20. Let $\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}$ be a fundamental matrix of the homogeneous system

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}(t).$$

Which of the following is a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 4e^t \end{pmatrix} \quad ?$$

ANS: $\vec{x}_p = \Phi(t) \int \Phi^{-1}(t) f(t)\, dt$

A. $\begin{pmatrix} 2t \\ -2t+1 \end{pmatrix} e^t$

B. $\begin{pmatrix} 2t+1 \\ 2t-1 \end{pmatrix} e^t$

C. $\begin{pmatrix} 2t+1 \\ -2t \end{pmatrix} e^t$

D. $\begin{pmatrix} -2t-1 \\ 2t-1 \end{pmatrix} e^t$

E. $\begin{pmatrix} 2t+2 \\ 2t-2 \end{pmatrix} e^t$

Recall the inverse of 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus $\Phi^{-1}(t) = \dfrac{1}{e^{4t}+e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix} = \frac{1}{2} e^{-4t} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix}$

$$\vec{x}_p = \Phi(t) \int \cancel{\tfrac{1}{2}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ \cancel{4}^{2} e^t \end{bmatrix} dt$$

$$= \Phi(t) \int \begin{bmatrix} -2 \\ 2e^{-2t} \end{bmatrix} dt$$

$$= \Phi(t) \begin{bmatrix} -\int 2\,dt \\ \int 2e^{-2t} dt \end{bmatrix}$$

$$= \Phi(t) \begin{bmatrix} -2t \\ -\int e^{-2t} d(-2t) \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \begin{bmatrix} -2t \\ -e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -2te^t - e^t \\ 2te^t - e^t \end{bmatrix}$$

$$= \begin{pmatrix} -2t-1 \\ 2t-1 \end{pmatrix} e^t$$

Nonhomogeneous Linear Systems

Consider

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

a general solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_c(t) + \vec{\mathbf{x}}_p(t)$.

Undetermined Coefficients

If $\vec{\mathbf{f}}(t)$ is a linear combination (with constant vector coefficients) of products of polynomials, exponential functions, and sines and cosines. We can make a guess to the general form of a particular solution $\vec{\mathbf{x}}_p$.

See illustrative examples from Lecture Notes Section 5.7.

Variation of Parameters

- Consider

$$\vec{\mathbf{x}}' = \mathbf{P}(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

Then a particular solution is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1} \mathbf{f}(t) dt,$$

where $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

- In particular, for the initial value problem

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$$

Then the solution is given by

$$\vec{\mathbf{x}}(t) = e^{\mathbf{A}t}\vec{\mathbf{x}}_0 + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}(s)} \vec{\mathbf{f}}(s) ds$$

Recall $e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$.

HW 5.7. #25

19. Let $\mathbf{x}(t)$ be the solution of the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

What is $\mathbf{x}(1)$?

A. $\begin{pmatrix} 3e \\ e \end{pmatrix}$

B. $\begin{pmatrix} 2e \\ e \end{pmatrix}$

C. $\begin{pmatrix} e \\ 0 \end{pmatrix}$

D. $\begin{pmatrix} 3e \\ 0 \end{pmatrix}$

E. $\begin{pmatrix} e \\ e \end{pmatrix}$

- $0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)+4 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

- Thus $\lambda = 1$ with multiplicity 2.

So we solve

$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then $\vec{v}_1 = (A - \lambda I)\vec{v}_2 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Thus we have

$\vec{x}_1 = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t, \quad \vec{x}_2 = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$

- So $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$

- As $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow c_1 = 0$ and $c_2 = 1$. Thus $\vec{x}(t) = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$ $(= \vec{x}_2(t))$

Then

$\vec{x}(1) = \begin{bmatrix} 3e \\ e \end{bmatrix}$

- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2 \vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

Matrix Exponentials and Linear Systems

Fundamental Matrix:

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix}$$

where $\vec{\mathbf{x}}_i$ are fundamental solutions to the system $\dfrac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$.

Exponential matrix: $e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!} + \cdots,$
$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$

Matrix Exponential Solutions:

Consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

then the solution is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0$. We can also use this eqn for the question

18. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 8 \\ -2 & \alpha \end{pmatrix} \mathbf{x},$$

find all the value of α so that the origin is an asymptotically unstable spiral point.

A. $\alpha > 0$

B. $\alpha > 8$

C. $\alpha < -8$

D. $0 < \alpha < 8$

E. $0 > \alpha > -8$

- Spiral point $\longleftrightarrow$ complex eigenvalues $\lambda = a \pm bi$ $\begin{cases} \text{stable}, & a < 0 \\ \text{unstable}, & a > 0 \end{cases}$

- We compute

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 8 \\ -2 & \alpha - \lambda \end{vmatrix} = \lambda(\lambda - \alpha) + 16 = 0$$

$$\Rightarrow \lambda^2 - \alpha\lambda + 16 = 0$$

$$\Rightarrow \lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 4 \times 16}}{2} = \frac{\alpha \pm \sqrt{\alpha^2 - 64}}{2}$$

- We need

  - complex root $\Rightarrow$ $\alpha^2 - 64 < 0$

  - real part > 0 $\Rightarrow$ $\frac{\alpha}{2} > 0$

$$\Rightarrow \begin{cases} \alpha^2 - 64 < 0 \\ \alpha > 0 \end{cases} \Rightarrow 0 < \alpha < 8$$

17. Find the Laplace transform of

$$e^t \int_0^t \sin\tau \cos(t-\tau)d\tau.$$

Compare this with

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

A. $\dfrac{s-1}{[(s-1)^2+1]^2}$

B. $e^s \dfrac{s}{(s^2+1)^2}$

C. $\dfrac{s^2}{(s^2+1)^2}$

D. $\dfrac{1}{s(s^2+1)}$

E. None of the above

We know

- $a = 1$
- $f(t) = \int_0^t \sin\tau \cos(t-\tau)dt$

$$= (\sin t) * (\cos t)$$

Recall

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Thus

$$\mathcal{L}\{e^t \int_0^t \sin\tau \cos(t-\tau)\,d\tau\}$$

$$= \frac{1}{(s-1)^2+1} \cdot \frac{s-1}{(s-1)^2+1}$$

$$= \frac{s-1}{[(s-1)^2+1]^2}$$

- Translation on the s-Axis: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

- Convolutions:
  - Definition: $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$
  - Property: $\mathcal{L}\{f(t)*g(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$

16. If $y(t)$ is the solution of the initial value problem

$$y'' + 2y' - 15y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0,$$

then $y(t) =$

A. $\frac{u_1(t)}{8}\left(e^{3t} - e^{-5t}\right)$

B. $\frac{u_1(t)}{8}\left(e^{-3t-1} - e^{-5t-1}\right)$

C. $\frac{u_1(t)}{8}\left(e^{3t-1} - e^{-5t-1}\right)$

D. $\frac{u_1(t)}{8}\left(e^{5t-5} - e^{-3t+3}\right)$

E. $\frac{u_1(t)}{8}\left(e^{3t-3} - e^{-5t+5}\right)$

Apply Laplace transform on both sides:

$$s^2 Y(s) + 2s Y(s) - 15 Y(s) = e^{-s}$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{s^2+2s-15}$$

$$= e^{-s} \frac{1}{(s+5)(s-3)}$$

Assume $\frac{1}{(s+5)(s-3)} = \frac{A}{s+5} + \frac{B}{s-3}$

$$= \frac{As - 3A + Bs + 5B}{(s+5)(s-3)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ -3A+5B=1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{8} \\ B = \frac{1}{8} \end{cases}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s-(-5)}\right\} = e^{-5t}$

$\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

Thus $Y(s) = \frac{1}{8} e^{-s}\left[-\frac{1}{s-(-5)} + \frac{1}{s-3}\right]$

Compare with

$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$,

we have

$y(t)$

$$= \frac{1}{8}u(t-1)\left[-e^{-5(t-1)} + e^{3(t-1)}\right]$$

$$= \frac{u_1}{8}\left(-e^{-5t+5} + e^{3t-3}\right)$$

- Transform of derivatives:

$$\mathcal{L}\{x\} = X, \quad \mathcal{L}\{x'\} = sX - x(0)$$
$$\mathcal{L}\{x''\} = s^2X - sx(0) - x'(0)$$
$$\mathcal{L}\{x'''\} = s^3X - s^2x(0) - sx'(0) - x''(0)$$

- Laplace Transform of $\delta(t-c)$: $\mathcal{L}\{\delta(t-c)\} = e^{-cs} \quad (c \geq 0)$

- Translation on the t-Axis: $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

15. Find the Laplace transform of $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ te^t, & t \ge 1. \end{cases}$

A. $\dfrac{e}{(s-1)^2} + \dfrac{e}{s-1}$

B. $\dfrac{e^{1-s}}{(s-1)^2} + \dfrac{e^{1-s}}{s-1}$

C. $\dfrac{e^{-s}}{(s-1)^2}$

D. $\dfrac{e^{-s}}{s^2(s-1)}$

E. $\dfrac{e^{1-s}}{s^2(s-1)}$

$$f(t) = u(t-1)\,\underline{te^t}$$

Want $(t-1)$ to appear

$$= u(t-1)\left[te^{t-1+1}\right]$$

$$= e\,u(t-1)\left[(t-1+1)e^{t-1}\right]$$

$g(t-1)$ with $g(t) = e^t$

$$= e\,u(t-1)\left[\underline{(t-1)e^{t-1}} + \underline{e^{t-1}}\right]$$

$f(t-1)$ with $f(t) = te^t$

Compare this with

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),$$

we have

$$\mathcal{L}\{f(t)\} = e \cdot e^{-s}\left(\frac{1}{(s-1)^2} + \frac{1}{s-1}\right)$$

$$= \frac{e^{1-s}}{(s-1)^2} + \frac{e^{1-s}}{s-1}$$

- Recall

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}$$

- Translation on the t-Axis: $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

Practice: Fall 2018 #17

17. Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{when } t < \pi, \\ t - 2\pi & \text{when } \pi \le t < 2\pi, \\ 0 & \text{when } t \ge 2\pi. \end{cases}$$

A. $e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\frac{1}{s^2} - e^{-\pi s}\frac{\pi}{s}$

B. $e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\frac{1}{s^2}$

C. $e^{\pi s}\frac{1}{s^2} - e^{2\pi s}\frac{1}{s^2} - \pi e^{2\pi s}\frac{1}{s^2}$

D. $\frac{1}{s}\left(e^{-\pi s} - e^{-2\pi s}\right)$

E. $e^{-\pi s}\frac{1}{s^2} + e^{-2\pi s}\frac{1}{s^2}$

ANS: We write $f(t)$ in terms of unit step function

$$f(t) = \begin{cases} 0, & t < \pi \\ t-2\pi, & t \ge \pi \end{cases} + \begin{cases} 0, & t < 2\pi \\ -(t-2\pi), & t \ge 2\pi \end{cases}$$

$$= u(t-\pi)\cdot(t-2\pi) + u(t-2\pi)[-(t-2\pi)]$$

$$= u(t-\pi)\cdot(t-\pi-\pi) - u(t-2\pi)\cdot(t-2\pi)$$

$f(t-\pi)$ with $f(t)=t$ $\qquad$ $f(t-2\pi)$ with $f(t)=t$

$$= u(t-\pi)\underline{(t-\pi)} - \pi\, u(t-\pi) - u(t-2\pi)\underline{(t-2\pi)}$$

Compare with $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

We have

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\cdot\frac{1}{s^2} - \pi e^{-\pi s} - e^{-2\pi s}\cdot\frac{1}{s^2}$$

So the correct answer is A

14. Find the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 5s + 7}{(s+2)(s^2 + 2s + 5)}.$$

$s^2+2s+1+4 = (s+1)^2+4$

A. $e^{-2t} + e^{-t}\sin(2t)$

B. $e^{-2t} + e^{-t}\cos(2t)$

C. $e^{-2t} - e^{-t}\cos(2t)$

D. $e^{-2t} - e^{-t}\sin(2t)$

E. $e^{2t} + e^{-t}\cos(2t)$

Assume

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{(s+1)^2+4}$$

$Bs^2+(2B+C)s+2$

$$= \frac{A\cdot(s^2+2s+5) + (Bs+C)(s+2)}{(s+2)(s^2+2s+5)}$$

$$= \frac{(A+B)s^2 + (2A+2B+C)\,s + 5A+2C}{(s+2)(s^2+2s+5)}$$

$$\Rightarrow \begin{cases} A+B=2 \\ 2A+2\overset{4}{B}+C=5 \\ 5A+2C=7 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=1 \end{cases}$$

Thus

$$F(s) = \frac{1}{s+2} + \frac{s+1}{(s+1)^2+4}$$

$$= \frac{1}{s-(-2)} + \frac{s-(-1)}{(s-(-1))^2+2^2}$$

$$\Rightarrow f(t) = e^{-2t} + e^{-t}\cos 2t$$

Rule 1. Linear Factor Partial Fractions

The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s-a$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}, \tag{2}$$

where $A_1, A_2, \cdots$, and A_n are constants.

Rule 2. Quadratic Factor Partial fractions

The portion of the partial fraction decomposition corresponding to the irreducible quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \cdots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}, \tag{3}$$

where $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots$, and B_n are constants.

13. A spring system with external forcing term is represented by the equation

$$y'' + 2y' + 2y = 4\cos(t) + 2\sin(t). \quad = f(t)$$

Then the steady state solution of the system is given by

x_p

A. $Y(t) = 2\sin(t)$

B. $Y(t) = \cos(t) - 2\sin(t)$

C. $Y(t) = -2\cos(t) + \sin(t)$

D. $Y(t) = -\cos(t)$

E. None of the above

$r^2 + 2r + 2 = 0$

$\Rightarrow \quad r = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

Thus $y_c = \underline{e^{-t}(c_1 \cos t + c_2 \sin t)}$

Thus $f(t)$ does not appear in y_c

Then we assume

$$y_p = A\cos t + B\sin t$$

$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t$$

Then $\underline{-A\cos t} - B\sin t - 2A\sin t + \underline{2B\cos t} + \underline{2A\cos t} + 2B\sin t$

$= 4\cos t + 2\sin t$

Thus

$$\begin{cases} 4 = -A + 2B + 2A = A + 2B \\ 2 = -B - 2A + 2B = B - 2A \end{cases}$$

$$\Rightarrow \begin{cases} A + 2B = 4 \Rightarrow 2A + 4B = 8 \\ B - 2A = 2 \end{cases}$$

$$\Rightarrow \begin{cases} A = 0 \\ B = 2 \end{cases}$$

- Damped Forced Oscillations ($c > 0$ and $F(t) \neq 0$)
  - transient solution $x_{tr}(t) = x_c(t)$, $\quad x_c(t) \to 0$ as $t \to \infty$
  - steady periodic solution $x_{sp}(t) = x_p(t)$

Undetermined Coefficients:
The general nonhomogeneous n th-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Find y_p by guessing a form and then plugging into DE (x^s is chosen so that y_1 and y_2 are not terms of y_p)

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + x_2 x^2 + \cdots + A_m x^m)\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$

12. A particular solution of the equation

$$y^{(4)} + 2y'' + y = 9\cos(t) + 2e^{-t} - 5t$$

is of the form

Note: We talked about this question in Practice Exams for Midterm 2.

A. $Y(t) = At\cos(t) + Bt\sin(t) + Ce^{-t} + Dt + E$

B. $Y(t) = A\cos(t) + B\sin(t) + Cte^{-t} + Dt^2 + Et$

C. $Y(t) = At\cos(t) + Bt\sin(t) + Ce^{-t} + Dt^2 + Et$

D. $Y(t) = At^2\cos(t) + Bt^2\sin(t) + Ce^{-t} + Dt + E$

E. $Y(t) = At^2\cos(t) + Bt^2\sin(t) + Cte^{-t} + Dt + E$

- We check the char. eqn for the homogeneous part:

$$r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2+1)^2 = 0$$

$$\Rightarrow r = i, i, -i, -i. \text{ Thus } y_c = (C_1 + C_2 t)\cos t + (C_3 + C_4)\sin t$$

- Thus we assume

$$\boxed{y_p = A\underline{t}^2\cos t + B\underline{t}^2\sin t + Ce^{-t} + Dt + E}$$

As cost appears twice in y_c

Undetermined Coefficients:
The general nonhomogeneous n th-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Find y_p by guessing a form and then plugging into DE (x^s is chosen so that y_1 and y_2 are not terms of y_c)

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$

11. In an electric circuit, a capacitor and an inductor are connected in series. The charge $Q(t)$ in the capacitor satisfies the equation

$$9Q'' + Q = 0,$$

with the initial condition $Q(0) = 1,\ Q'(0) = 1$. If we write $Q(t) = R\cos(\omega t - \delta)$, then find R and δ.

$9r^2 + 1 = 0 \Rightarrow r^2 = -\frac{1}{9} \Rightarrow r = \pm\frac{1}{3}i$

A. $R = 10,\ \delta = \arctan(2)$

B. $R = \sqrt{10},\ \delta = \arctan(3)$

C. $R = \sqrt{5},\ \delta = \frac{\pi}{4}$

D. $R = \sqrt{10},\ \delta = \arctan\left(\frac{1}{3}\right)$

E. $R = 5,\ \delta = -\frac{\pi}{3}$

Thus $Q(t) = A\cos\frac{1}{3}t + B\sin\frac{1}{3}t$

$Q(0) = 1 \Rightarrow A = 1.$

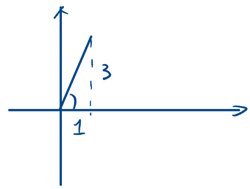
$Q'(0) = 0,\ Q'(t) = -\frac{1}{3}A\sin\frac{1}{3}t + \frac{1}{3}B\cos\frac{1}{3}t$

$Q'(0) = \frac{1}{3}B = 1 \Rightarrow B = 3$

Thus $Q(t) = \cos\frac{1}{3}t + 3\sin\frac{1}{3}t$.

Then

$R = \sqrt{10}$



$\delta = \arctan 3$

Related Topic:

Differential Equations as Vibrations

$$mx'' + cx' + kx = F(t) \begin{cases} m & \text{mass} \\ c & \text{dampening} \\ k & \text{spring constant} \\ F(t) & \text{forcing function} \end{cases}$$

- Free Undamped Motion ($c = 0$ and $F(t) = 0$)
  - General solution $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$, where $\omega_0 = \sqrt{\frac{k}{m}}$.
  - Need to know how to write $x(t) = C\cos(\omega_0 t - \alpha)$, where $C = \sqrt{A^2 + B^2}$ is the amplitude and α is the phase angle.
- Free Damped Motion ($c > 0$ and $F(t) = 0$)
  - Overdamped (two distinct real roots)
  - Critically damped (repeated real roots)
  - Underdamped (two complex roots)
    The solution can be written as $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$
- Undamped Forced Oscillations ($c = 0$ and $F(t) \neq 0$)

$$mx'' + kx = F_0\cos\omega t$$

- Damped Forced Oscillations ($c > 0$ and $F(t) \neq 0$)
  - transient solution $x_{tr}(t) = x_c(t),\quad x_c(t) \to 0$ as $t \to \infty$
  - steady periodic solution $x_{sp}(t) = x_p(t)$
  - practical resonance: Consider

$$mx'' + cx' + kx = F_0\cos\omega t$$

Practical resonance is the maximum value of $C(\omega)$. This may not exist.

10. Which of the following is a particular solution to the differential equation

$$y'' + 4y = \frac{1}{\cos(2t)} \quad ?$$

A. $t\cos(2t) + \ln|\cos(2t)|\sin(2t)$

B. $\frac{1}{2}t\cos(2t) - \frac{1}{2}\ln|\sin(2t)|\sin(2t)$

C. $-\frac{1}{2}t\cos(2t) + \frac{1}{4}\ln|\sin(2t)|\sin(2t)$

D. $\frac{1}{2}t\sin(2t) + \frac{1}{4}\ln|\cos(2t)|\cos(2t)$

E. $-\frac{1}{2}t\sin(2t) - \ln|\cos(2t)|\cos(2t)$

Note: We talked about this question in Practice Exam 2 for Midterm 2.

- We first compute y_c by solving $y''+4y=0$.

$$r^2+4=0 \Rightarrow r=\pm 2i.$$

Thus $y_c = C_1\cos 2t + C_2 \sin 2t$.

- Find $y_p = u_1y_1 + u_2y_2$, where $u_1 = -\int \frac{y_2 f}{W}dt$, $u_2 = \int \frac{y_1 f}{W}dt$.

$y_1 = \cos 2t$, $y_2 = \sin 2t$

Then $\cdot\, W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$

$\cdot\, u_1 = -\int \frac{y_2 f}{W}dt = -\int \frac{\sin 2t \cdot \frac{1}{\cos 2t}}{2}dt = -\frac{1}{2}\int \tan 2t\, dt = -\frac{1}{4}\int \tan 2t\, d2t$

$= \frac{1}{4}\ln|\cos 2t|$

$\cdot\, u_2 = \int \frac{y_1 f}{W}dt = \int \frac{\cos 2t \cdot \frac{1}{\cos 2t}}{2}dt$

$= \frac{1}{2}t$

- Thus

$y_p = u_1y_1 + u_2y_2$

$= \boxed{\frac{1}{4}\ln|\cos 2t|\cdot \cos 2t + \frac{1}{2}t\sin 2t}$

Variation of Parameters:

$$y'' + P(x)y' + Q(x)y = f(x)$$

homogeneous solution $y_c(x) = c_1y_1(x) + c_2y_2(x)$ known.

Then a particular solution is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

Wronskian: $W(x) = y_1y_2' - y_2y_1'$.

9. Given that the function $y_1 = t$ is a solution to the differential equation

$$t^2y'' - ty' + y = 0, \quad t > 0,$$

choose a function y_2 from the list below so that the pair $\{y_1, y_2\}$ is a fundamental set of the solutions to the differential equation above.

A. $y_2 = t^3$

B. $y_2 = t \ln t$

C. $y_2 = t \sin t$

D. $y_2 = t \cos t$

E. $y_2 = te^t$

Assume $y_2 = vy_1 = vt$

Then

$y_2 = vt$

$y_2' = v + v't$

$y_2'' = v' + v''t + v' = 2v' + v''t$

Then $t^2y_2'' - ty_2' + y_2 = 0$

$\Rightarrow t^2(2v' + v''t) - t(v + v't) + vt = 0$

$\Rightarrow 2v't^2 + v''t^3 - \cancel{tv} - v't^2 + \cancel{vt} = 0$

$\Rightarrow v''t^3 + v't^2 = 0$

$\Rightarrow v''t + v' = 0$

Let $u = v'$, then $u' = v''$. So $u't + u = 0 \Rightarrow \frac{du}{dt} \cdot t = -u$

$\Rightarrow \frac{du}{u} = -\frac{dt}{t} \Rightarrow \ln u = -\ln t \Rightarrow u = \frac{1}{t} \Rightarrow v' = \frac{1}{t}$

$\Rightarrow v = \ln t$

Reduction of Order

Consider

$$y'' + p(x)y' + q(x)y = 0,$$

with one solution $y = y_1(x)$ known.

Substitute:

$$y = \cancel{vy_1}$$
$$y' = \cancel{vy_1'} + v'y_1$$
$$y'' = \cancel{vy_1''} + 2v'y_1' + v''y_1$$

Diff. E. becomes

$$(2v'y_1' + v''y_1) + pv'y_1 = 0,$$

which is separable:

$$\frac{1}{(v')}(v')' = -\left(p + \frac{2y_1'}{y_1}\right).$$

Applications: Euler Equation: $ax^2y'' + bxy' + cy = 0$

8. The largest interval in which the solution of the initial value problem

$$\begin{cases}(5-t)y'' + (t-4)y' + 2y = \ln t \\ y(7) = 8\end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem is:

A. $(-\infty, 5)$

B. $(0, \infty)$

C. $(0, 5)$

D. $(0, 4)$

E. $(5, \infty)$

Recall in § 3.1

Theorem 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

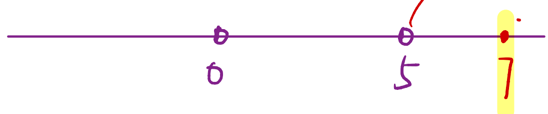
has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

$$y'' + \underline{\frac{t-4}{5-t}}\, y' + \underline{\frac{2}{5-t}}\, y = \underline{\frac{\ln t}{5-t}}$$

① All these functions of t should be continuous.

$$\Rightarrow \begin{cases} 5-t \neq 0 \\ t > 0 \end{cases}$$



② The interval should contain the initial pt.

Related Topic

Existence and Uniqueness Theorem

First Order, General Initial Value Problem:

$$y' = f(x, y), \quad y(x_0) = y_0$$

- Solution exists and is unique if f and $\frac{\partial}{\partial y} f$ are continuous at (x_0, y_0).
- Solutions are defined somewhere inside the region containing (x_0, y_0), where f and $\frac{\partial}{\partial y} f$ are continuous.

7. Use Euler's method to find approximate value of $y(0.2)$ for the following initial value problem with step size $h = 0.1$,

$$y' = y^2 + t^2, \qquad y(0) = 1.$$

$y' = f(t, y)$, $y(t_0) = y_0$

A. 1.1

B. 1.1122

C. 1.2

D. 1.22

E. 1.222

$h = 0.1$

n	$t_{n+1} = t_n + h$	$y_{n+1} = y_n + h \cdot f(t_n, y_n)$
0	$t_0 = 0$	$y_0 = 1$
1	$t_1 = 0.1$	$y_1 = y_0 + 0.1 \times f(0, 1) = 1 + 0.1 \times 1 = 1.1$
2	$t_2 = 0.2$	$y_2 = y_1 + 0.1 \times f(0.1, 1.1)$

$$= 1.1 + 0.1 \times (0.1^2 + 1.1^2)$$

$$= 1.1 + 0.1 \times (0.01 + 1.21)$$

$$= 1.1 + 0.1 \times 1.22$$

$$= 1.1 + 0.122$$

$$= 1.222$$

Euler's Method

Euler's Method: Consider $\dfrac{dy}{dx} = f(x, y)$, $\quad f(x_0) = y_0$

Euler's method with step size h: $\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot f(x_n, y_n) \end{cases}$

6. Which of the following statements is true about the equilibrium solutions of the autonomous equation

$$\frac{dy}{dt} = y(9 - y^2).$$

A. $y = -3$ is (asymptotically) stable, $y = 0$ is unstable, $y = 3$ is (asymptotically) stable

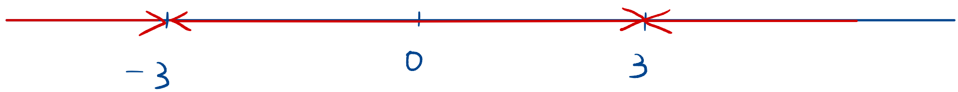
B. $y = -3$ is unstable, $y = 0$ is (asymptotically) stable, $y = 3$ is unstable

C. All of $y = -3$, $y = 0$ and $y = 3$ are semistable

D. $y = -3$ is unstable, $y = 0$ is semistable, $y = 3$ is (asymptotically) stable

E. $y = -3$ is (asymptotically) stable, $y = 0$ is semistable, $y = 3$ is unstable

Let $y' = y(9-y^2) = y(3-y)(3+y) = 0 \Rightarrow y = 0, 3, -3$



- $y > 3$, $y' = y(9-y^2) < 0$
- $0 < y < 3$, $y' = y(9-y^2) > 0$
- $-3 < y < 0$, $y' = y(9-y^2) < 0$
- $y < -3$, $y' = y(9-y^2) > 0$

☆ Additional Questions:

① If $y_0 = -4$, $\lim_{t\to\infty} y(t) = \underline{-3}$

② If $y_0 = 1$, $\lim_{t\to\infty} y(t) = \underline{3}$

③ If $y_0 = 4$, $\lim_{t\to\infty} y(t) = \underline{3}$

Autonomous Equations and Equilibrium Solutions

Autonomous Equations: $\frac{dx}{dt} = f(x)$

Critical points: values of x such that $f(x) = 0$.
$f(x_0) = 0 \Rightarrow$ equilibrium solution at $x = x_0$
$f(x_0) < 0 \Rightarrow$ solutions go down at $x = x_0$
$f(x_0) > 0 \Rightarrow$ solutions go up at $x = x_0$

Stability of Critical Points: Phase diagram method
unstable = solutions go away (either side)
stable = solutions go towards (both sides)
semi-stable = solutions mixed

5. If the following differential equation is exact, select the implicit solution to the initial value problem

$$\underbrace{(e^x \sin y - 2y \sin x)}_{M} + \underbrace{(e^x \cos y + 2\cos x + 2y)}_{N}y' = 0; \qquad y(0) = \pi.$$

If it is not exact, select "NOT EXACT".

A. $e^x \cos y - 2y \sin x = -1$

B. $e^x \sin y + 2y \cos x = 2\pi$

C. $e^x \sin y + 2y \cos x + y^2 = 2\pi + \pi^2$

D. $e^x \sin y - 2y \cos x + y^2 = -2\pi + \pi^2$

E. NOT EXACT

$$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x$$

$$\|$$

$$\frac{\partial N}{\partial x} = e^x \cos y - 2\sin x$$

Thus exact.

We find F such that

$$\frac{\partial F}{\partial x} = M = e^x \sin y - 2y \sin x$$

$$\Rightarrow F = \int e^x \sin y - 2y\sin x \, dx$$

$$\Rightarrow F = e^x \sin y + 2y\cos x + g(y)$$

Then $\frac{\partial F}{\partial y} = N \Rightarrow e^x \cos y + 2\cos x + g'(y)$

$$= e^x \cos y + 2\cos x + 2y$$

$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$$

Thus $F(x,y) = e^x \sin y + 2y\cos x + y^2 = C$ is a general solution.

As $F(0) = \pi$

Exact Equations

Exact Equations: $M(x,y)dx + N(x,y)dy = 0$, where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Solution: $F(x,y) = C$ such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$.

$$F(0,\pi) = 2\pi + \pi^2 = C$$

4. If $y = y(x)$ is a solution of

$$y' = \frac{y}{x} + \frac{x}{y}, \quad x > 0 \quad \text{and} \quad y(1) = 2,$$

then $y(e) =$

Note: We talked about this question in Practice Exam 1 for Midterm 1.

A. 0

B. $6\sqrt{e}$

C. $e\sqrt{6}$

D. $\sqrt{6}$

E. $\sqrt{6} - e$

Notice that the given equation is homogeneous

i.e. $y' = F(\frac{y}{x})$.

- Let $v = \frac{y}{x}$, then $y = vx$, $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$.

- So $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y} \Rightarrow v + x \cdot \frac{dv}{dx} = v + \frac{1}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2+1}{v} - v = \frac{v^2+1-v^2}{v} = \frac{1}{v}$

$\Rightarrow \int v\,dv = \int \frac{dx}{x} \Rightarrow \frac{1}{2}v^2 = \ln x + C$

$\Rightarrow \frac{1}{2}(\frac{y}{x})^2 = \ln x + C$. As $y(1) = 2$, $\frac{1}{2}(\frac{2}{1})^2 = \ln 1 + C$

$\Rightarrow C = 2$. Thus $\frac{1}{2}(\frac{y}{x})^2 = \ln x + 2$

- $y(e) = ?$ If $x = e$, then $\frac{1}{2}(\frac{y}{e})^2 = \cancel{\ln e}^{1} + 2 = 3$

$\Rightarrow y^2 = 6e^2 \Rightarrow y = \pm\sqrt{6}\,e$

Homogeneous Equations

Homogeneous Equations:	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
To identify:	All $x^n y^m$ have total power $(n+m)$ the same.
Solution:	Substitute $v = \frac{y}{x}$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (This converts equation to a separable Diff. E.)

3. Initially a tank holds 40 gallons of water with 10 lb of salt in solution. A salt solution containing $\frac{1}{2}$ lb of salt per gallon runs into the tank at the rate of 4 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $y(t)$ be the amount of salt in the tank after t minutes. Then $y(20) =$

A. 15 lb

B. 25 lb

C. 35 lb

D. 45 lb

E. 55 lb

$V_0 = 40$, $x(0)=10$

$C_i = \frac{1}{2}$, $r_i = 4$

$r_o = 2$, $x(20) = ?$

Recall $x'(t) = r_i C_i - r_o C_o$, where

$$C_o = \frac{x(t)}{V(t)}, \quad V(t) = V_0 + (r_i - r_o)t = 40+2t$$

Thus

$$x' = 2 - 2\cdot\frac{x}{40+2t} = 2 - \frac{x}{20+t}$$

$$\Rightarrow \quad x' + \frac{x}{20+t} = 2$$

$$\rho(t) = e^{\int \frac{1}{20+t}dt} = e^{\int \frac{1}{20+t} d(20+t)} = e^{\ln(20+t)} = 20+t$$

$$\rho x = (20+t)x = \int 2\cdot(20+t)\,dt = \int (40+2t)\,dt = 40t + t^2 + c$$

As $x(0)=10$, $(20+0)\times 10 = c \Rightarrow c = 200$

Thus $(20+t)x = 40t + t^2 + 200$, If $t=20$, $40x = 40\times 20 + 20^2 + 200$

$$\Rightarrow x = \frac{800+400+200}{40} = 20+10+5 = 35$$

Linear First-order Equations

Linear First-order Equations:	$\frac{dy}{dx} + P(x)y = Q(x)$
Solution:	$\rho y = \int \rho Q(x)dx$, where $\rho = e^{\int P(x)dx}$.
Applications:	Mixture Problems: $\frac{dx}{dt} = r_i c_i - r_o c_o$, where $c_o(t) = \frac{x(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_0)t$

2. Find the solution to the initial value problem

$$y' = \frac{3x^2+4x-4}{2y-4}, \quad y(1) = 3.$$ (separable)

A. $y^2 - 4y = x^3 + x^2 - x - 2$

B. $y^2 - 4y = x^3 + 2x^2 - 4x - 1$

C. $y^2 - 4y = x^3 + 2x^2 - 4x$

D. $y^2 - 4y - 2 = x^3 + 2x^2 - 4x$

E. $y^2 - 4y = x^3 + 2x^2 - 4x - 2$

Recall we talked about this question in Practice Exam 2 for Midterm 1.

$$\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$$

$$\Rightarrow (2y-4)\,dy = \int (3x^2+4x-4)\,dx$$

$$\Rightarrow y^2 - 4y = x^3 + 2x^2 - 4x + C$$

As $y(1)=3$, $9 - 4\cdot 3 = 1 + 2\cdot 1 - 4\cdot 1 + c$

$$\Rightarrow 9 - 12 = 1 + 2 - 4 + c$$

$$\Rightarrow c = -3 + 4 - 3 = -2$$

Thus $y^2 - 4y = x^3 + 2x^2 - 4x - 2$

Separable Equations

Separable Equations: $\dfrac{dy}{dx} = g(x)k(y)$

Solution: $\displaystyle\int \frac{dy}{k(y)} = \int g(x)dx + C$

Also check if $k(y) = 0$ is a solution

Applications: Newton's law of cooling: $\dfrac{dT}{dt} = k(A-T)$

Logistic equations: $\dfrac{dP}{dt} = kP(M-P) = aP - bP^2$

1. Let $y(t)$ be the solution to the initial value problem

$$ty' + y = 3t^2 - 2t + 2, \quad y(1) = 2.$$

Then $y(2) =$

A. 0

B. 2

C. 4

D. 6

E. 8

$y' + \frac{1}{t}y = 3t - 2 + \frac{2}{t}$, $y(1) = 2$.

$\rho(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

Recall $\rho y = \int \rho Q(t) dt$

$\Rightarrow ty = \int (3t^2 - 2t + 2)\, dt$

$\Rightarrow ty = t^3 - t^2 + 2t + c$

$y(1) = 2$.

$1 \cdot 2 = 1 - 1 + 2 + c \Rightarrow c = 0$

Thus $y = t^2 - t + 2$

$y(2) = 4 - 2 + 2 = 4$

Linear First-order Equations

Linear First-order Equations: $\frac{dy}{dx} + P(x)y = Q(x)$

Solution: $\rho y = \int \rho Q(x)dx$, where $\rho = e^{\int P(x)dx}$.

Applications: Mixture Problems: $\frac{dx}{dt} = r_i c_i - r_o c_o$,

where $c_o(t) = \frac{x(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_0)\, t$