# Manifold Alignment

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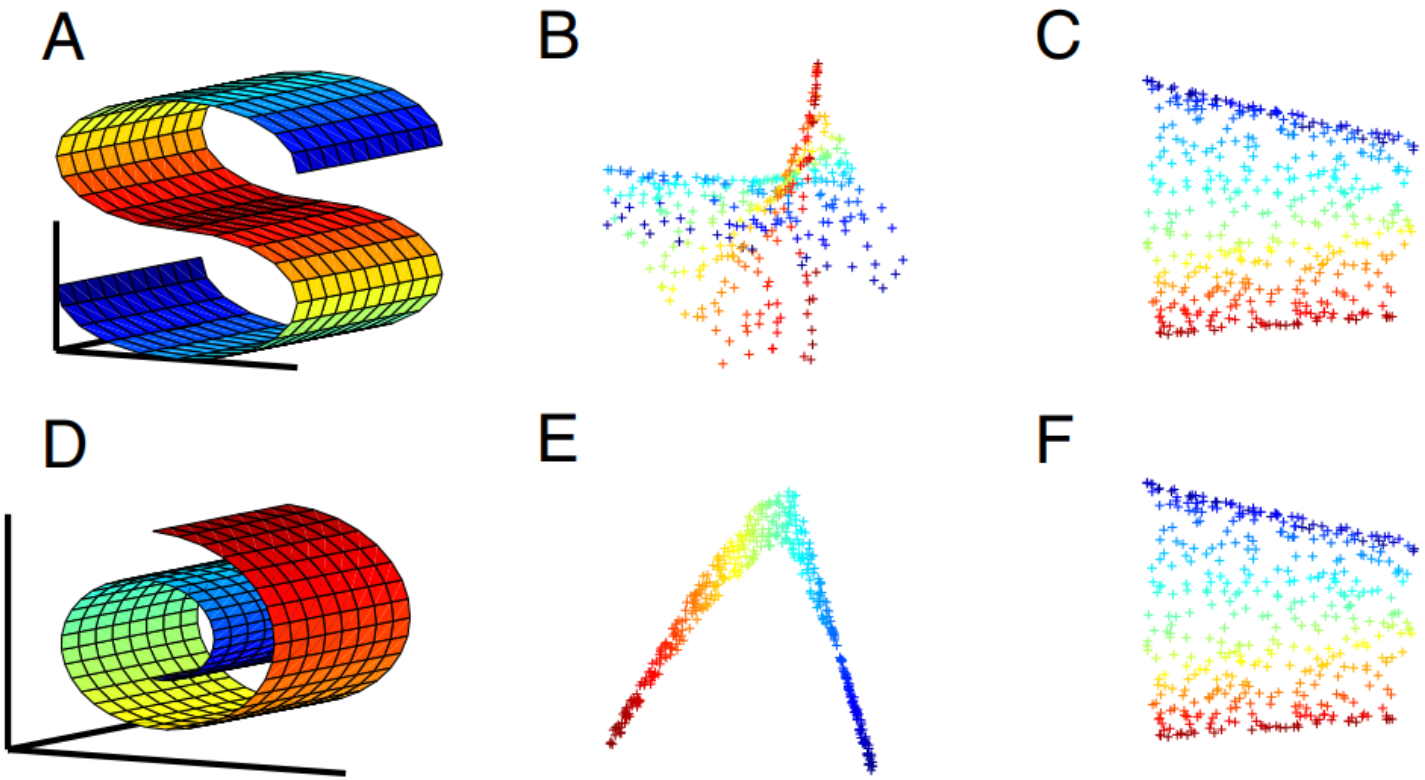
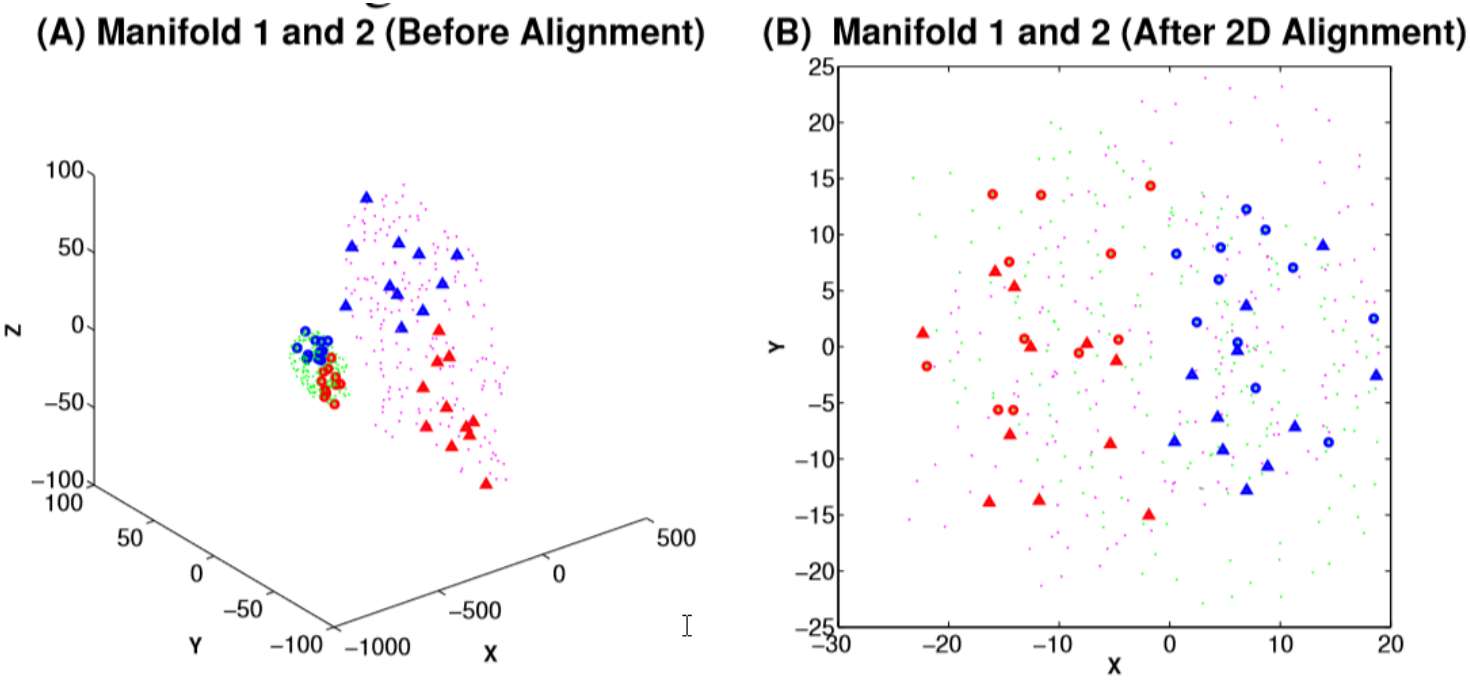
# Learning Correspondences



# Data Matching


Figure: Pose and angle matching

# Data Integration



# Domain Adaptation

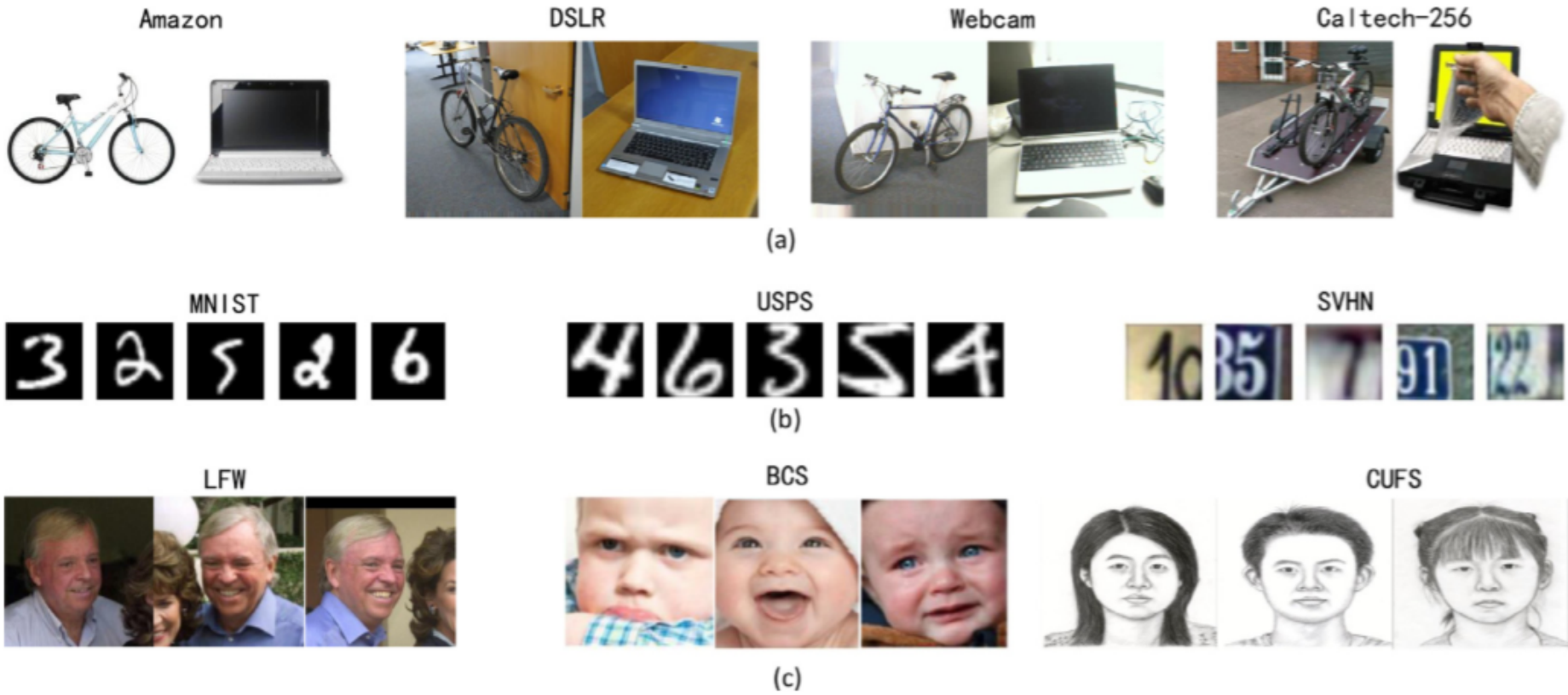



Figure: Distribution shift between domains can degrade the performance of a classifier trained on a source domain.

# Manifold Learning and Manifold Alignment

## Manifold learning (ML)

Aim to recover a low-dimensional manifold embedded in a high-dimensional ambient space.

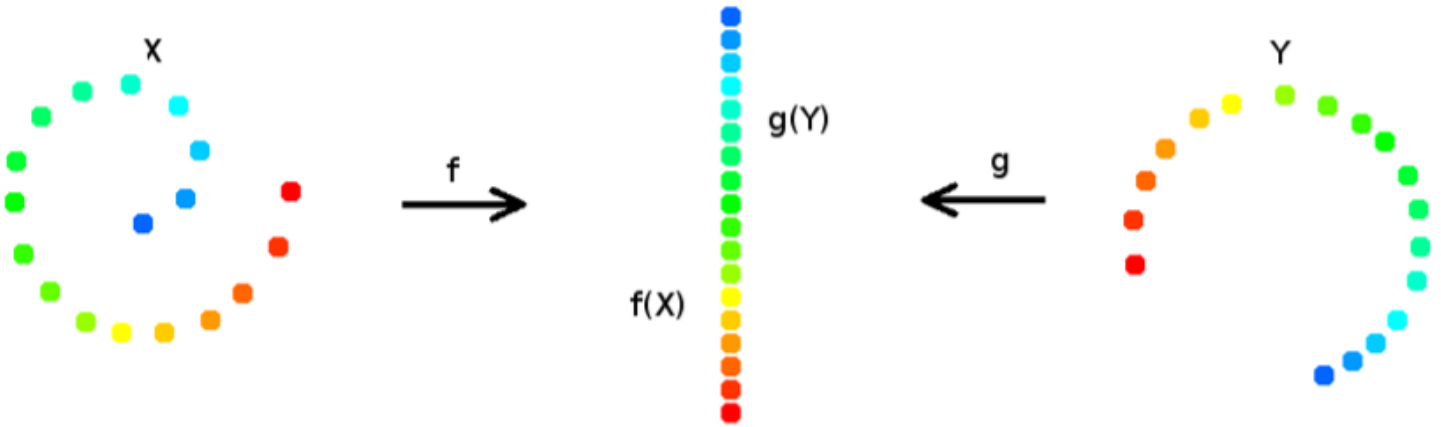
## Manifold alignment (MA)

Aim to project multiple sets of data into a common latent space, given that those original datasets lie on a common manifold.

- Most data sets can be modeled by manifolds.
- MA **aligns their underlying structures** and **transfers knowledge across** data sets.
- MA can be formalized as **dimensionality reduction with constraints** induced by the **correspondences** among data sets.

# Problem Statement

- Given datasets $X \in \mathbb{R}^{n \times p}$ and $Y \in \mathbb{R}^{m \times q}$ , whose instances lie on the same manifold $\mathcal{Z}$.
- MA is to find two functions $f : \mathbb{R}^p \to \mathbb{R}^k$, $g : \mathbb{R}^p \to \mathbb{R}^k$, such that $f(x_i)$ is close to $g(y_j)$ in terms of Euclidean distance if $x_i$ and $y_j$ are close w.r.t. geodesic distance along $\mathcal{Z}$.



# Problem Statement

- $x_i$ and $y_j$ are in exact correspondence $\Leftrightarrow f(x_i) = g(y_i)$.
  - **Prior correspondence** includes any information about the similarity of the instances in $X$ and $Y$, not just exact correspondence information.
- The union of the range of $f$ and $g$ is the joint latent space.
- The concatenation of the new coordinates $[f(X)^T, g(Y)^T]^T$ is the uniform representation of $X$ and $Y$ in the joint space.

# Practical Problems of MA

Many practical problems, ranging from bioinformatics to information retrieval and robotics, involve modeling multiple datasets that contain significant shared underlying structure.

- Data matching
- Protein alignment
- Cross-lingual retrieval
- Data integration
- Transfer learning

# Categories of MA

Instance-level vs. feature-level projections

- Linear (feature-level) alignment
- Non-linear (instance-level) alignment

Inter-data correspondences

- Supervised: with complete correspondence information.
- Semi-supervised: with incomplete correspondence information
- Unsupervised: with no correspondence information, some correspondences must be inferred.

# Two Key Ideas

- Consider local geometry and correspondence information.
- View multiple datasets as being samples on the same manifold.

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## Objective 1: Preserving Similarities

Preserve the local similarity and correspondence information.

## Objective 2: Embedding the Joint Laplacian

Form the joint Laplacian. MA is equivalent to Laplacian eigenmaps.

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## Objective 1: Preserving Similarities

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Form the joint Laplacian. MA is equivalent to Laplacian eigenmaps.

The two objectives are equivalent with each other.

# Notations

- $X^{(a)} \in \mathbb{R}^{n_a \times p_a}$: the $a$-th matrix ($n_a$ observations and $p_a$ features).

- $W^{(a)} \in \mathbb{R}^{n_a \times n_a}$: the similarity matrix.
- $D^{(a)} \in \mathbb{R}^{n_a \times n_a}$: a diagonal matrix with $D^{(a)}(i,i) = \sum_j W^{(a)}(i,j)$.
- $L^{(a)} = D^{(a)} - W^{(a)}$: the Laplacian associated with $X^{(a)}$.

- $W^{(a,b)} \in \mathbb{R}^{n_a \times n_b}$: the similarity matrix between $X^{(a)}$ and $X^{(b)}$. $W^{(a,b)}(i,j)$ is the similarity or the strength of correspondence of the two instances $X^{(a)}(i,\cdot)$ and $X^{(b)}(j,\cdot)$.
Typically, $W^{(a,b)}(i,j) = 1$ if the instances $X^{(a)}(i,\cdot)$ and $X^{(b)}(j,\cdot)$ are in correspondence and 0 otherwise.

# Objective 1: Preserve Similarities

Given $X^{(1)}, \ldots, X^{(c)}$, let $F^{(a)}$ be the embedding of the $a$-th one. The objective 1 has two parts:

- Preserve local similarity within each dataset $X^{(a)}$

$$C_\lambda\left(F^{(a)}\right) = \sum_{i,j} \left\| F^{(a)}(i,\cdot) - F^{(a)}(j,\cdot) \right\|^2 W^{(a)}(i,j)$$

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- Preserve local similarity within each dataset $X^{(a)}$

$$C_\lambda\left(F^{(a)}\right) = \sum_{i,j} \left\| F^{(a)}(i,\cdot) - F^{(a)}(j,\cdot) \right\|^2 W^{(a)}(i,j)$$

- Preserve correspondence information across datasets

$$C_\kappa\left(F^{(a)}, F^{(b)}\right) = \sum_{i,j} \left\| F^{(a)}(i,\cdot) - F^{(b)}(j,\cdot) \right\|^2 W^{(a,b)}(i,j)$$

# Objective 1: Preserve Similarities

The complete loss function is

$$
\begin{aligned}
& C_1\left(F^{(1)}, \ldots, F^{(c)}\right) \\
= \; & \nu \sum_a C_\lambda\left(F^{(a)}\right) + \mu \sum_{a \neq b} C_\kappa\left(F^{(a)}, F^{(b)}\right) \\
= \; & \nu \sum_a \sum_{i,j} \left\| F^{(a)}(i,\cdot) - F^{(a)}(j,\cdot) \right\|^2 W^{(a)}(i,j) \\
& + \mu \sum_{a \neq b} \sum_{i,j} \left\| F^{(a)}(i,\cdot) - F^{(b)}(j,\cdot) \right\|^2 W^{(a,b)}(i,j)
\end{aligned}
$$

where the scalers $\nu$ and $\mu$ are used to control how much the alignment should try to respect local similarity versus correspondence information. Typically, $\nu = \mu = 1$.

# Objective 2: Embedding the Joint Laplacian

$\mathbf{X}$ is the joint data matrix of $(\sum_i n_i) \times (\sum_i p_i)$
$\mathbf{W}$ is the joint similarity matrix of $(\sum_i n_i) \times (\sum_i n_i)$
$\mathbf{L} = \mathbf{D} - \mathbf{W}$ is the joint Laplacian

$$\mathbf{X} = \begin{pmatrix} X^{(1)} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & X^{(c)} \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} \nu W^{(1)} & \mu W^{(1,2)} & \cdots & \mu W^{(1,c)} \\ & & \cdots & \\ \mu W^{(c,1)} & \mu W^{(c,2)} & \cdots & \nu W^{(c)} \end{pmatrix}$$

where $\nu$ and $\mu$ are scalers that balance local similarity versus correspondence information, as defined before.

# Objective 2: Embedding the Joint Laplacian

The loss function for Laplacian eigenmaps:

$$C_2(\mathbf{F}) = \sum_{i,j} \|\mathbf{F}(i,\cdot) - \mathbf{F}(j,\cdot)\|^2 \mathbf{W}(i,j)$$

$$\begin{aligned} C_2(\mathbf{F}) &= \sum_{i,j}\sum_{k} [\mathbf{F}(i,k) - \mathbf{F}(j,k)]^2 \mathbf{W}(i,j) \\ &= \sum_{k}\sum_{i,j} [\mathbf{F}(i,k) - \mathbf{F}(j,k)]^2 \mathbf{W}(i,j) \\ &= \sum_{k} \mathrm{tr}\left(\mathbf{F}(\cdot,k)'\mathbf{L}\mathbf{F}(\cdot,k)\right) \\ &= \mathrm{tr}\left(\mathbf{F}'\mathbf{L}\mathbf{F}\right) \end{aligned}$$

# Equivalence of the Two Objectives

Check the definition of the joint Laplacian $\mathbf{L}$,

$$\mathbf{W}(i,j) = \begin{cases} \nu W^{(a)}(i,j) & \text{if } \mathbf{X}(i,\cdot) \text{ and } \mathbf{X}(j,\cdot) \text{ are both from } X^{(a)} \\ \mu W^{(a,b)}(i,j) & \text{if } \mathbf{X}(i,\cdot) \text{ and } \mathbf{X}(j,\cdot) \text{ are corresponding instances from } X^{(a)} \text{ and } X^{(b)} \\ 0 & \text{otherwise.} \end{cases}$$

The terms of $C_2(\mathbf{F})$ containing instances from the same dataset are exactly the $C_\lambda\left(F^{(a)}\right)$ terms, and the terms of containing instances from different datasets are exactly the $C_\kappa\left(F^{(a)}, F^{(b)}\right)$ terms.

$$C_1\left(F^{(1)}, \ldots, F^{(c)}\right) = C_2(\mathbf{F}) = C(\mathbf{F})$$

Embed the joint Laplacian $\Leftrightarrow$ Preserve local similarity and correspondence information.

# Optimization Problem

- An **additional constraint** is needed mathematically,

$$\mathbf{F}'\mathbf{D}\mathbf{F} = \mathbf{I}$$

  Without this constraint, the trivial solution of mapping all instances to zero would minimize the loss function.

- The final optimization formulation of MA is

$$\underset{\mathbf{F}:\mathbf{F}'\mathbf{D}\mathbf{F}=\mathbf{I}}{\arg\min} C(\mathbf{F}) = \underset{\mathbf{F}:\mathbf{F}'\mathbf{D}\mathbf{F}=\mathbf{I}}{\arg\min} \operatorname{tr}\left(\mathbf{F}'\mathbf{L}\mathbf{F}\right)$$

  - The optimal solution of it can be derived using the method of Lagrange multipliers.
  - Its solution $\Leftrightarrow$ that of Laplacian eigenmaps.

# Algorithm

## Joint Laplacian MA Algorithm

1. Find the adjacency matrices $W^{(1)}, \ldots, W^{(c)}$ of each dataset.
   - possibly only including a weight between two instances if one is in the k-nearest neighbors of the other.
2. Construct the joint Laplacian $\mathbf{L}$.
3. Compute the $d$ smallest nonzero eigenvectors of $\mathbf{L}f = \lambda \mathbf{D}f$.
4. The rows $1 + \sum_{l=0}^{g-1} n_l, 2 + \sum_{l=0}^{g-1} n_l, \ldots, n_g + \sum_{l=0}^{g-1} n_l$ of $\mathbf{F}$ are the new coordinates of $X^{(g)}$.

# Linear Restriction

## Nonlinear MA (as discussed above)

- The eigenvectors of the Laplacian are exactly the new coordinates of the embedded instances.
- No simple closed form for the mapping function.

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- The eigenvectors of the Laplacian are exactly the new coordinates of the embedded instances.
- No simple closed form for the mapping function.

## Linear MA

- Enforces an explicit, linear embedding function.
- Useful for making out-of-sample estimates.
- Finds relationships between the features of multiple datasets instead of just between instances.
- Identify a common linear subspace of the original datasets.

# Linear MA

## Problem Statement

- Given two datasets $X^{(1)} \in \mathbb{R}^{n \times p}$ and $X^{(2)} \in \mathbb{R}^{m \times q}$, whose instances lie on the same manifold $\mathcal{Z}$.
- Linear alignment is to find two matrices $\mathbf{F}^{(1)} \in \mathbb{R}^{p \times k}$ and $\mathbf{F}^{(2)} \in \mathbb{R}^{q \times k}$, such that $x_i \mathbf{F}^{(1)}$ is close to $x_j \mathbf{F}^{(2)}$ in terms of Euclidean distance if $x_i^{(1)}$ and $x_j^{(2)}$ are close w.r.t. geodesic distance along $\mathcal{Z}$.

## The Loss Function

$$C(\mathbf{F}) = \sum_{i \neq j} \|\mathbf{X}(i,\cdot)\mathbf{F} - \mathbf{X}(j,\cdot)\mathbf{F}\|^2 \mathbf{W}(i,j) = \mathrm{tr}\left(\mathbf{F}'\mathbf{X}'\mathbf{L}\mathbf{X}\mathbf{F}\right)$$

# Linear MA

## The optimization problem

$$\max C(\mathbf{F}) = \text{tr}\left(\mathbf{F}'\mathbf{X}'\mathbf{L}\mathbf{X}\mathbf{F}\right)$$
$$\text{s.t. } \mathbf{F}'\mathbf{X}'\mathbf{D}\mathbf{X}\mathbf{F} = \mathbf{I}$$

- Linear alignment reduces to locality preserving projections on the joint Laplacian of the datasets.
- The solution is the minimum eigenvectors of the generalized eigenvector problem:

$$\mathbf{X}'\mathbf{L}\mathbf{X}f = \lambda\mathbf{X}'\mathbf{D}\mathbf{X}f$$

# Canonical Correlation Analysis (CCA)

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- Inferring information from cross-covariance matrices.

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- First introduced by [HOTELLING, 1936].
- Inferring information from cross-covariance matrices.

- $\mathbf{X}$, $\mathbf{Y}$: two data matrix with $N$ observations (rows) and $p$, $q$ features (columns), respectively.
- $\Sigma_{XX}$, $\Sigma_{YY}$: the within-set covariance matrix of $\mathbf{X}$ and $\mathbf{Y}$.
- $\Sigma_{XY}$: the between-sets covariance matrix (cross-covariance).

CCA is to find the vectors $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$ that maximize the correlation between $\mathbf{X}u$ and $\mathbf{Y}v$.

# Optimization Problem of CCA

## Maximize the correlation after linear projection

$$\max_{u,v} \ \text{corr}(\mathbf{X}u, \mathbf{Y}v) = \frac{u^\top \Sigma_{XY} v}{\sqrt{(u^\top \Sigma_{XX} u)(v^\top \Sigma_{YY} v)}}$$

## The equivalent optimization problem

$$\max_{u,v} \ u^\top \Sigma_{XY} v$$

$$\text{subject to} \ \ u^\top \Sigma_{XX} u = 1 \text{ and } v^\top \Sigma_{YY} v = 1$$

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## The equivalent optimization problem

$$\max_{u,v} \ u^\top \Sigma_{XY} v$$

$$\text{subject to} \ \ u^\top \Sigma_{XX} u = 1 \text{ and } v^\top \Sigma_{YY} v = 1$$

The Lagrange function is

$$L(u, v) = u^\top \Sigma_{XY} v - \frac{\lambda}{2}(u^\top \Sigma_{XX} u - 1) - \frac{\mu}{2}(v^\top \Sigma_{YY} v - 1)$$

# CCA: The Optimal Solution

Differentiate w.r.t. $u$, $v$, $\lambda$ and $\mu$, and set to zeros:

$$\Sigma_{XY} v = \lambda \Sigma_{XX} u, \quad \Sigma_{XY} u = \lambda \Sigma_{YY} v$$
$$u^\top \Sigma_{XX} u = 1, \quad v^\top \Sigma_{YY} v = 1$$

which will gives: $\lambda = \mu$.
Find the vectors $u$ and $v$ by solving the eigenvalue problems:

$$\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY}^{-1} \Sigma_{XY} u = \lambda^2 u$$
$$\Sigma_{YY}^{-1} \Sigma_{XY} \Sigma_{XX}^{-1} \Sigma_{XY} v = \lambda^2 v$$

For $d > 1$, the optimal solution is the eigenvectors of the $d$ biggest eigenvalues of $\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY}^{-1} \Sigma_{XY}$ and $\Sigma_{YY}^{-1} \Sigma_{XY} \Sigma_{XX}^{-1} \Sigma_{XY}$.

# Summary

- Two ideas of manifold alignment: preserving the similarities and embedding the joint Laplacian.
- The consequent loss functions are equivalent.
- The non-linear embeddings generally produces more accurate alignments.
- Linear projections allow any new instances to be easily embedded in the manifold space.

# Correspondence Inference

How to infer the cross-datasets correspondences?

## Unsupervised case

- Using relations (rather than features) to represent local geometry [Chang and Mahadevan, 2009].
- Iterative methods [Cui et al., 2014].

## Semi-supervised case

- Building a bridge with known correspondences [Wang and Mahadevan, 2013].
- Utilize the class label information [Wang, 2010].

# Generalized Unsupervised MA (GUMA)

- Fully **unsupervised** matching without correspondences
- Formulated into an explicit integer optimization problem
- Efficient iterative alignment without computations in all permutation cases.

**Problem Setup:**

- Two datasets (normalized) $\mathbf{X} \in \mathbb{R}^{d_x \times n_x}, \mathbf{Z} \in \mathbb{R}^{d_z \times n_z}$(assume $n_x \leqslant n_z$), residing in two different manifolds $\mathcal{M}_x$ and $\mathcal{M}_z$
- The correspondences matrix $\mathbf{F} \in \{0,1\}^{n_x \times n_z}$

$$\Pi = \{\mathbf{F} | \mathbf{F} \in \{0,1\}^{n_x \times n_z}, \mathbf{F}\mathbf{1}_{n_z} = \mathbf{1}_{n_x}, \mathbf{1}_{n_x}^\top \mathbf{F} \leqslant \mathbf{1}_{n_z}^\top, n_x \leqslant n_z\}$$

- Linear projections $\mathbf{P}_x \in \mathbb{R}^{d \times d_x}$ and $\mathbf{P}_z \in \mathbb{R}^{d \times d_z} (d \leqslant d_x, d \leqslant d_z)$ from the two datasets to a mutual embedding space $\mathcal{M}$

# GUMA Problem

- **Overall objective function**

$$\min_{\mathbf{P}_x, \mathbf{P}_z, \mathbf{F}} E_s + \gamma_f E_f + \gamma_p E_p$$
$$\text{s.t. } \mathbf{F} \in \Pi, \ \mathbf{P}_x, \mathbf{P}_z \in \Theta$$

# GUMA Problem

- **Overall objective function**

$$\min_{\mathbf{P}_x, \mathbf{P}_z, \mathbf{F}} E_s + \gamma_f E_f + \gamma_p E_p$$
$$\text{s.t. } \mathbf{F} \in \Pi, \ \mathbf{P}_x, \mathbf{P}_z \in \Theta$$

- **The geometry matching term**

$$E_s = \|\mathbf{K}_x - \mathbf{F}\mathbf{K}_z\mathbf{F}^\top\|_F^2,$$

where $[\mathbf{K}]_{ij} = d(\mathbf{X}_{\cdot i}, \mathbf{X}_{\cdot j})$ is the full adjacency matrix, $d$ is geodesic distance.

- **The feature matching term**

$$E_f = \|\mathbf{P}_x^\top \mathbf{X} - \mathbf{P}_z^\top \mathbf{Z}\mathbf{F}^\top\|_F^2$$

which ensures the aligned data points have similar intrinsic feature representations in the mutual embedding space $\mathcal{M}$.

- **The feature matching term**

$$E_f = \|\mathbf{P}_x^\top \mathbf{X} - \mathbf{P}_z^\top \mathbf{Z}\mathbf{F}^\top\|_F^2$$

which ensures the aligned data points have similar intrinsic feature representations in the mutual embedding space $\mathcal{M}$.

- **The geometry preserving term**

$$\begin{aligned} E_p &= \sum_{i,j} \|\mathbf{P}_x^\top(\mathbf{x}_i - \mathbf{x}_j)\|^2 w_{ij}^x + \sum_{i,j} \|\mathbf{P}_z^\top(\mathbf{z}_i - \mathbf{z}_j)\|^2 w_{ij}^z \\ &= \operatorname{tr}(\mathbf{P}_x^\top \mathbf{X}\mathbf{L}_x\mathbf{X}^\top\mathbf{P}_x + \mathbf{P}_z^\top \mathbf{Z}\mathbf{L}_z\mathbf{Z}^\top\mathbf{P}_z). \end{aligned}$$

where $w_{ij}^x\,(w_{ij}^z)$ is the weight between the $i^{th}$ point and the $j^{th}$ point, $\mathbf{L}_x = \operatorname{diag}(\sum_j w_{1j}^x, \cdots, \sum_j w_{n_x j}^x) - \mathbf{W}_x$.

# GUMA Algorithm

1. Fix $\mathbf{P}_x$ and $\mathbf{P}_z \rightarrow$ solve a non-convex quadratic integer program
2. Fix $\mathbf{F} \rightarrow$ analytic solution for $\mathbf{P}_x$ and $\mathbf{P}_z$
3. Alternately optimize them

## Fixing $\mathbf{P}_x$ and $\mathbf{P}_z$

$$\min_{\mathbf{F} \in \Pi} \Psi_0(\mathbf{F}) = E_s + \gamma_f E_f$$

$$\Psi_0(\mathbf{F}) = \|\mathbf{K}_x\mathbf{F} - \mathbf{F}\mathbf{K}_z\|_F^2 + \mathrm{tr}(\mathbf{F}^\top \mathbf{1}\mathbf{1}^\top \mathbf{F}\mathbf{K}_{zz}) + \mathrm{tr}(\mathbf{F}^\top \mathbf{B})$$

where $\widehat{\mathbf{X}} = \mathbf{P}_x^\top \mathbf{X}$, $\widehat{\mathbf{Z}} = \mathbf{P}_z^\top \mathbf{Z}$, $\mathbf{K}_{zz} = \mathbf{K}_z \odot \mathbf{K}_z$ and $\mathbf{B} = \gamma_f(\mathbf{1}\mathbf{1}^\top(\widehat{\mathbf{Z}} \odot \widehat{\mathbf{Z}}) - 2\widehat{\mathbf{X}}^\top\widehat{\mathbf{Z}}) - \mathbf{1}\mathbf{1}^\top\mathbf{K}_{zz}$

# Subproblem about Learning Alignment

Relax this optimization problem

- Relax $\Pi$ into a compact convex set

$$\Pi' = \{\mathbf{F} | \mathbf{F} \geqslant 0, \mathbf{F}\mathbf{1}_{n_z} = \mathbf{1}_{n_x}, \mathbf{1}_{n_x}^\top \mathbf{F} \leqslant \mathbf{1}_{n_z}^\top, n_x \leqslant n_z\}$$

- Relax $\Psi_0$ into a convex function

$$\Psi(\mathbf{F}) = \|\mathbf{K}_x\mathbf{F} - \mathbf{F}\mathbf{K}_z\|_F^2 + \mathrm{tr}(\mathbf{F}^\top\mathbf{1}\mathbf{1}^\top\mathbf{F}\mathbf{K}_{zz} + \lambda\mathbf{F}^\top\mathbf{F}) + \mathrm{tr}(\mathbf{F}^\top\mathbf{B})$$

where $\lambda = n_x \times \max\{-\min(\mathrm{eig}(\mathbf{K}_{zz})), 0\}$

- $\Psi(\mathbf{F}_k)$ is non-increasing at each iteration and $\{\mathbf{F}_1, \mathbf{F}_2, \cdots\}$ will converge into a fixed point

# Subproblem about Learning Transformations

## Fixing **F**

The function to minimize

$$\begin{aligned} E_s &+ \gamma_p E_p \\ &= \operatorname{tr}(\mathbf{P}_x^\top \mathbf{X}(\gamma_f \mathbf{I} + \gamma_p \mathbf{L}_x)\mathbf{X}^\top \mathbf{P}_x + \mathbf{P}_z^\top \mathbf{Z}(\gamma_f \mathbf{F}^\top \mathbf{F} + \gamma_p \mathbf{L}_z)\mathbf{Z}^\top \mathbf{P}_z \\ &- 2\gamma_f \mathbf{P}_x^\top \mathbf{X}\mathbf{F}\mathbf{Z}^\top \mathbf{P}_z) \end{aligned}$$

Centralize $\mathbf{X}, \mathbf{Z}$ and reformulate the optimization problem

$$\begin{aligned} \max_{\mathbf{P}_x, \mathbf{P}_z} \quad & \operatorname{tr}\left(\mathbf{P}_x^\top \mathbf{X}\mathbf{F}\mathbf{Z}^\top \mathbf{P}_z\right) \\ \text{s.t.} \quad & \mathbf{P}_x^\top \mathbf{X}\left(\gamma_f \mathbf{I} + \gamma_p \mathbf{L}_x\right)\mathbf{X}^\top \mathbf{P}_x = \mathbf{I}, \mathbf{P}_z^\top \mathbf{Z}\left(\gamma_f \mathbf{F}^\top \mathbf{F} + \gamma_p \mathbf{L}_z\right)\mathbf{Z}^\top \mathbf{P}_z = \mathbf{I} \end{aligned}$$

Then use Canonical Correlation Analysis (CCA) to solve it.

# GUMA - Experiments

The balance parameters $\gamma_f, \gamma_p$ are set to 1. The embedding dimension $d$ is set to the minimal rank of two sets minus 1.

## GUMA for Set Matching (Expressions and Poses)

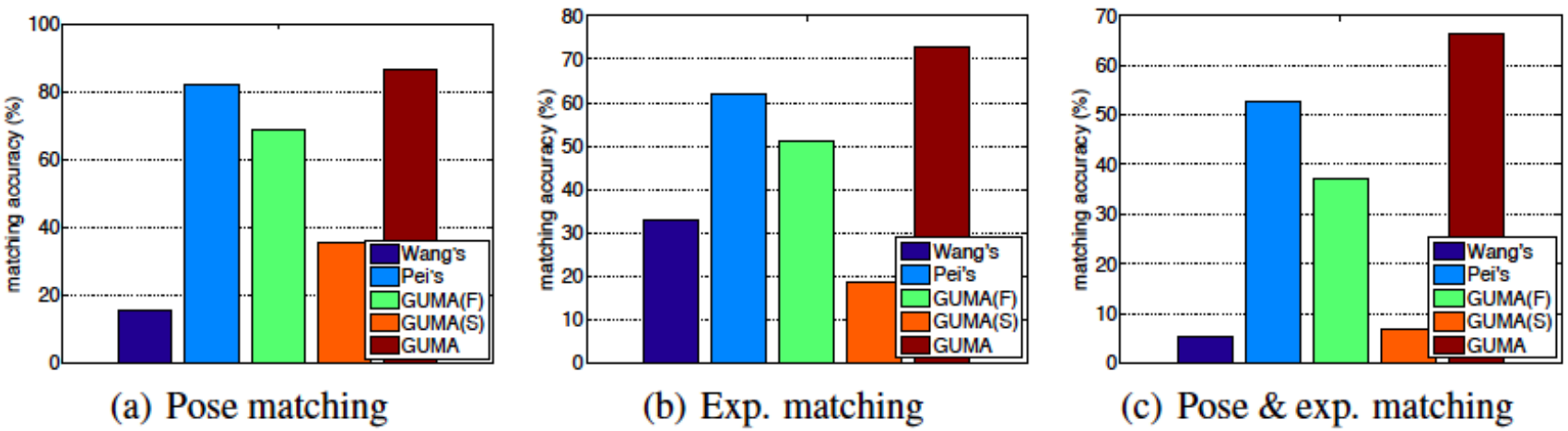
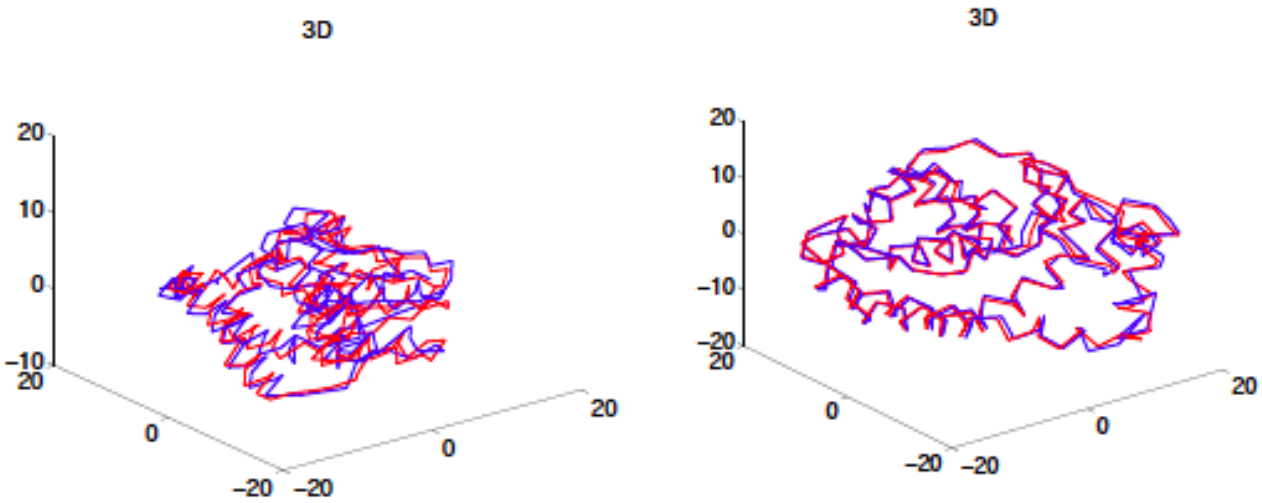



Figure: Alignment accuracy of face image sets from Multi-PIE [Cui et al., 2014] [Chang and Mahadevan, 2009] [Pei et al., 2012].

# GUMA - Experiments

## GUMA for Set Matching (Protein Structures)




(b) Wang's[29] (c) GUMA

Figure: The structure alignment results of two protein structures 1G7O-10 and 1G7O-21.

# GUMA - Experiments

## GUMA for Video-based Face Verification

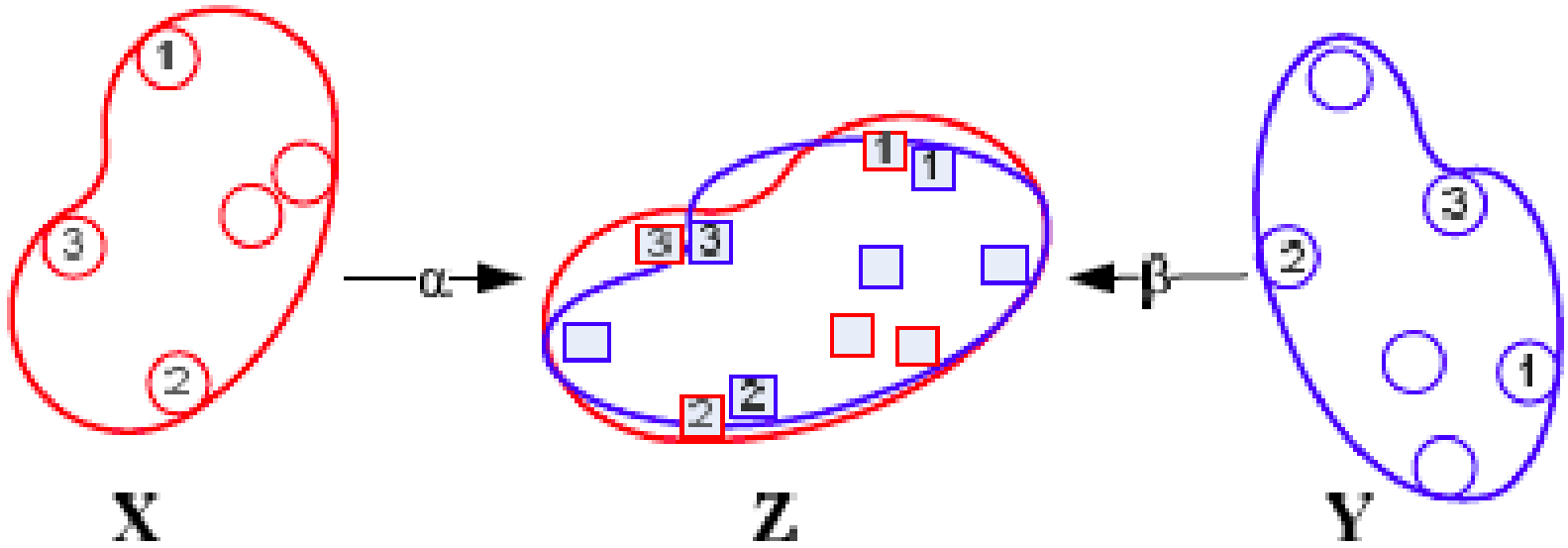
1. Align two videos by GUMA
2. Accumulate Euclidean distances of the counterparts as their dissimilarity

| Method | MSM[34] | MMD[31] | AHISD[4] | CHISD[4] | SANP[17] | DCC[18] | Unaligned | GUMA(un) | GUMA(su) |
|---|---|---|---|---|---|---|---|---|---|
| Mean Accuracy | 62.54 | 64.96 | 66.50 | 66.24 | 63.74 | 70.84 | 61.80 | 65.74 | 75.00 |
| Standard Error | ±1.47 | ±1.00 | ±2.03 | ±1.70 | ±1.69 | ±1.57 | ±2.29 | ±1.81 | ±1.56 |

Figure: The comparisons on YouTube faces dataset (%) [Cui et al., 2014].

# Motivation of Global MA

- The global manifold geometry needs to be respected.
- Directly represent the global geometry of the data.
- Using given correspondences, can deal with many to many correspondences.



# Global MA - Problem

## The Cost Function

$$C(\alpha, \beta, k) = \|\tau(\mathcal{D}) - \tau\left(\mathcal{D}_{X,Y,\alpha,\beta,k}\right)\|_2^2 =$$

$$\left\|\tau(\mathcal{D}) - k\left[\alpha^T X, \beta^T Y\right]^T \left[\alpha^T X, \beta^T Y\right]\right\|_2^2$$

where $\beta$ is a $d \times p$ matrix, $\alpha$ is a $d \times q$ matrix, $k$ is a positive number to rescale mapping functions.

# Global MA - Problem

## The Cost Function

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- $\mathcal{D} = \begin{pmatrix} D_{x,x} & D_{x,y} \\ D_{y,x} & D_{y,y} \end{pmatrix}$ is a $(m+n) \times (m+n)$ matrix and $D_{x,y}(i,j) = \min_{u \in [1,l]} \left(D_{x,x}\left(x_i, x_{a_u}\right) + D_{y,y}\left(y_j, y_{b_u}\right)\right)$

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- $\tau(A) = -HSH/2$ is a Gram matrix, $S_{i,j} = A_{i,j}^2$ is the matrix of squared distance, $H_{i,j} = \delta_{i,j} - 1/m$ is the "centering matrix".

# Global MA - Problem

## Theorem 1 [Wang and Mahadevan, 2013]

Let $Z = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, then the eigenvectors corresponding to the $d$ maximum eigenvalues of $Z\tau(\mathcal{D})Z^T\gamma = \lambda ZZ^T\gamma$ provide optimal mappings to minimize $C(\alpha, \beta, k)$.

# Global MA - Algorithm

1. **Rescale data set $Y$:**
   $Y = \eta Y$, where $\eta = \text{tr}\left(D_b^T D_a\right) / \text{tr}\left(D_b^T D_b\right)$.

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3. **Find the correspondence between $X$ and $Y$ :**
   Compute the eigenvectors $(\gamma_1, \gamma_2, \cdots, \gamma_d)$ corresponding to $d$ maximum eigenvalues of
   $$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \tau(\mathcal{D}) \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}^T \gamma = \lambda \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}^T \gamma.$$

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4. **Construct $\alpha$ and $\beta$ to map $X$ and $Y$ to the same $d$-dimensional space:**
   The $d$-dimensional representations of $X$ and $Y$ are columns of $\alpha^T X$ and $\beta^T Y$, where $[\alpha \quad \beta]^T = [\gamma_1, \cdots, \gamma_d]$.

# Global MA - Experiments

English Arabic Cross-Lingual Retrieval

- **Two collections**: one in English and one in Arabic (manually translated), each of the two document collections has 2,119 documents.
- **The features**: constructed by the language model.
- **Our testing scheme**: for each given English document, we retrieve its top $K$ most similar Arabic documents. The probability that the true match is among the top $K$ documents is used to show the goodness of the method.

# Global MA - Experiments

## English Arabic Cross-Lingual Retrieval

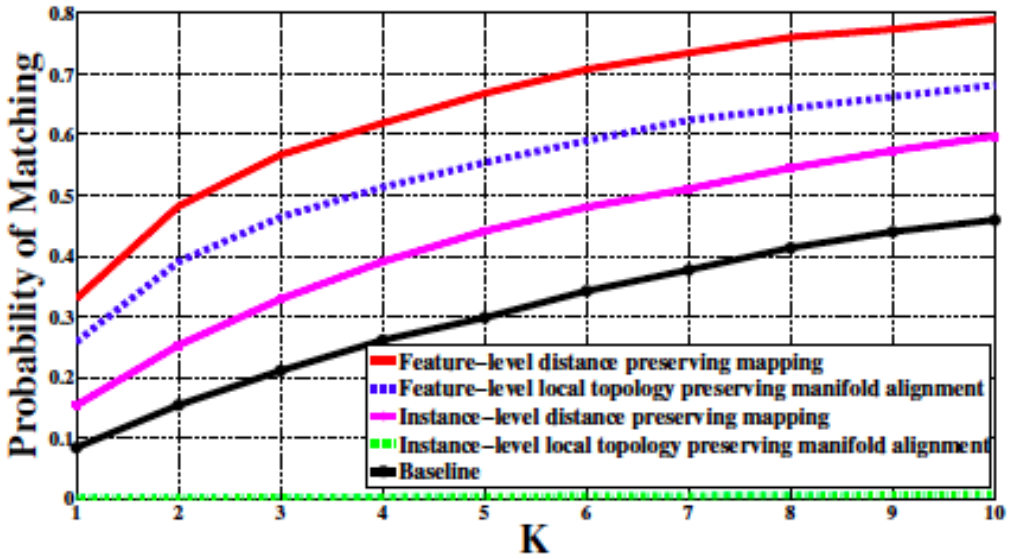


Figure: Test on English Arabic cross-lingual data (25% instances are in the given correspondence) [Wang and Mahadevan, 2013].

# Semi-supervised Manifold Alignment (SSMA)

- Make use of **class labels rather than correspondences** to align the manifolds.
- Handle the situation when the input domains do not share any common features or instances.

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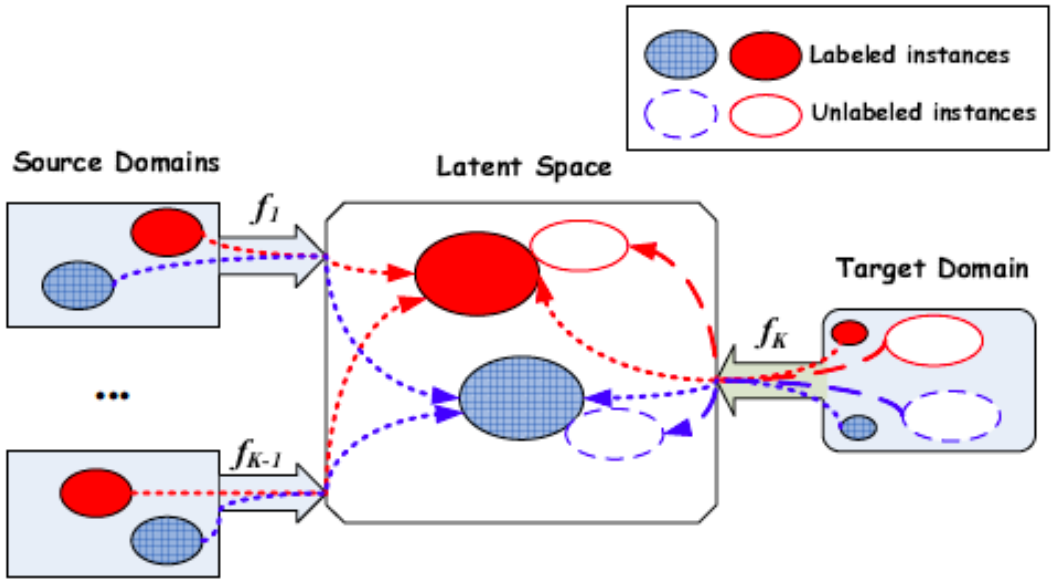



Figure: Illustration of manifold alignment using labels. Different colors represent different classes [Chang and Mahadevan, 2011].

# Idea of SSMA

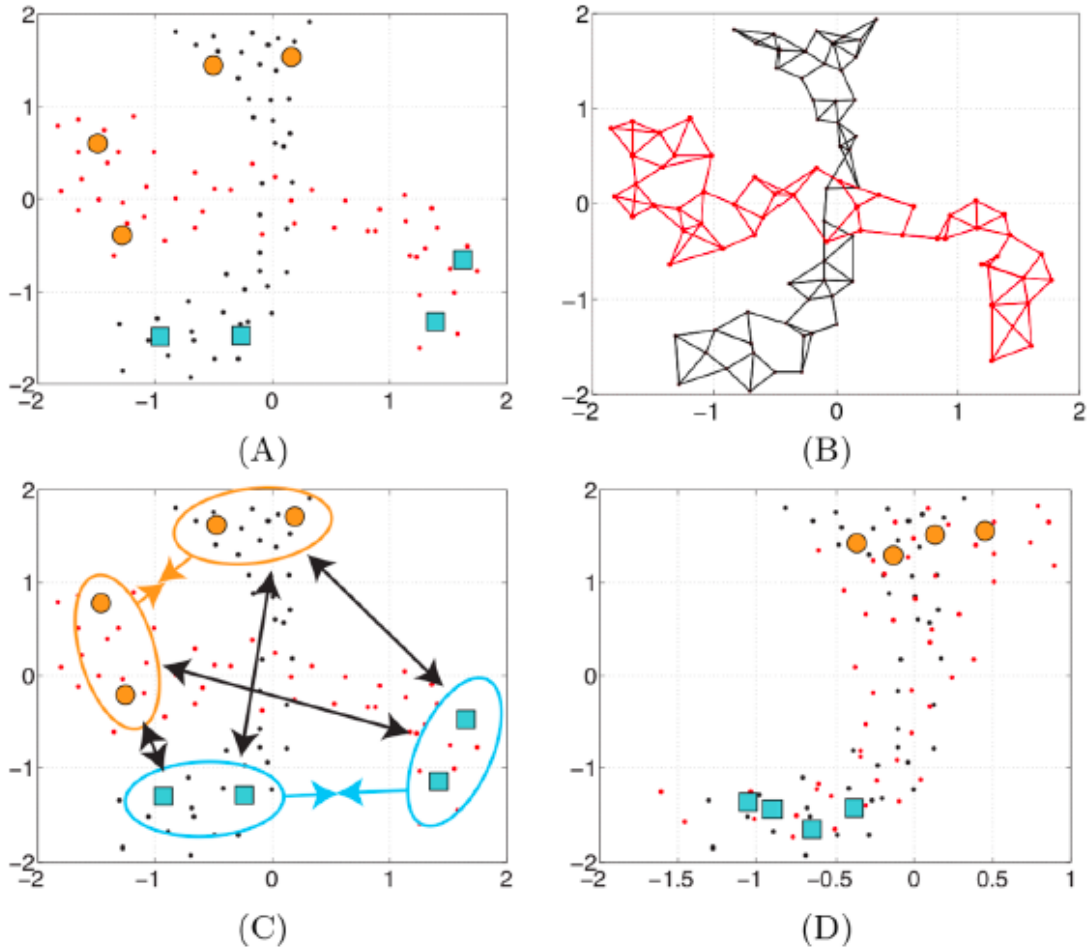
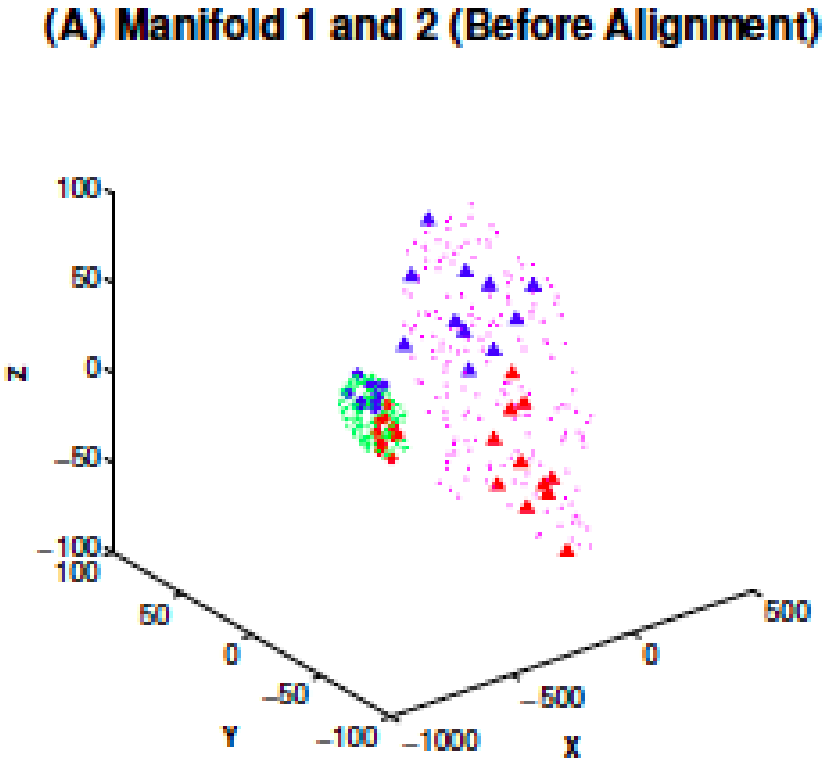
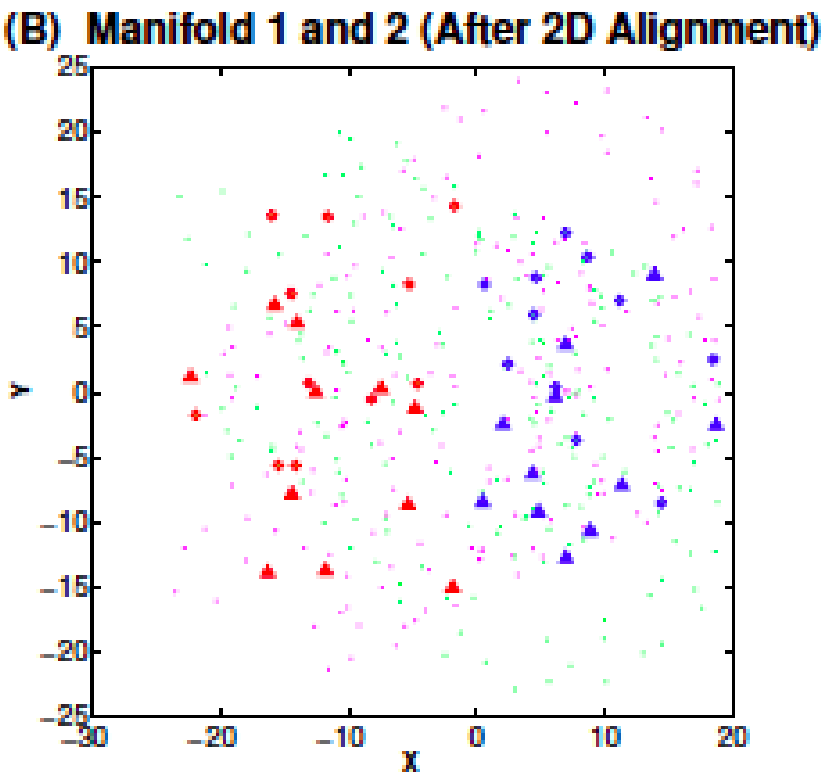



Figure: The idea behind SSMA [Devis et al., 2016].

# Illustration of SSMA



# SSMA - Problem

- Given $K$ input data sets $\mathbf{X}_k = (\mathbf{x}_k^1, \cdots, \mathbf{x}_k^{m_k})$, $k = 1, 2, \cdots, K$ with $p_k$ features each
- The labels for the first $l_k$ instances of $\mathbf{X}_k$ are given as $\mathbf{V}_k = (v_k^1, \cdots, v_k^{l_k})$.
- Similarity matrix $\mathbf{W}_s$ and dissimilarity matrix $\mathbf{W}_d$ are $(m_1 + \cdots + m_K) \times (m_1 + \cdots + m_K)$ matrix

$$\mathbf{W}_s = \begin{pmatrix} \mathbf{W}_s^{1,1} & \cdots & \mathbf{W}_s^{1,K} \\ & \cdots & \\ \mathbf{W}_s^{K,1} & \cdots & \mathbf{W}_s^{K,K} \end{pmatrix}, \mathbf{W}_d = \begin{pmatrix} \mathbf{W}_d^{1,1} & \cdots & \mathbf{W}_d^{1,K} \\ & \cdots & \\ \mathbf{W}_d^{K,1} & \cdots & \mathbf{W}_d^{K,K} \end{pmatrix},$$

where $\mathbf{W}_s^{a,b}(i,j) = 1$ ($\mathbf{W}_d^{a,b}(i,j) = 1$), if $\mathbf{x}_a^i$ and $\mathbf{x}_b^j$ are from the same class (different classes) and 0 otherwise.

# SSMA - Problem

- Let $\mathbf{W}_k(i,j)$ represent the similarity of $\mathbf{x}_k^i$ and $\mathbf{x}_k^j$. This similarity matrix can be computed as $e^{-\left\|\mathbf{x}_k^i - \mathbf{x}_k^j\right\|^2}$.
- The corresponding diagonal row sum matrix is defined as $\mathbf{D}_k(i,i) = \sum_j \mathbf{W}_k(i,j)$, $\mathbf{D}_s(i,i) = \sum_j \mathbf{W}_s(i,j)$ and $\mathbf{D}_d(i,i) = \sum_j \mathbf{W}_d(i,j)$ , and the combinatorial graph Laplacian matrix $\mathbf{L}_k = \mathbf{D}_k - \mathbf{W}_k$, $\mathbf{L}_s = \mathbf{D}_s - \mathbf{W}_s$ and $\mathbf{L}_d = \mathbf{D}_d - \mathbf{W}_d$.
- Matrices $\mathbf{L}$ and $\mathbf{Z}$ are defines as follows:
  - $\mathbf{L} = \text{diag}\{\mathbf{L}_1, \cdots, \mathbf{L}_K\}$ $((m_1 + \cdots + m_K) \times (m_1 + \cdots + m_K))$
  - $\mathbf{Z} = \text{diag}\{\mathbf{X}_1, \cdots, \mathbf{X}_K\}$ $((p_1 + \cdots + p_K) \times (m_1 + \cdots + m_K))$.

**Define** $A$, $B$ and $C$ with linear mapping

- The topology of each set is preserved ($\mu$ is a weight para.).

$$C = 0.5\mu \sum_{k=1}^{K} \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} \|\mathbf{f}_k^\top \mathbf{x}_k^i - \mathbf{f}_k^\top \mathbf{x}_k^j\|^2 \mathbf{W}_k(i,j),$$

# SSMA - Problem

- The instances from the same class (across the input sets) are mapped to similar locations.

$$A = 0.5 \sum_{a=1}^{K} \sum_{b=1}^{K} \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} ||\mathbf{f}_a^\top \mathbf{x}_a^i - \mathbf{f}_b^\top \mathbf{x}_b^j||^2 \mathbf{W}_s^{a,b}(i,j),$$

- The instances from different classes are well-separated from each other.

$$B = 0.5 \sum_{a=1}^{K} \sum_{b=1}^{K} \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} ||\mathbf{f}_a^\top \mathbf{x}_a^i - \mathbf{f}_b^\top \mathbf{x}_b^j||^2 \mathbf{W}_d^{a,b}(i,j),$$

# SSMA - Problem

SSMA simultaneously minimize $A$, $C$, and maximize $B$. So the overall cost function $\mathcal{C}(\mathbf{f}_1, \cdots, \mathbf{f}_K)$ to be minimized is

$$\mathcal{C}(\mathbf{f}_1, \cdots, \mathbf{f}_K) = (A + C)/B.$$

## Theorem 1 ([Chang and Mahadevan, 2011])

*The embedding that minimizes the cost function* $\mathcal{C}(\mathbf{f}_1, \cdots, \mathbf{f}_K)$ *is given by the eigenvectors corresponding to the smallest non-zero eigenvalues of the generalized eigenvalue decomposition* $\mathbf{Z}(\mu\mathbf{L} + \mathbf{L}_s)\mathbf{Z}^\top\mathbf{x} = \lambda\mathbf{Z}\mathbf{L}_d\mathbf{Z}^\top\mathbf{x}$.

# SSMA - Experiments

## Text Categorization

- One document **only appear in one category**.
- Divide the data set into two subsets **of the same size**.
- Learn LSI (Latent Semantic Analysis) topics from the first subset and LDA (Latent Dirichlet Allocation) topics from the second subset.
- The labels for **all documents** in the first subset and **5%** of the second subset are available.
- Apply domain adaptation algorithms in this **common latent space** to learn the categories for unlabeled documents.

# SSMA - Experiments

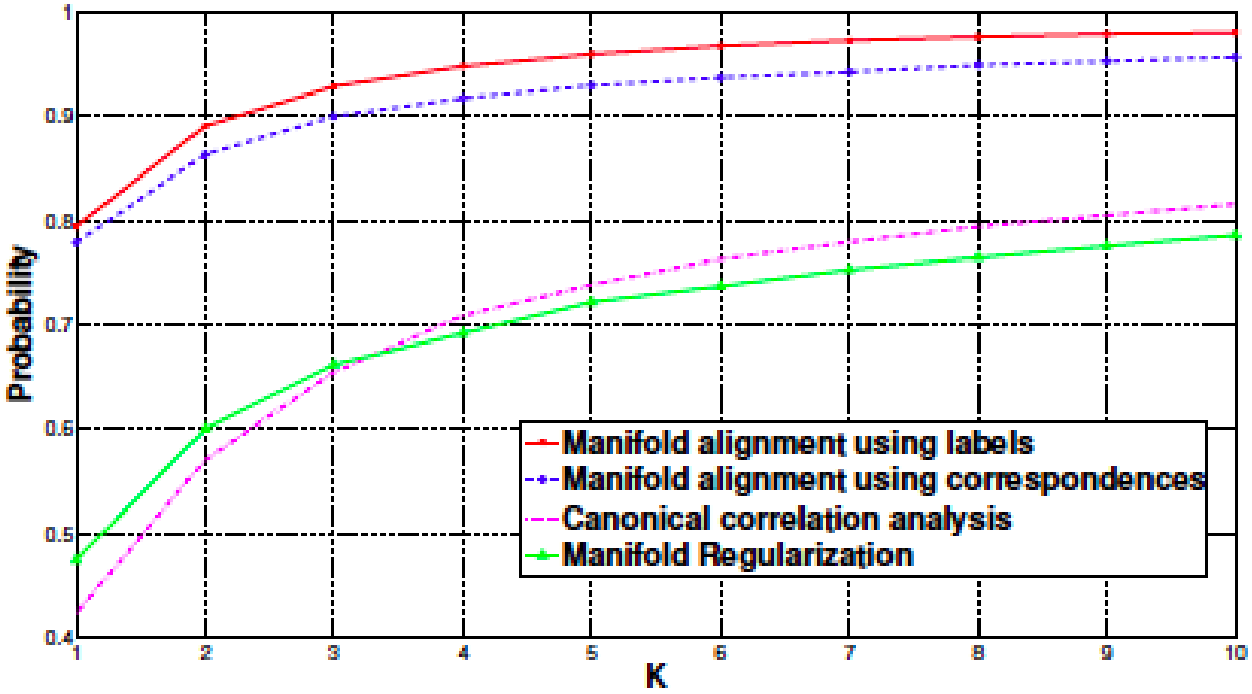



Figure: Text Categorization Test [Chang and Mahadevan, 2011].

# Kernel Manifold Alignment (KEMA)

- KEMA can align manifolds of very **different complexities** or data spaces of **different dimensionality**, and is robust to strong nonlinear feature deformations.

- KEMA generalizes other MA methods. When a linear kernel is used for all the domains, $\mathbf{K}_i = \mathbf{X}_i^\top \mathbf{X}_i$, KEMA reduces to SSMA.

- KEMA is especially well-suited to deal with **high-dimensional** problems (e.g., images and videos), and **under complicated distortions**, twists and warpings of the data manifolds.

# KEMA - Problem

- The original problem: $\mathbf{Z}(\mathbf{L} + \mu\mathbf{L}_s)\mathbf{Z}^\top\mathbf{x} = \lambda\mathbf{Z}\mathbf{L}_d\mathbf{Z}^\top\mathbf{x}$.
- $\phi_i(\cdot) : \mathbf{x} \to \phi(\mathbf{x}) \in \mathcal{H}_i$, $i = 1, 2, \cdots, D$.

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- $\phi_i(\cdot) : \mathbf{x} \to \phi(\mathbf{x}) \in \mathcal{H}_i$, $i = 1, 2, \cdots, D$.
- Replacing all the samples with their mapped feature vectors:

$$\mathbf{\Phi}(\mathbf{L}+\mu\mathbf{L}_s)\mathbf{\Phi}^\top\mathbf{U} = \lambda\mathbf{\Phi}\mathbf{L}_d\mathbf{\Phi}^\top\mathbf{U},$$

where $\mathbf{\Phi}$ is a block diagonal matrix containing the data matrices $\mathbf{\Phi}_i = (\phi_i(x_1), \cdots, \phi_i(x_{n_i}))^\top$.

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  where $\mathbf{\Phi}$ is a block diagonal matrix containing the data matrices $\mathbf{\Phi}_i = (\phi_i(x_1), \cdots, \phi_i(x_{n_i}))^\top$.
- The eigenvectors can be expressed by mapped samples: $\mathbf{u}_i = \mathbf{\Phi}_i\boldsymbol{\alpha}_i$, and in matrix notation $\mathbf{U} = \mathbf{\Phi}\mathbf{\Lambda}$. This leads to the problem:

$$\mathbf{\Phi}(\mathbf{L}+\mu\mathbf{L}_s)\mathbf{\Phi}^\top\mathbf{\Phi}\mathbf{\Lambda} = \lambda\mathbf{\Phi}\mathbf{L}_d\mathbf{\Phi}^\top\mathbf{\Phi}\mathbf{\Lambda},$$

# KEMA - Problem

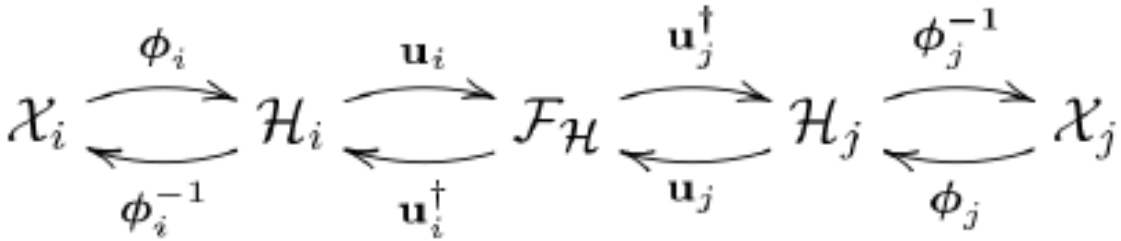
- Let $\mathbf{K}_i = \mathbf{\Phi}^\top \mathbf{\Phi}$, we obtain the final solution:

$$\mathbf{K}(\mathbf{L} + \mu \mathbf{L}_s)\mathbf{K}\mathbf{\Lambda} = \lambda \mathbf{K}\mathbf{L}_d\mathbf{K}\mathbf{\Lambda},$$

  where $\mathbf{K}$ is a block diagonal matrix containing the kernel $\mathbf{K}_i$.
- When a linear kernel is used for all the domains, $\mathbf{K}_i = \mathbf{X}_i^\top \mathbf{X}_i$, KEMA reduces to SSMA;
- Projection to the latent space:

$$P_{\mathcal{F}}(\mathbf{X}_i) = \mathbf{u}_i^\top \mathbf{\Phi}_i = \alpha_i \mathbf{\Phi}_i^\top \mathbf{\Phi}_i = \alpha_i \mathbf{K}_i.$$

- Can be depicted as:



# Learnable Manifold Alignment (LeMA)

- Multispectral (MS) & hyperspectral (HS) imagery.
- Collect a large amount of labeled and discriminative data.
- Problem: Can a limited amount of HS training data partially overlapping MS data improve the performance of a classification task using a large coverage of MS testing data?

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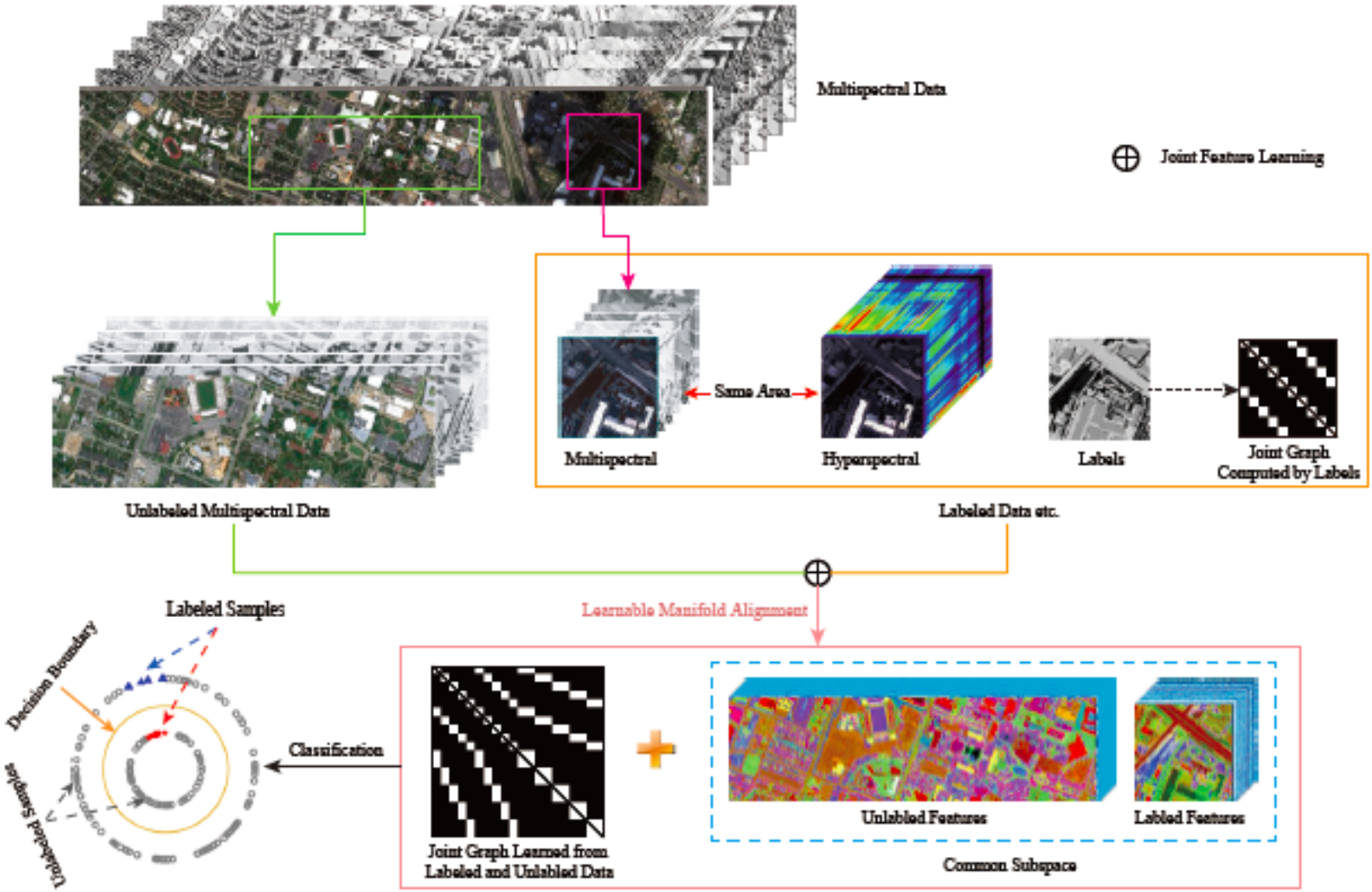
- Collect a large amount of labeled and discriminative data.
- Problem: Can a limited amount of HS training data partially overlapping MS data improve the performance of a classification task using a large coverage of MS testing data?

- Using a part of the data with labels to learn rest of the data (unlabeled).
- Simultaneously learn the projections, classification and Laplacian matrix.

# Illustration of LeMA [Hong et al., 2019]



# LeMA - Algorithm

**Algorithm 1:** Learnable Manifold Alignment (LeMA)

**Input:** $\widetilde{\mathbf{Y}}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}', \widetilde{\mathbf{L}}, \alpha, \beta, maxIter.$
**Output:** $\mathbf{P}, \mathbf{\Theta}, \widetilde{\mathbf{L}}$

$t = 1, \zeta = 1e-4;$
**Initializating** $\mathbf{P}$ and $\mathbf{\Theta}$
**while** *not converged* or $t > maxIter$ **do**
  Fix other variables to update $\mathbf{P}$ by Eq. (6)
  Fix other variables to update $\mathbf{\Theta}$ by **Algorithm 2**
  Fix other variables to update $\widetilde{\mathbf{L}}$ by equivalently optimizing $\widetilde{\mathbf{W}}$ in a distributed fashion:
    1. update $\widetilde{\mathbf{W}}_{HU}$ by **Algorithm 3**;
    2. update $\widetilde{\mathbf{W}}_{MU}$ by **Algorithm 3**;
    3. align $\widetilde{\mathbf{W}}_{HU}$ and $\widetilde{\mathbf{W}}_{MU}$ by $\max(\widetilde{\mathbf{W}}_{HU}, \widetilde{\mathbf{W}}_{MU})$;
    4. update $\widetilde{\mathbf{W}}_{UU}$ by **Algorithm 4**
    5. compute $\widetilde{\mathbf{L}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{W}}, \widetilde{\mathbf{D}}_{ii} = \sum_{i \neq j} \widetilde{\mathbf{W}}_{ij}$
  Compute the objective function value $E^{t+1}$ and check the convergence condition: **if** $|\frac{E^{t+1} - E^t}{E^t}| < \zeta$ **then**
    Stop iteration;
  **else**
    $t \leftarrow t + 1;$
  **end**
**end**

# LeMA - Problem

## The common subspace learning problem (CoSpace)

$$\min_{\mathbf{P},\Theta} \frac{1}{2}\|\widetilde{\mathbf{Y}} - \mathbf{P}\Theta\widetilde{\mathbf{X}}\|_{\mathrm{F}}^2 + \frac{\alpha}{2}\|\mathbf{P}\|_{\mathrm{F}}^2 + \frac{\beta}{2}\,\mathrm{tr}\left(\mathbf{ELE}^{\mathrm{T}}\right)$$

$$\text{s.t. } \mathbf{E} = \Theta\widetilde{\mathbf{X}}, \Theta\Theta^{\mathrm{T}} = \mathbf{I}$$

fidelity term + regularization term + supervised MA

| | |
|---|---|
| $\widetilde{\mathbf{Y}} = [\mathbf{Y}, \mathbf{Y}] \in \mathbb{R}^{d\times 2N}$, $\mathbf{Y} \in \mathbb{R}^{d\times N}$ | the label matrix (one-hot encoding) |
| $\widetilde{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_H & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_M \end{bmatrix} \in \mathbb{R}^{(d_H+d_M)\times 2N}$ | the data matrix |
| $\Theta_H = [\Theta_H, \Theta_M]$ | the common subspace projection |
| $\mathbf{P}$ | the linear projection of the common subspace and label information |
| $\mathbf{L} = \mathbf{D} - \mathbf{W}$ | joint Laplacian matrix |

# LeMA - Problem

$$\min_{\mathbf{P},\Theta,\widetilde{\mathbf{L}}} \frac{1}{2}\|\widetilde{\mathbf{Y}} - \mathbf{P}\Theta\widetilde{\mathbf{X}}\|_{\mathrm{F}}^2 + \frac{\alpha}{2}\|\mathbf{P}\|_{\mathrm{F}}^2 + \frac{\beta}{2}\,\mathrm{tr}\left(\mathbf{H}\widetilde{\mathbf{L}}\mathbf{H}^{\mathrm{T}}\right)$$
$$\text{s.t. } \mathbf{H} = \Theta\widetilde{\mathbf{X}}', \Theta\Theta^{\mathrm{T}} = \mathbf{I}, \widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}^{\mathrm{T}}, \widetilde{\mathbf{L}}_{i,j,i\neq j} \preceq 0, \widetilde{\mathbf{L}}_{i,j,i=j} \succeq 0, \mathrm{tr}(\widetilde{\mathbf{L}}) = s$$

where $\widetilde{\mathbf{X}}' = \begin{bmatrix} \mathbf{X}_H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_M & \mathbf{X}_U \end{bmatrix} \in \mathbb{R}^{(d_H+d_M)\times(2N+N_U)}$,
$\widetilde{\mathbf{L}} \in \mathbb{R}^{(2N+N_U)\times(2N+N_U)}$, $\mathbf{X}_U \in \mathbb{R}^{d_M\times N_U}$ represents the unlabeled MS samples, and $s > 0$ controls the scale.

# Model Optimization

## Optimization with respect to **P**

a typical least-squares problem

$$\min_{\mathbf{P}} \frac{1}{2}\|\widetilde{\mathbf{Y}} - \mathbf{P}\Theta\widetilde{\mathbf{X}}\|_{\mathrm{F}}^2 + \frac{\alpha}{2}\|\mathbf{P}\|_{\mathrm{F}}^2$$

which has a closed-form solution

$$\mathbf{P} = \left(\widetilde{\mathbf{Y}}\mathbf{E}^{\mathrm{T}}\right)\left(\mathbf{E}\mathbf{E}^{\mathrm{T}} + \alpha\mathbf{I}\right)^{-1},$$

where $\mathbf{E} = \Theta\widetilde{\mathbf{X}}$.

# Model Optimization

## Optimization with respect to $\Theta$

$$\min_{\Theta} \frac{1}{2}\|\widetilde{\mathbf{Y}} - \mathbf{P}\Theta\widetilde{\mathbf{X}}\|_{\mathrm{F}}^2 + \frac{\beta}{2}\operatorname{tr}\left(\mathbf{H}\widetilde{\mathbf{L}}\mathbf{H}^{\mathrm{T}}\right) \text{ s.t. } \mathbf{H} = \Theta\widetilde{\mathbf{X}}', \Theta\Theta^{\mathrm{T}} = \mathbf{I}$$

which has an equivalent form

$$\min_{\Theta,\mathbf{J},\mathbf{G}} \frac{1}{2}\|\widetilde{\mathbf{Y}} - \mathbf{P}\mathbf{J}\|_{\mathrm{F}}^2 + \frac{\beta}{2}\operatorname{tr}\left(\Theta\widetilde{\mathbf{X}}'\widetilde{\mathbf{L}}\left(\Theta\widetilde{\mathbf{X}}'\right)^{\mathrm{T}}\right)$$
$$\text{s.t. } \mathbf{J} = \Theta\widetilde{\mathbf{X}}, \mathbf{G} = \Theta, \mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{I}.$$

This problem can be solved with ADMM.

# Model Optimization

**Algorithm 2:** Solving the subproblem for $\boldsymbol{\Theta}$

**Input:** $\widetilde{\mathbf{Y}}, \mathbf{P}, \mathbf{J}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}', \widetilde{\mathbf{L}}, \beta, maxIter$.

**Output:** $\boldsymbol{\Theta}$.

**Initialization:** $\boldsymbol{\Theta} = \mathbf{0}, \mathbf{G} = \mathbf{0}, \boldsymbol{\Lambda}_1 = \mathbf{0}, \boldsymbol{\Lambda}_2 = \mathbf{0}, \mu = 10^{-3}, \mu_{\max} = 10^6, \rho = 1.5, \varepsilon = 10^{-6}, t = 1$.

**while** *not converged* or $t > maxIter$ **do**

  Fix other variables to update $\mathbf{J}$ by $\mathbf{J} = (\mathbf{P}^{\mathrm{T}}\mathbf{P} + \mu\mathbf{I})^{-1}(\mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{Y}} + \mu\boldsymbol{\Theta}\widetilde{\mathbf{X}} - \boldsymbol{\Lambda}_1)$.

  Fix other variables to update $\boldsymbol{\Theta}$ by

$$\boldsymbol{\Theta} = (\mu\mathbf{J}\widetilde{\mathbf{X}}^{\mathrm{T}} + \boldsymbol{\Lambda}_1\widetilde{\mathbf{X}}^{\mathrm{T}} + \mu\mathbf{G} + \boldsymbol{\Lambda}_2) \times (\mu\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^{\mathrm{T}} + \mu\mathbf{I} + \beta\widetilde{\mathbf{X}}'\widetilde{\mathbf{L}}\widetilde{\mathbf{X}}'^{\mathrm{T}})^{-1}.$$

  Fix other variables to update $\mathbf{G}$ by

$$[\mathbf{U}, \mathbf{S}, \mathbf{V}] = \mathrm{svd}(\boldsymbol{\Theta} - \boldsymbol{\Lambda}_2/\mu), \quad \mathbf{G} = \mathbf{U}\mathbf{I}_{n\times m}\mathbf{V}.$$

  Update Lagrange multipliers by

$$\boldsymbol{\Lambda}_1 \leftarrow \boldsymbol{\Lambda}_1 + \mu(\mathbf{J} - \boldsymbol{\Theta}\widetilde{\mathbf{X}}), \quad \boldsymbol{\Lambda}_2 \leftarrow \boldsymbol{\Lambda}_2 + \mu(\mathbf{G} - \boldsymbol{\Theta}).$$

  Update penalty parameter by $\mu = \min(\rho\mu, \mu_{\max})$.

  Check the convergence conditions: **if** $\|\mathbf{J} - \boldsymbol{\Theta}\widetilde{\mathbf{X}}\|_{\mathrm{F}} < \varepsilon$ *and* $\|\mathbf{G} - \boldsymbol{\Theta}\|_{\mathrm{F}} < \varepsilon$ **then**

    Stop iteration;

  **else**

    $t \leftarrow t + 1$;

  **end**

**end**

# Model Optimization

## Optimization with respect to $\widetilde{\mathbf{W}}$

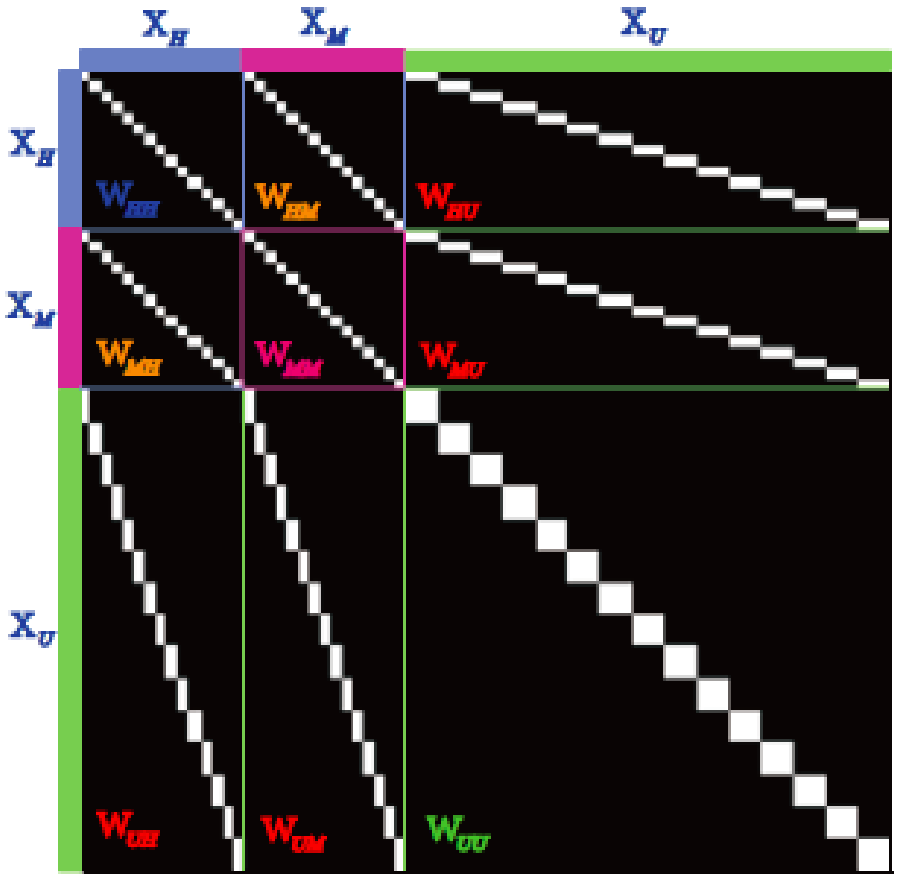



Figure: An example for the joint adjacency matrix $\widetilde{\mathbf{W}}$.

# Model Optimization

## The optimization problems on $\widetilde{\mathbf{W}}_{HU}$ and $\widetilde{\mathbf{W}}_{MU}$

$$\min_{\widetilde{\mathbf{w}}_{HU(MU)}} \frac{\beta}{4}\|\widetilde{\mathbf{W}} \odot \mathbf{Z}\|_{1,1} \text{ s.t. } 1/N_k \succeq \widetilde{\mathbf{W}}_{i,j} \succeq 0, \|\widetilde{\mathbf{W}}\|_{1,1} = s$$

which can be solved by ADMM.

## The optimization problems on $\widetilde{\mathbf{W}}_{UU}$

$$\min_{\widetilde{\mathbf{w}}_{UU}} \frac{\beta}{4}\|\widetilde{\mathbf{W}} \odot \mathbf{Z}\|_{1,1} \text{ s.t. } \widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}^{\mathrm{T}}, 1/N_k \succeq \widetilde{\mathbf{W}}_{i,j} \succeq 0, \|\widetilde{\mathbf{W}}\|_{1,1} = s.$$

# LeMA - Algorithm

**Algorithm 1:** Learnable Manifold Alignment (LeMA)

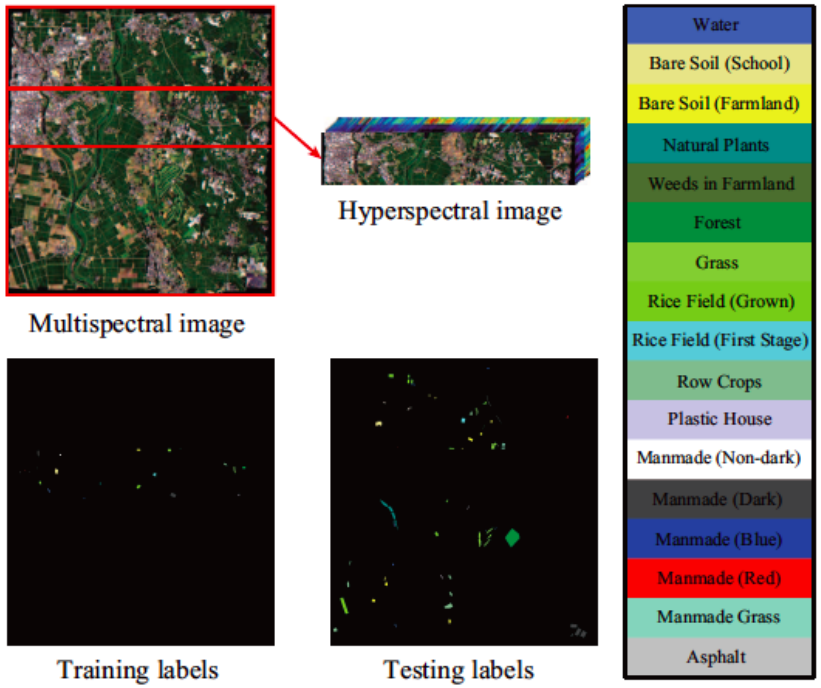
**Input:** $\widetilde{\mathbf{Y}}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}', \widetilde{\mathbf{L}}, \alpha, \beta, maxIter.$
**Output:** $\mathbf{P}, \mathbf{\Theta}, \widetilde{\mathbf{L}}$

$t = 1, \zeta = 1e-4;$
**Initializating** $\mathbf{P}$ and $\mathbf{\Theta}$
**while** *not converged* or $t > maxIter$ **do**
  Fix other variables to update $\mathbf{P}$ by Eq. (6)
  Fix other variables to update $\mathbf{\Theta}$ by **Algorithm 2**
  Fix other variables to update $\widetilde{\mathbf{L}}$ by equivalently optimizing $\widetilde{\mathbf{W}}$ in a distributed fashion:
    1. update $\widetilde{\mathbf{W}}_{HU}$ by **Algorithm 3**;
    2. update $\widetilde{\mathbf{W}}_{MU}$ by **Algorithm 3**;
    3. align $\widetilde{\mathbf{W}}_{HU}$ and $\widetilde{\mathbf{W}}_{MU}$ by $\max(\widetilde{\mathbf{W}}_{HU}, \widetilde{\mathbf{W}}_{MU})$;
    4. update $\widetilde{\mathbf{W}}_{UU}$ by **Algorithm 4**
    5. compute $\widetilde{\mathbf{L}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{W}}, \widetilde{\mathbf{D}}_{ii} = \sum_{i \neq j} \widetilde{\mathbf{W}}_{ij}$
  Compute the objective function value $E^{t+1}$ and check the convergence condition: **if** $|\frac{E^{t+1} - E^t}{E^t}| < \zeta$ **then**
    Stop iteration;
  **else**
    $t \leftarrow t + 1;$
  **end**
**end**

# LeMA - Experiments

## The Simulated MS-HS Datasets over Chikusei

- MS data: $2517 \times 2335 \times 10$ at a GSD of 2.5 m.
- HS data: 128 bands in the spectral range from 363nm to 1018nm with the 10nm spectral resolution.



# LeMA - Experiments

## The Simulated MS-HS Datasets over Chikusei

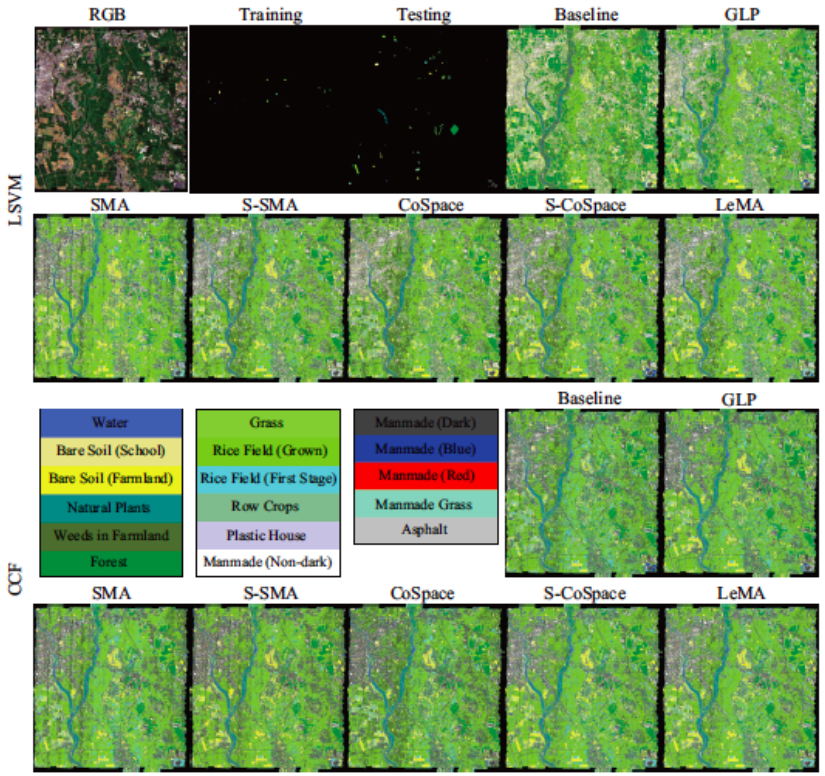



Figure: Classification maps of the different algorithms obtained using two kinds of classifiers on the Chikusei dataset [Hong et al., 2019].

# LeMA - Experiments

## The Simulated MS-HS Datasets over Chikusei

Table 2: Quantitative performance comparison with the different algorithms on the University of Houston data. The best one is shown in bold.

| Methods | Baseline (%) | | GLP (%) | | SMA (%) | | S-SMA (%) | | CoSpace (%) | | S-CoSpace (%) | | LeMA (%) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Parameter | $d$ | | $(k, \sigma, d)$ | | $d$ | | $(k, \sigma, d)$ | | $(\alpha, \beta, d)$ | | $(\alpha, \beta, d)$ | | $(\alpha, \beta, d)$ | |
| | 10 | | (10, 1, 10) | | 30 | | (10, 0.1, 30) | | (0.01, 0.01, 30) | | (0.1, 0.01, 30) | | (0.01, 0.01, 30) | |
| Classifier | LSVM | CCF | LSVM | CCF | LSVM | CCF | LSVM | CCF | LSVM | CCF | LSVM | CCF | LSVM | CCF |
| OA | 62.12 | 68.21 | 64.71 | 70.01 | 68.01 | 69.59 | 69.29 | 70.10 | 69.38 | 72.17 | 70.41 | 73.75 | 73.42 | **76.35** |
| AA | 65.97 | 70.47 | 68.18 | 72.18 | 70.50 | 71.02 | 72.00 | 72.88 | 71.69 | 73.56 | 73.12 | 75.61 | 74.76 | **77.18** |
| $\kappa$ | 0.5889 | 0.6543 | 0.6164 | 0.6728 | 0.6520 | 0.6695 | 0.6659 | 0.6754 | 0.6672 | 0.6975 | 0.6784 | 0.7146 | 0.7110 | **0.7428** |
| Class1 | 76.39 | 67.95 | 77.83 | 77.97 | 75.25 | 68.53 | 74.25 | 73.53 | 75.54 | 69.96 | **91.85** | 87.98 | 89.56 | 85.84 |
| Class2 | 80.59 | 78.08 | 93.85 | **98.01** | 97.57 | 77.9 | 97.57 | 93.67 | 73.74 | 77.99 | 90.12 | 91.59 | 93.67 | 93.85 |
| Class3 | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** | **100.00** |
| Class4 | 85.51 | 92.27 | 89.66 | 96.62 | 94.78 | 98.74 | 95.85 | 98.55 | 98.74 | 98.26 | 92.75 | 97.29 | 97.49 | **99.61** |
| Class5 | 99.06 | 99.4 | 99.49 | **99.66** | 98.97 | 99.14 | 99.32 | 99.4 | 99.4 | 99.4 | 99.4 | **99.66** | 99.49 | 99.57 |
| Class6 | 86.14 | 86.14 | 96.37 | 99.01 | 86.47 | 70.96 | **99.67** | 99.67 | 85.48 | 85.15 | **99.67** | 96.70 | 86.47 | 86.47 |
| Class7 | 50.62 | 63.76 | 48.63 | 64.01 | 72.32 | 77.14 | 72.15 | 69.66 | 73.98 | 80.05 | 75.06 | 80.96 | 83.21 | **88.03** |
| Class8 | 56.49 | 56.06 | 56.60 | 59.85 | 62.01 | 62.23 | **64.61** | 63.85 | 63.53 | 62.01 | 55.84 | 60.39 | 62.77 | 62.01 |
| Class9 | 56.22 | 70.58 | 69.63 | 69.02 | 49.96 | 61.27 | 50.57 | 45.00 | 59.79 | 64.93 | 65.8 | **71.54** | 64.49 | 61.88 |
| Class10 | 45.36 | 45.25 | 45.46 | 49.89 | 58.12 | 52.32 | 58.33 | 63.61 | **64.14** | 57.70 | 58.97 | 51.79 | 60.97 | 53.59 |
| Class11 | 27.43 | 43.88 | 22.45 | 38.65 | 28.86 | 36.46 | 36.46 | 34.77 | 36.54 | 47.26 | 35.78 | 38.65 | 41.27 | **49.96** |
| Class12 | 31.64 | 56.08 | 31.75 | 37.83 | 35.84 | 62.50 | 34.18 | 55.2 | 46.79 | 62.72 | 34.29 | 58.52 | 45.02 | **76.88** |
| Class13 | 0.00 | 0.67 | 0.00 | 1.11 | 0.00 | 0.00 | 0.00 | 0.45 | 0.00 | 0.45 | 0.00 | 0.89 | 0.00 | **1.78** |
| Class14 | 97.53 | 98.77 | 94.44 | 92.59 | **100.00** | **100.00** | 99.38 | 98.15 | **100.00** | 99.38 | 99.38 | **100.00** | 99.38 | **100.00** |
| Class15 | 96.59 | 98.16 | 96.59 | **98.43** | 97.38 | 98.16 | 97.64 | 97.64 | 97.64 | 98.16 | 97.90 | 98.16 | 97.64 | 98.16 |

# Comparison - Scenarios

- **KEMA (SSMA)**:
  - Making use of class labels rather than correspondences to align the manifolds.
  - Deal with strong distortions, or data from different dimensions.
  - Can adapt to different kernels to apply to various problems.

- **Global MA**:
  - Consider the global geometry instead of only the local one.
  - Based upon given correspondences, not class labels.

# Comparison - Scenarios

- **GUMA**:
  - Fully unsupervised matching without any pre-specified correspondences.
  - Take global geometry structures into consideration by using the geodesic distance.

- **LeMA**:
  - A large amount of low quality data and a small amount of high quality data can be obtained.
  - Linear case.

# Comparison - Correspondence

- **KEMA (SSMA)**:
  - Adjacency Matrix:
    Same data set: Gaussian kernel
    Cross data sets: Class labels
  - Joint Laplacian Matrix:
    Same data set: computed from the adjacency matrix
    Cross data sets: 0
- **LeMA**:
  - Joint Adjacency Matrix:
    Same data set: the LDA-like graph
    $$\widetilde{\mathbf{W}}_{i,j} = \begin{cases} 1/N_k, & \text{if } \mathbf{X}_i \text{ and } \mathbf{X}_j \text{ belong to the } k\text{-th class;} \\ 0, & \text{otherwise.} \end{cases}$$
    Cross data sets: iteratively updated using ADMM
  - Joint Laplacian Matrix:
    Computed from the adjacency matrix

# Comparison - Correspondence

- **GUMA**:
  - Adjacency Matrix:
    geodesic distance

- **Global MA**:
  - Joint Distance Matrix:
    Same data set: Euclidian distance or geodisic distance
    Cross data sets:
    $D_{x,y}(i,j) = \min_{u \in [1,l]} \left( D_{x,x}\left(x_i, x_{a_u}\right) + D_{y,y}\left(y_j, y_{b_u}\right) \right)$
  - Gram Matrix:
    Given the distance matrix $A$, $\tau(A) = -HSH/2$, $S_{i,j} = A^2_{i,j}$ is the matrix of squared distance, $H_{i,j} = \delta_{i,j} - 1/m$ is the "centering matrix"

# Unsupervised Domain Adaptation

## Notations and definitions

- $\mathcal{D}^s = \{\mathcal{X}^s, P(X)^s\}$: source domain and its marginal distributions.
- $\mathcal{T}^s = \{\mathcal{Y}^s, P(Y^s|X^s)\}$: source labels and its conditional distributions.
- $\mathcal{D}^t = \{\mathcal{X}^t, P(X)^t\}$: target domain and its marginal distributions.
- $\mathcal{T}^t = \{\mathcal{Y}^t, P(Y^t|X^t)\}$: target labels and its conditional distributions (to be learned).

## Unsupervised Domain Adaptation Problem

If a model (e.g. a classifier) is learned from a source domain, what is its capacity to correctly label data from the target domain?

# Unsupervised Domain Adaptation

## General Form of The Loss Function

$$\mathcal{L} = \mathcal{L}_{\mathrm{C}}\left(X^{L}, y\right) + \lambda \text{Distance}\left(X^{s}, X^{t}\right)$$

where

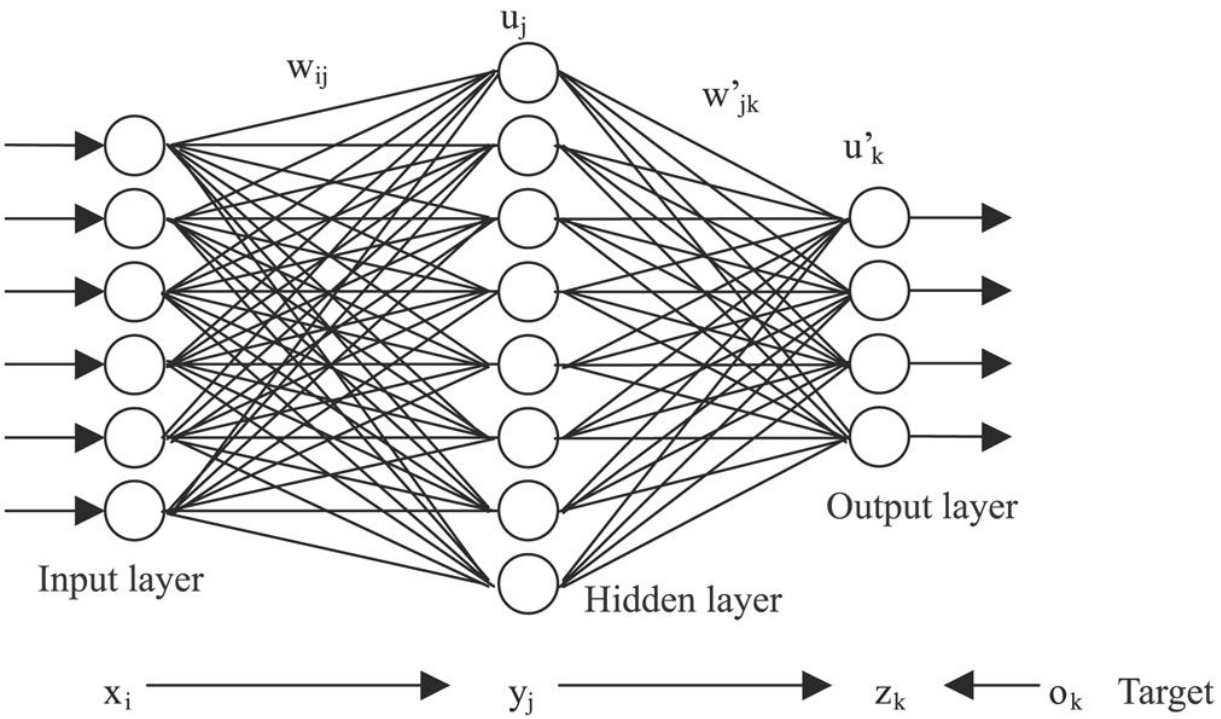
- $\lambda$: a hyperparameter for weighting penalty.
- $\mathcal{L} = \mathcal{L}_{\mathrm{C}}\left(X^{L}, y\right)$: the classification loss on the available labeled data $X^{L}$ and the ground-truth labels $y$.
- Distance $\left(X^{s}, X^{t}\right)$: the distance between the source and target data.

# Generalized MA for Domain Adaptation

## Two Key Ideas

- Using deep neural networks as the projecting function.
- Relax the correspondence-preserving constraint to distribution alignment.
  - Using the adversarial loss from a discriminator for measuring the distance between two distributions.

# Deep Neural Networks




- Strong power of feature representation.
- High-level abstraction.
- End-to-end learning with back-propagation algorithms.

# Align Manifolds or Distributions

## Manifold Alignment

- Accurate alignment between manifolds.
- The alignment results largely depend on the (either given or inferred) **correspondence information** [e.g., the similarities or distances between points, or the given class labels].

# Align Manifolds or Distributions

## Manifold Alignment

- Accurate alignment between manifolds.
- The alignment results largely depend on the (either given or inferred) **correspondence information** [e.g., the similarities or distances between points, or the given class labels].

## Distribution Alignment

- Only needs to minimize the **distance** between two distributions.
- More efficient while less accurate.

# Measure the Distance Between Distributions

How to measure the distance between distributions?

# Measure the Distance Between Distributions

How to measure the distance between distributions?

- KL-divergence.
- Maximum mean discrepancy (MMD).
- Wasserstein Distance.
- Adversarial loss from a discriminator.
- .....

# Adversarial Discriminative Domain Adaptation

- Pre-train a source encoder CNN using labeled source images.
- Perform adversarial adaptation by learning a target encoder CNN.
- During testing, target images are mapped with the target encoder to the shared feature space and classified by the source classifier.

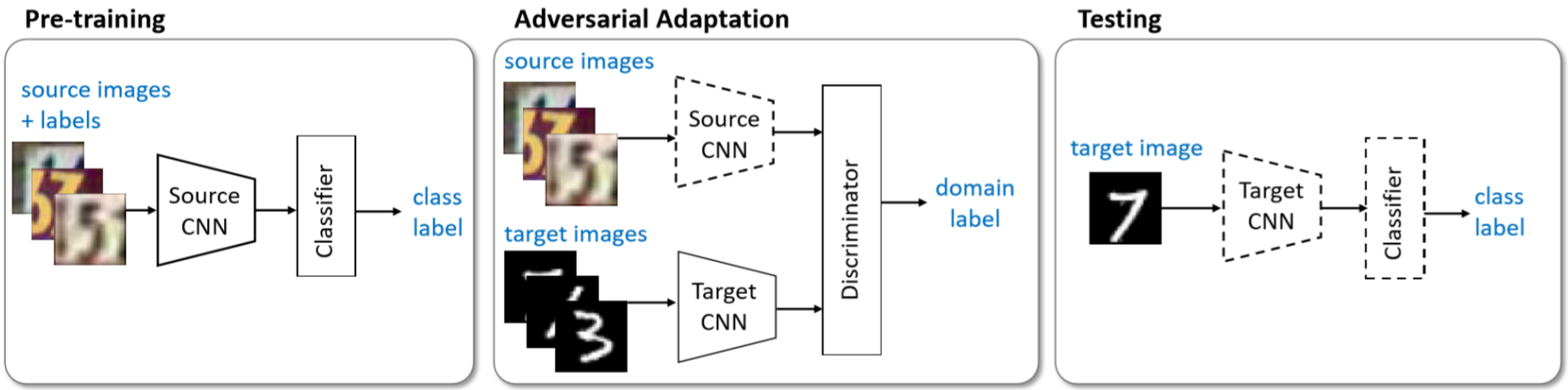



Figure: Overview of the ADDA approach. Dashed lines indicate fixed network parameters.

# Adversarial Discriminative Domain Adaptation

## Notations

- $M_s$: a source mapping (representation), projecting source data into a latent space.
- $M_t$: a target mapping (representation), projecting target data into a latent space.
- $C$: a common classifier for both source and target classification tasks.
- $D$: a domain discriminator for classifying whether a data point is drawn from the source or the target domain.

# Loss Function 1: Source Classifier

The source classification model is then trained using the standard supervised loss below:

$$\min_{M_s, C} \mathcal{L}_{\text{cls}}\left(\mathbf{X}_s,\, Y_s\right) = -\mathbb{E}_{(\mathbf{x}_s, y_s)\sim(\mathbf{X}_s, Y_s)} \sum_{k=1}^{K} \mathbf{1}_{[k=y_s]} \log C\left(M_s\left(\mathbf{x}_s\right)\right)$$

# Loss Function 2: Domain Discriminator

The domain discriminator $D$ is optimized according to a standard supervised loss:

$$\begin{aligned} \min_{D} \quad & \mathcal{L}_{\text{adv}_D}\left(\mathbf{X}_s, \mathbf{X}_t, M_s, M_t\right) \\ = \quad & -\mathbb{E}_{\mathbf{x}_s \sim \mathbf{X}_s}\left[\log D\left(M_s\left(\mathbf{x}_s\right)\right)\right] - \mathbb{E}_{\mathbf{x}_t \sim \mathbf{x}_t}\left[\log\left(1 - D\left(M_t\left(\mathbf{x}_t\right)\right)\right)\right] \end{aligned}$$

where the labels indicate the origin domain (1 for the source domain and 0 for the target domain).

# Loss Function 3: Adversarial Loss

## The GAN loss function for optimizing this mapping

Here the generator aims to mimic another unchanging distribution. The standard choice can be:

$$\min_{M_t} \mathcal{L}_{\text{adv}_M}\left(\mathbf{X}_s, \mathbf{X}_t, D\right) = -\mathbb{E}_{\mathbf{x}_t \sim \mathbf{X}_t}\left[\log D\left(M_t\left(\mathbf{x}_t\right)\right)\right]$$

This objective has the same fixed-point properties as the minimax loss, but provides stronger gradients to the target mapping.

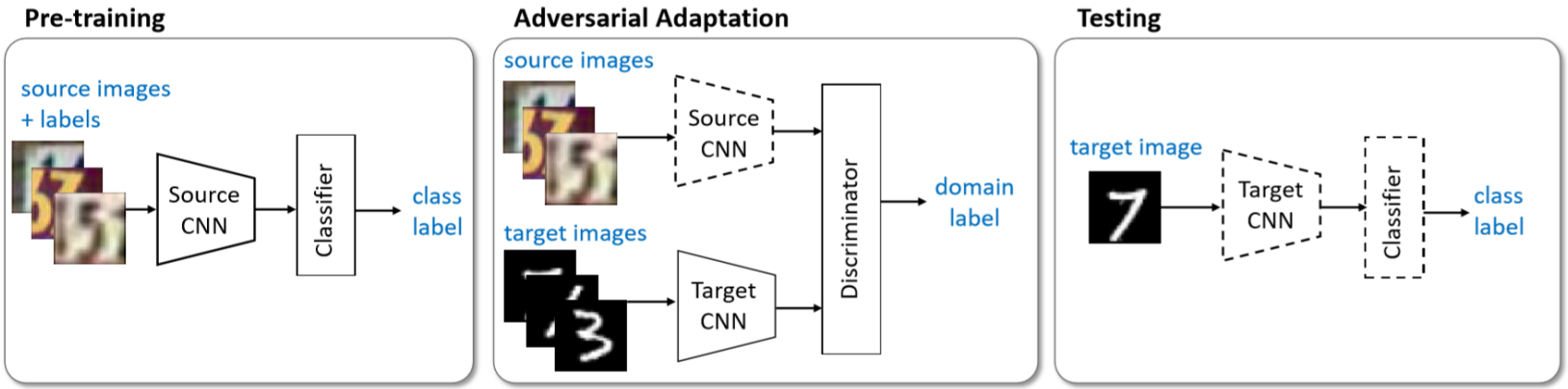
# Final Unconstrained Optimization

$$\min_{M_s, C} \quad \mathcal{L}_{\text{cls}}\left(\mathbf{X}_s, Y_s\right) = -\mathbb{E}_{(\mathbf{x}_s, y_s)\sim(\mathbf{X}_s, Y_s)} \sum_{k=1}^{K} \mathbf{1}_{[k=y_s]} \log C\left(M_s\left(\mathbf{x}_s\right)\right)$$

$$\min_{D} \quad \mathcal{L}_{\text{adv}_D}\left(\mathbf{X}_s, \mathbf{X}_t, M_s, M_t\right)$$

$$= \quad -\mathbb{E}_{\mathbf{x}_s\sim\mathbf{X}_s}\left[\log D\left(M_s\left(\mathbf{x}_s\right)\right)\right] - \mathbb{E}_{\mathbf{x}_t\sim\mathbf{X}_t}\left[\log\left(1 - D\left(M_t\left(\mathbf{x}_t\right)\right)\right)\right]$$

$$\min_{M_t} \quad \mathcal{L}_{\text{adv}_M}\left(\mathbf{X}_s, \mathbf{X}_t, D\right) = -\mathbb{E}_{\mathbf{x}_t\sim\mathbf{X}_t}\left[\log D\left(M_t\left(\mathbf{x}_t\right)\right)\right]$$



# Extract both Common and Specific Information

- Reconstruction-based approach [Bousmalis et al., 2016].
- Using an additional discriminator for common/specific classification [Tsai and Chien, 2017].

# Domain Separation Networks (DSN)

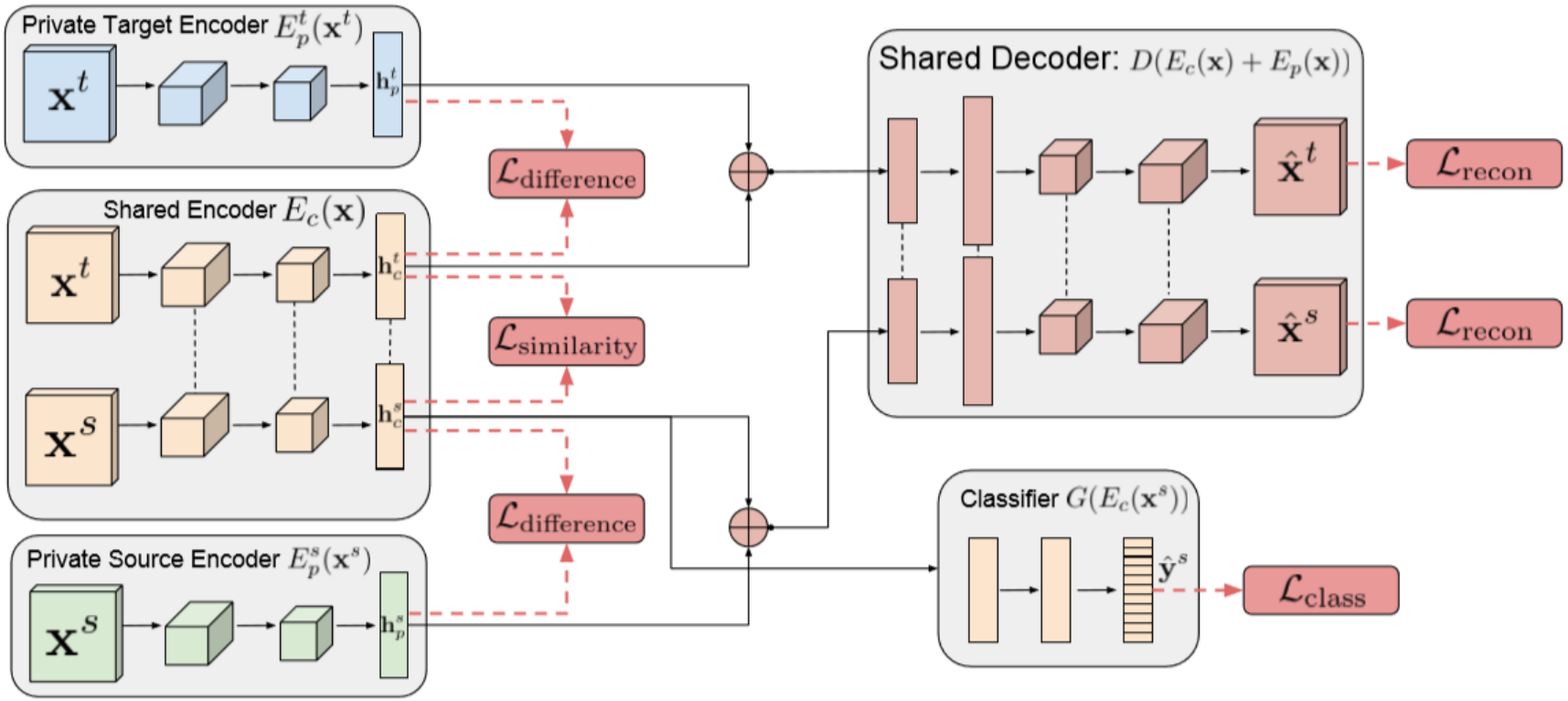



Figure: Training of Domain Separation Networks [Bousmalis et al., 2016].

# Performance Illustration

The reconstructed common and specific features by DSN [Bousmalis et al., 2016].

(a) MNIST (source) (b) MNIST-M (target) (c) Synth Objects (source) (d) LINEMOD (target)

Figure 2: Reconstructions for the representations of the two domains for "MNIST to MNIST-M" and for "Synth Objects to LINEMOD". In each block from left to right: the original image $\mathbf{x}_t$; reconstructed image $D(E_c(\mathbf{x}^t) + E_p(\mathbf{x}^t))$; shared only reconstruction $D(E_c(\mathbf{x}^t))$; private only reconstruction $D(E_p(\mathbf{x}^t))$.

# Adversarial Domain Separation and Adaptation

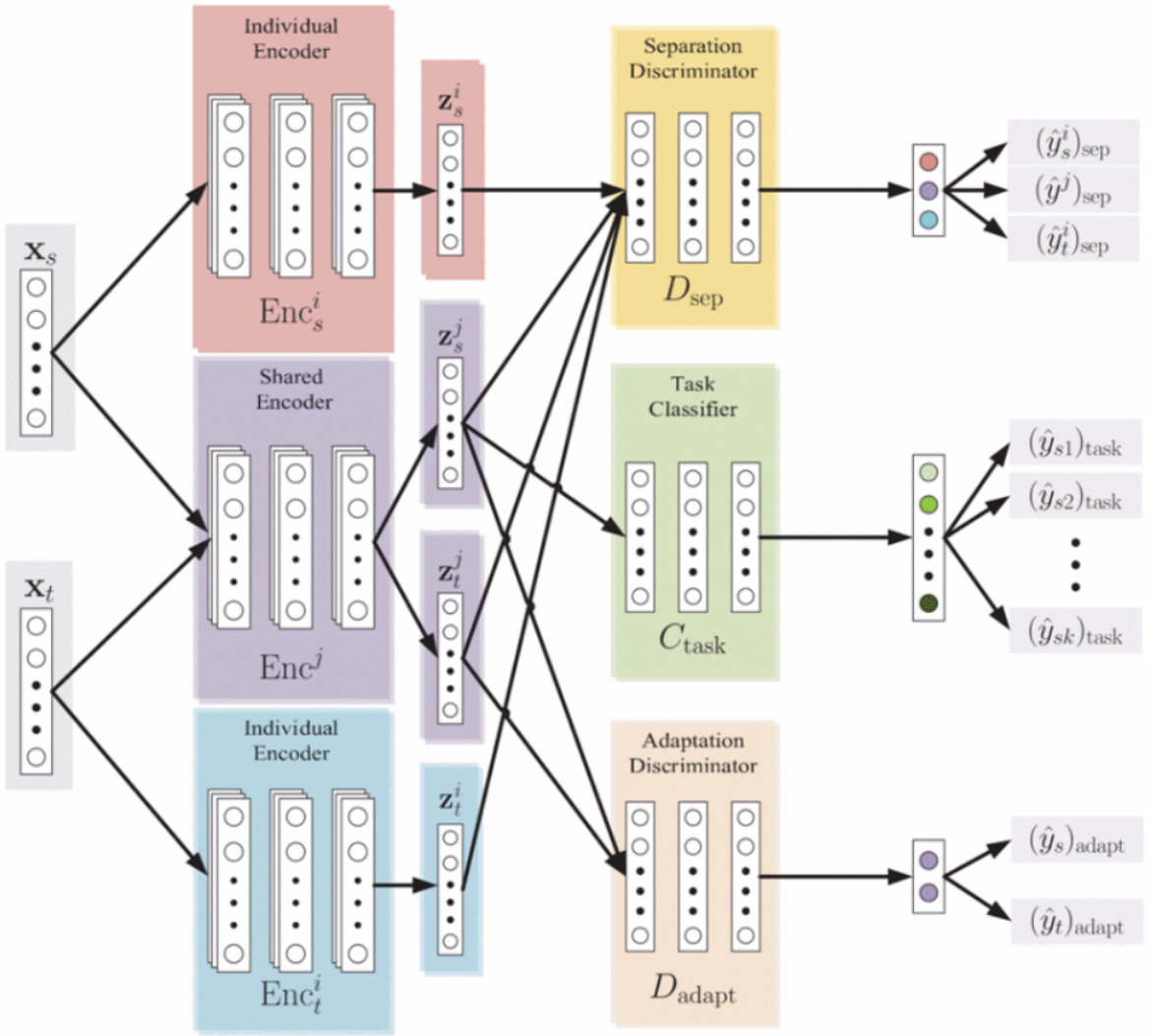



Figure: Overview of the ADSA structure [Tsai and Chien, 2017]

# Performance Comparison

ADSA jointly performed domain separation and adaptation to identify informative features for an improved classification performance [Tsai and Chien, 2017].

| | Baseline | VFAE | DAN | DSN | ADSA |
|---|---|---|---|---|---|
| B → D | 77.2 | 77.6 | 78.4 | 78.2 | **80.2** |
| B → E | 70.3 | 69.0 | 73.3 | 71.8 | **78.3** |
| B → K | 72.8 | 70.3 | 77.9 | 76.6 | **81.3** |
| D → B | 74.2 | 71.4 | 72.3 | 73.6 | **76.4** |
| D → E | 71.8 | 71.2 | 75.4 | 74.8 | **77.7** |
| D → K | 75.6 | 73.5 | 78.3 | 79.7 | **80.1** |
| E → B | 70.9 | 68.8 | 71.1 | 69.7 | **72.6** |
| E → D | 67.9 | 69.7 | 73.8 | 74.1 | **74.5** |
| E → K | 73.0 | 81.5 | 85.4 | **86.4** | 85.5 |
| K → B | 66.9 | 66.8 | 70.9 | 69.4 | **72.5** |
| K → D | 68.0 | 68.4 | 74.0 | 71.3 | **75.8** |
| K → E | 82.5 | 82.4 | **84.3** | 83.0 | 83.6 |
| Average | *73.4* | *72.5* | *76.3* | *76.1* | *78.2* |

Figure: Sentiment classification accuracies (%) for adaptation among different domains

# Overall Summary

- The basic manifold alignment problem
- Feature-level alignment (linear)
- Semi- or unsupervised MA: correspondence inference
- Manifold alignment with deep learning

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