Lecture 20. Direction Fields, Complex Eigenvalues

Direction Fields

In **Lecture 19**, **Example 4** , we observed that the direction fields (also known as slope fields) are quite useful in the study of ordinary differential equations (ODEs), especially for visualizing the behavior of solutions without solving the equations analytically.

Now we explain how to construct the direction field for a given a linear system. In practice, we use computer to generate the direction field but it is important to understand how we obtain them and why it is useful.

Example 1

$$\frac{dx}{dt} = x' = 6x - 7y$$
$$\frac{dy}{dt} = y' = x - 2y$$

We have $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}$.

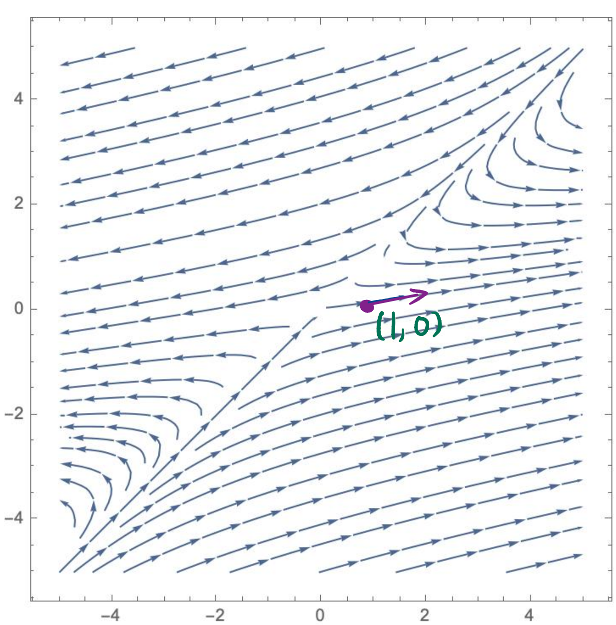
Then the general solution is

Eigenvalues: $\lambda_1 = 5, \lambda_2 = -1.$

Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 7 \\ 1 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Direction field:



$$\frac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}} = \boxed{\frac{dy}{dx} = \frac{x-2y}{6x-7y}}$$

At some point (x,y), we can compute the slope of the line tangent to the solution curve

For example, if $(x,y) = (1,0)$, then

$$\frac{dy}{dx} = \frac{1-2\cdot 0}{6-7\cdot 0} = \frac{1}{6}$$

Case 2. Complex Eigenvalues

Now we return to the discussion on the eigenvalue method for solving first-order linear system $\mathbf{x}' = A\mathbf{x}$.

It is understood that for a matrix A with real entries, the occurrence of complex eigenvalues is possible. Consequently, the eigenvectors associated with these complex eigenvalues will also be complex-valued.

🤔**Question:** How can we derive real-valued functions as solutions when matrix A has complex eigenvalues?

💡We will first examine **Example 2** before returning to the summary below.

Summary:

Assume we have complex eigenvalues $\lambda = p + q_i, \bar{\lambda} = p - qi$.

If $\mathbf{v}$ is an eigenvector associated with $\lambda = p + qi$, then $\mathbf{v}$ can be writien as $\mathbf{v} = \mathbf{a} + i\mathbf{b}$.

Then we have the solution

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = (\mathbf{a} + i\mathbf{b})e^{(p+qi)t}$$

$$\Rightarrow \qquad \mathbf{x}(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + ie^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

Then we get the real valued solutions

$$\begin{cases} \mathbf{x}_1(t) = \mathrm{Re}(\mathbf{x}(t)) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \\ \mathbf{x}_2(t) = \mathrm{Im}\left(\mathbf{x}(t)\right) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt) \end{cases}$$

Example 2 Apply the eigenvalue method to find a general solution of the given system. ~~Find also the corresponding particular solution to the given initial value problem.~~ Then use a computer system or graphing calculator to construct a direction field ~~and typical solution~~ curves for the given system.

$\vec{x}(t)' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $\qquad$ $\begin{aligned} x_1' &= 4x_1 - 3x_2 \\ x_2' &= 3x_1 + 4x_2 \end{aligned}$ $\qquad$ $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

ANS: Find the eigenvalues of A.

$(a-b)^2 = (b-a)^2$

$$0 = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0 \Rightarrow (\lambda-4)^2 + 9 = 0$$

$$\Rightarrow (\lambda-4)^2 = -9 \Rightarrow \lambda - 4 = \pm 3i \Rightarrow \lambda_{1,2} = 4 \pm 3i$$

So we have complex eigenvalues for A.

Rmk: It is still true that $\underbrace{\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t}}$ and $\underbrace{\vec{x}_2(t) = \vec{v}_2 e^{\lambda_2 t}}$ complex valued are two linearly independent solutions to the given system, where $\vec{v}_1, \vec{v}_2$ are the eigenvectors to λ_1, λ_2, resp.

However, we want to find real-valued solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ so that it is easier to discuss the general solutions.

To do this, we consider:

$\lambda = 4+3i$. We find the corresponding eigenvector.

Let $(A-\lambda I)\underset{\begin{bmatrix} a \\ b \end{bmatrix}}{\vec{v}} = \vec{0} \Rightarrow \begin{bmatrix} 4-(4+3i) & -3 \\ 3 & 4-(4+3i) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -3i & 3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -3ia + 3b = 0 \\ 3a - 3ib = 0 \end{cases} \Rightarrow \begin{cases} -ia - b = 0 & ① \\ a - ib = 0 & ② \end{cases}$$

Notice $i \times ① = ②$.

Let $a = 1$, then $① \Rightarrow b = -ia = -i$.

Thus $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector associated to $\lambda = 4+3i$.

Then by Thm 1, we know

$\vec{x}(t) = \vec{v} e^{\lambda t}$ is a solution to the given eqn. That is.

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(4+3i)t} \underset{\text{formula}}{\overset{\text{Using Euler's}}{=\!=}} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{4t}(\cos 3t + i \sin 3t)$$

$$= \begin{bmatrix} e^{4t}\cos 3t + \underline{i e^{4t}\sin 3t} \\ \underline{-i e^{4t}\cos 3t} + \overset{-i^2=1}{e^{4t}}\sin 3t \end{bmatrix} = \underbrace{\boxed{e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}}}_{\vec{x}_1(t)} + i\underbrace{\boxed{e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}}}_{\vec{x}_2(t)}$$

Real part of $\vec{x}(t)$

Imaginary part of $\vec{x}(t)$

We take $\vec{x}_1(t)$ $\vec{x}_2(t)$ as indicated in the green boxes.

Note $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are real-valued and they are linearly independent. Also $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions to the given eqn.

Scratch:

$\vec{x}(t)$ is a solution and $\vec{x}(t) = \vec{x}_1(t) + i\vec{x}_2(t)$

Plug it into $\vec{x}' = A\vec{x}$

We know

$\vec{x}_1'(t) + i\vec{x}_2'(t) = A\vec{x}_1(t) + iA\vec{x}_2(t)$

$\Rightarrow \begin{cases} \vec{x}_1'(t) = A\vec{x}_1(t) \\ \vec{x}_2'(t) = A\vec{x}_2(t) \end{cases}$

That is, $\vec{x}_1(t)$ $\vec{x}_2(t)$ are solutions to $\vec{x}' = A\vec{x}$

Thus the general solution is

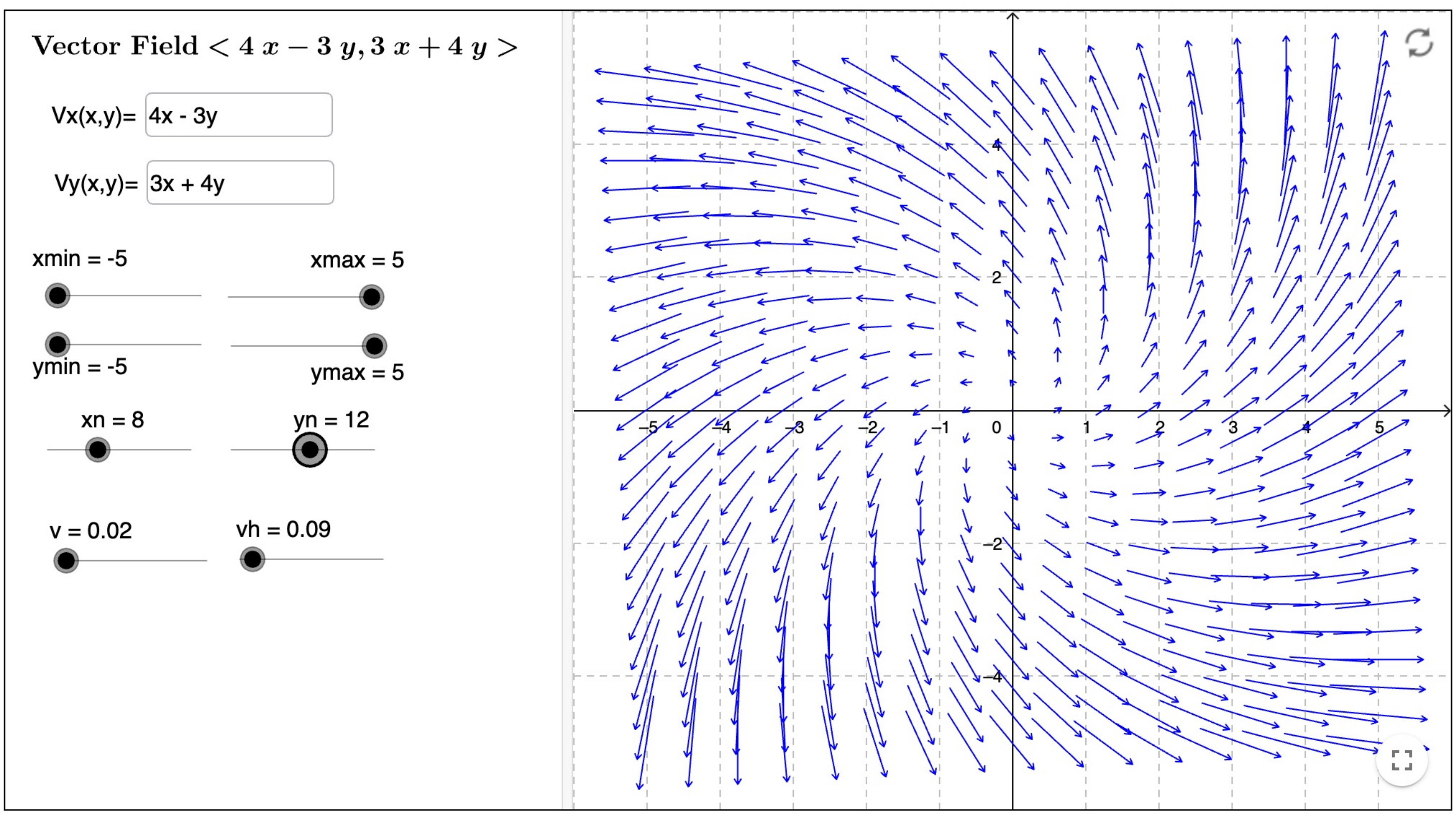
$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix} \Rightarrow \begin{cases} x_1(t) = c_1 e^{4t}\cos 3t + c_2 e^{4t}\sin 3t \\ x_2(t) = c_1 e^{4t}\sin 3t - c_2 e^{4t}\cos 3t \end{cases}$$

We use the online vector field generator (see the link in the lecture notes 18) to construct the following vector field.

Recall A has complex conjugate eigenvalues with positive real part.

In this case, the origin (point (0,0)) is called Spiral Source.



Similar to Q4 in HW14

Example 3. Suppose that the matrix A has the following eigenvalues and eigenvectors:

$$\lambda_1 = 2 + 3i \text{ with } \boldsymbol{v_1} = \begin{bmatrix} 3 \\ 1 - 5i \end{bmatrix}.$$

and

$$\lambda_2 = 2 - 3i \text{ with } \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 1 + 5i \end{bmatrix}.$$

Write the general real solution for the linear system $\boldsymbol{x}' = \mathrm{A}\boldsymbol{x}$, in the following forms:

(1) In eigenvalue/eigenvector form.

(2) In fundamental matrix form.

(3) As two equations.

ANS: Similar to the previous question (Example 2) we know a solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = \vec{v}_1 e^{\lambda_1 t} = \begin{bmatrix} 3 \\ 1-5i \end{bmatrix} e^{(2+3i)t}$$

$$= \begin{bmatrix} 3 \\ 1-5i \end{bmatrix} e^{2t} (\cos 3t + i \sin 3t)$$

Note all terms have e^{2t} so we can factor it out to simplify the calculation

$$= e^{2t} \begin{bmatrix} 3\cos 3t + 3i \sin 3t \\ \cos 3t + i \sin 3t - 5i \cos 3t + 5 \sin 3t \end{bmatrix}$$

$$= \boxed{e^{2t} \begin{bmatrix} 3\cos 3t \\ \cos 3t + 5 \sin 3t \end{bmatrix}} + i \boxed{e^{2t} \begin{bmatrix} 3 \sin 3t \\ \sin 3t - 5\cos 3t \end{bmatrix}}$$

$\vec{x}_1(t)$ $\vec{x}_2(t)$

Take $\vec{x}_1(t) = e^{2t}\begin{bmatrix} 3\cos 3t \\ \cos 3t + 5\sin 3t \end{bmatrix}$ and $\vec{x}_2(t) = e^{2t}\begin{bmatrix} 3\sin 3t \\ \sin 3t - 5\cos 3t \end{bmatrix}$

Then $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two linearly independent solutions to $\vec{x}' = A\vec{x}$.

(1) The eigenvalue/eigenvector form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\cos(3t) + \begin{bmatrix} 0 \\ 5 \end{bmatrix}\sin(3t)\right)e^{2t} + c_2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\sin(3t) + \begin{bmatrix} 0 \\ -5 \end{bmatrix}\cos(3t)\right)e^{2t}$$

(2) The fundamental matrix form:

We will talk about the standard def. for fundamental matrix later in this course. In this case, the fundamental matrix $\Phi(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)]$.

That is, $\Phi(t)$ is a 2×2 matrix with columns as $\vec{x}_1(t)$ and $\vec{x}_2(t)$

And

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3\cos(3t)e^{2t} & 3\sin(3t)e^{2t} \\ (\cos(3t) + 5\sin(3t))e^{2t} & (\sin(3t) - 5\cos(3t))e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(3) As two equations:

$$x(t) = 3\cos(3t)e^{2t}c_1 + 3\sin(3t)e^{2t}c_2$$
$$y(t) = (\cos(3t) + 5\sin(3t))e^{2t}c_1 + (\sin(3t) - 5\cos(3t))e^{2t}c_2$$

Exercise 4. Consider the linear system

$$\boldsymbol{y}' = \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix} \boldsymbol{y}.$$

(1) Find the eigenvalues and eigenvectors for the coefficient matrix.

(2) Find the real-valued solution to the initial value problem

$$\begin{cases} y_1' = -3y_1 - 2y_2, & y_1(0) = 0 \\ y_2' = 5y_1 + 3y_2, & y_2(0) = 5 \end{cases}$$

Solution.

(1) We have $A = \begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix}$, then

$$|A - \lambda I| = \begin{vmatrix} -\lambda - 3 & -2 \\ 5 & -\lambda + 3 \end{vmatrix} = \lambda^2 + 1 = 0$$

Thus we have $\lambda = \pm i$.

For $\lambda_1 = i$, we solve $(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0}$. We have

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} -3 - i & -2 \\ 5 & 3 - i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus we have

$$\begin{cases} (-3 - i)a - 2b = 0 \\ 5a + (3 - i)b = 0 \end{cases}$$

Note if we multiply the first equation by $\frac{1}{2}(-3 + i)$, we get the second equation. Thus the two equations are the same.

Let $b = 5$, then $a = -3 + i$. So we have $\mathbf{v}_1 = \begin{pmatrix} -3 + i \\ 5 \end{pmatrix}$, which is the eigenvector corresponding to the eigenvalue $\lambda_1 = i$.

Since the question is asking for all the eigenvectors, we look at $\lambda_2 = -i$. We solve $(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0}$.

Then

$$(A - \lambda_2 I)\mathbf{v}_2 = \begin{pmatrix} -3 + i & -2 \\ 5 & 3 + i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice if we multiply the first equation by $-\frac{1}{2}(3 + i)$, we obtain the second equation. Thus the two equation are the same.

Let $b = 5$, then $a = -(3+i)$. Thus we have $\mathbf{v}_2 = \begin{pmatrix} -3-i \\ 5 \end{pmatrix}$, which is the eigenvector corresponding to the eigenvalue $\lambda_2 = -i$.

(2) From (1), we know one solution is

$$\boldsymbol{y}(t) = \mathbf{v}_1 e^{\lambda_1 t} = \begin{pmatrix} -3+i \\ 5 \end{pmatrix} e^{it} = \begin{pmatrix} -3+i \\ 5 \end{pmatrix} e^{0t}(\cos t + i \sin t)$$
$$= \begin{pmatrix} -3\cos t - 3i \sin t + i\cos t - \sin t \\ 5\cos t + 5i \sin t \end{pmatrix} = \begin{pmatrix} -3\cos t - \sin t \\ 5\cos t \end{pmatrix} + i \begin{pmatrix} -3\sin t + \cos t \\ 5\sin t \end{pmatrix}$$

Similar to the discussion of **Example 2**, we assume

$$\boldsymbol{y}_1(t) = \begin{pmatrix} -3\cos t - \sin t \\ 5\cos t \end{pmatrix}, \quad \boldsymbol{y}_2(t) = \begin{pmatrix} -3\sin t + \cos t \\ 5\sin t \end{pmatrix}.$$

Thus the solution to the given equation is

$$\boldsymbol{y}(t) = c_1 \boldsymbol{y}_1(t) + c_2 \boldsymbol{y}_2(t) = c_1 \begin{pmatrix} -3\cos t - \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} -3\sin t + \cos t \\ 5\sin t \end{pmatrix}.$$

As

$$\boldsymbol{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix},$$

we have

$$\boldsymbol{y}(0) = c_1 \begin{pmatrix} -3\cos 0 - \sin 0 \\ 5\cos 0 \end{pmatrix} + c_2 \begin{pmatrix} -3\sin 0 + \cos 0 \\ 5\sin 0 \end{pmatrix} = \begin{pmatrix} -3c_1 + c_2 \\ 5c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

So we have $-3c_1 + c_2 = 0$, and $5c_1 = 5$.

Thus $c_1 = 1$ and $c_2 = 3$.

Therefore,

$$\boldsymbol{y}(t) = \begin{pmatrix} -3\cos t - \sin t \\ 5\cos t \end{pmatrix} + 3 \begin{pmatrix} -3\sin t + \cos t \\ 5\sin t \end{pmatrix},$$

Simplify this further, we have

$$\begin{aligned} y_1(t) &= -10\sin t \\ y_2(t) &= 5\cos t + 15\sin t \end{aligned}$$

Similar question to Q1 in HW14

Exercise 5. Suppose

$$\boldsymbol{y}(t) = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{t}\begin{bmatrix}1\\1\end{bmatrix},$$
$$\boldsymbol{y}(1) = \begin{bmatrix}0\\-1\end{bmatrix}.$$

(1) Find c_1 and c_2.

(2) Sketch the phase plane trajectory that satisfies the given initial condition.

ANS: (1) Plug in $\vec{y}(1) = \begin{bmatrix}0\\-1\end{bmatrix}$, we have

$$\vec{y}(1) = c_1 e^{-1}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{1}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}c_1e^{-1}+c_2e\\-c_1e^{-1}+c_2e\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1e^{-1}+c_2e = 0 & ① \\ -c_1e^{-1}+c_2e = -1 & ② \end{cases} \qquad ① \Rightarrow c_1e^{-1} = -c_2e \Rightarrow c_1 = -c_2e^2$$

Plug $c_1 = -c_2e^2$ into ②, we have

$$c_2e^2\cdot e^{-1} + c_2e = -1 \Rightarrow 2c_2e = -1 \Rightarrow c_2 = -\frac{1}{2e}$$

Then $c_1 = -c_2e^2 = +\frac{1}{2e}\cdot e^2 = \frac{1}{2}e$

Thus $\begin{cases} c_1 = \frac{1}{2}e \\ c_2 = -\frac{1}{2}e^{-1} \end{cases}$

(2) This question is easy in your webwork since there is only one graph passing the initial point $\begin{bmatrix}y_1(1)\\y_2(1)\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

To get the trajectory (solution curve) yourself, one way is to start from the solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tfrac{1}{2}e \cdot e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \tfrac{1}{2}e^{-1}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and try to eliminat the variable t.

we have

$$y_1 = \tfrac{1}{2}e \cdot e^{-t} - \tfrac{1}{2}e^{-1}e^{t} \quad ①$$

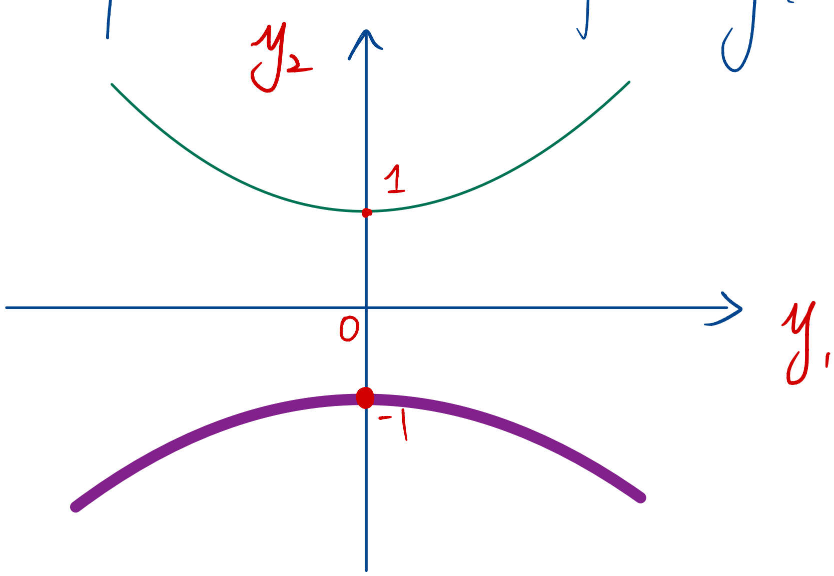
$$y_2 = -\tfrac{1}{2}ee^{-t} - \tfrac{1}{2}e^{-1}e^{t} \quad ②$$

$$①+② \Rightarrow \quad y_1 + y_2 = -e^{-1}e^{t} \quad ③$$

$$①-② \Rightarrow \quad y_1 - y_2 = ee^{-t} \quad ④$$

$$③\times④ \Rightarrow \quad y_1^2 - y_2^2 = -e^{-1}e^{t}ee^{-t} = -1$$

This is a parabola as the following:



As $\begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ we have the purple curve as the trajectory.