Models for Counts – Solutions

STAT-UB.0001 – Statistics for Business Control

Binomial Random Variables

1. A certain coin has a 25% of landing heads, and a 75% chance of landing tails.

   (a) If you flip the coin 4 times, what is the chance of getting exactly 2 heads?

   > **Solution:** There are 6 outcomes whith exactly 2 heads:
   >
   > $$HHTT, HTHT, HTTH, THHT, THTH, TTHH.$$
   >
   > By independence, each of these outcomes has probability $(.25)^2(.75)^2$. Thus,
   >
   > $$\text{P(exactly 2 heads out of 4 flips)} = 6(.25)^2(.75)^2.$$

   (b) If you flip the coin 10 times, what is the chance of getting exactly 2 heads?

   > **Solution:** Rather than list all outcomes, we will use a counting rule. There are ${}_{10}C_2$ ways of choosing the positions for the two heads; each of these outcomes has probability $(.25)^2(.75)^8$. Thus,
   >
   > $$\text{P(exactly 2 heads out of 10 flips)} = {}_{10}C_2\,(.25)^2(.75)^8.$$

2. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

> **Solution:** Let X be the number of times that we get the face with two spots. This is a binomial random variable with $n = 8$ and $p = \frac{1}{6}$. We compute
>
> $$\begin{aligned} P(X = 2) &= {}_nC_2\,p^2(1-p)^{n-2} \\ &= {}_8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 \\ &\approx 0.26. \end{aligned}$$

3. The probability is 0.04 that a person reached on a "cold call" by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale with result?

Solution: Let X be the number of sales. This is a binomial random variable with $n = 40$ and $p = 0.04$. Thus,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - {}_nC_0\, p^0 (1-p)^{n-0} \\ &= 1 - (0.96)^{40} \\ &\approx .805 \end{aligned}$$

4. A new restaurant opening in Greenwich village has a 30% chance of survival during their first year. If 16 restaurants open this year, find the probability that

   (a) exactly 3 restaurants survive.

   Solution: Let X be the number that survive. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

   $$\begin{aligned} P(X = 3) &= {}_{16}C_3\, (0.3)^3 (1 - 0.3)^{(16-3)} \\ &= .146 \end{aligned}$$

   (b) fewer than 3 restaurants survive.

   Solution:

   $$\begin{aligned} P(X < 3) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}_{16}C_0\, (0.3)^0 (0.7)^{16} + {}_{16}C_1\, (0.3)^1 (0.7)^{15} + {}_{16}C_2\, (0.3)^2 (0.7)^{14} \\ &= .099 \end{aligned}$$

   (c) more than 3 restaurants survive.

   Solution:

   $$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - (.099 + .146) \\ &= .754 \end{aligned}$$

5. The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

Solution: Let X be the number of times that we win. This is a binomial random variable with $n = 10$ and $p = 0.10$. We compute

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= {}_nC_0\, p^0(1-p)^{n-0} + {}_nC_1\, p^1(1-p)^{n-1} \\ &= {}_{10}C_0\,(0.10)^0(0.90)^{10} + {}_{10}C_1\,(0.10)^1(0.90)^9 \\ &= (0.90)^{10} + 10\,(0.10)(0.90)^9 \\ &\approx 0.736. \end{aligned}$$

6. The probability is 0.2 that an audit of a retail business will turn up irregularities in the collection of state sales tax. If 20 retail businesses are audited, find the probability that

(a) fewer than 2 will have irregularities in the collection of state sales tax.

Solution: Let X be the number audited. This is a binomial random variable with $n = 20$ and $p = 0.2$. Therefore,

$$P(X < 2) = {}_{20}C_0\,(0.2)^0(0.8)^{20} + {}_{20}C_1\,(0.2)^1(0.8)^{19} \qquad \approx .069$$

(b) more than 2 will have irregularities in the collection of state sales tax.

Solution:

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - \left[{}_{20}C_0\,(0.2)^0(0.8)^{20} + {}_{20}C_1\,(0.2)^1(0.8)^{19} + {}_{20}C_2\,(0.2)^2(0.8)^{18}\right] \\ &\approx .794. \end{aligned}$$

Poisson Random Variables

7. The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour.

   (a) What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

   Solution: Let X be the number of calls that arrive between 8:00 p.m. and 9:00 p.m. This is a Poisson random variable with mean

   $$\lambda = E(X) = (4.6 \text{ calls/hour})(1 \text{ hour}) = 4.6 \text{ calls}.$$

   Thus,

   $$P(X = 6) = \frac{\lambda^6}{6!}e^{-\lambda} = \frac{(4.6)^6}{6!}e^{-4.6}.$$

   (b) Find the probability that exactly 7 calls will come between 9:00 p.m. and 10:30 p.m.

   Solution: Let X be the number of calls that arrive between 9:00 p.m. and 10:30 p.m. This is a Poisson random variable with mean

   $$\lambda = E(X) = (4.6 \text{ calls/hour})(1.5 \text{ hours}) = 6.9 \text{ calls}.$$

   Thus,

   $$P(X = 7) = \frac{\lambda^7}{7!}e^{-\lambda} = \frac{(6.9)^7}{7!}e^{-6.9}.$$

8. Car accidents occur at a particular intersection in the city at a rate of about 2/year.

   (a) Estimate the probability of no accidents occurring in a 6-month period.

   Solution: Let X be the number of car accidents. This is Poisson random variable with mean

   $$\lambda = E(X) = (2 \text{ accidents/year})(0.5 \text{ years}) = 1 \text{ accident}.$$

   Thus,

   $$P(X = 0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-1} \approx .368.$$

   (b) Estimate the probability of two or more accidents occurring in a year.

   Solution: Let X be the number of car accidents. This is Poisson random variable with mean

   $$\lambda = E(X) = (2 \text{ accidents/year})(1.0 \text{ years}) = 2 \text{ accident}.$$

Thus,

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - \left[\frac{\lambda^0}{0!}e^{-\lambda} + \frac{\lambda^1}{1!}e^{-\lambda}\right] \\
&\approx .594.
\end{aligned}
$$

Empirical Rule with Binomial and Poisson Random Variables

9. If you flip a fair coin 100 times, would it be unusual to get 42 heads and 58 tails?

> **Solution:** Let X be the number of heads. Then, X is binomial with $n = 100$ and $p = 0.5$. Thus, its expecation and standard deviation are
>
> $$\mu = np = (100)(0.5) = 50,$$
>
> and
>
> $$\sigma = \sqrt{np(1-p)} = \sqrt{(100)(0.5)(1-0.5)} = 5.$$
>
> Since $np \geq 15$ and $n(1-p) \geq 15$, we can use the empirical rule to approximate the distribution of X. Thus, approximately 95% of the time, X will be in the range $\mu \pm 2\sigma$, or $(40, 60)$. So, it would not be unusual get observe $X = 42$.

10. If X is a Poisson random variable with $\lambda = 225$, would it be unusual to get a value of X which is less than 190?

> **Solution:** Set
>
> $$\mu = E(X) = \lambda = 255,$$
> $$\sigma = \text{sd}(X) = \sqrt{\lambda} = 15.$$
>
> Define z to be the number of standard deviations above the mean that 190 is, i.e.
>
> $$190 = \mu + \sigma z.$$
>
> Then,
>
> $$z = \frac{190 - \mu}{\sigma} = \frac{-35}{15} \approx -2.33.$$
>
> A value of X which is below 190 is more than 2.33 standard deviations below the mean of X. The empirical rule tells us that observations more than 2 standard deviations away from the mean are unusual (they occur less than 95% of the time). Therefore, values of X below 190 are unusual.

11. The probability is 0.10 that a person reached on a "cold call" by a telemarketer will make a purchase. If the telemarketer calls 200 people, would it be unusual for them to get 30 purchases?

Solution: Let X be the number of purchases. This is a Binomial random variable with size $n = 200$ and success probability $p = 0.10$. Thus, the expectation and standard deviation of X are

$$\mu = np = (200)(.10) = 20$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{(200)(.10)(.90)} = \sqrt{18} \approx 4.2$$

Since $np \geq 15$ and $np(1-p) \geq 15$, the distribution of X can be approximated by the empirical rule. Using the empirical rule approximation, 95% of the time, X will be in the range $\mu \pm 2\sigma$, or (11.6, 28.4), and 99.7% of the time, X will be in the range $\mu \pm 3\sigma$, or (7.4, 32.6). We would see $X \geq 30$ less than 5% of the time. It would be unusual to see X = 30, but not highly unusual.