该部分讨论IMU预积分的递推推导方式.

1. IMU运动模型:

$$\begin{cases} \dot{R}_{wb} = R_{wb} \omega_{wb}^{\wedge} \\ {}_w\dot{V} = {}_w a \\ {}_w\dot{P} = {}_w V \end{cases}$$

，其中 $w^{\wedge}$ 表示 w 的反对称矩阵

在t时刻的 Δt 范围上积分，得到 $t+\Delta t$ 时刻状态量

$$\begin{cases} R_{wb}(t+\Delta t) = R_{wb}(t)\, Exp\left(\int_t^{t+\Delta t} \omega_{wb}(\tau) d\tau\right) \\ {}_wV(t+\Delta t) = {}_wV(t) + \int_t^{t+\Delta t} {}_w a(\tau) d\tau \\ {}_wP(t+\Delta t) = {}_wP(t) + \int_t^{t+\Delta t} {}_wV(\tau) d\tau + \iint_t^{t+\Delta t} {}_w a(\tau) d\tau^2 \end{cases}$$

，其中 $Exp(w) = exp([w]_{\times}) = exp(w^{\wedge})$

取 Δt 为相邻图像时间间隔，则有相邻图像帧时之间，$[b_k, b_{k+1}]$

$$\begin{cases} P_{b_{k+1}}^w = P_{b_k}^w + V_{b_k}^w \Delta t_k + \iint_{t\in[t_k, t_{k+1}]} a^w(t) dt^2 \\ V_{b_{k+1}}^w = V_{b_k}^w + \int_{t\in[t_k,t_{k+1}]} a^w(t) dt \\ q_{b_{k+1}}^w = q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2} q^w(t) \otimes w^w(t) dt \\ \quad = q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2} [w^w(t)]_R\, q^w(t) dt = q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2}\Omega(w) q_t^{b_k} dt \end{cases}$$

2. IMU测量模型（加速度表示在body系下）

$$\begin{cases} \hat{a}_t = a_t + b_{a_t} + R_w^t g^w + n_a \\ \hat{\omega}_t = \omega_t + b_{w_t} + n_w \end{cases} \Rightarrow \begin{cases} a_t = \hat{a}_t - R_w^t g^w - b_{a_t} - n_a \\ \omega_t = \hat{\omega}_t - b_{w_t} - n_w \end{cases}$$

其中 R_w^t 为从w系到t系旋转.

$$\begin{cases} R_t^w a_t = a^w = R_t^w(\hat{a}_t - R_w^t g^w - b_{a_t} - n_a) \\ \quad = R_t^w(\hat{a}_t - b_{a_t} - n_a) - g^w \\ \omega_t = \hat{\omega}_t - b_{w_t} - n_w \end{cases}$$

$\longrightarrow$ 旋转在w系或t系一样，不需转换.

代入IMU离散运动模型

$$\begin{cases} P_{b_{k+1}}^w = P_{b_k}^w + V_{b_k}^w \Delta t_k + \iint_{t\in[t_k,t_{k+1}]} (R_t^w(\hat{a}_t - b_{a_t} - n_a) - g^w) dt^2 \\ V_{b_{k+1}}^w = V_{b_k}^w + \int_{t\in[t_k,t_{k+1}]} (R_t^w(\hat{a}_t - b_{a_t} - n_a) - g^w) dt \\ q_{b_{k+1}}^w = q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2}\Omega(\hat{\omega}_t - b_{w_t} - n_w) q_t^{b_k} dt \end{cases}$$

提取出常量重力加速度 g

$$\begin{cases} P_{b_{k+1}}^w = P_{b_k}^w + V_{b_k}^w \Delta t_k - \frac{1}{2} g^w \Delta t_k^2 + \iint_{t\in[t_k,t_{k+1}]} R_t^w(\hat{a}_t - b_{a_t} - n_a) dt^2 \\ V_{b_{k+1}}^w = V_{b_k}^w - g^w \Delta t_k + \int_{t\in[t_k,t_{k+1}]} R_t^w(\hat{a}_t - b_{a_t} - n_a) dt \\ q_{b_{k+1}}^w = q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2}\Omega(\hat{\omega}_t - b_{w_t} - n_w) q_t^{b_k} dt \end{cases}$$

乘上w系到body系在 b_k 时刻的旋转 $R_w^{b_k}$，得到IMU预积分

$$\begin{cases} R_w^{b_k} P_{b_{k+1}}^w = R_w^{b_k}(P_{b_k}^w + V_{b_k}^w \Delta t_k - \frac{1}{2} g^w \Delta t_k^2) + \iint_{t\in[t_k,t_{k+1}]} R_w^{b_k} R_t^w(\hat{a}_t - b_{a_t} - n_a) dt^2 \\ R_w^{b_k} V_{b_{k+1}}^w = R_w^{b_k}(V_{b_k}^w - g^w \Delta t_k) + \int_{t\in[t_k,t_{k+1}]} R_w^{b_k} R_t^w(\hat{a}_t - b_{a_t} - n_a) dt \\ q_w^{b_k} \otimes q_{b_{k+1}}^w = q_w^{b_k} \otimes q_{b_k}^w \otimes \int_{t\in[t_k,t_{k+1}]} \frac{1}{2}\Omega(\hat{\omega}_t - b_{w_t} - n_w) q_t^{b_k} dt \end{cases}$$

定义IMU预积分:

$$\begin{cases} \text{Position}: \ \alpha^{b_k}_{b_{k+1}} = \iint_{t\in[t_k,t_{k+1}]} R^{b_k}_t(\hat{a}_t - b_{a_t} - n_a)\,dt^2 \\ \text{Velocity}: \ \beta^{b_k}_{b_{k+1}} = \int_{t\in[t_k,t_{k+1}]} R^{b_k}_t(\hat{a}_t - b_{a_t} - n_a)\,dt \\ \text{Rotation}: \ \gamma^{b_k}_{b_{k+1}} = \int_{t\in[t_k,t_{k+1}]} \frac{1}{2}\Omega(\hat{w}_t - b_{w_t} - n_w)\gamma^{b_k}_t\,dt \end{cases}$$

$$\boxed{\begin{aligned} &\iint_t a\,dt^2 \quad \text{不定积分} \\ &= \int_t at + v_0\,dt \\ &= p_0 + v_0 t + \tfrac{1}{2}at^2 \end{aligned}}$$

由于IMU测量中的高斯噪声 n_a, n_w 假设为高斯白噪声(均值为0),则可得到相邻IMU帧之间,IMU预积分递推公式.

$$\begin{cases} \alpha^{b_k}_{i+1} = \alpha^{b_k}_i + \beta^{b_k}_i\delta t + \frac{1}{2}R(\gamma^{b_k}_i)(\hat{a}_i - b_{a_i})\delta t^2 \\ \beta^{b_k}_{i+1} = \beta^{b_k}_i + R(\gamma^{b_k}_i)(\hat{a}_i - b_{a_i})\delta t \\ \gamma^{b_k}_{i+1} = \gamma^{b_k}_i \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{w}_i - b_{w_i})\delta t \end{bmatrix} \end{cases}$$

其中 i 表示IMU测量时刻, δt 为IMU时间间隔.

接下来讨论IMU预积分的误差、协方差和雅克比.

首先给出IMU预积分误差的运动学模型

$$\begin{bmatrix} \delta\dot{\alpha}^{b_k}_t \\ \delta\dot{\beta}^{b_k}_t \\ \delta\dot{\theta}^{b_k}_t \\ \delta\dot{b}_{a_t} \\ \delta\dot{b}_{w_t} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & -R^{b_k}_t(\hat{a}_t - b_{a_t})^\wedge & -R^{b_k}_t & 0 \\ 0 & 0 & -(\hat{w}_t - b_{w_t})^\wedge & 0 & -I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha^{b_k}_t \\ \delta\beta^{b_k}_t \\ \delta\theta^{b_k}_t \\ \delta b_{a_t} \\ \delta b_{w_t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -R^{b_k}_t & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} n_a \\ n_w \\ n_{b_a} \\ n_{b_w} \end{bmatrix}$$

记作: $\delta\dot{z}^{b_k}_t = F_t\,\delta z^{b_k}_t + G_t n_t$

由IMU预积分定义和IMU运动模型可知.

$$\dot{\alpha}^{b_k}_t = \beta^{b_k}_t$$

参考《Quaternion kinematics for the error-state kalman filter》第5章中关于误差的定义.

真实状态(true state): $\dot{\alpha}_t = \beta_t$

理想状态(nominal state): $\dot{\alpha} = \beta$

误差状态(error state): $\delta\dot{\alpha}_t = \dot{\alpha}_t - \dot{\alpha} = \beta_t - \beta = \delta\beta_t$

则有 $\delta\dot{\alpha}^{b_k}_t = \delta\beta^{b_k}_t$

旋转 true-state: R_t

nominal-state: R

error-state: $\delta R = e^{[\delta\theta]_\times}$

$$R_t = R\,\delta R = R\cdot e^{[\delta\theta]_\times} = R(I + [\delta\theta]_\times) + O(\|\delta\theta\|^2)$$

为简化书写, 令 $\delta\dot{\beta}^{b_k}_t = \delta\dot{\beta}$, 则 $\delta\dot{\beta} = \dot{\beta}_{true} - \dot{\beta}_{nominal}$

~~由IMU预积分定义得~~

$\cancel{\dot{\beta}_{true} = R^{b_k}_t(\hat{a}_t - b_{a_t}) = R^{b_k}_{t\_true}(\hat{a}_{t\_true} - b_{a_t\_true})}$

~~因为~~ $\cancel{R^{b_k}_{t\_true} = R^{b_k}_t\,\delta R \approx R^{b_k}_t(I + [\delta\theta]_\times)}$

$\cancel{\hat{a}_{t\_true} =}$

由IMU预积分定义得:

$$\dot{\beta} = R^{b_k}_t(\hat{a}_t - b_{a_t} - n_a)$$

由《Quaternion kinematics for the error-state kalman filter》式(242)可得

$$\dot{\beta}_{true} = R_t^{b_k}\delta R(\hat{a}_t - (b_{a_t}+\delta b_{a_t}) - n_a)$$

$$\approx R_t^{b_k}(I+[\delta\theta]_\times)(\hat{a}_t - b_{a_t} - \delta b_{a_t} - n_a)$$

$$= R_t^{b_k}(\hat{a}_t - b_{a_t} - \delta b_{a_t} - n_a + [\delta\theta]_\times(\hat{a}_t - b_{a_t}) + \underbrace{[\delta\theta]_\times(-\delta b_{a_t}-n_a)}_{\text{忽略高阶小项}})$$

$$= R_t^{b_k}(\hat a_t - b_{a_t} - \delta b_{a_t} - n_a - [\hat a_t - b_{a_t}]_\times \delta\theta)$$

$$= R_t^{b_k}(\hat a_t - b_{a_t}) - R_t^{b_k}[\hat a_t-b_{a_t}]_\times\delta\theta - R_t^{b_k}\delta b_{a_t} - R_t^{b_k} n_a$$

由式(236b)可知

$$\dot\beta_{nominal} = R_t^{b_k}(\hat a_t - b_{a_t})$$

则

$$\delta\dot\beta = \dot\beta_{true} - \dot\beta_{nominal} = -R_t^{b_k}[\hat a_t - b_{a_t}]_\times\delta\theta - R_t^{b_k}\delta b_{a_t} - R_t^{b_k} n_a$$

由IMU预积分定义得：(令 $\delta\gamma_t^{b_k} = \delta\gamma$，$\delta\gamma = \gamma_{true}\otimes\gamma_{nominal}$)

$$\dot\gamma = \frac12 \gamma\otimes(\hat\omega_t - b_{w_t} - n_w)$$

由式(254)可知.

$$\dot\gamma_{true} = \frac12\gamma\otimes\delta\gamma\otimes(\hat\omega_t - (b_{w_t}+\delta b_{w_t}) - n_w)$$

$$= \frac12\gamma\otimes\delta\gamma\otimes(\hat\omega_t - b_{w_t} - \delta b_{w_t} - n_w)$$

又因为

$$\dot\gamma_{true} = (\gamma\otimes\delta\gamma)^{\cdot} = \dot\gamma\otimes\delta\gamma + \gamma\otimes\delta\dot\gamma$$

$$= \frac12\gamma\otimes(\hat\omega_t - b_{w_t})\otimes\delta\gamma + \gamma\otimes\delta\dot\gamma$$

所以有

$$\frac12\gamma\otimes\delta\gamma\otimes(\hat\omega_t - b_{w_t}-\delta b_{w_t}-n_w) = \frac12\gamma\otimes(\hat\omega_t-b_{w_t})\otimes\delta\gamma + \gamma\otimes\delta\dot\gamma$$

$$\Rightarrow \frac12\delta\gamma\otimes(\hat\omega_t - b_{w_t} - \delta b_{w_t} - n_w) = \frac12(\hat\omega_t - b_{w_t})\otimes\delta\gamma + \delta\dot\gamma$$

$$\Rightarrow 2\delta\dot\gamma = \delta\gamma\otimes(\hat\omega_t - b_{w_t}-\delta b_{w_t}-n_w) - (\hat\omega_t - b_{w_t})\otimes\delta\gamma$$

$$= ([\hat\omega_t - b_{w_t} - \delta b_{w_t} - n_w]_R - [\hat\omega_t - b_{w_t}]_L)\delta\gamma$$

$$= \begin{bmatrix} 0 & (\delta b_{w_t}+n_w)^T \\ -\delta b_{w_t} - n_w & -[2\hat\omega_t - 2b_{w_t} - n_w - \delta b_{w_t}]_\times\end{bmatrix}\delta\gamma$$

(注：$\delta\gamma = e^{\delta\theta/2} \approx I + \delta\theta/2 + O(\|\delta\theta\|^2)$)

$$\approx \begin{bmatrix} 0 & (\delta b_{w_t}+n_w)^T \\ -\delta b_{w_t} - n_w & -[2\hat\omega_t - 2b_{w_t} - n_w - \delta b_{w_t}]_\times\end{bmatrix}\begin{bmatrix}1\\ \delta\theta/2\end{bmatrix}$$

又因为

$$\delta\gamma = e^{\delta\theta\cdot u/2} = \cos\frac{\delta\theta}{2} + u\sin\frac{\delta\theta}{2} = \begin{bmatrix}\cos(\delta\theta/2)\\ u\sin(\delta\theta/2)\end{bmatrix}$$

$$\delta\dot\gamma = \begin{bmatrix}-\frac12\sin\frac{\delta\theta}{2}\cdot\delta\dot\theta \\ \frac{u}{2}\cos(\frac{\delta\theta}{2})\cdot\delta\dot\theta\end{bmatrix} \approx \begin{bmatrix}0\\ \frac u2\delta\dot\theta\end{bmatrix} = \begin{bmatrix}0\\ \frac{\delta\dot\theta}{2}\end{bmatrix}$$

$$\begin{bmatrix}0\\ \delta\dot\theta\end{bmatrix} = 2\delta\dot\gamma = \begin{bmatrix}(\delta b_{w_t}+n_w)^T\delta\theta/2 \\ -\delta b_{w_t} - n_w - [2\hat\omega_t - 2b_{w_t} - n_w - \delta b_{w_t}]_\times\delta\theta/2\end{bmatrix}$$

$$\overset{\text{忽略高阶小项}}{\approx} \begin{bmatrix}0\\ -\delta b_{w_t} - n_w - [\hat\omega_t - b_{w_t}]_\times\delta\theta\end{bmatrix}$$

$$\Rightarrow \delta\dot\theta = -[\hat\omega_t - b_{w_t}]_\times\delta\theta - \delta b_{w_t} - n_w$$

关于bias：

$\dot b_{a_t} = n_{b_a}$，$\dot b_a = 0 \Rightarrow \delta\dot b_{a_t} = \dot b_{a_t} - \dot b_a = n_{b_a}$

$\dot b_{w_t} = n_{b_w}$，$\dot b_w = 0 \Rightarrow \delta \dot b_{w_t} = \dot b_{w_t} - \dot b_w = n_{b_w}$

下面讨论IMU预积分误差模型的离散形式.

连续形式: $\delta \dot{z}_t^{b_k} = F_t \delta z_t^{b_k} + G_t n_t$

在δt时间上求解一阶微分方程有

$$F_d = \exp(F_t \delta t) \approx I + F_t \delta t$$

假设连续时间上高斯噪声 $Q_t = diag(\sigma_a^2, \sigma_w^2, \sigma_{b_a}^2, \sigma_{b_w}^2)$, 则离散形式.

$$Q_d = \int_0^{\delta t} F_d(\tau) G_t Q_t G_t^T F_d(\tau)^T d\tau$$
$$= \delta t F_d G_t Q_t G_t^T F_d^T$$
$$\approx \delta t G_t Q_t G_t^T$$

首先直接给出离散形式 (调整$\delta\beta$和$\delta\theta$顺序).

$$\begin{bmatrix} \delta\alpha_{i+1} \\ \delta\theta_{i+1} \\ \delta\beta_{i+1} \\ \delta b_{a_{i+1}} \\ \delta b_{w_{i+1}} \end{bmatrix} = \begin{bmatrix} I & f_{01} & \delta t & f_{03} & f_{04} \\ 0 & f_{11} & 0 & 0 & -\delta t \\ 0 & f_{21} & I & f_{23} & f_{24} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \delta\alpha_i \\ \delta\theta_i \\ \delta\beta_i \\ \delta b_{a_i} \\ \delta b_{w_i} \end{bmatrix} + \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} & 0 & 0 \\ 0 & -\frac{\delta t}{2} & 0 & -\frac{\delta t}{2} & 0 & 0 \\ -\frac{R_i \delta t}{2} & h_{21} & -\frac{R_{i+1}\delta t}{2} & h_{23} & 0 & 0 \\ 0 & 0 & 0 & \cancel{\delta t} & \delta t & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta t \end{bmatrix} \begin{bmatrix} n_{a_i} \\ n_{w_i} \\ n_{a_{i+1}} \\ n_{w_{i+1}} \\ n_{b_{a_i}} \\ n_{b_{w_i}} \end{bmatrix}$$

① 推导$\delta\theta_{i+1}$递推形式:

因为有 $\delta\dot\theta = -(\hat\omega_t - b_{w_t})^\wedge \delta\theta - n_w - \delta b_{w_t}$

若采用欧拉积分:

$$\delta\theta_{i+1} = \delta\theta_i + \delta\dot\theta \cdot \delta t$$
$$= \delta\theta_i + (-(\hat\omega_i - b_{w_i})^\wedge \delta\theta_i - n_{w_i} - \delta b_{w_i})\delta t$$
$$= (I - (\hat\omega_i - b_{w_i})^\wedge \delta t)\delta\theta_i - n_{w_i}\delta t - \delta b_{w_i}\delta t$$

$(\cdot)^\wedge = [\,]_\times$

若采用中值积分, 由于相邻两帧i, $i+1$之间, 假设bias不变 则

$$\delta\theta_{i+1} = (I - (\frac{\hat\omega_i + \hat\omega_{i+1}}{2} - b_{w_i})^\wedge \delta t)\delta\theta_i - \frac{n_{w_i} + n_{w_{i+1}}}{2}\delta t - \delta b_{w_i}\delta t$$

② 推导$\delta\beta_{i+1}$递推形式.

$$\delta\dot\beta = -R_t^{b_k}(\hat a_t - b_{a_t})^\wedge \delta\theta - R_t^{b_k}\delta b_{a_t} - R_t^{b_k} n_a$$

若采用中值积分

$$\delta\dot\beta_i = -\frac{R_{i+1}^{b_k}(\hat a_{i+1} - b_{a_i})^\wedge \delta\theta_{i+1} + R_i^{b_k}(\hat a_i - b_{a_i})^\wedge \delta\theta_i}{2} - \frac{R_i^{b_k} + R_{i+1}^{b_k}}{2}\delta b_{a_i}$$

$$\cancel{\frac{R_i^{b_k}+R_{i+1}^{b_k}}{2}}$$

$$- \frac{R_i^{b_k} n_{a_i} + R_{i+1}^{b_k} n_{a_{i+1}}}{2}$$

将$\delta\theta_{i+1}$代入$\delta\dot\beta_i$, 则有

$$\delta\beta_{i+1} = \delta\beta_i + \delta\dot\beta_i \delta t$$
$$= \delta\beta_i + \left[-\frac{1}{2}R_{i+1}^{b_k}(\hat a_{i+1} - b_{a_i})^\wedge \delta t\left[(I - (\overset{\frac{\hat\omega_i+\hat\omega_{i+1}}{2}}{\omega} - b_{w_i})^\wedge \delta t)\delta\theta_i - \overset{\frac{n_{w_i}+n_{w_{i+1}}}{2}}{n}\delta t - \delta b_{w_i}\delta t\right]\right]$$
$$+ \left[-\frac{1}{2}R_i^{b_k}(\hat a_i - b_{a_i})^\wedge \delta\theta_i\right]\delta t - \frac{R_i^{b_k} + R_{i+1}^{b_k}}{2}\delta b_{a_i}\delta t - \frac{R_i^{b_k} n_{a_i} + R_{i+1}^{b_k} n_{a_{i+1}}}{2}\delta t$$

$$\delta\beta_{i+1} = \delta\beta_i + \left[-\frac{1}{2}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\left(I-\left(\frac{\hat{\omega}_i+\hat{\omega}_{i+1}}{2}-b_{w_i}\right)^\wedge\delta t\right)\delta t - \frac{1}{2}R_i^{b_k}(\hat{a}_i-b_{a_i})^\wedge\delta t\right]\delta\theta_i$$

$$-\frac{R_i^{b_k}+R_{i+1}^{b_k}}{2}\delta t\cdot\delta b_{a_i} + \frac{1}{2}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\delta t^2\,\delta b_{w_i} - \frac{R_i^{b_k}}{2}\delta t\cdot n_{a_i}$$

$$+\frac{1}{4}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\delta t^2\cdot n_{w_i} - \frac{R_{i+1}^{b_k}}{2}\delta t\cdot n_{a_{i+1}} + \frac{1}{4}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\delta t^2\cdot n_{w_{i+1}}$$

所以有

$$f_{21} = -\frac{R_i^{b_k}}{2}(\hat{a}_i-b_{a_i})^\wedge\delta t - \frac{1}{2}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\left(I-\left(\frac{\hat{\omega}_i+\hat{\omega}_{i+1}}{2}-b_{w_i}\right)^\wedge\delta t\right)\delta t$$

$$f_{23} = -\frac{R_i^{b_k}+R_{i+1}^{b_k}}{2}\delta t$$

$$f_{24} = \frac{1}{2}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\delta t^2$$

$$h_{21} = h_{23} = \frac{1}{4}R_{i+1}^{b_k}(\hat{a}_{i+1}-b_{a_i})^\wedge\delta t^2$$

③ 推导 $\delta\alpha_{k+1}$ 递推形式：

$$\delta\dot{\alpha} = \delta\beta$$

离散形式：

$$\delta\dot{\alpha}_i = \frac{1}{2}(\delta\beta_i + \delta\beta_{i+1})$$

由于 $\delta\beta_{i+1} = \delta\beta_i + f_{21}\delta\theta_i - \frac{R_i^{b_k}+R_{i+1}^{b_k}}{2}\delta t\cdot\delta b_{a_i} + f_{24}\delta b_{w_i} - \frac{R_i^{b_k}}{2}\delta t\cdot n_{a_i} + h_{21}n_{w_i}$

$$-\frac{R_{i+1}^{b_k}}{2}\delta t\, n_{a_{i+1}} + h_{23}n_{w_{i+1}}$$

所以有

$$\delta\dot{\alpha}_i = \frac{1}{2}\left(2\delta\beta_i + f_{21}\delta\theta_i + f_{23}\delta b_{a_i} + f_{24}\delta b_{w_i} - \frac{R_i^{b_k}}{2}\delta t\, n_{a_i} + h_{21}n_{w_i} - \frac{R_{i+1}^{b_k}}{2}\delta t\, n_{a_{i+1}} + h_{23}n_{w_{i+1}}\right)$$

则有

$$\delta\alpha_{i+1} = \delta\alpha_i + \delta\dot{\alpha}_i\cdot\delta t$$

$$= \delta\alpha_i + \delta\beta_i\delta t + \frac{f_{21}}{2}\delta\theta_i\delta t + \frac{f_{23}}{2}\delta b_{a_i}\delta t + \frac{f_{24}}{2}\delta b_{w_i}\delta t - \frac{R_i^{b_k}}{4}\delta t^2 n_{a_i}$$

$$+\frac{h_{21}}{2}n_{w_i}\delta t - \frac{R_{i+1}^{b_k}}{4}\delta t^2 n_{a_{i+1}} + \frac{h_{23}}{2}n_{w_{i+1}}\delta t$$

所以有

$$f_{01} = \frac{\delta t}{2}f_{21}$$

$$f_{03} = \frac{\delta t}{2}f_{23}$$

$$f_{04} = \frac{\delta t}{2}f_{24}$$

$$h_{00} = -\frac{R_i^{b_k}}{4}\delta t^2$$

$$h_{01} = \frac{\delta t}{2}h_{21}$$

$$h_{02} = -\frac{R_{i+1}^{b_k}}{4}\delta t^2$$

$$h_{03} = \frac{\delta t}{2}h_{23}$$

将离散形式的 $i\rightarrow i+1$ 递推公式简写成.

$$\delta z_{i+1} = F\delta z_i + VB$$

则雅克比递推公式

$$J_{i+1} = FJ_i$$ ⟶ 仅用于提供对 bias 的 Jacobian，$J^{b_k} = I$

协方差迭代公式：

$$P_{i+1} = FP_iF^T + VQV^T \qquad P^{b_k} = 0$$

$$Q = \mathrm{diag}(\sigma_a^2, \sigma_w^2, \sigma_a^2, \sigma_w^2, \sigma_{b_a}^2, \sigma_{b_w}^2)$$

接下来讨论IMU预积分残差雅克比.

定义有预积分残差 = 预积分理想值 − 预积分测量值.

$$r_B(\hat{z}_{b_{k+1}}^{b_k}, X) = \begin{bmatrix} \delta\alpha_{b_{k+1}}^{b_k} \\ \delta\theta_{b_{k+1}}^{b_k} \\ \delta\beta_{b_{k+1}}^{b_k} \\ \delta b_a \\ \delta b_w \end{bmatrix} = \begin{bmatrix} R_w^{b_k}(P_{b_{k+1}}^w - P_{b_k}^w - V_{b_k}^w \Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha_{b_{k+1}}^{b_k} \\ 2\left[(\gamma_{b_{k+1}}^{b_k})^{-1} \otimes (q_{b_k}^w)^{-1} \otimes q_{b_{k+1}}^w\right]_{xyz} \\ R_w^{b_k}(V_{b_{k+1}}^w - V_{b_k}^w + g^w \Delta t_k) - \beta_{b_{k+1}}^{b_k} \\ ba_{b_{k+1}} - ba_{b_k} \\ bw_{b_{k+1}} - bw_{b_k} \end{bmatrix}$$

待优化变量:

k时刻PVQB: $P_{b_k}^w \quad V_{b_k}^w \quad q_{b_k}^w \quad b_{a_k} \quad b_{w_k}$

k+1时刻PVQB: $P_{b_{k+1}}^w \quad V_{b_{k+1}}^w \quad q_{b_{k+1}}^w \quad b_{a_{k+1}} \quad b_{w_{k+1}}$

由于:

$$f(x+\Delta x) \approx f(x) + \frac{\partial f(x)}{\partial x}\Delta x$$

$$\Rightarrow \frac{\partial f(x)}{\partial x} \approx \frac{\partial f(x+\Delta x)}{\partial \Delta x}$$

因而求Jacobian采用对增量Δx求偏导的方式.

$$P_{b_k}^w \leftarrow P_{b_k}^w + \delta P_{b_k}^w \qquad P_{b_{k+1}}^w \leftarrow P_{b_{k+1}}^w + \delta P_{b_{k+1}}^w$$

(注意此处未用$R\delta P$作增量)

$$V_{b_k}^w \leftarrow V_{b_k}^w + \delta V_{b_k}^w \qquad V_{b_{k+1}}^w \leftarrow V_{b_{k+1}}^w + \delta V_{b_{k+1}}^w$$

$$q_{b_k}^w \leftarrow q_{b_k}^w \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta_{b_k}^w}{2} \end{bmatrix} \qquad q_{b_{k+1}}^w \leftarrow q_{b_{k+1}}^w \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta_{b_{k+1}}^w}{2} \end{bmatrix}$$

$$b_{a_k} \leftarrow b_{a_k} + \delta b_{a_k} \qquad b_{a_{k+1}} \leftarrow b_{a_{k+1}} + \delta b_{a_{k+1}}$$

$$b_{w_k} \leftarrow b_{w_k} + \delta b_{w_k} \qquad b_{w_{k+1}} \leftarrow b_{w_{k+1}} + \delta b_{w_{k+1}}$$

首先讨论残差$r_{\delta\alpha}$的雅克比:

由于$r_{\delta\alpha}$与$q_{b_{k+1}}^w$, $V_{b_{k+1}}^w$无关, 因此关于它们的雅克比为0

$$r_{\delta\alpha}(P_{b_k}^w) \leftarrow r_{\delta\alpha}(P_{b_k}^w + \delta P_{b_k}^w)$$

$$= R_w^{b_k}(P_{b_{k+1}}^w - (P_{b_k}^w + \delta P_{b_k}^w) - V_{b_k}^w \Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha_{b_{k+1}}^{b_k}$$

$$\boxed{\text{其中 } \alpha_{b_{k+1}}^{b_k} \text{ 假设与bias是线性关系}, \ \alpha_{b_{k+1}}^{b_k} \approx \hat{\alpha}_{b_{k+1}}^{b_k} + J_{b_a}^{\alpha}\delta b_a + J_{b_w}^{\alpha}\delta b_w}$$

$$= R_w^{b_k}(P_{b_{k+1}}^w - P_{b_k}^w - V_{b_k}^w \Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - R_w^b \delta P_{b_k}^w - \hat{\alpha}_{b_{k+1}}^{b_k} - J_{b_a}^{\alpha}\delta b_a - J_{b_w}^{\alpha}\delta b_w$$

$$= -R_w^b \delta P_{b_k}^w + C$$

常数, 与δP无关

$$\frac{\partial r_{\delta\alpha}(P_{b_k}^w)}{\partial P_{b_k}^w} = \frac{\partial r_{\delta\alpha}(P_{b_k}^w + \delta P_{b_k}^w)}{\delta P_{b_k}^w} = -R_w^b$$

~~$r_{\delta\alpha}(P_{b_k}^w) \leftarrow$~~

$$r_{\delta\alpha}(V_{b_k}^w) \leftarrow r_{\delta\alpha}(V_{b_k}^w + \delta V_{b_k}^w)$$

$$= R_w^{b_k}(P_{b_{k+1}}^w - P_{b_k}^w - (V_{b_k}^w + \delta V_{b_k}^w)\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha_{b_{k+1}}^{b_k}$$

$$= -R_w^{b_k}\Delta t_k \delta V_{b_k}^w + D$$

$$\frac{\partial r_{\delta\alpha}(V_{b_k}^w)}{\partial V_{b_k}^w} = \frac{\partial r_{\delta\alpha}(V_{b_k}^w + \delta V_{b_k}^w)}{\partial \delta V_{b_k}^w} = -R_w^{b_k}\Delta t_k$$

$$r_{\delta\alpha}(q^w_{b_k}) \leftarrow r_{\delta\alpha}(R^w_{b_k} Exp(\delta\phi^w_{b_k}))$$

$$= (R^w_{b_k} Exp(\delta\phi^w_{b_k}))^T (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha^{b_k}_{b_{k+1}}$$

$$= Exp(-\delta\phi) R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha^{b_k}_{b_{k+1}}$$

$$\approx (I - \delta\phi^\wedge) R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha^{b_k}_{b_{k+1}}$$

$$= -\delta\phi^\wedge R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) + C$$

$$= [R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2)]^\wedge \delta\phi + C$$

$$\frac{\partial r_{\delta\alpha}(q^w_{b_k})}{\partial q^w_{b_k}} = \frac{\partial r_{\delta\alpha}(R^w_{b_k} Exp(\delta\phi))}{\partial \delta\phi} = [R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2)]^\wedge$$

$$\frac{\partial r_{\delta\alpha}(b_{a_k})}{\partial b_{a_k}} = \frac{\partial r_{\delta\alpha}(b_{a_k} + \delta b_{a_k})}{\partial \delta b_{a_k}} = -J^{\alpha}_{b_a}$$

$$\frac{\partial r_{\delta\alpha}(b_{w_k})}{\partial b_{w_k}} = \frac{\partial r_{\delta\alpha}(b_{w_k} + \delta b_{w_k})}{\partial \delta b_{w_k}} = -J^{\alpha}_{b_w}$$

$$r_{\delta\alpha}(P^w_{b_{k+1}}) \leftarrow r_{\delta\alpha}(P^w_{b_{k+1}} + \delta P^w_{b_{k+1}})$$

$$= R^{b_k}_w ((P^w_{b_{k+1}} + \delta P^w_{b_{k+1}}) - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2) - \alpha^{b_k}_{b_{k+1}}$$

$$= R^{b_k}_w P^w_{b_{k+1}} + C$$

$$\frac{\partial r_{\delta\alpha}(P^w_{b_{k+1}})}{\partial P^w_{b_{k+1}}} = \frac{\partial r_{\delta\alpha}(P^w_{b_{k+1}} + \delta P^w_{b_{k+1}})}{\partial \delta P^w_{b_{k+1}}} = R^{b_k}_w$$

由于 $r_{\delta\alpha}$ 与 $k+1$ 时刻 $V^w_{b_{k+1}}$, $q^w_{b_{k+1}}$, $b_{a_{k+1}}$, $b_{w_{k+1}}$ 无关，其 Jacobian 为 0

整理 $r_{\delta\alpha}$ 的雅克比：

$$\frac{\partial r_{\delta\alpha}}{\partial P^w_{b_k}} = -R^b_w$$

$$\frac{\partial r_{\delta\alpha}}{\partial (q^w_{b_k})} = [R^{b_k}_w (P^w_{b_{k+1}} - P^w_{b_k} - V^w_{b_k}\Delta t_k + \frac{1}{2} g^w \Delta t_k^2)]^\wedge$$

$$\frac{\partial r_{\delta\alpha}}{\partial V^w_{b_k}} = -R^{b_k}_w \Delta t_k$$

$$\frac{\partial r_{\delta\alpha}}{\partial b_{a_k}} = -J^{\alpha}_{b_a}$$

$$\frac{\partial r_{\delta\alpha}}{\partial b_{w_k}} = -J^{\alpha}_{b_w}$$

$$\frac{\partial r_{\delta\alpha}}{\partial P^w_{b_{k+1}}} = R^{b_k}_w$$

$$\frac{\partial r_{\delta\alpha}}{\partial V^w_{b_{k+1}}} = 0$$

$$\frac{\partial r_{\delta\alpha}}{\partial q^w_{b_{k+1}}} = 0$$

$$\frac{\partial r_{\delta\alpha}}{\partial b_{a_{k+1}}} = 0$$

$$\frac{\partial r_{\delta\alpha}}{\partial b_{w_{k+1}}} = 0$$

其次讨论残差 $r_{\delta\theta}$ 的雅克比：

由于 $r_{\delta\theta}$ 与 $P^w_{b_k}, V^w_{b_k}, b_{a_k}, P^w_{b_{k+1}}, V^w_{b_{k+1}}, b_{a_{k+1}}, b_{w_{k+1}}$ 无关，其雅克比为0

$$r_{\delta\theta}(q^w_{b_k}) \leftarrow r_{\delta\theta}\left(q^w_{b_k}\otimes\begin{bmatrix}1\\ \frac{\delta\theta}{2}\end{bmatrix}\right)$$

$$= 2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\left(q^w_{b_k}\otimes\begin{bmatrix}1\\ \frac{\delta\theta}{2}\end{bmatrix}\right)^{-1}\otimes q^w_{b_{k+1}}$$

$$\frac{\partial r_{\delta\theta}(q^w_{b_k})}{\partial q^w_{b_k}} = \frac{\partial r_{\delta\theta}\left(q^w_{b_k}\otimes\begin{bmatrix}1\\ \frac{\delta\theta}{2}\end{bmatrix}\right)}{\partial\delta\theta} = \lim_{\delta\theta\to0}\frac{2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\left(q^w_{b_k}\otimes\begin{bmatrix}1\\ \frac{\delta\theta}{2}\end{bmatrix}\right)^{-1}\otimes q^w_{b_{k+1}} - 2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\left(q^w_{b_k}\otimes\begin{bmatrix}1\\ 0\end{bmatrix}\right)^{-1}\otimes q^w_{b_{k+1}}}{\delta\theta}$$

$$= 2\lim_{\delta\theta\to0}\frac{\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\begin{bmatrix}1\\ -\frac{\delta\theta}{2}\end{bmatrix}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}} - \left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\begin{bmatrix}1\\ 0\end{bmatrix}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}}{\delta\theta}$$

$$= 2\lim_{\delta\theta\to0}\frac{\left[\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}\right]_R\left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\right]_L\begin{bmatrix}1\\ -\frac{\delta\theta}{2}\end{bmatrix} - \left[\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}\right]_R\left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\right]_L\begin{bmatrix}1\\ 0\end{bmatrix}}{\delta\theta}$$

$$= 2\lim_{\delta\theta\to0}\frac{\left[\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}\right]_R\left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\right]_L\begin{bmatrix}0\\ -\frac{\delta\theta}{2}\end{bmatrix}}{\delta\theta}$$

由于只是对纯虚四元数 $\delta\theta_{3\times1}$ 求导，可只考虑分子每一项的虚部，

另外有，对四元数 $q=[q_w\ q_x\ q_y\ q_z]$

右乘 $[q]_R = q_w I + \begin{bmatrix}0 & -q_v^T\\ q_v & -[q_v]_\times\end{bmatrix}$ ， $[q^{-1}]_R = q_w I + \begin{bmatrix}0 & q_v^T\\ -q_v & [q_v]_\times\end{bmatrix}$

左乘 $[q]_L = q_w I + \begin{bmatrix}0 & -q_v^T\\ q_v & [q_v]_\times\end{bmatrix}$

若只取3×3的虚部部分，则

$$([q^{-1}]_R)_{3\times3} = q_w I_{3\times3} + [q_v]_\times = ([q]_L)_{3\times3}$$

同理 $([q^{-1}]_L)_{3\times3} = ([q]_R)_{3\times3}$

则有 $$\frac{\partial r_{\delta\theta}(q^w_{b_k})}{\partial q^w_{b_k}} = 2\lim_{\delta\theta\to0}\frac{\left[\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}\right]_R\left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\right]_L\begin{bmatrix}0\\ -\frac{\delta\theta}{2}\end{bmatrix}}{\delta\theta}$$

$$= 2\lim_{\delta\theta\to0}\frac{\left\{\left[\left(q^w_{b_{k+1}}\right)^{-1}\otimes q^w_{b_k}\right]_L\left[\gamma^{b_k}_{b_{k+1}}\right]_R\begin{bmatrix}0\\ -\frac{\delta\theta}{2}\end{bmatrix}\right\}_{3\times3}}{\delta\theta}$$

$$= -\left[\left(q^w_{b_{k+1}}\right)^{-1}\otimes q^w_{b_k}\right]_L\left[\gamma^{b_k}_{b_{k+1}}\right]_R$$

$$r_{\delta\theta}(b_{w_k}) \leftarrow r_{\delta\theta}(b_{w_k}+\delta b_{w_k})$$

$$= 2\left(\gamma^{b_k}_{b_{k+1}}\otimes\begin{bmatrix}1\\ \frac{1}{2}J^{\gamma}_{b_w}\delta b_w\end{bmatrix}\right)^{-1}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}$$

$$\frac{\partial r_{\delta\theta}(b_{w_k})}{\partial b_{w_k}} = \frac{\partial r_{\delta\theta}(b_{w_k}+\delta b_{w_k})}{\partial\delta b_{w_k}}$$

$$= \lim_{\delta b_{w_k}\to0}\frac{2\left(\gamma^{b_k}_{b_{k+1}}\otimes\begin{bmatrix}1\\ \frac{1}{2}J^{\gamma}_{b_w}\delta b_w\end{bmatrix}\right)^{-1}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}} - 2\left(\gamma^{b_k}_{b_{k+1}}\otimes\begin{bmatrix}1\\ 0\end{bmatrix}\right)^{-1}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}}{\delta b_w}$$

$$= 2\lim_{\delta b_w\to0}\frac{\begin{bmatrix}0\\ -\frac{1}{2}J^{\gamma}_{b_w}\delta b_w\end{bmatrix}\otimes\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}}{\delta b_w}$$

$$= -\left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1}\otimes\left(q^w_{b_k}\right)^{-1}\otimes q^w_{b_{k+1}}\right]_R\begin{bmatrix}0\\ J^{\gamma}_{b_w}\end{bmatrix}$$

$$= -\left\{\left[\left(q^w_{b_{k+1}}\right)^{-1}\otimes q^w_{b_k}\otimes\gamma^{b_k}_{b_{k+1}}\right]_L\right\}_{3\times3} J^{\gamma}_{b_w}$$

$$r_{\delta\theta}(q^w_{b_{k+1}}) \leftarrow r_{\delta\theta}\left(q^w_{b_{k+1}} \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta}{2} \end{bmatrix}\right)$$

$$= 2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}} \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta}{2} \end{bmatrix}$$

$$\frac{\partial r_{\delta\theta}(q^w_{b_{k+1}})}{\partial q^w_{b_{k+1}}} = \frac{\partial r_{\delta\theta}\left(q^w_{b_{k+1}} \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta}{2} \end{bmatrix}\right)}{\partial \delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}} \otimes \begin{bmatrix} 1 \\ \frac{\delta\theta}{2} \end{bmatrix} - 2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{2\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}} \otimes \begin{bmatrix} 0 \\ \frac{\delta\theta}{2} \end{bmatrix}}{\delta\theta}$$

$$= \left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}}\right]_L$$

整理 $r_{\delta\theta}$ 的雅克比：

$$\frac{\partial r_{\delta\theta}}{\partial P^w_{b_k}} = 0 \qquad \frac{\partial r_{\delta\theta}}{\partial P^w_{b_{k+1}}} = 0$$

$$\frac{\partial r_{\delta\theta}}{\partial V^w_{b_k}} = 0 \qquad \frac{\partial r_{\delta\theta}}{\partial V^w_{b_k}} = 0$$

$$\frac{\partial r_{\delta\theta}}{\partial q^w_{b_k}} = -\left[\left(q^w_{b_{k+1}}\right)^{-1} \otimes q^w_{b_k}\right]_L \left[\gamma^{b_k}_{b_{k+1}}\right]_R \qquad \frac{\partial r_{\delta\theta}}{\partial q^w_{b_{k+1}}} = \left[\left(\gamma^{b_k}_{b_{k+1}}\right)^{-1} \otimes \left(q^w_{b_k}\right)^{-1} \otimes q^w_{b_{k+1}}\right]_L$$

$$\frac{\partial r_{\delta\theta}}{\partial b_{a_k}} = 0 \qquad \frac{\partial r_{\delta\theta}}{\partial b_{a_{k+1}}} = 0$$

$$\frac{\partial r_{\delta\theta}}{\partial b_{w_k}} = -\left\{\left[\left(q^w_{b_{k+1}}\right)^{-1} \otimes q^w_{b_k} \otimes \gamma^{b_k}_{b_{k+1}}\right]_L\right\}_{3\times 3} J^{\gamma}_{b_w} \qquad \frac{\partial r_{\delta\theta}}{\partial b_{w_{k+1}}} = 0$$

下面讨论残差 $r_{\delta\beta}$ 的雅克比：

由于 $r_{\delta\beta}$ 与 $P^w_{b_k}$, $P^w_{b_{k+1}}$, $q^w_{b_{k+1}}$, $b_{a_{k+1}}$, $b_{w_{k+1}}$ 无关，其雅克比为0.

$$r_{\delta\beta}(V^w_{b_k}) \leftarrow r_{\delta\beta}(V^w_{b_k} + \delta V^w_{b_k})$$

$$= R^{b_k}_w\left(V^w_{b_{k+1}} - (V^w_{b_k} + \delta V^w_{b_k}) + g^w \Delta t_k\right) - \beta^{b_k}_{b_{k+1}}$$

$$= -R^{b_k}_w \delta V^w_{b_k} + C$$

$$\frac{\partial r_{\delta\beta}(V^w_{b_k})}{\partial V^w_{b_k}} = \frac{\partial r_{\delta\beta}(V^w_{b_k} + \delta V^w_{b_k})}{\partial \delta V^w_{b_k}} = -R^{b_k}_w$$

$$r_{\delta\beta}(q^w_{b_k}) \leftarrow r_{\delta\beta}(R^w_{b_k} Exp(\delta\phi))$$

$$= \left(R^w_{b_k} Exp(\delta\phi)\right)^T \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right) - \beta^{b_k}_{b_{k+1}}$$

$$= Exp(-\delta\phi) R^{b_k}_w \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right) - \beta^{b_k}_{b_{k+1}}$$

$$\approx (I - \delta\phi^\wedge) R^{b_k}_w \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right) - \beta^{b_k}_{b_{k+1}}$$

$$= -\delta\phi^\wedge R^{b_k}_w \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right) + C$$

$$= \left[R^{b_k}_w \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right)\right]^\wedge \delta\phi + C$$

$$\frac{\partial r_{\delta\beta}(q^w_{b_k})}{\partial q^w_{b_k}} = \frac{\partial r_{\delta\beta}(R^w_{b_k} Exp(\delta\phi))}{\partial \delta\phi} = \left[R^{b_k}_w \left(V^w_{b_{k+1}} - V^w_{b_k} + g^w \Delta t_k\right)\right]^\wedge$$

同样假设 $\delta\beta^{b_k}_{b_{k+1}}$ 与 bias 是线性关系，$\beta^{b_k}_{b_{k+1}} \approx \hat{\beta}^{b_k}_{b_{k+1}} + J^{\beta}_{b_a}\delta b_a + J^{\beta}_{b_w}\delta b_w$

$$\frac{\partial r_{\delta\beta}(b_{a_k})}{\partial b_{a_k}} = \frac{\partial r_{\delta\beta}(b_{a_k} + \delta b_a)}{\partial \delta b_a} = -J^{\beta}_{b_a}$$

$$\frac{\partial r_{\delta\beta}(b_{w_k})}{\partial b_{w_k}} = \frac{\partial r_{\delta\beta}(b_{w_k} + \delta b_w)}{\partial \delta b_w} = -J^{\beta}_{b_w}$$

$$\gamma_{\delta\beta}(V^w_{b_{k+1}}) \leftarrow \gamma_{\delta\beta}(V^w_{b_{k+1}}+\delta V)$$
$$= R^{b_k}_w\left((V^w_{b_{k+1}}+\delta V) - V^w_{b_k} + g^w\Delta t_k\right) - \beta^{b_k}_{b_{k+1}}$$
$$= R^{b_k}_w \delta V + C$$

$$\frac{\partial \gamma_{\delta\beta}(V^w_{b_{k+1}})}{\partial V^w_{b_{k+1}}} = \frac{\partial \gamma_{\delta\beta}(V^w_{b_{k+1}}+\delta V)}{\partial \delta V} = R^{b_k}_w$$

整理 $\gamma_{\delta\beta}$ 的雅克比：

$$\frac{\partial \gamma_{\delta\beta}}{\partial P^w_{b_k}} = 0$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial V^w_{b_k}} = -R^{b_k}_w$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial q^w_{b_k}} = \left[R^{b_k}_w(V^w_{b_{k+1}} - V^w_{b_k} + g^w\Delta t_k)\right]^\wedge$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial b_{a_k}} = -J^{\beta}_{b_a}$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial b_{w_k}} = -J^{\beta}_{b_w}$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial P^w_{b_{k+1}}} = 0$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial V^w_{b_{k+1}}} = R^{b_k}_w$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial q^w_{b_{k+1}}} = 0$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial b_{a_{k+1}}} = 0$$

$$\frac{\partial \gamma_{\delta\beta}}{\partial b_{w_{k+1}}} = 0$$

~~下面讨论~~

下面讨论 γ_{b_a}，γ_{b_w} 雅克比。

由于 γ_{b_a}、γ_{b_w} 与 PVQ 无关，其雅克比为0

$$\frac{\partial \gamma_{b_a}}{\partial b_{a_{b_k}}} = -I \qquad \frac{\partial \gamma_{b_w}}{\partial b_{w_{b_k}}} = -I$$

$$\frac{\partial \gamma_{b_a}}{\partial b_{a_{b_{k+1}}}} = I \qquad \frac{\partial \gamma_{b_w}}{\partial b_{w_{b_{k+1}}}} = I$$