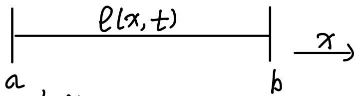
Lecture 01

Conservation laws



$$\frac{d}{dt}\int_a^b \rho(x)\,dx = f_a - f_b$$

assumption: $f_b = f(\rho(b,t))$ (There are also other assumptions).

$f_b = f(\rho(b,t))$: conservation laws, inviscid

$f_b = f(\rho(b,t), \rho_x(b,t))$: NS, viscid

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx + f(\rho(b,t)) - f(\rho(a,t)) = 0$$

Assume ρ is smooth (C^1).

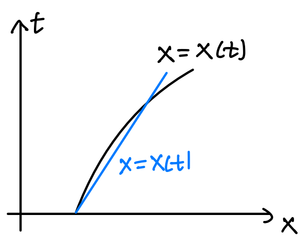
$$\int_a^b \rho_t(x,t)\,dx + \int_a^b f(\rho)_x\,dx = 0 \Rightarrow \int_a^b \rho_t + f(\rho)_x\,dx = 0$$

f is usually smooth $\Rightarrow$ $\rho_t + f(\rho)_x = 0$ p.t. wise

$u_t + f(u)_x = 0$ $\quad$ $f(u) = u \Rightarrow u_t + u_x = 0$

$f(u) = \frac{1}{2}u^2 \Rightarrow u_t + (\frac{u^2}{2})_x = 0$ Burgers' equation

Characteristics



$X = X(t)$ is the solution to the characteristic ODE:

$$\begin{cases} X'(t) = f'(u(X(t),t)) \\ X(0) = X^0 \end{cases}$$

The solution along characteristics is const.

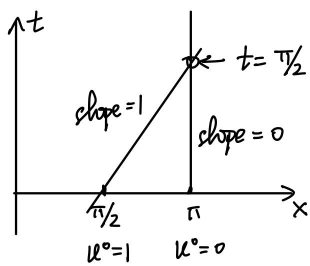
$$\frac{d}{dt}u(X(t),t) = u_x \cdot X'(t) + u_t = u_x f'(u(X(t),t)) + u_t = u_t + f(u(X(t),t))_x = 0.$$

$u(X(t),t) = u(X(0),0)$ is a constant. And $X'(t) = f'(u(X(0),0)) = \text{const}$.

So $X'(t)$ is in fact a straight line.

Burgers equation

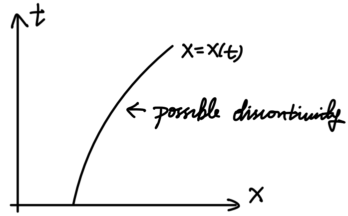
$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 & 0 \le x \le 2\pi \\ u(x,0) = \sin x. & \text{periodic} \end{cases}$$



We assumed that u is <u>differentiable</u>.

But this assumption is wrong.

Weak solutions



$0=\frac{d}{dt}\int_a^b u(x,t)\,dx + f(u(b,t)) - f(u(a,t))$

$=\frac{d}{dt}\left(\int_a^{X(t)} u(x,t)\,dx + \int_{X(t)}^b u(x,t)\,dx\right) + f(u(b,t)) - f(u(a,t)).$

$= u(X(t)^-,t)X'(t) + \int_a^{X(t)} u_t(x,t)\,dx$

$\quad - u(X(t)^+,t)X'(t) + \int_{X(t)}^b u_t(x,t)\,dx$

$\quad + f(u(b,t)) - f(u(a,t))$ Claim: $u_t + f(u)_x = 0$ on either left or right side of $X(t)$

$= X'(t)\left[u(X(t)^-,t) - u(X(t)^+,t)\right] - \int_a^{X(t)} f(u(x,t),t)_x\,dx - \int_{X(t)}^b f(u(x,t),t)_x\,dx$

$\quad + f(u(b,t)) - f(u(a,t))$

$= X'(t)\left[u(X(t)^-,t) - u(X(t)^+,t)\right] - \left(f(u(X(t)^-,t)) - \cancel{f(u(a,t))}\right) - \left(\cancel{f(u(b,t))} - f(u(X(t)^+,t))\right)$

$\quad + \cancel{f(u(b,t))} - \cancel{f(u(a,t))}$

$= X'(t)\left[u(X(t)^-,t) - u(X(t)^+,t)\right] - f(u(X(t)^-,t)) + f(u(X(t)^+,t)).$

Rankine-Hugoniot jump condition. $f(u^+) - f(u^-) = X'(t)(u^+ - u^-).$

Def 1: A p.w. smooth function $u(x,t)$ is a weak solution iff it is (1) smooth and satisfies the PDE p.t. wise except a few smooth curves, (2) at which the R-H jump condition is satisfied. Practical but not good mathematically

Def 2: $u(x,t)$ is a weak solution iff $\forall t_1, t_2$,

$$\int_a^b u(x,t_2)\,dx - \int_a^b u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(b,t))\,dt - \int_{t_1}^{t_2} f(u(a,t))\,dt = 0$$

$\int_0^\infty \int_{-\infty}^{+\infty} \varphi \cdot (u_t + f(u)_x)\,dx\,dt = 0$ assume $\varphi \in C_0^1(\mathbb{R}^2)$,

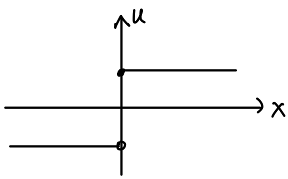
$-\int_0^\infty \int_{-\infty}^{+\infty} \varphi_t\, u + \varphi_x \cdot f(u)\,dx\,dt + \int_{-\infty}^{+\infty} \varphi\, u\,dx\,\Big|_0^{\cancel{\infty}} + \int_0^\infty \cancel{\varphi f(u)}\,dt\,\Big|_{-\infty}^{\infty} = 0$

$-\int_0^\infty \int_{-\infty}^{+\infty} \varphi_t \cdot u + \varphi_x \cdot f(u)\,dx\,dt - \int_{-\infty}^{+\infty} \varphi(x,0)\,u(x,0)\,dx = 0.$ $(*)$

Def 3: $u(x,t)$ is a weak solution iff $\forall \varphi \in C_0^1(\mathbb{R}^2)$, $(*)$ holds.

Examples:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases} \end{cases}$$

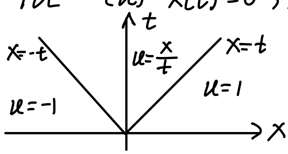


Riemann problem

(1) $u(x,t) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$ is a weak solution

check (i) const satisfies PDE (ii) $X(t)=0$, $X'(t)=0$. $f(u^+) - f(u^-) = \frac{1}{2}\cdot 1^2 - \frac{1}{2}(-1)^2 = 0.$

(2) $u(x,t) = \begin{cases} -1 & x<-t \\ x/t & -t \leq x < t \\ 1 & x \geq t \end{cases}$



(i): $u_t = -\frac{x}{t^2}$, $\left(\frac{u^2}{2}\right)_x = \frac{\partial}{\partial x}\left(\frac{x^2}{2t^2}\right) = \frac{x}{t^2}$ $\Rightarrow$ $u_t + f(u)_x = 0$.

(ii): In fact $u(x,t)$ is cts for $t>0$.

This is also a weak solution.

We want uniqueness, which leads to the entropy solution.

HW #1.

(1) (i) Find the smallest time t^* for which two characteristic lines of Burgers equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & 0 \le x < 2\pi \\ u(x,0) = \sin x & \text{periodic} \end{cases}$$

will intersect

(ii) Generalise the result to $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}$?

(2) (i) How does $u_x(x(t),t)$ behave along the characteristic line $x = x(t)$?

(ii) If $u_x = 0$ at $t=0$ (foot of characteristic line), what is the answer to (i) ?

(iii) If $u_x = 0$ at $t=0$, how does u_{xx} behave along the characteristic line ?

Lecture 02

Entropy Solution

Consider the viscous Burgers equation

$$\begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \\ u^\varepsilon(x,0) = u^0(x) \end{cases} \qquad \varepsilon>0,\ \varepsilon \ll 1.$$

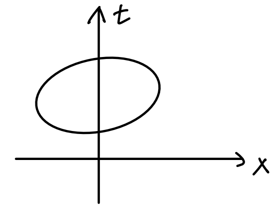
The transition region is $O(\varepsilon)$, so impractical to solve numerically

$u(x,t) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x,t)$ exists, so we define it as the entropy solution of conservation law.

Exercise: Prove $u(x,t)$ is a weak solution. (Prove existence).

This definition is not practical.

Other versions of entropy conditions



1. $U(u)$, $U''(u) \geq 0$ convex U: entropy function

$$U'(u^\varepsilon)(u^\varepsilon_t + f(u^\varepsilon)_x) = U'(u^\varepsilon)\,\varepsilon u^\varepsilon_{xx}$$

$$U(u^\varepsilon)_t + F(u^\varepsilon)_x = \varepsilon\left((U'(u^\varepsilon)u^\varepsilon_x)_x - U''(u^\varepsilon)(u^\varepsilon_x)^2\right)$$

where $F(u) \triangleq \int^u U'(u) f'(u)\,du$ (entropy flux).

$RHS = \varepsilon U(u^\varepsilon)_{xx} - \underline{\varepsilon U''(u^\varepsilon)(u^\varepsilon_x)^2} \leq \varepsilon U(u^\varepsilon)_{xx}$. $\quad u^\varepsilon_x \to \delta$ function $\to \delta^2 \notin L^1$.

$\forall \varphi \in C_0^2(\mathbb{R}\times\mathbb{R}^+)$, $\varphi \geq 0$,

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \varphi(U(u^\varepsilon)_t + F(u^\varepsilon)_x)\,dx\,dt \leq \int_0^{+\infty}\int_{-\infty}^{+\infty} \varphi\,\varepsilon\, U(u^\varepsilon)_{xx}\,dx\,dt$$

$$-\int_0^{+\infty}\int_{-\infty}^{+\infty} U(u^\varepsilon)\varphi_t + F(u^\varepsilon)\varphi_x\,dx\,dt \leq \varepsilon \int_0^{+\infty}\int_{-\infty}^{+\infty} U(u^\varepsilon)\varphi_{xx}\,dx\,dt$$

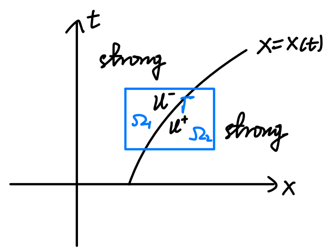
u^ε is bdd uniformly, then by DCT, let $\varepsilon \to 0^+$, ($u^\varepsilon \to u$ a.e.)

$$\underline{-\int_0^{+\infty}\int_{-\infty}^{+\infty} U(u)\varphi_t + F(u)\varphi_x\,dx\,dt \leq 0} \qquad \text{Integral entropy condition}$$

2. If u is a generic weak solution

In smooth region it holds immediately

$$U'(u)\cdot(u_t + f(u)_x) = 0.$$



Now take a region intersects $x(t)$, large enough.

$$-\iint_{\Omega_1} U(u)\varphi_t + F(u)\varphi_x\,dx\,dt - \int_{\Omega_2} U(u)\varphi_t + F(u)\varphi_x\,dx\,dt \leq 0.$$

Apply IBP and Stokes theorem (Check this!).

$$\iint_{\Omega_1} \overbrace{U(u)_t\varphi + F(u)_x\varphi}^{0}\,dx\,dt - \int_\Gamma (F(u^-)\varphi,\ U(u^-)\varphi)\,\vec{n}\,ds$$

$$+\iint_{\Omega_2} \overbrace{U(u)_t\varphi + F(u)_x\varphi}^{0}\,dx\,dt + \int_\Gamma (F(u^+)\varphi,\ U(u^+)\varphi)\cdot\vec{n}\,ds \leq 0 \qquad \vec{n} = \frac{(1, -x'(t))}{\sqrt{1+x'^2(t)}}$$

$$-\int_\Gamma (F(u^-) - x'(t)U(u^-))\frac{\varphi}{\sqrt{1+x'^2(t)}}\,ds + \int_\Gamma F(u^+) - x'(t)U(u^+))\frac{\varphi}{\sqrt{1+x'^2(t)}}\,ds \leq 0. \qquad dx = \frac{1}{\sqrt{\quad}}\,ds$$

$$\Rightarrow -F(u^-) + x'(t)U(u^-) + F(u^+) - x'(t)U(u^+) \leq 0.$$

$$F(u^+) - F(u^-) \leq x'(t)(U(u^+) - U(u^-)).$$

Like RH condition

Example: Burgers equation

① $u(x,t) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$

Take $U(u)=\frac{u^2}{2}$. $F(u)=\int U'(u)f'(u)\,du=\int u\cdot u\,du=\frac{1}{3}u^3$

$u^+=1,\ u^-=1 \Rightarrow U^+=\frac{1}{2},\ U^-=\frac{1}{2}.\quad F^+=\frac{1}{3}\quad F^-=-\frac{1}{3}.$

$F^+-F^-=\frac{2}{3}\neq 0=X'(t)\cdot(U^+-U^-).$

So this is not an entropy solution.

3. Oleinik entropy condition (How to derive)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to X'(t)$, RH condition

Along possible discontinuity curves, $s(u^-,z)\geq s(u^-,u^+)\geq s(z,u^+)$, where $s(a,b)=\frac{f(a)-f(b)}{a-b}$,

$\forall z$ within u^- & u^+. This still works for system.

4. If the conservation law is convex, i.e. $f''>0$, then Lax entropy condition is enough.

$\frac{f(u^-)-f(z)}{u^--z}\geq X''(t)\geq\frac{f(u^+)-f(z)}{u^+-z}$. Let $z\to u^\pm$, $f'(u^-)\geq X'(t)\geq f'(u^+)\Rightarrow \underline{u^-\geq u^+}$.

HW #2

Write a computer program to obtain the exact entropy solution of

$$\begin{cases} u_t+(\frac{u^2}{2})_x=0, & -\pi\leq 0<\pi. \\ u(x,0)=\alpha+\beta\sin x, & \text{periodic}. \end{cases}$$

α,β are constants

(1) Try to transfer the problem to

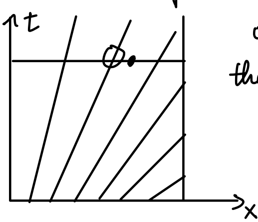
$$\begin{cases} v_t+(\frac{v^2}{2})_x=0, & -\pi\leq 0<\pi \\ v(x,0)=\sin x, & \text{periodic} \end{cases}$$

Consider $x'=ax+bt,\ t'=cx+dt,\ u=ev+f.$

(2) Antisymmetric w.r.t. $x=0$.

(3) Shock speed is 0 by (2), so the computational area is confined

(4) Draw 200 forward characteristics



o: as the initial guess

then apply Newton iteration

Lecture 03

Entropy solution:

1. $u(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$. $\quad u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$
2. Entropy function: $U(u)$, $U'' \geq 0$. $F(u)$: $F'(u) = U'(u) f'(u)$, $F(u) = \int U'(u) f'(u)\,du$.
   $\int_0^{+\infty}\int_{-\infty}^{+\infty} U(u)\varphi_t + F(u)\varphi_x \,dx\,dt \leq 0 \quad \forall \varphi \in C_0^1(\mathbb{R}^+\times\mathbb{R}),\ \varphi\geq 0.$
3. For piecewise smooth weak solutions, $(F(u^+) - F(u^-)) \leq x'(t)(U(u^+) - U(u^-))$.
4. Oleinik: $S(u^-, u) \geq S(u^-, u^+) \geq S(u, u^+)$ $\forall u$ between u^- and u^+ along T.
   $S(a,b) \stackrel{\triangle}{=} \dfrac{f(b)-f(a)}{b-a}$
5. Lax: $u^- \geq u^+$ (sufficient only if $f''(u) \geq 0$)

Entropy solution for Burgers equation

$$\begin{cases} u_x + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = \begin{cases} -1, & x<0 \\ 1, & x\geq 0\end{cases} \end{cases} \quad \text{rarefaction waves}$$

$u(x,t) = \begin{cases} -1, & x<0 \\ 1, & x\geq 0\end{cases}$ is not an entropy solution

$u(x,t) = \begin{cases} -1 & x<-t \\ x/t & -t<x<t \\ 1 & x\geq t\end{cases}$ is an entropy solution

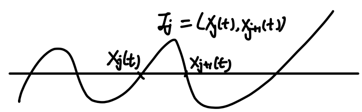
In fact, entropy solution is unique.

Theorem: The entropy solution evolution operator is L^1-contractive, i.e. if

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x)\end{cases} \qquad \begin{cases} v_t + f(v)_x = 0 \\ v(x,0) = v^0(x)\end{cases}$$

Then $\int_{-\infty}^{+\infty} |u(x,t) - v(x,t)|\,dx \leq \int_{-\infty}^{+\infty} |u(x,0) - v(x,0)|\,dx$.

Pf: We use the viscousity solution as the def of entropy solution.



For fixed $\varepsilon>0$, $u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$, $v^\varepsilon_t + f(v^\varepsilon)_x = \varepsilon v^\varepsilon_{xx}$.

$w^\varepsilon = u^\varepsilon - v^\varepsilon$. (Assume w^ε has countably many 0).

$\frac{d}{dt}\int_{-\infty}^{+\infty} |w(x,t)|\,dx = \frac{d}{dt}\sum_j \int_{I_j} |w(x,t)|\,dx = \frac{d}{dt}\sum_j \int_{I_j} s_j(t)\, w(x,t)\,dx$ $\qquad s_j(t) = w(x,t)$ on $(x_j(t), x_{j+1}(t))$.

$= \sum_j \frac{d}{dt}\int_{x_j(t)}^{x_{j+1}(t)} s_j(t)\, w(x,t)\,dx$

$= \sum_j \left[x_{j+1}'(t)\, s_j(t)\, \cancelto{0}{w(x_{j+1}(t),t)} - x_j'(t)\, s_j(t)\, \cancelto{0}{w(x_j(t),t)} + \int_{x_j(t)}^{x_{j+1}(t)} s_j(t)\, w_t(x,t)\,dx \right]$

$= \sum_j \int_{x_j(t)}^{x_{j+1}(t)} s_j(t)\,(u^\varepsilon_t(x,t) - v^\varepsilon_t(x,t))\,dx$ $\quad$ Plug in the PDEs.

$= \sum_j \int_{x_j(t)}^{x_{j+1}(t)} s_j(t)\,(-f(u^\varepsilon)_x + \varepsilon u^\varepsilon_{xx} + f(v^\varepsilon)_x - \varepsilon v^\varepsilon_{xx})\,dx$

$= \sum_j s_j(t)\left[-\cancel{f(u^\varepsilon(x_{j+1}(t),t))} + \cancel{f(u^\varepsilon(x_j(t),t))} + \cancel{f(v^\varepsilon(x_{j+1}(t),t))} - \cancel{f(v^\varepsilon(x_j(t),t))}\right.$ $\qquad w = u^\varepsilon - v^\varepsilon$

$\quad \left. + \varepsilon\,(u^\varepsilon_x(x_{j+1}(t),t) - u^\varepsilon_x(x_j(t),t) - v^\varepsilon_x(x_{j+1}(t),t) + v^\varepsilon_x(x_j(t),t))\right]$

$= \sum_j \varepsilon\, s_j(t)\,(w_x(x_{j+1}(t),t) - w_x(x_j(t),t)) \leq 0$

as $s_j(t) \geq 0 \Rightarrow w_x|_{x_{j+1}(t)} \leq 0$, $w_x|_{x_j(t)} \geq 0$, and vice versa.

So $\frac{d}{dt}\int_{-\infty}^{+\infty} |u^\varepsilon(x,t) - v^\varepsilon(x,t)|\,dx \leq 0$, let $\varepsilon \to 0$ (by DCT).

Generic space for entropy solutions

$BV \cap L^1$. $\|\cdot\|_{BV}$ is a semi norm.

L^1-contraction $\Rightarrow$ order preserving: $u(x,0) \geq v(x,0) \Rightarrow u(x,t) \geq v(x,t)$.

Pf: $\|u(x,t) - v(x,t)\|_1 \leq \|u(x,0) - v(x,0)\|_1 = \int_{-\infty}^{+\infty} u(x,0) - v(x,0)\, dx$.

$u_t + f(u)_x = 0 \Rightarrow \frac{d}{dt}\int_{-\infty}^{+\infty} u(x,t)\, dx = 0$ as $f(u)|_{\pm\infty} = 0$.

$\int_{-\infty}^{+\infty} u(x,0) - v(x,0)\, dx = \int_{-\infty}^{+\infty} u(x,t) - v(x,t)\, dx \leq \|u(x,t) - v(x,t)\|_1$

All "$\leq$" becomes "$=$", thus $u(x,t) \geq v(x,t)$.

Total Variation Diminishing (TVD).

Pf: $\begin{cases} v_t + f(v)_x = 0 \\ v(x,0) = u^0(x+\Delta x) \end{cases}$ $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x). \end{cases}$ $v(x) = u(x+\Delta x)$.

$\|v(x,t) - u(x,t)\|_1 \leq \|v(x,0) - u(x,0)\|_1$

$\int_{-\infty}^{+\infty} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}\, dx \leq \int_{-\infty}^{+\infty} \frac{u(x+\Delta x,0) - u(x,0)}{\Delta x}\, dx$ Let $\Delta x \to 0$. $TV(u(x,t)) \leq TV(u(x,0))$.

TVD doesn't guarantee uniqueness.

Maximum Principle:

If $m \leq u(x,0) \leq M$, then $m \leq u(x,t) \leq M$.

Pf: m (M) is a solution to $u_t + f(u)_x = 0$.

$m = v(x,0) \leq u(x,t) \Rightarrow m = v(x,t) \leq u(x,t)$.

Riemann Problem

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u_l, & \text{if } x < 0 \\ u_r, & \text{if } x \geq 0 \end{cases} \end{cases}$$

(1) $f''(u) > 0$

(1a) $u_l > u_r$, then the entropy solution is a shock $u(x,t) = \begin{cases} u_l, & x < st \\ u_r, & x \geq st \end{cases}$ where $s = \frac{f(u_r) - f(u_l)}{u_r - u_l}$ is shock speed

(1b) $u_l < u_r$, then the entropy solution is a rarefaction wave $u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ (f')^{-1}(\frac{x}{t}) & f'(u_l)t \leq x < f'(u_r)t \\ u_r & x \geq f'(u_r)t \end{cases}$

check it's a weak solution (cts $\Rightarrow$ RH, Lax hold).

HW#3

(1) Prove that $u(x,t) = (f')^{-1}(\frac{x}{t})$ is a solution to $u_t + f(u)_x = 0$. Here $f''(u) > 0$.

Rarefaction solution

Lecture 04

Irreversibility of entropy solution

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

(i) $u_0(x) = \begin{cases} 1 & x < -\frac{1}{2} \\ 0 & x \geq -\frac{1}{2} \end{cases}$ (ii) $u_0(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \leq x < 0 \\ 0 & x \geq 0 \end{cases}$

But $u(x,1) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$ for (i) & (ii).

Riemann Problem

...

(1b) PDE is scalar-scaling invariant: $x' = ax,\ t' = at\ \forall a > 0$.

$u(1,1) = u(2,2) \cdots$ Self-similarity: $u(x,t) = v(\frac{x}{t})$

(2) f is not convex: f'' changes sign. Osher

Let $\xi = \frac{x}{t}$, $u(x,t) = \begin{cases} -\frac{d}{d\xi}\left(\min\limits_{v \in [u_l, u_r]} (f(v) - \xi v)\right) & \text{if } u_r > u_l \\ -\frac{d}{d\xi}\left(\max\limits_{v \in [u_r, u_l]} (f(v) - \xi v)\right) & \text{if } u_l > u_r \end{cases}$

But if we only care about $f(u(0,t))$, then

$f(u(0,t)) = \begin{cases} \min\limits_{u \in [u_l, u_r]} f(u), & u_l < u_r \\ \max\limits_{u \in [u_r, u_l]} f(u), & u_r < u_l. \end{cases}$

Lax equivalence theorem:

Strang. Stability + dissapitive

Example. Riemann Problem for Burgers equation

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 \\ u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases} \end{cases} \Rightarrow u(x,t) = \begin{cases} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases}$$

$u_t + uu_x = 0$. Consider $u_j^{n+1} = u_j^n - u_j^n \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$ (Resemble $u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_j^n)$, $\frac{a\Delta t}{\Delta x} < 0$).

The scheme is good if the solution of the PDE is smooth.

$u_j^1 = u_j^0 - u_j^0 \frac{\Delta t}{\Delta x}(u_j^0 - u_{j-1}^0) = \begin{cases} 1, & j < 0 \\ 0, & j = 0 \\ 0, & j > 0 \end{cases}$, which is the initial condition. $u_j^n \longrightarrow \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$ is not a weak solution

Conservative scheme

$u_j^{n+1} = u_j^n - \lambda(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}})$ where $\hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j-p}, \cdots u_{j+q})$ satisfying

(1) Consistenty: $\hat{f}(u, \cdots, u) = f(u)$ for constant u

(2) Lipschitz continuity

$\hat{f}$ is numerical flux. $\lambda \triangleq \frac{\Delta t}{\Delta x}$

Lax-Wendroff Theorem

If a conservative scheme converges, then the limit is a week solution

Pf: $\sum_{j,n}\left(\frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{\hat{f}_{j+1/2}-\hat{f}_{j-1/2}}{\Delta x}\right)\varphi_j^n = 0 \quad \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+).\ \varphi_j^n = \varphi(x_j, t^n).$

Sum by parts, $\sum_j (a_{j+1}-a_j)b_j = \sum_j a_j(b_j - b_{j-1})$

$0 = \cdots = -\sum_{j,n}\left(u_j^n \frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t} + \hat{f}_{j-1/2}\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x}\right)\Delta t \Delta x$

$u_j^n \to u,\ \hat{f}_{j-1/2} \longrightarrow f(u)$ by consistenty & lipschitz continuity.

$\xrightarrow{DCT} \int_0^\infty \int_{-\infty}^{+\infty} u\varphi_t + f(u)\varphi_x \, dx\, dt.$

Examples of conservative schemes

(1) If $f'(u)>0$, then $\hat{f}_{j+1/2} = f(u_j)$. Less useful. (upwind)

(2) Lax-Friedrichs scheme: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = \frac{1}{2}\left[f(u_j) + f(u_{j+1}) - \alpha(u_{j+1}-u_j)\right]$ $\alpha = \max_u |f'(u)|$. (LF)

(3) Local Lax-Friedrichs: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = \frac{1}{2}\left(f(u_j) + f(u_{j+1}) - \alpha_{j+1/2}(u_{j+1}-u_j)\right)$ $\alpha_{j+1/2} = \max_{u\in[u_j, u_{j+1}]} |f'(u)|$. (LLF)

(4) Roe scheme: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = \begin{cases} f(u_j), & \text{if } s_{j+1/2} \geq 0 \\ f(u_{j+1}), & \text{if } s_{j+1/2} < 0 \end{cases}$. $s_{j+1/2} = \frac{f(u_{j+1}) - f(u_j)}{u_{j+1}-u_j}$ (can do multiplication to determine the sign when coding).

(5) Godunov scheme: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = f(v(0,t))$ where $v(0,t)$ is the solution to the Riemann problem

$$\begin{cases} v_t + f(v)_x = 0 \\ v(x,0) = \begin{cases} u_j & x<0 \\ u_{j+1} & x \geq 0 \end{cases} \end{cases}$$

$u_{j+1/2} = \frac{u_j+u_{j+1}}{2}$

$\frac{1}{2}f'(u_{j+1/2})(f(u_{j+1/2}) - f(u_j))$

(6) Lax-Wendroff: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = \frac{1}{2}\left(f(u_j) + f(u_{j+1}) - \lambda f'\left(\frac{u_j+u_{j+1}}{2}\right)(f(u_{j+1}) - f(u_j))\right)$

(7) MacCormack: $\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1}) = \frac{1}{2}\left[f(u_{j+1}) + f\left(u_j - \lambda(f(u_{j+1}) - f(u_j))\right)\right]$

HW #4

Code up (1) Roe (2) Godunov (3) LF (4) LW for

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = \frac{1}{3} + \frac{2}{3}\sin x \quad [0, 2\pi] \end{cases}$$

Get the result for $t=0.3$, $t=2$.

Take $N = 20, 40, 80, \cdots, 640$, error table:

$|f'(u)|\cdot\frac{\Delta t}{\Delta x} < 0.6$

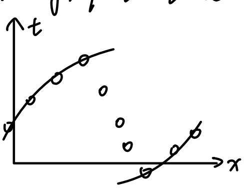
$\Delta x / N$	L^1, order; L^2, order; L^∞, order
$2\pi/20$	$\frac{1}{N}\sum\lvert e_n\rvert$ $\left(\frac{1}{N}\sum\lvert e_n\rvert^2\right)^{1/2}$
/640	2.41×10^{-4} 1.18 (three digits)

$e_h = ch^r,\ e_{2h} = c(2h)^r$

$\frac{e_{2h}}{e_h} = 2^r,\ r = \log_2 \frac{e_{2h}}{e_h}.$

For $t=2$, another error table for points $|x_j - x_{shock}| \geq 1$.

Finally, plot $t=2$ for $N=40$



Lecture 05

A non-conservative scheme may converge to a function which is not a weak solution

General upwind scheme

A 3-point scheme (2-point flux) is called an upwind scheme if

$$\hat{f}(u_j, u_{j+1}) = \begin{cases} f(u_j) & \text{if } f'(u) \geq 0 \text{ in } [u_j, u_{j+1}] \\ f(u_{j+1}) & \text{if } f'(u) \leq 0 \text{ in } [u_j, u_{j+1}] \end{cases}$$

Exercise: Roe, Godunov are upwind. LF, Local LF, LW are not upwind.

Consider $\begin{cases} u_t + (\frac{u^2}{2})_x = 0 \\ u(x,0) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases} \end{cases}$. The correct solution is $u(x,t) = \begin{cases} -1 & x<-t \\ x/t & -t \leq x < t \\ 1 & x \geq t \end{cases}$

For Roe scheme, $u_j^0 = \begin{cases} -1 & j<0 \\ 1 & j \geq 0 \end{cases}$. $\hat{f}_{j+1/2} = 1/2 \ \forall j$, $u_j^1 = u_j^0$, so it doesn't converge to entropy solution.

Monotone schemes

If $u_j^{n+1} = G(u_{j-p-1}^n, \dots, u_{j+q}^n) = u_j^n - \lambda(\hat{f}(u_{j-p}, \cdots u_{j+q}) - \hat{f}(u_{j-p-1}, \cdots u_{j+q-1}))$ and G is increasing (non-decreasing) w.r.t. every argument ($G(\uparrow, \uparrow, \cdots, \uparrow)$), then the scheme is called a monotone scheme.

For example, LF is monotone when $\alpha \frac{\Delta t}{\Delta x} \leq 1$.

$$u_j^{n+1} = u_j^n - \lambda(\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)) \text{ with}$$

$$\hat{f}(u_j, u_{j+1}) = \frac{1}{2}(f(u_j) + f(u_{j+1}) - \alpha(u_{j+1} + u_j)), \ f(u_{j-1}, u_j) = \frac{1}{2}(f(u_j) + f(u_{j-1}) - \alpha(u_j - u_{j-1})).$$

$$\frac{\partial G}{\partial u_{j-1}} = \lambda(\frac{1}{2}\alpha + \frac{1}{2}f'(u_{j-1})) \geq 0$$

$$\frac{\partial G}{\partial u_j} = 1 - \lambda(\frac{1}{2}f'(u_j) + \frac{\alpha}{2} - \frac{1}{2}f'(u_j) + \frac{\alpha}{2}) = 1 - \lambda\alpha$$

$$\frac{\partial G}{\partial u_{j+1}} = \lambda(\frac{1}{2}\alpha - \frac{1}{2}f'(u_{j+1})) \geq 0.$$

We consider only 3-point (2-point flux) monotone schemes from now on.

Suppose $\hat{f}(\uparrow, \downarrow)$ and $\lambda(\hat{f}_1 - \hat{f}_2) \leq 1 \Rightarrow$ monotone scheme

$$u_j^{n+1} = u_j^n - \lambda(\hat{f}(u_j^n, u_{j+1}^n) - \hat{f}(u_{j-1}^n, u_j^n)) = G(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$G_1 (\triangleq \frac{\partial G}{\partial u_{j-1}}) = \lambda \hat{f}_1 \geq 0, \quad G_2 (\triangleq \frac{\partial G}{\partial u_j}) = 1 - \lambda(\hat{f}_1 - \hat{f}_2) \geq 0, \quad G_3 (\triangleq \frac{\partial G}{\partial u_{j+1}}) = -\lambda \hat{f}_2 \geq 0.$$

Properties of monotone scheme

(1) Order preserving: If $u_j \leq v_j \ \forall j \Rightarrow G(u)_j \leq G(v)_j$. $G(u_{j-p}, \cdots u_{j+q}) \leq \cdots \leq G(v_{j-p}, \cdots, v_{j+q})$.

(2) Local maximum principle: $\min_{j-p \leq i \leq j+q} u_i \leq G(u)_j \leq \max_{j-p \leq i \leq j+q} u_i$ $(m_j \leq G(u)_j \leq M_j)$

Let $v_i = \begin{cases} M_i & j-p \leq i \leq j+q \\ u_i & \text{o.w.} \end{cases}$. $u_i \leq v_i$, by (1), $G(u)_i \leq G(v)_i$. In particular, $G(u)_j \leq G(v)_j \overset{\text{consistency}}{=} v_j = M_j$.

(3) l^1 contraction: $\|G(u) - G(v)\|_{l^1} \leq \|u - v\|_{l^1}$.

Let $w_j = u_j \vee v_j = v_j + (u_j - v_j)^+$

$G(u)_j \leq G(w)_j$, $G(v_j) \leq G(w)_j$. $G(w)_j - G(v)_j \geq \begin{cases} G(u)_j - G(v)_j \\ 0 \end{cases}$ (both).

$G(w)_j - G(v_j) \geq G(u)_j - G(v)_j = (G(u)_j - G(v)_j) \vee 0 = (G(u)_j - G(v)_j)^+$.

$\sum_j (G(u)_j - G(v)_j)^+ \leq \sum_j G(w)_j - G(v_j) \overset{\text{conservative (periodic / compact support)}}{=} \sum_j w_j - v_j = \sum_j (u_j - v_j)^+$

$\sum_j |G(u)_j - G(v)_j| = \sum_j (G(u)_j - G(v)_j)^+ + \sum_j (G(v)_j - G(u)_j)^+ \le \sum_j (u_j - v_j)^+ + \sum_j (v_j - u_j)^+ = \sum_j |u_j - v_j|.$

(Crandall-Tartar lemma).

(4) TVD: $TV(G(u)) \le TV(u)$, $TV(u) \triangleq \sum_j |u_{j+1} - u_j|$.

Take $v_j = u_{j+1}$, then $G(v)_j = G(u)_{j+1}$, and by (3).

(5) Entropy inequality:

$U(u) = |u-c|$, $F(u) = \mathrm{sgn}(u-c)(f(u)-f(c))$ $\forall c$.

$U'(u) = \begin{cases} 1 & u>c \\ -1 & u<c \end{cases}$, $U'' = 2\delta(x-c) \ge 0$.

WTS: $\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \le 0$ $\forall j$ Cell-entropy inequality

$\hat{F} = \hat{f}(c \vee u) - \hat{f}(c \wedge u)$. Verify $\hat{F}$ is consistent with F.

$|u_j^n - c| - \lambda(\hat{F}_{j+1/2} - \hat{F}_{j-1/2}) = G(c \vee u)_j - G(c \wedge u)_j$. check by def.

$c = G(c, \cdots, c) \le G(c \vee u^n)$, $u^{n+1} = G(u^n) \le G(c \vee u^n)$, $c \vee u^{n+1} \le G(c \vee u^n)$.

$c \wedge u^{n+1} \ge G(c \wedge u^n)$. $c \vee u^{n+1} - c \wedge u^{n+1} \le |u_j^n - c| - \lambda(\hat{F}_{j+1/2} - \hat{F}_{j-1/2})$

$c \vee u^{n+1} - c \wedge u^{n+1} = |u^{n+1} - c|$. So $\frac{|u_j^{n+1} - c| + |u_j^n - c|}{\Delta t} + \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \le 0$.

(6) Rate of convergence: $\|u_h - u\|_{L_1} \le C\sqrt{\Delta x}$ (general). Expect $\|u_h - u\|_{L} \le C\Delta x$.

HW#5

Prove that LF, LLF and Godunov schemes are monotone, but Roe and LW are not.

e.g. $\hat{f}(2,5) < \hat{f}(1,5)$

Flow chart

$N=40$, $\Delta x = \frac{2\pi}{N}$, $x(i) = i\Delta x$, $i = 0, 1, \cdots, N+1$

$u(i) = \alpha + \beta \sin(x(i))$, $i = 1, \cdots, N$.

$t = 0$, $t\_end = 0.3$. $ic = 0$.

Loop:

Boundary condition: $u(0) = u(N)$, $u(N+1) = u(1)$.

$am = \max |f'(u)|$.

$\Delta t = CFL \times \Delta x / am$, $CFL = 0.6$.

if $(t + \Delta t > tend)$ then

$\Delta t = t\_end - t$

$ic = 1$.

endif

$f(i) = u(i)^2/2$. $i = 0, 1, \cdots N+1$.

$flux(i+1/2) = \frac{1}{2}(f(i) + f(i+1)) - am(u(i+1) - u(i))$. $i = 0, 1, \cdots, N$

$u(i) = u(i) - \lambda(flux(i+1/2) - flux(i-1/2))$.

$t = t + \Delta t$. if $ic = 0$, go back (loop).

Lecture 06

Monotone scheme convergence rate:

$c\sqrt{\Delta x}$, (sharp for linear flux: $f(u)=u$)

Δx Some cases like burgers.

Theorem: Monotone schemes are at most first order accurate.

Theorem: TVD $\Rightarrow$ monotonicity preserving MP. If $u_{j+1}^n \geq u_j$ $\forall j$, then $u_{j+1}^{n+1} \geq u_j^{n+1}$

Pf. $TV(u^{n+1}) = \sum_j |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_j |u_{j+1}^n - u_j^n| = \sum_j u_{j+1}^n - u_j^n = \sum_j u_{j+1}^{n+1} - u_j^{n}$ $=$: conservation.

This implies $u_{j+1}^{n+1} - u_j^{n+1} \geq 0$.

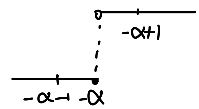
Theorem (Godunov): MP + linear scheme $\Rightarrow$ monotone

Def: A scheme is called a linear scheme if it is linear when applied to linear PDEs. ($u_t + a u_x = 0$).

In fact, Godunov = Roe = LF = LLF for linear PDEs: $u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n)$.

LW is a linear scheme

Pf: $u_t + u_x = 0$. $u_j^{n+1} = \sum_{\ell=-k_1}^{k_2} C_\ell(\lambda) u_{j-\ell}^n$. Consider $u_i^n = \begin{cases} 0 & i \leq -\alpha \\ 1 & i > -\alpha \end{cases}$. Then $u_{i+1}^n \geq u_i^n$.



$u_{j+1}^{n+1} = \sum_\ell C_\ell(\lambda) u_{j+1-\ell}^n$, $u_j^{n+1} = \sum_\ell C_\ell(\lambda) u_{j-\ell}^n$. Let $\Delta_+ u_j = u_{j+1} - u_j$

$\Delta_+ u_j^{n+1} = \sum_\ell C_\ell(\lambda) \Delta_+ u_{j-\ell}^n$. $\Delta_+ u_m^n = \begin{cases} 0 & \text{if } m \neq -\alpha \\ 1 & \text{if } m = -\alpha. \end{cases}$

$\Delta_+ u_0^{n+1} = \sum_\ell C_\ell(\lambda) \Delta_+ u_{-\alpha}^n = C_\ell(\lambda)$

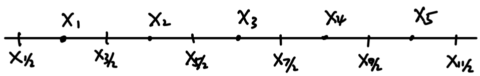
$\Delta_+ u_0^{n+1} \geq 0$ by MP. $C_\ell(\lambda) \geq 0$ $\forall \ell$ Then it's a monotone scheme.

Also called positive schemes. Linear schemes solving linear PDEs.

Finite volume scheme (FV)

$\bar{u}_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\,dx$

cell averages



For the conservation law, $I_j = (x_{j-1/2}, x_{j+1/2})$

$u_t + f(u)_x = 0$. Integrate, $\frac{d}{dt}\bar{u}_j + \frac{1}{\Delta x}\left(f(u(x_{j+1/2},t)) - f(u(x_{j-1/2},t))\right) = 0$.

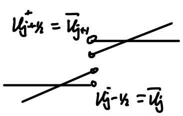
Initially, $\bar{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u^0(x)\,dx$, which is available.

$\bar{u}_j = \bar{u}_j^0 - \lambda\left(f(u(x_{j+1/2},t)) - f(u(x_{j-1/2},t))\right) = 0$. (Euler Forward for $\bar{u}_j$). $u(x_{j+1/2},t)$, $u(x_{j-1/2},t)$ are unknown.

Need a <u>reconstruction</u> procedure to get, from $\{\bar{u}_j\}$, approximations to $u_{j+1/2}$ ($\doteq u(x_{j+1/2},t)$).

Relation with monotone scheme:

Monotone scheme: $\frac{d\bar{u}_j}{dt} + (\hat{f}(\bar{u}_j, \bar{u}_{j+1}) - \hat{f}(\bar{u}_{j-1}, \bar{u}_j)) = 0$.



FV can use numerical flux after reconstruction

FV:

① reconstruction to get $u_{j+1/2}^-$ & $u_{j+1/2}^+$ at $x_{j+1/2}$.

② monotone flux: $\hat{f}_{j+1/2} = \hat{f}(u_{j+1/2}^-, u_{j+1/2}^+)$

③ discretize $\frac{d\bar{u}_j}{dt} + \frac{1}{\Delta x}\left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right) = 0$ in time

Reconstruction

$p(x)$ is a reconstruction polynomial in the stencil $\{I_{j-r}, \cdots, I_{j+q}\}$ if

$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x)\,dx = \bar{u}_i$. $i = j-r, \cdots, j+q$. $(*)$ $r+q+1$ conditions. $\deg p(x) = r+q$

(i) Choice of the stencil:

$S_0 = \{I_j\}$, $p(x) = \bar{u}_j$

$S_2 = \{I_{j-1}, I_j, I_{j+1}\}$, $p(x) = a + bx + cx^2$. $(*)$ for $i = j-1, j, j+1$. Give 3 linear equations for a, b, c.

$S_4 = \{I_{j-2}, I_{j-1}, I_j, I_{j+1}, I_{j+2}\}$.

Assume $\bar{u}_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x)\,dx$ is known for $i = j-r, \cdots j+q$.

Define $U(x) = \int_{x_{j-r-1/2}}^{x} u(\xi)\,d\xi$. $U(x_{i+1/2}) = \int_{x_{j-r-1/2}}^{x_{i+1/2}} u(\xi)\,d\xi = \sum_{l=j-r}^{i}\int_{l-1/2}^{l+1/2} u(\xi)\,d\xi = \sum_{l=j-r}^{i} \bar{u}_l\,\Delta x$

So we know the values of $U(x)$ at half grids: $i = j-r-1, j-r, \cdots, j+q$. $(U(x_{j-r-1+1/2}) = 0)$.

From $U(x)$, we can build $P(x)$ is a polynomial s.t. $P(x_{i+1/2}) = U(x_{i+1/2})$ (interpolation polynomial)

$\deg P(x) = r+q+1$ ($r+q+2$ values)

Claim: $p(x) = P'(x)$.

must be in $j-r-1, \cdots j+q$

Check: $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} p(x)\,dx = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} P'(x)\,dx = \frac{1}{\Delta x}\left(P(x_{i+1/2}) - P(x_{i-1/2})\right) = \frac{1}{\Delta x}\left(U(x_{i+1/2}) - U(x_{i-1/2})\right) = \bar{u}_i$ by def of $U(x)$.

HW#6

Prove the Godunov theorem: A positive linear scheme for solving $u_t + u_x = 0$ is at most 1^{st} order accurate.

Lecture 07

Finite volume schemes:

$$\frac{d\bar{u}_j}{dt} + \frac{1}{\Delta x}(\hat{f}_{j+\frac12} - \hat{f}_{j-\frac12}) = 0$$

$\bar{u}_j = \frac{1}{\Delta x}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u(x,t)\,dx$. $\hat{f}_{j+\frac12} = \hat{f}(u^-_{j+\frac12}, u^+_{j+\frac12})$. $\hat{f}$ is a monotone flux.

$\{\bar{u}_j\} \xrightarrow{\text{reconstruction}} \{u^\pm_{j+\frac12}\}$: build $p(x)$: $u \to U \to P \to p$. (P can be in Lagrange form / Newton form)

But in fact, $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\} \to u^-_{j+\frac12}$ is indeed linear, that is $u^-_{j+\frac12} = a\bar{u}_{j-1} + b\bar{u}_j + c\bar{u}_{j+1}$

$$u^-_{j+\frac12} = -\frac16\bar{u}_{j-1} + \frac56\bar{u}_j + \frac13\bar{u}_{j+1}. \quad u^+_{j+\frac12} = \frac13\bar{u}_j + \frac56\bar{u}_{j+1} - \frac16\bar{u}_{j+2}$$ Exercise: compute the numbers

If we use Euler forward for time marching, then the scheme is unconditionally unstable.

Runge-Kutta: RK3 as spatial order is 3. That is SSP-RK3:

$$u_j^{(1)} = \bar{u}_j^{\prime} - \lambda(\hat{f}_{j+\frac12} - \hat{f}_{j-\frac12}) = L(\bar{u}^n)_j$$

$$u_j^{(2)} = \frac34\bar{u}_j^n + \frac14 L\bar{u}_j^{(1)} = \frac34\bar{u}_j^n + \frac14[\bar{u}_j^{(1)} - \lambda(\hat{f}^{(1)}_{j+\frac12} - \hat{f}^{(1)}_{j-\frac12})]$$

$$\bar{u}_j^{n+1} = \frac13\bar{u}_j^n + \frac23 L\bar{u}_j^{(2)} = \frac13\bar{u}_j^n + \frac23[\bar{u}_j^{(2)} - \lambda(\hat{f}^{(2)}_{j+\frac12} - \hat{f}^{(2)}_{j-\frac12})]$$

Not monotone, monotone preserving, TVD, $\cdots$. Must have oscillations.

Is it possible to make the scheme TVD? (Preserve good properties like accuracy)

$TV(u) = \sum_j |u_{j+1} - u_j|$. $TV(u^{n+1}) \le TV(u^n) \Rightarrow$ no oscillation will be created

Lemma (Harten): A scheme written in the following form is TVD: $\bar{u}_j^{n+1} = \bar{u}_j^n + C_{j+\frac12}(\bar{u}^n_{j+1} - \bar{u}^n_j) - D_{j-\frac12}(\bar{u}^n_j - \bar{u}^n_{j-1})$

with $C_{j+\frac12} \ge 0$, $D_{j+\frac12} \ge 0$, $C_{j+\frac12} + D_{j+\frac12} \le 1$.

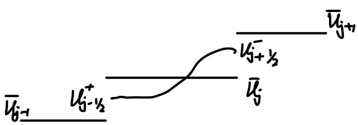
Pf: $\bar{u}^{n+1}_{j+1} = \bar{u}^n_{j+1} + C_{j+\frac32}\Delta_+\bar{u}^n_{j+1} - D_{j+\frac12}\Delta_+\bar{u}^n_j$.

$$\Delta_+\bar{u}_j^{n+1} = \Delta_+\bar{u}_j^n + C_{j+\frac32}\Delta_+\bar{u}^n_{j+1} - D_{j+\frac12}\Delta_+\bar{u}^n_j - C_{j+\frac12}\Delta_+\bar{u}^n_j + D_{j-\frac12}\Delta_+\bar{u}^n_{j-1}$$

$$= (1 - C_{j+\frac12} - D_{j+\frac12})\Delta_+\bar{u}^n_j + C_{j+\frac32}\Delta_+\bar{u}^n_{j+1} + D_{j-\frac12}\Delta_+\bar{u}^n_{j-1}$$

$$TV(u^{n+1}) = \sum|\Delta_+ u_j^{n+1}| \le \sum(1 - C_{j+\frac12} - D_{j+\frac12})|\Delta_+ u_j| + C_{j+\frac32}|\Delta_+\bar{u}_{j+1}| + D_{j-\frac12}|\Delta_-\bar{u}^n_{j-1}|$$

$$= \sum_j |\Delta_+ u^n_j| = TV(u^n).$$



$$\bar{u}_j^{n+1} = u_j^n - \lambda(\hat{f}(u^-_{j+\frac12}, u^+_{j+\frac12}) - \hat{f}(u^-_{j-\frac12}, u^+_{j-\frac12})).$$

Idea: using limiter

Let $\tilde{u}_j = u^-_{j+\frac12} - \bar{u}_j$, $\tilde{\tilde{u}}_j = \bar{u}_j - u^+_{j-\frac12}$ $\quad m$: minimod

$$\tilde{u}_j^{(mod)} = m(\tilde{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}), \quad \tilde{\tilde{u}}_j = m(\tilde{\tilde{u}}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}).$$

$$m(a_1, \cdots, a_n) = \begin{cases} s \cdot \min|a_i| & \text{if } \operatorname{sgn}(a_1) = \cdots = \operatorname{sgn}(a_n) = s. \\ 0 \end{cases}$$

Then $u^{-(mod)}_{j+\frac12} = \bar{u}_j + \tilde{u}_j^{(mod)}$, $u^{+(mod)}_{j-\frac12} = \bar{u}_j - \tilde{\tilde{u}}_j^{(mod)}$. The scheme becomes

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda(\hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j+\frac12}) - \hat{f}(u^{-(mod)}_{j-\frac12}, u^{+(mod)}_{j-\frac12}))$$

$$= \bar{u}_j^n - \lambda(\hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j+\frac12}) - \hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j-\frac12}) \longrightarrow C_{j+\frac12}(\bar{u}^n_{j+1} - \bar{u}^n_j)$$

$$+ \hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j-\frac12}) - \hat{f}(u^{-(mod)}_{j-\frac12}, u^{+(mod)}_{j-\frac12})). \longrightarrow D_{j-\frac12}(\bar{u}^n_j - \bar{u}^n_{j-1})$$

$$C_{j+\frac12} = -\lambda(\hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j+\frac12}) - \hat{f}(u^{-(mod)}_{j+\frac12}, u^{+(mod)}_{j-\frac12}))/(\bar{u}^n_{j+1} - \bar{u}^n_j)$$

$$= -\lambda\hat{f}_2(u^{-(mod)}_{j+\frac12}, \xi)(u^{+(mod)}_{j+\frac12} - u^{+(mod)}_{j-\frac12})/(\bar{u}^n_{j+1} - \bar{u}^n_j) \quad \text{(Mean value theorem).}$$

$$= -\lambda\hat{f}_2(\cdot, \xi)(\bar{u}^n_{j+1} - \tilde{\tilde{u}}^{(mod)}_{j+1} - \bar{u}^n_j + \tilde{\tilde{u}}^{(mod)}_j)/(\bar{u}^n_{j+1} - \bar{u}^n_j)$$

$$= -\lambda \hat{f}_2(\cdot,\xi)\Big[1 - \underbrace{\frac{\tilde{\bar{u}}_{j+1}^{(mod)}}{\bar{u}_{j+1}^n - \bar{u}_j^n}}_{\in[0,1]} + \underbrace{\frac{\tilde{\bar{u}}_j^{(mod)}}{\bar{u}_{j+1}^n - \bar{u}_j^n}}_{\in[0,1]}\Big], \quad [\cdots] \in [0,2].$$

So $G_{j+\frac{1}{2}} \geq 0$. So as $D_{j-\frac{1}{2}} \geq 0$.

Exercise: Show $G_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$ for $\lambda(\hat{f}_1 - \hat{f}_2) \leq \frac{1}{2}$.

This modification keeps accuracy to large extent.

Generalized MUSCL scheme.

HW#7

(1) Repeat HW#3 with LF-flux for third-order FVM scheme with SSP-RK3

(2) Repeat (1) with generalized MUSCL limiter.

Lecture 08

SSP-RK3 does not change the TVD property

Pf: $\bar{u}_j^{(1)} = \bar{u}_j^n - \lambda(\hat{f}^n_{j+\frac12} - \hat{f}^n_{j-\frac12})$. $TV(\bar{u}^{(1)}) \le TV(\bar{u}^n)$

$\bar{u}_j^{(2)} = \frac34 u_j^n + \frac14(\bar{u}_j^{(1)} - \lambda(\hat{f}^{(1)}_{j+\frac12} - \hat{f}^{(1)}_{j-\frac12})) = \frac34 \bar{u}_j^n + \frac14 \bar{w}$

$TV(\bar{u}^{(2)}) \le \frac34 TV(\bar{u}^n) + \frac14 TV(\bar{w}) \le \frac34 TV(\bar{u}^n) + \frac14 TV(\bar{u}^n) = TV(\bar{u}^n)$.

So $TV(\bar{u}^{n+1}) \le TV(\bar{u}^n)$.

Accuracy

$\check{u}_j = u_{j+\frac12} - \bar{u}_j = u_j + u_{x_j}\frac{\Delta x}{2} + O(\Delta x^2) - (u_j + O(\Delta x^2)) = u_{x_j}\cdot\frac{\Delta x}{2} + O(\Delta x^2)$.

$\bar{u}_{j+1} - \bar{u}_j = u_{j+1} - u_j + O(\Delta x^2) = u_{x_j}\Delta x + O(\Delta x^2)$, $\bar{u}_j - \bar{u}_{j-1} = u_{x_j}\Delta x + O(\Delta x^2)$.

$m(\check{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}) = \check{u}_j$ if $O(\Delta x^2)$ is small enough

So if $u_x \sim O(1)$, the scheme is of the same accuracy. (Smooth monotone region).

But when x is close to the extrema, then the order cannot be preserved. In fact, we have

Theorem: All TVD schemes degenerate to first order accuracy at smooth extrema. (Proof).

TVD schemes will be at most 2^{nd} order accurate in L^1 & first order accurate in L^∞.

High resolution schemes (some TVD)

To deal with this problem, there are two approaches

(1) TVB: $TV(\bar{u}^{n+1}) \le (1 + C\Delta t)TV(\bar{u}^n)$ or $\le TV(\bar{u}^n) + C\Delta t$ for fixed $C > 0$.

This implies $TV(\bar{u}^n) \le K(T)$ for $n\Delta t \le T$, K is a const w.r.t. T.

To make TVB from TVD, just need to change m (minimod) to $\tilde{m}$.

$$\tilde{m}(a_1, \cdots, a_p) = \begin{cases} a_1 & \text{if } |a_1| \le M\Delta x^2 \\ m(a_1, \cdots a_p) & \text{o.w.} \end{cases}$$

Exercise: Prove it's TVB

"Pf" of curing degeneration: $\check{u}_j = \frac12 u_x \Delta x + O(\Delta x^2)$, pick $M \ge \frac23 \max|u_{xx}|$

(Recall HW1, just choose M at $t=0$ as u_{xx} doesn't change along the characteristic lines).

(2) ENO: Essentially Non-oscillatory scheme

Intuition: Choose other stencils if there is discontinuous

ENO

ENO interpolation.

Newton form interpolation

Newton divided differences:

0th: $f[x_i] = x_i$

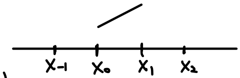
1st: $f[x_i, x_{i+1}] = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$

2nd: $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

$P_0(x) = f[x_i]$, $P_1(x) = P_0(x) + f[x_i, x_{i+1}](x - x_i)$, $P_2(x) = P_1(x) + f[x_i, x_{i+1}, x_{i+2}](x - x_i)(x - x_{i+1})$, $\cdots$

Start from a small stencil of 1 or 2 points

Start from 2 points:



$S_0 = \{x_0, x_1\}$. $P_0 = f[x_0] + f[x_0, x_1](x - x_0)$

$S_1^{(1)} = \{x_0, x_1, x_{-1}\}$, $P_1^{(1)}(x) = P_0(x) + f[x_0, x_1, x_{-1}](x-x_0)(x-x_1)$

$S_1^{(2)} = \{x_0, x_1, x_2\}$, $P_1^{(2)}(x) = P_0(x) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$.

$P_1^{(1)}$ and $P_1^{(2)}$ have the same shape. So to minimize the oscillation,

$$P_1 = \begin{cases} P_1^{(1)} & \text{if } |f[x_0, x_1, x_{-1}]| \le |f[x_0, x_1, x_2]| \\ P_1^{(2)} & \text{if } |f[x_0, x_1, x_2]| \le |f[x_0, x_1, x_{-1}]|. \end{cases}$$

For the next step, if we choose $S_1 = S_1^{(2)}$,

$P_2^{(1)} = P_1(x) + f[x_0, x_1, x_2, x_{-1}](x-x_0)(x-x_1)(x-x_2)$

$P_2^{(2)} = P_1(x) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

HW#8

Repeat HW#7 with TVB: $M=0$ (TVD), $M=1$, $M=5$, $M=10$.

Lecture 9

ENO intepolation

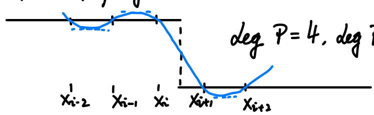
Feature:

(1) Always include the interval started

(2) Uniformly accurate for p.w. smooth function (Proof?)

(3) If there is a discontinuity between x_i & x_{i+1}, then $p(x)$ is monotone in (x_i, x_{i+1})

Partial proof:



deg $P=4$, deg $P'=3$. There are 3 zeros of P' by MVP. So there

Numerical observation: Even the discontinuities are close, still accurate (monotone).

Reconstruction:

$\{\bar{u}_j\} \longrightarrow U_{j+1/2} = \int_{x^*}^{x_{j+1/2}} u(\xi)\,d\xi.$

$S' = \{U_{j-1/2}, U_{j+1/2}\} \longrightarrow \text{ENO} \longrightarrow p(x) = P'(x).$

$U[x_{j-1/2}, x_{j+1/2}] = \dfrac{U_{j+1/2} - U_{j-1/2}}{x_{j+1/2} - x_{j-1/2}} = \bar{u}_j.$

For uniform mesh, only need to compute the undivided differences.

$\bar{u}[x_i] = \bar{u}_i$

$\bar{u}[x_i, x_{i+1}] = \bar{u}_{i+1} - \bar{u}_i$

$\bar{u}[x_i, x_{i+1}, x_{i+2}] = \bar{u}[x_{i+1}, x_{i+2}] - \bar{u}[x_i, x_{i+1}]$

Hence we choose the stencil as following

$S_1 = \{\bar{u}_j\}$

$S_2 = \begin{cases} \{\bar{u}_{j-1}, \bar{u}_j\} & \text{if } |\bar{u}[x_{j-1}, x_j]| < |\bar{u}[x_j, x_{j+1}]| \\ \{\bar{u}_j, \bar{u}_{j+1}\} & \text{otherwise} \end{cases}$

...

Then for r-th order, need $i = -r+1$ to $N+r$

Code:

```
f[i,0] = u(x_i),   i = -r+1, N+r
do k=1,r
   do i=-r+1, N+r-k
      f[i,k] = f[i+1,k-1] - f[i,k-1]
   enddo
enddo
is(i) = i    i=1, ..., N
do k=1,r
   do i=1,N
```

if $|f[is(i)-1,k]| \le |f[is(i),k]|$: $is(i) = is(i)-1$

enddo

enddo.

For example: $r=2$

$is(j) = j-2$ $\quad \bar{u}_{j-2}, \bar{u}_{j-1}, \bar{u}_j$ $\quad u^-_{j+1/2} = \frac{1}{3}\bar{u}_{j-2} - \frac{7}{6}\bar{u}_{j-1} + \frac{11}{6}\bar{u}_j$; $u^+_{j-1/2} = -\frac{1}{6}\bar{u}_{j-2} + \frac{5}{6}\bar{u}_{j-1} + \frac{1}{3}\bar{u}_j$

$is(j) = j-1$ $\quad \bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}$ $\quad u^-_{j+1/2} = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}$; $u^+_{j-1/2} = \frac{1}{3}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j - \frac{1}{6}\bar{u}_{j+1}$

$is(j) = j$ $\quad \bar{u}_j, \bar{u}_{j+1}, \bar{u}_{j+2}$ $\quad u^-_{j+1/2} = \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}$; $u^+_{j-1/2} = \frac{11}{6}\bar{u}_j - \frac{7}{6}\bar{u}_{j+1} + \frac{1}{3}\bar{u}_{j+2}$.

Weighted ENO (WENO)

ENO: $u^{-(1)}_{j+1/2} = \frac{1}{3}\bar{u}_{j-2} - \frac{7}{6}\bar{u}_{j-1} + \frac{11}{6}\bar{u}_j$, $u^{-(2)}_{j+1/2} = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}$, $u^{+(3)}_{j-1/2} = \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}$

ENO has the problem of losing accuracy at least 1 order.

$u_t + u_x = 0$ with e^{x-t} (e^x), $u_j = (-1)^j$, $u(x,0) = \sin^4 x$

Not using all stencils touched.

This motivates WENO.

$$u^-_{j+1/2} = w_1 u^{-(1)}_{j+1/2} + w_2 u^{-(2)}_{j+1/2} + w_3 u^{-(3)}_{j+1/2}, \quad w_i \ge 0, \ \sum w_i = 1.$$

How to choose w_i ?

(1) "Easy" to compute

(2) If all 3 stencils are smooth (u is smooth on the 3 stencils), "higher" accuracy : 5^{th}

(3) If one of the 3 stencils is not smooth, then that stencil gets a small w.

If we take $w_\ell = \gamma_\ell$ (linear weights), $\gamma_1 = \frac{1}{10}$, $\gamma_2 = \frac{3}{5}$, $\gamma_3 = \frac{3}{10}$, then (2) is achieved.

So we choose $\tilde{w}_\ell = \frac{\gamma_\ell}{(\beta_\ell + \varepsilon)^2}$. $\varepsilon = 10^{-6}$, β_ℓ. smoothness indicators : smaller $\rightarrow$ smoother

$$w_\ell = \frac{\tilde{w}_\ell}{\sum \tilde{w}_\ell} \quad \beta_\ell = \sum_{k=1}^{r} \Delta x^{2k-1} \int_{I_j} \left(\frac{d^k p_\ell(x)}{dx^k}\right)^2 dx$$

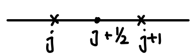
By Guangshan Jiang.

$$\beta_1 = \frac{13}{12}(\bar{u}_{j-2} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4}(\bar{u}_{j-2} - 4\bar{u}_{j-1} + 3\bar{u}_j)^2$$

$$\beta_2 = \frac{13}{12}(\bar{u}_{j-1} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4}(\bar{u}_{j-1} - \bar{u}_{j+1})^2$$

$$\beta_3 = \frac{13}{12}(\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2})^2 + \frac{1}{4}(3\bar{u}_j - 4\bar{u}_{j+1} + \bar{u}_{j+2})^2$$

Mirror symmetric w.r.t. $j+1/2$ for $u^-_{j+1/2}$ & $u^+_{j+1/2}$



HW#9

(1) Repeat HW#8 with 3^{rd} order ENO

(2) Repeat (1) with 5^{th} order WENO

(i) The same Δt as before

(ii) Take $\Delta t = C\Delta x^{\frac{5}{3}}$

Lecture 10

Review: $u_t + f(u)_x = 0$ (1D)

- weak solution. entropy condition
- L^1 contraction
- monotone schemes
- high-order FV schemes
  - reconstruction $\begin{cases} \text{TVD/TVB} \\ \text{ENO/WENO} \end{cases}$
  - monotone flux
  - time discretization: SSP-RK3

In 2D cases, FVM becomes expensive. Alternative: FDM.

FDM:

$u_j \approx u(x_j, t)$. $\frac{du_j}{dt} + f(u)_x = 0$

How to approximate $f(u)_x$?

① Use u_{j-1}, u_j, u_{j+1} to interpolate a polynomial, then take derivative. NOT conservative

② $f(u)_x = \frac{\hat{f}_{j+1/2} - \hat{f}_{j-1/2}}{\Delta x} + O(\Delta x^r)$. $f(u_{\pm 1/2})$ can be obtained by interpolation.

But even if $f(u_{\pm 1/2})$ is exact, r can be at most 2.

Suppose there is a function $h(x)$ which satisfies

$$f(u(x)) = \frac{1}{\Delta x}\int_{x-\Delta x/2}^{x+\Delta x/2} h(\xi)\, d\xi.$$

Then $\frac{d}{dx} f(u(x)) = \frac{1}{\Delta x}\left(h(x+\frac{\Delta x}{2}) - h(x - \frac{\Delta x}{2})\right)$. $(f(u(x)))_x\big|_{x=x_j} = \frac{1}{\Delta x}\left(h(x_j + \frac{\Delta x}{2}) - h(x_j - \frac{\Delta x}{2})\right)$.

This is Shu-Osher Lemma. So just need to get h.

We know $f(u_i) = \frac{1}{\Delta x}\int_{x_j - \frac{\Delta x}{2}}^{x_j+\frac{\Delta x}{2}} h(\xi)\, d\xi = \bar{h}_j$. Then $\bar{h}_j \Rightarrow h(x_j \pm \frac{\Delta x}{2})$ is the reconstruction.

This allows higher accuracy. ?

Stability:

Separable flux: $\hat{f}(u^-, u^+) = f^+(u^-) + f^-(u^+)$. $\frac{\partial f^+(u)}{\partial u} \geq 0$, $\frac{\partial f^-(u)}{\partial u} \leq 0$.

Lax-Friedrichs is separable.

Example:

$u_i \to f(u_i)$:

WENO5: $j-2 \quad j-1 \quad j \quad j+1 \quad j+2$

use $f^+(u_{j-2}), f^+(u_{j-1}), f^+(u_j), f^+(u_{j+1}), f^+(u_{j+2}) \xrightarrow{\text{reconstruct}} \hat{f}^-_{j+1/2}$

$f^-(u_{j-1}), f^-(u_{j-1}), f^-(u_j), f^-(u_{j+1}), f^-(u_{j+2}) \xrightarrow{\text{reconstruct}} \hat{f}^+_{j-1/2}$

$\hat{f}_{j+1/2} = \hat{f}^-_{j+1/2} + \hat{f}^+_{j+1/2}$

Flux-splitting: $f = f^+ + f^-$.

2D Case: $u_t + f(u)_x + g(u)_y = 0.$

- PDE properties: same as in 1D.
- monotone schemes: same as in 1D.

Notation. $\breve{\bar{u}}_{ij}$: $-$. x-integration, $\smile$: y-integration

$\breve{\bar{u}}_{ij} : \frac{1}{\Delta x_i}\frac{1}{\Delta y_j}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,y,t)\,dx\,dy$ $\quad \breve{\bar{u}}_{ij}$: operator on u.

Apply on both sides of PDE,

$$\breve{\bar{u}}_{ij} + \frac{\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}}{\Delta x_i} + \frac{\hat{g}_{i,j+1/2} - \hat{g}_{i,j-1/2}}{\Delta y_j} = 0.$$

where

$$\hat{f}_{i+1/2,j} = \frac{1}{\Delta y_j}\int_{y_{j-1/2}}^{y_{j+1/2}} f(u(x_{i+1/2,j}, y, t))\,dy = \breve{f}_{i+1/2,j}$$

$$\hat{g}_{i,j+1/2} = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} g(u(x, y_{i,j+1/2}, t))\,dx = \bar{g}_{i,j+1/2}.$$

Linear case:

If $f(u) = u$, $g(u) = u$ (const $\cdot u$ is also good), then $\breve{f}_{i+1/2,j} = \breve{u}_{i+1/2,j}$, $\bar{g}_{i,j+1/2} = \bar{u}_{i,j+1/2}$.

$\breve{\bar{u}}_{ij} \to \breve{u}_{i+1/2,j}$ is like 1-dim reconstruction. (Line by line).

The cost is 1D reconstruction.

Nonlinear case.

In general, $\hat{f}_{i+1/2,j} = \frac{1}{\Delta y_j}\int_{y_{j-1/2}}^{y_{j+1/2}} f(u(x_{i+1/2}, y, t))\,dy \neq f\left(\frac{1}{\Delta y}\int_{y_{j-1/2}}^{y_{j+1/2}} u(x_{i+1/2}, y, t)\,dy\right)$ cannot pull out f $\to \breve{u}$. comes from 1D reconstruction

So need to use two reconstruction and quadrature: $\breve{\bar{u}} \xrightarrow{r.} \bar{u} \xrightarrow{r.} u \to f(u) \xrightarrow{q.} \breve{f}$

In fact, $\breve{\bar{u}}_{ij} = u_{ij} + O(\Delta x^2)$ $(\Delta x = \Delta y)$. So we can identify $\breve{\bar{u}}_{ij}$ and u_{ij} up to 2-nd order.

H.W. #10

Use 5-th FD WENO with L-F flux-splitting to repeat HW#9.

$u_t + f(u)_x + g(u)_y = 0$

- FV: hard to program and expensive
- FD: Easy to generalize from 1D.

$u_{ij} \simeq u(x_i, y_j, t)$,

$$\frac{du_{ij}}{dt} + \frac{1}{\Delta x}\left(\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}\right) + \frac{1}{\Delta y}\left(\hat{g}_{i,j+1/2} - \hat{g}_{i,j-1/2}\right) = 0.$$

1D algorithm with fixed j $\quad\cdots$

Lecture 11

$u_t + f(u)_x = 0$. u is a vector, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$. $f(\cdot)$ is a vector function. $f(u) = \begin{pmatrix} f_1(u_1, \cdots, u_m) \\ \vdots \\ f_m(u_1, \cdots, u_m) \end{pmatrix}$

$Df \, (f') = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix}$ should have real eigenvalues and a complete set of eigenvectors.

This is called hyperbolic system

Linear case: $f(u) = Au$. A has eigenvalues $\lambda_1 \le \cdots \le \lambda_m$. and a complete set of eigenvectors $r_1, \cdots, r_m$.

$(Ar_1, \cdots, Ar_m) = (\lambda_1 r_1, \cdots, \lambda_m r_m)$. $AR = R\Lambda$. $(\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_m))$.

$A = R\Lambda R^{-1}$. $u_t + Au_x = 0 \quad u_t + R\Lambda R^{-1} u_x = 0 \Rightarrow (R^{-1}u)_t + \Lambda (R^{-1}u)_t = 0$.

Let $v = R^{-1}u$. Then $\begin{cases} (v_1)_t + \lambda_1 (v_1)_x = 0 \\ \cdots \\ (v_m)_t + \lambda_m (v_2)_x = 0 \end{cases}$. $u = Rv$.

General case:

f is smooth w.r.t. u. $u_t + f'(u) u_x = 0$. $f'(u) = A(u)$. $R^{-1}(u) u_t + \Lambda(u) R^{-1}(u) u_x = 0$.

① Assuming smoothness ② In general $(R^{-1}(u)u)_t \ne R^{-1}(u) u_t \cdots$

For the componentwise eqn. $\begin{cases} (u_1)_t + (f_1(u_1, \cdots, u_m))_x = 0 \\ \vdots \\ (u_m)_t + (f_m(u_1, \cdots, u_m))_x = 0. \end{cases}$

For FD, need flux spliting, $\frac{\partial f^+(u)}{\partial u} \ge 0$, $\frac{\partial f^-(u)}{\partial u} \le 0$. (All eigenvalues positive & negative).

If use L-F flux, $f^\pm(u) = \frac{1}{2}(f(u) \pm \alpha(u))$. Chose α large enough, then $\frac{\partial f^+(u)}{\partial u} \ge 0$, $\frac{\partial f^-(u)}{\partial u} \le 0$ are satisfied.

This is oscillatory b/c single shock is hard to happen (hard to follow R-H vector condition).

So discontinuity will propagate at different speed.

Local characteristic decomposition

$R^{-1}(u) = \begin{pmatrix} l_1(u) \\ \vdots \\ l_m(u) \end{pmatrix}$. l_i is left eigenvectors of $A(u)$. $l_m(u)A(u) = \lambda_m(u) l_m(u)$.

Given $\{u_j\}$, to compute $\hat{f}_{j+1/2}$, we proceed as follows

(1) Compute an average of u_j & u_{j+1}, denote by $u_{j+1/2}$, e.g. $u_{j+1/2} = \frac{1}{2}(u_j + u_{j+1})$, or the Roe average (by mean value thm)

$f(u_{j+1}) - f(u_j) = f'(u_{j+1/2})(u_{j+1} - u_j)$

(2) $f'(u_{j+1/2})$, $R_{j+1/2} = R(u_{j+1/2})$, $R^{-1}_{j+1/2} = R^{-1}(u_{j+1/2})$, $\Lambda_{j+1/2} = \Lambda(u_{j+1/2})$

(3) Project $f(u_i)$ & u_i to the local characteristic fields. $v_i = R^{-1}_{j+1/2} u_i$. $i = j-2, \cdots, j+3$ (depending on algorithm)

$g_i = R^{-1}_{j+1/2} f(u_i)$.

(4) Compute the flux splitting in each field

Let $g_i = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}$, then $(g_l)_i^\pm = \frac{1}{2}((g_l)_i \pm \alpha_l (v_l)_i)$. $l = 1, \cdots, m$. $i = j-2, \cdots, j+3$.

$\alpha_l = \max_u |\lambda_l(u)|$, $l = 1, \cdots, m$.

Feed it into the scalar routines. $(\hat{g}_l)^\mp_{j+1/2}$ $(\hat{g}_l)_{j+1/2} = (\hat{g}_l)^+_{j+1/2} + (\hat{g}_l)^-_{j+1/2}$.

(5) Project back to $\hat{f}_{j+1/2} = R_{j+1/2} \hat{g}_{j+1/2}$.

Example: Euler Equations of gas dynamics

$u = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}$ $m = \rho v$, momentum. $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$. p. pressure. Here we assume ideal gas.

$\gamma = 1.4$ for air at room temperature.

$$f(u) = \begin{pmatrix} m \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} = \begin{pmatrix} m \\ \frac{m^2}{\rho} + (\gamma-1)(E - \frac{1}{2}\frac{m^2}{\rho}) \\ \frac{m}{\rho}(E + (\gamma-1)(E - \frac{1}{2}\frac{m^2}{\rho})) \end{pmatrix}$$

$$\Lambda(u) = \begin{pmatrix} v-c & & \\ & v & \\ & & v+c \end{pmatrix},$$ $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed. (CW Shu. Applied Numerical Math, vol 9, 1992, pp45-71).

$$R(u) = \begin{pmatrix} 1 & 1 & 1 \\ v-c & v & v+c \\ H-vc & \frac{1}{2}v^2 & H+vc \end{pmatrix}$$ $H = \frac{E+p}{\rho}$ is the enthalpy.

$$R^{-1}(u) = \frac{1}{2}\begin{pmatrix} b_2 + \frac{v}{c} & b_1 v + \frac{1}{c} & b_1 \\ 2(1-b_2) & 2b_1 v & -2b_1 \\ b_2 - \frac{v}{c} & -(b_1 v - \frac{1}{c}) & b_1 \end{pmatrix},\quad b_1 = \frac{\gamma-1}{c^2},\quad b_2 = \frac{1}{2}v^2 b_1.$$

H.W. #11

Solve
$$\begin{cases} u_t + (\frac{u^2}{2})_x + (\frac{u^2}{2})_y = 0 & \text{on } [0,4\pi]\times[0,4\pi]. \\ u(x,y,0) = \frac{1}{3} + \frac{2}{3}\sin(\frac{x+y}{2}). & (\xi = \frac{x+y}{2}, \eta = \frac{x-y}{2}) \end{cases}$$

Error table & figures at $t=0.3$ & $t=2$. WENO5, FD.

Lecture 12

1D system: $u_t + f(u)_x = 0$.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \quad f(u) = \begin{pmatrix} f_1(u) \\ \vdots \\ f_m(u) \end{pmatrix} \quad f'(u) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix}$$

real eigenvalues $\lambda_1(u) \le \lambda_2(u) \le \cdots \le \lambda_m(u)$

complete set of eigenvectors: $\gamma_1(u), \cdots, \gamma_m(u)$.

Euler equations:

$$u = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \quad f(u) = \begin{pmatrix} m \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

ρ: density m: momentum E: total energy

v: velocity p: pressure

$v = \frac{m}{\rho}$ $\quad E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2$ $\quad \gamma = 1.4$ for air

Then for the $f'(u)$

$\lambda_1(u) = v - c$, $\lambda_2(u) = v$, $\lambda_3(u) = v + c$. $c = \sqrt{\frac{\gamma p}{\rho}}$. sound speed. $p > 0$, $\rho > 0$.

Algorithm flowchart of 5^{th} FD WENO

Given $\{u_j\}$, $\{f(u_j)\}$

1. Compute eigenvalues $\lambda_1(u), \lambda_2(u), \lambda_3(u)$ for all $u = u_j$. $\alpha = \max_{k, u} |\lambda_k(u)|$. Choose Δt s.t. $\alpha \frac{\Delta t}{\Delta x} \le CFL$.
2. Compute a "middle stage" $u_{j+1/2}$, an average of u_j & u_{j+1}, e.g. Roe average (P.L. Roe, JCP, 43, 1981, pp357-372).

   $t_0 = \frac{\sqrt{\rho_j}}{\sqrt{\rho_j} + \sqrt{\rho_{j+1}}}$. $t_1 = 1 - t_0 = \frac{\sqrt{\rho_{j+1}}}{\sqrt{\rho_j} + \sqrt{\rho_{j+1}}}$, $V_{xm} = t_0 v_j + t_1 v_{j+1}$ $\quad H_i = \frac{p_i + E_i}{\rho_i}$, $i = j, j+1$. $H_{xm} = t_0 H_j + t_1 H_{j+1}$.

   $q_m = \frac{1}{2} V_{xm}^2$, $C_m = \sqrt{(\gamma - 1)(H_{xm} - q_m)}$.

   Let $t_2 = V_{xm} = C_m$. $R = \begin{pmatrix} 1 & 1 & 1 \\ V_{xm} - C_m & V_{xm} & V_{xm} + C_m \\ H_{xm} - t_2 & q_m & H_{xm} + t_2 \end{pmatrix}$

   Let $\gamma_{cm} = \frac{1}{C_m}$. $b_1 = (\gamma - 1)\gamma_{cm}^2$, $b_2 = q_m \cdot b_1$. $t_0 = V_{xm} \cdot \gamma_{cm}$. $t_1 = b_1 \cdot V_{xm}$. $t_2 = 0.5\, b_1$.

   $$L = \begin{pmatrix} \frac{1}{2}(b_2 + t_0) & -\frac{1}{2}(t_1 + \gamma_{cm}) & t_2 \\ 1 - b_2 & t_1 & -b_1 \\ \frac{1}{2}(b_2 - t_0) & -\frac{1}{2}(t_1 - \gamma_{cm}) & t_2 \end{pmatrix}$$

   (or $u_{j+1/2} = u_j + u_{j+1}$)
3. Local characteristics decomposition:

   $w_i = L u_i$, $g_i = L f(u_i)$. $i = j-2, j-1, j, j, j+1, j+2, j+3$.
4. $g_i^{\pm} = \frac{1}{2}\left(g_i \pm \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} w_i\right)$: L-F flux, private α for each component: $\alpha_k = \max_u |\lambda_k(u)|$.

   $\alpha_1 = \max_i |v_i - c_i|$, $\alpha_2 = \max_i |v_i|$, $\alpha_3 = \max_i |v_i + c_i|$.
5. Use the scalar WENO to compute $\hat{g}^{\pm}_{i+1/2}$. $\Rightarrow \hat{g}_{i+1/2} = \hat{g}^{+}_{i+1/2} + \hat{g}^{-}_{i+1/2}$
6. $\hat{f}_{j+1/2} = R \hat{g}_{j+1/2}$

H.W. #12

Use 5^{th} FD WENO with SSP-RK3 to compute the Euler equation

(i) Initial condition: $\rho(x) = 1 + 0.2\sin x$, $v(x, 0) = 1$, $p(x, 0) = 1$, $\gamma = 1.4$. (Periodic BC).

Exact solution: $\rho(x, t) = 1 + 0.2\sin(x - vt)$, $v \equiv 1$, $p \equiv 1$.

(ii) $\gamma = 3$. $\rho(x,0) = 1+\frac{1}{2}\sin x$, $v(x,0) = \frac{1}{2}\sin x$, $p(x,0) = (1+\frac{1}{2}\sin x)^3$

Exact solution can be obtained by the fact that $w = v-\sqrt{3}\rho$ and $z = v+\sqrt{3}\rho$ both satisfy the standard Burgers equations

(iii) Shock tube problem: $\Omega = [-5,5]$

Sod: $t=2$. $\begin{pmatrix} \rho(x,0) \\ v(x,0) \\ p(x,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $x \le 0$, $\begin{pmatrix} 0.125 \\ 0 \\ 0.1 \end{pmatrix}$ $x>0$

Lax: $t=1.3$ $\cdots = \begin{pmatrix} 0.445 \\ 0.698 \\ 3.528 \end{pmatrix}$ $\cdots$ $\begin{pmatrix} 0.5 \\ 0 \\ 0.571 \end{pmatrix}$ $\cdots$

BC: Use ghost points, $u(j) = u(0)$, $j=-1,-2,\cdots$. or $u(j) \equiv$ init condition

2D Systems

$$u_t + f(u)_x + g(u)_y = 0$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad f(u) = \begin{pmatrix} f_1(u) \\ \vdots \\ f_m(u) \end{pmatrix}, \quad g(u) = \begin{pmatrix} g_1(u) \\ \vdots \\ g_m(u) \end{pmatrix}$$

Hyperbolic: $\xi_1 f'(u) + \xi_2 g'(u)$ has real eigenvalues and a complete set of eigenvectors $\forall \xi_1, \xi_2 \in \mathbb{R}$.

2D Euler.

$$u = \begin{pmatrix} \rho \\ m_1 \\ m_2 \\ E \end{pmatrix}, \quad f(u) = \begin{pmatrix} m_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ v_1(E+p) \end{pmatrix}, \quad g(u) = \begin{pmatrix} m_2 \\ \rho v_2 v_1 \\ \rho v_2^2 + p \\ v_2(E+p) \end{pmatrix}$$

$v_1 = \frac{m_1}{\rho}$, $v_2 = \frac{m_2}{\rho}$, $E = \frac{1}{\gamma-1}p + \frac{1}{2}\rho(v_1^2+v_2^2)$

$v = (v_1, v_2)$ is the velocity vector

Eigenvalues: $f'(u)$: $v_1 - c, v_1, v_1, v_1+c$; $g'(u) = v_2 - c, v_2, v_2, v_2+c$.

Eigenvectors: in the paper

FD WENO Scheme:

Given $\{u_{ij}\}$, hence also $\{f(u_{ij})\}$, $\{g(u_{ij})\}$.

For $j = 1:N_y$,

fix j. $w_i = w_{ij}$, $f_{xi} = f(u_{ij})$, use the 1D algorithm based on $\{w_i\}$ & $\{f_{xi}\}$.

we can get the approximation: $\frac{1}{\Delta x}(\hat{f}_{x\,i+\frac{1}{2},j} - \hat{f}_{x\,i-\frac{1}{2},j})$ to $f(u)_x$ at (x_i, y_j).

For $i = 1:N_x$

fix i. $w_j = u_{ij}$, $f_{yj} = g(u_{ij})$, use the 1D algorithm based on $\{w_j\}$ & $\{f_{yj}\}$.

we can get the approximation: $\frac{1}{\Delta y}(\hat{g}_{y\,i,j+\frac{1}{2}} - \hat{g}_{y\,i,j-\frac{1}{2}})$ to $g(u)_y$ at (x_i, y_j).

Theoretical properties of 1D system

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases} \quad \text{is hyperbolic}$$

$\lambda_\ell(u)$: eigenvalue, $r_\ell(u)$ eigenvector $(\nabla \lambda_\ell(u)) \cdot r_\ell$ is the directional derivative along the eigenvector direction.

$$(\nabla \lambda_\ell(u)) \cdot r_\ell = \begin{cases} > 0 \ (\text{or } < 0): \text{ genuinely non-linear} \\ \equiv 0: \text{ linearly degenerate} \end{cases}$$

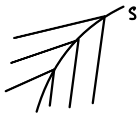
For the Euler equations in 1D,

1^{st} & 3^{rd} fields (with $v-c$, $v+c$) are genuinely nonlinear.

2^{nd} field (with v) is linearly degenerate.

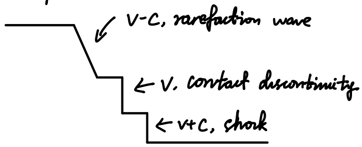
With these properties, we can solve the Riemann problems with entropy solutions for small data (in general) and big data for Euler.

Lax Entropy condition:



$$\lambda_1(u_\ell) \leq \cdots \leq \lambda_k(u_\ell) \leq \cdots \leq \lambda_m(u_\ell)$$
$$\leq s \leq$$
$$\lambda_1(u_r) \leq \cdots \leq \lambda_k(u_r) \leq \cdots \leq \lambda_m(u_r).$$

Example: Sod



(Smoller, procedure of computing exact solution).

Lecture 13

For the genuinely nonlinear characteristic field,

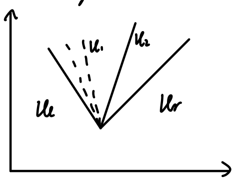
$$\lambda_1(u_r) \le \lambda_2(u_r) \le \cdots \le \lambda_k(u_r) \le \lambda_{k+1}(u_r) \le \cdots \le \lambda_m(u_r)$$

$$\le s \le$$

$$\lambda_1(u_l) \le \lambda_2(u_l) \le \cdots \le \lambda_k(u_l) \le \lambda_{k+1}(u_l) \le \cdots \le \lambda_m(u_l)$$

Lax condition: $\lambda_k(u_r) \le s \le \lambda_k(u_l)$.

Riemann problem:



Lax: Genuinely nonlinear or linearly degenerate, given small data, then $\exists\, u_1, u_2$ connecting u_l, u_r.

If we have the Riemann solver, then we can immediately write the Godunov scheme.

By divergence theorem, $\bar{u}_j^{n+1} = \bar{u}_j + \frac{\Delta t}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})$. Riemann solver.

Glimm proves $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}$, $u^0(x)$ is small w.r.t. TV-semi-norm has a solution.

The proof use numerical schemes to construct a convergent sequence, which converges to a solution. Sample randomly from the true solution.

Boundary condition

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = u^0(x) \\ u(0,t) = g(t). \end{cases}$$

For 1^{st} order scheme, $\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}) = u_j^n - \lambda(\bar{u}_j^n - \bar{u}_{j-1}^n)$, where $\hat{f}_{j+1/2} = \bar{u}_j^n$.

But if we want to use high order schemes, we need $0, -1, -2, n+1, n+2, n+3$.

Extrapolation: accuracy is fine. For stability, outflow extrapolation is fine, but not at inflow.

One order lower at boundary is usually fine. But alignment is a problem.

Inverse Lax-Wendroff procedure (ILW)

$u_t + u_x = 0$. RK3 use 7 points instead of 3. Not compact.

LW: $u_j^{n+1} = u_j^n + (u_t)_j^n \Delta t + (u_{tt})_j^n \frac{1}{2}\Delta t^2 = u_j^n - (u_x)_j \Delta t + (u_{xx})_j^n \frac{1}{2}\Delta t^2$.

Now suppose $x_{-1} = x_0 + \alpha\Delta x$. $u_{-1} = u_0 + (u_x)_0(\alpha\Delta x) + (u_{xx})_0 \frac{(\alpha\Delta x)^2}{2} = u_0 - (u_t)_0(\alpha\Delta x) + (u_{tt})(\frac{\alpha\Delta x}{2})^2$.

$u_{-1} = u_0 - g'(t)\alpha\Delta x + g''(t)\frac{(\alpha\Delta x)^2}{2} + \cdots$

(Simplified ILW: do extrapolation for high-order derivatives).

For parabolic PDEs, like $u_t = u_{xx}$. $u_1 = u_0 + (u_x)_1\Delta x + (u_{xx})_1 \frac{\Delta x^2}{2}$ u_x: extrapolation, u_{xx}: ILW.

For convection dominated PDEs. $u_x = \frac{1}{a}(u_t - u_{xx})$ $a \gg 1$. the other term use extrapolation

diffusion $\cdots$ $u_{xx} = u_t - au_x$ $a \ll 1$. $\cdots$

Lecture 14

Bound-preserving

$E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$, $\frac{p}{\gamma-1}$: internal energy , $\frac{1}{2}\rho v^2$: kinetic energy

When $Ma \gg 1$, i.e. $v \gg c$, $p = (\gamma-1)(E - \frac{1}{2}\rho v^2)$. "$-$" can cause trouble.

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u_0(x). \end{cases}$$

$m = \min_x u^0(x)$, $M = \max_x u^0(x)$. Then $m \le u(x,t) \le M$. Maximum principle.

Monotone schemes satisfies the maximum principle.

Local maximal principle : $m_j = \min(u_{j-p}^n, \cdots, u_{j+q}^n)$. $M_j = \max(u_{j-p}^n, \cdots u_{j+q}^n)$. $m_j \le u_{j+1}^n \le M_j$

But this restricted the accuracy to 2^{nd} order.

Pf: $\begin{cases} u_t + u_x = 0 \\ u(x,0) = \sin x. \end{cases}$

Consider $x = \frac{\pi}{2}$. Then $M_j = 1 - \Delta x^2$. $u_{j_0}^{n+1} \le 1 - \Delta x^2$.

Let $\Delta t = \frac{1}{2}\Delta x$. $u_{j_0,\text{exact}}^{n+1} = 1$.

Boundary preserving high-order scheme

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda(\hat{f}(u_{j+\frac12}^-, u_{j+\frac12}^+) - \hat{f}(u_{j-\frac12}^-, u_{j-\frac12}^+))$$
$$= G(\bar{u}_j^n, u_{j+\frac12}^-, u_{j+\frac12}^+, u_{j-\frac12}^-, u_{j-\frac12}^+) = G(\uparrow, \downarrow, \uparrow, \uparrow, \downarrow).$$

$\bar{u}_j^n$, $u_{j+\frac12}^-$, $u_{j-\frac12}^+$ belongs to the same polynomial.

Idea: use Gauss-Lobatto quadrature to compute $\bar{u}_j$.

For G-L quadrature, K-point is exact for polynomial degree $\le 2K-3$.

$$\bar{u}_j^n = w_1 u_{j-\frac12}^+ + w_K u_{j+\frac12}^- + \sum_{\ell=2}^{K-1} w_\ell\, p(x_j^\ell) - \lambda[\hat{f}(u_{j+\frac12}^-, u_{j+\frac12}^+) - \hat{f}(u_{j-\frac12}^+, u_{j+\frac12}^-) + \hat{f}(u_{j-\frac12}^+, u_{j+\frac12}^-) - \hat{f}(u_{j-\frac12}^-, u_{j-\frac12}^+)].$$
$$= w_1[u_{j-\frac12}^+ - \frac{\lambda}{w_1}(\hat{f}(u_{j-\frac12}^+, u_{j+\frac12}^-) - \hat{f}(u_{j-\frac12}^-, u_{j-\frac12}^+))] + w_K[u_{j+\frac12}^- - \frac{\lambda}{w_K}(\hat{f}(u_{j+\frac12}^-, u_{j+\frac12}^+) - \hat{f}(u_{j-\frac12}^+, u_{j+\frac12}^-)] + \sum_{\ell=2}^{K-1} w_\ell\, p(x_j^\ell).$$
$$= w_1 H_{\frac{\lambda}{w_1}}(u_{j-\frac12}^-, u_{j-\frac12}^+, u_{j+\frac12}^-) + w_K H_{\frac{\lambda}{w_K}}(u_{j-\frac12}^+, u_{j+\frac12}^-, u_{j+\frac12}^+) + \sum_{\ell=2}^{K-1} w_\ell\, p(x_j^\ell)$$

Require: $\frac{\lambda}{w_1} \le \lambda_0$, $\frac{\lambda}{w_K} \le \lambda_0$, and $p(x_j^\ell) \in [m, M]$ for all G-L points.

So we have to apply limiter to make the $p(\cdot)$ satisfies boundary preserving

Limiter: ① $p(x_j^\ell) \in [m, M]$ ② Do not lose order.

$p_j(x) = \bar{u}_j^n + (p_j(x) - \bar{u}_j^n)$, $\tilde{p}_j(x) = \bar{u}_j^n + \theta(p_j(x) - \bar{u}_j^n)$. $\theta = \min_\ell \left\{ \frac{M - \bar{u}_j^n}{p_j(x_j^\ell) - \bar{u}_j}, 1 \right\}$ (and θ for min).

Lemma: $\tilde{p}_j(x) - u(x) = O(\Delta x^r)$ if $p_j - u(x) = O(\Delta x^r)$.