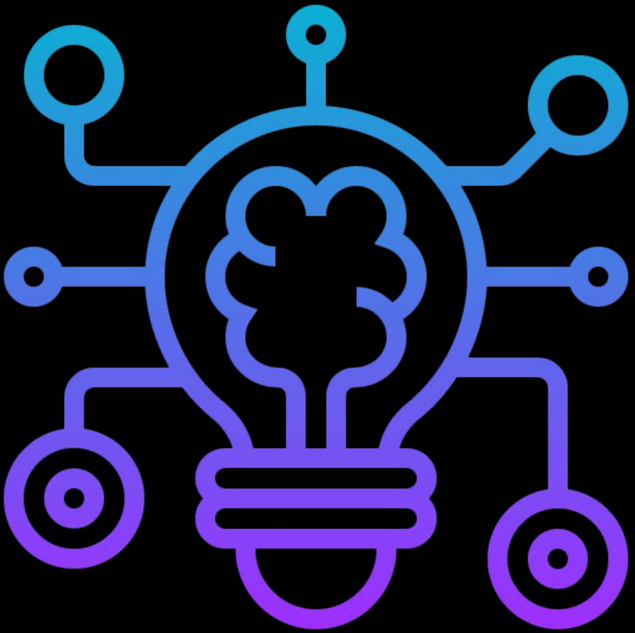

Search (I)

Mingsheng Long

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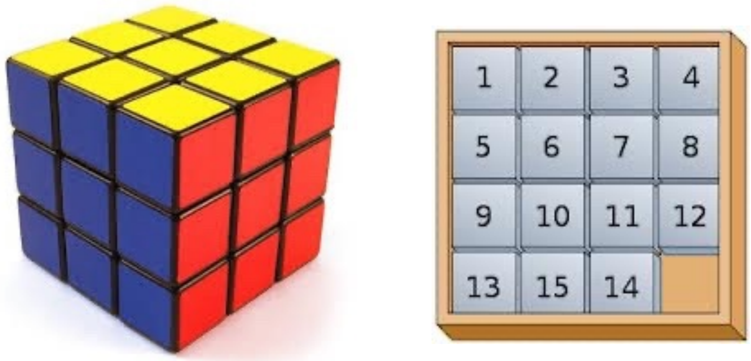
Spring 2023

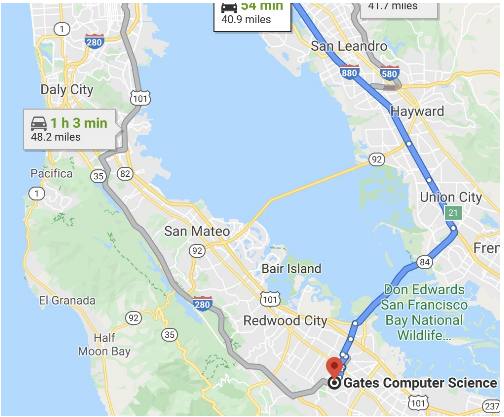
Outline

- **Search Problems**
- General Search
  - Tree Search, Graph Search
- Uninformed Search
  - Depth-First (DFS), Breadth-First (BFS), Uniform-Cost (UCS)
- Informed (Heuristic) Search
  - A$*$ Tree Search
  - A$*$ Graph Search

Search Problems in Real Applications



Puzzle solving



Route finding

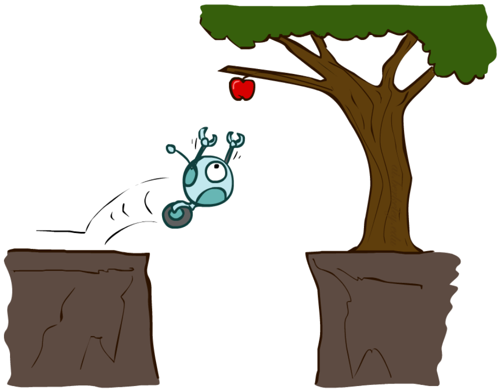
Robot motion planning

Multi-robot systems

Reflex Agents vs. Planning Agents

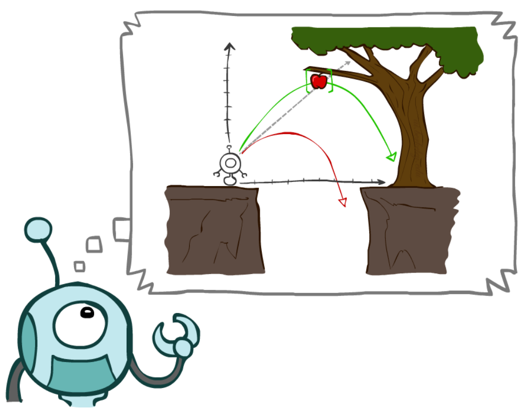
- Reflex Agents:

$$\text{percept } x \rightarrow \boxed{f} \rightarrow \text{single action } y$$

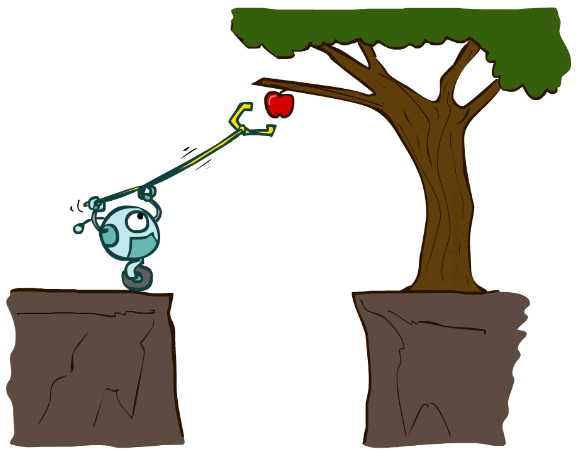


- Planning Agents:

$$\text{percept } x \rightarrow \boxed{f} \rightarrow \text{action sequence } (a_1, a_2, a_3, a_4, \cdots)$$



A model of how the world evolves in response to actions

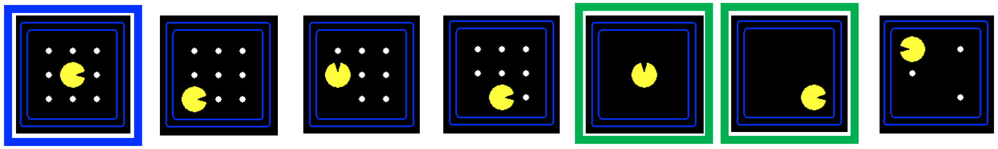


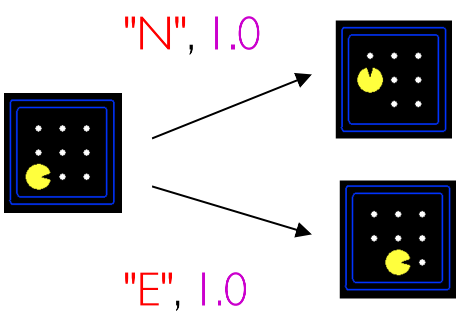
Action sequences to achieve a definite goal

Search Problems

- A search problem consists of:
  - A state space S
  - An initial state s_0
  - Actions $A(s)$ in each state
  - Transition model $\mathrm{Result}(s, a)$
    - A successor function
  - A goal test $G(s)$
  - Action cost $c(s, a, s')$

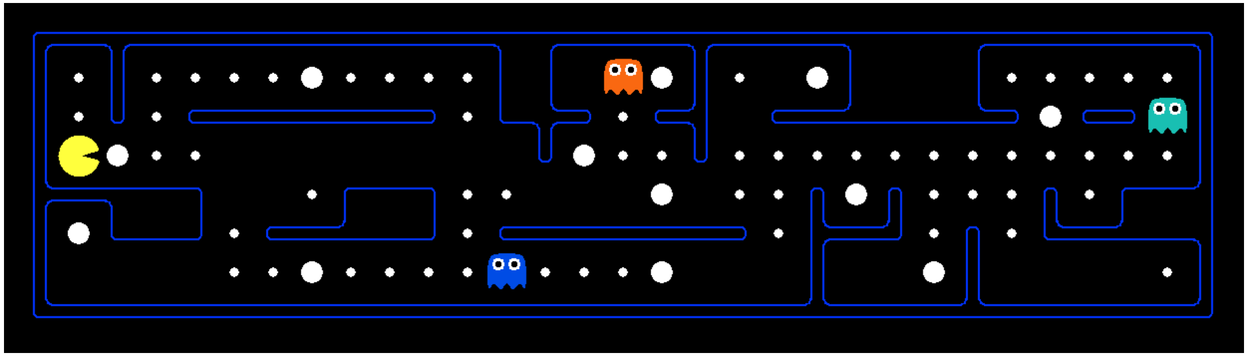
Pac-Man (吃豆人), 1980





- A solution is an action sequence that reaches a goal state
- An optimal solution has the least cost among all solutions

State Space



The world state includes every last detail of the environment

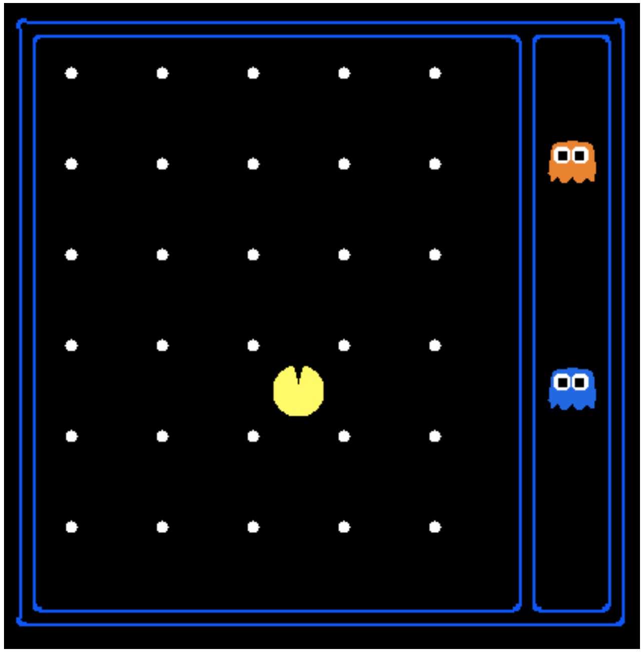
A search state abstracts away details not needed to solve the problem

- Problem: Pathing
  - States: (x,y) location
  - Actions: NSEW
  - Transition: update location only
  - Goal test: is (x,y)=END

- Problem: Eat-All-Dots
  - States: {(x,y), dot Booleans}
  - Actions: NSEW
  - Transition: update location and possibly a dot Boolean
  - Goal test: dots all false

State Space

- World state:
  - Agent positions: 120
  - Food count: 30
  - Ghost positions: 12
  - Agent facing: NSEW
- How many
  - World states: $120 \times (2^{30}) \times (12^{2}) \times 4$
  - States for pathing: 120
  - States for eat-all-dots: $120 \times (2^{30})$



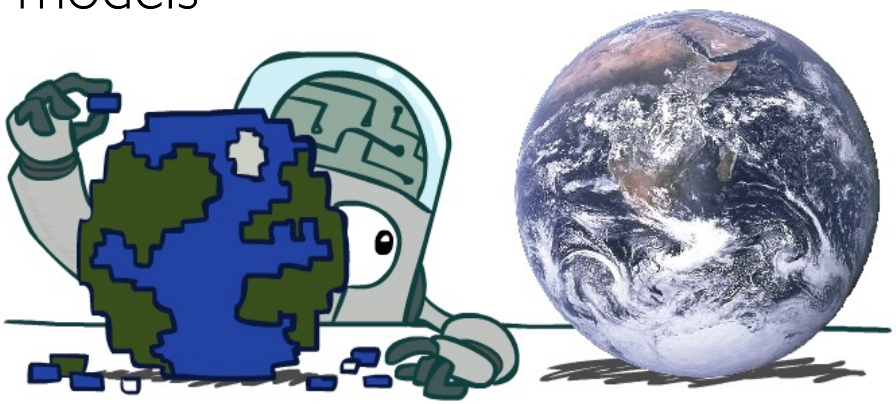
Usually exponential
(NP-hard)

- The size of the search space depends on the problem being solved

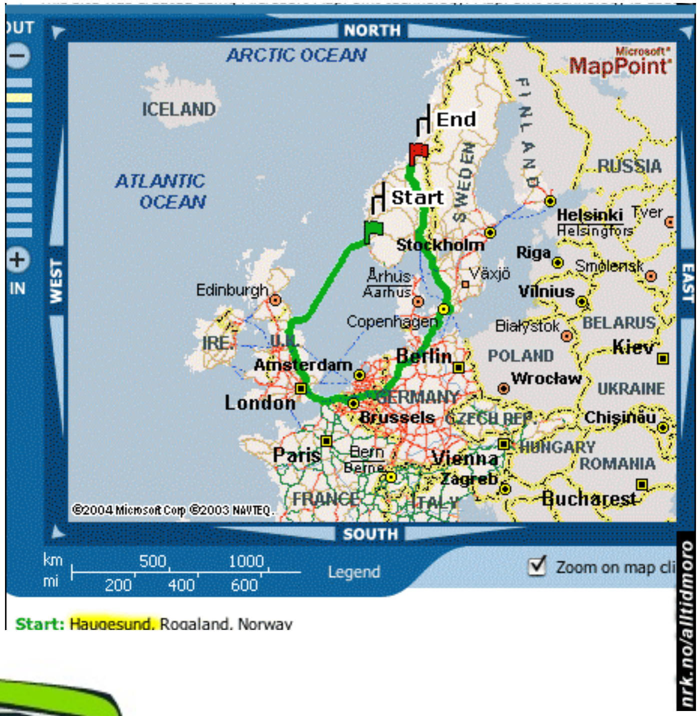
Search Models

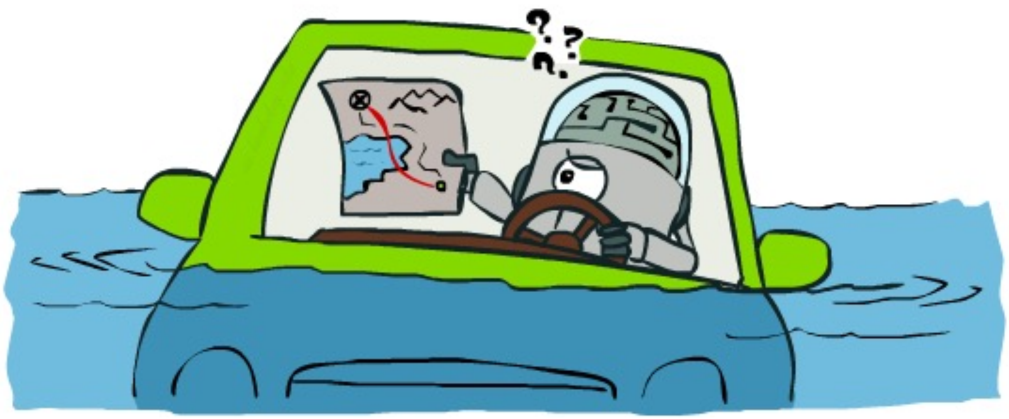
- A search problem is reasonable but not the real thing
- It is a model, an abstract mathematical description of real problems
- Search operates over models of the world
  - The agent doesn't actually try all the plans out in the real world
  - Planning is all in simulation (rollouts)
  - Search is only as good as your models

  - All models are wrong…
  - But some are useful 😄



Gone Wrong with Search Models





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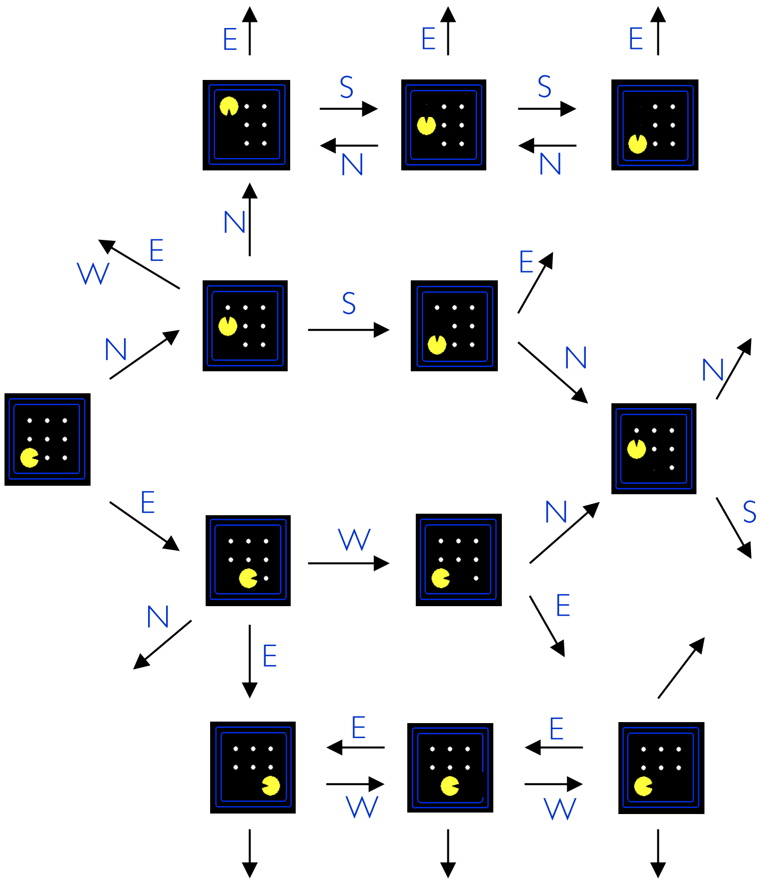
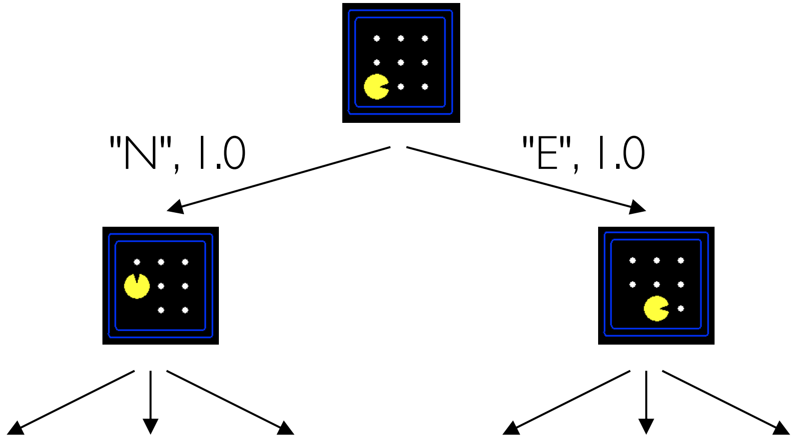
State Space Graph

- State space graph: A mathematical representation of a search problem
  - Nodes are states
  - Arcs represent transitions
  - The goal test is a set of goal nodes (maybe only one)
- Each state occurs only once
- We can rarely build the full graph in memory
  - It is too big but is a useful idea



Search Tree

- **Search tree**: A "what if" tree of plans and their outcomes
  - The start state is the **root node**
  - **Children** correspond to successors
  - **Nodes** show states, but correspond to plans that achieve those states
- For most problems, we can never actually build the whole tree



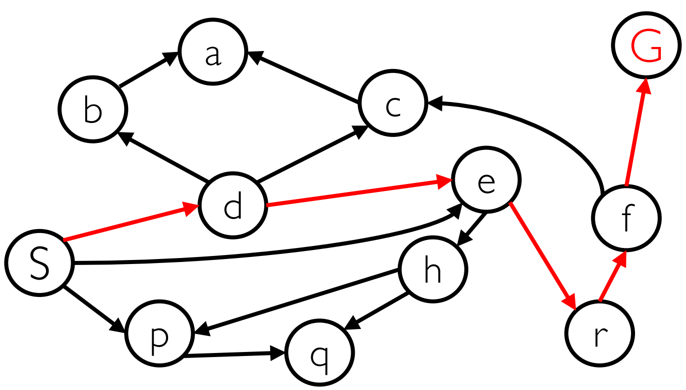
This is now / start

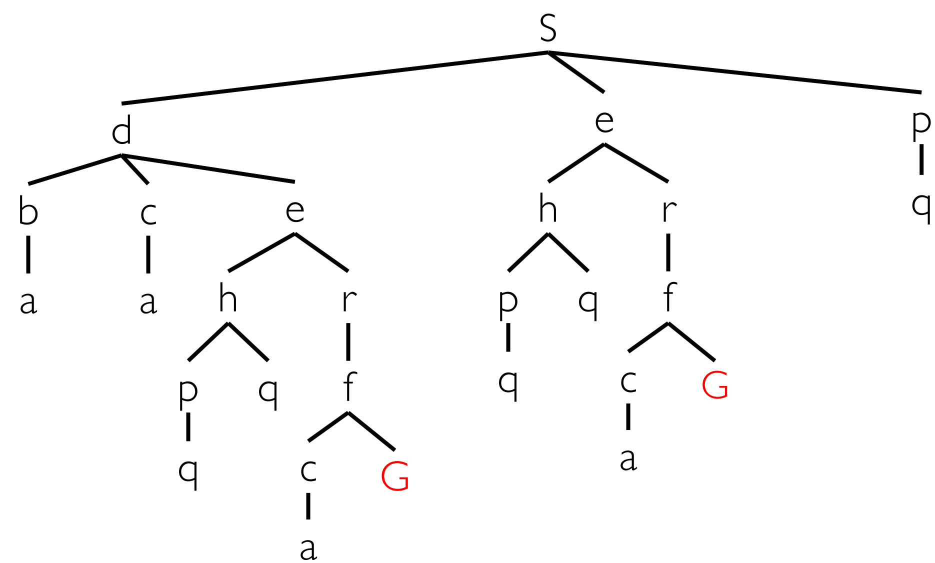
Possible futures

State Space Graph vs. Search Tree

- Each node in the search tree is an entire path in the state space graph.
- We construct the tree on demand – and we construct as little as possible.

How big is this search tree? ∞





General Tree Search

```
function TREE-SEARCH(problem) returns a solution, or failure
    initialize the frontier as a specific work list (stack, queue, priority queue)
    add initial state of problem to frontier
```

(1). Initialization

```
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution

```

(2). Selection

```
        for each resulting child from node
            add child to the frontier
```

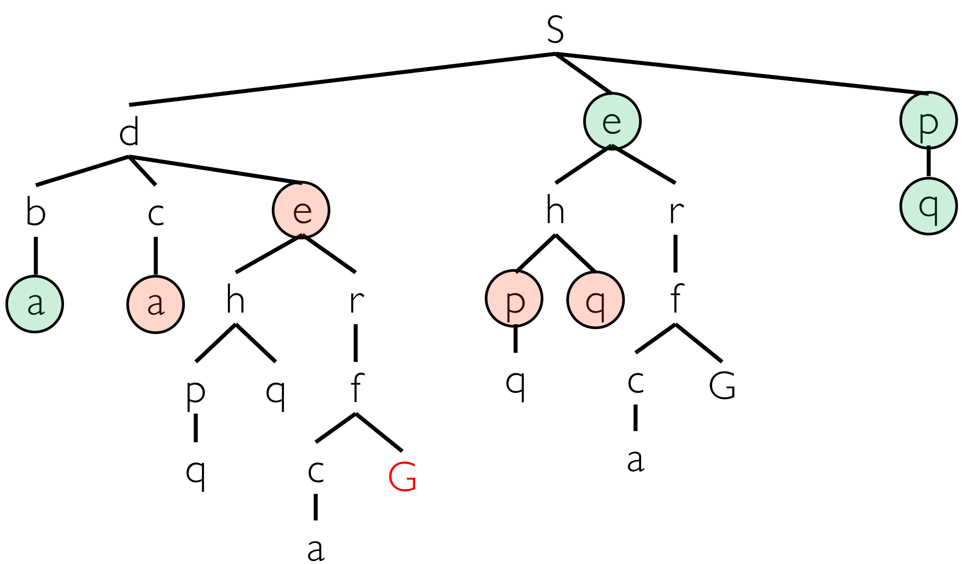
(3). Expansion

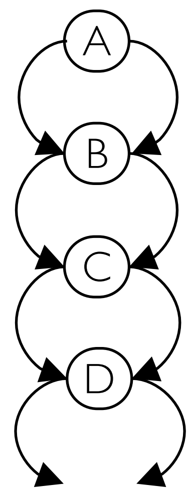
Frontier: leaf nodes for expansion

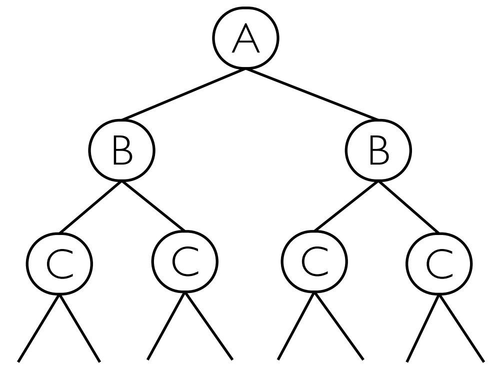
Strategy: How to choose a leaf node to expand

Tree Search: Extra Work

- Repeated states and redundant paths can cause a tractable problem to become intractable.







In BFS, for example, we should not bother expanding the red circled nodes

Failure to detect repeated states can cause exponentially more work

General Graph Search

```
function GRAPH-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a specific work list (stack, queue, priority queue)
    add initial state of problem to frontier
```

(1). Initialization

```
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
```

Strategy: How to choose a leaf node to expand

(2). Selection

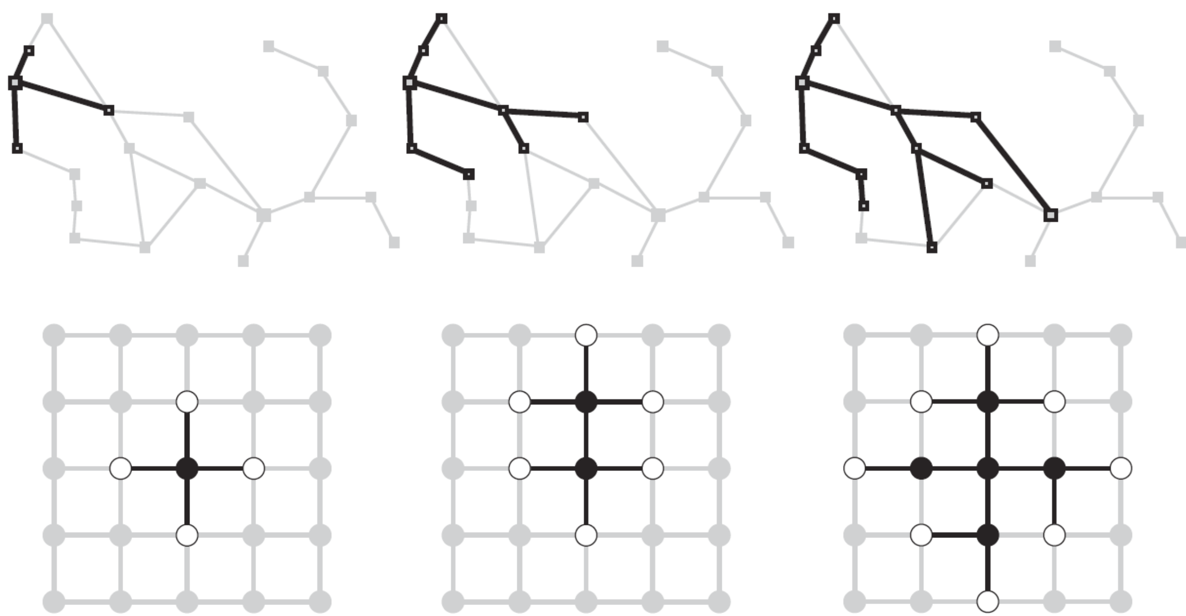
```
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
```

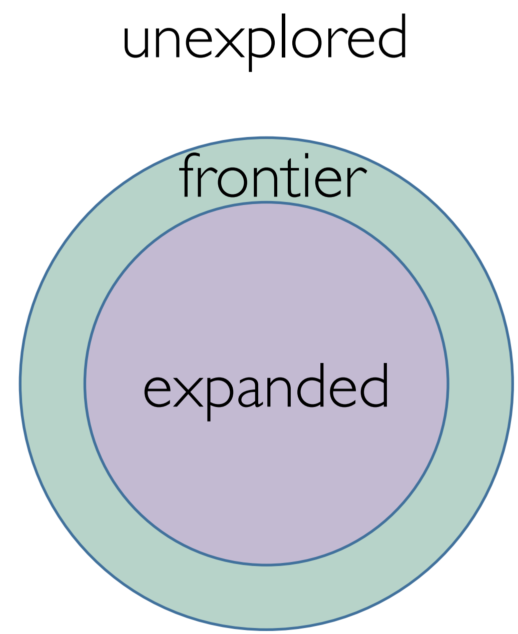
Idea: never expand a state twice

(3). Expansion

Graph Search: Frontier Separation

- This graph search algorithm overlays a growing tree on a graph
- Frontier separates expanded region from unexplored region of the state-space graph





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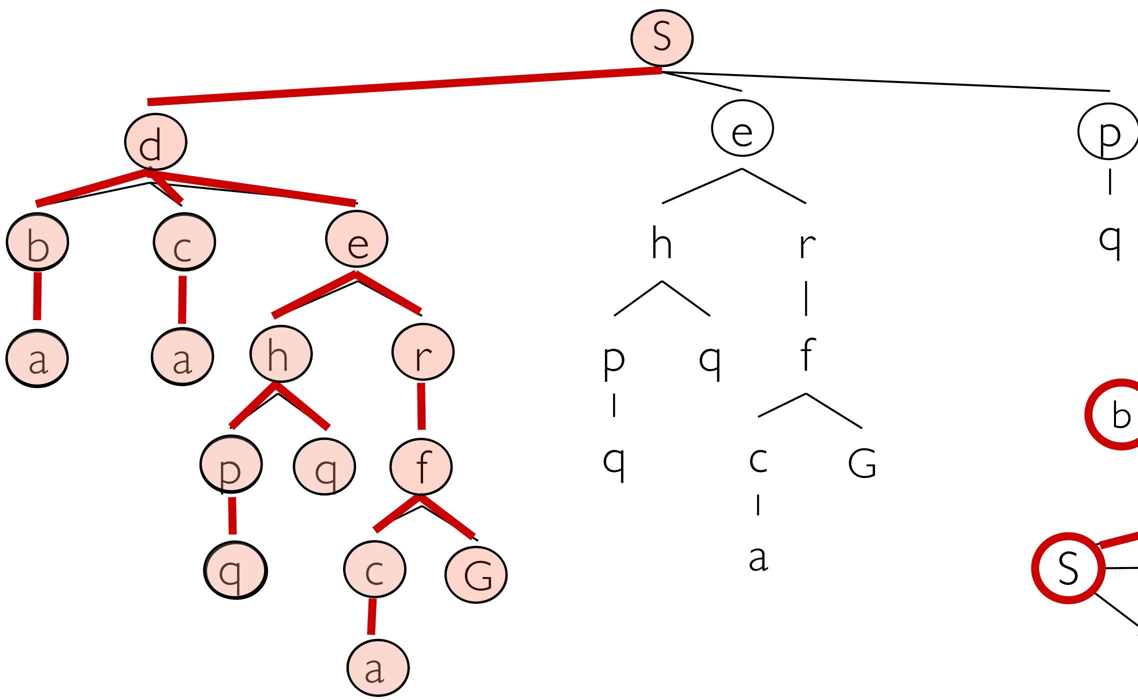
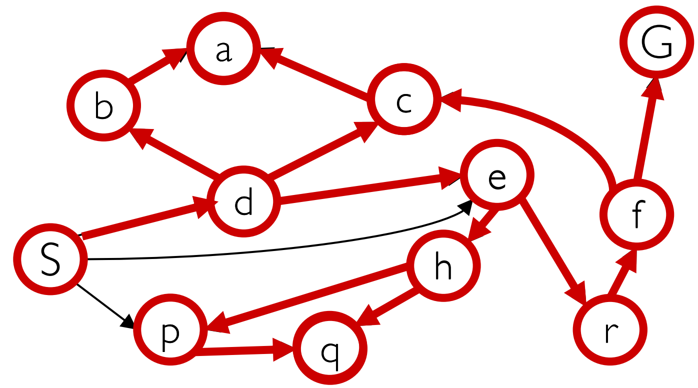
Depth-First Search (DFS)

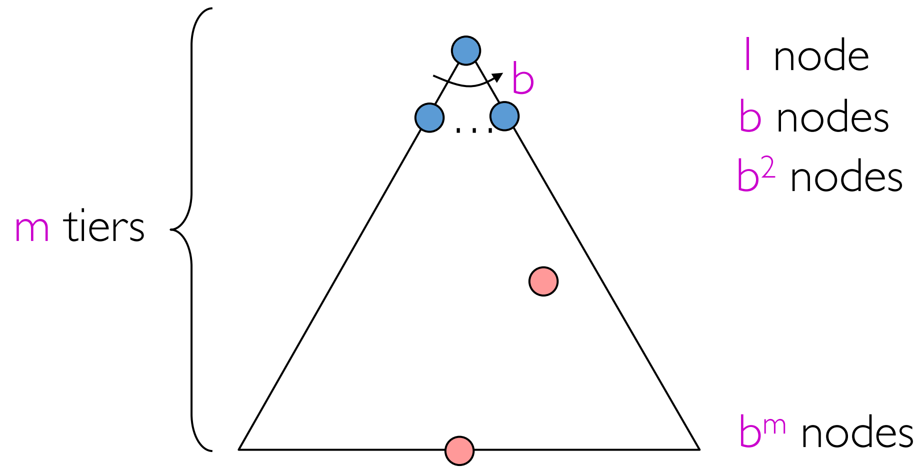
- **Strategy**: expand a deepest node first
- **Implementation:** Frontier is a LIFO stack





Search Algorithm Properties

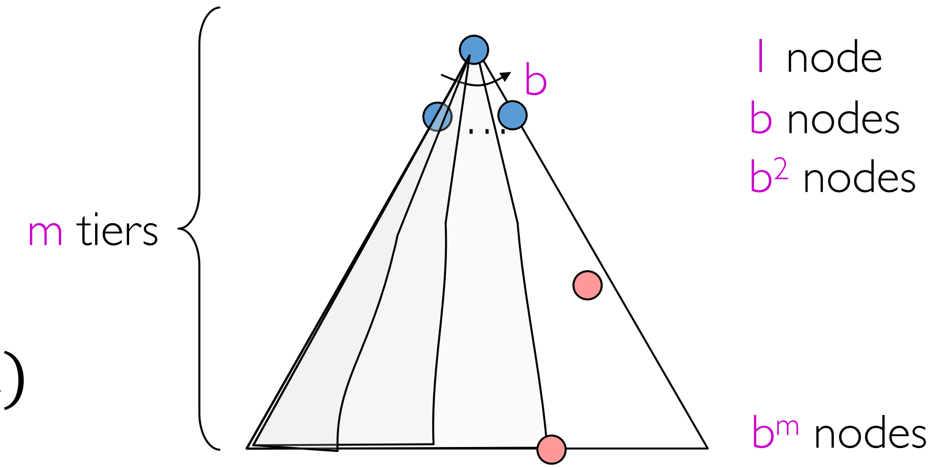
- Properties
  - Complete: Guaranteed to find a solution if one exists
  - Optimal: Guaranteed to find the least cost path
  - Time complexity
  - Space complexity
- Cartoon of search tree:
  - b is the branching factor
  - m is the maximum depth
  - Solutions at various depths
  - Number of nodes in entire tree



$$1 + b + b^2 + \ldots . b^m = O(b^m)$$

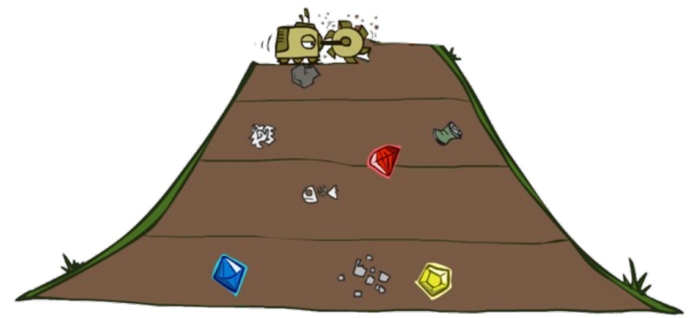
Depth-First Search Properties

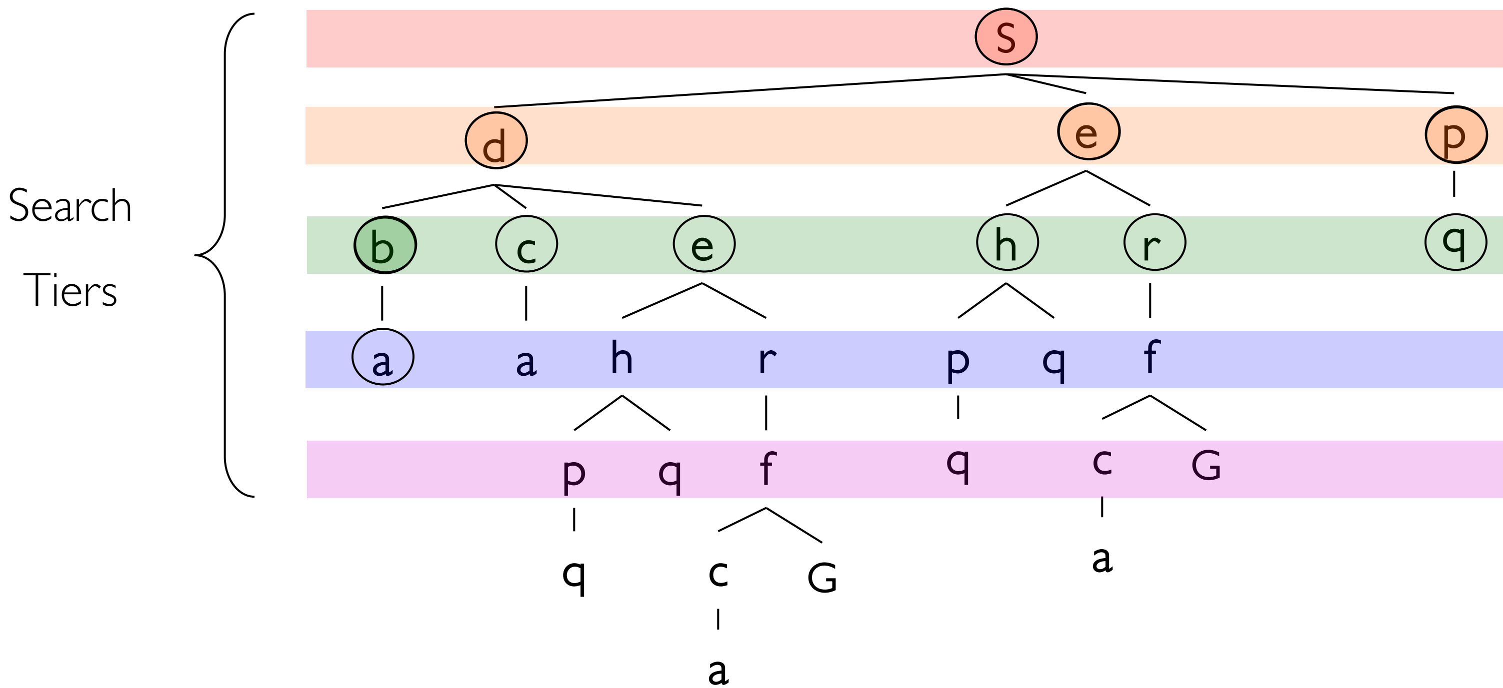
- Time complexity: what nodes DFS expand?
  - Some left prefix of the tree
  - Could process the whole tree
  - If m is finite, takes time $O(b^m)$
- Space complexity
  - Only siblings on path, so $O(bm)$
- Is it complete?
  - m could be infinite, so only if we prevent cycles (Graph Search)
- Is it optimal?
  - No, it finds the leftmost solution, regardless of depth or cost



Breadth-First Search (BFS)

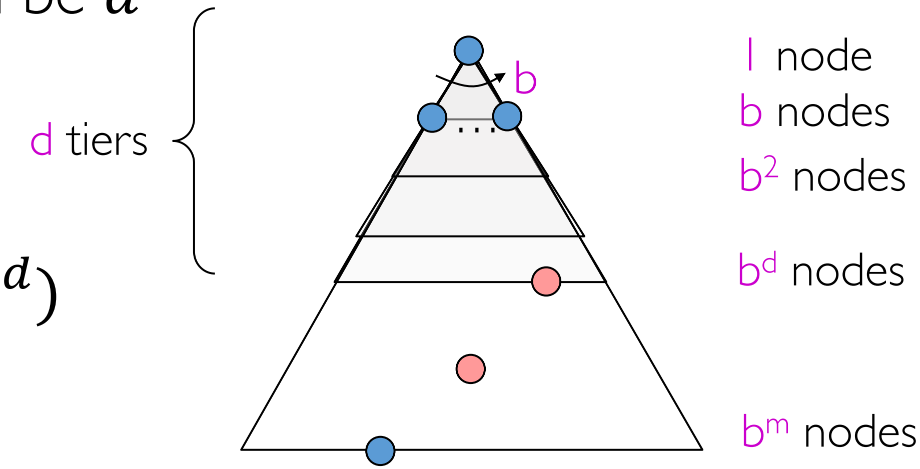
- Strategy: expand a shallowest node first
- Implementation: Frontier is a FIFO queue





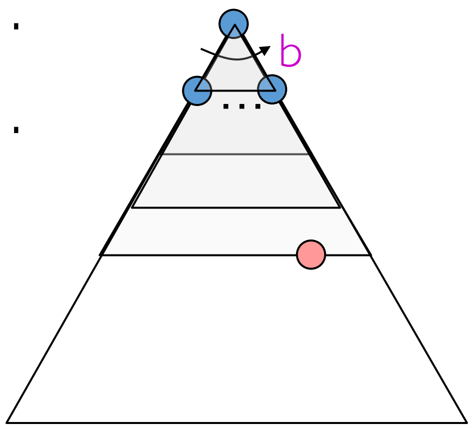
Breadth-First Search Properties

- Time complexity: what nodes does BFS expand?
  - Processes all nodes above shallowest solution
  - Let depth of shallowest solution be d
  - Search takes time $O(b^d)$
- Space complexity
  - Has roughly the last tier, so $O(b^d)$
- Is it complete?
  - d must be finite if a solution exists, so yes
- Is it optimal?
  - If costs are equal (e.g., 1)
  - If not, Uniform Cost Search (more later)



Iterative Deepening Search (IDS)

- **Idea:** get DFS's space advantage with BFS's time & shallow-solution advantages
  - Run a DFS with *depth limit 1*. If no solution…
  - Run a DFS with *depth limit 2*. If no solution…
  - Run a DFS with *depth limit 3*. …



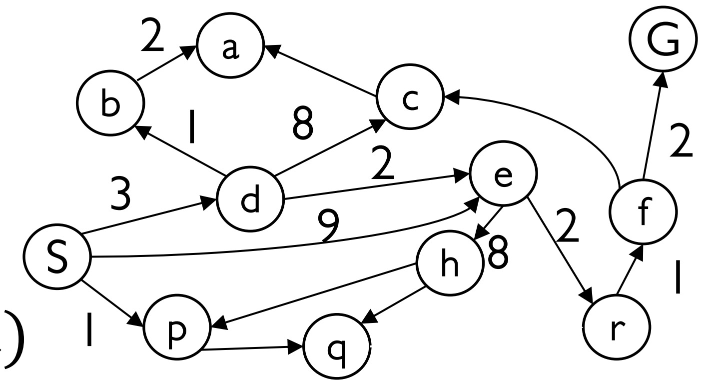
- Complete? Yes. Optimal? Yes.
- Isn't that wastefully redundant? Not so bad:

$$O(1) + O(b) + O(b^2) + \cdots + O(b^d) = O(b^d)$$

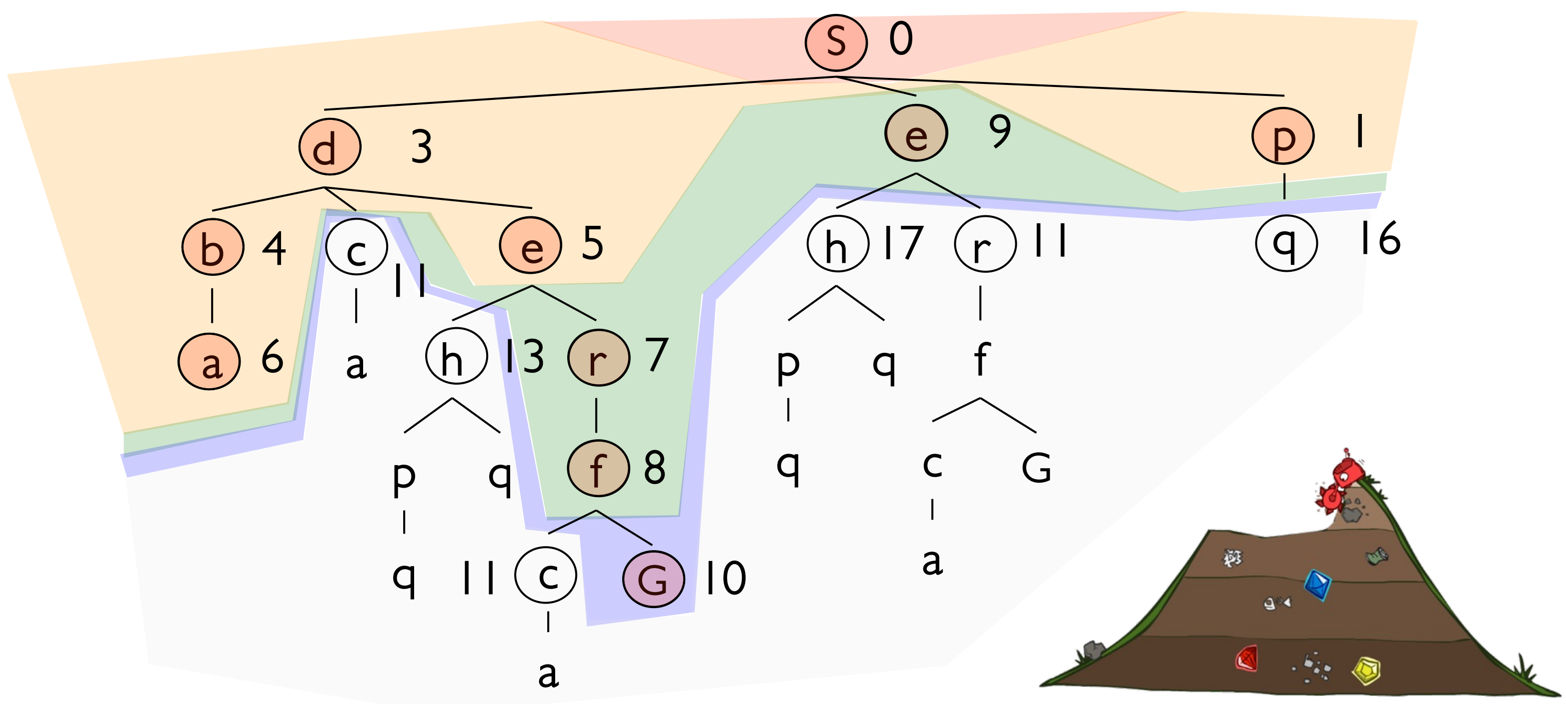
- Time complexity: $O(b^d)$
- Space complexity: $O(bd)$

[Dijkstra, 1956] Uniform Cost Search (UCS)

- **Strategy**: expand lowest $g(n)$
  - $g(n)$ = cost from root to n
- Frontier is a priority queue sorted by $g(n)$

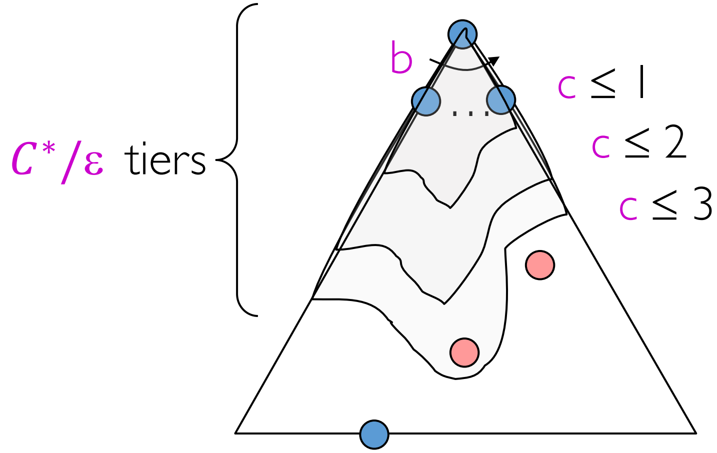


Cost contours



Uniform Cost Search Properties

- Time complexity: what nodes does UCS expand?
  - Processes all nodes with cost less than the cheapest solution
  - If that solution costs C^* and arcs cost at least ε , then the "effective depth" is roughly C^*/ϵ
  - Takes time $O(b^{C^*/\varepsilon})$
- Space complexity
  - Has roughly the last tier, so $O(b^{C^*/\varepsilon})$



- Is it complete?
  - Assuming C^* is finite and $\varepsilon > 0$, yes
- Is it optimal?
  - Yes (Proof via A∗)

Uniform Cost (Graph) Search

```
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using node path-cost as the priority
    add initial state of problem to frontier with path-cost = 0
```

(1). Initialization

```
    loop do
        if the frontier is empty then return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
```

(2). Selection

```
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher path-cost then
                replace that frontier node with child
```

(3). Expansion

Key idea: extra check

A shorter path to a frontier state is discovered.

Uninformed Search: Summary

- Tree Search

Criterion	Action Costs	Complete?	Optimal?	Time	Space
Depth-First	$= c$	No	No	$O(b^m)$	$O(bm)$
Breadth-First	$= c$	Yes	Yes	$O(b^d)$	$O(b^d)$
Iterative Deepening	$= c$	Yes	Yes	$O(b^d)$	$O(bd)$
Uniform-Cost	$\geq \varepsilon$	Yes	Yes	$O(b^{C^*/\varepsilon})$	$O(b^{C^*/\varepsilon})$

- Graph Search
  - Depth-first search is complete for finite state spaces
  - The space and time complexities are bounded by the size of the state space
- All these search algorithms are the same except for frontier strategies
  - Can code one implementation that takes a variable queuing object

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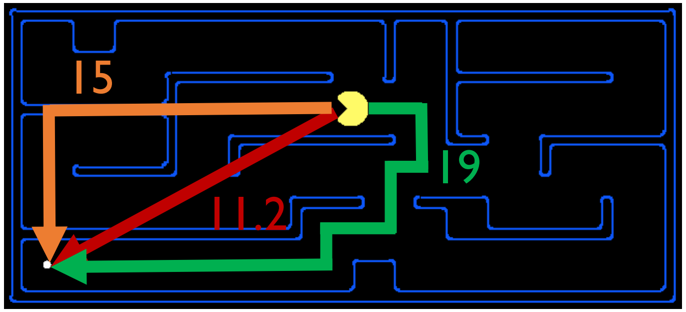
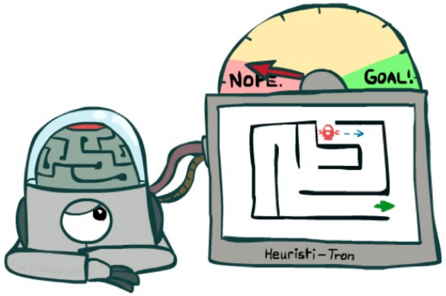
  - A∗ Tree Search
  - A∗ Graph Search

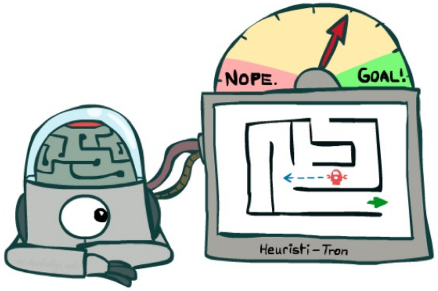
Informed (Heuristic) Search

- Issues of Uniform Cost Search
  - Explores options in every "direction"
  - No information about goal location



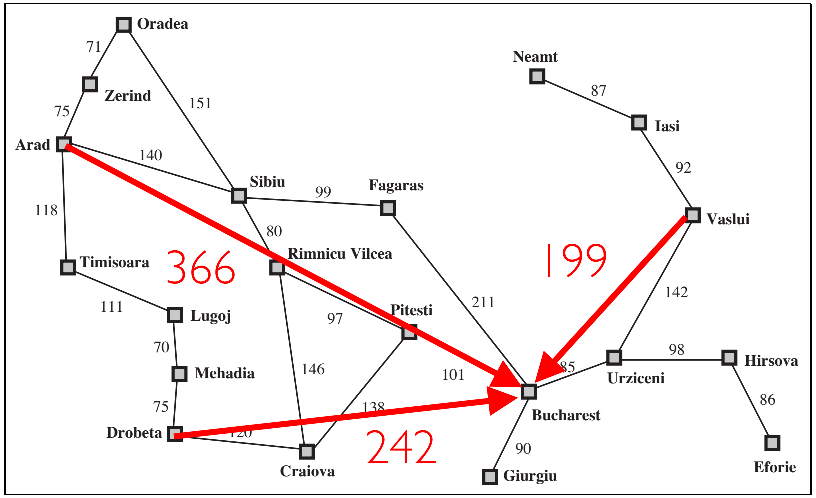
- A heuristic is:
  - A function that estimates how close a state is to a goal
  - Designed for a particular search problem
  - Examples: Manhattan distance, Euclidean distance for pathing





Example: Heuristic Function

- Route finding: straight-line distance



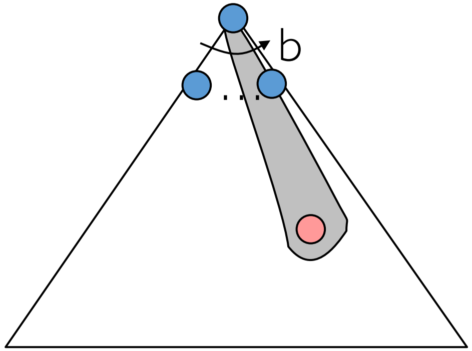
Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

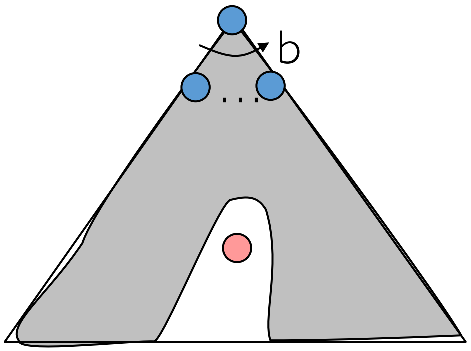
Figure 3.22 Values of h_{SLD}—straight-line distances to Bucharest.

Greedy Search

- Strategy: expand a node that you think is closest to a goal state
  - $h(n)$ = heuristic of state n
- Frontier is an ascending order priority queue by $h(n)$
- **Best-first** takes you straight to the (wrong) goal
  - A common case

- Worst-case: like a badly-guided DFS

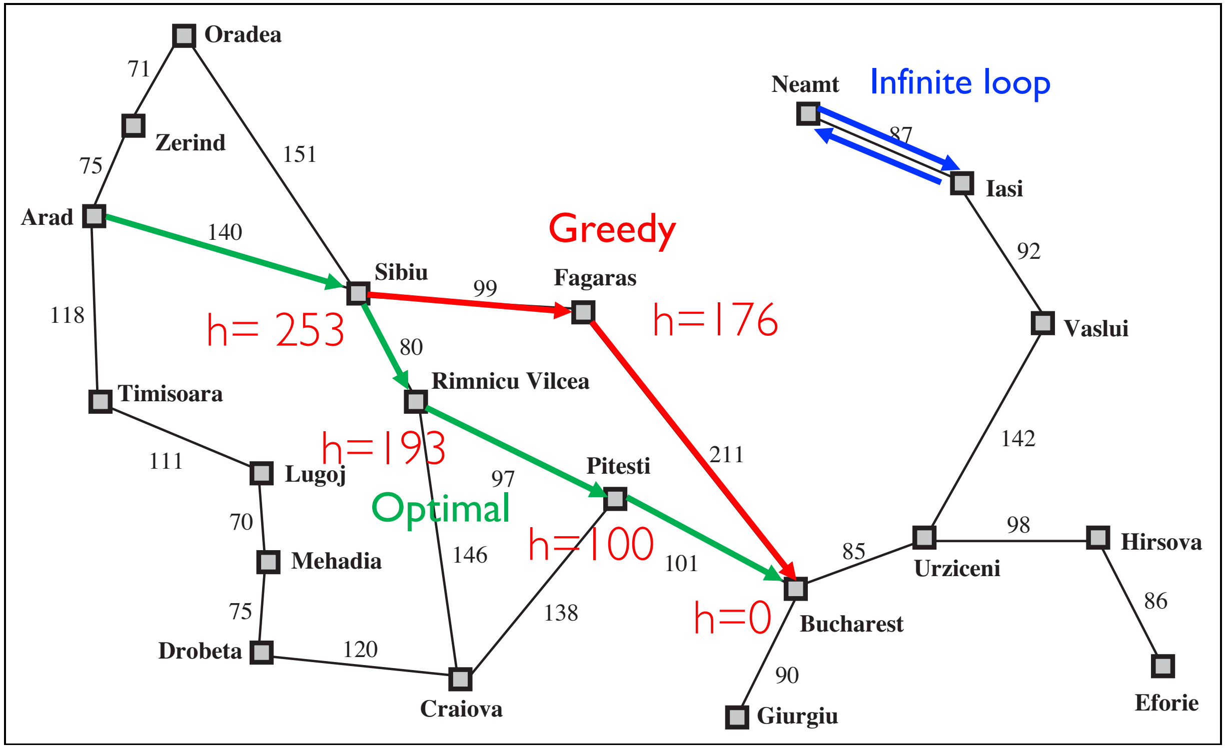




Greedy Search Properties

- Greedy Tree Search
  - Is it complete? No. Example: Isai → Fagaras, leads to infinite loop
- Greedy Graph Search: avoid repeated state
  - Is it optimal? No.

Sibiu-Fagaras-Bucharest =
99+211 = 310
Sibiu-Rimnicu Vilcea-Pitesti-Bucharest =
80+97+101 = 278



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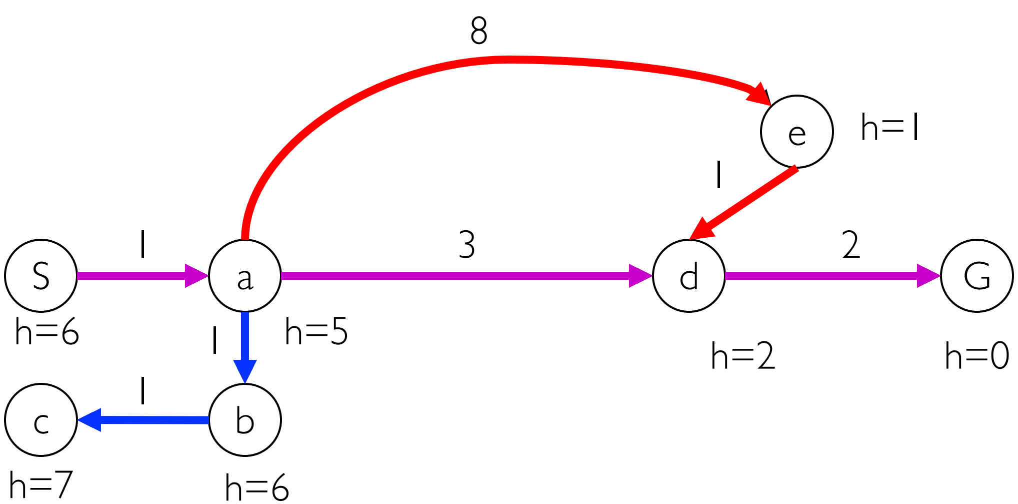
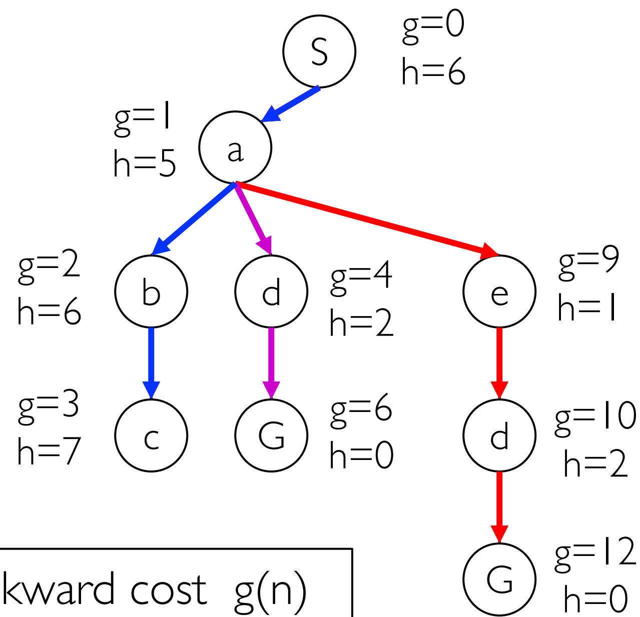
[Hart/Nilsson/Raphael, 1968]

A∗ Search

- **Strategy**: Combining UCS and Greedy Search
  - Sorted by $f(n) = g(n) + h(n)$



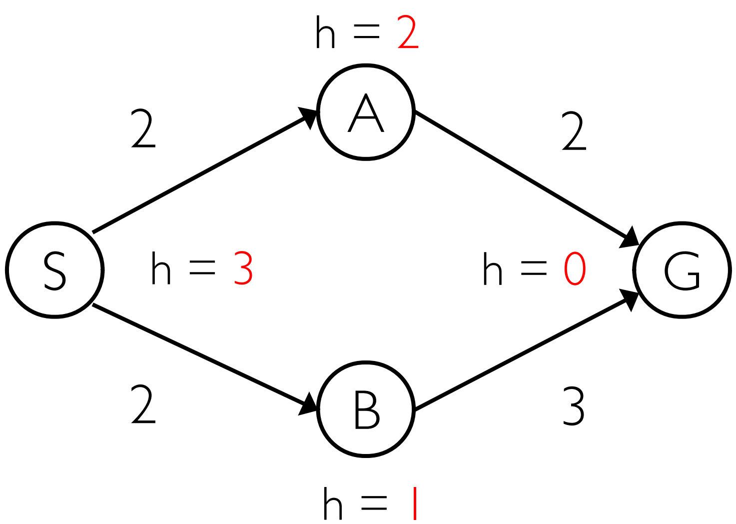


Uniform-cost orders by path cost, or backward cost g(n)
Greedy orders by goal proximity, or forward cost h(n)
A∗ Search orders by the sum: f(n) = g(n) + h(n)

When Should A* Terminate?

- Should we stop when we enqueue a goal?

State space graph



Queue

S(0+3)

S-B(2+1), S-A(2+2)

S-A(2+2), S-B-G(5+0)

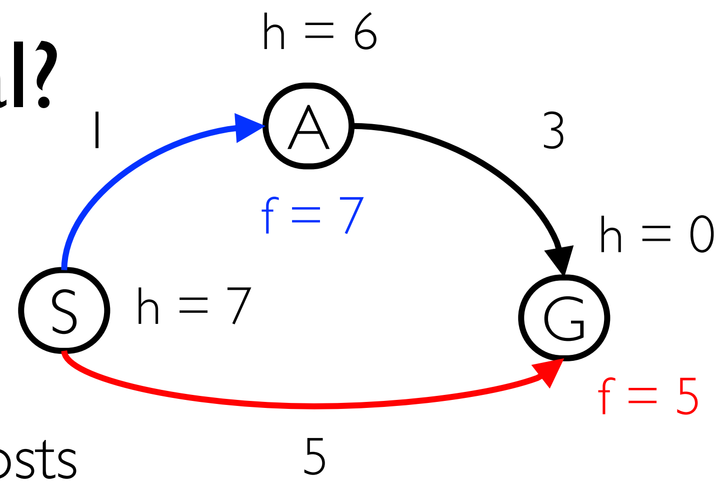
S-A-G(4+0), S-B-G(5+0)

First goal enqueued

First goal dequeued

No: only stop when we dequeue a goal

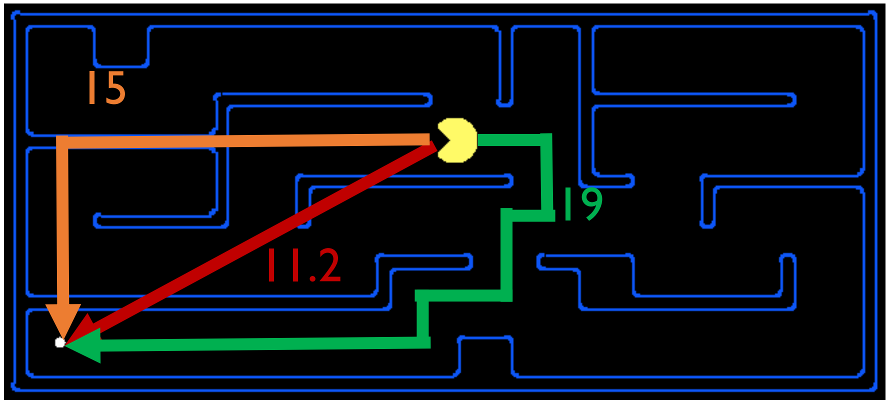
Is A∗ Optimal?



- Example: What went wrong?
  - Over-estimate good goal cost
  - Need estimates to be less than actual costs
- Admissible Heuristics: A heuristic h is admissible (可采纳) if:

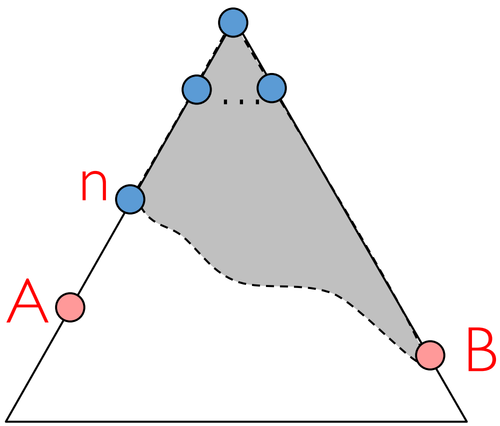
$$0 \leq h(n) \leq h^*(n)$$

  - where $h^*(n)$ is the true cost to a nearest goal
- Example:



Optimality of A* Tree Search

- Assume:
  - A is an optimal goal node, B is a suboptimal goal node
  - h is admissible
- Claim: A will exit the frontier before B
- Proof:
  - Imagine B is on the frontier
  - Some ancestor n of A on the optimal path (maybe A itself) is also on the frontier
  - Claim: n will be expanded before B
  - All ancestors of A expand before B
  - A expands before B



Optimality of A* Tree Search

- Claim: n will be expanded before B
- Proof:
  - $f(n)$ is less or equal to $f(A)$:

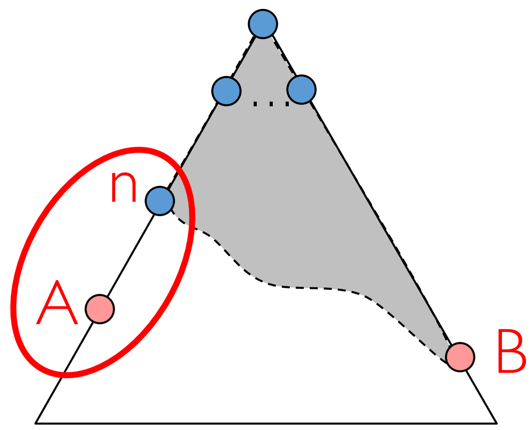
$$\underbrace{f(n) = g(n) + h(n)}_{\text{Definition of } f} \underbrace{\leq}_{\text{Admissibility}} \underbrace{g(A) = f(A)}_{h=0 \text{ at a goal}}$$

  - $f(A)$ is less than $f(B)$:

$$\underbrace{f(A) = g(A)}_{h=0 \text{ at a goal}} \underbrace{<}_{B \text{ is suboptimal}} \underbrace{g(B) = f(B)}_{h=0 \text{ at a goal}}$$

  - n expands before B:

$$f(n) \leq f(A) < f(B)$$



Efficiency of A∗

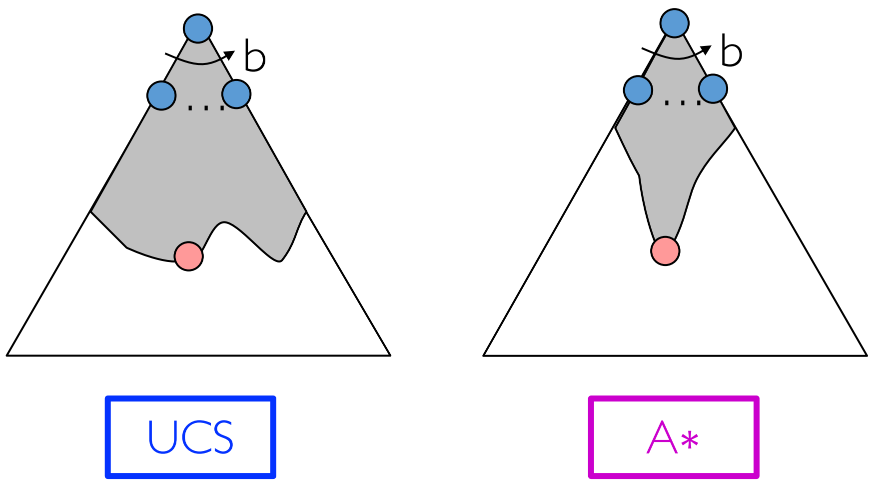
- **Theorem**: efficiency of A∗
  - A∗ explores all states s satisfying

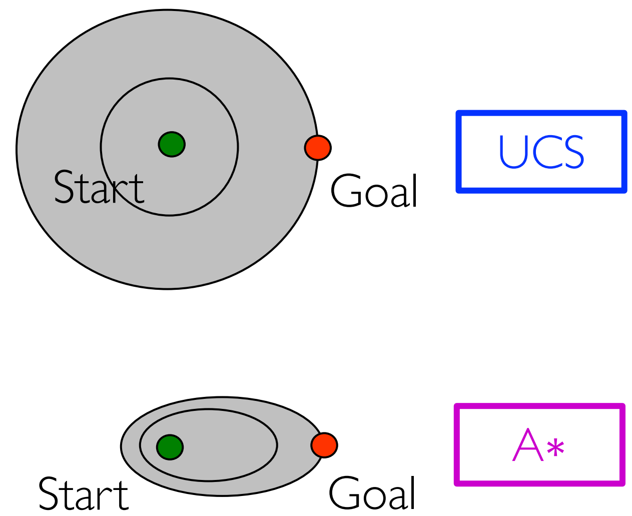
$$g(s) \leq g(s_{\text{goal}}) - h(s)$$

- **Interpretation**: the larger $h(s)$, the better

Key idea: distortion

A∗ distorts edge costs to favor goal states



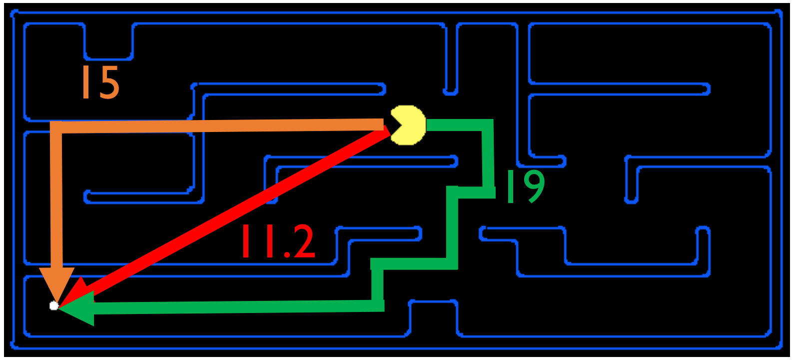


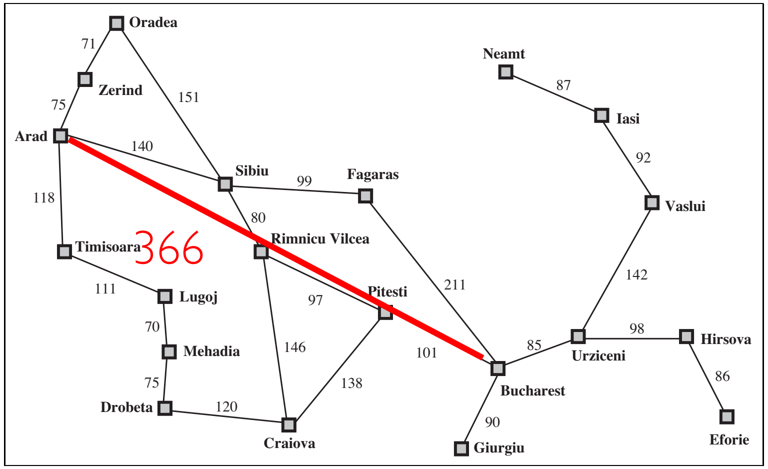
Creating Heuristics

- Admissible heuristics are often solutions to relaxed problems

Definition: relaxed problems
Problem P_2 is a relaxed version of P_1 if $A_2(s) \supseteq A_1(s)$ for every s

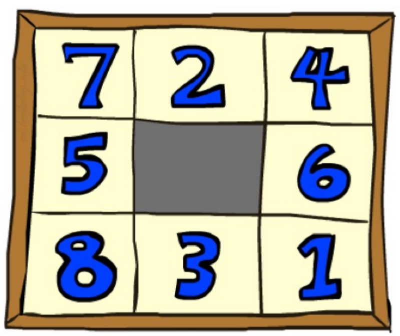
Theorem:
$h_2^*(s) \leq h_1^*(s)$ for every s, so $h_2^*(s)$ is admissible for P_1



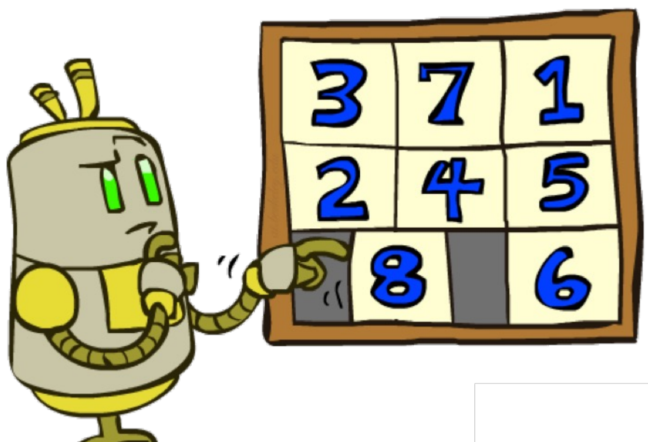


Example: 8 Puzzle

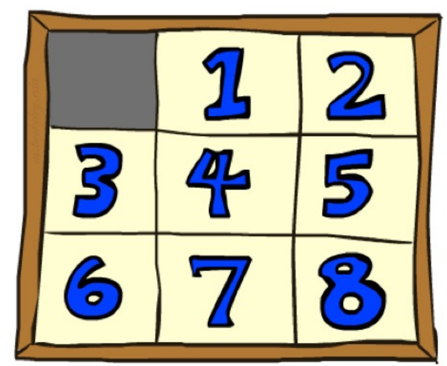
- The 8-puzzle actions:
  - A tile can move from square A to square B if A is horizontally or vertically adjacent to B and B is blank.



Start State



Actions

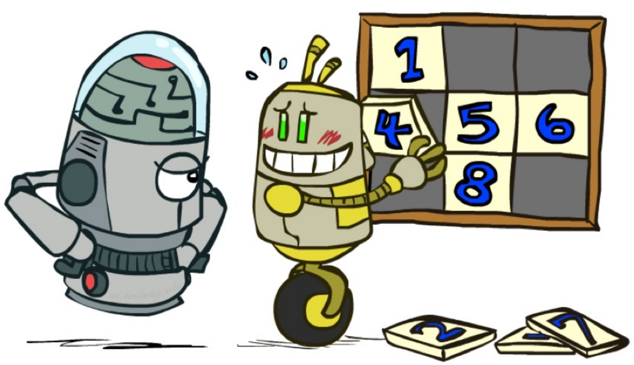


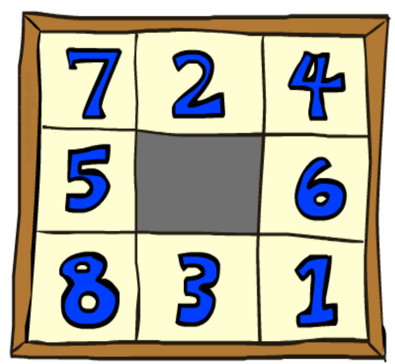
Goal State

Example: 8 Puzzle

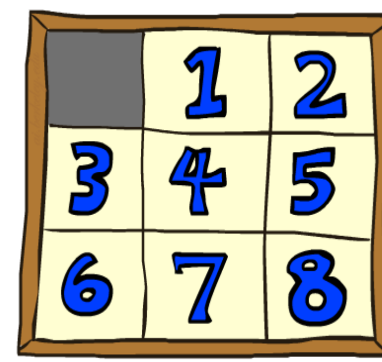
- Relaxation I: A tile can move directly from square A to square B.
  - h_1 = Number of tiles misplaced, e.g. $h_1(\text{start}) = 8$

- Relaxation II: A tile can move from square A to square B if A is adjacent to B ~~and B is blank~~.
  - h_2 = Total Manhattan distance, e.g. $h_2(\text{start}) = 18$





Start State



Goal State

Example: 8 Puzzle

- As heuristics get closer to the true cost, you will expand fewer nodes but usually do more work per node to compute the heuristic itself

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
UCS	112	6,300	3.6×10^6
TILES h_1	13	39	227
MANHATTAN h_2	12	25	73

Key Idea: a **trade-off** between quality of estimate and work per node

- Combining heuristics: If a collection of admissible heuristics $h_1 \cdots h_m$ is available and none of them dominates any of the others, then take

$$h(n) = \max\{h_1(n), \cdots, h_m(n)\}$$

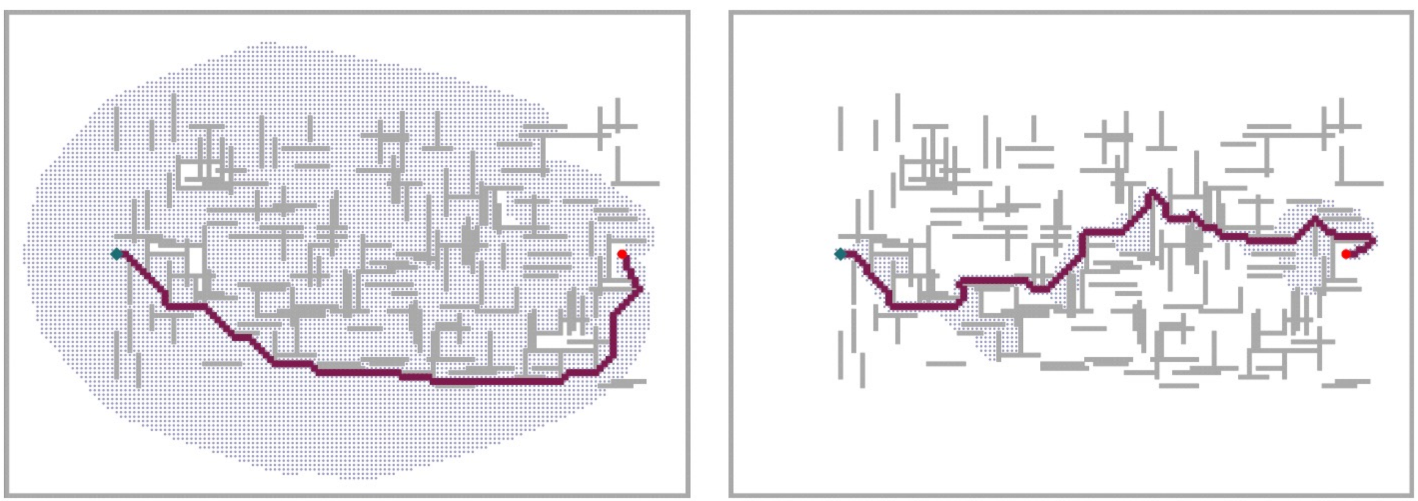
Definition: **dominance**

$h_1 \geq h_2$ if

$\forall n \;\; h_1(n) \geq \; h_2(n)$

Weighted A∗ Search

- Evaluate states by various ways $f(n) = g(n) + W * h(n)$
  - A∗ Search: $W = 1$
  - Uniform-cost search: $W = 0$
  - Greedy best-first search: $W = \infty$
  - Weighted A∗ search: $1 < W < \infty$



Outline

- Search Problems
- General Search
  - Tree Search, Graph Search
- Uninformed Search
  - Depth-First (DFS), Breadth-First (BFS), Uniform-Cost (UCS)
- Informed (Heuristic) Search
  - A$*$ Tree Search
  - **A$*$ Graph Search**

Uniform Cost (Graph) Search

```
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using node path-cost as the priority
    add initial state of problem to frontier with path-cost = 0
                                                              (1). Initialization
    loop do
        if the frontier is empty then return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
                                                              (2). Selection
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher path-cost then
                replace that frontier node with child
                                                              (3). Expansion
```

Key idea: extra check

A shorter path to a frontier state is discovered.

A∗ Graph Search

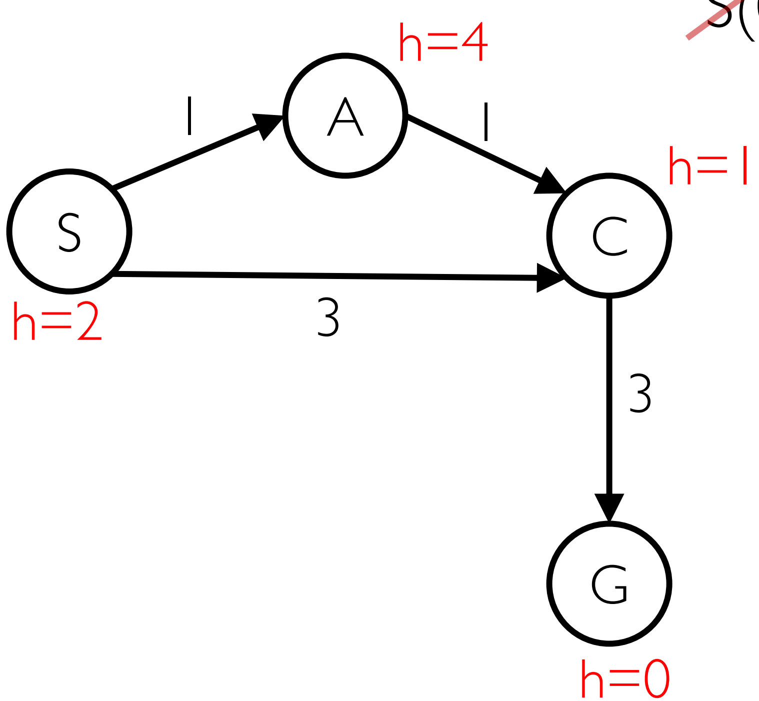
```
function A-STAR-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using f(n) = g(n) + h(n) as the priority
    add initial state of problem to frontier with priority f(S) = 0 + h(S)
                                                                (1). Initialization
    loop do
        if the frontier is empty then return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
                                                                (2). Selection
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher f(n) then
                replace that frontier node with child
                                                                (3). Expansion
```

Key idea: extra check

A path with shorter estimation to a frontier state is discovered

A∗ Graph Search Gone Wrong

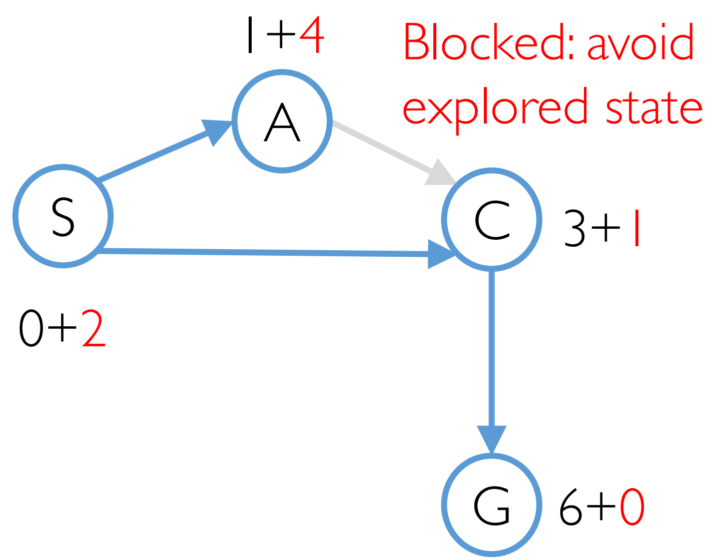
State space graph



Queue

~~S~~(0+2), ~~S-C~~(3+1), ~~S-A~~(1+4), S-C-G(6+0)

Search Tree



Consistency of Heuristics

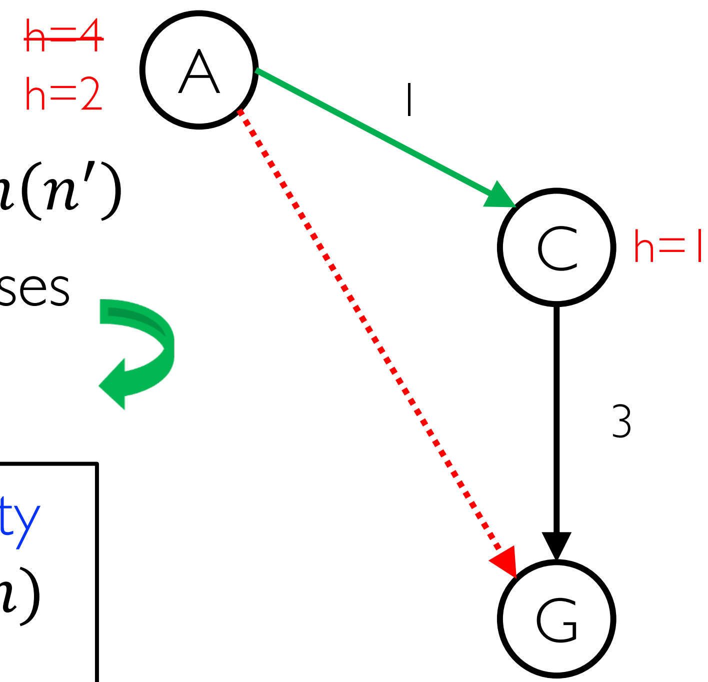
- **Admissibility** (可采纳性): heuristic cost ≤ actual cost to goal
- **Consistency** (一致性): heuristic "arc" cost ≤ actual cost for each arc

$$h(n) - h(n') \leq c(n, a, n')$$

- Consequences of consistency
  - Triangle inequality

$$h(n) \leq c(n, a, n') + h(n')$$

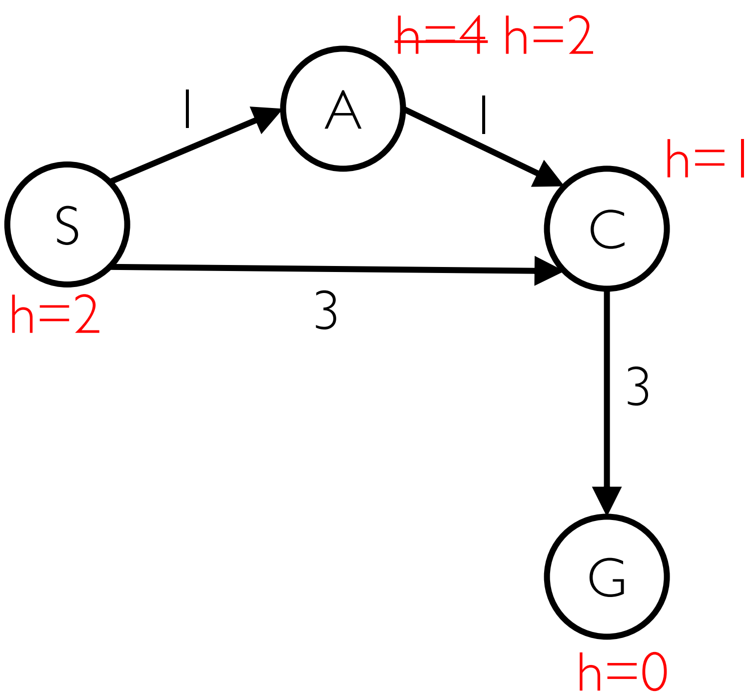
  - The f value along a path never decreases
  - A∗ graph search is optimal



Theorem: consistency implies admissibility
If a heuristic $h(n)$ is consistent, then $h(n)$ is admissible.

A* Graph Search Gone Right

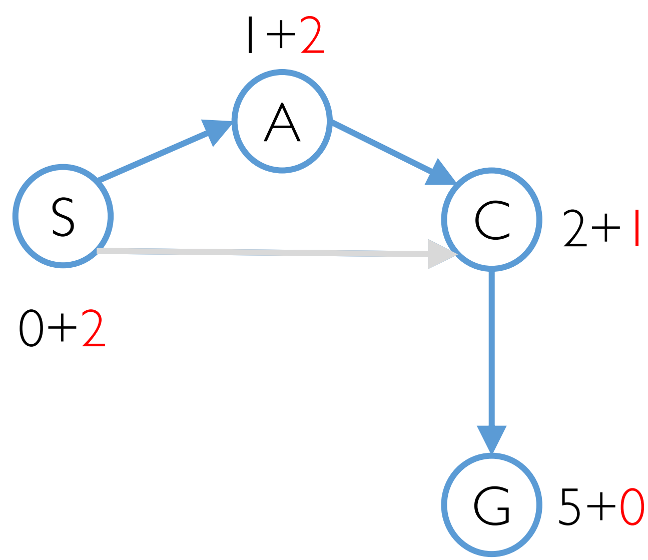
State space graph



Queue

~~S(0+2)~~, ~~S-A(1+2)~~ , ~~S-A-C(2+1)~~
, S-A-C-G(5+0)

Search Tree

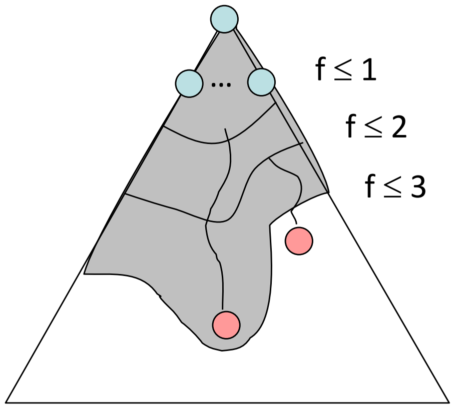


Optimality of A∗ Graph Search

- **Step I**: if $h(n)$ is consistent, then the values of $f(n)$ along any path are nondecreasing.

$$f(n') = g(n') + h(n') = g(n) + c(n, a, n') + h(n') \geq g(n) + h(n) = f(n)$$

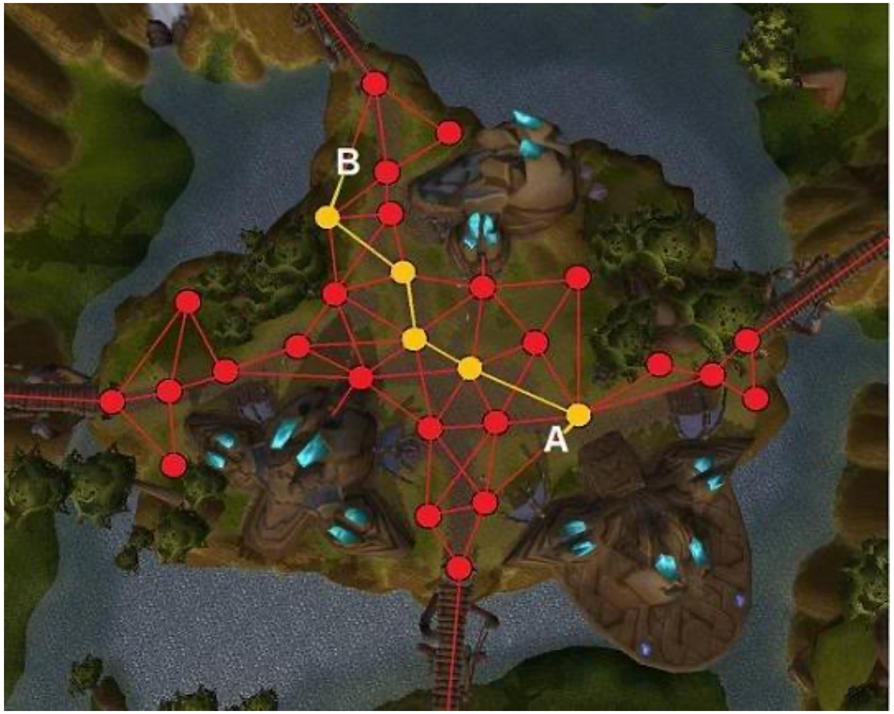
- **Step II**: whenever A∗ selects any node n for expansion, the optimal path to that node has been found.
  - Intuition: replace g in UCS with f
  - A node n with minimum $f(n)$ corresponds to a path to n with minimum $g(n)$, because $h(n)$ are always the same



- **Step III**: the first goal node selected for expansion must be an optimal solution because f is the true cost for goal nodes which have $h = 0$

Applications of A∗ Search

- Video games
- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- Database query optimization
- …



Example: pathfinding in games

A$*$ Search: Summary

- A$*$ search uses both backward costs and (estimates of) forward costs
- A$*$ search is optimal with admissible / consistent heuristics
  - Admissibility guarantees optimality of A$*$ Tree Search
  - Consistency guarantees optimality of A$*$ Graph Search
- Implementation: replace path cost in UCS with $f = g + h$
- Heuristic design is key: use relaxed problem or learn from experience

Thank You

Questions?

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[Some slides adapted from Dan Klein and Pieter Abbeel]