CSE101: Design and Analysis of Algorithms

Ragesh Jaiswal, CSE, UCSD

Recap.

- Growth rates:
  - Arrange the following functions in ascending order of growth rate:
    - n
    - $2^{\sqrt{\log n}}$
    - $n^{\log n}$
    - $2^{\log n}$
    - $n / \log n$
    - n^n

Introduction

- Algorithm: A step-by-step way of solving a problem.
- Design of Algorithms:
  - *"Algorithm is more of an art than science"*
  - However, we will learn some basic tools and techniques that have evolved over time. These tools and techniques enable you to effectively design and analyse algorithms.
- Analysis of Algorithms:
  - Proof of correctness: An argument that the algorithm works correctly for all inputs.
    - Proof: A valid argument that establishes the truth of a mathematical statement.
  - Analysis of worst-case running time as a function of the input size.

Introduction

- Proof: A valid argument that establishes the truth of a mathematical statement.
  - The statements used in a proof can include axioms, definitions, the premises, if any, of the theorem, and previously proven theorems and uses rules of inference to draw conclusions.
- A proof technique very commonly used when proving correctness of Algorithms is *Mathematical Induction*.

Definition (Strong Induction)

To prove that $P(n)$ is true for all positive integers, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We show that $P(1)$ is true.
- Inductive step: We show that for all k, if $P(1), P(2), ..., P(k)$ are true, then $P(k+1)$ is true.

Definition (Strong Induction)

To prove that $P(n)$ is true for all positive integers, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We show that $P(1)$ is true.
- Inductive step: We show that for all k, if $P(1), P(2), ..., P(k)$ are true, then $P(k+1)$ is true.

- Question: Show that for all $n > 0$, $1 + 3 + ... + (2n-1) = n^2$.

Introduction

- Question: Show that for all $n > 0$, $1 + 3 + ... + (2n - 1) = n^2$.

Proof

- Let $P(n)$ be the proposition that $1 + 3 + 5 + ... + (2n - 1)$ equals n^2.
- Basis step: $P(1)$ is true since the summation consists of only a single term 1 and $1^2 = 1$.
- Inductive step: Assume that $P(1), P(2), ..., P(k)$ are true for any arbitrary integer k. Then we have:

$$\begin{aligned} 1 + 3 + ... + (2(k+1) - 1) &= 1 + 3 + ... + (2k - 1) + (2k + 1) \\ &= k^2 + 2k + 1 \quad (\text{since } P(k) \text{ is true}) \\ &= (k+1)^2 \end{aligned}$$

  This shows that $P(k + 1)$ is true.
- Using the principle of Induction, we conclude that $P(n)$ is true for all $n > 0$. $\square$

Introduction

- Algorithm: A step-by-step way of solving a problem.
- Design of Algorithms:
  - *"Algorithm is more of an art than science"*
  - However, we will learn some basic tools and techniques that have evolved over time. These tools and techniques enable you to effectively design and analyse algorithms.
- Analysis of Algorithms:
  - Proof of correctness: An argument that the algorithm works correctly for all inputs.
    - Proof: A valid argument that establishes the truth of a mathematical statement.
  - Analysis of worst-case running time as a function of the input size.

Introduction

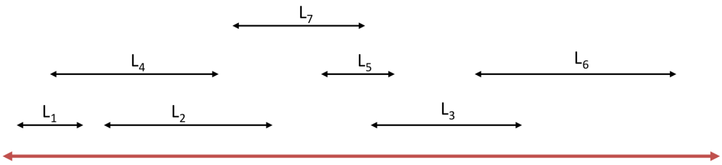
- Algorithm Design Techniques
  - Divide and Conquer
  - Greedy Algorithms
  - Dynamic Programming
  - Network Flows

- Material that will be covered in the course:
  - Basic graph algorithms
  - Algorithm Design Techniques
    - Divide and Conquer
    - Greedy Algorithms
    - Dynamic Programming
    - Network Flows
  - Computational intractability

- Some examples of Divide and Conquer Algorithms:
  - Binary Search
  - Median finding
  - Multiplying numbers
  - Merge sort, quick sort.

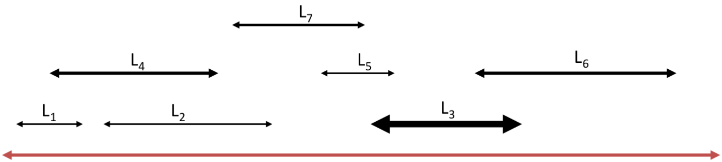
Problem

Interval scheduling: You have a lecture room and you get n requests for scheduling lectures. Each request has a start time and an end time. The goal is to maximise the number of lectures.



Problem

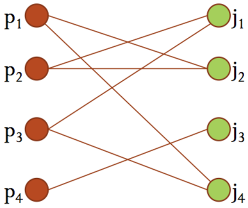
Interval scheduling: You have a lecture room and you get n requests for scheduling lectures. Each request has a start time, an end time, and a price (that you will get in case the lecture is scheduled). The goal is to maximise your earnings.



Problem

<u>Job assignment</u>: There are n people and n jobs. Each person has a list of jobs he/she could possibly do. Find a job assignment so that:

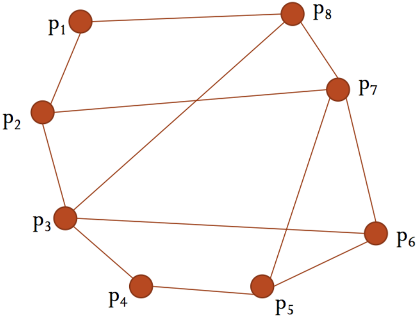
1. each job is assigned to a different person, and
2. each person is assigned a job from his/her list.



- Is it always possible to find a fast algorithm for any problem?

Problem

Given a social network, find the largest subset of people such that no two people in the subset are friends.



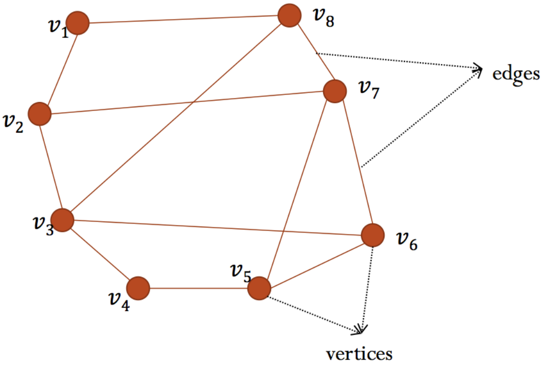
- The problem in the previous slide is called the Independent Set problem and no one knows if it can be solved in polynomial time (quickly).
- There is a whole class of problems to which Independent Set belongs.
- If you solve one problem in this class quickly, then you can solve all the problems in this class quickly.
- You can also win a million dollars!!
- We will see techniques of how to show that a new problem belongs to this class:
  - Why: *because then you can say to your boss that the new problem belongs to the difficult class of problems and even the most brilliant people in the world have not been able to solve the problem so do not expect me to do it. Also, if I can solve the problem there is no reason for me to work for you!*

Course Overview

- Material that will be covered in the course:
  - Basic graph algorithms
  - Algorithm Design Techniques
    - Divide and Conquer
    - Greedy Algorithms
    - Dynamic Programming
    - Network Flows
  - Computational intractability

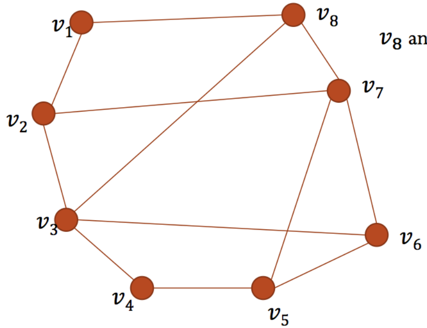
Graphs

- A way to represent a set of objects with pair-wise relationships among them.
- The objects are represented as vertices and the relationships are represented as edges.



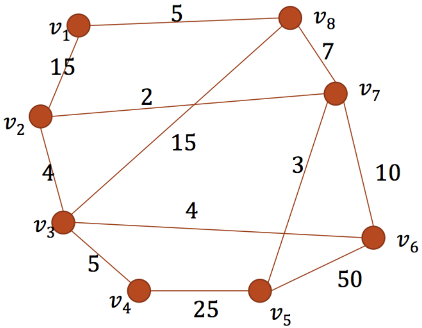
$$
\begin{aligned}
G &= (V, E) \\
V &= \{v_1, \ldots, v_8\} \\
E &= \{(v_1, v_8), \ldots\}
\end{aligned}
$$

- Examples
  - Social networks
  - Communication networks
  - Transportation networks
  - Dependency networks

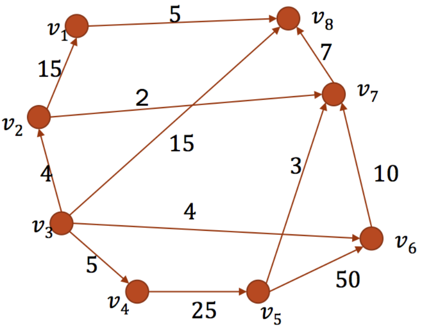


v_8 and v_7 are friends

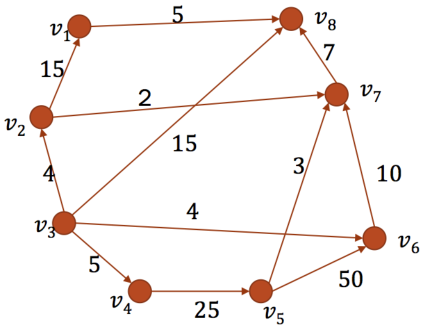
- Weighted graphs: There are weights associated with each edge quantifying the relationship. For example, delay in communication network.



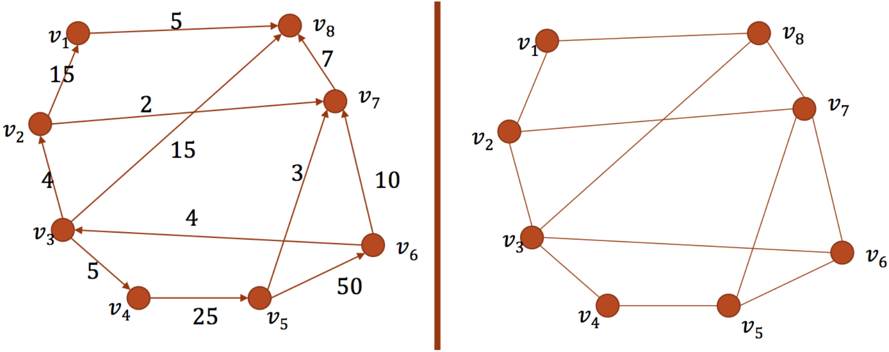
- Directed graphs: Asymmetric relationships between the objects. For example, one way streets.



- Path: A sequence of vertices $v_1, v_2, ..., v_k$ such that for any consecutive pair of vertices v_i, v_{i+1}, (v_i, v_{i+1}) is an edge in the graph. It is called a path from v_1 to v_k.
- Cycle: A cycle is a path where $v_1 = v_k$ and $v_1, ..., v_{k-1}$ are distinct vertices.

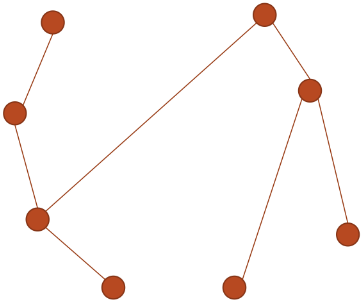


- Strongly connected: A graph is called strongly connected iff for any pair of vertices u, v, there is a path from u to v and a path from v to u.



- Tree: A strongly connected, undirected graph is called a tree if it has no cycles.
- How many edges does a tree have?

if there has two sub
children
then 2n-1



- Let $P(n)$ be the statement

  Any tree with n nodes has exactly $n-1$ edges.

- An inductive proof will have the following steps:
  - Base case: Show that $P(1)$ is true.
  - Inductive step: Show that if $P(1), P(2)..., P(k)$ are true, then so is $P(k+1)$.

- Let $P(n)$ be the statement

  Any tree with n nodes has exactly $n-1$ edges.

- An inductive proof will have the following steps:
  - Base case: Show that $P(1)$ is true.
  - Inductive step: Show that if $P(1), P(2)..., P(k)$ are true, then so is $P(k+1)$.

Proof outline

- Base case: $P(1)$ is true since any tree with 1 vertex has 0 edges.
- Inductive step: Assume that $P(1), ..., P(k)$ are true.
  - Now, consider any tree T with $k+1$ vertices.
  - Claim 1: There is a vertex v in T that has exactly 1 edge.
  - Consider the tree T' obtained by removing v and its edge from T.
  - Claim 2: T' is a tree with k vertices.
  - As per the induction hypothesis, T' has $k-1$ edges. This implies that T has k edges.

Proof

- Base case: $P(1)$ is true since any tree with 1 vertex has 0 edges.
- Inductive step: Assume that $P(1), ..., P(k)$ are true.
  - Now, consider any tree T with $k + 1$ vertices.
  - Claim 1: There is a vertex v in T that has exactly 1 edge.
    - Proof: For the sake of contradiction, assume that there does not exist such a vertex in T. Then this means that all vertices have at least two edges incident on them. Start with an arbitrary vertex u_1 in T. Starting from u_1 use one of the edges incident on u_1 to visit its neighbor u_2. Since u_2 also has at least two incident edges, take one of the other edges to visit its neighbor u_3. On repeating this, we will (in finite number of steps) visit a vertex that was already visited. This implies that there is a cycle in T. This is a contradiction. □
  - Consider the tree T' obtained by removing v and its edge from T.
  - Claim 2: T' is a tree with k vertices.
    - Proof: T' clearly has k vertices. T' is strongly connected since otherwise T is not strongly connected. Also, T' does not have a cycle since otherwise T has a cycle. □
  - As per the induction hypothesis, T' has $k - 1$ edges. This implies that T has k edges. □

End