Homework #3 – Due Thursday, Jul. 26
STAT-UB.0001 – Statistics for Business Control

Problem 1

A multiple-choice quiz has 20 questions. Each question has five possible answers, of which only one is correct.

(a) What is the probability that sheer guesswork will yield exactly 10 correct answers?

Solution: The number of correct answers, X, is binomial with $n = 20$ and $p = 1/5 = 0.2$.

$$P(X = 10) = \binom{20}{10}(.20)^{10}(.80)^{10} = 0.00203$$

(b) What is the expected number of correct answers by sheer guesswork?

Solution:

$$E[X] = (20)(.2) = 4.$$

(c) Suppose 5 points are awarded for a correctly answered question. How many points should be deducted for an incorrectly answered question, so that for a student guessing randomly, the expected score on a question is zero? (Most standardized tests use this method to set penalties for guessing.)

Solution: If we let c denote the cost for guessing incorrectly, and if Y denotes the score for a question on which a student guesses randomly, then Y has the PDF:

y	$-c$	5
$p(y)$	0.80	0.20

For the expected score to be zero, we must have

$$\mathrm{E}[Y] = (0.80)(-c) + (0.20)(5) = 0,$$

so that

$$c = \frac{(0.20)(5)}{(0.80)} = 1.25.$$

(d) If a student is able to correctly eliminate one option as a possible correct answer but is still guessing randomly, what happens to his/her expected score for that question? Use your answer to (c) as the number of points being deducted for an incorrect answer.

Solution: In this case Y has the PDF

y	-1.25	5
$p(y)$	0.75	0.25

The expected score is

$$\mathrm{E}[Y] = (0.75)(-1.25) + (0.25)(5) = 0.3125.$$

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Problem 2

A Motel has 16 bedrooms. From past experience, the manager knows that 20% of the people who make room reservations don't show up. The manager accepts 20 reservations. If a customer with a reservation shows up and the motel has run out of rooms, it is the motel's policy to pay \$100 as compensation to the customer.

(a) What is the expected number of customers that will show up?

Solution: Let X denote the number of customers that show up. This is binomial with $n = 20$ and $p = 0.80$.

We can compute

$$\mathrm{E}[X] = (20)(0.80) = 16.$$

(b) What is the expected value of the compensation that the motel must pay?

Solution: The manager will have to pay compensation if $X > 16$. We can compute

$$\begin{aligned}
P(X = 17) &= \binom{20}{17}(.80)^{17}(.20)^3 = 0.2054 \\
P(X = 18) &= \binom{20}{18}(.80)^{18}(.20)^2 = 0.1369 \\
P(X = 19) &= \binom{20}{19}(.80)^{19}(.20)^1 = 0.0576 \\
P(X = 20) &= \binom{20}{20}(.80)^{20}(.20)^0 = 0.0115
\end{aligned}$$

The probability that the manager will not pay compensation is

$$P(X \leq 16) = 1 - (0.2054 + 0.1369 + 0.0576 + 0.0115) = 0.5886.$$

Let Y denote the amount of compensation that the manager has to pay. We can use the probabilities above to compute the PDF of Y:

x	≤ 16	17	18	19	20
y	0	100	200	300	400
$p(y)$	0.5886	0.2054	0.1369	0.0576	0.0115

The expected value of Y is

$$\begin{aligned}
\mathrm{E}[Y] &= (0.5886)(0) + (0.2054)(100) + (0.1369)(200) + (0.0576)(300) + (0.0115)(400) \\
&= 69.8.
\end{aligned}$$

The expected compensation is \$69.8.

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Problem 3

An automatic car wash takes exactly 5 minutes to wash a car. On average, 8 cars per hour arrive at the car wash. Suppose the number of cars arrive follows a Poisson distribution. Now suppose 15 minutes before closing time, 3 cars are in line. If the car wash is in continuous use until closing time, what is the chance that anyone will be in line at closing time?

Solution: Let X denote the number of cars that arrive in the 15 mins. This is a Poisson with $\lambda = \frac{(8)(15)}{(60)} = 2$. If one or more cars arrive at the car wash within the 15 mins, then there will be someone in line at closing time. Thus, we can compute

$$\begin{aligned} \mathrm{P}(X > 0) &= 1 - \mathrm{P}(X = 0) \\ &= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} \\ &= 1 - \frac{(1)(e^{-2})}{1} \\ &= 0.864. \end{aligned}$$

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Problem 4

Suppose that you throw two dice. Each die can come up as 1, 2, 3, 4, 5 or 6, and the results from the two dice are independent of each other. We are interested in the random variable X, the sum of the two numbers that land face up. The possible values for X are $2, 3, \ldots, 12$.

(a) Make a table giving the probability distribution of X. Explain briefly how you did the calculations.

Solution: The first and second rolls are each equally likely to be any of the numbers from 1–6. We can compute a table of the values of X for each pair (first roll, second roll):

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Each of the 36 sample points is equally likely. There is one sample point with $X = 2$, two sample points with $X = 3$, etc. Thus, we have the pdf

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b) Show that $E(X) = 7$ and $\mathrm{var}(X) = 210/36 = 5.833$

Solution:

$$\begin{aligned}
\mathrm{E}[X] &= (1/36)(2) + (2/36)(3) + (3/36)(4) + (4/36)(5) + (5/36)(6) + (6/36)(7) \\
&\quad + (5/36)(8) + (4/36)(9) + (3/36)(10) + (2/36)(11) + (1/36)(12) \\
&= 7. \\
\mathrm{var}[X] &= (1/36)(2-7)^2 + (2/36)(3-7)^2 + (3/36)(4-7)^2 + (4/36)(5-7)^2 + (5/36)(6-7)^2 \\
&\quad + (6/36)(7-7)^2 + (5/36)(8-7)^2 + (4/36)(9-7)^2 + (3/36)(10-7)^2 \\
&\quad\quad + (2/36)(11-7)^2 + (1/36)(12-7)^2 \\
&= \frac{210}{36} \\
&= 5.833.
\end{aligned}$$

(c) The distribution of X would look somewhat bell-shaped. (This is not a coincidence. The more dice you toss, the closer the distribution of the sum comes to a normal distribution. More on this later in the course.) For now, let's see how well the empirical rule works. Show that the probability that X is within $E(X) \pm \mathrm{sd}(X)$ is $24/36 = 0.667$. Show that the probability that X is within $E(X) \pm 2\mathrm{sd}(X)$ is $34/36 = 0.944$.

Solution: We have that

$$\begin{aligned}\mu &= E[X] = 7\\ \sigma &= \sqrt{\mathrm{var}(X)} = \sqrt{210/36} = 2.415.\end{aligned}$$

We can compute the z scores for the different values of x using the formula $z = (x - \mu)/\sigma$.

x	2	3	4	5	6	7	8	9	10	11	12
z	−2.07	−1.66	−1.24	−0.83	−0.41	0.00	0.41	0.83	1.24	1.66	2.07

With this table, we can see that

$$\begin{aligned}P(-1 < Z < 1) &= P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9)\\ &= \tfrac{4}{36} + \tfrac{5}{36} + \tfrac{6}{36} + \tfrac{5}{36} + \tfrac{4}{36}\\ &= \tfrac{24}{36}\\ &= 0.6667.\end{aligned}$$

Also,

$$\begin{aligned}P(-2 < Z < 2) &= 1 - \{P(X = 2) + P(X = 12)\}\\ &= 1 - (\tfrac{1}{36} + \tfrac{1}{36})\\ &= \tfrac{34}{36}\\ &= 0.9444.\end{aligned}$$

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