Plane Isometries

YSC3237 Modern Algebra Exploration 1

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Prerequisites

Definition (1)

The *Euclidean space* is a finite-dimensional vector space over R, with an inner product $\langle v1, v2\rangle$. Euclidean n-space $\mathbb{R}^n$ is the space of all n-tuples of real numbers $(x_1, x_2, \ldots, xn)$. $\mathbb{R}^2$ is called the *Euclidean plane*.

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Definition (2)

The *Euclidean distance* is a function $d : \mathbb{R}^2 \mapsto \mathbb{R}^2$defined as

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

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Definition (3)

An isometry of $\mathbb{R}^2$ is the transformation f on the plane that preserves the distances, that is, it is a map $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ satisfying that for any pair of points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$d(f(x_1, y_1), f(x_2, y_2)) = d((x_1, y_1), (x_2, y_2)).$$

Definition (4)

A line in the Euclidean plane is a set

$$L_{(a_0,b_0),(a_1,b_1)} := \{(x, y) \in \mathbb{R}^2 | d((x, y), (a_0, b_0)) = d((x, y), (a_1, b_1)$$

for some $(a_0, b_0), (a_1, b_1) \in \mathbb{R}^2$ with $(a_0, b_0) \neq (a_1, b_1)$ with d defined as the Euclidean distance.

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Definition (5)

Planar isometries are:

- Reflections
- Rotations
- Translations
- Glide Reflections

Is the set of isometries a group?

Theorem (6)

The set of plane isometries is a group under composition.

Proof.

- Closure under composition.
  S, T preserves distance $\implies S \circ T$ preserves distance.

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- The binary operation function composition is always associative.

$\square$

D_3 is a group of isometries

	e	r	r^2	s	rs	r^2s
e	e	r	r^2	s	rs	r^2s
r	r	r^2	e	rs	r^2s	s
r^2	r^2	e	r	r^2s	s	rs
s	s	r^2s	rs	e	r^2	r
rs	rs	s	r^2s	r	e	r^2
r^2s	r^2s	rs	s	r^2	r	e

The Three Reflection Theorem

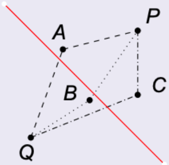
Theorem (7)

Any Euclidean isometry can be written as the composition of at most three reflections.

Lemma

Any point P is uniquely determined by its distances to three non-collinear points A, B, C.

Consequently, any isometry is completely determined by the images of any three non-collinear points.



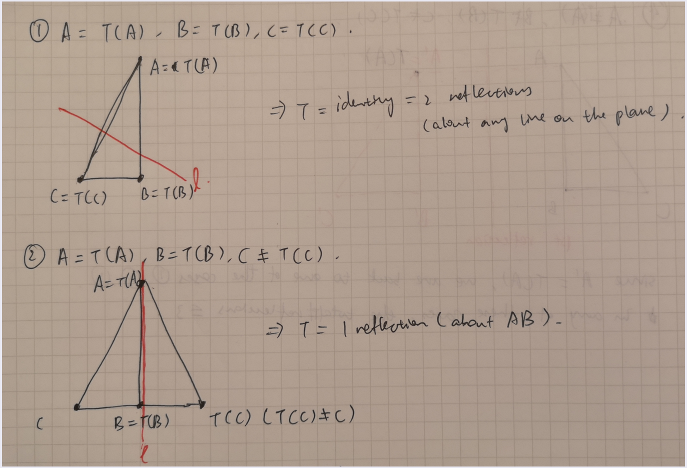
Proof.

Suppose Q has the same distances to A, B, C.
Then A, B, C must lie on the line equidistant from P and Q, contradicting the fact they are not collinear. $\square$

The Three Reflection Theorem

Proof.

Consider the points A, B, C non-collinear and their f-images A', B', C'.



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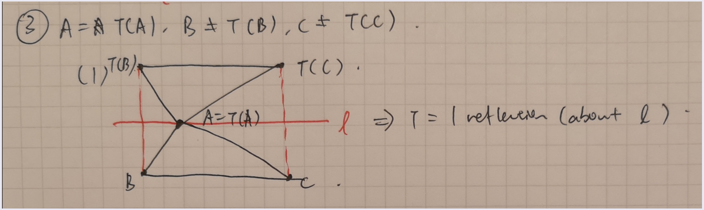

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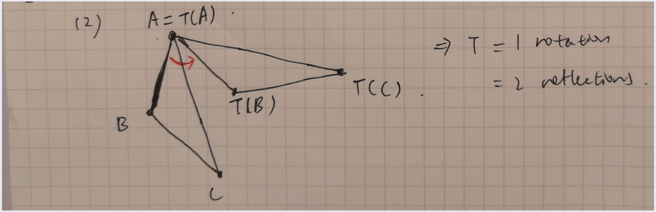

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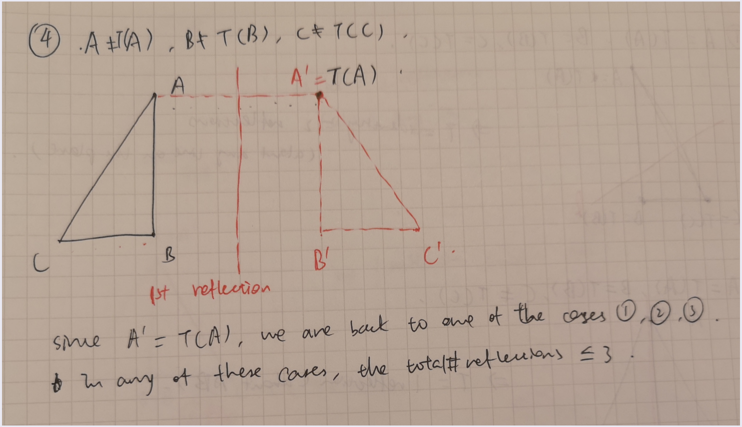


Classification of Plane Isometries

Theorem (8)

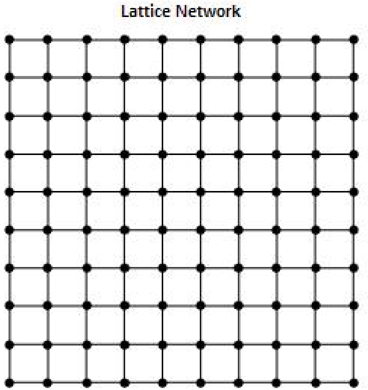
Theorem 8: Every isometry of the plane, other than the identity, is either a translation, a rotation, a reflection, or a glide-reflection.

For any isometry $T(x) = Ax + b$ of $\mathbb{R}^2$, the only cases are:

1. $A = \mathrm{Rot}_0 = I$: T is a translation.
2. $A = \mathrm{Rot}_\theta \in (0, 2\pi)$: T is a clockwise rotation by θ degrees about the point $(I - A)^{-1}b$.
3. $A = \mathrm{Ref}_\theta, b$ perpendicular to the axis of reflection l : T is a reflection in the axis $l + b/2$.
4. $A = \mathrm{Ref}_\theta, b$ parallel to the axis of reflection l : T is a glide reflection.
5. $A = \mathrm{Ref}_\theta, b$ neither parallel nor perpendicular to axis of reflection l : T is a glide-reflection.

A concrete example: the square lattice!

The symmetry group of the square lattice has generators $\mathrm{Rot}_{\frac{\pi}{2}}$, Ref_0, $T_(1,0)$, $T(0,1)$.



Generalizations: crystallographic groups

Definition (9)

The set of all symmetries of a crystal pattern is called the crystallographic group of the crystal pattern.

- Their order is always infinite because there are infinitely many translations.
- A crystallographic group is a subgroup of the isometry group of Euclidean space $\mathbb{R}^n$.

Generalizations: crystallographic groups

Some examples of crystallographic groups in $\mathbb{R}^2$:

- frieze groups patterns that are repetitive in one direction
- wallpaper groups: patterns that are repetitive in two directions

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Everything we have said can be generalized to the Euclidean space of n dimensions $\mathbb{R}^n$!

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Food for thought: what about non-Euclidean space?