 h - \partial_\nu(E^{\nu\mu}\partial_\mu h) = \underbrace{\tilde{J}\partial_t\tilde{a}^{\nu\alpha}\partial_\nu v_\alpha}_{:=F_0} - e''(h)(\partial_t h)^2
$$

that

$$
\partial_{33}h = -\frac{1}{E_{33}}\left(F_0 - e''(h)(\partial_t h)^2 - \sum_{\nu+\mu\leq 5}\partial_\nu(E^{\nu\mu}\partial_\mu h) - e'(h)\partial_t^2 h\right),
$$

because the above identity shows that the second order normal derivative $\partial_{33}h$ can be bounded by the terms containing h with the same or lower order derivates and less normal derivatives. Hence, one can apply the same method to inductively control terms containing h with more normal derivatives. For example, $\partial_{3333}h$ can be controlled in the same way by taking ∂^2 in (3.31) and then express $\partial_{3333}h$ in terms of the terms with same/lower order and ≤ 3 normal derivatives.

Therefore, combining with (3.7), (3.32), (3.37) and (3.51), one has the control for the Sobolev norm of enthalpy h and its time derivatives after taking $\epsilon > 0$ in (3.51) sufficiently small to be absorbed by $\|h\|_4^2$

$$
\sum_{k=0}^3 \|\partial_t^k h\|_{4-k}^2 + \|\sqrt{e'(h)}\partial_t^4 h\|_0^2\Big|_{t=0}^{t=T} \lesssim \mathcal{P}_0 + \int_0^T P(\mathcal{E}_\kappa(t))\, dt.
\tag{3.52}
$$

3.4 Div-Curl estimates for v

In this section we are going to do the div-curl estimates for v and its time derivatives in order to reduce the estimates of $\mathcal{E}_\kappa$ to the tangential estimates. Recall the Hodge-type decomposition in (2.5):

$$\forall s \geq 1: \quad \|X\|_s \lesssim \|X\|_0 + \|\text{curl } X\|_{s-1} + \|\text{div } X\|_{s-1} + |X \cdot N|_{s-0.5}.$$

Let $X = v, \partial_t v, \partial_t^2 v, \partial_t^3 v$ and $s = 4, 3, 2, 1$, respectively. We get

$$\begin{aligned}
\|v\|_4 &\lesssim \|v\|_0 + \|\text{div } v\|_3 + \|\text{curl } v\|_3 + |\overline{\partial}^3(v \cdot N)|_{0.5} \\
\|\partial_t v\|_3 &\lesssim \|\partial_t v\|_0 + \|\text{div } \partial_t v\|_2 + \|\text{curl } \partial_t v\|_2 + |\overline{\partial}^2(\partial_t v \cdot N)|_{0.5} \\
\|\partial_t^2 v\|_2 &\lesssim \|\partial_t^2 v\|_0 + \|\text{div } \partial_t^2 v\|_1 + \|\text{curl } \partial_t^2 v\|_1 + |\overline{\partial}(\partial_t^2 v \cdot N)|_{0.5} \\
\|\partial_t^3 v\|_1 &\lesssim \|\partial_t^3 v\|_0 + \|\text{div } \partial_t^2 v\|_0 + \|\text{curl } \partial_t^2 v\|_0 + |\partial_t^3 v \cdot N|_{0.5}.
\end{aligned} \tag{3.53}$$

First, the L^2-norm of v is controlled by:

$$\|v(T)\|_0 \leq \|v_0\|_0 + \int_0^T \|\partial_t v(t)\| \, dt, \tag{3.54}$$

while for $\|v_t\|_0, \|v_{tt}\|_0$ and $\|v_{ttt}\|_0$, we commute ∂_t through $\partial_t v = -\nabla_{\tilde{a}} h + g e_3$ and obtain

$$\begin{aligned}
\|\partial_t v(T)\|_0 &\lesssim \|\partial_t v(0)\|_0 + \int_0^T \|\partial_t^2 v(t)\|_0 \, dt \lesssim \|\partial_t v(0)\|_0 + \int_0^T P(\|\partial^2\eta\|_1, \|\partial\eta\|_{L^\infty}, \|v\|_3, \|h\|_1, \|\partial_t h\|_1) \, dt \\
\|\partial_t^2 v(T)\|_0 &\lesssim \|\partial_t^2 v(0)\|_0 + \int_0^T P(\|\partial^2\eta\|_1, \|\partial\eta\|_{L^\infty}, \|\partial_t v\|_1, \|h\|_2, \|\partial_t h\|_1, \|\partial_t^2 h\|_1) \, dt \\
\|\partial_t^3 v(T)\|_0 &\lesssim \|\partial_t^3 v(0)\|_0 + \int_0^T P(\|\partial^2\eta\|_1, \|\partial\eta\|_{L^\infty}, \|v\|_3, \|\partial_t v\|_2, \|\partial_t^2 v\|_1, \|h\|_3, \|\partial_t h\|_2, \|\partial_t^2 h\|_1, \|\partial_t^3 h\|_0) \, dt.
\end{aligned} \tag{3.55}$$

Now we are going to control the curl term. Recall that $-\nabla_{\tilde{a}} h$ is the Eulerian gradient of h whose Eulerian curl is 0. This motivates us to take Eulerian curl in the equation $\partial_t v = -\nabla_{\tilde{a}} h + g e_3$ to get

$$\partial_t (\text{curl}_{\tilde{a}} v)_\lambda = \epsilon_{\lambda\mu\alpha} \partial_t \tilde{a}^{\nu\mu} \partial_\nu v^\alpha, \tag{3.56}$$

where $(\text{curl}_{\tilde{a}} X)_\lambda := \epsilon_{\lambda\mu\alpha} \tilde{a}^{\nu\mu} \partial_\nu X^\alpha$ is the Eulerian curl of X and $\epsilon_{\lambda\mu\alpha}$ is the sign of the 3-permutation $(\lambda\mu\alpha) \in S_3$. Taking ∂^3 in the last equation and then taking inner proudct with $\partial^3 \text{curl}_{\tilde{a}} v$, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\partial^3 \text{curl}_{\tilde{a}} v|^2 = \int_\Omega (\partial^3 \text{curl}_{\tilde{a}} v^\lambda) \partial^3 (\epsilon_{\lambda\mu\alpha} \partial_t \tilde{a}^{\nu\mu} \partial_\nu v^\alpha) \, dy \lesssim P(\|v\|_4, \|\partial^2\eta\|_2), \tag{3.57}$$

and thus

$$\|\text{curl}_{\tilde{a}} v(T)\|_3 \lesssim P(\|v_0\|_4) + \int_0^T P(\|v\|_4, \|\partial^2\eta\|_2) \, dt. \tag{3.58}$$

The Lagrangian curl only differs the Eulerian curl from a sufficiently small term which shall be absorbed in the LHS

$$\begin{aligned}
\|\text{curl } v(T)\|_3 &= \|\text{curl}_{I-\tilde{a}} v(T)\|_3 + \|\text{curl}_{\tilde{a}} v\|_3 \\
&\lesssim \|\text{Id} - \tilde{a}\|_3 \|v\|_4 + P(\|v_0\|_4) + \int_0^T P(\|v\|_4, \|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}) \, dt \\
&\lesssim \epsilon \|v\|_4 + P(\|v_0\|_4) + \int_0^T P(\|v\|_4, \|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}) \, dt.
\end{aligned} \tag{3.59}$$

Commuting ∂_t^k $(k = 1, 2, 3)$ though (3.56), we get the evolution equation for $\text{curl}_{\tilde{a}} \partial_t^k v$

$$\partial_t (\text{curl}_{\tilde{a}} \partial_t^k v)_\lambda = \epsilon_{\lambda\mu\alpha} \partial_t^k (\partial_t \tilde{a}^{\nu\mu} \partial_\nu v^\alpha) - \partial_t ([\partial_t^k, \text{curl}_{\tilde{a}}] v)_\lambda.$$

Commuting ∂^{3-k} through the above equation, and then taking L^2 inner product with $\partial^{3-k}(\text{curl}_{\tilde{a}}\partial_t^k v)$, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{3-k}\text{curl}_{\tilde{a}}\partial_t^k v|^2\,dy &= \int_\Omega \partial^{3-k}\left(\epsilon_{\lambda\mu\alpha}\partial_t^k(\partial_t\tilde{a}^{\nu\mu}\partial_\nu v^\alpha) - \partial_t([\partial_t^k,\text{curl}_{\tilde{a}}]v)_\lambda\right)\partial^{3-k}(\text{curl}_{\tilde{a}}\partial_t^k v)^\lambda\,dy \\
&\lesssim \|\text{curl}_{\tilde{a}}\partial_t^k v\|_0 \cdot \underbrace{\left\|\partial^{3-k}\left(\epsilon_{\lambda\mu\alpha}\partial_t^k(\partial_t\tilde{a}^{\nu\mu}\partial_\nu v^\alpha) - \partial_t([\partial_t^k,\text{curl}_{\tilde{a}}]v)_\lambda\right)\right\|_0}_{=:D_k}.
\end{aligned}
\tag{3.60}
$$

One can use $\tilde{a} \approx \partial\tilde{\eta}$, $\partial_t\eta = v + \psi$ and the estimates for ψ in Lemma 3.2 to control D_k directly. Note that the leading order terms in D_k are $\partial^{3-k}\partial_t^{k+1}\tilde{a}$ and $\partial^{4-k}\partial_t^k v$. Therefore,

$$
\sum_{k=1}^{3} D_k \lesssim \sum_{k=1}^{3} P(\|\partial_t^k v\|_{4-k}, \|\partial_t^k\psi\|_{4-k}, \|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}) \lesssim P(\mathcal{E}_\kappa),
$$

which implies

$$
\|\text{curl}\ \partial_t^k v\|_{3-k} \lesssim \epsilon\|\partial_t^k v\|_{4-k} + \mathcal{P}_0 + \int_0^T P(\mathcal{E}_\kappa(t))\,dt. \tag{3.61}
$$

For boundary terms in (3.53), we know by the trace lemma that

$$
|\overline{\partial}^3(v\cdot N)|_{0.5} = |\overline{\partial}^3 v^3|_{0.5} \lesssim \|\overline{\partial}^3\partial v^3\|_0,
$$

in which the term ∂v contains a non-tangential derivative $\partial_3 v^3$. But from the definition of divergence, we can express $\partial_3 v_3$ in terms of div v and $\overline{\partial}v$:

$$
\partial_3 v^3 = \text{div}\ v - \partial_1 v^1 - \partial_2 v^2,
$$

which gives

$$
|\overline{\partial}^3(v\cdot N)|_{0.5} \lesssim \|\overline{\partial}^4 v\|_0 + \|\overline{\partial}^3\text{div}\ v\|_0. \tag{3.62}
$$

Similarly we have

$$
|\overline{\partial}^2(\partial_t v\cdot N)|_{0.5} \lesssim \|\overline{\partial}^3\partial_t v\|_0 + \|\overline{\partial}^2\text{div}\ \partial_t v\|_0 \tag{3.63}
$$

$$
|\overline{\partial}(\partial_t^2 v\cdot N)|_{0.5} \lesssim \|\overline{\partial}^2\partial_t^2 v\|_0 + \|\overline{\partial}\text{div}\ \partial_t^2 v\|_0 \tag{3.64}
$$

$$
|\partial_t^3 v\cdot N|_{0.5} \lesssim \|\overline{\partial}\partial_t^3 v\|_0 + \|\text{div}\ \partial_t^3 v\|_0. \tag{3.65}
$$

Therefore the boundary estimates are all reduced to divergence and tangential estimates.

Now we come to estimate the divergence. Recall that the Eulerian divergence $\text{div}_{\tilde{a}}X = \text{div}\ X + (\tilde{a}^{\mu\alpha} - \delta^{\mu\alpha})\partial_\mu X_\alpha$, which together with (3.7) implies

$$
\begin{aligned}
\forall s > 2.5:\ \|\text{div}\ X\|_{s-1} &\lesssim \|\text{div}_{\tilde{a}}X\|_{s-1} + \|I-\tilde{a}\|_{s-1}\|X\|_s \lesssim \|\text{div}_{\tilde{a}}X\|_{s-1} + \epsilon\|X\|_s \\
\forall 1 \le s \le 2.5:\ \|\text{div}\ X\|_{s-1} &\lesssim \|\text{div}_{\tilde{a}}X\|_{s-1} + \|I-\tilde{a}\|_{L^\infty}\|X\|_s \lesssim \|\text{div}_{\tilde{a}}X\|_{s-1} + \epsilon\|X\|_s.
\end{aligned}
\tag{3.66}
$$

The ϵ-terms can be absorbed by $\|X\|_s$ on LHS by choosing $\epsilon > 0$ sufficiently small. So it suffices to estimate the Eulerian divergence which satisfies $\text{div}_{\tilde{a}}v = -\partial_t e(h)$. Taking time derivatives in this equation, we get

$$
\text{div}_{\tilde{a}}\partial_t^k v = -\partial_t^{k+1}e(h) - [\partial_t^k, \tilde{a}^{\mu\alpha}]\partial_\mu v_\alpha,\ \ k = 0, 1, 2, 3.
$$

The leading order terms in $\text{div}_{\tilde{a}}\partial_t^k v$ are $e'(h)\partial_t^k h\partial_t h$, $\partial_t^k\tilde{a}^{\mu\alpha}\partial_\mu v_\alpha$ and $\partial_t\tilde{a}^{\mu\alpha}\partial_\mu\partial_t^{k-1}v_\alpha$ when $k \ge 1$. Therefore, we have

$$
\begin{aligned}
\|\mathrm{div}_{\tilde a} v\|_3 &\lesssim \|e'(h)\partial_t h\|_3\\
\|\mathrm{div}_{\tilde a}\partial_t v\|_2 &\lesssim \|e'(h)\partial_t^2 h\|_2\|\partial_t h\|_2 + \|\partial v\|_2^2 \lesssim P(\|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2, \|v\|_3)\\
&\lesssim P(\|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2) + P(\|v_0\|_3) + \int_0^T P(\|\partial_t v(t)\|_3)\, dt
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|\mathrm{div}_{\tilde a}\partial_t^2 v\|_1 &\lesssim P(\|e'(h)\partial_t^3 h\|_1, \|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2, \|v\|_3, \|\partial_t v\|_2)\\
&\lesssim P(\|e'(h)\partial_t^3 h\|_1, \|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2) + \mathcal{P}_0 + \int_0^T P(\|\partial_t v\|_3, \|\partial_t^2 v\|_2)\, dt\\
\|\mathrm{div}_{\tilde a}\partial_t^3 v\|_0 &\lesssim P(\|e'(h)\partial_t^4 h\|_0, \|e'(h)\partial_t^3 h\|_1, \|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2, \|v\|_3, \|\partial_t v\|_2, \|\partial_t^2 v\|_1)\\
&\lesssim P(\|e'(h)\partial_t^4 h\|_0, \|e'(h)\partial_t^3 h\|_1, \|e'(h)\partial_t^2 h\|_2, \|\partial_t h\|_2) + \mathcal{P}_0 + \int_0^T P(\|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1)\, dt.
\end{aligned}
\tag{3.67}
$$

Combining (3.66) and (3.67), we know the divergence estimates are all be reduced to the estimates of h which has been done in Section 3.3. By choosing $\epsilon > 0$ in (3.66) to be sufficiently small, and using the estimates of h in (3.52), we finally finish the divergence estimates

$$
\sum_{k=0}^{3} \|\mathrm{div}\, \partial_t^k v\|_{3-k} \lesssim \mathcal{P}_0 + \int_0^T P(\mathcal{E}_\kappa(t))\, dt. \tag{3.68}
$$

3.5 Estimates for time derivatives of v

As a result of div-curl estimates, it suffices to estimate the L^2-norm of $\overline{\partial}^4 v, \overline{\partial}^3\partial_t v, \cdots, \partial_t^4 v$. In this part we are going to do the tangential estimates for the time derivatives of v, in order to finish the control $\|\partial_t^k v\|_{4-k}$ with $k \geq 1$. The fact that $\partial^2\eta$ and $\partial_t\partial^2\eta$ are of the same spatial regularity in Sobolev norms is essential for us to close the estimates.

Let $\mathfrak{D}^4 = \partial_t^4, \partial_t^3\overline{\partial}, \partial_t^2\overline{\partial}^2, \partial_t\overline{\partial}^3$. First we compute

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega |\mathfrak{D}^4 v|^2\, dy &= \int_\Omega \mathfrak{D}^4 v_\alpha \mathfrak{D}^4\partial_t v^\alpha\, dy\\
&= -\int_\Omega \mathfrak{D}^4 v_\alpha \mathfrak{D}^4(\tilde a^{\mu\alpha}\partial_\mu h)\\
&= -\int_\Omega (\mathfrak{D}^4 v_\alpha)\tilde a^{\mu\alpha}(\partial_\mu \mathfrak{D}^4 h)\, dy - \underbrace{\int_\Omega \mathfrak{D}^4 v_\alpha([\mathfrak{D}^4, \tilde a^{\mu\alpha}]\partial_\mu h)\, dy}_{L_1}.
\end{aligned}
\tag{3.69}
$$

In the first integral above, we integrate ∂_μ by parts and invoking the equation $\mathrm{div}_{\tilde a} v = -e'(h)\partial_t h$ to obtain:

$$
\begin{aligned}
&-\int_\Omega (\mathfrak{D}^4 v_\alpha)\tilde a^{\mu\alpha}(\partial_\mu \mathfrak{D}^4 h)\, dy\\
&= -\int_\Gamma \mathfrak{D}^4 v_\alpha \tilde a^{\mu\alpha} N_\mu \underbrace{\mathfrak{D}^4 h}_{=0}\, dS \underbrace{-\int_\Omega ([\mathfrak{D}^4, \tilde a^{\mu\alpha}]\partial_\mu v_\alpha)\mathfrak{D}^4 h\, dy}_{L_2} + \underbrace{\int_\Omega \mathfrak{D}^4 v_\alpha \partial_\mu \tilde a^{\mu\alpha}\mathfrak{D}^4 h\, dy}_{L_3} + \int_\Omega \mathfrak{D}^4 \mathrm{div}_{\tilde a} v \mathfrak{D}^4 h\, dy\\
&= -\int_\Omega \mathfrak{D}^4(e'(h)\partial_t h)\mathfrak{D}^4 h\, dy + L_2 + L_3\\
&= -\frac{d}{dt}\frac{1}{2}\int_\Omega e'(h)|\mathfrak{D}^4 h|^2\, dy + \underbrace{\int_\Omega e''(h)\partial_t h|\mathfrak{D}^4 h|^2 - [\mathfrak{D}^4, e'(h)]\partial_t h \mathfrak{D}^4 h\, dy}_{L_4} + L_2 + L_3.
\end{aligned}
\tag{3.70}
$$

It is not difficult to see L_3 and L_4 can be controlled directly:

$$L_3 \lesssim \|\mathfrak{D}^4 v\|_0 \|\partial a\|_2 \|\mathfrak{D}^4 h\|_0 \lesssim P(\mathcal{E}_\kappa(t)), \tag{3.71}$$

$$\sum_{\mathfrak{D}^4} L_4 \lesssim \|\sqrt{e'(h)}\mathfrak{D}^4 h\|_0 P\left(\sum_{k\geq 1} \|\sqrt{e'(h)}\partial_t^k \partial^{4-k} h\|_0\right) \lesssim P(\mathcal{E}_\kappa(t)). \tag{3.72}$$

To estimate L_1 and L_2, it suffices to control the commutator $[\mathfrak{D}^4, \tilde{a}]f$ in L^2-norm.

$$\begin{aligned}\|[\mathfrak{D}^4, \tilde{a}]f\|_0 &= \|(\mathfrak{D}^4 a)f + 4(\mathfrak{D}^3 a)(\mathfrak{D}f) + 6(\mathfrak{D}^2 a)(\mathfrak{D}^2 f) + 4(\mathfrak{D}a)(\mathfrak{D}^3 f)\|_0 \\ &\lesssim \|\mathfrak{D}^4 a\|_0 \|f\|_2 + \|\mathfrak{D}^3 a\|_1 \|\mathfrak{D}f\|_1 + \|\mathfrak{D}^2 a\|_1 \|\mathfrak{D}^2 f\|_1 + \|\mathfrak{D}a\|_2 \|\mathfrak{D}^3 f\|_0.\end{aligned}$$

Let $f = \partial v$ and ∂h respectively (corresponding to L_1 and L_2), and recall $a = [\partial\eta]^{-1}$. By Lemma 3.3, we have

$$\begin{aligned}L_1 &\lesssim \|\mathfrak{D}^4 v\|_0 (\|\mathfrak{D}^4 a\|_0 \|\partial v\|_2 + \|\mathfrak{D}^3 a\|_1 \|\mathfrak{D}\partial v\|_1 + \|\mathfrak{D}^2 a\|_1 \|\mathfrak{D}\partial v\|_1 + \|\mathfrak{D}a\|_2 \|\mathfrak{D}^3 \partial v\|_0) \\ &\lesssim P(\sum_{k=1}^4 \|\partial_t^k v\|_{4-k}, \|\partial^2\eta\|_2, \|v\|_4) \lesssim P(\mathcal{E}_\kappa(t)); \\ L_2 &\lesssim \|\mathfrak{D}^4 h\|_0 (\|\mathfrak{D}^4 a\|_0 \|\partial h\|_2 + \|\mathfrak{D}^3 a\|_1 \|\mathfrak{D}\partial h\|_1 + \|\mathfrak{D}^2 a\|_1 \|\mathfrak{D}\partial h\|_1 + \|\mathfrak{D}a\|_2 \|\mathfrak{D}^3 \partial h\|_0) \\ &\lesssim P(\sum_{k=0}^3 \|\partial_t^k v\|_{4-k}, \|h\|_2, \|\partial_t h\|_1, \|\partial_t^2 h\|_1, \|\partial_t^3 h\|_0, \|\partial^2\eta\|_2) \lesssim P(\mathcal{E}_\kappa(t)).\end{aligned} \tag{3.73}$$

Summing up (3.69)-(3.73), we are able to get the energy bound

$$\frac{d}{dt}\frac{1}{2}\left(\sum_{k=1}^4 \|\partial_t^k \overline{\partial}^{4-k} v\|_0^2 + \|\sqrt{e'(h)}\partial_t^k \overline{\partial}^{4-k} h\|_0^2\right) \lesssim P(\mathcal{E}_\kappa(t)). \tag{3.74}$$

3.6 Estimates for spatial derivatives of v: Alinhac good unknown method

Now it remains to control $\|\overline{\partial}^4 v\|_0^2$ to close the a priori estimates of the approximation system (3.1). It should be emphasized here that our method in Section 3.5 cannot be used in the full spatial derivatives, because the L^2 norm of commutator $[\overline{\partial}^4, \tilde{a}](\partial v)$ and $[\overline{\partial}^4, \tilde{a}](\partial h)$ is out of control due to the lack of time derivatives. To overcome such difficulty, we introduce Alinhac good unknowns for both v and h, which actually uncover that the essential leading order terms in $\overline{\partial}^4 \nabla_{\tilde{a}} v$ and $\overline{\partial}^4 \nabla_{\tilde{a}} h$ is exactly the covariant derivative $\nabla_{\tilde{a}}$ of their Alinhac good unknowns. As a result, one can commute $\overline{\partial}^4$ and $\nabla_{\tilde{a}}$ without producing any higher order commutator but $\overline{\partial}^4(\text{div}_{\tilde{a}} v)\overline{\partial}^4 h$.

Our idea is motivated by Gu-Wang [24] in the study of local wellposedness of incompressible free-boundary MHD equations. Actually, this method was first introduced by Alinhac [3], and has been frequently used in the study of free-boundary problems of incompressible fluid because the incompressibility condition (Eulerian divergence-free) eliminates the only extra term $\overline{\partial}^r(\text{div}_{\tilde{a}} v) = 0$, e.g., Masmoudi-Rousset [46], Alazard-Burq-Zuily [1], Wang-Xin [59], etc. However, there is very few work using this technique when doing interior energy control in the study of compressible fluids. Luckily, for Euler equations, $\overline{\partial}^4(\text{div}_{\tilde{a}} v)\overline{\partial}^4 h = -\overline{\partial}^4(e'(h)\partial_t h)\overline{\partial}^4 h$ exactly gives the energy term $-\frac{1}{2}\frac{d}{dt}\int_\Omega \|\overline{\partial}^4 h\|_0^2$ and thus no extra higher order term appears. This method also avoids the control of 5-th order wave equation of h together with delicate elliptic estiamtes, e.g., Lindblad-Luo [42], Luo [44], Ginsberg-Lindblad-Luo [22].

3.6.1 Introducing Alinhac good unknowns

For simplicity we replace $\overline{\partial}^4$ by $\overline{\partial}^2\overline{\Delta}$ which is more convenient for us to deal with the correction term ψ on the boundary. For a function g, we define its "Alinhac good unknown" (for the 4-th order derivative) to be

$$\mathbf{G} := \overline{\partial}^2\overline{\Delta} g - \overline{\partial}^2\overline{\Delta}\tilde{\eta} \cdot \nabla_{\tilde{a}} g = \overline{\partial}^2\overline{\Delta} g - \overline{\partial}^2\overline{\Delta}\eta_\beta \tilde{a}^{\mu\beta}\partial_\mu g. \tag{3.75}$$

Then

$$
\begin{aligned}
\overline{\partial}^2\overline{\Delta}(\nabla_{\tilde{a}}^{\alpha} g) &= \nabla_{\tilde{a}}^{\alpha}(\overline{\partial}^2\overline{\Delta} g) + (\overline{\partial}^2\overline{\Delta}\tilde{a}^{\mu\alpha})\partial_\mu g + [\overline{\partial}^2\overline{\Delta}, \tilde{a}^{\mu\alpha}, \partial_\mu g] \\
&= \nabla_{\tilde{a}}^{\alpha}(\overline{\partial}^2\overline{\Delta} g) - \overline{\partial}\overline{\Delta}(\tilde{a}^{\mu\gamma}\overline{\partial}\partial_\beta\tilde{\eta}_\gamma\tilde{a}^{\beta\alpha})\partial_\mu g + [\overline{\partial}^2\overline{\Delta}, \tilde{a}^{\mu\alpha}, \partial_\mu g] \\
&= \nabla_{\tilde{a}}^{\alpha}(\overline{\partial}^2\overline{\Delta} g) - \tilde{a}^{\beta\alpha}\partial_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma\tilde{a}^{\mu\gamma}\partial_\mu g - ([\overline{\partial}\overline{\Delta}, \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu g + [\overline{\partial}^2\overline{\Delta}, \tilde{a}^{\mu\alpha}, \partial_\mu g] \\
&= \underbrace{\nabla_{\tilde{a}}^{\alpha}(\overline{\partial}^2\overline{\Delta} g - \overline{\partial}^2\overline{\Delta}\eta_\gamma\tilde{a}^{\mu\gamma}\partial_\mu g)}_{=\nabla_{\tilde{a}}^{\alpha}\mathbf{G}} + \underbrace{\overline{\partial}^2\overline{\Delta}\eta_\gamma\nabla_{\tilde{a}}^{\alpha}(\nabla_{\tilde{a}}^{\gamma} g) - ([\overline{\partial}\overline{\Delta}, \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu g + [\overline{\partial}^2\overline{\Delta}, \tilde{a}^{\mu\alpha}, \partial_\mu g]}_{=:C^\alpha(g)},
\end{aligned}
$$

where $[\overline{\partial}^2\overline{\Delta}, f, g] := \overline{\partial}^2\overline{\Delta}(fg) - \overline{\partial}^2\overline{\Delta}(f)g - f\overline{\partial}^2\overline{\Delta}(g)$. A direct computation yields that

$$
\begin{aligned}
\|\overline{\partial}^2\overline{\Delta}\eta_\gamma\nabla_{\tilde{a}}^{\alpha}(\nabla_{\tilde{a}}^{\gamma} g)\|_0 &\lesssim \|\tilde{\eta}\|_4\|\nabla_{\tilde{a}}^{\alpha}(\nabla_{\tilde{a}}^{\gamma} g)\|_{L^\infty} \lesssim P(\|\partial^2\eta\|_2)\|\partial g\|_3; \\
\|([\overline{\partial}\overline{\Delta}, \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu g\|_0 &\lesssim \|[\overline{\partial}\overline{\Delta}, \tilde{a}^{\mu\gamma}\tilde{a}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma\|_0\|g\|_{W^{1,\infty}} \lesssim P(\|\partial\tilde{a}\|_{L^\infty}, \|\tilde{\eta}\|_4)\|g\|_3 \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})\|g\|_3 \\
\|[\overline{\partial}^2\overline{\Delta}, \tilde{a}^{\mu\alpha}, \partial_\mu g]\|_0 &\lesssim (\|\partial\tilde{a}\|_2 + \|\tilde{a}\|_{L^\infty})\|g\|_4 \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})\|\partial g\|_3.
\end{aligned}
$$

Therefore, Alinhac good unknown enjoys the following important properties:

$$\overline{\partial}^2\overline{\Delta}(\nabla_{\tilde{a}}^{\alpha} g) = \nabla_{\tilde{a}}^{\alpha}\mathbf{G} + C^\alpha(g) \tag{3.76}$$

with

$$\|C^\alpha(g)\| \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})\|\partial g\|_3. \tag{3.77}$$

3.6.2 Tangential estimates of v: Interior part

Now we introduce the Alinhac good unknowns for v and h

$$\mathbf{V} := \overline{\partial}^2\overline{\Delta} v - \overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} v \tag{3.78}$$

$$\mathbf{H} := \overline{\partial}^2\overline{\Delta} h - \overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} h. \tag{3.79}$$

Applying $\overline{\partial}^2\overline{\Delta}$ to the second equation in system (3.1) and then using (3.78), (3.79) to get

$$\partial_t\mathbf{V} = -\nabla_{\tilde{a}}\mathbf{H} + \underbrace{\partial_t(\overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}} v) - C(h)}_{=:\mathbf{F}}, \tag{3.80}$$

subject to the boundary condition

$$\mathbf{H} = -\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\tilde{a}^{3\beta}\partial_3 h \quad \text{on } \Gamma, \tag{3.81}$$

with the continuity equation

$$\nabla_{\tilde{a}}\cdot\mathbf{V} = \overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\tilde{a}} v) - C^\alpha(v_\alpha) \quad \text{in } \Omega. \tag{3.82}$$

Note that for any function g with its Alinhac good unknown $\mathbf{G}$ defined by (3.75), one has

$$
\begin{aligned}
\|\overline{\partial}^4 g(T)\|_0 \approx \|\overline{\partial}^2\overline{\Delta} g(T)\|_0 &\lesssim \|\mathbf{G}(T)\|_0 + \|\overline{\partial}^2\overline{\Delta}\tilde{\eta}\tilde{a}\partial g\|_0 \\
&\lesssim \|\mathbf{G}(T)\|_0 + \|\overline{\partial}^2\overline{\Delta}\eta(T)\|_0\|a\|_{L^\infty}\|\partial g(T)\|_2 \\
&\lesssim \|\mathbf{G}(T)\|_0 + P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})(\|g(0)\|_3 + \int_0^T \|\partial_t g(t)\|_3\, dt)
\end{aligned}
\tag{3.83}
$$

The time integral $\int_0^T P(\|\partial_t g(t)\|_3)\, dt$ can be controlled by $\int_0^t P(\mathcal{E}_\kappa)$ by plugging $g = v$ and $g = h$. Therefore it suffices to bound $\|\mathbf{V}\|_0^2 + \|\mathbf{H}\|_0^2$ to close the estimates for $\|\overline{\partial}^4 v\|_0^2 + \|\overline{\partial}^4 h\|_0^2$. Taking L^2 inner product between (3.80) and $\mathbf{V}$, one gets

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2\,dy = -\int_\Omega \nabla_{\tilde a}\mathbf{H}\cdot\mathbf{V}\,dy + \int_\Omega \mathbf{F}\cdot\mathbf{V}\,dy. \tag{3.84}$$

The second term on the RHS of (3.84) can be directly controlled

$$\begin{aligned}
\int_\Omega \mathbf{F}\cdot\mathbf{V}\,dy &\le (\|\partial_t(\overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}v)\|_0 + \|C(h)\|_0)\|\mathbf{V}\|_0\\
&\lesssim (P(\|\overline{\partial}^2\overline{\Delta}\eta\|_0, \|\overline{\partial}^2\overline{\Delta}v\|_0, \|\partial\eta\|_{L^\infty}, \|v\|_3, \|\partial_t v\|_3) + P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})\|\partial h\|_3)\|\mathbf{V}\|_0\\
&\lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|\partial_t v\|_3, \|\partial h\|_3)P(\|\tilde\eta\|_4, \|v\|_4)\\
&\lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|\partial_t v\|_3, \|\partial h\|_3).
\end{aligned} \tag{3.85}$$

For the first term in LHS of (3.84), we integrate by part and use (3.81), (3.82) to get

$$\begin{aligned}
-\int_\Omega \nabla_{\tilde a}\mathbf{H}\cdot\mathbf{V} &= -\int_\Omega \tilde a^{\mu\alpha}\partial_\mu\mathbf{H}\cdot\mathbf{V}_\alpha\,dy\\
&= -\int_\Gamma \mathbf{H}(\tilde a^{\mu\alpha}N_\mu\mathbf{V}_\alpha)dS + \int_\Omega \mathbf{H}(\nabla_{\tilde a}\cdot\mathbf{V})\,dy + \int_\Omega(\partial_\mu\tilde a^{\mu\alpha})\mathbf{H}\mathbf{V}_\alpha\,dy\\
&= \int_\Gamma \partial_3 h\overline{\partial}^2\overline{\Delta}\tilde\eta_\beta\tilde a^{3\beta}\tilde a^{3\alpha}\mathbf{V}_\alpha dS + \int_\Omega \mathbf{H}\overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\tilde a}v)\,dy \underbrace{-\int_\Omega \mathbf{H}C^\alpha(v_\alpha) + \int_\Omega(\partial_\mu\tilde a^{\mu\alpha})\mathbf{H}\mathbf{V}_\alpha\,dy}_{L_0}\\
&=: I + K + L_0.
\end{aligned} \tag{3.86}$$

First, L_0 can be directly controlled by $P(\mathcal{E}_\kappa)$ by using (3.77)

$$L_0 \lesssim \|\mathbf{H}\|_0 P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty})\|v\|_4 + \|\partial\tilde a\|_2\|\mathbf{H}\|_0\|\mathbf{V}\|_0 \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_4, \|h\|_4). \tag{3.87}$$

Then we use $\mathrm{div}_{\tilde a}v = -e'(h)\partial_t h$ to bound K

$$\begin{aligned}
K &:= \int_\Omega \mathbf{H}\overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\tilde a}v)\,dy = -\int_\Omega(\overline{\partial}^2\overline{\Delta}h - \overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}h)\overline{\partial}^2\overline{\Delta}(e'(h)\partial_t h)\,dy\\
&= -\frac{d}{dt}\frac{1}{2}\int_\Omega e'(h)|\overline{\partial}^2\overline{\Delta}h|^2\,dy + \frac{1}{2}\int_\Omega e''(h)\partial_t h|\overline{\partial}^2\overline{\Delta}h|^2\,dy + \int_\Omega \mathbf{H}([\overline{\partial}^2\overline{\Delta}, e'(h)]\partial_t h)\,dy\\
&\quad \underbrace{-\int_\Omega e'(h)(\overline{\partial}^2\overline{\Delta}\partial_t h)\overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}h\,dy}_{:=K^*}\\
&\lesssim -\frac{d}{dt}\frac{1}{2}\int_\Omega e'(h)|\overline{\partial}^2\overline{\Delta}h|^2\,dy + K^* + P(\|\partial^2\eta\|_2, \|v\|_4, \|h\|_4, \|\partial_t h\|_3).
\end{aligned} \tag{3.88}$$

K^* cannot be bounded directly because it contains a higher order term $\overline{\partial}^2\overline{\Delta}\partial_t h$, but we can consider its time integral and integrate ∂_t by parts, then using ϵ-Young inequality to absorb the ϵ-term.

$$\begin{aligned}
\int_0^T K^*(t)\,dt &= -\int_0^T\int_\Omega e'(h)(\overline{\partial}^2\overline{\Delta}\partial_t h)\overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}h\,dy\,dt\\
&= \int_\Omega e'(h)\overline{\partial}^2\overline{\Delta}h\overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}h\,dy\bigg|_{t=0}^{t=T} + \int_0^T\int_\Omega e'(h)\overline{\partial}^2\overline{\Delta}h\partial_t(\overline{\partial}^2\overline{\Delta}\tilde\eta\cdot\nabla_{\tilde a}h)\,dy\\
&\lesssim \|e'(h)\overline{\partial}^2\overline{\Delta}h\|_0\|\partial^2\eta\|_2\|\nabla_{\tilde a}h\|_{L^\infty}\bigg|_{t=T} + P(\|h(0)\|_4, \|v_0\|_4) + \int_0^T P(\|h\|_4, \|\partial^2\eta\|_2, \|v\|_4)\,dt.
\end{aligned} \tag{3.89}$$

Using ϵ-Young's inequality, we have

$$
\begin{aligned}
&\|e'(h)\overline{\partial}^2\overline{\Delta}h\|_0\|\partial^2\eta\|_2\|\nabla_{\tilde{a}}h\|_{L^\infty}\Big|_{t=T}\\
&\lesssim \epsilon\|e'(h)\overline{\partial}^2\overline{\Delta}h(T)\|_0^2+\frac{1}{8\epsilon}(\|\partial^2\eta\|_2^4+\|\nabla_{\tilde{a}}h\|_{L^\infty}^4)\\
&\lesssim \epsilon\|e'(h)\overline{\partial}^2\overline{\Delta}h(T)\|_0^2+\left(\mathcal{P}_0+\int_0^T P(\|v\|_4,\|\partial\eta\|_{L^\infty},\|h\|_3,\|\partial_t h\|_3)\,dt\right).
\end{aligned}
$$

Therefore we get the estimate of K by choosing $\epsilon>0$ sufficiently small:

$$
\int_0^T K(t)\,dt\lesssim -\|e'(h)\overline{\partial}^2\overline{\Delta}h(T)\|_0^2+\mathcal{P}_0+\int_0^T P(\|v\|_4,\|\partial^2\eta\|_2,\|\partial\eta\|_{L^\infty},\|h\|_4,\|\partial_t h\|_3)\,dt. \tag{3.90}
$$

3.6.3 Tangential estimates of v: Boundary part

Now it remains to control the boundary term I, where the Taylor sign boundary term in $\mathcal{E}_\kappa$ is produced and the correction term ψ exactly eliminates the extra out-of-control terms produced by the tangential smoothing (these terms are 0 if $\kappa=0$).

$$
\begin{aligned}
I&=\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\mathbf{V}_\alpha\,dS\\
&=\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta(\overline{\partial}^2\overline{\Delta}v_\alpha-\overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}}v_\alpha)\,dS\\
&=\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta(\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha-\overline{\partial}^2\overline{\Delta}\psi-\overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}}v_\alpha)\,dS.
\end{aligned}
\tag{3.91}
$$

We construct the Taylor-sign term in the energy functional $\mathcal{E}_\kappa$ from the first term.

$$
\begin{aligned}
&\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha\,dS\\
=&\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\overline{\partial}^2\overline{\Delta}\partial_t\Lambda_\kappa\eta_\alpha\,dS\\
&+\int_\Gamma(\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta)([\Lambda_\kappa,\partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}]\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha)\,dS\\
=&\frac{d}{dt}\frac{1}{2}\int_\Gamma \partial_3 h|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|^2\,dS-\frac{1}{2}\int_\Gamma \partial_t\partial_3 h|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|^2\,dS\\
&\underbrace{-\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\partial_t\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS}_{B_1}+\underbrace{\int_\Gamma(\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta)([\Lambda_\kappa,\partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}]\overline{\partial}^2\overline{\Delta}\partial_t\eta_\alpha)\,dS}_{LB_1}.
\end{aligned}
\tag{3.92}
$$

In LB_1, we integrate $\overline{\partial}^{0.5}$ by parts, use Sobolev trace lemma, (2.12) and Lemma 3.3 to get

$$
\begin{aligned}
LB_1&=\int_\Gamma(\overline{\partial}^{1.5}\overline{\Delta}\Lambda_\kappa\eta_\beta)\overline{\partial}^{0.5}([\Lambda_\kappa,\partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}]\overline{\partial}(\overline{\partial}\overline{\Delta}\partial_t\eta_\alpha))\,dS\\
&\lesssim\|\partial^2\eta\|_2|\partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}|_{W^{1,\infty}}|\overline{\partial}\overline{\Delta}\partial_t\eta_\alpha|_{0.5}\\
&\lesssim\|\partial^2\eta\|_2\|h\|_4\|\partial\tilde{a}\|_{L^\infty}\|v+\psi\|_4\lesssim P(\|\partial^2\eta\|_2,\|\partial\eta\|_{L^\infty},\|v\|_3,\|h\|_4).
\end{aligned}
\tag{3.93}
$$

Next, we plug $\partial_t\tilde{a}^{3\alpha}=-\tilde{a}^{3\gamma}\partial_\mu\partial_t\tilde{\eta}_\gamma\tilde{a}^{\mu\alpha}$ into B_1 and then separate the norm derivative of $\tilde{\eta}_\gamma$ from tangential derivatives.

$$\begin{aligned}
B_1 &= \underbrace{\int_\Gamma \partial_3 h \tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\partial_3\partial_t\tilde{\eta}_\gamma\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS}_{LB_2} + \int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\overline{\partial}_i\partial_t\tilde{\eta}_\gamma\tilde{a}^{i\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS \\
&= LB_2 + \underbrace{\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\overline{\partial}_i\partial_t\Lambda_\kappa^2\psi_\gamma\tilde{a}^{i\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS}_{LB_3} \\
&\quad + \underbrace{\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\gamma\tilde{a}^{i\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\,dS}_{B_1^*}.
\end{aligned} \tag{3.94}$$

LB_2 can be directly bounded

$$\begin{aligned}
LB_2 &\lesssim |\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0^2|\partial_3 h\tilde{a}^{3\gamma}\partial_3\partial_t\tilde{\eta}_\gamma|_{L^\infty} \\
&\lesssim |\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0^2 P(\|h\|_3,\|v\|_3,\|\partial\eta\|_{L^\infty}) \lesssim P(\mathcal{E}_\kappa).
\end{aligned} \tag{3.95}$$

In LB_3, the term $\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha$ cannot be directly bounded, but we can use (2.8) in Lemma 2.4 to control this term by $(1/\sqrt{\kappa})|\eta|_{3.5}$.

$$\begin{aligned}
LB_3 &\lesssim |\partial_3 h\tilde{a}^{3\gamma}\tilde{a}^{i\alpha}|_{L^\infty}|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0|\overline{\partial}\Lambda_\kappa^2\psi|_{L^\infty}\frac{1}{\sqrt{\kappa}}|\overline{\partial}\overline{\Delta}\eta|_{0.5} \\
&\lesssim \frac{1}{\sqrt{\kappa}}P(\|\partial^2\eta\|_2,\|v\|_3,\|h\|_3)|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0|\overline{\partial}\psi|_{L^\infty}
\end{aligned}$$

The factor $1/\sqrt{\kappa}$ can be eliminated by plugging the expression of ψ in (3.2). We apply Sobolev embedding $W^{1,4}(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$ first, and note that $\overline{\partial}\psi = P_{\geq 1}(\overline{\partial}\overline{\Delta}^{-1}(\cdots))$ does not contain the low-frequency part, which (actually follows from the Littlewood-Paley characterization of $W^{1,4}$ and $\dot{W}^{1,4}$) implies $|\overline{\partial}\psi|_{W^{1,4}} \approx |\overline{\partial}\psi|_{\dot{W}^{1,4}} \approx |\overline{\Delta}\psi|_{L^4}$. Hence, we have

$$|\overline{\partial}\psi|_{L^\infty} \lesssim |\overline{\Delta}\psi|_{L^4} = \left|\mathbb{P}\underbrace{\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)}_{f}\right|_{L^4}.$$

According to the Littlewood-Paley characterization of $L^4(\mathbb{R}^2)$ and the almost orthogonality property, we know

$$|\mathbb{P}f|_{L^4} \approx \left|\left(\sum_{N\in\mathbb{Z}}|\tilde{P}_N P_{\geq 1}f|^2\right)^{1/2}\right|_{L^4} \approx \left|\left(\sum_{N\geq 0}|\tilde{P}_N f|^2\right)^{1/2}\right|_{L^4} \lesssim \left|\left(\sum_{N\in\mathbb{Z}}|\tilde{P}_N f|^2\right)^{1/2}\right|_{L^4} \approx |f|_{L^4},$$

where $\tilde{P}$ is the Littlewood-Paley projection with respect to $\tilde{\chi}(\cdot) := \chi(2\cdot)$.

Remark. For more details of Littlewood-Paley characterization of Sobolev spaces, we refer readers to Chapter 1.3 in Grafakos [23] or Appendix A in Tao [54].

Hence, we have

$$\begin{aligned}
|\overline{\partial}\psi|_{L^\infty} &\lesssim \left|\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right|_{L^4} \\
&\lesssim \left|\overline{\Delta}(\eta_\beta - \Lambda_\kappa^2\eta_\beta)\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i(v - \Lambda_\kappa^2 v)\right|_{L^4} \\
&\lesssim |\overline{\Delta}(\eta_\beta - \Lambda_\kappa^2\eta_\beta)|_{L^\infty}|\tilde{a}^{i\beta}|_{L^\infty}|\overline{\partial}_i\Lambda_\kappa^2 v|_{0.5} + |\overline{\Delta}\tilde{\eta}_\beta|_{0.5}|\tilde{a}^{i\beta}|_{L^\infty}|\overline{\partial}(v - \Lambda_\kappa v)|_{L^\infty},
\end{aligned}$$

where in the last step we use $H^{0.5} \hookrightarrow L^4$ in $\mathbb{R}^2$. Now, recall (2.9) in Lemma 2.4 that we are able to control $|\overline{\Delta}\eta_\beta - \Lambda_\kappa^2\overline{\Delta}\eta|_{L^\infty}$ by $\sqrt{\kappa}|\overline{\Delta}\eta|_{1.5} \le \sqrt{\kappa}\|\partial^2\eta\|_2$. Similarly, $|\overline{\partial}(v - \Lambda_\kappa v)|_{L^\infty} \lesssim \sqrt{\kappa}|\overline{\partial}v|_{1.5} \lesssim \sqrt{\kappa}\|v\|_3$. Therefore, one has

$$|\overline{\partial}\psi|_{L^\infty} \lesssim \sqrt{\kappa}P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_3),$$

and

$$LB_3 \lesssim P(\|\partial^2\eta\|_2, \|\partial\eta\|_{L^\infty}, \|v\|_3, \|h\|_3)|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0 \lesssim P(\mathcal{E}_\kappa). \tag{3.96}$$

As for B_1^*, it cannot be directly bounded, but together with another out-of-control term will be exactly eliminated by the correction term in (3.91).

Now we start to control the third term in (3.91). Again we separate the normal derivative of v from tangential derivatives

$$\begin{aligned}
&-\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}\cdot\nabla_{\tilde{a}}v_\alpha\, dS\\
&= -\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma\tilde{a}^{3\gamma}\partial_3 v_\alpha\, dS \underbrace{-\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma\tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha\, dS}_{B_2^*}\\
&= \int_\Gamma (-\partial_3 h\tilde{a}^{3\alpha}\partial_3 v_\alpha)(\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta)(\tilde{a}^{3\gamma}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma)\, dS + B_2^*\\
&\lesssim |\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta|_0^2 P(\|v\|_3, \|\partial\eta\|_{L^\infty}, \|h\|_3) + B_2^* \lesssim P(\mathcal{E}_\kappa) + B_2^*,
\end{aligned} \tag{3.97}$$

where in the last step we control $|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta|_0$ as follows

$$\begin{aligned}
|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta|_0 &\le |\Lambda_\kappa(\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta)|_0 + |[\Lambda_\kappa, \tilde{a}^{3\beta}]\overline{\partial}(\overline{\partial}\overline{\Delta}\Lambda_\kappa\eta_\beta)|_0\\
&\lesssim |\Lambda_\kappa(\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta)|_0 + |\tilde{a}|_{W^{1,\infty}}|\overline{\partial}^3\eta|_0\\
&\lesssim P(\|\partial\eta\|_{L^\infty}, \|\partial^2\eta\|_2) \lesssim P(\mathcal{E}_\kappa).
\end{aligned} \tag{3.98}$$

So far, what remains to be bounded is the second term in RHS of (3.91)

$$I_2 := -\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\psi, \tag{3.99}$$

and

$$B_1^* = \int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta\tilde{a}^{3\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\gamma\tilde{a}^{i\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha\, dS \tag{3.100}$$

and

$$B_2^* = -\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\gamma\tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha\, dS \tag{3.101}$$

Plugging the expression of ψ in (3.2) into (3.99), one has

$$I_2 = -\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2(\overline{\Delta}\eta_\gamma\tilde{a}^{ir}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha)\, dS \tag{3.102}$$

$$+\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2\tilde{\eta}_\gamma\tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha\, dS \tag{3.103}$$

$$+\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta([\overline{\partial}^2, \tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha]\overline{\Delta}\tilde{\eta}_\gamma)\, dS \tag{3.104}$$

$$+\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta\overline{\partial}^2 P_{<1}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v - \overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right). \tag{3.105}$$

It is clear that (3.103) exactly cancels with B_2^* in (3.101), and (3.104) can be directly bounded

$$
\begin{aligned}
(3.104) &\lesssim |\partial_3 h\tilde{a}^{3\alpha}|_{L^\infty}|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0|[\overline{\partial}^2,\tilde{a}^{i\gamma}\overline{\partial}_i v_\alpha]\overline{\Delta}\tilde{\eta}_\gamma|_0 \\
&\lesssim P(\|h\|_3,\|\partial^2\eta\|_2,\|\partial\eta\|_{L^\infty},\|v\|_4,|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0)\lesssim P(\mathcal{E}_\kappa).
\end{aligned}
\tag{3.106}
$$

For (3.105), one can apply Bernstein's inequality (2.3) in Lemma 2.3 and (3.98) to get

$$
\begin{aligned}
(3.105) &\lesssim |\partial_3 h\tilde{a}^{3\alpha}|_{L^\infty}|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0\left|P_{<1}\left(\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v-\overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right)\right|_{\dot{H}^2} \\
&\lesssim |\partial_3 h\tilde{a}^{3\alpha}|_{L^\infty}|\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta|_0\cdot\left|\overline{\Delta}\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i\Lambda_\kappa^2 v-\overline{\Delta}\Lambda_\kappa^2\eta_\beta\tilde{a}^{i\beta}\overline{\partial}_i v\right|_0 \\
&\lesssim P(\mathcal{E}_\kappa).
\end{aligned}
\tag{3.107}
$$

For (3.102), we try to move one Λ_κ on η_β to η_α to produce the cancellation with B_1^* in (3.100):

$$
(3.102) = -\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta(\tilde{a}^{3\alpha}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha)(\tilde{a}^{i\gamma}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\gamma)\,dS \tag{3.108}
$$

$$
-\int_\Gamma \partial_3 h\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\beta([\Lambda_\kappa,\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\tilde{a}^{ir}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha]\overline{\partial}^2\overline{\Delta}\eta_\gamma)\,dS \tag{3.109}
$$

$$
-\int_\Gamma \partial_3 h\tilde{a}^{3\alpha}\tilde{a}^{3\beta}\overline{\partial}^2\overline{\Delta}\tilde{\eta}_\beta([\overline{\partial}^2,\tilde{a}^{i\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha]\overline{\Delta}\eta_\gamma)\,dS \tag{3.110}
$$

Now we see that (3.108) exactly cancels with B_1^* in (3.100). The terms in (3.109) can be controlled by using the mollifier property (2.12) after integrating $\overline{\partial}^{0.5}$ by part (similar to the estimates of LB_1), and (3.110) can be directly controlled by using Sobolev trace lemma. We omit the detailed computation here.

$$
(3.109)+(3.110)\lesssim P(\mathcal{E}_\kappa). \tag{3.111}
$$

Finally, summing up (3.92)-(3.101), (3.111) and plugging it into (3.91), we get the estimate for the boundary term I after using the Taylor sign condition $\partial_3 h\le -c_0/2<0$:

$$
\int_0^T I(t)\,dt\lesssim\frac{1}{2}\int_\Gamma\partial_3 h|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|^2\,dS+\int_0^T P(\mathcal{E}_\kappa(t))\,dt\lesssim-\frac{c_0}{4}|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|_0^2+\int_0^T P(\mathcal{E}_\kappa(t))\,dt \tag{3.112}
$$

Now, summing up (3.84), (3.85), (3.86), (3.87), (3.90) and (3.112), we get the estimates for the Alinhac good unknowns

$$
\|\mathbf{V}(T)\|_0^2+\|e'(h)\overline{\partial}^2\overline{\Delta}h(T)\|_0^2+|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|_0^2\lesssim\mathcal{P}_0+\int_0^T P(\mathcal{E}_\kappa(t))\,dt. \tag{3.113}
$$

Finally, from the property of Alinhac good unknowns (3.83), we can get the estimates of $\overline{\partial}^4 v$ that

$$
\|\overline{\partial}^4 v(T)\|_0^2+\|e'(h)\overline{\partial}^4 h(T)\|_0^2+|\tilde{a}^{3\alpha}\overline{\partial}^2\overline{\Delta}\Lambda_\kappa\eta_\alpha|_0^2\lesssim\mathcal{P}_0+\int_0^T P(\mathcal{E}_\kappa(t))\,dt. \tag{3.114}
$$

3.7 Closing the κ-independent a priori estimates

We conclude this section by deriving the uniform-in-κ a priori bound for the energy functional $\mathcal{E}_\kappa$ of approximation system (3.1). Let $\mathcal{T}(t):=-\frac{1}{\partial_3 h(t)}$. Then

$$
\frac{d}{dt}\|\mathcal{T}(t)\|_{L^\infty}=\|\mathcal{T}(t)\|_{L^\infty}^2\|\partial_3\partial_t h(t)\|_{L^\infty}\le\|\mathcal{T}(t)\|_{L^\infty}^2\mathcal{E}_\kappa. \tag{3.115}
$$

This implies that the physical sign condition can be propagated if $\mathcal{E}_\kappa$ remains finite. Next, by plugging (3.53), (3.54), (3.55), (3.59), (3.61), (3.62)-(3.65), (3.68), (3.74) and (3.114) into (3.3), with $\epsilon > 0$ chosen sufficiently small, together with the estimates for $\|\partial\eta\|_{L^\infty}$ and $\|\partial^2\eta\|_2$, i.e.,

$$\|\partial\eta\|_{L^\infty} \le \underbrace{\|\partial\eta_0\|_{L^\infty}}_{=1} + \int_0^T \|v(t) + \psi(t)\|_{L^\infty}\,dt \lesssim 1 + \int_0^T \|v(t)\|_2 + \|\psi(t)\|_2\,dt, \tag{3.116}$$

$$\|\partial^2\eta\|_2 \le \|\partial^2\eta_0\|_2 + \int_0^T \|v(t)\|_4\,dt, \tag{3.117}$$

we get

$$\mathcal{E}_\kappa(T) \lesssim \mathcal{P}_0 + \int_0^T P(\mathcal{E}_\kappa(t))\,dt. \tag{3.118}$$

Now, (3.4) follows from (3.118) and the Gronwall-type inequality in Tao [54], which finishes the proof of Proposition 3.1.

4 Construction of the solution to the approximation system

The goal of this section is to construct the solution to the κ-approximation (nonlinear) system (3.1) by an iteration of the approximate solutions $\{(v^{(n)}, h^{(n)}, \eta^{(n)})\}_{n=0}^\infty$. We start with $(v^{(0)}, h^{(0)}, \eta^{(0)}) = (v^{(1)}, h^{(1)}, \eta^{(1)}) = (0, 0, \mathrm{Id})$. Inductively, given $(v^{(n)}, h^{(n)}, \eta^{(n)})$ for some $n \ge 1$, we construct the $(n+1)$-th approximate solutions $(v^{(n+1)}, h^{(n+1)}, \eta^{(n+1)})$ from the linearization of (3.1) near $a^{(n)} := (\partial\eta^{(n)})^{-1}$:

$$\begin{cases} \partial_t \eta^{(n+1)} = v^{(n+1)} + \psi^{(n)} & \text{in } \Omega, \\ \partial_t v^{(n+1)} = -\nabla_{\tilde{a}^{(n)}} h^{(n+1)} - g e_3 & \text{in } \Omega, \\ \mathrm{div}_{\tilde{a}^{(n)}} v^{(n+1)} = -e'(h^{(n)})\partial_t h^{(n+1)} & \text{in } \Omega, \\ h^{(n+1)} = 0 & \text{on } \Gamma, \\ (\eta^{(n+1)}, v^{(n+1)}, h^{(n+1)})|_{t=0} = (\mathrm{Id}, v_0, h_0). \end{cases} \tag{4.1}$$

Here $\tilde{a}^{(n)} := (\partial\tilde{\eta}^{(n)})^{-1}$ and the correction term $\psi^{(n)}$ is determined by (3.2) with $\eta = \eta^{(n)}, v = v^{(n)}, \tilde{a} = \tilde{a}^{(n)}$ in that equation. Specifically, we need following facts for the linearized approximation system (4.1) to construct a solution to the κ-approximation (nonlinear) system (3.1):

- System (4.1) has a (unique) solution (in a suitable function space).
- The solution of (4.1) constructed in the last step has an energy estimate uniformly in n.
- The approximate solutions $\{(v^{(n)}, h^{(n)}, \eta^{(n)})\}_{n=0}^\infty$ converge strongly (in some Sobolev spaces).

4.1 A priori estimates for the Linearized approximation system

Before we construct the solution of (4.1), we would like to derive the uniform-in-n a priori estimates for this system. Define the energy functional for (4.1) to be

$$\mathcal{E}^{(n+1)}(t) := \|\partial^2\eta^{(n+1)}(t)\|_2^2 + \|\partial\eta^{(n+1)}(t)\|_{L^\infty}^2 + \sum_{k=0}^4 \|\partial_t^{4-k} v^{(n+1)}(t)\|_k^2 + \|\partial_t^{4-k} h^{(n+1)}(t)\|_k^2 + W^{(n+1)}(t), \tag{4.2}$$

where $W^{(n)}$ is the energy functional for the 5-th order wave equation of $h^{(n)}$

$$W^{(n+1)}(t) := \sum_{k=0}^4 \|\partial_t^{5-k} h^{(n+1)}(t)\|_k^2 + \|\partial_t^{4-k} \nabla_{\tilde{a}^{(n)}} h^{(n+1)}(t)\|_k^2. \tag{4.3}$$

Our conclusion is

Proposition 4.1. For the solution $(v^{(n+1)}, h^{(n+1)}, \eta^{(n+1)})$ of (4.1), there exists $T_\kappa > 0$ sufficiently small, depending only on $\kappa > 0$ such that

$$\sup_{0\le t\le T_\kappa} \mathcal{E}^{(n+1)}(t) \lesssim \mathcal{P}_0. \tag{4.4}$$

Remark. As we will see in the following computation, the control of 4-th order derivatives of v and h does not need the energy of 5-th order wave equation of h; the only important difference from the a priori estimates for (3.1) is the boundary term (4.40) for which we apply the property of tangential smoothing to give a direct control with an extra factor $1/\kappa$, instead of producing subtle cancellation as in Section 3.6. **However, we included $W^{(n)}$ in $\mathcal{E}^{(n)}$ since we need this constraint when constructing the function space when proving the existence of the solution to the linearized system.**

We prove Proposition 4.1 by induction on n. First, when $n = -1, 0$, then the conclusion automatically holds because of $(v^{(0)}, h^{(0)}, \eta^{(0)}) = (v^{(1)}, h^{(1)}, \eta^{(1)}) = (0, 0, \mathrm{Id})$. Suppose uniform bound holds for all positive integers$\le n-1$. Then from the induction hypothesis, one has

$$\forall k \le n, \quad \sup_{0\le t\le T_\kappa} \mathcal{E}^{(k)}(t) \lesssim \mathcal{P}_0. \tag{4.5}$$

We would like to first simplify our notation before we derive the energy estimate for $(v^{(n+1)}, h^{(n+1)}, \eta^{(n+1)})$. We denote $(v^{(n)}, h^{(n)}, \eta^{(n)})$ by $(\mathring{v}, \mathring{h}, \mathring{\eta})$ and $\mathring{a} := [\partial\mathring{\eta}]^{-1}$, $\mathring{J} := \det[\partial\mathring{\eta}]$; and $(v^{(n+1)}, h^{(n+1)}, \eta^{(n+1)})$ by (v, h, η). The smoothed version of $\mathring{a}, \mathring{\eta}, \mathring{J}$ are denoted by $\tilde{\mathring{a}}, \tilde{\mathring{\eta}}, \tilde{\mathring{J}}$ respectively. Besides, we define $\sigma := e'(\mathring{h})$. Now, the linearized system (4.1) becomes

$$\begin{cases} \partial_t \eta = v + \mathring{\psi} & \text{in } \Omega, \\ \partial_t v = -\nabla_{\tilde{\mathring{a}}} h - g e_3 & \text{in } \Omega, \\ \mathrm{div}_{\tilde{\mathring{a}}} v = -\sigma \partial_t h & \text{in } \Omega, \\ h = 0 & \text{on } \Gamma, \\ (\eta, v, h)|_{t=0} = (\mathrm{Id}, v_0, h_0). \end{cases} \tag{4.6}$$

4.1.1 Uniform-in-n bounds for the coefficients

The energy functional for $\mathring{v}, \mathring{h}, \mathring{\eta}$ is

$$\mathring{\mathcal{E}} := \|\partial^2 \mathring{\eta}\|_2^2 + \|\partial\mathring{\eta}\|_{L^\infty}^2 + \sum_{k=0}^{4} \|\partial_t^{4-k} \mathring{v}\|_k^2 + \|\partial_t^{4-k} \mathring{h}\|_k^2 + \mathring{W} \lesssim \mathcal{P}_0, \tag{4.7}$$

where $\mathring{W}$ is the energy functional for the 5-th order wave equation of $\mathring{h}$, i.e.,

$$\mathring{W} := \sum_{k=0}^{4} \|\partial_t^{5-k} \mathring{h}\|_k^2 + \|\partial_t^{4-k} \nabla_{\tilde{a}^{(n-1)}} \mathring{h}\|_k^2. \tag{4.8}$$

We have the following bounds for $\mathring{a}, \mathring{\eta}, \mathring{J}$ provided they hold for $(v^{(k)}, h^{(k)}, \eta^{(k)})$ for $k \le n-1$. The control of these quantities are important when we do the uniform-in-n a priori estimates and construct the solution for system (4.6).

Lemma 4.2. Let $T \in (0, T_\kappa)$. There exists some $0 < \epsilon << 1$ and $N > 0$ such that

$$\mathring{\psi} \in L_t^\infty([0,T]; H^4(\Omega)), \ \ \partial_t^l \mathring{\psi} \in L_t^\infty([0,T]; H^{5-l}(\Omega)), \ \ \forall 1 \le l \le 4; \tag{4.9}$$

$$\|\mathring{J} - 1\|_3 + \|\mathring{\tilde{J}} - 1\|_3 + \|\mathrm{Id} - \mathring{\tilde{a}}\|_3 + \|\mathrm{Id} - \mathring{a}\|_3 \le \epsilon; \tag{4.10}$$

$$\partial_t \mathring{\eta} \in L^\infty([0,T]; H^4(\Omega)), \tag{4.11}$$

$$\partial_t^{l+1} \mathring{\eta} \in L^\infty([0,T]; H^{5-l}(\Omega)), \ \ \forall 1 \le l \le 4; \tag{4.12}$$

$$\mathring{J} \in L_t^\infty([0,T]; L^\infty(\Omega)), \ \ \partial \mathring{J} \in L_t^\infty([0,T]; H^2(\Omega)), \tag{4.13}$$

$$\partial_t \mathring{J} \in L_t^\infty([0,T]; H^3(\Omega)), \ \ \partial_t^{1+l} \mathring{J} \in L_t^\infty([0,T]; H^{4-l}(\Omega)), \ \ \forall 1 \le l \le 4; \tag{4.14}$$

$$1/N \le \sigma \le N, \, \partial_t^l \sigma \in L^\infty([0,T]; H^{5-l}(\Omega)), \ \ \forall 1 \le l \le 5. \tag{4.15}$$

Proof. First, the bound for $\mathring{\psi}$ and $\partial_t^l \mathring{\psi}$ for $1 \le l \le 3$ directly follows from (3.12)-(3.15) in Lemma 3.3. Then the identity

$$\mathrm{Id} - \mathring{a} = -\int_0^t \partial_t \mathring{a} = \int_0^t \mathring{a} : (\partial \partial_t \mathring{\eta}) : \mathring{a} = \int_0^t \mathring{a} : (\partial(\mathring{v} + \psi^{(n-1)})) : \mathring{a}$$

yields (4.10) by choosing ϵ suitably small (depending on T_κ). Similar results hold for $\mathring{J}$.

As for $\mathring{\eta}$, $\partial_t \mathring{\eta} = \mathring{v} + \psi^{(n-1)}$ gives the bound (4.11). Taking ∂_t^l in this equation and combining the induction hypothesis on $\mathring{E}$ and $\psi^{(n-1)}$ we can get the bound for $\partial_t^{l+1}\mathring{\eta}$ in (4.12). For $\mathring{J}$, recall $\mathring{J} := \det[\partial \mathring{\eta}]$ which equals a multi-linear funtion of its elements $\partial \mathring{\eta}$. So the bound for $\mathring{\eta}$ and $\partial_t^l \mathring{\eta}$ yields the bounds for $\partial_t^l \mathring{J}$.

To conclude the proof, it suffices to control $\|\partial_t^4 \mathring{\psi}\|_1$. From (3.20), we know

$$\begin{cases} \Delta \partial_t^4 \mathring{\psi} = 0 & \text{in } \Omega, \\ \partial_t^4 \mathring{\psi} = \overline{\Delta}^{-1} \mathbb{P} \partial_t^4 \left(\overline{\Delta} \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 \mathring{v} - \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \mathring{v} \right) & \text{on } \Gamma. \end{cases} \tag{4.16}$$

Using Lemma 2.2 for harmonic functions, we know

$$\begin{aligned} \|\partial_t^4 \mathring{\psi}\|_1 \lesssim |\partial_t^4 \mathring{\psi}|_{0.5} &= \left| \overline{\Delta}^{-1} \mathbb{P} \partial_t^4 \left(\overline{\Delta} \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 \mathring{v} - \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \mathring{v} \right) \right|_{0.5} \\ &\lesssim |\mathbb{P} \partial_t^4 \left(\overline{\Delta} \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 \mathring{v} - \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \mathring{v} \right) |_{\dot{H}^{-1.5}} \\ &\lesssim |\mathbb{P} \partial_t^4 \left(\overline{\Delta} \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 \mathring{v} - \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \mathring{v} \right) |_{\dot{H}^{-0.5}}, \end{aligned}$$

where we used the Bernstein inequality (2.4) and the definition of $\mathbb{P}$ (restrict $|\xi| \gtrsim 1$ to get the last inequality).

The most difficult terms appear when ∂_t^4 falls on $\overline{\Delta}\mathring{\eta}$ or $\overline{\partial}\mathring{v}$. Here we only show how to control $\overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \partial_t^4 \mathring{v}$ and the rest highest order terms can be controlled in the same way. For any test function $\phi \in \dot{H}^{0.5}(\mathbb{R}^2)$ with $|\phi|_{\dot{H}^{0.5}} \le 1$, we consider

$$\begin{aligned} |\langle \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \partial_t^4 \mathring{v}, \phi \rangle| &= |\langle \overline{\partial}_i \partial_t^4 \mathring{v}, \overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \phi \rangle| \\ &= |\langle \overline{\partial}^{0.5} \partial_t^4 \mathring{v}, \overline{\partial}^{0.5} (\overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \phi) \rangle| \\ &\lesssim |\partial_t^4 v|_{\dot{H}^{0.5}} |\overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \phi|_{\dot{H}^{0.5}} \\ &\lesssim \|\partial_t^4 v\|_1 (|\phi|_{\dot{H}^{0.5}} |\overline{\Delta} \Lambda_\kappa^2 \mathring{\eta} \mathring{\tilde{a}}|_{L^\infty} + |\phi|_{L^4} |\overline{\Delta} \mathring{\eta}|_{\dot{W}^{0.5,4}} |\mathring{a}|_{L^\infty}) \\ &\lesssim (\|\partial_t^4 v\|_1 \|\partial^2 \mathring{\eta}\|_2 \|\partial \mathring{\eta}\|_{L^\infty}) |\phi|_{\dot{H}^{0.5}}, \end{aligned}$$

where we used $\dot{H}^{-0.5}$-$\dot{H}^{0.5}$ duality and Kato-Ponce inequality (2.1). Taking supremum over all $\phi \in \dot{H}^{0.5}(\mathbb{R}^2)$ with $|\phi|_{\dot{H}^{0.5}} \le 1$, we obtain

$$\|\overline{\Delta} \Lambda_\kappa^2 \mathring{\eta}_\beta \mathring{\tilde{a}}^{i\beta} \overline{\partial}_i \partial_t^4 \mathring{v}\|_{\dot{H}^{-0.5}} \lesssim \|\partial_t^4 v\|_1 \|\partial^2 \mathring{\eta}\|_2 \|\partial \mathring{\eta}\|_{L^\infty},$$

and thus gives the bound for $\|\partial_t^4\mathring{\psi}\|_1$. From the second equation of (4.1) we know that $\partial_t^4\mathring{v} = -\partial_t^3\nabla_{\tilde{a}^{(n-1)}}\mathring{h}$, of which the H^1-norm of the RHS is exactly in the energy functional $\mathcal{E}^{(n-1)}$ as in (4.2)-(4.3). So $\|\partial_t^4\mathring{\psi}\|_1$ is bounded by the induction hypothesis.

It remains to control $\|\partial_t^5\mathring{\eta}\|_1$ which also gives the bounds for $\partial_t^5\mathring{\tilde{J}}$. Taking ∂_t^4 in the first equation of (4.6) we have $\partial_t^5\mathring{\eta} = \partial_t^4\mathring{v} + \partial_t^4\psi^{(n-1)}$ and $\partial_t^4\psi^{(n-1)}$ can be bouned in H^1 in the same way as above. □

4.1.2 Uniform-in-n a priori estimates for the linearized system

With the inductive hypothesis (4.5) and Lemma 4.2, we are now going to control the energy functional for (v, h, η) which solves the system (4.6):

$$\mathcal{E}^{(n+1)} := \|\partial^2\eta\|_2^2 + \|\partial\eta\|_{L^\infty}^2 + \sum_{k=0}^{4}\|\partial_t^{4-k}v\|_k^2 + \|\partial_t^{4-k}h\|_k^2 + W^{(n+1)}, \tag{4.17}$$

where $W^{(n+1)}$ is the energy functional for the 5-th order wave equation of h

$$W^{(n+1)} := \sum_{k=0}^{4}\|\partial_t^{5-k}h\|_k^2 + \|\partial_t^{4-k}\nabla_{\mathring{\tilde{a}}}h\|_k^2. \tag{4.18}$$

The estimates for $\mathcal{E}^{(n+1)} - W^{(n+1)}$ is quite similar (actually a bit easier) to what we have done in Sect. 3, so we will not go over all the details, but still point out the different steps, especially the boundary term control, because we no longer need κ-independent estimates.

Step 1: Estimates for h

The L^2-norm can be directly bounded by $\|h(T)\|_0 \lesssim \|h(0)\|_0 + \int_0^T \|\partial_t h(t)\|\,dt$. To control $\|\partial_t^{4-k}h\|_k$ for $0 \le k \le 4$, we can mimic the proof in Section 3.3: Taking $\mathring{\tilde{J}}\mathrm{div}_{\mathring{\tilde{a}}}$ in the second equation of (4.6), we get

$$\mathring{\tilde{J}}\sigma\partial_t^2 h - \partial_\mu(\mathring{E}^{\nu\mu}\partial_\mu h) = \mathring{\tilde{J}}\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \mathring{\tilde{J}}\partial_t\sigma\partial_t h, \tag{4.19}$$

where $\mathring{E}^{\nu\mu} := \mathring{\tilde{J}}\mathring{\tilde{a}}^{\nu\alpha}\mathring{\tilde{a}}^{\mu}_{\alpha}$. Similarly as in (3.37), one has

$$\frac{1}{2}\int_\Omega \sigma|\partial_t h|^2 + |\partial h|^2\,dy\bigg|_{t=T} \lesssim \mathcal{P}_0 + \int_0^T P(\|\partial_t h\|_0, \|\partial h\|_0, \|\partial\mathring{\eta}\|_{L^\infty}, \|\partial v\|_2)\,dt. \tag{4.20}$$

Let $\mathfrak{D} = \overline{\partial}$ or ∂_t. We take $\mathfrak{D}^3$ in (4.19) to get

$$\mathring{\tilde{J}}\sigma\partial_t^2\mathfrak{D}^3 h - \partial_\nu(\mathring{E}^{\nu\mu}\mathfrak{D}^3\partial_\mu h) = \mathfrak{D}^3(\mathring{\tilde{J}}\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha) - [\mathfrak{D}^3, \mathring{\tilde{J}}\sigma]\partial_t^2 h + \mathfrak{D}^3(\mathring{\tilde{J}}\partial_t\sigma\partial_t h) + \partial_\nu([\mathfrak{D}^3, \mathring{E}^{\nu\mu}]\partial_\mu h). \tag{4.21}$$

Compared with (3.38), we only replace $\tilde{a}$ and $\tilde{J}$ by $\mathring{\tilde{a}}$ and $\mathring{\tilde{J}}$ respectively. Using the same method and the a priori bound in Lemma 4.2, one can similarly get

$$\begin{aligned}&\sum_{\mathfrak{D}^3}\int_\Omega \tilde{J}e'(h)|\mathfrak{D}^3\partial_t h|^2 + |\partial\mathfrak{D}^3 h|^2\,dy\bigg|_{t=T}\\ &\lesssim \mathcal{P}_0 + \epsilon\|\overline{\partial}^3 h(T)\|_1^2 + \sum_{\mathfrak{D}^3}\int_0^T P(\|\partial^2\mathring{\eta}\|_2, \|\partial\mathring{\eta}\|_{L^\infty}, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_3, \|h\|_3, \|\partial_t h\|_3, \|\partial_t^2 h\|_1)\|\mathfrak{D}^3\partial h\|_0\,dt,\end{aligned} \tag{4.22}$$

where $\epsilon > 0$ can be chosen sufficiently small such that $\epsilon\|\overline{\partial}^3 h(T)\|_1^2$ can be absorbed by LHS.

Finally, the full Sobolev norm $\|\partial_t^{4-k}h\|_k,\ k = 0, 1, 2, 3, 4$ can be bounded by analogous argument as in Section 3.3.4.

Step 2: Div-Curl estimates for v

From (4.10), (4.11) and (4.12) in Lemma 4.2, we know all the steps can be copied as in Section 3.4 after replace $\tilde{a}$ by $\mathring{\tilde{a}}$, η by $\mathring{\eta}$ and $e'(h)$ by σ. We omit the computation and list the results here:

- L^2-estimates:

$$\|v(T)\|_0 \le \|v_0\|_0 + \int_0^T \left(\|\partial_t v(0)\|_0 + \int_0^t \|\partial_t^2 v(\tau)\|_0^2\, d\tau\right) dt$$
$$\lesssim \|v_0\|_0 + T\|\partial_t v(0)\|_0 + T\int_0^T \|a(t)\|_{L^\infty}\|\partial\partial_t h(t)\|_0 + \|\partial_t a(t)\|_{L^\infty}\|\partial_t h\|_0\, dt,$$
$$\|\partial_t v(T)\|_0 \lesssim \|\partial_t v(0)\|_0 + \int_0^T \|\partial_t^2 v(t)\|_0\, dt \lesssim \|\partial_t v(0)\|_0 + \int_0^T P(\|\partial^2\mathring{\eta}\|_1, \|\partial\mathring{\eta}\|_{L^\infty}, \|v\|_3, \|h\|_1, \|\partial_t h\|_1)\, dt,$$
$$\|\partial_t^2 v(T)\|_0 \lesssim \|\partial_t^2 v(0)\|_0 + \int_0^T P(\|\partial^2\mathring{\eta}\|_1, \|\partial\mathring{\eta}\|_{L^\infty}, \|\partial_t v\|_1, \|h\|_2, \|\partial_t h\|_1, \|\partial_t^2 h\|_1)\, dt,$$
$$\|\partial_t^3 v(T)\|_0 \lesssim \|\partial_t^3 v(0)\|_0 + \int_0^T P(\|\partial^2\mathring{\eta}\|_1, \|\partial\mathring{\eta}\|_{L^\infty}, \|v\|_3, \|\partial_t v\|_2, \|\partial_t^2 v\|_1, \|h\|_3, \|\partial_t h\|_2, \|\partial_t^2 h\|_1, \|\partial_t^3 h\|_0)\, dt. \tag{4.23}$$

- Boundary estimates

$$|\overline{\partial}^2(\partial_t v\cdot N)|_{0.5} \lesssim \|\overline{\partial}^3\partial_t v\|_0 + \|\overline{\partial}^2 \mathrm{div}\, \partial_t v\|_0 \tag{4.24}$$
$$|\overline{\partial}(\partial_t^2 v\cdot N)|_{0.5} \lesssim \|\overline{\partial}^2\partial_t^2 v\|_0 + \|\overline{\partial} \mathrm{div}\, \partial_t^2 v\|_0 \tag{4.25}$$
$$|\partial_t^3 v\cdot N|_{0.5} \lesssim \|\overline{\partial}\partial_t^3 v\|_0 + \|\mathrm{div}\, \partial_t^3 v\|_0. \tag{4.26}$$

- div-curl estimates

$$\sum_{k=0}^3 \|\mathrm{div}\, \partial_t^k v\|_{3-k} \lesssim \mathcal{P}_0 + \int_0^T P(\mathcal{E}^{(n+1)}(t))\, dt. \tag{4.27}$$
$$\sum_{k=0}^3 \|\mathrm{curl}\, \partial_t^k v\|_{3-k} \lesssim \epsilon\sum_{k=0}^3 \|\partial_t^k v\|_{4-k} + \mathcal{P}_0 + \mathcal{P}_0\int_0^T P(\mathcal{E}^{(n+1)}(t) - W^{(n+1)}(t))\, dt. \tag{4.28}$$

Combining with Hodge's decomposition inequality in Lemma 2.5, to estimate the full Sobolev norm of $\partial_t^k v$, it suffices to control $\|\overline{\partial}^k\partial_t^{4-k}\|_0$.

Step 3: Tangential estimates for time derivatives of v

This part also follows in the same way as Section 3.5. Let $\mathfrak{D}^4 = \partial_t^4, \overline{\partial}\partial_t^3, \overline{\partial}^2\partial_t^2, \overline{\partial}^3\partial_t$. One can directly compute $\frac{d}{dt}\frac{1}{2}\int_\Omega |\mathfrak{D}^4 h|\, dy$ and follow the same method in (3.69)-(3.73) to get the analogous conclusion as (3.74):

$$\sum_{k=1}^4 \frac{d}{dt}\left(\|\partial_t^k\overline{\partial}^{4-k} v\|_0^2 + \|\sqrt{\sigma}\partial_t^k\overline{\partial}^{4-k} h\|_0^2\right) \lesssim \mathcal{P}_0\cdot P(\mathcal{E}^{(n+1)}(t) - W^{(n+1)}(t)). \tag{4.29}$$

Step 4: Tangential estimates for v

In this step we still mimic the proof as in Section 3.6. We replace $\overline{\partial}^4$ by $\overline{\partial}^2\overline{\Delta}$ for convenience in dealing with the correction term $\mathring{\psi}$ and then introduce the Alinhac good unknown $\mathring{\mathbf{V}}$ and $\mathring{\mathbf{H}}$ for v and H:

$$\mathring{\mathbf{V}} := \overline{\partial}^2\overline{\Delta} v - \overline{\partial}^2\overline{\Delta}\mathring{\tilde{\eta}}\cdot\nabla_{\mathring{\tilde{a}}} v, \tag{4.30}$$
$$\mathring{\mathbf{H}} := \overline{\partial}^2\overline{\Delta} h - \overline{\partial}^2\overline{\Delta}\mathring{\tilde{\eta}}\cdot\nabla_{\mathring{\tilde{a}}} h. \tag{4.31}$$

Applying $\overline{\partial}^2\overline{\Delta}$ to the second equation in the linearization system (4.6), one gets

$$\partial_t\mathring{\mathbf{V}} = -\nabla_{\mathring{\tilde{a}}}\mathring{\mathbf{H}} + \underbrace{\partial_t(\overline{\partial}^2\overline{\Delta}\mathring{\tilde{\eta}}\cdot\nabla_{\mathring{\tilde{a}}} v) - \mathring{C}(h)}_{=:\mathring{\mathbf{F}}}, \tag{4.32}$$

subject to the boundary condition

$$\mathring{\mathbf{H}} = -\overline{\partial}^2\overline{\Delta}\mathring{\tilde{\eta}}_\beta \mathring{\tilde{a}}^{3\beta}\partial_3 h \quad \text{on } \Gamma, \tag{4.33}$$

and the corresponding compressibility condition

$$\nabla_{\mathring{\tilde{a}}} \cdot \mathring{\mathbf{V}} = \overline{\partial}^2\overline{\Delta}(\mathrm{div}_{\mathring{\tilde{a}}} v) - \mathring{C}^\alpha(v_\alpha), \quad \text{in } \Omega. \tag{4.34}$$

Here for any function g, the comuutator $\mathring{C}(g)$ is defined in the same way as in (3.76) but replacing $\tilde{a}$ by $\mathring{\tilde{a}}$:

$$\overline{\partial}^2\overline{\Delta}(\nabla_{\mathring{\tilde{a}}} g) = \nabla_{\mathring{\tilde{a}}}\mathring{\mathbf{G}} + \mathring{C}(g), \tag{4.35}$$

with

$$\|\mathring{C}(g)\|_0 \lesssim P(\|\partial^2\mathring{\tilde{\eta}}\|_2, \|\partial\mathring{\tilde{\eta}}\|_{L^\infty})\|\partial g\|_3. \tag{4.36}$$

Here $\mathring{\mathbf{G}}$ is the Alinhac good unknown for g.

Similar to (3.83), one also has

$$\|\overline{\partial}^4 g(t)\|_0 \lesssim \|\mathring{\mathbf{G}}\|_0 + P(\|g(0)\|_3) + P(\|\partial\mathring{\tilde{\eta}}\|_{L^\infty}, \|\partial^2\mathring{\tilde{\eta}}\|_2)\int_0^t P(\|\partial_t g(\tau)\|_3)\, d\tau. \tag{4.37}$$

Now we take L^2 inner product between (4.32) and $\mathring{\mathbf{V}}$ to get analogous result to (3.84).

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathring{\mathbf{V}}|^2\, dy = -\int_\Omega \nabla_{\mathring{\tilde{a}}}\mathring{\mathbf{H}}\cdot\mathring{\mathbf{V}}\, dy + \int_\Omega \mathring{\mathbf{F}}\cdot\mathring{\mathbf{V}}\, dy, \tag{4.38}$$

where $\|\mathring{\mathbf{F}}\|_0$ can be directly controlled as in (3.85). As for the first term, we integrate by parts to get

$$-\int_\Omega \nabla_{\mathring{\tilde{a}}}\mathring{\mathbf{H}}\cdot\mathring{\mathbf{V}}\, dy = -\int_\Gamma \mathring{\tilde{a}}^{3\alpha}\mathring{\mathbf{V}}_\alpha\mathring{\mathbf{H}}\, dS + \int_\Omega \mathring{\mathbf{H}}(\nabla_{\mathring{\tilde{a}}}\cdot\mathring{\mathbf{V}})\, dy + \int_\Omega \partial_\mu\mathring{\tilde{a}}^{\mu\alpha}\mathring{\mathbf{H}}\mathring{\mathbf{V}}_\alpha\, dy, \tag{4.39}$$

where the second and the third term can be controlled in the same way as in (3.86)-(3.90).

For the boundary term in (4.39), we no longer need to plug the precise form of $\mathring{\psi}$ into it and find the subtle cancelltaion as in Section 3.6 because the energy estimate is not required to be κ-independent. Instead, we integrate $\overline{\partial}^{0.5}$ by parts, apply Kato-Ponce inequality (2.1) and Sobolev embedding $H^{0.5}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ to get

$$\begin{aligned}
-\int_\Gamma \mathring{\tilde{a}}^{3\alpha}\mathring{\mathbf{V}}_\alpha\mathring{\mathbf{H}}\, dS &= \int_\Omega \partial_3 h\overline{\partial}^2\overline{\Delta}(\Lambda_\kappa^2\mathring{\eta}_\beta)\mathring{\tilde{a}}^{3\beta}\mathring{\tilde{a}}^{3\alpha}\mathring{\mathbf{V}}_\alpha\, dS \\
&\lesssim (|\partial_3 h\mathring{\tilde{a}}^{3\beta}\mathring{\tilde{a}}^{3\alpha}|_{L^\infty}|\overline{\partial}^2\overline{\Delta}(\Lambda_\kappa^2\mathring{\eta}_\beta)|_{0.5}| + |\partial_3 h\mathring{\tilde{a}}^{3\beta}\mathring{\tilde{a}}^{3\alpha}|_{W^{0.5,4}}|\overline{\partial}^2\overline{\Delta}(\Lambda_\kappa^2\mathring{\eta}_\beta)|_{L^4})|\mathring{\mathbf{V}}|_{\dot{H}^{-0.5}}\, dS \\
&\lesssim \|\partial_3 h\|_2(|\mathring{\tilde{a}}|_{L^\infty} + |\overline{\partial}\mathring{\tilde{a}}|_{0.5}^2)\frac{1}{\kappa}\|\overline{\partial}^3\eta\|_1(|\overline{\partial}^3 v|_{\dot{H}^{0.5}} + |\overline{\partial}^3\mathring{\tilde{\eta}}|_{\dot{H}^{0.5}}|\mathring{\tilde{a}}\partial v|_{L^\infty} + |\overline{\partial}^3\mathring{\tilde{\eta}}|_{L^4}|\mathring{\tilde{a}}\partial v|_{\dot{W}^{0.5,4}}) \\
&\lesssim \frac{1}{\kappa}P(\|\partial h\|_2, \|v\|_4, \|\partial^2\mathring{\tilde{\eta}}\|_2, \|\partial\mathring{\tilde{\eta}}\|_{L^\infty}).
\end{aligned} \tag{4.40}$$

Combining (4.37) with the estimates above, we have

$$\|\overline{\partial}^4 v(T)\|_0 \lesssim \mathcal{P}_0 + \mathcal{P}_0\int_0^T P_\kappa(\mathcal{E}^{(n+1)}(t) - W^{(n+1)}(t))\, dt. \tag{4.41}$$

Summing up the estimates for h, div-curl estimates and tangential estimates, we get

$$\sum_{k=0}^4 \|\partial_t^{4-k} v\|_k + \|\partial_t^{4-k} h\|_0 \lesssim \mathcal{P}_0 + \mathcal{P}_0\int_0^T P_\kappa(\mathcal{E}^{(n+1)}(t) - W^{(n+1)}(t))\, dt. \tag{4.42}$$

4.1.3 Estimates for $W^{(n+1)}$: 5-th order wave equation of h

We would like to control

$$W^{(n+1)} = \sum_{k=0}^{4} \|\sqrt{\sigma}\partial_t^{5-k}h\|_k^2 + \|\partial_t^{4-k}\nabla_{\mathring{\tilde{a}}}h\|_k^2.$$

We take $\text{div}_{\mathring{\tilde{a}}}$ in the second equation of (4.6) to get the wave equation for h:

$$\sigma\partial_t^2 h - \Delta_{\mathring{\tilde{a}}}h = \partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h. \tag{4.43}$$

Before we derive the higher order wave equation, we would like to reduce the estimates of $W^{(n+1)}$ to that of $\|\sqrt{\sigma}\partial_t^5 h\|_0^2 + \|\partial_t^4\nabla_{\mathring{\tilde{a}}}h\|_0^2$ via (4.43) and the elliptic estimate Lemma 2.6.

We start with $\|\partial^4\nabla_{\mathring{\tilde{a}}}h\|_0$ and $\|\partial^4\partial_t h\|_0$. By the elliptic estimate Lemma 2.6, we have

$$\|\partial^4\nabla_{\mathring{\tilde{a}}}h\|_0 \lesssim C(\|\partial\mathring{\eta}\|_{L^\infty}, \|\overline{\partial}^2\mathring{\eta}\|_2)(\sum_{r\le 3}\|\partial^r\Delta_{\mathring{\tilde{a}}}h\|_0 + \|\overline{\partial}\partial\mathring{\eta}\|_3\|h\|_4), \tag{4.44}$$

in which the term $\|\overline{\partial}\partial\mathring{\eta}\|_3 \lesssim \kappa^{-1}\|\partial^2\mathring{\eta}\|_2$ by the property of tangential smoothing. The term $\partial^3\Delta_{\mathring{\tilde{a}}}h$ can be expressed as follows by using (4.43)

$$\partial^3\Delta_{\mathring{\tilde{a}}}h = \partial^3(\sigma\partial_t^2 h) - \partial^3(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h), \tag{4.45}$$

which produces one more time derivative and thus reduce the control of $\partial^4\nabla_{\mathring{\tilde{a}}}h$ to $\partial^3\partial_t^2 h$:

$$\|\partial^3\Delta_{\mathring{\tilde{a}}}h\|_0 \le \|\sigma\partial_3\partial_t^2 h\|_0 + \|[\partial^3,\sigma]\partial_t^2 h\|_0 + \|\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha\|_3 + \|\partial_t\sigma\partial_t h\|_3. \tag{4.46}$$

As for $\partial^4\partial_t h$, we note that for any $1 \le r \le 4$,

$$(\partial^r f)_\alpha = \partial^{r-1}\partial_\alpha f = \partial^{r-1}(\mathring{\tilde{a}}^{\mu\alpha}\partial_\mu f) + \partial^{r-1}((\delta^{\mu\alpha} - \mathring{\tilde{a}}^{\mu\alpha})\partial_\mu f)$$

together with $\|\mathring{\tilde{a}} - 1\| \le \epsilon$ gives

$$\|\partial^r f\|_0 \lesssim \|\partial^{r-1}\nabla_{\mathring{\tilde{a}}}f\|_0 + \epsilon\|\partial^r f\|_0, \tag{4.47}$$

where the last term can be absorbed by LHS after choosing $\epsilon > 0$ sufficently small.

Therefore we have

$$\partial^3\partial_\alpha\partial_t h = \partial^3(\mathring{\tilde{a}}^{\mu\alpha}\partial_\mu\partial_t h) + \partial^3(\underbrace{(\delta^{\mu\alpha} - \mathring{\tilde{a}}^{\mu\alpha})}_{\|\cdot\|_3\le\epsilon}\partial_\mu\partial_t h), \tag{4.48}$$

which gives

$$\|\partial^4\partial_t h\|_0 \lesssim \|\partial^3\nabla_{\mathring{\tilde{a}}}\partial_t h\|_0 + \epsilon\|\partial^4\partial_t h\|_0, \tag{4.49}$$

where the last term can be absorbed by LHS after choosing $\epsilon > 0$ sufficiently small. So we are able to reduce the estimates for $\|\partial^4\nabla_{\mathring{\tilde{a}}}h\|_0$ and $\|\partial^4\partial_t h\|_0$ to $\|\partial^3\partial_t^2 h\|_0$ and $\|\partial^3\nabla_{\mathring{\tilde{a}}}\partial_t h\|_0$, respectively, plus lower order terms. In other words, we replace one spatial derivative by one time derivative via the elliptic estimate and wave equation (4.43).

Next, since $\partial_t h|_\Gamma = 0$, we apply the elliptic estimate in Lemma (2.6) to $\nabla_{\mathring{\tilde{a}}}\partial_t h$ to get

$$\|\partial^3\nabla_{\mathring{\tilde{a}}}\partial_t h\|_0 \lesssim C(\|\partial\mathring{\eta}\|_{L^\infty}, \|\overline{\partial}^2\mathring{\eta}\|_2)(\sum_{r\le 2}\|\partial^r\Delta_{\mathring{\tilde{a}}}\partial_t h\|_0 + \|\overline{\partial}\partial\mathring{\eta}\|_3\|\partial_t h\|_3). \tag{4.50}$$

The term $\partial^r\Delta_{\mathring{\tilde{a}}}\partial_t h$ can be re-expressed as follows by commuting ∂_t through (4.43):

$$\partial^2\Delta_{\mathring{\tilde{a}}}\partial_t h = \sigma\partial^2\partial_t^3 h - \partial^2\partial_t(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h) + [\partial^2\partial_t,\sigma]\partial_t^2 h - \partial^2([\partial_t,\Delta_{\mathring{\tilde{a}}}]h), \tag{4.51}$$

and thus the control of $\partial^2 \Delta_{\mathring{\tilde{a}}} \partial_t$ is reduced to $\sigma\partial^2\partial_t^3 h$ plus the other terms on the RHS of the last inequality

$$\|\partial^2 \Delta_{\mathring{\tilde{a}}} \partial_t h\|_0 \lesssim \|\sigma\partial^2\partial_t^3 h\|_0 + \|\partial_t(\partial_t \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h)\|_2 + \|[\partial^2\partial_t, \sigma]\partial_t^2 h\|_0 + \|[\partial_t, \Delta_{\mathring{\tilde{a}}}]h\|_2. \tag{4.52}$$

As for $\partial^3\partial_t^2 h$, we again rewrite one Lagrangian spatial derivative in terms of one Eulerian spatial derivative plus an error term:

$$\partial^2\partial_\alpha\partial_t^2 h = \partial^2(\mathring{\tilde{a}}^{\mu\alpha}\partial_\mu\partial_t^2 h) + \partial^2((\delta^{\mu\alpha} - \mathring{\tilde{a}}^{\mu\alpha})\partial_\mu\partial_t^2 h), \tag{4.53}$$

which gives

$$\|\partial^3\partial_t^2 h\|_0 \lesssim \|\partial^2\nabla_{\mathring{\tilde{a}}}\partial_t^2 h\|_0 + \epsilon\|\partial^3\partial_t^2 h\|_0, \tag{4.54}$$

where the last term can be again absorbed by LHS after choosing $\epsilon > 0$ sufficiently small.

The reduction mechanism above can be summarized as the following diagram

$$\begin{aligned} &\partial^4\partial_t h \xrightarrow{(4.47)} \partial^3\nabla_{\mathring{\tilde{a}}} h \xrightarrow[(4.43)]{\text{Lem 2.6}} \partial^2\partial_t^3 h \xrightarrow{(4.47)} \partial\nabla_{\mathring{\tilde{a}}}\partial_t^3 h \xrightarrow[(4.43)]{\text{Lem 2.6}} \partial_t^5 h;\\ &\partial^4\nabla_{\mathring{\tilde{a}}} h \xrightarrow[(4.43)]{\text{Lem 2.6}} \partial^3\partial_t^2 h \xrightarrow{(4.47)} \partial^2\nabla_{\mathring{\tilde{a}}}\partial_t^2 h \xrightarrow[(4.43)]{\text{Lem 2.6}} \partial\partial_t^4 h \xrightarrow{(4.47)} \nabla_{\mathring{\tilde{a}}}\partial_t^4 h. \end{aligned} \tag{4.55}$$

As is shown above, we are able to replace one spatial derivative by one time derivative after using the elliptic estimate and wave equation (4.43). Repeat the steps above, we can reduce the estimates of $W^{(n+1)}$ to $\|\partial_t^5 h\|_0$ and $\|\partial_t^4\nabla_{\mathring{\tilde{a}}} h\|_0$ which can be controlled via the 5-th order wave equation of h (i.e., taking ∂_t^4 in (4.43)) plus commutator terms.

$$\begin{aligned} \sum_{k=1}^4 \|\partial_t^{5-k}\partial^k h\|_0 + \|\partial_t^{4-k}\partial^k\nabla_{\mathring{\tilde{a}}} h\|_0 &\lesssim C(\|\partial\mathring{\eta}\|_{L^\infty}, \|\overline{\partial}^2\mathring{\eta}\|_2)(\sigma+\sigma^2)(\|\partial_t^5 h\|_0 + \|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h\|_0) && (4.56)\\ &+ \frac{1}{\kappa} C(\|\partial\mathring{\eta}\|_{L^\infty}, \|\partial^2\mathring{\eta}\|_2)(\sigma+\sigma^2)(\|h\|_0 + \|\partial_t h\|_0 + \cdots + \|\partial_t^3 h\|_0) && (4.57)\\ &+ \|[\partial^3, \sigma]\partial_t^2 h\|_0 + \|\partial_t \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha\|_3 + \|\partial_t\sigma\partial_t h\|_3 && (4.58)\\ &+ \|\partial_t(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h)\|_2 + \|[\partial^2\partial_t, \sigma]\partial_t^2 h\|_0 + \|[\partial_t, \Delta_{\mathring{\tilde{a}}}]h\|_2 && (4.59)\\ &+ \|\partial_t^2(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h)\|_1 + \|[\partial\partial_t^2, \sigma]\partial_t^2 h\|_0 + \|[\partial_t^2, \Delta_{\mathring{\tilde{a}}}]h\|_1 && (4.60)\\ &+ \|\partial_t^3(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h)\|_0 + \|[\partial_t^3, \sigma]\partial_t^2 h\|_0 + \|[\partial_t^3, \Delta_{\mathring{\tilde{a}}}]h\|_0 && (4.61)\\ &+ \sum_{k=1}^4 \|\partial^k([\partial_t^{4-k}, \mathring{\tilde{a}}^{\mu\alpha}]\partial_\mu h)\|_0^2 && (4.62) \end{aligned}$$

Here, all the commutator and error terms (4.57)-(4.62) are of ≤ 4 derivatives of v, η, h and $\partial_t^4\mathring{\tilde{a}}$ which have no problem to bound. Thus,

$$(4.57) + \cdots + (4.62) \lesssim \mathcal{E}^{(n+1)}(t) - W^{(n+1)}(t). \tag{4.63}$$

It remains to control $\|\sqrt{\sigma}\partial_t^5 h\|_0 + \|\partial_t^4\nabla_{\mathring{\tilde{a}}} h\|_0$. We apply ∂_t^4 to (4.43) to get:

$$\sigma\partial_t^6 h - \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu\partial_t^4 h) = \underbrace{\partial_t^4(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha) - \partial_t^4(\partial_t\sigma\partial_t h) - [\partial_t^4, \sigma]\partial_t^2 h + [\partial_t^4, \Delta_{\mathring{\tilde{a}}}]h}_{=:F_5}. \tag{4.64}$$

Multiplying (4.64) by $\partial_t^5 h$ and integrate over Ω, we get

$$\int_\Omega \sigma\partial_t^5 h\partial_t^6 h\, dy - \int_\Omega \partial_t^5 h\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu\partial_t^4 h)\, dy = \underbrace{\int_\Omega F_5\partial_t^5 h\, dy}_{LW_1}. \tag{4.65}$$

The first term in (4.65) is

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\sigma|\partial_t^5 h|^2\,dy-\frac{1}{2}\int_{\Omega}\partial_t\sigma|\partial_t^5 h|^2\,dy. \tag{4.66}$$

For the second term in (4.65), we integrate ∂_ν by parts and note that $\partial_t^5 h|_\Gamma=0$ makes the boundary integral vanish.

$$\begin{aligned}&-\int_{\Omega}\partial_t^5 h\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy\\ &=\int_{\Omega}(\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu\partial_t^5 h)(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy+\underbrace{\int_{\Omega}\partial_t^5 h\partial_\nu\mathring{\tilde{a}}^{\nu\alpha}(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy}_{LW_2}\\ &=\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h|^2\,dy+\underbrace{\int_{\Omega}([\mathring{\tilde{a}}^{\nu\alpha},\partial_t]\partial_\nu\partial_t^4 h)(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy}_{LW3}+LW_2.\end{aligned} \tag{4.67}$$

Plugging (4.66) and (4.67) into (4.65), we have

$$\frac{d}{dt}\left(\int_{\Omega}\sigma|\partial_t^5 h|^2+|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h|^2\,dy\right)=\frac{1}{2}\int_{\Omega}\partial_t\sigma|\partial_t^5 h|^2\,dy+LW_1+LW_2+LW_3. \tag{4.68}$$

Now we come to estimate the RHS of (4.68): First, invoking (1.7), we have

$$\frac{1}{2}\int_{\Omega}\partial_t\sigma|\partial_t^5 h|^2\,dy\lesssim\|\partial_t\sigma\|_{L^\infty}\|\partial_t^5 h\|_0^2\lesssim\|\sqrt{\sigma}\partial_t^5 h\|_0^2. \tag{4.69}$$

$$LW_2\lesssim\|\partial_t^5 h\|_0\|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h\|_0\|\partial^2\mathring{\eta}\|_2. \tag{4.70}$$

$$LW_3=-\int_{\Omega}(\partial_t\mathring{\tilde{a}}^{\nu\alpha})(\partial_\nu\partial_t^4 h)(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy\lesssim\|\partial_t\partial\mathring{\eta}\|_2\|\partial\partial_t^4 h\|_0\|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h\|_0. \tag{4.71}$$

Again, we can write $\partial_\alpha\partial_t^4 h=\mathring{\tilde{a}}^{\mu\alpha}\partial_\mu\partial_t^4 h+(\delta^{\mu\alpha}-\mathring{\tilde{a}}^{\mu\alpha})\partial_\mu\partial_t^4 h$ and invoke $|\mathring{\tilde{a}}-\mathrm{Id}|\le\epsilon$ to get $\|\partial\partial_t^4 h\|_0\approx\|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h\|_0$. Therefore the bound for LW_3 is

$$LW_3=-\int_{\Omega}(\partial_t\mathring{\tilde{a}}^{\nu\alpha})(\partial_\nu\partial_t^4 h)(\mathring{\tilde{a}}^{\mu}_{\alpha}\partial_\mu\partial_t^4 h)\,dy\lesssim\|\partial_t\partial\mathring{\eta}\|_2\|\nabla_{\mathring{\tilde{a}}}\partial_t^4 h\|_0^2. \tag{4.72}$$

It remains to estimate LW_1, i.e., $\|F_5\|_0$.

- $\|\partial_t^4(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha)\|_0$: There are two terms containing 5 derivatives: $\partial_t^5\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha$ and $\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_t^4\partial_\nu v_\alpha$. The rest terms are of ≤ 4 derivatives and have been controlled in the previous proof. By Lemma 4.2 we know $\|\partial_t^5\mathring{\tilde{a}}\|_0\lesssim\mathcal{P}_0$, which gives $\|\partial_t^5\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha\|_0\lesssim\mathcal{P}_0\|\partial v\|_2$. As for the second term, we invoke the second equation of (4.6) to get $\partial_t^4\partial v=-\partial_t^3\partial(\nabla_{\mathring{\tilde{a}}}h+ge_3)=-\partial_t^3\partial(\nabla_{\mathring{\tilde{a}}}h)$, so we have

$$\|\partial_t^4(\partial_t\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha)\|_0\lesssim\mathcal{P}_0+\|\partial v\|_2^2+\|\partial_t\partial\mathring{\eta}\|_2\|\partial_t^3\partial(\nabla_{\mathring{\tilde{a}}}h)\|_0. \tag{4.73}$$

- $\|\partial_t^4(\partial_t\sigma\partial_t h)\|_0$: Expanding all the terms, and then use the previous estimates for ≤ 4 derivative and invoking Lemma 4.2, we have

$$\begin{aligned}&\|\partial_t^4(\partial_t\sigma\partial_t h)\|_0\\ &\lesssim\|\partial_t^5\sigma\|_0\|\partial_t h\|_2+\|\partial_t^4\sigma\|_0\|\partial_t^2 h\|_2+\|\partial_t^3\sigma h\|_1\|\partial_t^3 h\|_1+\|\partial_t^2\sigma\|_{L^\infty}\|\partial_t^4 h\|_0+\|\partial_t\sigma\|_{L^\infty}\|\partial_t^5 h\|_0\\ &\lesssim\mathcal{P}_0(1+\|\partial_t^5 h\|_0).\end{aligned} \tag{4.74}$$

  Similarly one can control $[\partial_t^4,\sigma]\partial_t^2 h$ in exactly the same way, so we omit the details.

- $[\partial_t^4, \Delta_{\mathring{\tilde{a}}}]h$: Direct computation gives

$$
\begin{aligned}
[\partial_t^4, \Delta_{\mathring{\tilde{a}}}]h &= \partial_t^4(\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h)) - \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu\partial_t^4 h)\\
&= \partial_t^4(\mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h)) - \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu\partial_t^4(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h) + \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu\partial_t^4(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h) - \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu\partial_t^4 h)\\
&= [\partial_t^4, \mathring{\tilde{a}}^{\nu\alpha}]\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h) + \mathring{\tilde{a}}^\nu_\alpha\partial_\nu([\partial_t^4, \mathring{\tilde{a}}^\mu_\alpha]\partial_\mu h)\\
&= \sum_{l=1}^4(\partial_t^l\mathring{\tilde{a}}^{\nu\alpha})(\partial_t^{4-l}\partial_\nu(\mathring{\tilde{a}}^\mu_\alpha\partial_\mu h)) + \mathring{\tilde{a}}^\nu_\alpha\partial_\nu\left((\partial_t^l\mathring{\tilde{a}}^\mu_\alpha)(\partial_t^{4-l}\partial_\mu h)\right).
\end{aligned}
$$

Therefore, we have the control for $[\partial_t^4, \sigma]\partial_t^2 h$

$$
\begin{aligned}
\|[\partial_t^4, \Delta_{\mathring{\tilde{a}}}]h\|_0 &\lesssim \|\partial_t^4\mathring{\tilde{a}}\|_0\|\nabla_{\mathring{\tilde{a}}}h\|_3 + \|\partial_t^3\mathring{\tilde{a}}\|_1\|\partial_t\nabla_{\mathring{\tilde{a}}}h\|_2 + \|\partial_t^2\mathring{\tilde{a}}\|_2\|\partial_t^2\nabla_{\mathring{\tilde{a}}}h\|_1 + \|\partial_t\mathring{\tilde{a}}\|_1\|\partial_t^3\nabla_{\mathring{\tilde{a}}}h\|_1\\
&\quad + \|\mathring{\tilde{a}}\|_{L^\infty}(\|\partial\partial_t^4\mathring{\tilde{a}}\|_0\|\partial h\|_2 + \|\partial\partial_t^3\mathring{\tilde{a}}\|_0\|\partial\partial_t h\|_2 + \|\partial\partial_t^2\mathring{\tilde{a}}\|_1\|\partial\partial_t^2 h\|_1 + \|\partial\partial_t\mathring{\tilde{a}}\|_2\|\partial\partial_t^3 h\|_0)\\
&\lesssim \mathcal{P}_0(\text{terms of } \le 4 \text{ derivatives } + \|\partial_t^3\nabla_{\mathring{\tilde{a}}}h\|_1)\\
&\lesssim \mathcal{P}_0(P(\mathcal{E}^{(n+1)} - W^{(n+1)}) + \|\partial_t^3\nabla_{\mathring{\tilde{a}}}h\|_1).
\end{aligned}
\tag{4.75}
$$

Combining (4.73)-(4.75), one has

$$
LW_1 \lesssim \frac{1}{\sqrt{\sigma}}\mathcal{P}_0\cdot\left(P(\mathcal{E}^{(n+1)} - W^{(n+1)}) + \|\partial_t^3\nabla_{\mathring{\tilde{a}}}h\|_1\right)\|\partial_t^5 h\|_0. \tag{4.76}
$$

Summing up (4.68), (4.69),(4.70), (4.72) and (4.69), we get the estimates for the wave equation (4.64)

$$
\frac{d}{dt}\left(\int_\Omega \sigma|\partial_t^5 h|^2 + |\nabla_{\mathring{\tilde{a}}}\partial_t^4 h|^2\,dy\right) \lesssim \mathcal{P}_0\cdot P(\mathcal{E}^{(n+1)}(t))\|\partial_t^5 h\|_0 \tag{4.77}
$$

Therefore we finish the control of $W^{(n+1)}$ by (4.56),(4.63) and (4.77)

$$
\frac{d}{dt}W^{(n+1)}(t) \lesssim \mathcal{P}_0\cdot P_\kappa(\mathcal{E}^{(n+1)}(t)). \tag{4.78}
$$

4.1.4 Uniform-in-n a priori estimates for the linearized approximation system

From (4.17), (4.18), (4.42) and (4.78), we get

$$
\mathcal{E}^{(n+1)}(T) \lesssim \mathcal{E}^{(n+1)}(0) + \mathcal{P}_0\int_0^T P_\kappa(\mathcal{E}^{(n+1)}(t))\,dt, \tag{4.79}
$$

which gives the uniform-in-n a priori estimates

$$
\sup_{0\le t\le T_\kappa}\mathcal{E}^{(n+1)}(t) \lesssim_\kappa \mathcal{P}_0
$$

for the linearized approximation system (4.6) (also for (4.1)) with the help of Gronwall-type inequality in Tao [54].

□

4.2 Construction of the solutions to the linearized approximation system

In this subsection we are going to construct the solutions to the linearized approximation system (4.6):

$$
\begin{cases}
\partial_t\eta = v + \mathring{\psi} & \text{in } \Omega;\\
\partial_t v = -\nabla_{\mathring{\tilde{a}}}h - ge_3 & \text{in } \Omega;\\
\mathrm{div}_{\mathring{\tilde{a}}}v = -\sigma\partial_t h & \text{in } \Omega;\\
h = 0 & \text{on } \Gamma;\\
(\eta, v, h)|_{t=0} = (\mathrm{Id}, v_0.h_0),
\end{cases}
$$

given that $\mathring{\eta}, \mathring{a}, \mathring{\psi}, \sigma$ satisfying Lemma 4.2.

4.2.1 Function space and Solution map

Definition (Norm, Function space and Contraction)

We define the norm

$$\|\cdot\|_{Z^r} := \sum_{s=0}^{r} \sum_{k+l=s} \|\partial_t^k \partial^l \cdot\|_0$$

and define the function space $\mathbb{X}(M,T) :=$

$$\left\{(w,\pi,\xi) : (w,\xi)|_{t=0} = (v_0, \mathrm{id}),\ \sup_{0\le T}\left(\|w, \partial_t\pi, \nabla_{\mathring{a}}\pi\|_{Z^4} + \|\partial_t\xi\|_{Z^3} + \|\partial^2\xi\|_{Z^2} + \|\partial\xi\|_{L^\infty}\right) \le M\right\}. \tag{4.80}$$

Note that for given $M > 0, T > 0$, $\mathbb{X}(M,T)$ is a Banach space. We then define the solution map $\Xi : \mathbb{X}(M,T) \to \mathbb{X}(M,T)$ by

$$\begin{aligned} \Xi : \mathbb{X}(M,T) &\to \mathbb{X}(M,T) \\ (w,\pi,\xi) &\mapsto (v,h,\eta). \end{aligned} \tag{4.81}$$

The image (v,h,η) is defined as follows:

1. Define η by
$$\partial_t\eta = w + \mathring{\psi}, \quad \eta(0) = \mathrm{Id}. \tag{4.82}$$

2. Define v by
$$\partial_t v = -\nabla_{\mathring{a}}\pi - ge_3, \quad v(0) = v_0. \tag{4.83}$$

3. Define h by the solution of the following wave equation
$$\begin{cases} \sigma\partial_t^2 h - \Delta_{\mathring{a}} h = \partial_t \mathring{\tilde{a}}^{\nu\alpha}\partial_\nu v_\alpha - \partial_t\sigma\partial_t h & \text{in } \Omega, \\ h = 0 & \text{on } \Gamma, \\ (h, \partial_t h)|_{t=0} = (h_0, h_1). \end{cases} \tag{4.84}$$

Note that the existence of this wave equation is standard in the PDE theory and so we omit the proof.

4.2.2 Construct the solution: Contraction Mapping Theorem

Now we need to verify

1. Ξ is a self-mapping of $\mathbb{X}$,
2. Ξ is a contraction on $\mathbb{X}$.

Once these two properties are proved, we can apply the Contraction Mapping Theorem to Ξ to get there exists a unique fixed point (v,h,η) of Ξ which solves the linearized system (4.6).

First we verify Ξ is a self-mapping of $\mathbb{X}$.

Estimates for η: A direct computation gives

$$\|\partial\eta\|_{L^\infty} \le \|\partial\eta(0)\|_{L^\infty} + \int_0^T \|\partial(w+\mathring{\psi})\|_{L^\infty}\,dt \le 1 + \int_0^T \|w\|_{Z^4} + \|\partial\mathring{\psi}\|_2\,dt, \tag{4.85}$$

$$\|\partial^2\eta\|_{Z^2} \le \|\partial^2\eta(0)\|_{Z^2} + \int_0^T \|\partial^2(w+\mathring{\psi})\|_{Z^2} \le \int_0^T \|w\|_{Z^4} + \|\mathring{\psi}\|_{Z^4}\,dt, \tag{4.86}$$

$$\|\partial^{3-k}\partial_t^k\partial_t\eta\|_0 \lesssim \|\partial^{3-k}\partial_t^k\partial_t\eta(0)\|_0 + \int_0^T \|\partial^{3-k}\partial_t^{k+1}(w+\mathring{\psi})\|_0 \lesssim \int_0^T \|w\|_{Z^4} + \|\mathring{\psi}\|_{Z^4}\,dt. \tag{4.87}$$

Estimates for v: First we have for $l = 1,2,3,4$:

$$\|v\|_0 \le \|v_0\|_0 + \int_0^T \|\partial_t v(0)\|_0 + \left(\int_0^t \|\partial_t^2 v(\tau)\|_0 d\tau\right) dt \le \|v_0\|_0 + T\|\partial_t v(0)\|_0 + \int_0^T \|\nabla_{\mathring{a}}\pi\|_{Z^4}\, dt \tag{4.88}$$

$$\|\partial_t^l v\|_0 \le \|\partial_t^l v(0)\|_0 + \int_0^T \|\partial_t^l \nabla_{\mathring{a}}\pi\|_0\, dt \le \|\partial_t^l v(0)\|_0 + \int_0^T \|\nabla_{\mathring{a}}\pi\|_{Z^4}\, dt. \tag{4.89}$$

For the space-time derivatives, we also have

$$\|\partial^4 v\|_0 \le \|v_0\|_4 + \int_0^T \|\partial_t\partial^4 v\|_0\, dt \le \|v_0\|_4 + \int_0^T \|\partial^4\nabla_{\mathring{a}}\pi\|_0\, dt \le \|v_0\|_4 + \int_0^T \|\nabla_{\mathring{a}}\pi\|_{Z^4}\, dt \tag{4.90}$$

$$\|\partial_t^l\partial^{4-l} v\|_0 \le \|\partial_t^l\partial^{4-l} v\|_0 + \int_0^T \|\partial_t^l\partial^{4-l}\nabla_{\mathring{a}}\pi\|_0\, dt \le \|v(0)\|_{Z^4} + \int_0^T \|\nabla_{\mathring{a}}\pi\|_{Z^4}\, dt. \tag{4.91}$$

Therefore,

$$\|v\|_{Z^4} \le \|v_0\|_{Z^4} + T\|\partial_t v(0)\|_0 + \int_0^T \|\nabla_{\mathring{a}}\pi\|_{Z^4} \lesssim (1+T)\mathcal{P}_0 + TM. \tag{4.92}$$

Estimates for h: It suffices to estimate $\|\partial_t h\|_{Z^4}$ and $\|\nabla_{\mathring{a}} h\|_{Z^4}$ via the wave equation of h (4.84). Again we can apply the same method as in Section 4.1.3 to derive

$$\|\partial_t h\|_{Z^4} + \|\nabla_{\mathring{a}} h\|_{Z^4} \lesssim_M \mathcal{P}_0 + \mathcal{P}_0 \int_0^T P_\kappa(\|\partial\eta\|_{L^\infty}, \|\partial_t^2\eta\|_2, \|v\|_{Z^4}, \|h\|_{Z^4}, \|\nabla_{\mathring{a}} h\|_{Z^4}, \|\partial_t h\|_{Z^4})\, dt. \tag{4.93}$$

Combining (4.85)-(4.87), (4.92) and (4.93), we obtain that the solution map Ξ is a self-map of $\mathbb{X}$ after applying the Gronwall's inequality.

Next we prove $\Xi : \mathbb{X}(M,T) \to \mathbb{X}(M,T)$ is a contraction. Given $(w_1,\pi_1,\xi_1),\ (w_2,\pi_2,\xi_2) \in \mathbb{X}(M,T)$ and their images under Ξ $(v_1,h_1,\eta_1),\ (v_2,h_2,\eta_2)$, we define

$$[w] := w_1 - w_2,\ [\pi] := \pi_1 - \pi_2,\ [\xi] := \xi_1 - \xi_2;\ \ [v] := v_1 - v_2,\ [h] := h_1 - h_2,\ [\eta] := \eta_1 - \eta_2.$$

From (4.82), (4.83) and (4.84), we can derive the equations for $([v],[h],[\eta])$ with initial data $(\mathbf{0},0,\mathbf{0})$:

$$\begin{aligned} \partial_t[\eta] &= [w], \\ \partial_t[v] &= \nabla_{\mathring{a}}[\pi], \\ \sigma\partial_t^2[h] - \Delta_{\mathring{a}}[h] &= \partial_t \mathring{a}^{\nu\alpha} - \partial_t\sigma\partial_t[h]. \end{aligned}$$

Similarly as above we can derive the estimates

$$\begin{aligned} &\|\partial[\eta]\|_{L^\infty} + \|\partial^2[\eta]\|_{Z^2} + \|\partial_t[\eta]\|_{Z^3} + \|\partial_t[h]\|_{Z^4} + \|\nabla_{\mathring{a}}[h]\|_{Z^4} + \|[v]\|_{Z^4} \\ \lesssim_M\ & \mathcal{P}_0 \int_0^T P_\kappa(\|\partial[\xi]\|_{L^\infty}, \|\partial^2[\xi]\|_{Z^2}, \|\partial_t[\xi]\|_{Z^3}, \|\nabla_{\mathring{a}}[\pi]\|_{Z^4}, \|\partial_t[\pi]\|_{Z^4}, \|[w]\|_{Z^4})\, dt. \end{aligned} \tag{4.94}$$

Therefore, choosing $T_\kappa > 0$ sufficiently small such that RHS of (4.94) is bounded by

$$\frac{1}{2}\left(\|\partial[\xi]\|_{L^\infty} + \|\partial^2[\xi]\|_{Z^2} + \|\partial_t[\xi]\|_{Z^3} + \|\nabla_{\mathring{a}}[\pi]\|_{Z^4} + \|\partial_t[\pi]\|_{Z^4} + \|[w]\|_{Z^4}\right),$$

we prove that $\Xi : \mathbb{X}(M,T_\kappa) \to \mathbb{X}(M,T_\kappa)$ is a contraction self-map. By the Contraction Mapping Theorem, we know Ξ has a unique fixed point $(v,h,\eta) \in \mathbb{X}(M,T_\kappa)$ which is the solution to the linearized approximation system (4.6).

4.3 Iteration and convergence of the solutions to the linearized system

Up to now we have constructed a sequence of solutions $\{(v^{(n)}, h^{(n)}, \eta^{(n)})\}_{n=1}^{\infty}$ which solves the n-th linearized κ-approximation system (4.1). The last step in this section is to prove that $\{(v^{(n)}, h^{(n)}, \eta^{(n)})\}_{n=1}^{\infty}$ converges in some strong Sobolev norm, and thus produce a solution (v, h, η) to the nonlinear κ-approximation system (3.1).

Let $\hat{n} \geq 3$, and define

$$[v]^{(n)} := v^{(n+1)} - v^{(n)},\ \ [h]^{(n)} := h^{(n+1)} - h^{(n)},\ \ [\eta]^{(n)} := \eta^{(n+1)} - \eta^{(n)}, \tag{4.95}$$

and

$$[a]^{(n)} := a^{(n)} - a^{(n-1)},\ \ [\psi]^{(n)} := \psi^{(n)} - \psi^{(n-1)}. \tag{4.96}$$

Then these quantities satisfy the following system

$$\begin{cases} \partial_t[\eta]^{(n)} = [v]^{(n)} + [\psi]^{(n)} & \text{in } \Omega \\ \partial_t[v]^{(n)} = -\nabla_{\tilde{a}^{(n)}}[h]^{(n)} - \nabla_{[\tilde{a}]^{(n)}} h^{(n)} & \text{in } \Omega \\ \operatorname{div}_{\tilde{a}^{(n)}}[v]^{(n)} = -\operatorname{div}_{[\tilde{a}]^{(n)}} v^{(n)} - e'(h^{(n)})\partial_t[h]^{(n)} - (e'(h^{(n)}) - e'(h^{(n-1)}))\partial_t h^{(n)} & \text{in } \Omega \\ [h]^{(n)} = 0 & \text{on } \Gamma \end{cases} \tag{4.97}$$

We will prove the following energy converges to 0 as $n \to \infty$ for all $t \in [0, T]$

$$[\mathcal{E}]^{(n)}(t) := \sum_{k=0}^{3} \|\partial_t^{3-k}[v]^{(n)}\|_k^2 + \|\partial_t^{3-k}[h]^{(n)}\|_k^2 + \|[\eta]^{(n)}\|_3^2 + \|[a]^{(n)}\|_2^2. \tag{4.98}$$

4.3.1 Estimates of $[a]$, $[\psi]$ and $[\eta]$

By definition, we have

$$\begin{aligned} [a]^{(n)\mu\nu}(T) &= \int_0^T \partial_t(a^{(n)\mu\nu} - a^{(n-1)\mu\nu})\,dt \\ &= -\int_0^T [a]^{(n)\mu\gamma}\partial_\beta\partial_t\eta_\gamma^{(n)} a^{(n)\beta\nu} + a^{(n-1)\mu\gamma}\partial_\beta\partial_t[\eta]_\gamma^{(n-1)} a^{(n)\beta\nu} + a^{(n-1)\mu\gamma}\partial_\beta\partial_t\eta_\gamma^{(n-1)}[a]^{(n)\beta\nu}, \end{aligned}$$

which gives

$$\|[a]^{(n)}(T)\|_2 \lesssim \mathcal{P}_0 \int_0^T \|[a]^{(n)}(t)\|_2^2 \|\partial_t[\eta]^{(n-1)}\|_3\,dt \lesssim \mathcal{P}_0 \int_0^T \|[a]^{(n)}(t)\|_2^2 (\|[v]^{(n-1)}\|_3 + \|[\psi]^{(n-1)}\|_3))\,dt. \tag{4.99}$$

As for $[\psi]^{(n)}$, it satisfies $-\Delta[\psi]^{(n)} = 0$ subject to the following boundary condition

$$\begin{aligned} [\psi]^{(n)} = \overline{\partial}^{-1}\mathbb{P}\Big(& \overline{\Delta}[\eta]_\beta^{(n-1)}\tilde{a}^{(n)i\beta}\overline{\partial}_i\Lambda_\kappa^2 v^{(n)} + \overline{\partial}\eta_\beta^{(n-1)}[\tilde{a}]^{(n)i\beta}\overline{\partial}_i\Lambda_\kappa^2 v^{(n)} + \overline{\partial}\eta_\beta^{(n-1)}\tilde{a}^{(n-1)i\beta}\overline{\partial}_i\Lambda_\kappa^2[v]^{(n-1)} \\ & - \overline{\Delta}\Lambda_\kappa^2[\eta]_\beta^{(n-1)}\tilde{a}^{(n)i\beta}\overline{\partial}_i v^{(n)} - \overline{\Delta}\Lambda_\kappa^2\eta_\beta^{(n-1)}[\tilde{a}]^{(n)i\beta}\overline{\partial}_i v^{(n)} - \overline{\Delta}\Lambda_\kappa^2\eta_\beta^{(n-1)}\tilde{a}^{(n-1)i\beta}\overline{\partial}_i[v]^{(n-1)}\Big). \end{aligned}$$

By the standard elliptic estimates, we have the control for $[\psi]^{(n)}$

$$\|[\psi]^{(n)}\|_3^2 \lesssim |[\psi]^{(n)}|_{2.5} \lesssim \mathcal{P}_0\left(\|[\eta]^{(n-1)}\|_3^2 + \|[v]^{(n-1)}\|_2^2 + \|[\tilde{a}]^{(n)}\|_1^2\right). \tag{4.100}$$

Therefore we get

$$\sup_{[0,T]}\|[a]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 T^2\left(\|[a]^{(n)}, [a]^{(n-1)}\|_{L_t^\infty H^2} + \|[v]^{(n-1)}, [v]^{(n-2)}, [\eta]^{(n-2)}\|_{L_t^\infty H^3}^2\right), \tag{4.101}$$

and the bound for $[\eta]$ combining with $\partial_t[\eta]^{(n)} = [v]^{(n)} + [\psi]^{(n)}$:

$$\sup_{[0,T]} \|[\eta]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 T^2 \left(\|[a]^{(n)}\|_{L_t^\infty H^2} + \|[v]^{(n)}, [v]^{(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2 \right) \tag{4.102}$$

Similar as in Lemma 3.3 and Lemma 4.2, one can get estimates for the time derivatives of $[\eta]$ and $[\psi]$

$$\|[\partial_t\psi]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 \left(\|[a]^{(n)}\|_2^2 + \|[\partial_t v]^{(n-1)}\|_2^2 + \|[v]^{(n-1)}, [\eta]^{(n-1)}\|_3^2 \right) \tag{4.103}$$

$$\|[\partial_t^2\psi]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 \left(\|[a]^{(n)}\|_2^2 + \|[\partial_t^2 v]^{(n-1)}\|_1^2 + \|[\partial_t v]^{(n-1)}\|_2^2 + \|[v]^{(n-1)}, [\eta]^{(n-1)}\|_3^2 \right) \tag{4.104}$$

$$\|[\partial_t\eta]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 T^2 \left(\|[a]^{(n)}, [\partial_t v]^{(n)}, [\partial_t v]^{(n-1)}\|_{L_t^\infty H^2} + \|[v]^{(n)}, [v]^{(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2 \right) \tag{4.105}$$

$$\|[\partial_t^2\eta]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 T^2 \left(\|[\partial_t^2 v]^{(n),(n-1)}\|_{L_t^\infty H_1}^2 + \|[a]^{(n)}, [\partial_t v]^{(n),(n-1)}\|_{L_t^\infty H^2} + \|[v]^{(n),(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2 \right) \tag{4.106}$$

$$\|[\partial_t^3\eta]^{(n)}\|_1^2 \lesssim \mathcal{P}_0 \left(\|[\partial_t^2 v]^{(n),(n-1)}\|_{L_t^\infty H_1}^2 + \|[a]^{(n)}, [\partial_t v]^{(n),(n-1)}\|_{L_t^\infty H^2} + \|[v]^{(n),(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2 \right). \tag{4.107}$$

4.3.2 Estimates of $[h]$

Taking $\tilde{J}^{(n)}\mathrm{div}_{\tilde{a}^{(n)}}$ in the second equation of (4.97), we get an analogous wave equation for $[h]$:

$$\begin{aligned} e'(h^{(n)})\tilde{J}^{(n)}\partial_t^2[h]^{(n)} - \partial_\nu(\tilde{E}^{\nu\mu}\partial_\mu[h]^{(n)}) = {} & \tilde{J}^{(n)}\tilde{a}^{(n)\nu\alpha}\partial_\nu([\tilde{a}]^{(n)\mu}{}_\alpha\partial_\mu h^{(n)}) \\ & - \tilde{J}^{(n)}\partial_t\left((e'(h^{(n)}) - e'(h^{(n-1)}))\partial_t h^{(n)}\right) \\ & - \tilde{J}^{(n)}(\partial_t\tilde{a}^{(n)\nu\alpha})\partial_\nu[v]_\alpha^{(n)}, \end{aligned} \tag{4.108}$$

where $\tilde{E}^{\nu\mu} := \tilde{J}^{(n)}\tilde{a}^{(n)\nu\alpha}\tilde{a}_\alpha^{(n)\mu}$.

One can apply the similar method in Section 3.3 and use the estimates of $[\eta], [\psi]$ to obtain the following energy estimates

$$\sum_{k=1}^{3} \|\partial_t^k[h]^{(n)}\|_{3-k}^2 + \|\sqrt{e'(h^{(n)})}\partial_t^3[h]^{(n)}\|_0^2 \lesssim \int_0^T P([\mathcal{E}]^{(n),(n-1)}(t))\,dt. \tag{4.109}$$

4.3.3 Div-Curl estimates

From Hodge's decomposition inequality Lemma 2.5, we have

$$\begin{aligned} \|[v]^{(n)}\|_3^2 &\lesssim \|[v]^{(n)}\|_0^2 + \|\mathrm{div}\,[v]^{(n)}\|_2^2 + \|\mathrm{curl}\,[v]^{(n)}\|_2^2 + |[v]^{(n)}\cdot N|_{2.5} \\ \|[\partial_t v]^{(n)}\|_2^2 &\lesssim \|[\partial_t v]^{(n)}\|_0^2 + \|\mathrm{div}\,[\partial_t v]^{(n)}\|_1^2 + \|\mathrm{curl}\,[\partial_t v]^{(n)}\|_1^2 + |[\partial_t v]^{(n)}\cdot N|_{1.5} \\ \|[\partial_t^2 v]^{(n)}\|_1^2 &\lesssim \|[\partial_t^2 v]^{(n)}\|_0^2 + \|\mathrm{div}\,[\partial_t^2 v]^{(n)}\|_0^2 + \|\mathrm{curl}\,[\partial_t^2 v]^{(n)}\|_0^2 + |[\partial_t v]^{(n)}\cdot N|_{0.5} \end{aligned}$$

The L^2-norm can be bounded in the same way as in Section 3.4 and the boundary term can be reduced to the tangential estimates for $[v]$ and its time derivative. As for the curl part, we apply curl $_{\tilde{a}^{(n)}}$ to the second equation of (4.97) to get the evolution equation of curl $_{\tilde{a}^{(n)}[v]^{(n)}}$

$$\partial_t(\mathrm{curl}\,_{\tilde{a}^{(n)}[v]^{(n)}})_\lambda = \epsilon_{\lambda\mu\alpha}\partial_t\tilde{a}^{(n)\nu\mu}\partial_\nu[v]_\alpha^{(n)} - \epsilon_{\lambda\mu\alpha}\partial_t[\tilde{a}]^{(n)\nu\mu}\partial_\nu v_\alpha^{(n)}. \tag{4.110}$$

Applying $D^2 = \partial^2, \partial\partial_t$ or ∂_t^2 to (4.110), and mimicing the proof in Section 3.4, one can get

$$
\begin{aligned}
\|\text{curl } [v]^{(n)}\|_2^2 &\lesssim \epsilon \|[v]^{(n)}\|_3^2 + P_\kappa(\mathcal{P}_0)T^2 \left(\|[v]^{(n),(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3} + \|[\tilde{a}]^{(n),(n-1)}\|_{L_t^\infty H^2} \right) \\
&\lesssim \epsilon \|[v]^{(n)}\|_3^2 + P_\kappa(\mathcal{P}_0)T^2 \sup_{[0,T]} [\mathcal{E}]^{(n),(n-1)}(t) \\
\|\text{curl } [\partial_t v]^{(n)}\|_1^2 &\lesssim \epsilon \|[\partial_t v]^{(n)}\|_2^2 + P_\kappa(\mathcal{P}_0)T^2 \sup_{[0,T]} [\mathcal{E}]^{(n),(n-1)}(t) \\
\|\text{curl } [\partial_t^2 v]^{(n)}\|_0^2 &\lesssim \epsilon \|[\partial_t^2 v]^{(n)}\|_1^2 + P_\kappa(\mathcal{P}_0)T^2 \sup_{[0,T]} [\mathcal{E}]^{(n),(n-1)}(t).
\end{aligned}
\tag{4.111}
$$

Similar results hold for div control by using the same method as in Section 3.4, so we only list the result here

$$
\begin{aligned}
&\|\text{div } [v]^{(n)}\|_2^2 + \|\text{div } [\partial_t v]^{(n)}\|_1^2 \|\text{div } [\partial_t^2 v]^{(n)}\|_0^2 \\
\lesssim\ & P_\kappa(\mathcal{P}_0)T^2 \sup_{[0,T]} [\mathcal{E}]^{(n),(n-1)}(t).
\end{aligned}
\tag{4.112}
$$

4.3.4 Tangential estimates of $[\partial_t^k v]$ for $k \geq 1$

Let $\mathfrak{D}^3 = \overline{\partial}^2 \partial_t, \overline{\partial}\partial_t^2, \partial_t^3$. Using the same method as in Section 3.5 and Section 4.1.2 (Step 4), we can derive the estimates

$$
\sum_{k=1}^{3} \|\overline{\partial}^{3-k} \partial_t^k [v]^{(n)}\|_0^2 + \|\overline{\partial}^{3-k} \partial_t^k [h]^{(n)}\|_0^2 \lesssim \int_0^T P([\mathcal{E}]^{(n),(n-1),(n-2)}(t))\, dt. \tag{4.113}
$$

4.3.5 Tangential estimates of $[v]$: Alinhac good unknown

We adopt the same method as in Section 3.6. For each n, we define the Alinhac good unknowns by

$$
\mathbf{V}^{(n+1)} = \overline{\partial}^3 v^{(n+1)} - \overline{\partial}^3 \tilde{\eta}^{(n)} \cdot \nabla_{\tilde{a}^{(n)}} v^{(n+1)}, \quad \mathbf{H}^{(n+1)} = \overline{\partial}^3 h^{(n+1)} - \overline{\partial}^3 \tilde{\eta}^{(n)} \cdot \nabla_{\tilde{a}^{(n)}} h^{(n+1)}. \tag{4.114}
$$

Their difference is denoted by

$$
[\mathbf{V}]^{(n)} := \mathbf{V}^{(n+1)} - \mathbf{V}^{(n)}, [\mathbf{H}]^{(n)} := \mathbf{H}^{(n+1)} - \mathbf{H}^{(n)}.
$$

Similarly as in Section 3.6, we can derive the analogous version of (3.80) as

$$
\partial_t [\mathbf{V}]^{(n)} + \nabla_{\tilde{a}^{(n)}} [\mathbf{H}]^{(n)} = -\nabla_{[\tilde{a}]^{(n)}} \mathbf{H}^{(n)} + \mathbf{F}^{(n)}, \tag{4.115}
$$

subject to the boundary data

$$
[\mathbf{H}]^{(n)}|_\Gamma = -\left(\overline{\partial}^3 \tilde{\eta}_\beta^{(n)} \tilde{a}^{(n)3\beta} + \overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \tilde{a}^{(n)3\beta} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} [\tilde{a}]^{(n)3\beta} \right), \tag{4.116}
$$

and the compressibility equation

$$
\nabla_{\tilde{a}^{(n)}} \cdot [\mathbf{V}]^{(n)} = -\nabla_{[\tilde{a}]^{(n)}} \cdot \mathbf{V}^{(n)} + \mathbf{G}^{(n)}, \tag{4.117}
$$

where

$$
\begin{aligned}
\mathbf{F}^{(n)\alpha} = \partial_t &\left(\overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \tilde{a}^{(n)\mu\beta} \partial_\mu v_\alpha^{(n+1)} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} [\tilde{a}]^{(n)\mu\beta} \partial_\mu v_\alpha^{(n+1)} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} \tilde{a}^{(n)\mu\beta} \partial_\mu [v]_\alpha^{(n)} \right) \\
&+ [\tilde{a}]^{(n)\mu\beta} \partial_\mu (\tilde{a}^{(n)\gamma\alpha} \partial_\gamma h^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{a}^{(n-1)\mu\beta} \partial_\mu ([\tilde{a}]^{(n)\gamma\alpha} \partial_\gamma h^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} \\
&+ \tilde{a}^{(n-1)\mu\beta} \partial_\mu (\tilde{a}^{(n-1)\gamma\alpha} \partial_\gamma [h]^{(n)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{a}^{(n-1)\mu\beta} \partial_\mu ([\tilde{a}]^{(n)\gamma\alpha} \partial_\gamma h^{(n)}) \overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \\
&- \left[\overline{\partial}^2, [\tilde{a}]^{(n)\mu\beta} \tilde{a}^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu h^{(n+1)} - \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta} [\tilde{a}]^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu h^{(n+1)} \\
&- \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta} \tilde{a}^{(n-1)\gamma\alpha} \overline{\partial} \right] \partial_\gamma [\tilde{\eta}]_\beta^{(n-1)} \partial_\mu h^{(n+1)} - \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta} \tilde{a}^{(n-1)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n-1)} \partial_\mu [h]^{(n)} \\
&- \left[\overline{\partial}^3, [\tilde{a}]^{(n)\mu\alpha}, \partial_\mu h^{(n+1)} \right] - \left[\overline{\partial}^3, \tilde{a}^{(n-1)\mu\alpha}, \partial_\mu [h]^{(n)} \right],
\end{aligned}
$$

and

$$\begin{aligned}
\mathbf{G}^{(n)} =& \overline{\partial}^3(\mathrm{div}_{\tilde{a}^{(n)}}[v]^{(n)} - \mathrm{div}_{[\tilde{a}]^{(n)}} v^{(n)}) \\
&- \left[\overline{\partial}^2, [\tilde{a}]^{(n)\mu\beta}\tilde{a}^{(n)\gamma\alpha}\overline{\partial}\right]\partial_\gamma\tilde{\eta}_\beta^{(n)}\partial_\mu v_\alpha^{(n+1)} - \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta}[\tilde{a}]^{(n)\gamma\alpha}\overline{\partial}\right]\partial_\gamma\tilde{\eta}_\beta^{(n)}\partial_\mu v_\alpha^{(n+1)} \\
&- \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta}\tilde{a}^{(n-1)\gamma\alpha}\overline{\partial}\right]\partial_\gamma[\tilde{\eta}]_\beta^{(n-1)}\partial_\mu v_\alpha^{(n+1)} - \left[\overline{\partial}^2, \tilde{a}^{(n-1)\mu\beta}\tilde{a}^{(n)\gamma\alpha}\overline{\partial}\right]\partial_\gamma\tilde{\eta}_\beta^{(n-1)}\partial_\mu[v]_\alpha^{(n)} \\
&- \left[\overline{\partial}^3, [\tilde{a}]^{(n)\mu\alpha}, \partial_\mu v_\alpha^{(n+1)}\right] - \left[\overline{\partial}^3, \tilde{a}^{(n-1)\mu\alpha}, \partial_\mu[v]_\alpha^{(n)}\right] \\
&+ [\tilde{a}]^{(n)\mu\beta}\partial_\mu(\tilde{a}^{(n)\gamma\alpha}\partial_\gamma v_\alpha^{(n+1)})\overline{\partial}^3\tilde{\eta}_\beta^{(n)} + \tilde{a}^{(n-1)\mu\beta}\partial_\mu([\tilde{a}]^{(n)\gamma\alpha}\partial_\gamma v_\alpha^{(n+1)})\overline{\partial}^3\tilde{\eta}_\beta^{(n)} \\
&+ \tilde{a}^{(n-1)\mu\beta}\partial_\mu(\tilde{a}^{(n-1)\gamma\alpha}\partial_\gamma[v]_\alpha^{(n)})\overline{\partial}^3\tilde{\eta}_\beta^{(n)} + \tilde{a}^{(n-1)\mu\beta}\partial_\mu([\tilde{a}]^{(n)\gamma\alpha}\partial_\gamma v_\alpha^{(n)})\overline{\partial}^3[\tilde{\eta}]_\beta^{(n-1)}.
\end{aligned}$$

Multiplying $[\mathbf{V}]^{(n)}$ in (4.115) and integrate by parts in the $[\mathbf{H}]$ term, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{V}^{(n)}\|_0^2 =& \int_\Omega [\mathbf{H}]^{(n)}\left(\nabla_{\tilde{a}^{(n)}}\cdot[\mathbf{V}]^{(n)} - \partial_\mu\tilde{a}^{\mu\alpha}[\mathbf{V}]_\alpha^{(n)}\right)dy + \int_\Omega(\mathbf{F}^{(n)} - \nabla_{[\tilde{a}]^{(n)}}\mathbf{H}^{(n)})\cdot[\mathbf{V}]^{(n)}\,dy \\
&- \int_\Gamma [\mathbf{H}]^{(n)}\tilde{a}^{(n)3\alpha}[\mathbf{V}]_\alpha^{(n)}\,dS.
\end{aligned}$$

Similarly as in (3.85)-(3.90), we are able to control the first three terms by

$$-\frac{1}{2}\frac{d}{dt}\|e'(h^{(n)})\overline{\partial}^4[h]^{(n)}\|_0^2 + \mathcal{P}_0 P([\mathcal{E}]^{(n),(n-1)}(t)).$$

As for the boundary term, we can mimic the proof in (4.40), i.e., integrate $\overline{\partial}^{0.5}$ by parts, to get

$$\begin{aligned}
&- \int_\Gamma [\mathbf{H}]^{(n)}\tilde{a}^{(n)3\alpha}[\mathbf{V}]_\alpha^{(n)}\,dS \\
=& \int_\Gamma \partial_3 h\tilde{a}^{(n)3\alpha}[\mathbf{V}]_\alpha^{(n)}\left(\overline{\partial}^3\tilde{\eta}_\beta^{(n)}\tilde{a}^{(n)3\beta} + \overline{\partial}^3[\tilde{\eta}]_\beta^{(n-1)}\tilde{a}^{(n)3\beta} + \overline{\partial}^3\tilde{\eta}_\beta^{(n-1)}[\tilde{a}]^{(n)3\beta}\right) \\
\lesssim& |[\mathbf{V}]^{(n)}|_{\dot{H}^{-0.5}}\left(\frac{1}{\kappa}\mathcal{P}_0|\overline{\partial}^2[\eta]^{(n-1)}|_{\dot{H}^{0.5}} + \|[\tilde{a}]\|_2\right)
\end{aligned}$$

Summing up all the estimates above and using the analogue of (3.83), we get

$$\|\overline{\partial}^3[v]^{(n)}\|_0^2 + \|e'(h^{(n)})\overline{\partial}^3[h]^{(n)}\|_0^2 \lesssim P_\kappa(\mathcal{P}_0)T^2 P\left(\sup_{t\in[0,T]}[\mathcal{E}]^{(n),(n-1),(n-2)}(t)\right). \tag{4.118}$$

Combining the estimates for $[\eta]$, $[a]$, $[h]$, $[v]$ and div-curl estimates above, we finally get

$$\forall t\in[0,T],\ \ [\mathcal{E}]^{(n)}(t) \lesssim P_\kappa(\mathcal{P}_0)T^2 P\left(\sup_{t\in[0,T]}[\mathcal{E}]^{(n),(n-1),(n-2)}(t)\right).$$

Therefore by choosing $T = T_\kappa > 0$ sufficiently small, we can get

$$\sup_{[0,T_\kappa]}[\mathcal{E}]^{(n)}(t) \le \frac{1}{8}\left(\sup_{t\in[0,T]}[\mathcal{E}]^{(n-1)}(t) + \sup_{t\in[0,T]}[\mathcal{E}]^{(n-2)}(t)\right), \tag{4.119}$$

which implies

$$\sup_{[0,T_\kappa]}[\mathcal{E}]^{(n)}(t) \le \frac{1}{2^n}P_\kappa(\mathcal{P}_0) \to 0 \quad \text{as } n\to\infty.$$

4.4 Construction of the solution to the approximation system (3.1)

Proposition 4.3. Suppose the initial data $(v_0, h_0) \in H^4$ satisfying the compatibility conditions. Given $\kappa > 0$, there exists a $T_\kappa > 0$ such that the nonlinear κ-approximation system (3.1) has a unique solution $(v(\kappa), h(\kappa), \eta(\kappa))$ in $[0, T_\kappa]$ satisfying the estimates

$$\sup_{0\le t\le T_\kappa} \mathcal{E}(t) \lesssim \mathcal{P}_0, \tag{4.120}$$

where

$$\begin{aligned} \mathcal{E}(t) &:= \|\partial^2\eta(t)\|_2^2 + \|\partial\eta(t)\|_{L^\infty}^2 + \sum_{k=0}^{4} \|\partial_t^{4-k} v\|_k^2 + \|\partial_t^{4-k} h\|_k^2 + W, \\ W &:= \sum_{k=0}^{4} \|\partial_t^{5-k} h\|_k^2 + \|\partial_t^{4-k}\nabla_{\tilde a} h\|_k^2. \end{aligned} \tag{4.121}$$

Proof. Using the above estimates, we prove that the strong convergence of the approximation solutions $\{(v^{(n)}, h^{(n)}, \eta^{(n)})\}$ to the linearized system (4.1). The limit $(v(\kappa), h(\kappa), \eta(\kappa))$ solves the nonlinear κ-approximation system (3.1). The corresponding energy estimates can be derived by passing to the limit $n \to \infty$ in (4.17) which is uniform-in-n. □

5 Local well-posedness of the compressible gravity water wave system

From Proposition 4.3, given $\kappa > 0$, we have constructed a solution $(v(\kappa), h(\kappa), \eta(\kappa))$ to the nonlinear κ-approximation system (3.1). Proposition 3.1 gives a κ-independent estimate (3.4) on some time interval $[0, T_0]$, which yields a strong convergence to a limit (v, h, η) for every $t \in [0, T_0]$. This limit (v, h, η) is a solution to the compressible gravity water wave system (1.11) with energy estimate (1.15) in Theorem 1.1 if we set $\kappa \to 0+$ in (3.1). Therefore, the existence has been proved.

To prove the uniqueness, we suppose $(v^1, h^1, \eta^1), (v^2, h^2, \eta^2)$ to be two solutions to the compressible gravity water wave system (1.11) which satisfies the energy estimate (1.15) in Theorem 1.1. Denote the difference by $([v], [h], [\eta]) := (v^1 - v^2, h^1 - h^2, \eta^1 - \eta^2)$ and $a^i := (\partial\eta^i)^{-1}$ with $[a] := a^2 - a^1$. Then $([v], [h], [\eta])$ solves the following system **with vanishing initial data**:

$$\begin{cases} \partial_t[\eta] = [v] & \text{in } \Omega, \\ \partial_t[v] = -\nabla_{a^1}[h] + \nabla_{[a]}h^2 & \text{in } \Omega, \\ \operatorname{div}_{a^1}[v] = \operatorname{div}_{[a]}v^2 - e'(h^2)\partial_t[h] - (e'(h^1) - e'(h^2))\partial_t h^2 & \text{in } \Omega \\ [h] = 0 & \text{on } \Gamma. \end{cases} \tag{5.1}$$

We define the energy functional of (5.1) by

$$[\mathcal{E}] = \|[\eta]\|_2^2 + \sum_{k=0}^{2} \|\partial_t^{2-k}[v]\|_k^2 + \|\partial_t^{2-k}[h]\|_k^2 + |(a^1)^{3\alpha}\overline{\partial}^2[\eta]_\alpha|_0^2. \tag{5.2}$$

This looks very similar to (4.97). The only essential difference is the boundary term

$$\int_\Gamma [\mathbf{H}](a^1)^{3\alpha}[\mathbf{V}]_\alpha \, dS,$$

where we define the Alinhac good unknowns

$$\mathbf{V}^i = \overline{\partial}^2 v^i - \overline{\partial}^2\eta^i \cdot \nabla_{a^i} v^i, \quad \mathbf{H}^i = \overline{\partial}^2 h^i - \overline{\partial}^2\eta^i \cdot \nabla_{a^i} h^i,$$

and

$$[\mathbf{V}] := \mathbf{V}^1 - \mathbf{V}^2, \quad [\mathbf{H}] := \mathbf{H}^1 - \mathbf{H}^2.$$

The boundary terms then becomes

$$\begin{aligned}
\int_\Gamma [\mathbf{H}](a^1)^{3\alpha}[\mathbf{V}]_\alpha &= -\int_\Gamma \partial_3[h]\overline{\partial}^2\eta^2_\beta (a^2)^{3\beta}(a^2)^{3\alpha}[\mathbf{V}]_\alpha\, dS - \int_\Gamma \partial_3 h^1(\overline{\partial}^2[\eta]_\beta (a^1)^{3\beta} + \overline{\partial}^2\eta^2_\beta [a]^{3\beta})(a^1)^{3\alpha}[\mathbf{V}]_\alpha\, dS \\
&\lesssim -\frac{1}{2}\frac{d}{dt}\int_\Gamma \partial_3 h^1 |(a^1)^{3\alpha}\overline{\partial}^2[\eta]_\alpha|_0^2\, dS \\
&\quad -\int_\Gamma \partial_3 h^1 (a^1)^{3\gamma}\overline{\partial}^2[\eta]_\gamma (\overline{\partial}^2\eta^2_\beta [a]^{\mu\beta}\partial_\mu v^1_\alpha - \overline{\partial}^2\eta^2_\beta (a^2)^{\mu\beta}\partial_\mu [v]_\alpha)(a^1)^{3\alpha}\, dS \\
&\quad -\int_\Gamma \partial_3 h^1(\overline{\partial}^2[\eta]_\beta (a^1)^{3\beta} + \overline{\partial}^2\eta^2_\beta [a]^{3\beta})(a^1)^{3\alpha}[\mathbf{V}]_\alpha\, dS \\
&\lesssim -\frac{c_0}{2}\frac{d}{dt}\int_\Gamma |(a^1)^{3\alpha}\overline{\partial}^2[\eta]_\alpha|_0^2\, dS + P(\text{initial data})P([\mathcal{E}](t)).
\end{aligned}$$

Here in the second step we use the precise formula of $[\mathbf{V}]$, and in the third step we apply the physical sign condition for h^1. Therefore we have

$$\sup_{t\in[0,T_0]} [\mathcal{E}](t) \le P(\text{initial data}) + \int_0^{T_0} P([\mathcal{E}](t))\, dt.$$

Since the initial data of (5.1) is 0, then we know $[\mathcal{E}](t) = 0$ for all $t \in [0, T_0]$ which gives the uniqueness of the solution to the compressible gravity water wave system (1.11).

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