Problem:

Let ABC be a triangle inscribed in circle $\\omega$. Let the tangents to $\\omega$ at B and C intersect at point D, and let $\\overline{AD}$ intersect $\\omega$ at P. If $AB=5$, $BC=9$, and $AC=10$, AP can be written as the form $\\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n$.

Response:

To solve the problem, we start by noting that \\(D\\) is the intersection of the tangents to the circumcircle \\(\\omega\\) at \\(B\\) and \\(C\\). This means that \\(D\\) is the excenter of the triangle \\(ABC\\) opposite to \\(A\\). By the properties of tangents from a point to a circle, \\(DB = DC\\). [score=0.20]

We use the fact that \\(AD\\) is the angle bisector of \\(\\angle BAC\\) when extended to intersect the circumcircle again at \\(P\\). By the Angle Bisector Theorem, we know that the ratio of the segments created by the angle bisector on \\(BC\\) is equal to the ratio of the other two sides of the triangle. Therefore, we have:\n\\[\n\\frac{BD}{DC} = \\frac{AB}{AC} = \\frac{5}{10} = \\frac{1}{2}\n\\]\nSince \\(BD = DC\\), we can use the Power of a Point theorem at \\(D\\). The power of point \\(D\\) with respect to the circle is:\n\\[\nDB^2 = DC^2 = DA \\cdot DP\n\\]\nSince \\(D\\) is the intersection of the tangents at \\(B\\) and \\(C\\), we have:\n\\[\nDB^2 = DC^2 = BD^2 = BD \\cdot BP\n\\]\nLet \\(BD = x\\). Then \\(DC = x\\) and \\(BC = BD + DC = 2x = 9\\), so \\(x = \\frac{9}{2}\\). Therefore, \\(BD = DC = \\frac{9}{2}\\). [score=0.66]

Using the Power of a Point theorem again, we have:\n\\[\nBD^2 = DA \\cdot DP \\implies \\left(\\frac{9}{2}\\right)^2 = DA \\cdot DP \\implies \\frac{81}{4} = DA \\cdot DP\n\\]\nWe need to find the length of \\(DA\\). By the Angle Bisector Theorem, the ratio of the segments \\(BD\\) and \\(DC\\) is the same as the ratio of \\(AB\\) and \\(AC\\), which is \\(\\frac{1}{2}\\). Therefore, \\(DA\\) can be found using the formula for the length of the angle bisector:\n\\[\nAD = \\frac{2 \\cdot AB \\cdot AC \\cdot \\cos \\left(\\frac{A}{2}\\right)}{AB + AC}\n\\]\nHowever, a simpler approach is to use the fact that \\(AD \\cdot DP = \\left(\\frac{9}{2}\\right)^2 = \\frac{81}{4}\\). Since \\(AD + DP = AP\\), we need to find \\(AP\\). Using the fact that \\(DA \\cdot DP = \\frac{81}{4}\\) and knowing that \\(DA = \\frac{90}{13}\\) (from the Angle Bisector Theorem and the length of the angle bisector formula), we can solve for \\(AP\\):\n\\[\nAP = \\frac{81}{4 \\cdot DA} = \\frac{81}{4 \\cdot \\frac{90}{13}} = \\frac{81 \\cdot 13}{4 \\cdot 90} = \\frac{1053}{360} = \\frac{117}{40}\n\\]\nThus, \\(AP = \\frac{117}{40}\\), and the sum \\(m + n\\) is:\n\\[\n117 + 40 = 157\n\\]\nTherefore, the answer is:\n\\[\n\\boxed{157}\n\\] [score=0.92]