Midterm 2 Review of Common Ordinary Differential Equations

2nd Order, Homogeneous Linear, Constant Coefficients

2nd Order, Homogeneous Linear, Constant Coefficients:

$$ay'' + by' + cy = 0$$

Characteristic Equation:

$$ar^2 + br + c = 0$$

Solution depends on the type of roots:

- $r = r_1, r_2$(real, not repeated),
  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
- $r = r_1 = r_1$ (repeated root),
  $y = (c_1 + c_2 x) e^{r_1 x}$.
- $r = r_{1,2} = A \pm Bi$(complex conjugates),
  $y = e^{Ax} \left(c_1 \cos Bx + c_2 \sin Bx \right)$

Higher Order, Homogeneous Linear, Constant Coefficients

Higher Order, Homogeneous Linear, Constant Coefficients:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

Characteristic Equation:

$$a_n r^n \cdots + a_1 r + a_0 = 0$$

- Solution generalized from 2nd order case.
- Long division method can be used when solving char. eqn.

Solutions to Nonhomogeneous Equations

Consider the nonhomogeneous equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = f(x)$$

with homogeneous solution $y_c = c_1 y_1(x) + \cdots + c_n y_n$ known.

Then the general solution is $y = y_c + y_p$, where y_p is a particular solution.

Undetermined Coefficients:

The general nonhomogeneous n th-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Find y_p by guessing a form and then plugging into DE (x^s is chosen so that y_i's are not terms of y_c)

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s \left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m \right)$
$a \cos kx + b \sin kx$	$x^s (A \cos kx + B \sin kx)$
$e^{rx} (a \cos kx + b \sin kx)$	$x^s e^{rx} (A \cos kx + B \sin kx)$
$P_m(x) e^{rx}$	$x^s \left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m \right) e^{rx}$
$P_m(x)(a \cos kx + b \sin kx)$	$x^s [(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) \cos kx$ $+ (B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) \sin kx]$

Variation of Parameters:

$$y'' + P(x) y' + Q(x) y = f(x)$$

homogeneous solution $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ known.

Then a particular solution is

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx$$

Wronskian: $W(x) = y_1 y_2' - y_2 y_1'$.

Remark: Let $u_1 = -\int \frac{y_2(x) f(x)}{W(x)} dx$ and $u_2 = \int \frac{y_1(x) f(x)}{W(x)} dx$, then the above equation becomes

$$y_p(x) = u_1 y_1 + u_2 y_2$$

Differential Equations as Vibrations

$$mx'' + cx' + kx = F(t) \begin{cases} m & \text{mass} \\ c & \text{dampening} \\ k & \text{spring constant} \\ F(t) & \text{forcing function} \end{cases}$$

- Free Undamped Motion ($c = 0$ and $F(t) = 0$)
  - General solution $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$, where $\omega_0 = \sqrt{\frac{k}{m}}$.
  - Need to know how to write $x(t) = C\cos(\omega_0 t - \alpha)$, where $C = \sqrt{A^2 + B^2}$ is the amplitude and α is the phase angle.
- Free Damped Motion ($c > 0$ and $F(t) = 0$)
  - Overdamped (two distinct real roots)

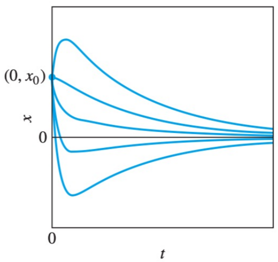

FIGURE 3.4.7. Overdamped motion: $x(t) = c_1e^{r_1t} + c_2e^{r_2t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

  - Critically damped (repeated real roots)

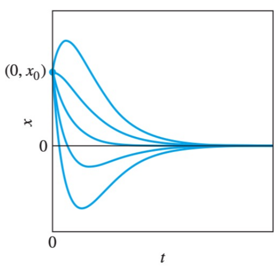

FIGURE 3.4.8. Critically damped motion: $x(t) = (c_1 + c_2t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

Differential Equations as Vibrations (continued)

  - Underdamped (two complex roots)

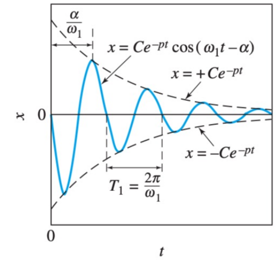

FIGURE 3.4.9. Underdamped oscillations: $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.

The solution can be written as $x(t) = C_1e^{-pt}\cos(\omega_1 t - \alpha_1)$

- Undamped Forced Oscillations ($c = 0$ and $F(t) \neq 0$)

$$mx'' + kx = F_0\cos\omega t$$

- Damped Forced Oscillations ($c > 0$ and $F(t) \neq 0$)
  - transient solution $x_{\text{tr}}(t) = x_c(t)$, $\quad x_c(t) \to 0$ as $t \to \infty$
  - steady periodic solution $x_{\text{sp}}(t) = x_p(t)$
  - practical resonance: Consider

$$mx'' + cx' + kx = F_0\cos\omega t$$

Practical resonance is the maximum value of $C(\omega)$. This may not exist.

From Higher-order Equation to 1st-order System

Consider the single nth-order equation

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)}),$$

we introduce the independent variables

$$x_1 = x,\ x_2 = x',\ x_3 = x'',\ \cdots, x_n = x^{(n-1)}$$

Then we have the following system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \cdots \\ x_n' = f(t, x_1, x_2, \cdots, x_n) \end{cases}$$

The Method of Elimination

Examples: $\begin{cases} x' = -3x - 4y \\ y' = 2x + y \end{cases}$, and other examples in Lecture Notes in 4.2.

$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$, show solutions are ellipses. See Notes in 4.1.

$\begin{cases} x' = y \\ y' = 2x \end{cases}$, show solutions are hyperbolas. See Notes in 4.1.

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$

Solution: $\vec{\mathbf{x}} = c_1\vec{\mathbf{x}}_1 + c_2\vec{\mathbf{x}}_2 + \cdots$,
where $\vec{\mathbf{x}}_i$ are fundamental solutions
from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\vec{\mathbf{v}}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

Additional Notes Summarized by Yourself

You can fill in this empty block to summarize the course contents that are not listed in this file.