

Ch7 习题

$$1. \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t=0\}. \end{cases} \quad \text{至多一个光滑解}$$

Proof: ~~全~~ 设 u_1, u_2 为原方程 2 个光滑解. $v = u_1 - u_2$ 要证 $v = 0$

$$v \text{ 满足 } \begin{cases} v_t - \Delta v = 0 & \text{in } U_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial U \times [0, T] \\ v = 0 & \text{on } U \times \{t=0\} \end{cases}$$

两边乘以 v , 积分得

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 - \int_U v \Delta v = 0$$

分部积分

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 = - \int_U |\nabla v|^2 dx \leq 0$$

$$\text{又: } \|v(0)\|_2^2 = 0 \quad \text{故 } \forall t \in (0, T], \quad \frac{1}{2} \|v(t)\|_2^2 \leq 0$$

$$\Rightarrow \|v(t)\|_2^2 = 0 \quad \Rightarrow v = 0 \quad \text{in } [0, T]$$

\uparrow
 $v \in C^\infty$

□

2. 设 u 是下方程的光滑解.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$

$$\text{证明: } \|u(\cdot, t)\|_2(U) \leq e^{-\lambda_1 t} \|g\|_2(U)$$

$\lambda_1 > 0$ 是 $-\Delta$ 的主特征值

Proof: 方程两边乘以 u , 积分得

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 = - \int_U |\nabla u|^2 dx$$

$$\text{又: } \lambda_1 = \inf_{\substack{u \in H_0^1(U) \\ u \neq 0}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}$$

$$\text{故 } \frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq -\lambda_1 \|u(t)\|_2^2$$

由 Gronwall 不等式

$$\begin{aligned} \|u(t)\|_2^2 &\leq e^{-2\lambda_1 t} \|u(0)\|_2^2 \\ &= e^{-2\lambda_1 t} \|g\|_2^2 \end{aligned}$$

□

3. 设 u 为
$$\begin{cases} \partial_t u + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$
 的光滑解, 其中 L 为二阶椭圆算子

v 为
$$\begin{cases} \partial_t v + L^* v = 0 & \text{in } U_T \\ v = 0 & \text{on } \partial U \times [0, T] \\ v = h & \text{on } U \times \{t=T\} \end{cases}$$
 的光滑解.

求证: $\int_U g(x) v(x, 0) dx = \int_U u(x, T) h(x) dx.$

证明: $\int_U u(x, T) h(x) - g(x) v(x, 0) dx = \int_U u(x, T) v(x, T) - u(x, 0) v(x, 0) dx$
显然, 这是关于 t 分部积分出来的边界项

$$\begin{aligned} \text{上式} &= \int_0^T \int_U (\partial_t u) \cdot v + \partial_t v \cdot u \, dx \, dt \\ \langle L^* u, v \rangle &= \langle u, L^* v \rangle \rightarrow \int_0^T \int_U (\partial_t u) \cdot v + \phi L u \cdot v - u L^* v + \partial_t v \cdot u \, dx \, dt \\ &= \int_0^T \int_U \underbrace{(\partial_t u + Lu)}_0 v + u \cdot \underbrace{(\partial_t v - L^* v)}_0 \, dx \, dt = 0 \end{aligned}$$

□

4. (Galerkin Method for Poisson).

$f \in L^2(U)$. $u_m = \sum_{k=1}^m d_m^k w_k$ solves $\int_U Du_m \cdot Dw_k \, dx = \int_U f w_k \, dx$ $1 \leq k \leq m$.

问: $\{u_m\}$ 是否存在. 在 $H_0^1(U)$ 中收敛于 $\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的解.

Pf: Step 1: $\{u_m\}$ 在 H_0^1 中一致有界

$\int_U Du_m \cdot Dw_k \, dx = \int_U f w_k \, dx$. 两边乘 d_m^k , 对 k 求和得:

$$\begin{aligned} \int_U \|Du_m\|_{L^2}^2 &= \int_U f u_m \, dx \\ C \|u_m\|_{H_0^1}^2 &\leq \|f\|_2 \|u_m\|_{H_0^1} \leq \varepsilon \|u_m\|_{H_0^1}^2 + C(\varepsilon) \|f\|_2^2 \\ \varepsilon \text{ 充分小} &\Rightarrow \|u_m\|_{H_0^1}^2 \leq C \|f\|_2^2 \quad \checkmark \end{aligned}$$

Step 2: \exists 子列 $u_{m_k} \rightarrow u$ in $H_0^1(U)$.

$Du_{m_k} \rightarrow v$ in $L^2(U)$

$v \stackrel{a.e.}{=} Du$?

$\forall \varphi \in C_c^\infty \quad \int_U Du_{m_k} \cdot D\varphi = \int_U f \varphi$

$= \int_U u_{m_k} \cdot D^2 \varphi$

$\downarrow k \rightarrow \infty$

$= \int_U u \cdot D^2 \varphi = \int_U Du \cdot D\varphi$

~~$\int_U u_{m_k} \cdot D\varphi$~~

□

[7.5] 设

$$\begin{cases} \mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } L^2(0, T; H_0^1(U)), \\ \mathbf{u}'_k \rightharpoonup \mathbf{v} \text{ in } L^2(0, T; H^{-1}(U)). \end{cases}$$

证明: $\mathbf{u}' = \mathbf{v}$ in $L^2(0, T; H^{-1}(U))$.

证明: 我们断言:

Claim: 对任意 $\phi \in C_c^\infty(0, T)$, $w \in H_0^1(U)$, 成立:

$$\left\langle \int_0^T \phi'(t) \mathbf{u}(t) dt, w \right\rangle = \left\langle - \int_0^T \mathbf{v}(t) \phi(t) dt, w \right\rangle,$$

其中 $\langle \cdot, \cdot \rangle$ 代表 $H^{-1}(U), H_0^1(U)$ 中元素之间的作用(pairing).

若Claim获证, 那么在 $H^{-1}(U)$ 中(即作为 $H_0^1(U)$ 上的连续线性泛函)成立:

$$\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \mathbf{v}(t) \phi(t) dt.$$

再由时间弱导数定义知

$$\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \mathbf{u}'(t) \phi(t) dt.$$

这样就有 $\mathbf{u}' = \mathbf{v}$ in $L^2(0, T; H^{-1}(U))$.

Claim的证明仍然由直接计算可得: 注意到 $t \mapsto \pi(t)w \in L^2(0, T; H_0^1)$, 那么:

$$\begin{aligned} \left\langle \int_0^T \phi'(t) \mathbf{u}(t) dt, w \right\rangle &= \int_0^T \langle \phi'(t) \mathbf{u}(t), w \rangle dt \\ &= \int_0^T \langle \mathbf{u}(t), \phi'(t) w \rangle dt \\ (\mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } L^2(0, T; H_0^1(U))) &= \lim_{k \rightarrow \infty} \int_0^T \langle \mathbf{u}_k(t), \phi'(t) w \rangle dt \\ &= \lim_{k \rightarrow \infty} \int_0^T \langle \mathbf{u}_k(t) \phi'(t), w \rangle dt \\ &= \lim_{k \rightarrow \infty} \left\langle \int_0^T \mathbf{u}_k(t) \phi'(t) dt, w \right\rangle \\ &= - \lim_{k \rightarrow \infty} \left\langle \int_0^T \mathbf{u}'_k(t) \phi(t) dt, w \right\rangle \\ &= - \lim_{k \rightarrow \infty} \int_0^T \langle \mathbf{u}'_k(t) \phi(t), w \rangle dt \\ (\mathbf{u}'_k \rightharpoonup \mathbf{v} \text{ in } L^2(0, T; H^{-1}(U))) &= - \int_0^T \langle \mathbf{v}(t), \phi(t) w \rangle dt \\ &= \left\langle - \int_0^T \mathbf{v}(t) \phi(t) dt, w \right\rangle \end{aligned}$$

6. 设 H 是 Hilbert 空间, $u_k \rightarrow u$ in $L^2(0, T; H)$, $\operatorname{ess\,sup}_{0 \leq t \leq T} \|u_k(t)\| \leq C \quad \forall k \in \mathbb{Z}_+$

证明: $\operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\| \leq C$

Proof: 先证: $\forall 0 \leq a \leq b \leq T, v \in H$, 有:

(hint). $\int_a^b (v, u_k(t)) dt \leq C \|v\| (b-a)$ (显然).

若 hint 成立, 则

$$\frac{1}{b-a} \int_a^b (v, u_k(t)) dt \leq C \|v\|$$

这样, 由 Lebesgue 微分定理, $t \in (a, b)$ 中点.

对 a.e. $t \in [0, T]$ 有: $|\langle u_k(t), v \rangle| \leq C \|v\|$

$k \rightarrow \infty$, 由弱收敛知, $|\langle u, v \rangle| \leq C \|v\| \quad a.e. t \in [0, T]$

$\Rightarrow \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\| \leq C$

6. H 为 Hilbert 空间, $u_k \rightarrow u$ in $L^2(0, T; H)$, 且 $\operatorname{ess\,sup}_{0 \leq t \leq T} \|u_k(t)\| \leq C \quad \forall k \in \mathbb{Z}_+$.

求证: $\operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\| \leq C$

证明: 设 $f_{a,b}(v) = \int_a^b (v, u_k(t)) dt$.

显然, $f_{a,b} \in L^2(0, T; H)$.

故 $\lim_{k \rightarrow \infty} f_{a,b}(u_k) = f_{a,b}(u) = \int_a^b (u(t), u(t)) dt$.

而 $f_{a,b}(u_k) = \int_a^b (u_k(t), u(t)) dt$
 $\leq C \int_a^b \|u(t)\|_H dt$

$\leq C \sqrt{b-a} \left\| \|u(t)\|_H \right\|_{L^2_t(a,b)}$
 $\Rightarrow \int_a^b \|u(t)\|_H^2 dt \leq C^2 (b-a) \quad \forall a, b \in [0, T]$

$\Rightarrow \|u(t)\|_H \leq C \quad a.e. t \in [0, T]$

↑
 证 $b \rightarrow a$. 用 Lebesgue 微分定理即可
 令 $b \rightarrow a$

□

7. 设 u 是光滑解:
$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$

且函数 c 满足 $c \geq \gamma > 0$.

证明: $|u(x, t)| \leq Ce^{-\gamma t}$.

Proof: 设 $v = e^{\gamma t} u$.

k) $\partial_t v - \Delta v + cv = \gamma e^{\gamma t} u + e^{\gamma t} u_t - e^{\gamma t} \Delta u + ce^{\gamma t} u$

$$= \gamma v + \underbrace{(e^{\gamma t} (\partial_t - \Delta + c) u)}_{=0} e^{\gamma t} = \gamma v$$

$$\Rightarrow \begin{cases} \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0 & \text{in } U \times (0, \infty) \\ v = 0 & \text{on } \partial U \times [0, \infty) \\ v = g & \text{on } U \times \{t=0\} \end{cases}$$

由弱极大值原理:

$\forall (x, t) \in U_T$,

$$|v(x, t)| = e^{\gamma t} |u(x, t)| \leq \sup_{\Gamma_T} |v(x, t)| = \sup_{x \in U} |g(x)|.$$

$$\Rightarrow |u(x, t)| \leq e^{-\gamma t} \|g\|_{L^\infty}$$

8. 若 u 是 7 中方程的光滑解: $g \geq 0$, c 有界但不一定非负. 证明 $u \geq 0$

Proof: 令 $v = e^{-(\|c\|_{L^\infty} + 1)t} u$.

同 7.
$$\Rightarrow \begin{cases} \partial_t v - \Delta v + (c + \|c\|_{L^\infty} + 1) v = 0 & \text{in } U \times (0, \infty) \\ v = 0 & \text{on } \partial U \times [0, \infty) \\ v = g & \text{on } U \times \{t=0\} \end{cases}$$

由弱极大值原理. $\min_{\overline{U_T}} v \geq -\max_{\Gamma_T} u^- = -\max_{U \times \{t=0\}} g^-$.

$$g \geq 0 \Rightarrow g^- \equiv 0 \Rightarrow \min_{\overline{U_T}} v \geq 0 \Rightarrow \min_{\overline{U_T}} u \geq 0.$$

□

9. 证明 7.1.3 中 (54).
7.2.3 中 (59).

Proof:

(59) 是什么? $\forall u \in H^1(\Omega) \cap H_0^1(\Omega)$. 成立: $\beta \|u\|_{H^1}^2 \leq (Lu, -\Delta u) + \gamma \|u\|_{L^2(\Omega)}^2$.
($\exists \beta > 0, \gamma > 0$).

实际用上只是对 Galerkin 逼近序列因此不难.

在此, 为了方便, 我们设 $Lu = -\sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u)$ 在实际构造中
 $\{u \in C^\infty, u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0, \frac{\Delta u}{\partial n}|_{\partial\Omega} = 0\}$.

$$\text{要证: } -\int_{\Omega} \sum_{i,j} a^{ij} \partial_i u \partial_j (\Delta u) dx \geq \frac{\theta}{2} \int_{\Omega} |D^2 u|^2 dx - C \int_{\Omega} |u|^2 dx$$

希望对 I 有何控制?

问题: 计算出 u 的二阶导数 (along $\partial\Omega$).

• 如何处理 $\frac{\partial u}{\partial n}$. (边界的法向导数)?

手段: 边界拉直 (on 边界 $\{x_n = 0\}$), 写成 \mathbb{R}^n 坐标.

• 直接计算. 先求 $\partial_{nn} u$. 再求 $\partial_{\alpha n} u$. 再求 $\partial_{\alpha\beta} u$.

• 希望的结果: $|I| \lesssim \int_{\Omega} |D^2 u|^2 + C \int_{\Omega} |u|^2 dx$.

$$\text{这 } \iff |I| \lesssim \int_{\Omega} |\nabla u|^2 dx$$

因 I 本身是 \rightarrow 迹定理
 $\partial\Omega$ 的积分;

转化为 Ω 中的积分;
且有非“迹定理”

局部坐标下, 法向为 e_n

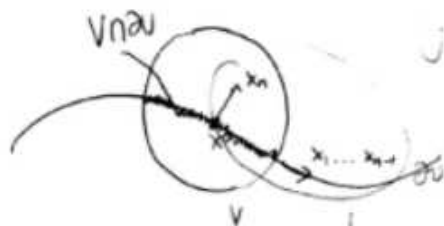
$$\iff |I| \lesssim \int_{\Sigma} (\partial_{nn} u)^2 d\mathcal{H}^{n-1} \leftarrow \text{于是这成为了我们的目标.}$$

即: 设法用 $\partial_{nn} u$ 等, 来给出 I 中各个导数 (尤其是二阶的) 的估计.
圈 $\partial_{nn} \partial_{nn} u$.

于是, 我们现在要做的是, 将 ∂U 上的各阶导数用 $\partial_n u, \partial_{\alpha_n} u$ 表示出来.

Step 1: 由单位分解 (因 ∂U 紧), 可以假设 a^j 的支持 $\text{supp } a^j$ 包含于某一点 $x_0 \in \partial U$ 的邻域 V 内.

不妨设: x^0 是 ~~原点~~
 ∂U 在 x^0 处的切向量 $x_n \text{ 轴 } (e_n)$.



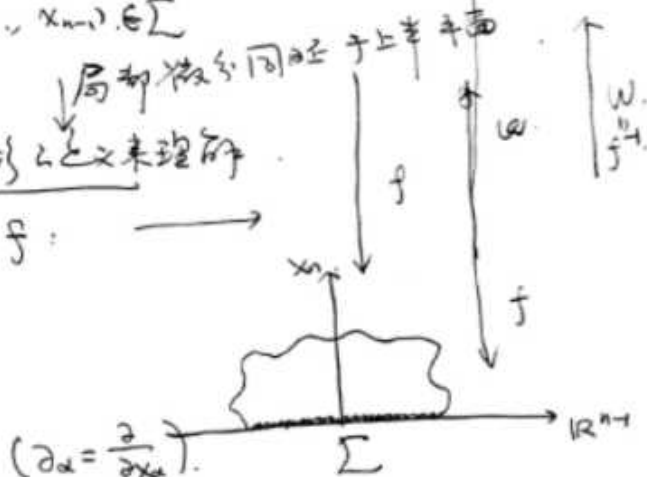
$\Sigma = V \cap \partial U$ 在 x_n 轴上的投影.

记 $x_n = w(x')$, $x' = (x_1, \dots, x_{n-1}) \in \Sigma$
 $w \in C^2(\Sigma)$.

这一段文字其实可以用 带边流形 的定义来理解.

如同 \mathbb{R}^n 存在 微分同胚 号:

w 即是 该微分同胚 的逆



Step 2: 计算 $\partial_\alpha \partial_\beta u$. ($\partial_\alpha = \frac{\partial}{\partial x_\alpha}$).

令 $v(x') = \frac{\partial}{\partial x_n} u(x', w(x'))$. 即 $V \cap \partial U$ 任一点, 用局部坐标可写成 $(x', w(x'))$.

① ~~计算~~: 求 $\partial_\alpha \partial_\beta u$ 和 $\partial_n^2 u$.

对上式求导 ($\alpha = 1, 2, \dots, n-1$).

$$\text{有: } \frac{\partial}{\partial x_\alpha} \partial_\alpha u = \partial_\alpha \partial_n u + \partial_n^2 u \cdot \partial_\alpha w. \quad \dots \textcircled{1}$$

$$\text{又: } u|_{\partial U} = 0. \quad \text{故: } u(x', w(x')) = 0. \quad \forall x' \in \Sigma.$$

$$\text{对 } x_\alpha \text{ 求导: } \partial_\alpha u + \partial_n u \partial_\alpha w = \partial_\alpha u + v \cdot \partial_\alpha w = 0. \quad \text{记 } v = \frac{\partial u}{\partial x_n} \text{ 在 } V \cap \partial U \text{ 上.}$$

$$\text{对 } x_\beta \text{ 求导: } \partial_\alpha \partial_\beta u + \partial_\alpha \partial_n u \partial_\beta w + \partial_\beta v \partial_\alpha w + v \cdot \partial_\alpha \partial_\beta w = 0. \quad \dots \textcircled{2}$$

$1 \leq \alpha, \beta \leq n-1$.

$$\text{取 } \alpha = \beta \text{ 有: } \partial_\alpha^2 u + \partial_\alpha \partial_n u \partial_\alpha w + \partial_\alpha v \partial_\alpha w + v \cdot \partial_\alpha^2 w = 0. \quad \dots \textcircled{3}$$

对 α 从 1 到 $n-1$ 求和:

$$\text{注意到 } \Delta u|_{\partial\Omega} = 0 \Rightarrow -\partial_n^2 u = \sum_{1 \leq \alpha \leq n-1} \partial_\alpha^2 u.$$

$$\text{有: } -\partial_n^2 u + \sum_{1 \leq \alpha \leq n-1} \partial_\alpha \partial_n u \cdot \partial_\alpha w + \partial_\alpha v \partial_\alpha w + v \partial_\alpha^2 w = 0 \quad \dots (4)$$

① 代入④. 有: (重指标代表求和).

$$-\partial_n^2 u + (\partial_\alpha v - \partial_n^2 u \partial_\alpha w) \partial_\alpha w + \partial_\alpha v \partial_\alpha w + v \partial_\alpha^2 w = 0$$

$$\Rightarrow \partial_n^2 u (1 + \sum_\alpha (\partial_\alpha w)^2) = v \Delta_n w + 2 \partial_\alpha v \partial_\alpha w$$

$$\Rightarrow \partial_n^2 u = \frac{2 \partial_\alpha w}{\sqrt{1 + |\nabla_n w|^2}} \partial_\alpha v + \frac{v \Delta_n w}{\sqrt{1 + |\nabla_n w|^2}}$$

这样, 对 $\partial_n^2 u$, 我们达到了目的, 即用 $\partial_\alpha v, v$ (i.e. $\partial_\alpha \partial_n u, \partial_n u$) 表示.

$$\text{方便起见, 令 } \sigma_{nn} = \frac{2 \partial_\alpha w}{\sqrt{1 + |\nabla_n w|^2}} \in C^1(\Sigma)$$

$$T_{nn} = \frac{\Delta_n w}{\sqrt{1 + |\nabla_n w|^2}} \in C(\Sigma).$$

$$\Rightarrow \partial_n^2 u = \sigma_{nn} \partial_\alpha v + T_{nn} v.$$

上下指标表示求和

... (5)

② 求 $\partial_\alpha \partial_n u$.

⑤ 代入④. 即有:

$$\partial_\alpha v = \partial_\alpha \partial_n u + \partial_\alpha w (\sigma_{nn} \partial_\beta v + T_{nn} v).$$

$$\Rightarrow \partial_\alpha \partial_n u = \partial_\alpha v - \partial_\alpha w (\sigma_{nn} \partial_\beta v + T_{nn} v).$$

$$= \partial_\beta v (\delta_\alpha^\beta - \sigma_{nn} \partial_\alpha w) - T_{nn} \partial_\alpha w \cdot v.$$

$$\text{令 } T_{nn} = T_{nn} \partial_\alpha v \in C(\Sigma) \quad \sigma_{nn}^\beta = \delta_\alpha^\beta - \sigma_{nn} \partial_\alpha w \in C^1(\Sigma)$$

$$\text{有: } \partial_\alpha \partial_n u = \sigma_{nn}^\beta \partial_\beta v - T_{nn} v. \quad \dots (6)$$

(3). 求 $\partial_\beta \partial_\alpha u$.

②代入②有:

$$\partial_\beta u + (\sigma_{\alpha n}^\nu \partial_\nu u + \tau_{\alpha n} u) \partial_\beta w + \partial_\beta u \partial_\alpha w + u \partial_\alpha \partial_\beta w = 0$$

$$\Rightarrow \partial_\beta u = -(\sigma_{\alpha n}^\nu \partial_\nu u + \tau_{\alpha n} u) \partial_\beta w - \partial_\beta u \partial_\alpha w - \partial_\alpha \partial_\beta w \cdot u$$

$$= \partial_\beta u (-\sigma_{\alpha n}^\nu \partial_\beta w - \partial_\beta \partial_\alpha w)$$

$$+ u (-\partial_\alpha \partial_\beta w - \tau_{\alpha n} \partial_\beta w)$$

$$\hat{=} \sigma_{\alpha\beta}^\nu = -\sigma_{\alpha n}^\nu \partial_\beta w - \partial_\beta \partial_\alpha w$$

$$\tau_{\alpha\beta} = -\partial_\beta w - \tau_{\alpha n} \partial_\beta w$$

$$\Rightarrow \partial_\beta u = \sigma_{\alpha\beta}^\nu \partial_\nu u + \tau_{\alpha\beta} u$$

$$\sigma_{\alpha\beta}^\nu \in C^1(\Sigma)$$

$$\tau_{\alpha\beta} \in C(\Sigma)$$

Step 3: 完成估计:

如今, 可以用一些 $C^1(\Sigma)$ 的 g^α 与 $C^0(\Sigma)$ 的 h .

表 I 如下:

$$I = \int_\Sigma v (g^\alpha \partial_\alpha u + h u) d\mathcal{H}^{n-1} \cdot \underbrace{|\det w|}_{1}$$

$$|I| = \left| \int_\Sigma \frac{v (g^\alpha \partial_\alpha u^2 + h u^2)}{2} \right|$$

$$\left| \int_\Sigma \left(\frac{1}{2} g^\alpha \partial_\alpha u^2 + h u^2 \right) d\mathcal{H}^{n-1} \right|$$

$$\leq \int_\Sigma \left(h - \frac{1}{2} \partial_\alpha g^\alpha \right) v^2 d\mathcal{H}^{n-1} \quad \text{逆处理}$$

$$\lesssim \int_\Sigma v^2 d\mathcal{H}^{n-1} \lesssim \int_\Sigma \left| \frac{\partial u}{\partial n} \right|^2 dS \lesssim \int_\Sigma |\nabla u|^2 dx$$

再由 Chs. T9 有 $\int_\Sigma |\nabla u|^2 dx \lesssim \|u\|_{L^2} \|D^2 u\|_{L^2} \lesssim \varepsilon \|D^2 u\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^2}^2$ □

10. 求证: 如下方程至多一个光滑解 $d > 0$ ← 书上应该漏条件

$$\begin{cases} u_t + d u_t - u_{xx} = f & \text{in } (0,1) \times (0,T) \\ u = 0 & \text{on } \{0\} \times [0,T] \cup \{1\} \times [0,T] \\ u(0) = g \quad u(1) = h \end{cases}$$

证明: 若有2个光滑解 ~~u_1, u_2~~ , 令 $V = u_1 - u_2$,

$$\text{则 } V \text{ 满足 } \begin{cases} V_t + d V_t - V_{xx} = 0 & \text{in } (0,1) \times (0,T) \\ V = 0 & \text{on } \{0\} \times [0,T] \cup \{1\} \times [0,T] \\ V(0) = g, \quad V(1) = h. \end{cases}$$

$$\text{令 } E(t) = \frac{1}{2} \|V_t\|_{L^2}^2 + \frac{1}{2} \|V_x\|_{L^2}^2$$

$$\text{则 } E'(t) = \int_0^1 V_t V_{tt} + V_x V_{xt} dx$$

第2项分部积分.

$$= \int_0^1 V_t V_{tt} - V_{xx} V_t dx$$

$$= \int_0^1 (-d V_t) V_t dx$$

$$= -d \int_0^1 V_t^2 dx \leq 0$$

$$\text{又 } E(0) = 0, \quad \therefore E(t) \equiv 0 \Rightarrow V(t) = \text{const.}$$

$$\text{又 } V(0) = 0 \quad \therefore V \equiv 0$$

□

$$11. \begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } (0,1) \times (0,T) \\ u = \partial_x u = 0 & \text{on } (\{0\} \times [0,T]) \cup (\{1\} \times [0,T]) \\ u = g, \quad u_t = h & \text{on } [0,1] \times \{t=0\} \end{cases}$$

存在唯一光滑解

Proof:

$$v = u_1 - u_2$$

$$\Rightarrow \partial_t^2 v + \partial_x^4 v = 0$$

$$\text{乘 } v_t \Rightarrow \partial_t v \partial_t^2 v + \partial_t v \partial_x^4 v = 0$$

$$\text{而: } \partial_t (\partial_t v)^2 = 2 \partial_t^2 v \partial_t v$$

$$\partial_t (\partial_x^2 v)^2 = 2 (\partial_x^2 v \cdot \partial_t \partial_x^2 v)$$

$$\text{所以上式} \Rightarrow \frac{1}{2} \partial_t (\|v_t\|_{L^2}^2 + \|\partial_x^2 v\|_{L^2}^2) = 0$$

积分
(对 \$t\$ 求导)

$$\Rightarrow \|v_t\|^2 + \|\partial_x^2 v\|^2 = \text{const.} = t=0 \text{ 时的值} = 0$$

$$\Rightarrow v = 0$$

□

12: 设 \$A\$ 为实 Banach 空间 \$X\$ 上的闭算子, 定义域为 \$D(A)\$ ~~预解式 \$R_\lambda\$~~ 若 \$\lambda, \nu \in \rho(A)\$

求证: (i) \$R_\lambda - R_\nu = (\nu - \lambda) R_\lambda R_\nu\$

$$(ii) R_\lambda R_\nu = R_\nu R_\lambda$$

证明: 不妨 \$\lambda \neq \nu\$

$$R_\lambda - R_\nu = R_\lambda \cdot \underbrace{(\nu I - A)}_{Id} R_\nu - \underbrace{(\lambda I - A)}_{Id} R_\lambda R_\nu$$

$$= (\nu - \lambda) R_\lambda R_\nu$$

$$(2) \text{ 调换 } \lambda, \nu. \text{ 有 } R_\nu R_\lambda (\lambda - \nu) = R_\nu - R_\lambda$$

$$= -(R_\lambda - R_\nu)$$

$$= -(\nu - \lambda) R_\lambda R_\nu$$

约掉 \$\lambda - \nu\$

$$\Rightarrow R_\lambda R_\nu = R_\nu R_\lambda$$

□

13. Justify the equality

$$A \int_0^\infty e^{-\lambda t} S(t) u \, dt = \int_0^\infty e^{-\lambda t} A S(t) u \, dt$$

used in (16) of §7.4.1. (Hint: Approximate the integral by a Riemann sum and recall A is a closed operator.)

先证明: 对 \int_0^M , A 可与之换序:

$\forall M \in \mathbb{R}_+$, $\frac{1}{2^k} [0, M]$ 的分割:

$$[0, M] = \bigcup_{j=0}^{2^k-1} \left[\frac{j}{2^k} M, \frac{j+1}{2^k} M \right]$$

$$\text{由 } I_k(t) := \sum_{j=0}^{2^k-1} e^{-\lambda t_j} S(t_j) u, \quad t \in [t_j, t_{j+1}), \quad \Rightarrow e^{-\lambda t} S(t) u. \quad \text{as } k \rightarrow \infty$$

Simple functions

及 A 的线性:

$$\begin{aligned} A \int_0^M e^{-\lambda t} S(t) u \, dt &= A \left(\lim_{k \rightarrow \infty} \int_0^M I_k(t) \, dt \right) \\ &\stackrel{A \text{ 的线性}}{=} \lim_{k \rightarrow \infty} A \int_0^M I_k(t) \, dt. \\ &\stackrel{I_k \text{ simple}}{=} \lim_{k \rightarrow \infty} \int_0^M A I_k(t) \, dt \\ &\stackrel{A \text{ 的线性}}{=} \lim_{k \rightarrow \infty} \int_0^M A e^{-\lambda t_j} S(t_j) u \, dt \\ &= \int_0^M A e^{-\lambda t} S(t) u \, dt \stackrel{A \text{ 的线性}}{=} \int_0^M e^{-\lambda t} A S(t) u \, dt \end{aligned}$$

$$\begin{aligned} \text{又: } e^{-\lambda t} &= \text{rapidly decays} \\ \|S(t)\| &\leq 1. \\ A \text{ 的线性} \end{aligned}$$

$$\Rightarrow \int_0^M e^{-\lambda t} S(t) u \, dt \xrightarrow{M \rightarrow \infty} \int_0^\infty e^{-\lambda t} S(t) u \, dt.$$

$$\Rightarrow A \left(\int_0^M e^{-\lambda t} S(t) u \, dt \right) \xrightarrow{M \rightarrow \infty} A \left(\int_0^\infty e^{-\lambda t} S(t) u \, dt \right).$$

therefore "II" holds.

$$\int_0^M e^{-\lambda t} A S(t) u \, dt \xrightarrow{M \rightarrow \infty} \int_0^\infty e^{-\lambda t} A S(t) u \, dt.$$

$$\int_0^M e^{-\lambda t} S(t) A u \, dt \xrightarrow{M \rightarrow \infty} \int_0^\infty e^{-\lambda t} S(t) A u \, dt$$

□.

14.

设 ϕ 是热方程的基本解, 即 $\phi(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$.

$\forall t > 0$, 令

$$[S(t)g](x) = \int_{\mathbb{R}^n} \phi(x-y, t) g(y) dy, \quad x \in \mathbb{R}^n.$$

$$g: \mathbb{R}^n \rightarrow \mathbb{R}.$$

$$S(0)g = g.$$

则: $\{S(t)\}_{t \geq 0}$ 是 $L^2(\mathbb{R}^n)$ 上的压缩半群.
不是 $L^\infty(\mathbb{R}^n)$ 上的压缩半群.

$S(t)$ 是 L^2 上的压缩半群

$$\|S(t)g\|_{L^2} = \|\phi * g\|_{L^2}$$

$$\leq \|\phi\|_1 \|g\|_{L^2} = \|g\|_{L^2} \Rightarrow \|S(t)\| \leq 1.$$

$$S(t+s)g = \int_{\mathbb{R}^n} \phi(x-y, t+s) g(y) dy = \int_{\mathbb{R}^n} \hat{\phi}(\xi, t+s) \hat{g}(\xi) d\xi.$$

$$= \hat{\phi}(\xi, t) \hat{\phi}(\xi, s) \hat{g}(\xi) = S(t)S(s)g$$

$t \mapsto S(t)g$ 连续性:

$$\|S(t+h)g - S(t)g\|_{L^2} \leq \|S(h)g - g\|_{L^2}.$$

$$= \left\| \int_{\mathbb{R}^n} \phi(x-y, h) g(y) dy - g(x) \right\|_{L^2_x}.$$

$$= \left\| \int_{\mathbb{R}^n} \phi(x-y, h) (g(y) - g(x)) dy \right\|_{L^2_x}.$$

$$= \left\| \int_{\mathbb{R}^n} \phi(y, h) (g(x-y) - g(x)) dy \right\|_{L^2_x}$$

$$\leq \int_{\mathbb{R}^n} \phi(y, h) \|g(x-y) - g(x)\|_{L^2_x} dy.$$

$\xrightarrow{DCT} 0$

\downarrow_0 (平移连续性).

$S(t)$ 不是 L^∞ 上的压缩半群, 因在 $t=0$ 处不连续

令 $g(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$ $g(0) = 0$

$$(S(t)g)(x) = \int_{-\infty}^x \frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}} dy \quad (S(t)g)(0) = \frac{1}{2} \quad \forall t > 0$$

$$\Rightarrow \|S(t)g - g\|_{L^\infty} \geq \frac{1}{2}$$

□

15.

设 X 上有以 A 为无穷小生成元的压缩半群 $\{S(t)\}_{t \geq 0}$

定义 $D(A^k) := \{u \in D(A^{k-1}) \mid A^{k-1}u \in D(A)\} \quad (k \geq 2)$

证明: 若 $\exists k, u \in D(A^k)$, 则 $\forall t \geq 0, S(t)u \in D(A^k)$

Pf: $\forall j \in \{1, 2, \dots, k-1\}$, 要证: $A^j S(t)u \in D(A)$

i.e. $\lim_{s \rightarrow 0^+} \frac{S(s) A^j S(t)u - A^j S(t+s)u}{s}$

$$= \lim_{s \rightarrow 0^+} \frac{S(s) A^j S(t)u - A^j S(t+s)u}{s}$$

$$= \frac{S(s) S(t) A^j u - S(t) A^j u}{s}$$

$$= \frac{S(t+s) (A^j u) - S(t) (A^j u)}{s}$$

归纳假设 $A^j u \in D(A)$

$$= \text{exists} \quad (u \text{ 及 } S(t) = Id)$$

□

16. 用 T15 证明: 若 u 是 $\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$ 在 $X = L^2(U)$ 中的半群解

且 $g \in C_0^\infty(U)$, 则 $u(\cdot, t) \in C_0^\infty(U)$ $0 \leq t \leq T$

Pf. 先证 $\{S(t)\}_{t \geq 0}$ 压缩, 此为显见. 因在方程两边同乘 u , 分部积分有

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 = 0 \Rightarrow \frac{d}{dt} \|u\|_{L^2}^2 \leq 0$$

$\therefore \|S(t)g\|_{L^2} \leq \|g\|_{L^2}$

$$\text{而 } g \in C_0^\infty(U) \subset H^{2k}(U) \cap H_0^{2k-1}(U) = D(A^k) \Rightarrow \|S(t)g\|_{L^2} \leq \|g\|_{L^2} \quad \forall g \in L^2$$

由 T15 知, $\forall t \geq 0$

$$u(\cdot, t) = S(t)g \in H^{2k}(U) \cap H_0^{2k-1}(U) \quad \forall k \in \mathbb{Z}_+$$

由 Sobolev 嵌入知 $u(\cdot, t) \in C^\infty(U)$

□