Comparison between Bayesian and Frequentist Regularization in Confirmatory Factor Analysis

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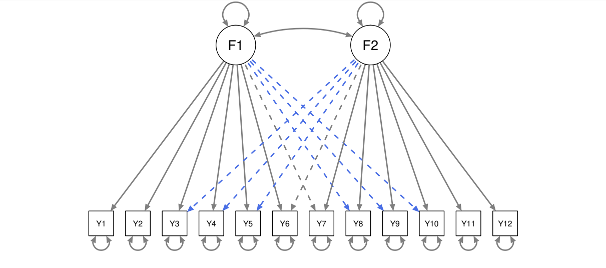
Introduction

Cross-loadings in Confirmatory Factor Analysis

A general confirmatory factor analysis (CFA) model is specified as:

$$\mathbf{y}_i = \boldsymbol{\mu} + \boldsymbol{\Lambda}\mathbf{F}_i + \boldsymbol{\epsilon}_i, i = 1, 2, ..., N, \tag{1}$$



Dashed line: Cross-loading; Solid line: Main loading

Regularization

Ridge, Lasso, and Adaptive Lasso (alasso)

- Frequentist (Li & Jacobucci, 2021; Yuan & Liu, 2021)
- Bayesian (Chen et al., 2021; Lu et al., 2016; Muthén & Asparouhov, 2012)

Comparison

- Equivalence.
- Uncertainty quantification.
- Estimation of penalty parameters.
- Model complexity and small samples.

Frequentist Regularization for Selecting Cross-loadings

For a model with J cross-loadings $\boldsymbol{\lambda}$ to be regularized, Ridge, Lasso and Alasso can be formulated as follows:

$$
\begin{aligned}
F_{ridge} &= F_{ML} + \gamma \sqrt{\sum_{j=1}^{J} \lambda_j^2} \\
F_{lasso} &= F_{ML} + \gamma \sum_{j=1}^{J} |\lambda_j| \\
F_{alasso} &= F_{ML} + \gamma_j \sum_{j=1}^{J} |\lambda_j|
\end{aligned}
\tag{2}
$$

- A greater value of penalty parameters $(\gamma, \gamma_j \geq 0)$ leads to increased penalization.
- Models were selected using BIC or cross-validation.

Bayesian Regularization: Penalty Priors

For the cross-loadings:

- Ridge:

$$\lambda_j^{c_k} \sim N(0, \sigma^2), \sigma^2 \sim Uniform(0, 1) \quad (3)$$

- Lasso:

$$\lambda_j^{c_k} \sim N(0, \psi_{jj}\tau_j^2), \psi_{jj}^{-1} \sim Gamma(\alpha_j, \beta_j) \quad (4)$$

$$\tau_j^2 \sim Gamma(1, \frac{\gamma^2}{2}), \gamma^2 \sim Gamma(a_l, b_l) \quad (5)$$

where $\alpha_j = \beta_j = 0.01$, $a_l = 1$, and $b_l = 0.01$.

- Adaptive Lasso:

$$\tau_j^2 \sim Gamma(1, \frac{\gamma_j^2}{2}), \gamma_j^2 \sim Gamma(a_l, b_l) \quad (6)$$

Ridge, Lasso, Adaptive Lasso

- Frequentist (Jacobucci & Grimm, 2018; Liang & Jacobucci, 2020; McNeish, 2015)
  - Ridge is advantageous in handling high collinearity.
  - Lasso performed better in shrinking nuisance parameters into zero compared to ridge.
  - Alasso can reduce the bias compared to Ridge and Lasso
- Bayesian
  - Ridge is advantageous in handling high collinearity (Zhang & Liang, 2023)
  - Lasso performed better in maintaining a simple loading structure compared to Ridge (Chen et al., 2021)
  - Alasso performed better in variable selection and estimation accuracy compared to Lasso (Feng et al., 2017)
  - Alasso performed better in detecting non-zero residual correlations compared to Lasso (Pan et al., 2021)

Purpose

- Investigate the similarity and differences in point estimates and variable selection when applying ridge, lasso, and alasso in both frequentist and Bayesian frameworks.
- Explore the strengths and limitations of various regularization methods under frequentist and Bayesian frameworks.

Simulation Study

Design

- Model structure: two- and three-factor models.

The loading matrices for the two- and three-factor models:

$$\Lambda'_{true} = \begin{pmatrix} M & M & M & M & M & 0 & 0 & 0 & C_2 & C_1 \\ 0 & 0 & 0 & C_2 & C_1 & M & M & M & M & M \end{pmatrix}$$

$$\Lambda'_{true} = \begin{pmatrix} M & M & M & M & M & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_2 & 0 & M & M & M & M & M & 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & C_2 & 0 & M & M & M & M & M \end{pmatrix}$$

- Major loadings: 1
- Number of non-zero cross-loadings per factor: 1 or 2
- Magnitude of non-zero cross-loadings: 0.1, 0.2, 0.3
- Factor correlation: 0.3, 0.5, 0.7
- Sample size: 200, 500, 800

Frequentist Regularization

The regsem package (Jacobucci, 2017): models were selected using BIC for γ, γ_j from 0 to 0.29 with a .01 increment.

Penalty cannot be assigned to all cross-loadings, otherwise the model would not converge.

So we assume that there are at least one item per factor loaded on only one factor, and assign penalty on the cross-loadings with $C-$marks:

$$\mathbf{\Lambda}_{est}^{'} = \begin{pmatrix} 1 & M & M & M & M & 0 & C_0 & C_0 & C_2 & C_1 \\ 0 & C_0 & C_0 & C_2 & C_1 & 1 & M & M & M & M \end{pmatrix}$$

$$\mathbf{\Lambda}_{est}^{'} = \begin{pmatrix} 1 & M & M & M & M & 0 & C_0 & C_0 & C_0 & C_1 & 0 & C_0 & C_0 & C_2 & C_0 \\ 0 & C_0 & C_0 & C_2 & C_0 & 1 & M & M & M & M & 0 & C_0 & C_0 & C_0 & C_1 \\ 0 & C_0 & C_0 & C_0 & C_1 & 0 & C_0 & C_0 & C_2 & C_0 & 1 & M & M & M & M \end{pmatrix}$$

Bayesian Regularization

Bayesian Regularization: Penalty priors were assigned for the cross-loadings:

- Ridge:

$$\lambda_j^{c_k} \sim N(0, \sigma^2), \sigma^2 \sim Uniform(0, 1) \tag{7}$$

- Lasso:

$$\lambda_j^{c_k} \sim N(0, \psi_{jj}\tau_j^2), \psi_{jj}^{-1} \sim Gamma(\alpha_j, \beta_j) \tag{8}$$

$$\tau_j^2 \sim Gamma(1, \frac{\gamma^2}{2}), \gamma^2 \sim Gamma(a_l, b_l) \tag{9}$$

  where $\alpha_j = \beta_j = 0.01$, $a_l = 1$, and $b_l = 0.01$.

- Adaptive Lasso:

$$\tau_j^2 \sim Gamma(1, \frac{\gamma_j^2}{2}), \gamma_j^2 \sim Gamma(a_l, b_l) \tag{10}$$

Bayesian Estimation

For the main loadings, intercepts, and factor variance-covariance matrix, defuse priors were assigned:

$$\lambda_j^m \sim N(\lambda_0, H_{\lambda 0}), \mu_j \sim N(\mu_0, H_{\mu 0}), \mathbf{\Phi}^{-1} \sim Wishart(\mathbf{R}_0, \rho_0) \tag{11}$$

where $\lambda_0 = \mu_0 = 0$, $H_{\lambda 0} = H_{\mu 0} = 1000$, $\mathbf{R}_0$ is the identity matrix and ρ_0 is the number of factors plus one.

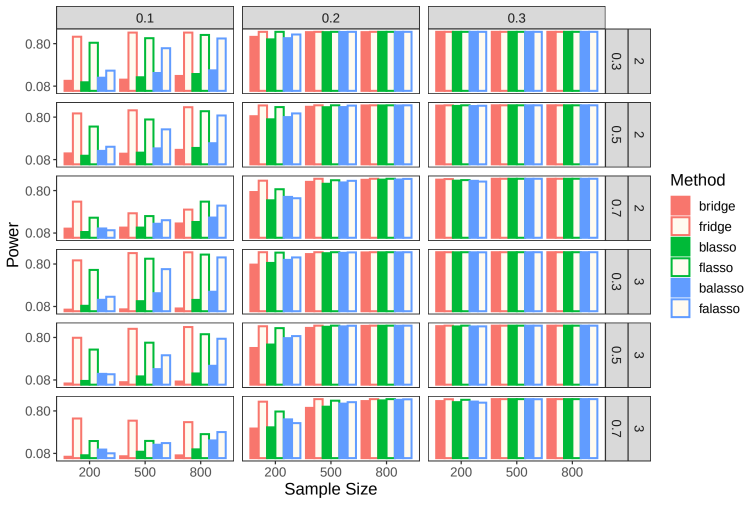
- Number of burn-in iterations were set between 5,000 - 20,000.
- Model Convergence Criteria: EPSR value $<$ 1.05.
- Software: R, JAGS (Plummer, 2003)

Evaluation Criteria

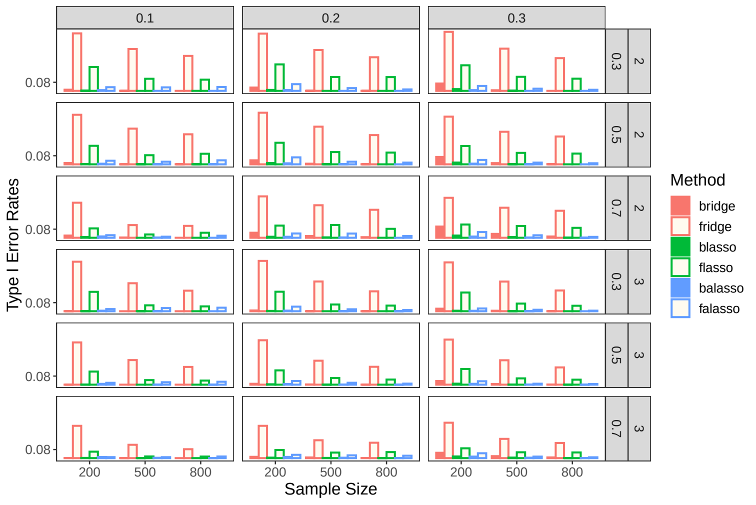
200 datasets per condition.

- Parameter Identification (Thresholding rule: for frequentist regularization: $|\beta| > 0$; for Bayesian regularization: $|\beta| > 0.1$; Zhang, Pan, & Ip, 2021):
  - Power
  - Type I error rate
- Parameter Estimation (For Bayesian estimation, median of posterior samples was used as point estimate):
  - Relative bias
  - Root mean square error

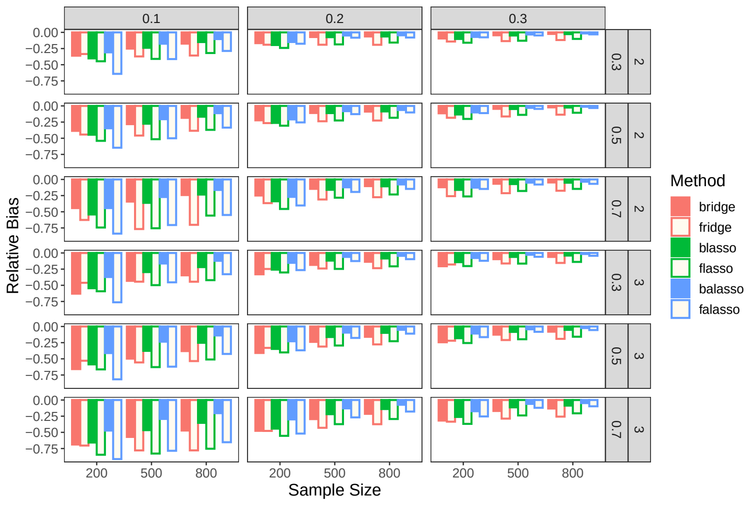
Results: Power



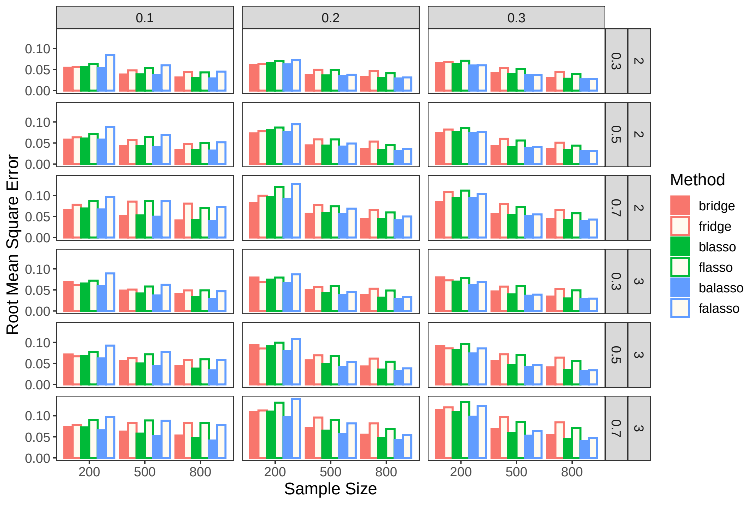
Results: Type I Error Rates



Results: Relative Bias



Results: Root Mean Square Error



Discussion

Method Comparison

- Frequentist methods generally provide greater power but may also encounter inflated type I error rates, particularly for ridge.
  - However, note that different thresholds were used for frequentist and Bayesian methods.
- Bayesian regularization generally performed better in parameter estimation (lower bias and RMSE).
- Frequentist alasso exhibited lower power in general, while showed improved parameter estimates with increasing sample size, compared to ridge and lasso.
- Bayesian alasso outperformed Bayesian ridge and lasso in most conditions, particularly in the three-factor model.

Takeaways

- Frequentist methods generally offer higher power in detecting cross-loadings, while Bayesian methods excel at controlling type I errors
- Use Bayesian Alasso for Lower Bias and RMSE
- Exercise caution about Type I error rates when using Frequentist Ridge and Lasso.
- Findings may vary depending on sample size, model complexity, and the chosen threshold.

Thank you for listening!

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