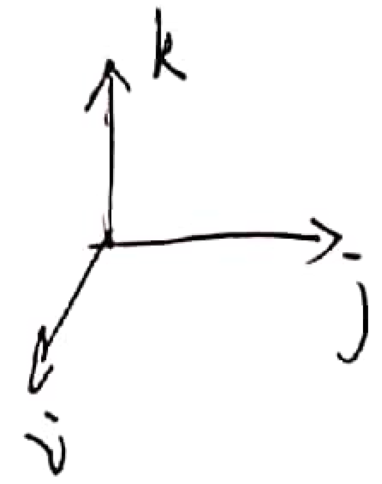
四元数 (JPL)

1. 定义:

$$Q = q_w + q_x i + q_y j + q_z k \iff Q = q_w + q_v$$

其中 $q_v = q_x i + q_y j + q_z k = (q_x, q_y, q_z)^T$

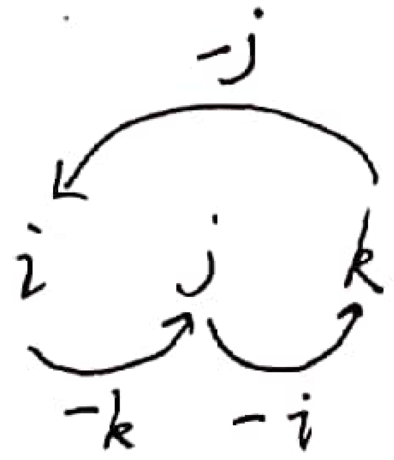
$$q \triangleq \begin{bmatrix} q_v \\ q_w \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix}$$



左手

另外 $i^2 = j^2 = k^2 = kji = -1$

$$ij = -ji = -k, \quad jk = -kj = -i, \quad ki = -ik = -j$$



2. 乘法:

$$q \otimes p = (q_w + q_x i + q_y j + q_z k)(p_w + p_x i + p_y j + p_z k)$$

$$\begin{aligned}
&= q_w p_w - q_x p_x - q_y p_y - q_z p_z \\
&\quad + (q_w p_x + q_x p_w - q_y p_z + q_z p_y) i \\
&\quad + (q_w p_y + q_x p_z + q_y p_w - q_z p_x) j \\
&\quad + (q_w p_z - q_x p_y + q_y p_x + q_z p_w) k
\end{aligned}$$

$$= \begin{bmatrix} p_w & -p_z & p_y & p_x \\ p_z & p_w & -p_x & p_y \\ -p_y & p_x & p_w & p_z \\ -p_x & -p_y & -p_z & p_w \end{bmatrix} \begin{pmatrix} q_x \\ q_y \\ q_z \\ q_w \end{pmatrix} = \begin{bmatrix} p_w I_3 \overset{\text{正号}}{+} [p_v]_\times & p_v \\ -p_v^T & p_w \end{bmatrix} \cdot q = [p]_R \, q$$

令 $[p]_R = [\Xi(p), p]$

$$= \begin{bmatrix} q_w & q_z & -q_y & q_x \\ -q_z & q_w & q_x & q_y \\ q_y & -q_x & q_w & q_z \\ -q_x & -q_y & -q_z & q_w \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ p_w \end{pmatrix} = \begin{bmatrix} q_w I_3 \overset{\text{负号}}{-} [q_v]_\times & q_v \\ -q_v^T & q_w \end{bmatrix} \cdot p = [q]_L \, p$$

令 $[q]_L = [\Psi(q), q]$

3. JPL四元数表示旋转. (Rodrigues Rotation formula)

$${}^L p = {}^L_G q \otimes {}^G p \otimes {}^L_G q^{-1}$$

$$= [{}^L_G q]_L \cdot {}^G p \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w I_3 - [{}^L_G q_v]_\times & {}^L_G q_v \\ -{}^L_G q_v^T & q_w \end{bmatrix} \cdot \begin{bmatrix} p \\ 0 \end{bmatrix} \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w p - [{}^L_G q_v]_\times p \\ -{}^L_G q_v^T p \end{bmatrix} \otimes {}^L_G q^{-1}$$

$$= \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix} \otimes \begin{bmatrix} -{}^L_G q_v \\ q_w \end{bmatrix} = [{}^L_G q^{-1}]_R \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix}$$

$$= \begin{bmatrix} q_w I_3 - [{}^L_G q_v]_\times & -{}^L_G q_v \\ {}^L_G q_v^T & q_w \end{bmatrix} \cdot \begin{bmatrix} q_w p - {}^L_G q_v \times p \\ -{}^L_G q_v^T p \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - q_w \, {}^L_G q_v \times p - q_w [{}^L_G q_v]_\times p + [{}^L_G q_v]_\times ({}^L_G q_v \times p) + {}^L_G q_v \cdot {}^L_G q_v^T \cdot p \\ -{}^L_G q_v^T \cdot ({}^L_G q_v \times p) \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - 2 q_w \, {}^L_G q_v \times p + {}^L_G q_v \cdot {}^L_G q_v^T p - {}^L_G q_v \times p \times {}^L_G q_v \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_w^2 p - 2 q_w [{}^L_G q_v]_\times \cdot p + {}^L_G q_v \cdot {}^L_G q_v^T p + [{}^L_G q_v]_\times^2 p \\ 0 \end{bmatrix}$$

其中 $[{}^L_G q_v]_\times^2 = {}^L_G q_v \cdot {}^L_G q_v^T - |q_v|^2 I_3 = {}^L_G q_v \cdot {}^L_G q_v^T - (1 - q_w^2) I_3$

$$= \begin{bmatrix} (2q_w^2 - 1) I_3 - 2 q_w [{}^L_G q_v]_\times + 2 \, {}^L_G q_v \cdot {}^L_G q_v^T & \\ 0 & \end{bmatrix} \begin{bmatrix} {}^G p \\ 0 \end{bmatrix}$$

$${}^L_G C({}^L_G q) = (2 q_w^2 - 1) I_3 - 2 q_w [{}^L_G q_v]_\times + 2 \, {}^L_G q_v \cdot {}^L_G q_v^T$$

$$= \Xi({}^L_G q)^T \Psi({}^L_G q)$$

因此有四元数形式旋转转为旋转矩阵：

$$ {}^L_G C({}^L_G q) = (2q_w^2 - 1) I_3 - 2q_w [{}^L_G q_v]_\times + 2\, {}^L_G q_v \cdot {}^L_G q_v^T $$

另外四元数形式旋转可以表示为欧拉公式（Euler formula）形式

$$ q = \begin{bmatrix} u\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix} = \begin{bmatrix} q_v \\ q_w \end{bmatrix} $$

（注意该处的u为右手旋转轴）
可能是习惯表达式。

其中为什么是 $\phi/2$ 而不是 ϕ，可参考《Quaternion kinematics for the error-state Kalman filter》中式(54)和(109)。

因此有，

$$ {}^L_G C({}^L_G q) = (2\cos^2(\phi/2) - 1) I_3 - 2\cos(\phi/2)\sin(\phi/2)[u]_\times + 2\sin^2(\phi/2)\, u u^T $$

$$ = \cos\phi \cdot I_3 - \sin\phi \cdot [u]_\times + (1-\cos(\phi))\, u u^T $$

参考《Quaternion kinematics ……》中式(78)可知

$$ {}^L_G C({}^L_G q) = \cos\phi \cdot I_3 + [-u]_\times \sin\phi + (-u)(-u)^T (1-\cos\phi) $$

$$ = e^{-\phi[u]_\times} $$

$$ \approx I - \phi[u]_\times $$

JPL和Hamilton在该处有一负号差别，是由于JPL是左手系，而Hamilton是右手系。

MSCKF 推导 (JPL)

1. 状态表示:

IMU state:

$$X_{IMU} = \left[{}^G p_I^T,\ {}^G V_I^T,\ {}^I_G \bar{q}^T,\ b_a^T,\ b_g^T \right]^T$$

其中 ${}^I_G\bar{q}^T$ 为单位四元数, 从{G}到{I}

IMU error-state:

$$\tilde{X}_{IMU} = \left[{}^G \tilde{p}_I^T,\ {}^G \tilde{V}_I^T,\ \delta\theta_I^T,\ \tilde{b}_a^T,\ \tilde{b}_g^T \right]^T$$

由《Quaternion kinematics for the error-state kalman filter》表3可知

true-state: x_t 表示带有噪声的运动学模型 (从测量中恢复)

norminal-state: $\hat{x}_t$ 表示不带噪声的运动学模型 (求期望时消去了)

对于error-state的表示, 除了旋转, 其它状态量为

$$\tilde{x} = x_t - \hat{x}_t$$

假设 $\hat{\bar{q}}$ 为 $\bar{q}$ 的估计值.

$$\bar{q} = \delta\bar{q} \otimes \hat{\bar{q}}, \quad \delta\bar{q} \approx \left[\tfrac{1}{2}\delta\theta^T \ \ 1\right]^T$$

假设有N帧相机pose也被包含在状态中

$$\hat{X}_k = \left[\hat{X}_{IMU_k}^T,\ {}^{C_1}_G\hat{\bar{q}}^T,\ {}^G\hat{p}_{C_1}^T \cdots {}^{C_N}_G\hat{\bar{q}}^T,\ {}^G\hat{p}_{C_N}^T \right]^T$$

error-state:

$$\tilde{X}_k = \left[\tilde{X}_{IMU_k}^T,\ \delta\theta_{C_1}^T,\ {}^G\tilde{p}_{C_1}^T \cdots \delta\theta_{C_N}^T,\ {}^G\tilde{p}_{C_N}^T \right]^T$$

2. 连续形式IMU运动模型

true-state:

$$\begin{aligned} {}^I_G\dot{\bar{q}}(t) &= \tfrac{1}{2} w(t) \otimes {}^I_G\bar{q}(t) = \tfrac{1}{2}[w(t)]_L \cdot {}^I_G\bar{q}(t) = \tfrac{1}{2}\begin{bmatrix} -[w]_\times & w \\ -w^T & 0 \end{bmatrix} {}^I_G\bar{q}(t) \\ &= \tfrac{1}{2}\Omega(w(t))\,{}^I_G\bar{q}(t) \end{aligned}$$

$${}^G\dot{V}_I(t) = {}^G a(t)$$

$${}^G\dot{p}_I(t) = {}^G V_I(t)$$

$$\dot{b}_a(t) = n_{wa}(t)$$

$$\dot{b}_g(t) = n_{wg}(t)$$

IMU 测量:

$$w_m = w + C({}^I_G\bar{q})\,w_G + b_g + n_g$$

$$a_m = C({}^I_G\bar{q})\left({}^G a - {}^G g + 2[w_G]_\times {}^G V_I + [w_G]_\times^2\, {}^G p_I\right) + b_a + n_a$$

其中 w_G 是考虑地球自转的影响

norminal-state (estimate-state):

$${}^I_G\dot{\hat{\bar{q}}} = \tfrac{1}{2}\hat{w}(t) \otimes {}^I_G\hat{\bar{q}}(t) = \tfrac{1}{2}\Omega(\hat{w}(t))\,{}^I_G\hat{\bar{q}}(t)$$

$$\begin{aligned} {}^G\dot{\hat{V}}_I &= E[{}^G\dot{V}_I] = E[{}^G a(t)] \\ &= E\left[C^T({}^I_G\bar{q})(a_m - b_a - n_a) + {}^G g - 2[w_G]_\times {}^G V_I - [w_G]_\times^2\, {}^G p_I\right] \\ &= C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) + {}^G g - 2[w_G]_\times {}^G\hat{V}_I - [w_G]_\times^2\, {}^G\hat{p}_I \end{aligned}$$

$${}^G\dot{\hat{p}}_I = {}^G\hat{V}_I$$

$$\dot{\hat{b}}_a = 0_{3\times1}$$

$$\dot{\hat{b}}_g = 0_{3\times1}$$

The linearized continuous-time model for IMU error-state:

$$\dot{\tilde{x}}_{IMU} = F\tilde{x}_{IMU} + G n_{IMU}$$

其中 $n_{IMU} = [n_g^T \ n_{wg}^T \ n_a^T \ n_{wa}^T]^T$

首先给出 F 和 G 的形式，再作推导：

$$F = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & I_3 & 0_{3\times3} & 0_{3\times3} \\ -C_{\hat q}^T[\hat a]_\times & 0_{3\times3} & -2[\omega_G]_\times & -C_{\hat q}^T & -[\omega_G]_\times^2 \\ -[\hat\omega]_\times & -I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

错误!!! 在后面补充.

$$G = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & -C_{\hat q}^T & 0_{3\times3} \\ -I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_3 \\ 0_{3\times3} & I_3 & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

其中 $C_{\hat q} = C({}^I_G\hat{q})$

$\hat a = a_m - \hat b_a$

$\hat\omega = \omega_m - \hat b_g - C_{\hat q}\omega_G$

$${}^G\dot{\tilde{p}}_I = {}^G\dot{p}_I - {}^G\dot{\hat{p}}_I = {}^G v_I - {}^G\hat{v}_I = {}^G\tilde{v}_I$$

$$\dot{\tilde b}_a = \dot b_a - \dot{\hat b}_a = n_{wa}$$

$$\dot{\tilde b}_g = \dot b_g - \dot{\hat b}_g = n_{wg}$$

下面推导 ${}^G\dot{\tilde{v}}_I$ 和 $\dot{\delta\theta}_I$

$${}^I_G\dot{\bar q} = \frac12 \omega\otimes{}^I_G\bar q$$

$$(\delta q\otimes{}^I_G\hat q)^{\cdot} = {}^I_G\dot{\bar q} = \frac12\omega\otimes{}^I_G\bar q$$

$$\dot{\delta q}\otimes{}^I_G\hat q + \delta q\otimes{}^I_G\dot{\hat q} = {}^I_G\dot{\bar q} = \frac12\omega\otimes\delta q\otimes{}^I_G\hat q$$

$$\dot{\delta q}\otimes{}^I_G\hat q + \delta q\otimes\frac12\hat\omega\otimes{}^I_G\hat q = {}^I_G\dot{\bar q} = \frac12\omega\otimes\delta q\otimes{}^I_G\hat q$$

$$\dot{\delta q}\otimes{}^I_G\hat q = \frac12\omega\otimes\delta q\otimes{}^I_G\hat q - \frac12\delta q\otimes\hat\omega\otimes{}^I_G\hat q$$

$$\dot{\delta q} = \frac12(\omega\otimes\delta q - \delta q\otimes\hat\omega)$$

$$\begin{bmatrix}\frac{\dot{\delta\theta}}{2}\\ 0\end{bmatrix} = \frac12([\omega]_L - [\hat\omega]_R)\delta q$$

$$= \frac12\left(\begin{bmatrix}-[\omega]_\times & \omega \\ -\omega^T & 0\end{bmatrix} - \begin{bmatrix}[\hat\omega]_\times & \hat\omega \\ -\hat\omega^T & 0\end{bmatrix}\right)\delta q$$

$$= \frac12\begin{bmatrix}-[\omega+\hat\omega]_\times & \omega-\hat\omega \\ -(\omega-\hat\omega)^T & 0\end{bmatrix}\delta q$$

$$= \frac12\begin{bmatrix}-[\omega+\hat\omega]_\times & \omega-\hat\omega \\ -(\omega-\hat\omega)^T & 0\end{bmatrix}\begin{bmatrix}\frac{\delta\theta}{2}\\ 1\end{bmatrix}$$

$$\Rightarrow \dot{\delta\theta} = -[\omega+\hat\omega]_\times\frac{\delta\theta}{2} + \omega - \hat\omega$$

因为 $\omega = \omega_m - b_g - n_g - C({}^I_G\bar q)\omega_G$ 其中 $C({}^I_G\bar q) = e^{-[\delta\theta]_\times}C({}^I_G\hat q) \approx (1 \mp [\delta\theta]_\times)C({}^I_G\hat q) = C({}^I_G\hat q) \mp [\delta\theta]_\times C({}^I_G\hat q)$

$\hat\omega = \omega_m - \hat b_g - C({}^I_G\hat q)\omega_G$

$$\omega+\hat\omega = 2\omega_m - b_g - \hat b_g - n_g - 2C({}^I_G\hat q) + [\delta\theta]_\times C({}^I_G\hat q)\omega_G$$

$$= 2\omega_m - (\hat b_g + \tilde b_g) - \hat b_g - n_g - 2C({}^I_G\hat q)\omega_G + [\delta\theta]_\times C({}^I_G\hat q)\omega_G$$

$$= 2\omega_m - 2\hat b_g - 2C({}^I_G\hat q)\omega_G \underbrace{- \tilde b_g - n_g + [\delta\theta]_\times C({}^I_G\hat q)\omega_G}_{\text{小量}}$$

$$[\omega+\hat{\omega}]_\times \frac{\delta\theta}{2} = [2\omega_m - 2\hat{b}_g - 2C({}^I_G\hat{\bar{q}})\omega_G]_\times \frac{\delta\theta}{2} - \underbrace{[\tilde{b}_g - n_g + [\delta\theta]_\times C({}^I_G\hat{\bar{q}})\omega_G]_\times \frac{\delta\theta}{2}}_{\text{忽略高阶小项}}$$

$$= [\omega_m - \hat{b}_g - C({}^I_G\hat{\bar{q}})\omega_G]_\times \delta\theta$$

$$= [\hat{\omega}]_\times \delta\theta$$

$$\omega - \hat{\omega} = (\omega_m - b_g - n_g - C({}^I_G\hat{\bar{q}})\omega_G + [\delta\theta]_\times C({}^I_G\hat{\bar{q}})\omega_G) - (\omega_m - \hat{b}_g - C({}^I_G\hat{\bar{q}})\omega_G)$$

$$= -(b_g - \hat{b}_g) - n_g + \underbrace{[\delta\theta]_\times C({}^I_G\hat{\bar{q}})\omega_G}_{\text{忽略小项}}$$

$$= -\tilde{b}_g - n_g$$

$$\Rightarrow \dot{\delta\theta} = -[\hat{\omega}]_\times \delta\theta - \tilde{b}_g - n_g$$

$${}^G\dot{v}_I(t) = {}^G a(t)$$

$$= C^T({}^I_G\bar{q})(a_m - b_a - n_a) + {}^G g - 2[\omega_G]_\times {}^G v_I - [\omega_G]_\times^2 {}^G p_I$$

$${}^G\dot{\hat{v}}_I = C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) + {}^G g - 2[\omega_G]_\times {}^G\hat{v}_I - [\omega_G]_\times^2 {}^G\hat{p}_I$$

$${}^G\dot{\tilde{v}}_I = {}^G\dot{v}_I(t) - {}^G\dot{\hat{v}}_I$$

$$= C^T({}^I_G\bar{q})(a_m - b_a - n_a) - C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) - 2[\omega_G]_\times {}^G\tilde{v}_I - [\omega_G]_\times^2 {}^G\tilde{p}_I$$

$$= (C(\delta\bar{q})C({}^I_G\hat{\bar{q}}))^T(a_m - b_a - n_a) - C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) - 2[\omega_G]_\times {}^G\tilde{v}_I - [\omega_G]_\times^2 {}^G\tilde{p}_I$$

$$\approx ((I - [\delta\theta]_\times)C({}^I_G\hat{\bar{q}}))^T(a_m - b_a - n_a) - C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) - X$$

$$= C^T({}^I_G\hat{\bar{q}})(I - [\delta\theta]_\times^T)(a_m - b_a - n_a) - C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) - X$$

$$= C^T({}^I_G\hat{\bar{q}})(a_m - b_a - n_a) + C^T({}^I_G\hat{\bar{q}})[\delta\theta]_\times(a_m - b_a - n_a) - C^T({}^I_G\hat{\bar{q}})(a_m - \hat{b}_a) - X$$

$$= C^T({}^I_G\hat{\bar{q}})(-\tilde{b}_a - n_a) + C^T({}^I_G\hat{\bar{q}})[\delta\theta]_\times(a_m - \hat{b}_a - \tilde{b}_a - n_a) - X$$

$$= C^T({}^I_G\hat{\bar{q}})(-\tilde{b}_a - n_a) + C^T({}^I_G\hat{\bar{q}})[\delta\theta]_\times(a_m - \hat{b}_a) + \underbrace{C^T({}^I_G\hat{\bar{q}})[\delta\theta]_\times(-\tilde{b}_a - n_a)}_{\text{高阶小量}} - X$$

$$= -C^T({}^I_G\hat{\bar{q}})\tilde{b}_a - C^T({}^I_G\hat{\bar{q}})n_a - C^T({}^I_G\hat{\bar{q}})[a_m - \hat{b}_a]_\times \delta\theta - 2[\omega_G]_\times {}^G\tilde{v}_I - [\omega_G]_\times^2 {}^G\tilde{p}_I$$

综上可得

$${}^G\dot{\tilde{p}}_I = {}^G\tilde{v}_I$$

$${}^G\dot{\tilde{v}}_I = -[\omega_G]_\times^2 {}^G\tilde{p}_I - 2[\omega_G]_\times {}^G\tilde{v}_I - C^T({}^I_G\hat{\bar{q}})[a_m - \hat{b}_a]_\times \delta\theta - C^T({}^I_G\hat{\bar{q}})\tilde{b}_a - C^T({}^I_G\hat{\bar{q}})n_a$$

$$\dot{\delta\theta} = -[\hat{\omega}]_\times \delta\theta - \tilde{b}_g - n_g$$

$$\dot{\tilde{b}}_a = n_{wa}$$

$$\dot{\tilde{b}}_g = n_{wg}$$

$$F = \begin{bmatrix} 0_{3\times3} & I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\omega_G]_\times^2 & -2[\omega_G]_\times & -C^T({}^I_G\hat{\bar{q}})[\,]_\times & -C^T({}^I_G\hat{\bar{q}}) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & -[\hat{\omega}]_\times & 0_{3\times3} & -I_3 \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

$$G = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & -C^T({}^I_G\hat{\bar{q}}) & 0_{3\times3} \\ -I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_3 \\ 0_{3\times3} & I_3 & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

3. 离散形式IMU运动学模型

从连续形式的运动学模型 得到离散形式. 分为两部分.

①IMU状态的估计状态离散化 (norminal 或 estimate state)

② IMU状态对应的协方差离散化, 该协方差离散化主要从连续形式的 error-state 运动模型得到.

首先讨论 IMU估计状态的离散化.

对于运动学模型:

$$ {}_G^I\dot{\hat{\bar{q}}} = \frac{1}{2}\Omega(\hat{\omega}(t))\,{}_G^I\hat{\bar{q}}(t) $$

$$ {}^G\dot{\hat{v}}_I = C^T({}_G^I\hat{\bar{q}})(a_m-\hat{b}_a) + {}^Gg - 2[\omega_G]_\times {}^G\hat{v}_I - [\omega_G]_\times^2\,{}^G\hat{p}_I $$

$$ {}^G\dot{\hat{p}}_I = {}^G\hat{v}_I $$

$$ \dot{\hat{b}}_a = 0_{3\times1} $$

$$ \dot{\hat{b}}_g = 0_{3\times1} $$

利用四阶 Runge-Kutta 数值积分 (4th-order Runge-Kutta numerical integration), 得到每个时刻离散的IMU估计状态。其中RK积分可参考《Quaternion kinematics ……》公式(334)

接下来讨论IMU状态的协方差离散化.

对于连续 error-state 模型

$$ \dot{\tilde{X}}_{IMU} = F\tilde{X}_{IMU} + Gn_{IMU} $$

该一阶微分方程~~有~~解的形式为

$$ \tilde{X}_{t_{k+\Delta t}} = \Phi(t_{k+\Delta t}, t_k)\tilde{X}_{t_k} + \int_{t_k}^{t_{k+\Delta t}} \Phi(t_{k+\Delta t}, \tau)Gn_{IMU}\,d\tau $$

其中状态传递矩阵 $\Phi(t_{k+\Delta t}, t_k)$

$$ \dot{\Phi}(t_{k+\Delta t}, t_k) = F(t)\Phi(t_{k+\Delta t}, t_k) \quad \text{其中 } \Phi(t_k, t_k) = I $$

$$ \Rightarrow \Phi(t_{k+\Delta t}, t_k) = \exp\left(\int_{t_k}^{t_{k+\Delta t}} F(t)\,dt\right) = \Phi_k \approx e^{F\Delta t} \approx I + F\Delta t $$

另外 error-state 的噪声协方差为

$$ Q_{t_k} = \int_{t_k}^{t_{k+\Delta t}} \Phi(t_{k+\Delta t}, \tau)GQG^T\Phi(t_{k+\Delta t}, \tau)^T\,d\tau = Q_k $$

则IMU状态的协方差离散传递形式为.

$$ P_{II_{k+1|k}} = \Phi_k P_{II_{k|k}} \Phi_k^T + Q_k $$

若包含IMU和Camera的全状态协方差为

$$ P_{k|k} = \begin{pmatrix} P_{II_{k|k}} & P_{IC_{k|k}} \\ P_{IC_{k|k}}^T & P_{CC_{k|k}} \end{pmatrix} $$

则传递协方差为

$$ P_{k+1|k} = \begin{pmatrix} P_{II_{k+1|k}} & \Phi_k P_{IC_{k|k}} \\ P_{IC_{k|k}}^T \Phi_k^T & P_{CC_{k|k}} \end{pmatrix} $$

接下来讨论状态扩展 (state Augmentation):

当有一帧新图像被处理时, 相机位姿可通过IMU位姿得到:

$$ {}_G^C\hat{\bar{q}} = {}_I^C\bar{q} \otimes {}_G^I\hat{\bar{q}} $$

$$ {}^G\hat{p}_C = {}^G\hat{p}_I + C^T({}_G^I\hat{\bar{q}})\,{}^Ip_C $$

其中 ${}_I^C\bar{q}$ 为相机与IMU之间旋转, ${}^Ip_C$ 为相机在IMU系下坐标

最新相机位姿的协方差可通过下式扩展

$$P_{k|k} \leftarrow \begin{bmatrix} I_{6N+15} \\ J_I^C \end{bmatrix} P_{k|k} \begin{bmatrix} I_{6N+15} \\ J_I^C \end{bmatrix}^T$$

其中 J_I^C 为协方差从 IMU 转到 Camera 的雅克比，推导如下

$${}^C_G\hat{\bar q} = {}^C_I\bar q \otimes {}^I_G\hat{\bar q}$$

$$C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)\, C({}^I_G\hat{\bar q})$$

$$C(\delta q^C)\, C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)\, C(\delta q^I)\, C({}^I_G\hat{\bar q})$$

$$(1-[\delta\theta^C]_\times)\, C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)\,(1-[\delta\theta^I]_\times)\, C({}^I_G\hat{\bar q})$$

$$C({}^C_G\hat{\bar q}) - [\delta\theta^C]_\times C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)\, C({}^I_G\hat{\bar q}) - C({}^C_I\bar q)[\delta\theta^I]_\times C({}^I_G\hat{\bar q})$$

$$[\delta\theta^C]_\times C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)[\delta\theta^I]_\times C({}^I_G\hat{\bar q})$$

$$\begin{aligned}[\delta\theta^C]_\times &= C({}^C_I\bar q)[\delta\theta^I]_\times C({}^I_G\hat{\bar q})\, C^T({}^C_G\hat{\bar q}) \\ &= C({}^C_I\bar q)[\delta\theta^I]_\times C({}^I_G\hat{\bar q})\, C^T({}^I_G\hat{\bar q})\, C^T({}^C_I\bar q) \\ &= C({}^C_I\bar q)[\delta\theta^I]_\times C^T({}^C_I\bar q)\end{aligned}$$

$$[\delta\theta^C]_\times = [C({}^C_I\bar q)\,\delta\theta^I]_\times \qquad \boxed{[Ca]_\times = C[a]_\times C^T}$$

$$\delta\theta^C = C({}^C_I\bar q)\,\delta\theta^I$$

$$\Rightarrow \frac{\partial\delta\theta^C}{\partial\delta\theta^I} = C({}^C_I\bar q)$$

以上推导假设相机与IMU之间外参 $C({}^C_I\bar q)$ 为常数，若需标定该外参

$$C({}^C_G\hat{\bar q}) = C({}^C_I\bar q)\, C({}^I_G\hat{\bar q})$$

$$C(\delta q^C)\, C({}^C_G\hat{\bar q}) = C(\delta q^C_I)\, C({}^C_I\bar q)\, C({}^I_G\hat{\bar q})$$

$$(1-[\delta\theta^C]_\times)\, C({}^C_G\hat{\bar q}) = (1-[\delta\theta^C_I]_\times)\, C({}^C_I\bar q)\, C({}^I_G\hat{\bar q})$$

$$\delta\theta^C = \delta\theta^C_I$$

$$\Rightarrow \partial\delta\theta^C / \partial\delta\theta^C_I = I_3$$

$${}^G\hat p_C = {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\,{}^I p_C$$

$${}^G\hat p_C + {}^G\tilde p_C = {}^G\hat p_I + {}^G\tilde p_I + C^T({}^I_G\hat{\bar q})\,{}^I p_C$$

$$\Rightarrow \frac{\partial\, {}^G\tilde p_C}{\partial\, {}^G\tilde p_I} = I_3$$

$$\begin{aligned}{}^G\hat p_C + {}^G\tilde p_C &= {}^G\hat p_I + \left(C(\delta q^I)\, C({}^I_G\hat{\bar q})\right)^T {}^I p_C \\ &= {}^G\hat p_I + \left((1-[\delta\theta^I]_\times)\, C({}^I_G\hat{\bar q})\right)^T {}^I p_C \\ &= {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\,(1-[\delta\theta^I]_\times^T)\,{}^I p_C \\ &= {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\,(1+[\delta\theta^I]_\times)\,{}^I p_C \\ &= {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\, p_C + C^T({}^I_G\hat{\bar q})[\delta\theta^I]_\times {}^I p_C\end{aligned}$$

$$\begin{aligned}{}^G\tilde p_C &= C^T({}^I_G\hat{\bar q})[\delta\theta^I]_\times {}^I p_C \\ &= -C^T({}^I_G\hat{\bar q})[{}^I p_C]_\times \delta\theta^I\end{aligned}$$

$$\Rightarrow \frac{\partial\, {}^G\tilde p_C}{\partial\delta\theta^I} = -C^T({}^I_G\hat{\bar q})[{}^I p_C]_\times$$

若需标定相机和IMU外参

$$\begin{aligned}{}^G\hat p_C + {}^G\tilde p_C &= {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\,({}^I\hat p_C + {}^I\tilde p_C) \\ &= {}^G\hat p_I + C^T({}^I_G\hat{\bar q})\,{}^I\hat p_C + C^T({}^I_G\hat{\bar q})\,{}^I\tilde p_C\end{aligned}$$

$${}^G\tilde p_C = C^T({}^I_G\hat{\bar q})\,{}^I\tilde p_C$$

$$\Rightarrow \frac{\partial\, {}^G\tilde p_C}{\partial\, {}^I\tilde p_C} = C^T({}^I_G\hat{\bar q})$$

至此，那先比 J_I^c 可得

$$J_I^c=\left[\begin{array}{ccccc|c}\frac{\partial\delta\theta^c}{\partial^G\tilde{p}_I} & \frac{\partial\delta\theta^c}{\partial^G\tilde{v}_I} & \frac{\partial\delta\theta^c}{\partial\delta\theta_I} & \frac{\partial\delta\theta^c}{\partial\tilde{b}_a} & \frac{\partial\delta\theta^c}{\partial\tilde{b}_g} & 0_{3\times6N}\\ \frac{\partial^G\tilde{p}_c}{\partial^G\tilde{p}_I} & \frac{\partial^G\tilde{p}_c}{\partial^G\tilde{v}_I} & \frac{\partial^G\tilde{p}_c}{\partial\delta\theta_I} & \frac{\partial^G\tilde{p}_c}{\partial\tilde{b}_a} & \frac{\partial^G\tilde{p}_c}{\partial\tilde{b}_g} & 0_{3\times6N}\end{array}\right]$$

$$=\left[\begin{array}{ccccc|c}0_{3\times3} & 0_{3\times3} & C(^C_I\bar{q}) & 0_{3\times3} & 0_{3\times3} & 0_{3\times6N}\\ I_3 & 0_{3\times3} & -C(^I_G\hat{\bar{q}})[^Ip_c]_\times & 0_{3\times3} & 0_{3\times3} & 0_{3\times6N}\end{array}\right]$$

其中 $\frac{\partial^G\tilde{p}_c}{\partial\delta\theta_I}=-C(^I_G\hat{\bar{q}})[^Ip_c]_\times$ 与原始论文《A multi-state constraint Kalman filter for vision-aided inertial navigation》

$\frac{\partial^G\tilde{p}_c}{\partial\delta\theta_I}=[C^T(^I_G\hat{\bar{q}})\,^Ip_c]_\times$ 不同，若旋转采用右扰动。

$$^G\hat{p}_c+{}^G\tilde{p}_c={}^G\hat{p}_I+\left(C(^I_G\hat{\bar{q}})C(\delta q^G)\right)^T\,{}^Ip_c$$

$$={}^G\hat{p}_I+\left(C(^I_G\hat{\bar{q}})(1-[\delta\theta^G]_\times)\right)^T\,{}^Ip_c$$

$$={}^G\hat{p}_I+(1-[\delta\theta^G]_\times)^T\,C^T(^I_G\hat{\bar{q}})\,{}^Ip_c$$

$$={}^G\hat{p}_I+C^T(^I_G\hat{\bar{q}})\,{}^Ip_c-[\delta\theta^G]_\times^T\,C^T(^I_G\hat{\bar{q}})\,{}^Ip_c$$

$$^G\tilde{p}_c=[\delta\theta^G]_\times\,C^T(^I_G\hat{\bar{q}})\,{}^Ip_c$$

$$=-[C^T(^I_G\hat{\bar{q}})\,{}^Ip_c]_\times\,\delta\theta^G$$

$$\Rightarrow\frac{\partial^G\tilde{p}_c}{\partial\delta\theta^G}=-[C^T(^I_G\hat{\bar{q}})\,{}^Ip_c]_\times$$

与原文仍然相差一个负号，可能是原文书写有误。

另外 $\frac{\partial\delta\theta^c}{\partial\delta\theta_I}=C(^C_I\bar{q})$ 与论文《Robust stereo visual inertial odometry for fast autonomous flight》 $\frac{\partial\delta\theta^c}{\partial\delta\theta_I}=C(^I_G\bar{q})$，~~是因为~~

该处确认为原文书写有误，因为在其开源代码中为 $C(^C_I\bar{q})$

下面讨论视觉观测模型

假设在第i个相机的归一化平面上，第j个地图点的观测为 $z_i^{(j)}$，

则

$$z_i^{(j)}=\frac{1}{^{C_i}Z_j}\begin{bmatrix}^{C_i}X_j\\ ^{C_i}Y_j\end{bmatrix}+n_i^{(j)}$$

其中 $n_i^{(j)}$ 为 2×1 图像噪声，方差为 $R_i^{(j)}=\sigma_{im}^2 I_2$

另外，地图点 P_{f_j} 在第i个相机下坐标为 $^{C_i}P_{f_j}$

$$^{C_i}P_{f_j}=\begin{bmatrix}^{C_i}X_j\\ ^{C_i}Y_j\\ ^{C_i}Z_j\end{bmatrix}=C(^{C_i}_G\bar{q})(^GP_{f_j}-{}^GP_{C_i})$$

其中 $^GP_{f_j}$ 是地图点在世界坐标系下位置，$^GP_{C_i}$ 是相机在G系下位置。

则第j个地图点在第i个相机观测残差

$$r_i^{(j)}=z_i^{(j)}-\hat{z}_i^{(j)}$$

其中 $\hat{z}_i^{(j)}=\frac{1}{^{C_i}\hat{Z}_j}\begin{bmatrix}^{C_i}\hat{X}_j\\ ^{C_i}\hat{Y}_j\end{bmatrix}$，$^{C_i}\hat{P}_{f_j}=\begin{bmatrix}^{C_i}\hat{X}_j\\ ^{C_i}\hat{Y}_j\\ ^{C_i}\hat{Z}_j\end{bmatrix}=C(^{C_i}_G\hat{\bar{q}})(^G\hat{P}_{f_j}-{}^G\hat{P}_{C_i})$

由于视觉观测 $z_i^{(j)}$ 相对于状态 ${}^{C_i}_{G}\bar{q}$，${}^{G}p_{C_i}$，${}^{G}p_{f_j}$ 是非线性的，利用EKF估计器则需要线性化，接下来讨论线性化过程.

假设 $z_i^{(j)} = h(X_{C_i}, p_{f_j}) = h({}^{C_i}_{G}\bar{q}, {}^{G}p_{C_i}, {}^{G}p_{f_j})$，则

又因为 $z_i^{(j)} = \dfrac{1}{{}^{C_i}Z_j}\begin{bmatrix}{}^{C_i}X_j\\ {}^{C_i}Y_j\end{bmatrix} = \begin{bmatrix}u\\ v\end{bmatrix}$ $\quad {}^{C_i}p_{f_j} = \begin{bmatrix}{}^{C_i}X_j\\ {}^{C_i}Y_j\\ {}^{C_i}Z_j\end{bmatrix} = C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$

根据链式求导可知

$$H_{X_i}^{(j)} = \frac{\partial z_i^{(j)}}{\partial X_{C_i}} = \frac{\partial z_i^{(j)}}{\partial {}^{C_i}p_{f_j}}\cdot\frac{\partial {}^{C_i}p_{f_j}}{\partial X_{C_i}}, \quad H_{f_j}^{(j)} = \frac{\partial z_i^{(j)}}{\partial {}^{G}p_{f_j}} = \frac{\partial z_i^{(j)}}{\partial {}^{C_i}p_{f_j}}\cdot\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{G}p_{f_j}}$$

$$\frac{\partial z_i^{(j)}}{\partial {}^{C_i}p_{f_j}} = \begin{bmatrix}\frac{\partial u}{\partial {}^{C_i}X_j} & \frac{\partial u}{\partial {}^{C_i}Y_j} & \frac{\partial u}{\partial {}^{C_i}Z_j}\\ \frac{\partial v}{\partial {}^{C_i}X_j} & \frac{\partial v}{\partial {}^{C_i}Y_j} & \frac{\partial v}{\partial {}^{C_i}Z_j}\end{bmatrix} = \begin{bmatrix}\frac{1}{{}^{C_i}Z_j} & 0 & -\frac{{}^{C_i}X_j}{{}^{C_i}Z_j^2}\\ 0 & \frac{1}{{}^{C_i}Z_j} & -\frac{{}^{C_i}Y_j}{{}^{C_i}Z_j^2}\end{bmatrix}$$

$${}^{C_i}p_{f_j} = C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$$

$${}^{C_i}p_{f_j} + {}^{C_i}\tilde{p}_{f_j} = C(\delta\bar{q}^{C_i})C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$$

$$= (I - [\delta\theta^{C_i}]_\times)C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$$

$${}^{C_i}\tilde{p}_{f_j} = -[\delta\theta^{C_i}]_\times C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$$

$$= [C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})]_\times \delta\theta^{C_i}$$

$$\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{C_i}_{G}\bar{q}} = \frac{\partial {}^{C_i}\tilde{p}_{f_j}}{\partial \delta\theta^{C_i}} = [C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})]_\times = [{}^{C_i}p_{f_j}]_\times$$

$${}^{C_i}p_{f_j} + {}^{C_i}\tilde{p}_{f_j} = C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - ({}^{G}p_{C_i} + {}^{G}\tilde{p}_{C_i}))$$

$$= C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i}) - C({}^{C_i}_{G}\bar{q}){}^{G}\tilde{p}_{C_i}$$

$${}^{C_i}\tilde{p}_{f_j} = -C({}^{C_i}_{G}\bar{q}){}^{G}\tilde{p}_{C_i}$$

$$\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{G}p_{C_i}} = \frac{\partial {}^{C_i}\tilde{p}_{f_j}}{\partial {}^{G}\tilde{p}_{C_i}} = -C({}^{C_i}_{G}\bar{q})$$

$${}^{C_i}p_{f_j} + {}^{C_i}\tilde{p}_{f_j} = C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} + {}^{G}\tilde{p}_{f_j} - {}^{G}p_{C_i})$$

$$= C({}^{C_i}_{G}\bar{q}){}^{G}\tilde{p}_{f_j} + C({}^{C_i}_{G}\bar{q})({}^{G}p_{f_j} - {}^{G}p_{C_i})$$

$${}^{C_i}\tilde{p}_{f_j} = C({}^{C_i}_{G}\bar{q}){}^{G}\tilde{p}_{f_j}$$

$$\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{G}p_{f_j}} = \frac{\partial {}^{C_i}\tilde{p}_{f_j}}{\partial {}^{G}\tilde{p}_{f_j}} = C({}^{C_i}_{G}\bar{q})$$

因此有

$$\frac{\partial {}^{C_i}p_{f_j}}{\partial X_{C_i}} = \begin{bmatrix}\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{C_i}_{G}\bar{q}} & \frac{\partial {}^{G}p_{f_j}}{\partial {}^{G}p_{C_i}}\end{bmatrix} = \begin{bmatrix}[{}^{C_i}p_{f_j}]_\times & -C({}^{C_i}_{G}\bar{q})\end{bmatrix}$$

$$\frac{\partial {}^{C_i}p_{f_j}}{\partial {}^{G}p_{f_j}} = C({}^{C_i}_{G}\bar{q})$$

则有测量残差

$$r_i^{(j)} = z_i^{(j)} - \hat{z}_i^{(j)} \approx H_{X_i}^{(j)}\tilde{X} + H_{f_j}^{(j)}\,{}^{G}\tilde{p}_{f_j} + n_i^{(j)}$$

将同一地图点j在所有相机下的观测堆放一起

$$r^{(j)} = H_X^{(j)}\tilde{X} + H_{f_j}^{(j)}\,{}^{G}\tilde{p}_{f_j} + n^{(j)}$$ （噪声$n^{(j)}$方差为$R^{(j)} = \sigma^2 I$）

由于地图点 ${}^{G}p_{f_j}$ 的计算含有状态量 X 的参与，因此 $\tilde{X}$ 与 ${}^{G}\tilde{p}_{f_j}$ 相关。测量残差 $r^{(j)}$ 没法直接用在 EKF 中，因此将残差 $r^{(j)}$ 投影到 $H_{f_j}^{(j)}$ 的左零空间 (left nullspace)，$A^T H_{f_j}^{(j)} {}^{G}\tilde{p}_{f_j} = 0$，则

$$r_0^{(j)} = A^T(z^{(j)} - \hat{z}^{(j)}) \approx A^T H_X^{(j)} \tilde{X} + A^T n^{(j)} = H_0^{(j)} \tilde{X}^{(j)} + n_0^{(j)}$$

其中矩阵 A 不用显式计算出来，可利用 Givens rotations 将 $r^{(j)}$ 和 $H_X^{(j)}$ 投影到 $H_{f_j}^{(j)}$ 的左零空间。另外 $E[n_0^{(j)} n_0^{(j)T}] = \sigma^2 A^T A = \sigma^2 I$

将多个地图点的观测堆放到一起：

$$r_0 = H_X \tilde{X} + n_0$$

为减少计算量，对稀疏矩阵 QR 分解：

$$H_X = [Q_1 \ Q_2] \begin{bmatrix} T_H \\ 0 \end{bmatrix}$$

则

$$r_0 = [Q_1 \ Q_2] \begin{bmatrix} T_H \\ 0 \end{bmatrix} \tilde{X} + n_0$$

$$\Rightarrow \begin{bmatrix} Q_1^T r_0 \\ Q_2^T r_0 \end{bmatrix} = \begin{bmatrix} T_H \\ 0 \end{bmatrix} \tilde{X} + \begin{bmatrix} Q_1^T n_0 \\ Q_2^T n_0 \end{bmatrix}$$

$$\Rightarrow r_n = Q^T r_0 = T_H \tilde{X} + n_n$$

其中测量噪声 $n_n = Q_1^T n_0$ 方差为 $R_n = Q_1^T R_0 Q_1 = \sigma^2 I$

最后 EKF 状态更新有：

Kalman 增益：$K = P T_H^T (T_H P T_H^T + R_n)^{-1}$

状态更新：$X_{k+1|k+1} = X_{k+1|k} + K \cdot r_n$

协方差更新：$P_{k+1|k+1} = (I - K T_H) P_{k+1|k} (I - K T_H)^T + K R_n K^T$

其中 Q_1 不用显式计算出来，r_n 和 T_H 可利用 Givens rotations 获取。