Conditional Probability and Counting Rules – Solutions
STAT-UB.0001 – Statistics for Business Control

Multiplicative Rule

1. Suppose you run a lottery in the class of 20 students. You put 1 red ball and 19 green balls into a blackbox. One by one, each student randomly picks a ball from the blackbox, and whoever gets the red ball wins the lottery.

   (a) What is the probability for the first student to win the lottery?

   Solution:
   $$\frac{1}{20}$$

   (b) What is the probability for the second student to win the lottery?

   Solution: Let A="the first student wins", and B="the second student wins". Then,
   $$\begin{aligned} P(B) &= P(A^c \cap B) \\ &= P(A^c)P(B \mid A^c) \\ &= \frac{19}{20}\frac{1}{19} \\ &= \frac{1}{20} \end{aligned}$$

   (c) What is the probability for the last student to win the lottery?

   Solution: Let
   $$\begin{aligned} A_1 &= \text{the first student wins} \\ A_2 &= \text{the second student wins} \\ &\vdots \\ A_{20} &= \text{the 20-th student wins} \end{aligned}$$
   Then,
   $$\begin{aligned} P(A_{20}) &= P(A_1^c \cap A_2^c \cap \cdots \cap A_{19}^c \cap A_{20}) \\ &= P(A_1^c)P(A_2^c \mid A_1^c)P(A_3^c \mid A_1^c \cap A_2^c) \\ &\cdots P(A_{19}^c \mid A_1^c \cap A_2^c \cap \cdots A_{18}^c)P(A_{20} \mid A_1^c \cap A_2^c \cap \cdots A_{19}^c) \\ &= \frac{19}{20}\frac{18}{19}\frac{17}{18}\cdots\frac{1}{2} \\ &= \frac{1}{20} \end{aligned}$$

Independence

2. Suppose that you flip two fair coins. Let $A =$"the first coin shows Heads," $B =$ "The second coin shows Heads." Find the probability of getting Heads on both coins, i.e. find $P(A \cap B)$.

Solution: The long way to solve this problem is to write out the elementary outcomes and their probabilities:

Outcome	Probability
HH	$\frac{1}{4}$
HT	$\frac{1}{4}$
TH	$\frac{1}{4}$
TT	$\frac{1}{4}$

Since $A \cap B = \{\text{HH}\}$, it follows that

$$\mathrm{P}(A \cap B) = \tfrac{1}{4}.$$

We can solve this problem much more expediently using the independence of A and B:

$$\begin{aligned}\mathrm{P}(A \cap B) &= \mathrm{P}(A)\,\mathrm{P}(B \mid A)\\ &= \mathrm{P}(A)\,\mathrm{P}(B)\\ &= (\tfrac{1}{2})(\tfrac{1}{2})\\ &= \tfrac{1}{4}.\end{aligned}$$

3. Suppose that you roll two dice. What is the probability of getting exactly one 6?

Solution: Define the following events:

$$\begin{aligned}A &= \text{“6 on the first roll,”}\\ B &= \text{“6 on the second roll,”}\end{aligned}$$

Using the shorthand $\bar{A} = A^c$ and $A\bar{B} = A \cap \bar{B}$, the event "exactly one 6" can be written as

$$\text{“exactly one 6”} = A\bar{B} \cup \bar{A}B$$

These events are mutually exclusive, so

$$\mathrm{P}(\text{exactly one 6}) = \mathrm{P}(A\bar{B}) + \mathrm{P}(\bar{A}B)$$

Using the independence of events A and B, we get

$$\mathrm{P}(A\bar{B}) = \mathrm{P}(A)\,\mathrm{P}(\bar{B}) = (\tfrac{1}{6})(\tfrac{5}{6})$$
$$\mathrm{P}(\bar{A}B) = \mathrm{P}(\bar{A})\,\mathrm{P}(B) = (\tfrac{5}{6})(\tfrac{1}{6})$$

Note that these two expressions are equal. Thus,

$$\mathrm{P}(\text{exactly one } 6) = 2 \cdot \tfrac{1}{6} \cdot \tfrac{5}{6} \approx 28\%$$

4. Suppose that you sell fire insurance policies to two different buildings in Manhattan, located in different neighborhoods. You estimate that the buildings have the following chances of being damaged by fire in the next 10 years: 5%, and 1%. Assume that fire damages to the two buildings are independent events. Compute the probability that exactly one building gets damaged by fire in the next 10 years.

Solution:

$$(.05)(.99) + (.95)(.01) = .059 = 5.9\%$$

5. Consider the following experiment. A hat contains two coins:

   - one coin, the "fair" coin, has 50% chance of heads and 50% chance of tails on every flip;
   - the other coin, the "heads" coin, has heads on both sides, so it always lands heads on every flip.

   You reach into the hat and pull out a random coin, equally likely to get the fair coin or the heads coin. Then, you flip this coin twice.

   Define events A and B as

$$\begin{aligned} A &= \text{the first flip lands heads} \\ B &= \text{the second flip lands heads.} \end{aligned}$$

   (a) Based on your intuition, do you think that A and B independent events?

   (b) Compute $P(A)$.

   Solution: There are two possibilities with equal chances: either we pick the fair coin, or we pick the heads coin. We know that

$$P(\text{fair coin}) = P(\text{heads coin}) = 0.5$$

   Given the coin, it is easy to compute the probabilities of heads:

$$\begin{aligned} P(\text{heads on first flip} \mid \text{fair coin}) &= 0.5 \\ P(\text{heads on first flip} \mid \text{heads coin}) &= 1.0 \end{aligned}$$

   Finally,

$$\begin{aligned} P(A) &= P(\text{fair coin})P(\text{heads on first} \mid \text{fair coin}) \\ &\quad + P(\text{heads coin})P(\text{heads on first} \mid \text{heads coin}) \\ &= (0.5)(0.5) + (0.5)(1.0) \\ &= 0.75. \end{aligned}$$

   (c) Compute $P(A \cap B)$.

   Solution: Given the coin, the first and the second flips are independent:

$$\begin{aligned} P(\text{heads on both flips} \mid \text{fair coin}) &= P(\text{heads on first flip} \mid \text{fair coin}) \\ &\qquad \cdot P(\text{heads on second flip} \mid \text{fair coin}) \\ &= (0.5)(0.5) \\ &= 0.25. \end{aligned}$$

   Similarly,

$$P(\text{heads on both flips} \mid \text{heads coin}) = 1.0.$$

Now,

$$\begin{aligned}P(A) &= P(\text{fair coin})P(\text{heads on both} \mid \text{fair coin}) \\ &\qquad + P(\text{heads coin})P(\text{heads on both} \mid \text{heads coin}) \\ &= (0.5)(0.25) + (0.5)(1.0) \\ &= 0.625.\end{aligned}$$

(d) Use your answers to parts (b) and (c) to either prove or disprove that A and B are independent.

Solution: To check for independence, we compare the product $P(A)P(B)$ with $P(A \cap B)$. Noting that $P(A) = P(B)$, we have

$$\begin{aligned}P(A)P(B) &= (0.75)(0.75) \\ &= 0.5625.\end{aligned}$$

Clearly, $P(A)P(B) \neq P(A \cap B)$. Thus, the events are not independent.

To get some more intuition for what is happening here, note that

$$\begin{aligned}P(B) &= 0.75, \\ P(B \mid A) &= P(B \cap A)/P(A) \\ &= 0.625/0.75 \\ &= 0.833.\end{aligned}$$

That is, before performing the experiment, we have a 75% chance of getting a heads on the second flip. In the middle of the experiment, if we see that the first flip is heads, then we have an 83.3% chance of getting heads on the next flip. Why is this? After we see event A, we gain some information relevant to event B, namely that it is more likely we have selected the heads coin.

Bayes' Rule

6. With probability 0.15, a person will pass the job interview for a Data Analyst position. Among those who passed the interview, 60% had taken college courses in Statistics. It happens also that 30% of all those who interviewed had college courses in Statistics. Find the probability that a person with college courses in Statistics will pass the job interview.

Solution: The information in the problem is

$$\begin{aligned} \mathrm{P}(\mathrm{Pass}) &= .15 \\ \mathrm{P}(\mathrm{Stats}) &= .30 \\ \mathrm{P}(\mathrm{Stats} \mid \mathrm{Pass}) &= .60 \end{aligned}$$

The problem is asking us to compute the quantity $\mathrm{P}(\mathrm{Pass} \mid \mathrm{Stats})$. Using Bayes' rule,

$$\begin{aligned} \mathrm{P}(\mathrm{Pass} \mid \mathrm{Stats}) &= \mathrm{P}(\mathrm{Stats} \mid \mathrm{Pass}) \cdot \frac{\mathrm{P}(\mathrm{Pass})}{\mathrm{P}(\mathrm{Stats})} \\ &= (.60) \cdot \frac{(.15)}{(.30)} \\ &= .30. \end{aligned}$$

That is, there is a 30% chance that a person with college courses in statistics will pass the exam.

7. Amazon.com maintains a list of all registered customers, along with their email addresses. During July, they sent coupons to 20% of their customers. They recorded that 5% of their customers made purchases in July, and 40% of all purchases were made with coupons. In this problem we will compute the proportion of customers sent a coupon in July who made a purchase in that month. For simplicity, we will assume that customers either make 0 or 1 purchases in July.

(a) Consider a random customer, and define two events:

$$\begin{aligned}\text{Coupon} &= \text{the customer received a coupon in July,}\\ \text{Purchase} &= \text{the customer made a purchase in July.}\end{aligned}$$

Express all percentages given in the problem statement as probabilities or conditional probabilities of these two events. Example: P(Coupon) = 0.20.

Solution: The information in the problem is

$$\begin{aligned}\text{P(Coupon)} &= .20\\ \text{P(Purchase)} &= .05\\ \text{P(Coupon} \mid \text{Purchase)} &= .40\end{aligned}$$

(b) Use Bayes' rule to compute the proportion of custoers sent a coupon in July who made a purchase that month.

Solution: The problem is asking us to compute the quantity P(Purchase | Coupon). Using Bayes' rule,

$$\begin{aligned}\text{P(Purchase} \mid \text{Coupon)} &= \text{P(Coupon} \mid \text{Purchase)} \cdot \frac{\text{P(Purchase)}}{\text{P(Coupon)}}\\ &= (.40) \cdot \frac{(.05)}{(.20)}\\ &= .10.\end{aligned}$$

8. Suppose that 1% of population have a special disease. A blood test detects the disease with probability 0.95 when it is present, but also falsely detects it when it's not present with probability 0.02. Test shows that a person has the disease; what is the probability that he indeed has it?

Solution: The information in the problem is

$$\begin{aligned} \mathrm{P}(\text{Disease}) &= .01 \\ \mathrm{P}(\text{Diagnosed} \mid \text{Disease}) &= .95 \\ \mathrm{P}(\text{Diagnosed} \mid \text{not Disease}) &= .02 \end{aligned}$$

The problem is asking us to compute P(Disease | Diagnosed). With Bayes' Rule,

$$\begin{aligned} &\mathrm{P}(\text{Disease} \mid \text{Diagnosed}) \\ &= \frac{\mathrm{P}(\text{Disease})\,\mathrm{P}(\text{Diagnosed} \mid \text{Disease})}{\mathrm{P}(\text{Disease})\,\mathrm{P}(\text{Diagnosed} \mid \text{Disease}) + \mathrm{P}(\text{not Disease})\,\mathrm{P}(\text{Diagnosed} \mid \text{not Disease})} \\ &= \frac{(.01)(.95)}{(.01)(.95) + (.99)(.02)} \\ &\approx 32.4\% \end{aligned}$$

9. A desk lamp produced by a company was found to be defective (D). There are three factories (A, B, C) where such desk lamps are manufactured. A Quality Control Manager is responsible for investigating the source of found defects. This is what the manager knows about the company's desk lamp production and the possible source of defects:

Factory	% of total production	Probability of defective lamps
A	0.35	0.015
B	0.35	0.010
C	0.30	0.020

If a randomly selected lamp is defective, what is the probability that the lamp was manufactured in factory C?

Solution:

$$\begin{aligned} \mathrm{P}(C \mid D) &= \frac{\mathrm{P}(D \cap C)}{\mathrm{P}(D)} \\ &= \frac{\mathrm{P}(C)\,\mathrm{P}(D \mid C)}{\mathrm{P}(A)\,\mathrm{P}(D \mid A) + \mathrm{P}(B)\,\mathrm{P}(D \mid B) + \mathrm{P}(C)\,\mathrm{P}(D \mid C)} \\ &= \frac{(0.30)(0.02)}{(0.35)(0.015) + (0.35)(0.01) + (0.30)(0.02)} \\ &= 0.407 \end{aligned}$$

The Multiplication Rule

10. A man has 4 pair of pants, 6 shirts, 8 pairs of socks, and 3 pairs of shoes. How many ways can he get dressed?

> **Solution:** Using the multiplication rule, there are
>
> $$4 \cdot 6 \cdot 8 \cdot 3 = 576$$
>
> ways for the man to get dressed.

11. A restaurant offers soup or salad to start, and has 11 entrées to choose from, each of which is served with rice, baked potato, or zucchini. How many meals can you have if you can choose to eat one of their 4 desserts or have no desert?

> **Solution:**
>
> $$2 \cdot 11 \cdot 3 \cdot 5 = 330$$
>
> Note that there are 5 choices for the final course (4 desserts or no dessert).

12. How many answer sheets are possible for a true/false test with 15 questions?

> **Solution:**
>
> $$2^{15} = 32768$$

Permutations

13. How many ways can 5 people stand in line?

> **Solution:**
>
> $$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$$

14. How many different batting orders are possible for 9 baseball players?

Solution:

$$9! = 362880$$

15. How many ways can 8 books be put on a shelf?

Solution:

$$8! = 40320$$

More Permutations

16. Twelve people belong to a club. How many ways can they pick a president, vice-president, secretary, and treasurer?

Solution:

$$12 \cdot 11 \cdot 10 \cdot 9 = \frac{12!}{8!} = 11880$$

17. In a horse race the first three finishers are said to win, place, and show. How many finishes are possible for a race with 11 horses?

Solution:

$$11 \cdot 10 \cdot 9 = \frac{11!}{8!} = 990$$

18. Five different awards are to be given to a class of 30 students. How many ways can this be done if (a) each student can receive any number of awards, (b) each sutent can receive at most one award?

Solution: (a) $30^5 = 24300000$
(b) $30!/(25!) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = 17100720$

Combinations

19. A club has 12 members.

    (a) A club has 12 people. How many ways can they pick 2 people to be on a committee to plan a party?

    Solution:

    $$\binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66.$$

    (b) How many ways can they pick 4 people to be on a committee to plan a party?

Solution:

$$\binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495.$$

20. A restaurant offers 15 possible toppings for its pizza. How many different pizzas with 3 toppings can be ordered?

Solution:

$$\binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455$$

21. We are going to pick 5 cards out of a deck of 52. In how many ways can this be done?

Solution:

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2598960.$$