Advanced Probability Theory, MATH5007P, Midterm Exam

Student ID: Name:

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1. (10 points) Assume that $\mathcal{B}$ is the generated σ-field of $\mathcal{A}$. Prove that $f^{-1}(\mathcal{A})$ is the generating class of $f^{-1}(\mathcal{B})$.

2. (10 points) Let μ be a finite measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$. Show that

$$\int (F(x+c) - F(x))dx = c\mu(\mathbb{R})$$

3. (10 points) Let $X_1, X_2, \ldots, X_n, \ldots$ be uncorrelated with $EX_i = \mu_i$ and $\text{var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \ldots X_n$ and $\nu_n = ES_n/n$ then as $n \to \infty$, $S_n/n - \nu_n \to 0$ in L^2 and in probability.

4. (15 points) Let X_n be independent Poisson r.v.'s with $EX_n = \lambda_n$, and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum \lambda_n = \infty$ then $S_n/ES_n \to 1$ a.s.

5. (15 points) Suppose the ith bulb burns for an amount of time X_i and then remains burned out for time Y_i before being replaced. Suppose the X_i, Y_i are positive and independent with the X's having distribution F and the Y's having distribution G, both of which have finite mean. Let R_t be the amount of time in $[0, t]$ that we have a working light bulb. Show that $R_t/t \to EX_i/(EX_i + EY_i)$ almost surely.

6. (20 points) Let $X, X_1, \cdots, X_n, \cdots$ be i.i.d. Suppose B is a Borel set on $\mathbb{R}$ with $P(X \in B) \in (0, 1)$. Let $N(\omega) := \inf\{n : X_n(\omega) \in B\}$ and $N(\omega) = \infty$ if there is no such n. Show that $N < \infty$ almost surely. Let $B' \subset B$ be another Borel set. Show that the distribution of X_N is the distribution of X conditioned on B, i.e.

$$P(X_N \in B') = \frac{P(X \in B')}{P(X \in B)}$$

7. (10 points) Recall that the L^2 metric space is complete. Let $X_1, X_2, \ldots$ be independent r.v. Show that $\sum X_i$ converges in L^2 if and only if $\sum EX_i$ and $\sum \mathrm{var}(X_i)$ converges.

8. (10 points) Construct a sequence of independent random variables so that their sum converge a.s. but not in L^2