Conditional Probability and Counting Rules

STAT-UB.0001 Statistics for Business Control

Ningshan Zhang

IOMS Department
nzhang@stern.nyu.edu

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Review

- Random experiment, sample point, sample space
- Probability
- Events, union and intersection, mutually exclusive events, complementary event
- Additive rule
- Conditional probability, multiplicative rule

Statistical Independence

A and B are independent events if the occurrence of A doesnt change the probability that B occurs. Formally,

- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
- Equivalently, $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
- Equivalently, $\mathbb{P}(B \mid A) = \mathbb{P}(B)$

Example:

- Are mutually exclusive events independent?
- Flip two coins, A = "the first coin shows Heads", B = "The second coin shows Heads".

Statistical Independence

Statistical independence simplifies calculation of probabilities:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Example: Suppose that you flip two fair coins. Let A = "the first coin shows Heads", B = "The second coin shows Heads". Find the probability of getting Heads on both coins, i.e. find $\mathbb{P}(A \cap B)$.

Bayes' Rule

A tool to relate $\mathbb{P}(A \mid B)$ to $\mathbb{P}(B|A)$:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Example: With probability 0.15, a person will pass the job interview for a Data Analyst position. Among those who passed the interview, 60% had taken college courses in Statistics. It happens also that 30% of all those who interviewed had college courses in Statistics. Find the probability that a person with college courses in Statistics will pass the job interview.

Bayes' Rule

More generally, given k mutually exclusive events $B_1, B_2, \ldots, B_k$ such that $\mathbb{P}(B_1) + \mathbb{P}(B_2) + \cdots + \mathbb{P}(B_k) = 1$, then

$$\begin{aligned}\mathbb{P}(B_i \mid A) &= \frac{\mathbb{P}(A \cap B_i)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A \mid B_k)\mathbb{P}(B_k)}\end{aligned}$$

Example: Suppose that 1% of population have a special disease. A blood test detects the disease with probability 0.95 when it is present, but also falsely detects it when it's not present with probability 0.02. Test shows that a person has the disease; what is the probability that he indeed has it?

Counting

In some probability problems, all sample points are equally likely. Then

$$\mathbb{P}(A) = \frac{\text{no. of sample points in } A}{\text{no. of sample points in } \Omega}$$

- In simple experiments, we can list out all the sample points. Example: flip a coin twice, then $\Omega = \{HH, HT, TH, TT\}$.
- In more complex experiments, we need to use counting rules to help us.

The Multiplication Rule

Example: A man has 4 pair of pants, 6 shirts, 8 pairs of socks, and 3 pairs of shoes. How many ways can he get dressed?

- How many ways can he pick a pair of pants?
- How many ways can he pick a pair of pants and a shirt?
- How many ways can he pick a pair of pants, a shirt, and a pair of socks?
- How many ways can he get dressed?

The Multiplication Rule

Suppose k experiments are performed in order, where the first experiment has n_1 outcomes, the second experiment has n_2 outcomes, ..., the k-th experiment has n_k outcomes. The total number of different outcomes is the product

$$n_1 \cdot n_2 \cdots n_k.$$

Permutations Rule

Example: How many ways can 5 people stand in line?

- How many ways to chose the first person?
- How many ways to lineup the first two person?
- $\dots$
- How many ways to lineup 5 people?

Arrange n objects

Number of ways to arrange n objects is $n!$.

- "!" is the factorial symbol, $n! = n(n-1)(n-2)\cdots 1$.
- Define $0! = 1$

Permutations Rule

Example: Twelve people belong to a club. How many ways can they pick a president, vice-president, secretary, and treasurer?

Arrange k out of n objects

The number of ways to arrange k out of n objects is

$$P(n, k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Combinations Rule

Example: Twelve people belong to a club. How many ways can they pick 2 people to be on a committee to plan a party?

Pick unordered k out of n objects

Number of ways to pick *unordered* k out of n objects is

$$C(n,k) = \frac{\text{no. of ways to pick } \textit{ordered } k \text{ objects out of } n}{\text{no. of ways to order } k \text{ objects}} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Also written as $\binom{n}{k}$. Define $C(n,0) = 1$.

Permutations vs Combinations

Permutations are for lists (order matters) and combinations are for groups (order does not matter). They are related by

$$P(n,k) = C(n,k)P(k,k).$$

- Permutations: A club consists of five members. How many ways are there of selecting three officers: president, secretary and treasurer?
- Combinations: A club consists of five members. How many ways are there of selecting a group of three people?

Probability Problems with Counting Rules

There are 5 messages on your answering machine, from A, B, C, D, and E, left in a random order.

- What is the probability that they are in alphabetical order (ABCDE)?
- What is the probability that A is immediately after D?
- If you only listen to the first three messages, what is the probability that these are from person A, B and C?
- If you only listen to the first three messages, what is the probability that E is among them?

Summary

- Independent Events
- Bayes' Rule
- Counting Rules: permutations and combinations