Evans Ch12 习题

本节习题中，除特别声明，我们均假设函数是光滑的，记号 $\Box_d = \partial_t^2 - \Delta_{d}$

[12.1] 设 $u$ 紧支，且是拟线性方程 $u_{tt} - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i} = 0$ in $\mathbb{R}^d \times (0,\infty)$ 的解

请构造合适的能量 $E(t)$，使得 $E'(t) = 0$

证明：令 $E(t) = \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} L(\nabla u)\, dx$.

$$\Rightarrow E'(t) = \int_{\mathbb{R}^d} u_t u_{tt}\, dx + \int_{\mathbb{R}^d} \sum_{i=1}^{n} L_{p_i}(\nabla u) \cdot \partial_t u_{x_i}\, dx$$

$$\overset{\text{分部积分}}{\Longrightarrow} E'(t) = \int_{\mathbb{R}^d} u_t u_{tt}\, dx - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i}\, u_t\, dx$$

$$= \int_{\mathbb{R}^d} u_t \underbrace{\left(u_{tt} - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i}\right)}_{=0} dx = 0.$$

□

[12.2] 设 $u$ 为 Klein-Gordon 方程 $\begin{cases} u_{tt} - \Delta u + m^2 u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(0) = g,\ u_t(0) = h \end{cases}$ 的解

求证：(1) $E(t) = \frac{1}{2}\int_{\mathbb{R}^d} u_t^2 + |\nabla u|^2 + m^2 u^2\, dx$ 关于 $t$ 不变

(2) $\lim_{t\to\infty} \int_{\mathbb{R}^d} |\nabla u|^2 + m^2 u^2\, dx = E(0)$. (Hint: 模仿 §4.3.1)

证明：(1) $E'(t) = \int_{\mathbb{R}^d} u_t u_{tt} + \nabla u \cdot \nabla u_t + m^2 u \cdot u_t\, dx$

$$\overset{\text{第2项分部积分}}{=} \int_{\mathbb{R}^d} u_t \underbrace{(u_{tt} - \Delta u + m^2 u)}_{=0} dx = 0$$

(2) 不妨设 $m = 1$，令 $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$

原方程作 Fourier 变换可得：

$$\begin{cases} \hat{u}_{tt} + \langle \xi \rangle^2 \hat{u} + \hat{u} = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ \hat{u}(0) = \hat{g},\ \hat{u}_t(0) = \hat{h} \end{cases}$$

$$\Rightarrow \hat{u}(\xi, t) = \hat{g}(\xi)\cos(t\langle\xi\rangle) + \frac{\hat{h}}{\langle \xi \rangle}\sin(t\langle\xi\rangle)$$

待估计的式子是 $I = \int_{\mathbb{R}^d} |\nabla u|^2 +$

由 Plancherel 恒等式:

$$I = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \langle\xi\rangle^2 d\xi \quad \leftarrow \text{不妨 } g, h \text{ 实值吧.}$$

$$= \int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2 \cos^2(t\langle\xi\rangle)\, d\xi \quad \leftarrow I_1$$

$$+ \int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \sin^2(t\langle\xi\rangle)\, d\xi \quad \leftarrow I_2$$

$$\underbrace{+ 2\int_{\mathbb{R}^d} \cos(t\langle\xi\rangle) \sin(t\langle\xi\rangle) \langle\xi\rangle \hat{g}(\xi) \hat{h}(\xi)\, d\xi}_{=:J}$$

$\boxed{\text{Claim}}$ $J \longrightarrow 0$ as $t \longrightarrow +\infty$

$$J = \int_{\mathbb{R}^d} \sin(2t\langle\xi\rangle) f(\xi)\, d\xi. \quad \text{其中 } f(\xi) = \langle\xi\rangle \hat{g}(\xi) \hat{h}(\xi) \in \mathcal{S}(\mathbb{R}^d).$$

$$= \int_0^\infty \sin(2t\sqrt{\rho^2+1}) \left( \int_{\partial B(0,\rho)} f \, dS \right) d\rho$$

令 $F(\rho) := \int_{\partial B(0,\rho)} f \, dS \in C^\infty(\mathbb{R})$.

而 $f \in \mathcal{S}(\mathbb{R}^d) \Longrightarrow F(\rho) \in L^1(\mathbb{R})$.

于是 $\exists \{F_n\} \subseteq C_c^\infty(0,+\infty)$ s.t. $F_n \longrightarrow F$ in $L^1(\mathbb{R})$

$|$ 0 的一个小邻域 $\cap\, \mathrm{spt}\, F_n = \emptyset$

(相当于 $F_n$ 的支集总离原点差一些)

对 $F_n$ 而言.

令 $J_n = \int_0^\infty \sin(2t\sqrt{\rho^2+1}) F_n(\rho)\, d\rho$

$$\overset{u=\sqrt{\rho^2+1}}{=\!=} \int_1^\infty \frac{u F_n(\sqrt{u^2-1})}{\sqrt{u^2-1}} \sin(2tu)\, du.$$

$$= \int_{\underbrace{\mathrm{spt}\, F_n}_{\text{远离 } u=1 \text{ (因 } \rho=0 \text{ 远离 } \mathrm{spt}\, F_n)}} \boxed{\frac{u}{\sqrt{u^2-1}} F_n(\sqrt{u^2-1})}\, \sin(2tu)\, du. \quad \cancel{\Rightarrow \in L^1(\mathbb{R}^+)}$$

$$= \int_1^\infty \underbrace{\left( \chi_{\mathrm{spt}\, F_n} \frac{u}{\sqrt{u^2-1}} F_n(\sqrt{u^2-1}) \right)}_{\in L^1(\mathbb{R}^+)} \sin(2tu)\, du$$

于是由Riemann-Lebesgue引理知, $J_n \to 0$ as $t\to\infty$

而 $|J-J_n| \underset{\uparrow \atop \sin\text{有界}}{\le} \|F_n - F\|_{L^1(\mathbb{R}^d)} \to 0$ as $n\to\infty$ $\forall t>0$ 均成立

故 $J=\int_{\mathbb{R}^d} \sin(2t\langle\xi\rangle) f(\xi)\, d\xi \to 0$ as $t\to\infty$.

~~Rmk~~ $\boxed{\text{Warning:}}$ ①这里不可使用Riemann-Lebesgue引理, 因为sin里面是 $2t\langle\xi\rangle$, 不是 $2t\xi$

②这里不可直接变量替换 $u=\sqrt{|\xi|^2+1}$, 因 u 值不保持 $(0,+\infty)$, 故端奇点0处有奇性

Claim 证毕.

对 $I_1 = \int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2 \cos^2(t\langle\xi\rangle)\, d\xi$

$$= \frac{1}{2}\underbrace{\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2 \cos(2t\langle\xi\rangle)\, d\xi}_{\downarrow\ \text{同}J\to 0\text{的证明}\atop 0\ \text{as}\ t\to\infty} + \frac{1}{2}\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2\, d\xi$$

$$\overset{t\to\infty}{=\!=} \frac{1}{2}\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2\, d\xi \ \overset{\text{Plancherel}}{=\!=}\ \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)}$$

$$I_2 \xrightarrow[t\to\infty]{\text{同}I_1} \frac{1}{2}\|h\|^2_{L^2(\mathbb{R}^d)}$$

$$\therefore I \longrightarrow \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g|^2 + |h|^2\, dx = E(u) \quad \text{as } t\to\infty$$

□

$\boxed{\text{Rmk}}$: 此题也可直接用Stein的"Harmonic Analysis" Chapter 8.1 Prop 3 (震荡积分渐近展开) 去证明 $J \sim O(t^{-\frac{1}{2}})$ as $t\to\infty$.

□

[12.3] 设 $u$ 满足 [12.2] 中的方程.

$\bar{u}(\bar{x}, t) = u(x,t)\cos(m x_{d+1})$.

(1) 求证: $\Box_{d+1}\bar{u} = 0$ in $\mathbb{R}^{d+1}\times(0,\infty)$.

(2) $d=1$ 时. 求解 Klein-Gordon 方程.

证明: (1) $\Box_{d+1}\bar{u} = \partial_t^2(u(x,t))\cos(m x_{d+1}) - \sum_{i=1}^{d}\partial_{x_i}^2(u(x,t))\cos(m x_{d+1})$

$+ m^2 u(x,t)\cos(m x_{d+1})$.

$$= \cos(m x_{d+1})\underbrace{(\Box_d u + m^2 u)}_{=0} = 0$$

(2) 实际上 (1) 是 "$\Longleftrightarrow$" 的.

于是考虑
$$\begin{cases} \Box_2 \bar{u} = 0 \\ \bar{u}(0) = \bar{g}(\bar{x}) = g(x_1)\cos(m x_2) \\ \bar{u}_t(0) = \bar{h}(\bar{x}) = h(x_1)\ \cancel{\cos}\,(m x_2). \end{cases}$$

由Poisson 公式:

$$\bar{u}(x_1, x_2, t) = \frac{1}{2}\partial_t\left(t^2 \int_{(\bar{x},t)} \frac{\bar{g}(y)}{(t^2+|y-x|^2)^{\frac{1}{2}}}\,dy\right)$$

$$+ \frac{t^2}{2}\int_{B(\bar{x},t)} \frac{\bar{h}(y)}{(t^2+|y-x|^2)^{\frac{1}{2}}}\,dy$$

$u(x_1, t) = \bar{u}(x_1, 0, t)$.

□

[12.4] 设 $u$ 为 $\Box u + \lambda u_t = 0$ 在 $\mathbb{R}^d\times(0,\infty)$ 中的解. $\lambda>0$.

试求一个指数项, 该项乘u之后给出了 $\Box v + cv = 0$ 的解. 这个操作将 Klein-Gordon 方程的系数反号.

证明: 令 $v = e^{\frac{\lambda}{2}t}u(x,t)$.

是 $u(x,t) = e^{-\frac{\lambda t}{2}}v(x,t)$

$$\Box u + \lambda u_t \underset{\substack{\uparrow \\ \text{直接计算}}}{=} e^{-\lambda t/2}\left(\Box v - \frac{\lambda^2}{4}v\right) = 0$$

$\Rightarrow \Box v - \frac{\lambda^2}{4}v = 0$

□

[12.5] 证明：对任一给定的 $y \in \mathbb{R}^d \setminus \{0\}$，~~表~~ $u = e^{i(x\cdot y - \sigma t)}$ 为 Klein-Gordon 方程 $u_{tt} - \Delta u + m^2 u = 0$ 的解，$\sigma = \sqrt{|y|^2 + m^2}$

证明：$u_{tt} = (-i\sigma)^2 \, e^{i((x\cdot y) - \sigma t)}$

$= -(|y|^2 + m^2) \, e^{i(x\cdot y - \sigma t)}$

$u_{x_j x_j} = (-i y_j)^2 e^{i(x\cdot y - \sigma t)} \Rightarrow -\Delta u = |y|^2 e^{i((x\cdot y) - \sigma t)}$

$\Rightarrow (\Box + m^2) u = 0$ .

□

[12.6] 设 $\begin{cases} \Box u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u(0) = g, \ u_t(0) = h & \text{on } \mathbb{R}^3 \times \{0\} \end{cases}$ $g, h \in C_c^\infty(\mathbb{R}^3)$.

求证：$\exists C > 0$ s.t. $|u(x,t)| \le \frac{C}{t}$ ← 这应是 t 较大的时候

一般应是 $|u(x,t)| \lesssim \frac{C}{\langle t \rangle}$.

证明：由 Ch 2.4 的 Kirchoff 公式有

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t h(y) + g(y) + \nabla g(y) \cdot (y-x) \, dS_y .$$

① $t < 1$ 时，$|h(y)| \lesssim 1$

$\therefore \left| \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t h(y) \, dS_y \right| \lesssim |t| \le 1.$

后面两项同理（注意 $|y-x| = t$）

② $t \ge 1$ 时，由 Spt g，Spt h 紧知，$\mathcal{H}^2(\partial B(x,t) \cap \mathrm{Spt}\, h) \lesssim 1$.

~~而相关函数~~ 被积函数 $L^\infty$ norm $\lesssim t$ $\quad$ Spt g

$\therefore |u(x,t)| \lesssim \frac{1}{t}$

□

[12.~~7~~] 设 $\begin{cases} \Box u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty). \\ u(0) = g & u_t(0) = h \ \text{ on } \mathbb{R}^2 \times \{0\}. \end{cases}$ $g, h \in C_c^\infty$

求证：$|u(x,t)| \lesssim \frac{1}{\langle t \rangle^{\frac{1}{2}}}$.

证明：由 Poisson 公式

$$u(t,x) = \underbrace{\frac{1}{2\pi} \partial_t \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} \, dy}_{I_1} + \underbrace{\frac{1}{2\pi} \int_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}} \, dy}_{I_2}.$$

~~证明~~ $I_2 = \frac{1}{2\pi}\int_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy$

设 $g, h$ 支于 $B(0,R)$ 中

则 $|I| \underset{\rho=|y-x|}{\le} \frac{\|h\|_{L^\infty}}{2\pi}\int_0^t \int_{\partial B(a,\rho)} \frac{dS}{\sqrt{t^2-\rho^2}}\,d\rho$

$$\le \frac{\|h\|_{L^\infty}}{2\pi}\int_0^t \frac{\mathcal{H}^2(\partial B(x,\rho)\cap B(0,R))}{\sqrt{t^2-\rho^2}}\,d\rho$$

$$\lesssim \frac{1}{\sqrt{t}}\underbrace{\int_{\max\{0,|x|-R\}}^{\min\{t,|x|+R\}} \frac{4\pi R^2}{\sqrt{t-\rho}}\,d\rho}_{\lesssim R^{5/2}} \lesssim \frac{1}{\sqrt{t}}$$

对 $I_1$: 变量替换 ~~知~~

$$\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy \overset{y=x+tz}{=} t\int_{B(0,1)}\frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz$$

$$\Rightarrow I_1 = \frac{1}{2\pi}\partial_t \int_{B(x,t)} \frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}}\,dy$$

$$= \frac{1}{2\pi}\int_{B(0,1)}\frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz + \frac{t}{2\pi}\int_{B(0,1)}\frac{\nabla g(x+tz)\cdot z}{\sqrt{1-|z|^2}}\,dz$$

换回 $y=x+tz$

$$= \frac{1}{2\pi t}\int_{B(x,y)}\frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy + \frac{1}{2\pi t}\int_{B(x,t)}\frac{|\nabla g(y)|\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy$$

第一项同 $I_2$

$$\lesssim \frac{1}{t^{3/2}} + \frac{\overset{\le t}{\max\|y-x\|}}{t}\int_{B(x,t)}\frac{\|\nabla g\|_{L^\infty}}{\sqrt{t^2-|y-x|^2}}\,dy$$

$$\lesssim \frac{1}{t^{3/2}} + \int_{B(x,t)}\frac{dy}{\sqrt{t^2-|y-x|^2}}\,dy \lesssim \frac{1}{\sqrt{t}} \qquad \therefore I_1+I_2 \lesssim \frac{1}{\sqrt{t}}$$

□

[12.8] 设 $u^\varepsilon$ 为波方程 $\begin{cases} \Box_2 u^\varepsilon = 0 \quad u_t^\varepsilon(\cdot)=0 \\ u^\varepsilon(0) = g^\varepsilon(\cdot) = \begin{cases} e^{-\frac{\varepsilon(r-2)^2}{(r-1)(3-r)}} & 1<r<3 \quad r=|x| \\ 0 & \text{otherwise} \end{cases} \end{cases}$ 的解

求证：$\sup\limits_{\mathbb{R}^2\times[0,4]} |u^\varepsilon(x,t)| \longrightarrow \infty$ as $\varepsilon\to 0^+$

证明：由Poisson公式

$$u(x,t) = \frac{1}{2\pi}\partial_t \int_{B(x,t)} \frac{g^\varepsilon(y)\,dy}{\sqrt{t^2-|y-x|^2}}$$

$$\Rightarrow u(0,t) = \frac{\partial_t}{2\pi}\int_{B(0,t)} \frac{g^\varepsilon(y)\,dy}{\sqrt{t^2-|y|^2}}$$

$$= \frac{\partial_t}{2\pi}\int_1^3 \int_{\partial B(0,\rho)} \frac{e^{-\varepsilon\frac{(\rho-2)^2}{(\rho-1)(3-\rho)}}}{\sqrt{t^2-\rho^2}}\,dS\,d\rho$$

$$= \frac{1}{2\pi}\int_1^3 \frac{t\rho}{(t^2-\rho^2)^{\frac{3}{2}}} e^{-\frac{\varepsilon(\rho-2)^2}{(\rho-1)(3-\rho)}}\,d\rho$$

$$\xrightarrow{\varepsilon\to 0^+} \frac{1}{2\pi}\int_1^3 \frac{t\rho}{(t^2-\rho^2)^{\frac{3}{2}}}\,d\rho = -t\left(\frac{1}{\sqrt{t^2-9}} - \frac{1}{\sqrt{t^2-1}}\right)$$

$$\longrightarrow \infty \quad \text{as } t\to 3.$$

□

*[12.9] 波方程Kelvin变换不变性

设 $u:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$，其双曲Kelvin变换为 $\bar u(x,t) = u(\bar x,\bar t)\,\left|t^2-|\bar x|^2\right|^{\frac{d-1}{2}}$

$$= u\left(\frac{x}{|x|^2-t^2}, \frac{t}{|x|^2-t^2}\right)\frac{1}{\left||x|^2-t^2\right|^{\frac{d-1}{2}}}$$

$\forall |x|^2\neq t^2$. $\bar x = \frac{x}{|x|^2-t^2}$ $\quad \bar t = \frac{t}{|x|^2-t^2}$

求证：$\Box u = 0 \Longrightarrow \Box\bar u = 0$.

证明：设 $r=|x|$. ~~$\bar X = \frac{X}{}$~~

令 $X = \frac{x}{t^2-r^2}$，$T = \frac{t}{t^2-r^2}$. $\quad 0<r<t$.

则 $K: (\mathbb{R}^{1+d}, g=-dt^2+dx_1^2+\cdots+dx_d^2) \longrightarrow (\mathbb{R}^{1+d}, \bar g = -dT^2+dX_1^2+\cdots+dX_d^2)$

是共形变换（直接验证将Minkowski space-time 映到Minkowski space-time）

且可记成极坐标，那么 $K:(t,r,w)\to(T,R,w)$

有 $T = \frac{t}{t^2-r^2}$ $\quad r<t$ $\quad$ ($r>t$ 同理可算，略)

$R = \frac{r}{t^2-r^2}$

$w = w$

① 我们先证

$$dT^2 - dR^2 = \Omega^2(dt^2 - dr^2), \quad \Omega = \frac{1}{t^2-r^2}$$

直接计算:

$$dT = \partial_t\left(\frac{t}{t^2-r^2}\right)dt + \partial_r\left(\frac{t}{t^2-r^2}\right)dr$$

$$= -\frac{r^2+t^2}{(t^2-r^2)^2}dt + \frac{2rt}{(t^2-r^2)^2}dr$$

$$dT^2 = \left(\frac{-(r^2+t)^2}{(t^2-r^2)^2}dt + \frac{2rt}{(t^2-r^2)^2}dr\right)$$

$$dR = \cancel{\frac{(r^2+t^2)}{(t^2-r^2)^2}}\ \partial_t\left(\frac{r}{t^2-r^2}\right)dt + \partial_r\left(\frac{r}{t^2-r^2}\right)dr$$

$$= -\frac{2tr}{(t^2-r^2)^2}dt + \frac{t^2+r^2}{(t^2-r^2)^2}dr$$

$$\cancel{(dR)^2}\ dR^2 = \left(\frac{-2tr}{(t^2-r^2)^2}dt + \frac{t^2+r^2}{(t^2-r^2)^2}dr\right)^2$$

$$dT^2 - dR^2 \underset{拆括号}{=} \left(\frac{(r^2+t^2)^2}{(t^2-r^2)^4} - \frac{4r^2t^2}{(t^2-r^2)^4}\right)(dt^2-dr^2)$$

$$= \Omega^2(dt^2-dr^2) \qquad \Omega = \frac{1}{t^2-r^2}$$

$$\Rightarrow \bar{g} = \Omega^2 g$$

② 设R为数量曲率 (Scalar curvature)

则 $\Box_g u + \frac{d-1}{4d}Ru = 0$

在变换 $g \mapsto \bar{g} := e^{2\Phi}g$, $p = -\frac{4}{d-1}$ 下不变

$u \mapsto e^{\frac{2\Phi}{p}}u =: \bar{u}$

特别，由$\Omega>0$知，取 $e^{2\Phi} = \Omega^2$ 即契合原题

check: $\cancel{\Box_{\bar g}\bar{u} = \bar{g}^{\mu\nu}\partial_\mu\partial_\nu\bar{u} - \bar{g}^{\mu\nu}\Gamma^\eta_{\mu\nu}\partial_\eta\bar{u}}$

注意 $\Box_g = g^{\mu\nu}\partial_\mu\partial_\nu$)

$\cancel{= \exp\left(-\frac{d+3}{2}\Phi\right)\left(\Box_g u + \left(\frac{4}{p}\Delta\right.\right.}$

(直接check.

证明见 Yvonne Choquet-Bruhat, Cecile Dewitt-Morrette: Analysis, Manifolds and Physics, Vol.1 Page 352.

~~而 $\bar g = \Omega^{-2} g$~~

在(3)的证明中，最后的结果为

$$\Box_g u - \frac{d-1}{4d} R u = \Omega^{\cancel{\ }}\left(\Box_{\bar g}\bar u - \frac{d-1}{4d}\bar R \bar u\right)\Omega^{-\frac{3+d}{2}}$$

而 $\mathbb{R}^{1+d}$ 数量曲率为0，故 $\Box_g u = \Box_{\bar g}\bar u \cdot \Omega^{-\frac{3+d}{2}}$

~~i.e. $\Box \bar u = (t^2-|x|^2)^{-\frac{d+}{2}}$~~

$\Rightarrow$ $\Box_g u = 0$ implies $\Box_{\bar g}\bar u = 0$

□

Rmk：证明参考了 Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, Analysis, Manifolds and Physics, Vol 2. Page 260.

□

[2.10] 假设 u, v 满足

$$\begin{cases}(u-v)_t = 2a\sin\left(\frac{u+v}{2}\right) & a\neq 0\\ (u+v)_x = \frac{2}{a}\sin\left(\frac{u-v}{2}\right)\end{cases}$$

求证：$w:=u$，$w:=v$ 均是 sine-Gordon 方程 $w_{xt}=\sin w$ 的解。

并解释为何类似于 $\Box w = \sin w$

证明 (1) 第一个方程对x求导，第二个方程对t求导，可得：

$$u_{xt} - v_{xt} = 2a\cos\left(\frac{u+v}{2}\right)(u_x+v_x) = a\frac{2}{a}\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$u_{xt} + v_{xt} = \frac{1}{a}\cos\left(\frac{u-v}{2}\right)(u_t - v_t)$$

$$\overset{\text{代入第一个方程}}{=} \frac{1}{a}\cos\left(\frac{u-v}{2}\right)\cdot 2a\sin\left(\frac{u+v}{2}\right) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

~~$= \sin(u+v)$~~

相加 $\Rightarrow u_{xt} = \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) + \cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \sin u$.

相减 $\Rightarrow v_{xt} = \sin v$

$\therefore$ 均为 $w_{xt}=\sin w$ 的解

(2) 为何类似于 $\Box w = \sin w$？

现有 $w_{xt} = \sin w$. ~~令 $x = \frac{\alpha+\beta}{2}$，$t = \frac{\alpha-\beta}{2}$~~

~~则 $\partial_{x}\left(\frac{\alpha+\beta}{2}\right)$~~ ~~$\alpha = \frac{x+t}{2}$. $\beta = \frac{x-t}{2}$~~

~~$\partial_{xt}\left(w\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right)\right) = \left(\frac12\partial_\alpha + \frac12\partial_\beta\right)\left(\frac12\partial_\alpha - \frac12\partial_\beta\right)w$~~

令 $x = \frac{\alpha+\beta}{\cancel{2}}$. $t = \frac{\alpha-\beta}{\cancel{2}}$ ~~$= \frac14(\Box w)\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right)$~~

则 $\alpha = \frac{x+t}{2}$. $\beta = \frac{x-t}{2}$

□

$\partial_x = \partial_\alpha + \partial_\beta$. $\partial_t = \partial_\alpha - \partial_\beta$ $\Rightarrow$ $\partial_x\partial_t w = \sin w$ 变成 $\Box_{\alpha\beta} w = \sin w$

[12.11] 接着T10，给定 sine-Gordon 方程的一个解 $v$. 我们可以通过解T10中的方程组得到另一个解$u$. 此过程称作 Bäcklund 变换.

从 $v=0$ 开始，用 Bäcklund 变换去计算，$a$不同时 $u=$?

解. $v=0$代入，便有

$$\begin{cases} u_t = 2a\sin\frac{u}{2} \\ u_x = \frac{2}{a}\sin\frac{u}{2}. \end{cases}$$

首先有：$u_t = a^2 u_x$，这是一个传输方程

所以 $u$ 必有形式 $f(at+\frac{x}{a})$ （见课本ch2.1）.

而又注意到 $u_t = 2a\sin\frac{u}{2}$

$$= 4a\sin\frac{u}{4}\cos\frac{u}{4}$$

$$\Rightarrow \frac{u_t}{4\cos^2\frac{u}{4}} = a\tan\frac{u}{4}$$

$$\Rightarrow \partial_t\left(\tan\frac{u}{4}\right) = a\tan\frac{u}{4}$$

直接计算$\Rightarrow$ $\tan\frac{u}{4} = Ce^{at+\frac{x}{a}}$

$u = 4\arctan\left(Ce^{at+\frac{x}{a}}\right)$ 为所求.

$\square$

[12.12] 求证Sobolev不等式：$\|\partial^{\beta_1}u_1\cdots\partial^{\beta_m}u_m\|_{L^2(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m}\|u_j\|_{H^k(\mathbb{R}^d)}$, $k>\frac{d}{2}$

证明：我们先证：

① $\|\partial^\beta u\|_{L^\infty} \lesssim \|u\|_{H^k}$. 若 $\frac{1}{2}-\frac{k}{d}+\frac{|\beta|}{d}<0$.

此为显然. 因为此时 $k-|\beta|>\frac{d}{2}$. $H^{k-|\beta|}\hookrightarrow L^\infty$

$\Rightarrow \|\partial^\beta u\|_{L^\infty} \lesssim \|\partial^\beta u\|_{H^{k-|\beta|}} \lesssim \|u\|_{H^k(\mathbb{R}^d)}$.

② 若 $\frac{1}{2}-\frac{k-|\beta|}{d}=\frac{1}{p}>0$. 则 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^k}$

这由 Gagliardo-Nirenberg-Sobolev 不等式是显然的

③ 若 $\frac{1}{2}=\frac{k-|\beta|}{d}$ 则 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^k}$ $\forall\, 2\le p<\infty$

这仍是易见的. 因为只用证 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^{\frac{d}{2}}}$.

$\forall\varepsilon>0$. $H^{\frac{d}{2}} \hookrightarrow H^{\frac{d}{2}-\varepsilon}$. 再由GNS不等式知.

$\forall p\in[2,\infty)$ $\exists\varepsilon$. s.t. $H^{\frac{d}{2}-\varepsilon}(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$

故该式成立.

下面开始凑Hölder不等式

将 $\beta_1 \cdots \beta_m$ 分成三组：　　　　　　Hölder之后的可积指标为

$\lambda_1, \cdots, \lambda_r$　　满足①　　　　$L^\infty$

$\mu_1, \cdots, \mu_s$　　满足②　　　　$L^{p_i}$　　$1 \le i \le s$　$2 \le p_i < \infty$ 待定

$\nu_1, \cdots, \nu_t$　　满足③　　　　$L^{q_j}$　　$1 \le j \le t$　$\frac{1}{q_j} = \frac{1}{2} - \frac{k-|\nu_j|}{d}$

若能做到

$$\| (\partial^{\lambda_1} u_{\lambda_1} \cdots \partial^{\lambda_r} u_{\lambda_r}) \cdot (\partial^{\mu_1} u_{\mu_1} \cdots \partial^{\mu_s} u_{\mu_s}) \cdot (\partial^{\nu_1} u_{\nu_1} \cdots \partial^{\nu_t} u_{\nu_t}) \|_{L^2}$$

$$\lesssim \|\partial^{\lambda_1} u_{\lambda_1}\|_{L^\infty} \cdots \|\partial^{\lambda_r} u_{\lambda_r}\|_{L^\infty} \cdot \|\partial^{\mu_1} u_{\mu_1}\|_{L^{p_1}} \cdots \|\partial^{\mu_s} u_{\mu_s}\|_{L^{p_s}}$$

$$\cdot \|\partial^{\nu_1} u_{\nu_1}\|_{L^{q_1}} \cdots \|\partial^{\nu_t} u_{\nu_t}\|_{L^{q_t}}.$$

则再由①～③即有上式 $\lesssim \prod_{j=1}^m \|u_j\|_{H^k(\mathbb{R}^d)}$.

下面只用验证 Hölder 不等式的指标的确能做到.

i.e. 可选取 $p_i \in [2, +\infty)$. s.t. $\frac{1}{2} = \sum_{i=1}^s \frac{1}{p_i} + \sum_{j=1}^t \frac{1}{q_j} + \frac{1}{\infty} + \cdots + \frac{1}{\infty}$

这因为

$$\sum_{j=1}^t \frac{1}{q_j} = \frac{t}{2} - \frac{kt}{d} + \sum_{j=1}^t \frac{|\nu_j|}{d}$$

$$\le \frac{t}{2} - \frac{kt}{d} + \sum \frac{k}{d}$$

而右边 $< \frac{1}{2} \iff \frac{t}{2} - \frac{(t-1)k}{d} < \frac{1}{2}$

$\iff \frac{t-1}{2} < \frac{(t-1)k}{d} \iff k > \frac{d}{2}$　✓ 说明可以做到

于是可选取恰当的 $p_1, \cdots, p_s$ s.t. $\frac{1}{2} = \frac{1}{\infty} + \cdots + \frac{1}{\infty} + \frac{1}{p_1} + \cdots + \frac{1}{p_s} + \frac{1}{q_1} + \cdots + \frac{1}{q_t}$.

证毕!　　　　　　$\square$.

12.13 光滑映射 $u: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$. $u = (u^1, \dots, u^m)$ 被称作射到 $S^{m-1}$ 的波映射 (wave map into $S^{m-1}$), 是指 $\begin{cases} |u| = 1. \text{ in } \mathbb{R}^n \times [0, \infty). \\ u_{tt} - \Delta u \perp S^{m-1}. \text{ at } u. \end{cases}$

求证: $u_{tt} - \Delta u = (|\nabla u|^2 - |u_t|^2) u$.

<u>证明</u>: 显见 $u_{tt} - \Delta u \,//\, u$. 因 $u_{tt} - \Delta u \perp S^{m-1}$ at $u \in S^{m-1}$.

~~[illegible]~~ 而 $|u|^2 = 1 \Rightarrow \partial_t |u|^2 = 2u \cdot u_t = 0 \Rightarrow u u_t = 0$

$\Rightarrow u_{tt} \cdot u = \partial_t (u_t \cdot u) - u_t \cdot u_t = -|u_t|^2$.

~~$\therefore (u_{tt} - \Delta u) \cdot u =$~~

同样. $\Delta u \cdot u = \sum_{j=1}^{n} u_{x_j x_j} \cdot u$

$$= \underbrace{\sum_{j=1}^{n} (u \cdot u_{x_j})_{x_j}}_{\text{因 } u_{x_j} \cdot u = \frac{\partial_{x_j} |u|^2}{2} = 0} - \sum_{j=1}^{n} u_{x_j}^2$$

$$= 0 - |\nabla u|^2$$

故 $\begin{cases} (\Box u) \cdot u = |\nabla u|^2 - |u_t|^2 \\ \Box u // u \end{cases} \Rightarrow \Box u = (|\nabla u|^2 - |u_t|^2)\, u.$

$\Box$

[12.14] ~~求证~~: 若u是映入 $S^{m-1}$ 的波映射(如13所述), 且是有紧支集(关于x).

求证. $\frac{d}{dt} \int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 \, dx = 0$

证明: $\frac{d}{dt} \int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 \, dx$

$\cancel{= \int_{\mathbb{R}^n} u_t \cdot u_{tt} + \nabla u \cdot \nabla u_t}$

$$= \sum_{i=1}^{m} \frac{d}{dt} \int_{\mathbb{R}^n} |u_t^i|^2 + |\nabla u_t^i|^2 \, dx$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^n} u_t^i u_{tt}^i + \nabla u_t^i \cdot \nabla u^i \, dx$$

$$\overset{\text{分部积分}}{=} \sum_{i=1}^{m} \int_{\mathbb{R}^n} u_t^i (u_{tt}^i - \Delta u^i) \, dx$$

$$= \int_{\mathbb{R}^n} u_t \cdot \Box u \, dx.$$

$$= \int_{\mathbb{R}^n} (|\nabla u|^2 - |u_t|^2) \underbrace{(u \cdot u_t)}_{=0} \, dx$$

$$= 0$$

15. 跳过. (书上 §12.4 明明有更强的结论).

$\Box$

关于NLS:

[12.16] 设u为复值函数.满足NLS $iu_t+\Delta u=f(|u|^2)u$ in $\mathbb{R}^d\times(0,\infty)$. ---(*)

$f:\mathbb{R}\to\mathbb{R}$. 求证: 若$\xi\in\mathbb{R}^d$. 则 $w(x,t):=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}u(x-\xi t,t)$

也满足NLS.(称作 Galilean Invariance).

证明: $\partial_t w=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(\frac{-i|\xi|^2}{4}\right)e^{-\frac{i}{4}|\xi|^2t}u(x-\xi t,t)+e^{-\frac{i}{4}|\xi|^2t}u_t(x-\xi t,t)-\sum_{j=1}^d u_{x_j}(x-\xi t,t)\cdot\xi_j)$

$$\partial_{x_j}w=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(\frac{i}{2}\xi_j\,u(x-\xi t)+u_{x_j}(x-\xi t,t)\right).$$

$$\partial_{x_j}^2w=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left[\frac{i}{2}\xi_j\left(\frac{i}{2}\xi_j u(x-\xi t)+u_{x_j}(x-\xi t,t)\right)+\frac{i}{2}\xi_j u_{x_j}(x-\xi t,t)+u_{x_jx_j}(x-\xi t,t)\right]$$

$$\Rightarrow\ \Delta w=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(-\frac{|\xi|^2}{4}u(x-\xi t)+\left(\sum_{j=1}^d i\cdot u_{x_j}(x-\xi t,t)\cdot\xi_j\right)+\Delta u(x-\xi t,t)\right)$$

$$\Rightarrow\ iw_t-\Delta w=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\underbrace{(iu_t-\Delta u)}_{\text{“}f(|u|^2)u}$$

$$\underset{|w|=|u|}{=}f(|w|^2)\cdot w$$

□

[12.17-12.18] 设u为(*)的速降解.

求证: (1) $\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2dx=0$ (质量守恒).

(2) $\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla u|^2+F(|u|^2)\,dx=0$. $F'=f$ (能量守恒).

(3) $\frac{d}{dt}\int_{\mathbb{R}^d}\frac{\bar u\nabla u-u\nabla\bar u}{2i}dx=0$ (动量守恒).

(4) $\frac{d^2}{dt^2}\int_{\mathbb{R}^d}|u|^2|x|^2dx=8\int_{\mathbb{R}^d}|\nabla u|^2+4d\int_{\mathbb{R}^d}f(|u|^2)u^2-F(|u|^2)dx$

(5) 用(4)去证(NLS) $iu_t+\Delta u=-|u|^2u$ in $\mathbb{R}^d\times(0,\infty)$ 在

$E(0)=\int_{\mathbb{R}^d}|\nabla u(\cdot,0)|^2-\frac{|u(\cdot,0)|^4}{2}dx<0$ 时无整体解. 其中$d\geq2$

证明: (1) $\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2dx=2\mathrm{Re}\int_{\mathbb{R}^d}\bar u u_t\,dx$

(*)两边乘以 $2\bar u$ 得. $2\bar u u_t-i\Delta u\cdot\bar u=-i\bar u u f(|u|^2)=-i|u|^2f(|u|^2)$.

$$\therefore\ \frac{d}{dt}\int_{\mathbb{R}^d}=2\mathrm{Re}\int_{\mathbb{R}^d}i\bar u\Delta u-i\underbrace{|u|^2f(|u|^2)}_{实}dx=2\mathrm{Re}\int_{\mathbb{R}^d}i\bar u\Delta u$$

$$=-2\mathrm{Im}\int_{\mathbb{R}^d}\bar u\Delta u=\underbrace{2\mathrm{Im}\int_{\mathbb{R}^d}|\nabla u|^2dx}_{=0}$$

(2) $$\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 + F(|u|^2)\, dx$$

$$= \underbrace{\cancel{\frac{d}{dt}}\,\mathrm{Re}\int_{\mathbb{R}^d} 2\nabla u_t \cdot \nabla \bar{u}}_{I_1} + \underbrace{\frac{d}{dt}\int_{\mathbb{R}^d} F(|u|^2)\, dx}_{I_2}$$

$$I_1 = 2\,\mathrm{Re}\int_{\mathbb{R}^d} \nabla \bar{u} \cdot \nabla u_t \, dx$$

$$\overset{\text{分部积分}}{=} -2\,\mathrm{Re}\int_{\mathbb{R}^d} u_t \cdot \Delta \bar{u}\, dx.$$

$$= -2\,\mathrm{Re}\int_{\mathbb{R}^d} \Delta \bar{u}\,(-i)\,(-\Delta u + f(|u|^2)u)\, dx$$

$$= -2\,\mathrm{Im}\int_{\mathbb{R}^d} \underbrace{|\Delta u|^2}_{\mathrm{Im}=0} + \Delta \bar{u} \cdot u \cdot f(|u|^2)$$

$$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} \bar{u} \cdot \Delta u\, f(|u|^2)\, dx$$

$$I_2 = \int_{\mathbb{R}^d} f(|u|^2)\, \partial_t(|u|^2)$$

$$\overset{|u|^2 = u\bar{u}}{=} 2\,\mathrm{Re}\int_{\mathbb{R}^d} f(|u|^2)\, \bar{u}\, u_t\, dx$$

代入 $iu_t + \Delta u = f(|u|^2)u$

$$= 2\,\mathrm{Re}\int_{\mathbb{R}^d} f(|u|^2)\, \bar{u}\, (f(|u|^2)u - \Delta u)\cdot(-i)\, dx$$

$$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} f(|u|^2)\, \bar{u}\, (f(|u|^2)\, u - \Delta u)\, dx$$

$$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} \underbrace{|u|^2 f^2(|u|^2)}_{\mathrm{Im}=0} - f(|u|^2)\, \bar{u}\, \Delta u$$

$$= -2\,\mathrm{Im}\int_{\mathbb{R}^d} \bar{u}\, \Delta u\, f(|u|^2)\, dx = -I_1$$

$$\therefore\ I_1 + I_2 = 0 = \frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 + F(|u|^2)\, dx$$

(3) $\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\bar{u}\nabla u - u\nabla\bar{u}}{2i}\,dx$

$$= \frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{Im}(\bar{u}\nabla u)\,dx = \mathrm{Im}\int_{\mathbb{R}^d} \frac{d}{dt}(\bar{u}\nabla u)\,dx$$

$$= \mathrm{Im}\int_{\mathbb{R}^d} \bar{u}_t\nabla u + u\nabla\bar{u}_t\,dx$$

第2项分部积分 $\underline{=}$ $\mathrm{Im}\int_{\mathbb{R}^d} \bar{u}_t\nabla u - \nabla u\cdot\bar{u}_t\,dx = 0.$

(4). 先求 $\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2|u|^2\,dx$

$$= \int_{\mathbb{R}^d} |x|^2\,\partial_t(u\bar{u})\,dx = 2\int_{\mathbb{R}^d}|x|^2\,\mathrm{Re}(\bar{u}u_t)$$

$$= 2\int_{\mathbb{R}^d}|x|^2\,\mathrm{Re}\Big(\bar{u}(-i)\big(f(|u|^2)u - \Delta u\big)\Big)$$

$$= 2\int_{\mathbb{R}^d} \underbrace{\mathrm{Re}\left(|x|^2|u|^2 f(|u|^2)\cdot(-i)\right)}_{=0\ (\text{因}f\text{实值})} + |x|^2\,\mathrm{Re}\left((-i)\bar{u}\Delta u\right)\,dx.$$

$$= -2\int_{\mathbb{R}^d} |x|^2\,\mathrm{Im}(\bar{u}\Delta u)\,dx$$

$$= -2\int_{\mathbb{R}^d} |x|^2\,\mathrm{Im}(\bar{u}\Delta u + \underbrace{|\nabla u|^2}_{\text{实值,故 } \mathrm{Im}=0})\,dx$$

$$= -2\int_{\mathbb{R}^d} |x|^2\,\mathrm{Im}\Big(\sum_{i=1}^d \bar{u}\,\partial_i^2 u + \partial_i u\,\partial_i\bar{u}\Big)\,dx.$$

$$= -2\int_{\mathbb{R}^d} |x|^2\,\mathrm{Im}\sum_{i=1}^d \partial_i(\bar{u}\,\partial_i u)\,dx$$

$$= -2\int |x|^2\cdot\nabla\cdot(\mathrm{Im}\,\bar{u}\nabla u)\,dx.$$

(分部积分) $= +2\int (\nabla|x|^2)\cdot\mathrm{Im}(\bar{u}\nabla u)\,dx.$ $\quad \nabla|x|^2 = (2x_1,\dots,2x_d) = 2x$

$$= 4\int x\cdot\mathrm{Im}(\bar{u}\nabla u)\,dx = 4\,\mathrm{Im}\int \bar{u}\,(x\cdot\nabla u)\,dx$$

$$= 4\,\mathrm{Im}\int \bar{u}\cdot|x|\left(\nabla u\cdot\frac{x}{|x|}\right)dx \qquad \text{令 } u_r = \frac{\nabla u\cdot x}{|x|}$$

$$= 4\,\mathrm{Im}\int r\bar{u}u_r\,dx. \qquad r = |x|$$

再求一导：

这里面好像有一个地方正负号反了，但是最后结论是对的。懒得找了。

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d}|x|^2|u|^2\,dx$$

$$= \frac{d}{dt}\left(4\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{u}(x\cdot\nabla u)\,dx\right)$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{u}_t(x\cdot\nabla u)+\bar{u}(x\cdot\nabla u_t)\,dx$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j\cdot u_{x_j}+\bar{u}\,x_j\cdot\partial_{x_j}u_t\,dx.$$

第2项分部积分

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j u_{x_j} - (\bar{u}\,x_j)_{x_j}\,u_t\,dx$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j u_{x_j} - \bar{u}_{x_j}x_j u_t - \bar{u}u_t\,dx$$

$$= \underbrace{-8\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}u_t\bar{u}_{x_j}x_j}_{J_1} - 4d\,\underbrace{\mathrm{Im}\int_{\mathbb{R}^d}\bar{u}u_t\,dx}_{J_2}$$

$$J_2 = \mathrm{Im}\int_{\mathbb{R}^d}\bar{u}u_t\,dx = \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}(-iu_t)\,dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}\left(\Delta u - f(|u|^2)u\right)dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}\Delta u - f(|u|^2)|u|^2\,dx.$$

(分部积分) $= \mathrm{Re}\int_{\mathbb{R}^d}-|\nabla u|^2 - f(|u|^2)|u|^2\,dx$

$$\Rightarrow -4dJ_2 = 4d\int_{\mathbb{R}^d}|\nabla u|^2\,dx + 4d\int_{\mathbb{R}^d}f(|u|^2)|u|^2\,dx$$

$$J_1 = \mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}u_t\bar{u}_{x_j}x_j.$$

$$= \mathrm{Im}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(i\Delta u - iu f(|u|^2)\right)\bar{u}_{x_j}x_j\,dx$$

$$= \underbrace{-\mathrm{Re}\int_{\mathbb{R}^d}\Delta u\,(\nabla\bar{u}\cdot x)\,dx}_{K_1} + \underbrace{\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}u\,\bar{u}_{x_j}x_j f(|u|^2)\,dx.}_{K_2}$$

$$k_2 = \mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} u\bar{u}_{x_j}\, x_j f(|u|^2)\,dx$$

$$= \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} \underbrace{\partial_j(|u|^2)}_{=2\mathrm{Re}\,\bar{u}_{x_j}u} \cdot x_j f(|u|^2)\,dx.$$

$F' = f$

$$= \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} \partial_j(|u|^2)\, F'(|u|^2)\cdot x_j\,dx = \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\partial_j(F(|u|^2))\cdot x_j\,dx$$

分部积分

$$= -\frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} F(|u|^2)\,\underbrace{\partial_j x_j}_{=1}\,dx = -\frac{d}{2}\int_{\mathbb{R}^d} F(|u|^2)\,dx.$$

$$k_1 = -\mathrm{Re}\int_{\mathbb{R}^d} \Delta u\,(\nabla\bar{u}\cdot x)\,dx.$$

分部积分

$$= \mathrm{Re}\int_{\mathbb{R}^d} \nabla u\cdot\nabla(\nabla\bar{u}\cdot x)\,dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^d}\sum_j\sum_k u_{x_k}\overline{u_{x_j}}\,\delta_{jk} + u_{x_k}\overline{u_{x_j x_k}}\,x_j\,dx$$

$$= \mathrm{Re}\sum_k\int_{\mathbb{R}^d} u_{x_k}\overline{u_{x_k}}\,dx + \sum_j\sum_k\mathrm{Re}\int_{\mathbb{R}^d} u_{x_k}\overline{u_{x_j x_k}}\,x_j\,dx.$$

$$= \int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{1}{2}\sum_j\sum_k\mathrm{Re}\int_{\mathbb{R}^d}\left(u_{x_k}\overline{u_{x_j x_k}} + u_{x_j x_k}\overline{u_{x_k}}\right)x_j\,dx.$$

$$= \int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{1}{2}\sum_j\mathrm{Re}\int_{\mathbb{R}^d} x_j\,\partial_j(|\nabla u|^2)\,dx.$$

分部积分

$$= \int_{\mathbb{R}^d}|\nabla u|^2\,dx - \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}|\nabla u|^2\,dx$$

$$= \left(1-\frac{d}{2}\right)\int_{\mathbb{R}^d}|\nabla u|^2\,dx$$

$$\therefore \frac{d^2}{dt^2}\int_{\mathbb{R}^d}|x|^2|u|^2\,dx = \cancel{-8\left(-\frac{d}{2}+\left(1-\frac{d}{2}\right)\right)\int_{\mathbb{R}^d}|\nabla u|^2\,dx}.$$

$$+8\left(1-\frac{d}{2}\right)\int_{\mathbb{R}^d}|\nabla u|^2\,dx - 4d\int_{\mathbb{R}^d}F(|u|^2)\,dx$$

$$+4d\int_{\mathbb{R}^d}|\nabla u|^2\,dx + 4d\int_{\mathbb{R}^d}f(|u|^2)|u|^2\,dx$$

$$= 8\int_{\mathbb{R}^d}|\nabla u|^2\,dx + 4d\int_{\mathbb{R}^d} f(|u|^2)|u|^2 - F(|u|^2)\,dx$$

(4) 得证

(5) 此处不需要加条件，下面的证明不对。
实际上：当E(0)<0时，已经可以得到
I(t)=4E(0)t^2+I(0)t+I(0)是开口向下的抛物线，当t很大时该项必为负，与I非负矛盾。

(5). 暂时加条件 $\mathrm{Im}\int r\bar{\varphi}\varphi_r\,dx \leq 0$ $\varphi=u(0,\cdot)$

φ为初值 u(0)

令 $I(t)=\int_{\mathbb{R}^d}|x|^2|u|^2\,dx$

由(4)证明 $I'(t) = 4\,\mathrm{Im}\int r\bar{u}u_r\,dx < 0$

知. $I(t) \leq I(0) =: A_0$. $I(0)>0$.

$I''(t) \overset{(4)}{\underset{(2)}{=}} \cancel{(8-4d)} \overset{(4)}{=} 8\int_{\mathbb{R}^d}|\nabla u|_2^2\,dx - 4d\int_{\mathbb{R}^d}\frac{|u|^4}{2}\,dx$

$= (8-4d)\|\nabla u\|_{L^2}^2 + \underbrace{4dE(0)}_{<0}$

~~d≥3时~~ $\Rightarrow I''(t) \leq (8-4d)\|\nabla u\|_{L^2}^2 \leq 0$

$\Rightarrow -I''(t) \geq (4d-8)\|\nabla u\|_{L^2}^2$

~~而 I'(t)~~ 先设 $d\geq 3$

$(I'(t))^2 = 4^2(\mathrm{Im}\int r\bar{u}u_r\,dx)^2$

$\leq 16\left(\int |x|^2|u|^2\,dx\right)\cdot\|\nabla u\|_{L^2}^2$

$I''\leq 0$ . $I'(0)<0 \Rightarrow I'(t)<0$

$\Rightarrow I(t) < I(0) = A_0$

$\therefore (I'(t))^2 \leq 16A_0\|\nabla u\|_{L^2}^2$

$\leq \frac{-4A_0}{d-2}\cdot I''(t)$

$\Rightarrow I''(t) \leq -\frac{d-2}{4A_0}(I'(t))^2$.

$\Rightarrow \frac{d}{dt}\left(\frac{1}{I'(t)}\right) \geq \frac{d-2}{4A_0}$.

$\Rightarrow \frac{1}{I'(t)} \geq \frac{1}{I'(0)} + \frac{d-2}{4A_0}t. = \frac{4A_0+(d-2)t\,I'(0)}{4A_0 I'(0)}$

$\Rightarrow I'(t) \leq \frac{4A_0 I'(0)}{4A_0+(d-2)I'(0)t}$. $\Rightarrow \|\nabla u\|_{L^2} \geq \frac{-\sqrt{A_0}\,I'(0)}{4A_0+(d-2)I'(0)t}$.

$\geq -4\sqrt{A_0}\,\|\nabla u\|_{L^2}$

故 $t\to -\frac{4A_0}{(d-2)I'(0)}+0$ 时

$\|\nabla u\|_{L^2} \geq +\infty$

$\Rightarrow$ 无整体解.

$d=2$时: $I''(t) = 8E(0)$

故 $I'(t) = 8E(0)t + I'(0)$.

$I(t) = 4E(0)t^2 + I'(0)t + I(0)$.

$E(0)<0$ 知. $t$充分大. $I(t)<0$. 这与 $I(t) = \int_{\mathbb{R}^d} |x|^2|u|^2 \geq 0$ 矛盾!

$\square$

Rmk: 此题(根据书上Hint)参考了 R.T. Glassey: On the blowing up of solutions to the Cauchy problem for NLS, Journal of Mathematical Physics 18, 1794 (1977)

但原文方程为 $iu_t - \Delta u = F(|u|^2)u$. 记后符号与此题有出入.

同时. 原文对(5)中添加的条件成立是假设. 但Evans书上没有. 实际上. 若 $\varphi(x) = u(x,0)$.

有形式 $\varphi(x) = e^{-i|x|^2}\underbrace{\psi(x)}_{\text{实值. 非零}}$, 则所加条件满足.

$\square$