四元数 (Hamilton)

1. 定义：

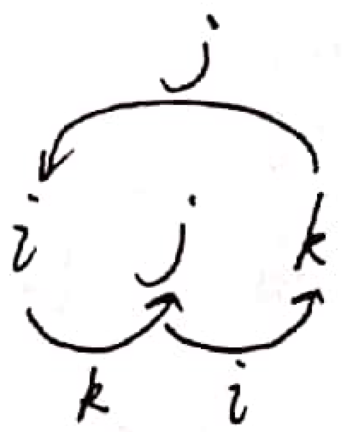
假设有2个复数 $A=a+bi$, $C=c+di$. 构造

$$Q = A + Cj = a+bi+cj+dij \quad (令\ k \triangleq ij)$$

$$\triangleq a+bi+cj+dk \in \mathbb{H}$$

其中 $i^2=j^2=k^2=ijk=-1$,

$ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$



四元数另一种表示形式：scalar + vector

$$Q = q_w + q_x i + q_y j + q_z k \iff Q = q_w + q_v$$

其中 $q_v = q_x i + q_y j + q_z k = (q_x, q_y, q_z)^T$

$$q \triangleq \begin{bmatrix} q_w \\ q_v \end{bmatrix} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix}$$

2. 四元数运算

① Sum

$$p \pm q = \begin{bmatrix} p_w \\ p_v \end{bmatrix} \pm \begin{bmatrix} q_w \\ q_v \end{bmatrix} = \begin{bmatrix} p_w \pm q_w \\ p_v \pm q_v \end{bmatrix}$$

② Product

$$p \otimes q = \begin{bmatrix} p_w q_w - p_v^T q_v \\ p_w q_v + q_w p_v + p_v \times q_v \end{bmatrix} \qquad p_v \times q_v = \begin{bmatrix} i & j & k \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{bmatrix}$$

$$p \otimes q \neq q \otimes p$$

$$(p \otimes q) \otimes r = p \otimes (q \otimes r)$$

$$p \otimes (q+r) = p \otimes q + p \otimes r, \quad (p+q) \otimes r = p \otimes r + q \otimes r$$

$$q_1 \otimes q_2 = [q_1]_L q_2 \qquad [q]_L = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & -q_z & q_y \\ q_y & q_z & q_w & -q_x \\ q_z & -q_y & q_x & q_w \end{bmatrix} = q_w I + \begin{bmatrix} 0 & -q_v^T \\ q_v & [q_v]_\times \end{bmatrix}$$

$$q_1 \otimes q_2 = [q_2]_R q_1 \qquad [q]_R = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & q_z & -q_y \\ q_y & -q_z & q_w & q_x \\ q_z & q_y & -q_x & q_w \end{bmatrix} = q_w I + \begin{bmatrix} 0 & -q_v^T \\ q_v & -[q_v]_\times \end{bmatrix}$$

反对称运算：

$$[a]_\times \triangleq \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \qquad [a]_\times^T = -[a]_\times \qquad [a]_\times b = a \times b$$

$$(q \otimes x) \otimes p = [p]_R [q]_L x, \quad q \otimes (x \otimes p) = [q]_L [p]_R x$$

$$\Rightarrow [p]_R [q]_L = [q]_L [p]_R$$

③ Identity

$$q_1 \triangleq \begin{bmatrix} 1 \\ 0_v \end{bmatrix} = 1 \qquad q_1 \otimes q = q \otimes q_1 = q$$

④ Conjugate

$$q^* \triangleq q_w - q_v = \begin{bmatrix} q_w \\ -q_v \end{bmatrix}$$

$$q \otimes q^* = q^* \otimes q = q_w^2 + q_x^2 + q_y^2 + q_z^2$$

$$(p \otimes q)^* = q^* \otimes p^*$$

⑤ Norm

$$\|q\| \triangleq \sqrt{q\otimes q^*} = \sqrt{q^*\otimes q} = \sqrt{q_w^2+q_x^2+q_y^2+q_z^2}$$

$$\|p\otimes q\| = \|q\otimes p\| = \|p\|\|q\|$$

⑥ Inverse

$$q\otimes q^{-1} = q^{-1}\otimes q = q_1 \quad , \quad q^{-1} = q^*/\|q\|^2$$

⑦ Unit Quaternion

$\|q\|=1$, $q^{-1}=q^*$, $q=\begin{bmatrix}\cos\theta \\ u\sin\theta\end{bmatrix}$, $u = u_x i + u_y j + u_z k$ 是单位向量

$$\|q\| = \sqrt{\cos^2\theta + (u_x\sin\theta)^2 + (u_y\sin\theta)^2 + (u_z\sin\theta)^2} = 1$$

⑧ Quaternion commutator

$$[p,q] \triangleq p\otimes q - q\otimes p = 2\,p_v\times q_v$$

⑨ Product of pure quaternions

$$p_v\otimes q_v = \begin{bmatrix}-p_v^T q_v \\ p_v\times q_v\end{bmatrix}, \quad q_v\otimes q_v = -q_v^T q_v = -\|q_v\|^2$$

$\|u\|=1$, $u\otimes u = -1 \Rightarrow i^2=-1$

⑩ Natural powers of pure quaternions

纯四元数 $v = u\theta$, $\theta=\|v\|$, $\|u\|=1$

$v^2=-\theta^2$, $v^3=-u\theta^3$, $v^4=\theta^4$, $v^5=u\theta^5$, $v^6=-\theta^6$, …

⑪ Exponential of pure quaternions

定义四元数指数: $e^q \triangleq \sum_{k=0}^{\infty}\frac{1}{k!}q^k \in \mathbb{H}$

纯四元数 $v = v_x i + v_y j + v_z k$ 的指数.

$e^v = \sum_{k=0}^{\infty}\frac{1}{k!}v^k$，令 $v=u\theta$, $\theta=\|v\|$, $\|u\|=1$

$$e^{u\theta} = \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\cdots\right) + \left(u\theta - \frac{u\theta^3}{3!} + \frac{u\theta^5}{5!}+\cdots\right)$$

$$= \cos\theta + u\sin\theta$$

$$= \begin{bmatrix}\cos\theta \\ u\sin\theta\end{bmatrix} \longrightarrow \text{unit quaternion.}$$

☆纯四元数的指数为单位四元数，则

$$e^{-v} = (e^v)^*$$

欧拉公式: $e^{i\theta} = \cos\theta + i\sin\theta$. 令 $\theta=\pi$, $\boxed{e^{\pi i}+1=0}$

⑫ Exponential of general quaternions

$$e^q = e^{q_w+q_v} = e^{q_w}e^{q_v} = e^{q_w}\begin{bmatrix}\cos\|q_v\| \\ \frac{q_v}{\|q_v\|}\sin\|q_v\|\end{bmatrix}$$

⑬ Logarithm of unit quaternions

$\|q\|=1$, $\log q = \log(\cos\theta + u\sin\theta) = \log(e^{u\theta}) = u\theta = \begin{bmatrix}0 \\ u\theta\end{bmatrix}$

单位四元数的对数为纯四元数.

$u = q_v/\|q_v\|$, $q_w=\cos\theta$, $\|q_v\|=\sin\theta \Rightarrow \theta = \arctan(\|q_v\|, q_w)$

⑭ logarithm of general quaternions

$$\log q = \log\left(\|q\|\frac{q}{\|q\|}\right) = \log\|q\| + \log\frac{q}{\|q\|} = \log\|q\| + u\theta = \begin{bmatrix}\log\|q\| \\ u\theta\end{bmatrix}$$

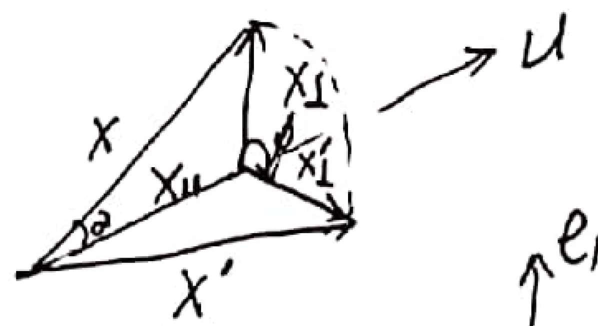
⑮ Exponential forms of the type q^t

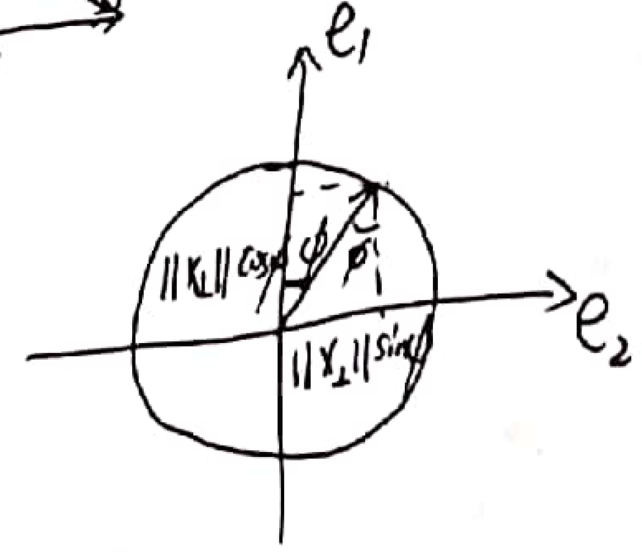
$$q^t = \exp(\log(q^t)) = \exp(t\log(q))$$

如果 $\|q\|=1$, $q^t = \exp(tu\theta) = \exp(ut\theta) = \begin{bmatrix}\cos t\theta \\ u\sin t\theta\end{bmatrix}$

3.旋转

① 3D向量旋转表示





$$x = x_\parallel + x_\perp$$

$$x_\parallel = u\cdot(\|x\|\cos\alpha) = uu^T x$$

$$x_\perp = x - x_\parallel = x - uu^T x$$

$$e_1 = x_\perp,\quad e_2 = u\times x_\perp = u\times x$$

$$x'_\perp = \frac{e_1}{\|x_\perp\|}\cdot\|x_\perp\|\cos\phi + \frac{e_2}{\|x_\perp\|}\cdot\|x_\perp\|\sin\phi$$

$$= e_1\cos\phi + e_2\sin\phi$$

$$= x_\perp\cos\phi + (u\times x)\sin\phi$$

$$x' = x_\parallel + x_\perp\cos\phi + (u\times x)\sin\phi$$

② 旋转李群定义.

旋转操作 $r: \mathbb{R}^3 \to \mathbb{R}^3 : v \to r(v)$

性质1: Rotation preserves the vector norm

$$\|r(v)\| = \sqrt{\langle r(v), r(v)\rangle} = \sqrt{\langle v, v\rangle} \triangleq \|v\|$$

等价

性质2: Rotation preserves angle between vectors

$$\langle r(v), r(w)\rangle = \langle v, w\rangle = \|v\|\|w\|\cos\alpha$$

性质3: Rotation preserves the relative orientation of vectors

$$u\times v = w \iff r(u)\times r(v) = r(w)$$

旋转群 rotation group SO(3) 定义:

$$SO(3): \{r: \mathbb{R}^3\to\mathbb{R}^3 \,/\, \forall v,w\in\mathbb{R}^3,\ \|r(v)\| = \|v\|,\ r(v)\times r(w) = r(v\times w)\}$$

③ 旋转李群与旋转矩阵

$$r(v) = Rv$$

$$(Rv)^T(Rv) = v^T R^T R v = v^T v$$

$$\Rightarrow R^T R = I = RR^T \quad\Rightarrow R^{-1} = R^T$$

$$\det(R) = r_1^T(r_2\times r_3) = r_1^T r_1 = 1 \quad\Rightarrow \det(R) = 1$$

The exponential map

$$\frac{d}{dt}(R^T R) = \dot{R}^T R + R^T\dot{R} = 0$$

$$R^T\dot{R} = -(R^T\dot{R})^T \quad\Rightarrow R^T\dot{R} \text{ 为反对称矩阵.}$$

The set of skew-symmetric 3x3 matrices is denoted so(3), and receives the name of Lie algebra (李代数) of SO(3).

假设有向量 $\omega = (\omega_x, \omega_y, \omega_z)\in\mathbb{R}^3$, $[\omega]_\times \in so(3)$

$$R^T\dot{R} = [\omega]_\times$$

$$\dot{R} = R[\omega]_\times$$

$$R(t) = R(0)\,e^{[\omega]_\times t} = R(0)\,e^{[\omega t]_\times}$$

$$e^{[\omega t]_\times} = R(0)^T R(t)$$

令 $v \triangleq \omega\Delta t$,

$$R = e^{[v]_\times}$$

$$\exp: so(3)\to SO(3)\ ;\ [v]_\times \to \exp([v]_\times) = e^{[v]_\times} = R$$

定义 $\mathrm{Exp}(v) \triangleq \exp([v]_\times)$

其中v是旋转向量，可表示为 $v = \underset{\text{旋转角度}}{\phi}\,\underset{\text{旋转轴}}{u}$

The Rodrigues rotation formula：旋转矩阵与旋转向量

令 $v=\phi u$ ，$\|u\|=1$

$$R=e^{[v]_\times}=e^{\phi[u]_\times}$$
$$=I+\phi[u]_\times+\frac{1}{2}\phi^2[u]_\times^2+\frac{1}{3!}\phi^3[u]_\times^3+\frac{1}{4!}\phi^4[u]_\times^4+\cdots$$
$$=I+\phi[u]_\times+\frac{1}{2}\phi^2[u]_\times^2-\frac{1}{3!}\phi^3[u]_\times-\frac{1}{4!}\phi^4[u]_\times^2+\frac{1}{5!}\phi^5[u]_\times+\frac{1}{6!}\phi^6[u]_\times^2+\cdots$$

其中 $[u]_\times$ 满足：

$$[u]_\times^2=uu^T-I \qquad [u]_\times^5=[u]_\times \qquad \cdots$$
$$[u]_\times^3=-[u]_\times \qquad [u]_\times^6=[u]_\times^2$$
$$[u]_\times^4=-[u]_\times^2 \qquad [u]_\times^7=-[u]_\times$$

得到 Rodrigues rotation formula：

$$R=I+\sin\phi\cdot[u]_\times+(1-\cos\phi)[u]_\times^2$$
$$=I\cos\phi+[u]_\times\sin\phi+uu^T(1-\cos\phi)$$

The logarithmic maps:

$\log: SO(3)\to so(3)$ ； $R\to\log(R)=[\phi u]_\times$

$$\phi=\arccos\left(\frac{trace(R)-1}{2}\right),\quad u=\frac{(R-R^T)^\vee}{2\sin\phi},\quad ([v]_\times)^\vee=v$$

$\mathrm{Log}: SO(3)\to\mathbb{R}^3$， $R\to\mathrm{Log}(R)=u\phi$

其中 $\mathrm{Log}(R)\triangleq(\log(R))^\vee$

旋转： $x'=Rx=(I\cos\phi+[u]_\times\sin\phi+uu^T(1-\cos\phi))x$
$$=x\cos\phi+[u]_\times x\sin\phi+uu^Tx(1-\cos\phi)$$
$$=x_\parallel+(u\times x)\sin\phi+x\cos\phi-uu^Tx\cos\phi$$
$$=x_\parallel+(u\times x)\sin\phi+x_\perp\cos\phi$$

④ 旋转李群与四元数.

$$r(v)=q\otimes v\otimes q^*$$
$$\|q\otimes v\otimes q^*\|=\|q\|^2\|v\|=\|v\|$$

$\Rightarrow \|q\|^2=1 \rightarrow$ 单位四元数：

$$q^*\otimes q=1=q\otimes q^*$$

$$r(v)\times r(w)=(q\otimes v\otimes q^*)\times(q\otimes w\otimes q^*)$$
$$=\frac{1}{2}((q\otimes v\otimes q^*)\otimes(q\otimes w\otimes q^*)-(q\otimes w\otimes q^*)\otimes(q\otimes v\otimes q^*))$$
$$=\frac{1}{2}(q\otimes v\otimes w\otimes q^*-q\otimes w\otimes v\otimes q^*)$$
$$=\frac{1}{2}(q\otimes(v\otimes w-w\otimes v)\otimes q^*)$$
$$=q\otimes(v\times w)\otimes q^*$$
$$=r(v\times w)$$

The exponential map

$$\frac{d}{dt}(q^*\otimes q)=\dot{q}^*\otimes q+q^*\otimes\dot{q}=0$$
$$\Rightarrow q^*\otimes\dot{q}=-(\dot{q}^*\otimes q)=-(q^*\otimes\dot{q})^*$$

$\Rightarrow q^*\otimes\dot{q}$ 是纯四元数.

令 $q^*\otimes\dot{q}=\Omega=\begin{bmatrix}0\\ \Omega\end{bmatrix}\in H_p$

$$\dot{q}=q\otimes\Omega$$
$$q(t)=q(0)\otimes e^{\Omega t}$$

令 $V\triangleq\Omega\Delta t$

$$q=e^V$$

$\exp: H_p\to S^3$ ； $V\to\exp(V)=e^V$

$V=v/2$

令 $V=\theta u=\phi u/2$ ， $v=\phi u$

$\mathrm{Exp}: R^3\to S^3$， $v\to\mathrm{Exp}(v)=e^{v/2}$

$\mathrm{Exp}(v)\triangleq\exp(v/2)$

$$V=\Omega\Delta t=\phi u/2=\omega\Delta t/2$$
$$\Rightarrow \omega=2\Omega\Rightarrow\Omega=\omega/2$$
$$\Rightarrow \dot{q}=\frac{1}{2}q\otimes\omega,\quad q=e^{\omega t/2}$$

Quaternion and rotation vector:

$V = \theta u = \phi u/2 = v/2$

$q \triangleq \exp(\theta u) = Exp(\phi u) = e^{\phi u/2} = \cos\phi/2 + u\sin\phi/2 = \begin{bmatrix} \cos(\phi/2) \\ u\sin(\phi/2) \end{bmatrix}$

The logarithmic maps:

$\log: S^3 \to \mathbb{H}_p : \quad q \to \log(q) = \theta u$

$Log: S^3 \to \mathbb{R}^3 : \quad q \to Log(q) = \phi u$

$Log(q) \triangleq 2\log(q)$

$\phi = 2\arctan(\|q_v\|, q_w), \quad u = q_v/\|q_v\|$

旋转操作:

$x' = q \otimes x \otimes q^*, \quad q = Exp(\phi u), \quad x = xi + yj + zk = \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathbb{H}_p$

$x' = q \otimes x \otimes q^*$

$= (\cos\frac{\phi}{2} + u\sin\frac{\phi}{2}) \otimes (0 + x) \otimes (\cos\frac{\phi}{2} - u\sin\frac{\phi}{2})$

$= x\cos^2\frac{\phi}{2} + (u\otimes x - x\otimes u)\sin\frac{\phi}{2}\cos\frac{\phi}{2} - u\otimes x\otimes u\sin^2\frac{\phi}{2}$

$= x\cos^2\frac{\phi}{2} + 2(u\times x)\sin\frac{\phi}{2}\cos\frac{\phi}{2} - (x - 2(u^Tx)u)\sin^2\frac{\phi}{2}$

$= x(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) + (u\times x)\sin\phi + \underbrace{(u^Tx)}_{\text{数}}u\sin^2\frac{\phi}{2}\cdot 2$

$= x\cos\phi + (u\times x)\sin\phi + u\cdot(u^Tx)(1-\cos\phi)$

$= (x - uu^Tx)\cos\phi + (u\times x)\sin\phi + uu^Tx$

$= x_\perp\cos\phi + (u\times x)\sin\phi + x_\parallel$

$u\otimes x\otimes u = (-u^Tx + u\times x)\otimes u$

$= -(u^Tx)u + (u\times x)\otimes u$

$= -(u^Tx)u + \frac{1}{2}(u\otimes x\otimes u - x\otimes u\otimes u)$

$= -(u^Tx)u + \frac{1}{2}u\otimes x\otimes u + \frac{1}{2}x$

$\Rightarrow u\otimes x\otimes u = x - 2(u^Tx)u$

⑤旋转矩阵与四元数.

旋转向量 $v = \phi u$, $q = Exp(\phi u)$, $R = Exp(\phi u)$

$\forall v, x \in \mathbb{R}^3$, $q = Exp(v)$, $R = Exp(v)$

then,

$q\otimes x\otimes q^* = Rx$

$\Rightarrow R = [q^*]_R[q]_L$

$= (q_w^2 - q_v^Tq_v)I + 2q_vq_v^T + 2q_w[q_v]_\times$

⑥复合旋转

$q_{AC} = q_{AB}\otimes q_{BC}$

$x_A = q_{AB}\otimes x_B\otimes q_{AB}^*$

$= q_{AB}\otimes(q_{BC}\otimes x_C\otimes q_{BC}^*)\otimes q_{AB}^*$

$= (q_{AB}\otimes q_{BC})\otimes x_C\otimes(q_{BC}^*\otimes q_{AB}^*)$

$= (q_{AB}\otimes q_{BC})\otimes x_C\otimes(q_{AB}\otimes q_{BC})^*$

$= q_{AC}\otimes x_C\otimes q_{AC}^*$

$R_{AC} = R_{AB}R_{BC}$

$x_A = R_{AB}x_B$

$= R_{AB}(R_{BC}x_C)$

$= (R_{AB}R_{BC})x_C$

$= R_{AC}x_C$

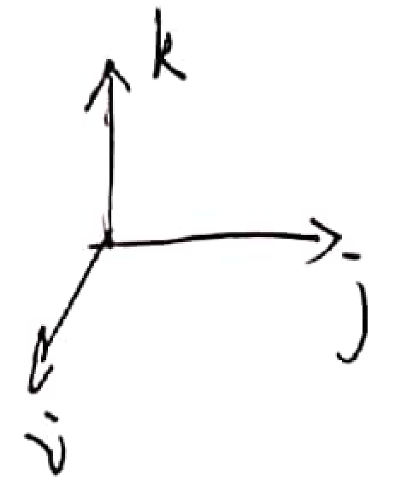
四元数（JPL）

1. 定义：

$$Q = q_w + q_x i + q_y j + q_z k \iff Q = q_w + q_v$$

其中 $q_v = q_x i + q_y j + q_z k = (q_x, q_y, q_z)^T$

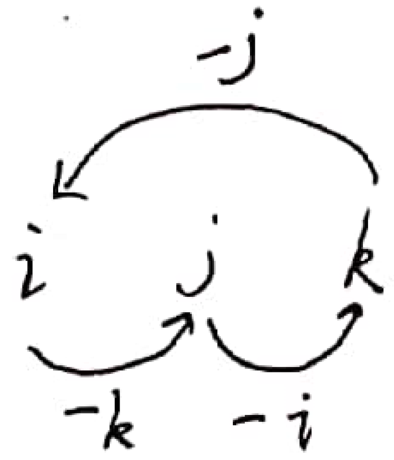
$$q \triangleq \begin{bmatrix} q_v \\ q_w \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix}$$



左手

另外 $i^2 = j^2 = k^2 = kji = -1$

$$ij = -ji = -k, \quad jk = -kj = -i, \quad ki = -ik = -j$$



2. 乘法：

$$q \otimes p = (q_w + q_x i + q_y j + q_z k)(p_w + p_x i + p_y j + p_z k)$$

$$= q_w p_w - q_x p_x - q_y p_y - q_z p_z$$
$$+ (q_w p_x + q_x p_w - q_y p_z + q_z p_y) i$$
$$+ (q_w p_y + q_x p_z + q_y p_w - q_z p_x) j$$
$$+ (q_w p_z - q_x p_y + q_y p_x + q_z p_w) k$$

$$= \begin{bmatrix} p_w & -p_z & p_y & p_x \\ p_z & p_w & -p_x & p_y \\ -p_y & p_x & p_w & p_z \\ -p_x & -p_y & -p_z & p_w \end{bmatrix} \begin{pmatrix} q_x \\ q_y \\ q_z \\ q_w \end{pmatrix} = \begin{bmatrix} p_w I_3 \overset{\text{正号}}{+} [p_v]_\times & p_v \\ -p_v^T & p_w \end{bmatrix} \cdot q = [p]_R q$$

令 $[p]_R = [\Xi(p), p]$

$$= \begin{bmatrix} q_w & q_z & -q_y & q_x \\ -q_z & q_w & q_x & q_y \\ q_y & -q_x & q_w & q_z \\ -q_x & -q_y & -q_z & q_w \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ p_w \end{pmatrix} = \begin{bmatrix} q_w I_3 \overset{\text{负号}}{-} [q_v]_\times & q_v \\ -q_v^T & q_w \end{bmatrix} \cdot p = [q]_L p$$

令 $[q]_L = [\Psi(q), q]$

3. JPL四元数表示旋转.（Rodrigues Rotation formula）

$${}^L p = {}^L_G q \otimes {}^G p \otimes {}^L_G q^{-1}$$

$$= [{}^L_G q]_L \cdot {}^G p \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w I_3 - [{}^L_G q_v]_\times & {}^L_G q_v \\ -{}^L_G q_v^T & q_w \end{bmatrix} \cdot \begin{bmatrix} p \\ 0 \end{bmatrix} \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w p - [{}^L_G q_v]_\times p \\ -{}^L_G q_v^T p \end{bmatrix} \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix} \otimes \begin{bmatrix} -{}^L_G q_v \\ q_w \end{bmatrix} = [{}^L_G q^{-1}]_R \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix}$$

$$= \begin{bmatrix} q_w I_3 - [{}^L_G q_v]_\times & -{}^L_G q_v \\ {}^L_G q_v^T & q_w \end{bmatrix} \cdot \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - q_w {}^L_G q_v \times p - q_w [{}^L_G q_v]_\times p + [{}^L_G q_v]_\times ({}^L_G q_v \times p) + {}^L_G q_v \cdot {}^L_G q_v^T \cdot p \\ -{}^L_G q_v^T \cdot ({}^L_G q_v \times p) \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - 2 q_w {}^L_G q_v \times p + {}^L_G q_v \cdot {}^L_G q_v^T p - {}^L_G q_v \times p \times {}^L_G q_v \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - 2 q_w [{}^L_G q_v]_\times \cdot p + {}^L_G q_v \cdot {}^L_G q_v^T p + [{}^L_G q_v]_\times^2 p \\ 0 \end{bmatrix}$$

其中 $[{}^L_G q_v]_\times^2 = {}^L_G q_v {}^L_G q_v^T - |q|^2 I_3$
$= {}^L_G q_v {}^L_G q_v^T - (1 - q_w^2) I_3$

$$= \begin{bmatrix} (2q_w^2 - 1) I_3 - 2 q_w [{}^L_G q_v]_\times + 2\, {}^L_G q_v \cdot {}^L_G q_v^T \\ 0 \end{bmatrix} \begin{bmatrix} {}^G p \\ 0 \end{bmatrix}$$

$${}^L_G C({}^L_G q) = (2q_w^2 - 1) I_3 - 2 q_w [{}^L_G q_v]_\times + 2\, {}^L_G q_v \cdot {}^L_G q_v^T$$

$$= \Xi({}^L_G q)^T \Psi({}^L_G q)$$

因此有四元数形式旋转转为旋转矩阵:

$$ {}^{L}_{G}C({}^{L}_{G}q) = (2q_w^2-1)I_3 - 2q_w[{}^{L}_{G}q_v]_\times + 2\,{}^{L}_{G}q_v \cdot {}^{L}_{G}q_v^T $$

另外四元数形式旋转可以表示为欧拉公式(Euler formula)形式

$$ q = \begin{bmatrix} u\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix} = \begin{bmatrix} q_v \\ q_w \end{bmatrix} $$

(注意该处的u为右手旋转轴)
可能是习惯表达方式。

其中为什么是$\phi/2$而不是ϕ,可参考<<Quaternion kinematics for the error-state Kalman filter>>中式(54)和(109)。

因此有。

$$ {}^{L}_{G}C({}^{L}_{G}q) = (2\cos^2(\phi/2)-1)I_3 - 2\cos(\phi/2)\sin(\phi/2)[u]_\times + 2\sin^2(\phi/2)uu^T $$

$$ = \cos\phi \cdot I_3 - \sin\phi \cdot [u]_\times + (1-\cos(\phi))uu^T $$

参考<<Quaternion kinematics ···>>中式(78)可知

$$ {}^{L}_{G}C({}^{L}_{G}q) = \cos\phi \cdot I_3 + [-u]_\times \sin\phi + (-u)(-u)^T(1-\cos\phi) $$

$$ = e^{-\phi[u]_\times} $$

$$ \approx I - \phi[u]_\times $$

JPL和Hamilton在该处有一负号差别,是由于JPL是左手系,而Hamilton是右手系。