Random Variables – Solutions
STAT-UB.0001 – Statistics for Business Control

Permutations and Combinations

1. There are 5 messages on your answering machine, from A, B, C, D, and E, left in a random order.

   (a) What is the probability that they are in alphabetical order (ABCDE)?

   Solution: There are 5! number of ways to arrange 5 people, and ABCDE is one of them. So the probability is

   $$\frac{1}{\#\{\text{arrange five people}\}} = \frac{1}{5!}$$

   (b) What is the probability that A is immediately after D?

   Solution: Treat "DA" as a single person F, and there are 4! number of ways to arrange the 4 people B, C, E, and F. So the probability is

   $$\frac{\#\{\text{arrange five people, A is immediately after D}\}}{\#\{\text{arrange five people}\}} = \frac{4!}{5}$$

   (c) If you only listen to the first three messages, what is the probability that these are from person A, B and C?

   Solution: There are $\binom{5}{3}$ ways to chose a combination of three messages, and $\{A, B, C\}$ is one of the combinations. So the probability is

   $$\frac{1}{\#\{\text{choose a combination of three people out of five}\}} = \frac{1}{\binom{5}{3}}$$

   (d) If you only listen to the first three messages, what is the probability that E is among them?

   Solution: In order to chose a combination of three people such at E is among them, the first step is to select E, and the second step is to chose two people out of the remaining 4. So the probability is

   $$\frac{\#\{\text{choose a combination of three people out of five, E is among them}\}}{\#\{\text{choose a combination of three people out of five}\}} = \frac{1 \cdot \binom{4}{2}}{\binom{5}{3}}$$

2. **New York state lotto.** You pick six of the numbers 1 through 54, and then in a televised drawing six of the numbers are selected. If all six of your numbers are selected then you win a share of the first place prize. If five or four of your numbers are selected you win a share of the second or third prize.

   (a) How many ways are there to select 6 numbers for the lotto ticket?

   Solution:
   $$\binom{54}{6} = \frac{54 \cdot 53 \cdot 52 \cdot 51 \cdot 50 \cdot 49}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 25827165$$

   (b) How many ways are there to select a first prize number?

   Solution:
   $$1$$

   (c) What is the probability of selecting a first prize number?

   Solution:
   $$\begin{aligned} P(\text{first prize}) &= \frac{\#\{\text{lotto tickets that match all six numbers}\}}{\#\{\text{lotto tickets}\}} \\ &= \frac{1}{\binom{54}{6}} \\ &= 1/25827165 \\ &= 0.000004\% \end{aligned}$$

   (d) What is the probability of selecting a second prize number?

   Solution: How many lotto tickets have five matches and one mismatch from our pick? Consider the following sequence of experiments for enumerating all such possibilities: (1) pick 5 numbers out of our six to match; (2) pick 1 number to mismatch. There are $\binom{6}{5}$ possible outcomes for experiment (1), and $\binom{48}{1}$ possible outcomes for experiment (2). (Note that $48 = 54 - 6$; there are 48 numbers in the range 1–54 which aren't on our lotto ticket.) Thus,
   $$\begin{aligned} P(\text{second prize}) &= \frac{\#\{\text{lotto tickets that match exactly five numbers}\}}{\#\{\text{lotto tickets}\}} \\ &= \frac{\binom{6}{5}\binom{48}{1}}{\binom{54}{6}}. \end{aligned}$$

   The computation for third prize is similar to the computation for second prize:

$$
\begin{aligned}
P(\text{third prize}) &= \frac{\#\{\text{lotto tickets that match exactly four numbers}\}}{\#\{\text{lotto tickets}\}} \\
&= \frac{\binom{6}{4}\binom{48}{2}}{\binom{54}{6}}.
\end{aligned}
$$

3. **Quality assurance.** Suppose we have a batch of 100 light bulbs, which contains 5 defective bulbs. If we pick 10 for testing, what is the probability that no bulbs in the sample are defective? We can answer this question in three steps.

   (a) How many ways are there of picking 10 bulbs for testing out of 100?

   Solution:
   $$\binom{100}{10}$$

   (b) How many ways are there of picking 10 non-defective bulbs?

   Solution:
   $$\binom{95}{10}$$

   (c) What is the probability that there are no defective bulbs in your sample of 10?

   Solution:
   $$\begin{aligned} P(\text{no defects in sample}) &= \frac{\binom{95}{10}}{\binom{100}{10}} \\ &= 58\%. \end{aligned}$$

4. **The Birthday Problem.** A class has 20 students. What is the probability that at least two students have the same birthday? Assume that each person in the class was assigned a random birthday between January 1 and December 31.

Solution: Assume that everyone in the class is randomly assigned a birthday, which corresponds to number between 1 and 365 representing the day of the year. It turns out to be much easier to compute the probability using the complement rule, as

$$\text{P(at least 2 people have the same birthday)} = 1 - \text{P(all 20 birthdays are different)}.$$

We use counting rules to compute the probability that all 20 birthdays are different. We first compute the size of sample space. That is 365^{20}.

Next, we compute the number of ways to assign 365 birthdays to 20 students, such that no two students can have the same birthday. That is $P(365, 20) = \frac{365!}{345!}$.

Thus the probability of all 20 birthdays are different is

$$\frac{P(365,20)}{365^{20}} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-19}{365}.$$

Probability Distribution Function and Expectation

5. Consider the following game:

   1. You pay \$6 to flip a coin.
   2. If the coin lands heads, you get \$10; otherwise, you get nothing.

   (a) Would you play this game? Why or why not?

   Solution: It will usually be beneficial to play the game when our expected winnings are positive. In this situation, if we play the game many times, then in the long run we will make a profit.

   (b) What is the random experiment involved in the game? What are the sample space? What are the probabilities of the sample points?

   Solution: The random experiment is the coin flip. The sample points for the coin flip are H and T; each of these has probability $\frac{1}{2}$.

   (c) Let W be the random variable equal to the amount of money you win from playing the game. If you lose money, W will be negative. Find the value of W for each of the sample points.

   Solution:
   The values of the random variable corresponding to the sample points are as follow:

Outcome	W
H	4
T	-6

   (d) Describe W in terms of its probability distribution function (PDF).

   Solution: The PDF of W is given by the table:

w	-6	4
$p(w)$	0.5	0.5

   (e) What are your expected winnings? That is, what is μ, the expectation of W?

   Solution:
   Using the PDF computed in part (c), the expected value of W is

$$\begin{aligned}\mu &= (.5)(-6) + (.5)(4)\\ &= -1.\end{aligned}$$

On averate, we lose \$1 every time we play the game.

6. Suppose you flip two coins. Let X be the random variable which counts the number of heads on the two tosses.

   (a) List all of the sample points of the experiment, along with the corresponding values of X.

   Solution:

Outcome	X
HH	2
HT	1
TH	1
TT	0

   (b) Compute the probability distribution function of X.

   Solution:

x	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

   (c) Compute the expectation of X.

   Solution: Using the PDF we computed in part (b), the expectation of X is

   $$\begin{aligned} \mathrm{E}(X) &= (\tfrac{1}{4})(0) + (\tfrac{2}{4})(1) + (\tfrac{1}{4})(2) \\ &= 1. \end{aligned}$$

   (d) What is the interpretation of the expectation of X?

   Solution: If we were to repeat the experiment many times, getting a different value of X each time, then the average value of X will be close to 1.

7. Let X be a random variable describing the number of cups of coffee a randomly-chosen member of the class drinks on a typical day. There is a 22% chance that the student has one cup, a 16% chance that the student has two cups, a 16% chance that the student has three cups, an 11% chance that the student has four cups, and a 3% chance that the student has five cups. Also, there is a 32% chance that the student doesn't drink any coffee.

   (a) Let $p(x)$ be the probability distribution function of X. Fill in the following table:

x	0	1	2	3	4	5
$p(x)$						

Solution:

x	0	1	2	3	4	5
$p(x)$	.32	.22	.16	.16	.11	.03

(b) Find $\mathrm{E}(X)$, the expectation of X.

Solution:

$$\mathrm{E}(X) = (.32)(0) + (.22)(1) + (.16)(2) + (.16)(3) + (.11)(4) + (.03)(5) \quad = 1.61.$$

(c) What is the interpretation of the expectation of X?

Solution: The long-run sample mean. If we performed the random experiment upon which X is measured many times, getting a different value of X each time, then the sample mean would be very close to the expectation of X. In particular, if every day we sample a student from the class and measure how many cups of coffee they drink, then after awhile, the average number of cups of coffee will be close to the expectation (1.61).

Variance and Standard Deviation

8. This is a continuation of problem 7.

   (a) Find $\text{var}(X)$ and $\text{sd}(X)$, the variance and standard deviation of X, the number of cups of coffee that a random student from the class drinks on a typical day.

   Solution:

   $$\begin{aligned}\text{var}(X) &= (.32)(0-1.61)^2 + (.22)(1-1.61)^2 + (.16)(2-1.61)^2 + (.16)(3-1.61)^2 \\ &\quad + (.11)(4-1.61)^2 + (.03)(5-1.61)^2 \\ &= 2.2179\end{aligned}$$

   The standard deviation of X is given by

   $$\text{sd}(X) = \sqrt{\text{var}(X)} = \sqrt{2.2179} = 1.48.$$

   (b) What is the interpretation of the standard deviation of X?

   Solution: The long-run sample standard deviation. If we performed the random experiment upon which X is measured many times, getting a different value of X each time, then the sample standard deviation would be very close to the standard deviation of X. In particular, if every day we sample a student from the class and measure how many cups of coffee they drink, then after awhile, the sample standard deviation of the number of cups of coffee will be close to the 1.48.

9. Consider the following game:

   1. You pay \$6 to pick a card from a standard 52-card deck.
   2. If the card is a diamond ($\diamondsuit$), you get \$22; if the card is a heart ($\heartsuit$), you get \$6; otherwise, you get nothing.

   Perform the following calculations to decide whether or not you would play this game.

   (a) Let W be the random variable equal to the amount of money you win from playing the game. If you lose money, W will be negative. Find the PDF of W.

   Solution: The sample points corresponding to the suit of the card are $\spadesuit$, $\heartsuit$, $\clubsuit$, and $\diamondsuit$; each of these has probability $\frac{1}{4}$. The values of the random variable W corresponding to the sample points are as follow:

Outcome	W
$\spadesuit$	-6
$\heartsuit$	0
$\clubsuit$	-6
$\diamondsuit$	16

Thus, the PDF of W is given by the table:

w	-6	0	16
$p(w)$	0.50	0.25	0.25

(b) What are your expected winnings? That is, what is μ, the expectation of W?

Solution:
Using the PDF computed in part (a), the expected value of W is

$$\begin{aligned}\mu &= (.50)(-6) + (.25)(0) + (.25)(16) \\ &= 1.\end{aligned}$$

On average, we win \$1 every time we play the game.

(c) What is the standard deviation of W?

Solution:
Using the PDF computed in part (a), and the expected value computed in part (b), we compute the variance of W as

$$\begin{aligned}\sigma^2 &= (.50)(-6-1)^2 + (.25)(0-1)^2 + (.25)(16-1)^2 \\ &= 81.\end{aligned}$$

Thus, the standard deviation of W is

$$\sigma = \sqrt{81} = 9.$$

(d) What are the interpretations of the expectation and standard deviation of W?

Solution: If we played the game many many times, then the average of our winnings over all times we played would be close to the \$1, and the standard deviations of our winnings over all times we played would be close to \$9.

Properties of Expectation

10. **Affine Transformations.** Let X be a random variable with expectation $\mu_X = 2$. What is the expectation of $5X + 2$?

Solution:
$$5\mu_X + 2 = 12.$$

11. **Sums of Random Variables.** Let X and Y be random variables with $\mu_X = 1$, $\mu_Y = -5$. What is $\mathrm{E}(X + Y)$?

Solution:
$$\mathrm{E}(X + Y) = \mu_X + \mu_Y = 1 + (-5) = -4.$$

12. Let X and Y be random variables with $\mu_X = -2$, $\mu_Y = 3$.

    (a) Find the expectation of $-3X + 2$.

    Solution:
    $$\mathrm{E}(-3X + 2) = -3\mu_X + 2 = -3(-2) + 2 = 8.$$

    (b) Find the expectation of $X + Y$.

    Solution:
    $$\mathrm{E}(X + Y) = \mu_X + \mu_Y = 1.$$

13. You invite four people to go out to dinner on Friday night. The attendance probabilities for the four potential guests are 50%, 20%, 30%, and 90%.

    (a) Find the expected number of guests.

    Solution: Let X be the number of guests. Then, X can be written as
    $$X = Y_1 + Y_2 + Y_3 + Y_4,$$
    where
    $$Y_i = \begin{cases} 1 & \text{if guest } i \text{ attends,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathrm{E}[Y_1] = .50$, $\mathrm{E}[Y_2] = .20$, $\mathrm{E}[Y_3 = .30]$, and $\mathrm{E}[Y_4] = .90$, so

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[Y_1 + Y_2 + Y_3 + Y_4] \\ &= \mathrm{E}[Y_1] + \mathrm{E}[Y_2] + \mathrm{E}[Y_3] + \mathrm{E}[Y_4] \\ &= .50 + .20 + .30 + .90 \\ &= 1.9. \end{aligned}$$

(b) The dinner will be a *prix fixe* meal, costing \$50 per person. What is the expected total cost for yourself and your guests?

Solution: The total cost is $C = 50 + 50X$. Thus,

$$\begin{aligned} \mathrm{E}[C] &= \mathrm{E}[50 + 50X] \\ &= 50 + 50\,\mathrm{E}[X] \\ &= 50 + 50(1.9) \\ &= 145. \end{aligned}$$

The expected total cost is \$145.

(c) What is the interpretation of your answer to part (b)?

Solution: If there were many similar nights with the same circumstances, then the average cost of all of the dinners would be \$145.