# Review Questions for Material after Midterm 2

**Please refer to our previous Midterm 1 and Midterm 2 Reviews via Brightspace for other associated questions. The questions below are mostly about the material after Midterm 2.**

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There will be 25 multiple-choice questions for the final exam. The exam is cumulative, which tests the material covered throughout the semester.

Below are some selected questions from the past final exams related to **the material after Midterm 2**. You can find more exercises and answers from the past exam archive.

1. Spring 2019 #22
2. Spring 2019 #21
3. Spring 2018 #14
4. Fall 2019 #19
5. Fall 2018 #16
6. Spring 2017 #17
7. Fall 2015 #14
8. Midterm 2 #8

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We list the above questions here for convenience. The complete notes for solving these questions will be posted on Tuesday, April 26.

1. Spring 2019 #22

   **Keywords:** $\text{Proj}_W\,\mathbf{y}$**, Best Approximation Theorem**

   **Related questions:** Fall 2019 #17

   Find the distance from the vector $\mathbf{y}$ to the subspace $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

A. $12$.
B. $2\sqrt{2}$.
C. $3\sqrt{3}$.
D. $8$.
E. $3\sqrt{5}$.

Recall

① The closest point in $W = \text{span}\{\vec{u}, \vec{v}\}$ is $\text{Proj}_W \vec{y}$

② The distance from $\vec{y}$ to $W$ is $\|\vec{y} - \text{Proj}_W \vec{y}\|$

The formula for finding $\text{Proj}_W \vec{y}$ requires an orthogonal basis for W. But the given $\vec{u}$ and $\vec{v}$ are not orthogonal, so we use Gram-Schmidt process to find $\{\vec{u}_1, \vec{u}_2\}$ as orthogonal basis.

- $\vec{u}_1 = -\frac{1}{2}\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- $\vec{u}_2 = \vec{v} - \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle}\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

Then

$$\text{Proj}_W \vec{y} = \frac{\langle \vec{y}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle}\vec{u}_1 + \frac{\langle \vec{y}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle}\vec{u}_2$$

$\vec{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$

$$= \frac{-1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-20}{4+1}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\|\vec{y} - \text{Proj}_W \vec{y}\| = \sqrt{0^2 + (-5+8)^2 + (10-4)^2} = \sqrt{3^2 + 6^2} = \sqrt{49} = 3\sqrt{5}$$

2. Spring 2019 #21

**Keywords:** **Least-squares Solution**

**Related questions:** Spring 2017 #15, Fall 2019 #23,

Find the least-squares solution to

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$$

A. $(0,1)$
B. $(1,1)$
C. $(1,2)$
D. $(0,2)$
E. $(2,1)$

Recall the least-squares solution is the solution to

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 12 & 38 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 50 \end{bmatrix}$$

Then

$$\left[\begin{array}{cc|c} 6 & 12 & 18 \\ 12 & 38 & 50 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 6 & 19 & 25 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 7 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

3. Spring 2018 #14

**Keywords:** **Gram-Schmidt Process**

**Related questions:** Fall 2019 #25, Fall 2019 #22, Fall 2018 #25,

What are the resulting orthonormal vectors after applying the Gram-Schmidt process to the vectors

$$\underset{\vec{v}_1}{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} ?$$

A. $\begin{bmatrix} \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{21}} \\ -\frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \end{bmatrix}$

B. $\begin{bmatrix} \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 2 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \qquad \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1$$

$$= \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} - \cancel{\frac{6-1+2}{9+1+4}}^{\,\frac{1}{2}} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix} \xrightarrow{\times 2} \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \vec{u}_2$$

$$= \vec{v}_3 - \frac{0}{14} \vec{u}_1 - \frac{-2}{10} \vec{u}_2$$

$$= \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 2 \end{bmatrix} \sim \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}$$

4. Fall 2019 #19

**Keywords:** **Inner Product Space, Orthogonal Projection (for inner product space other than $\mathbb{R}^n$)**

**Related questions:** Fall 2018 #23, Fall 2017 #19, Fall 2015 #13

Let $C[-1,1]$ be the vector space of all continuous functions defined on $[-1,1]$. Define with the inner product on $C[-1,1]$ by

$$\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Find the orthogonal projection of $10t^3-5$ onto the subspace spanned by 1 and $t$ (with respect to the above inner product on $C[-1,1]$).

A. $6t-10$

B. $6t+5$

C. $10t^3-6t$

D. $10t^3-5$

E. $6t-5$

Let $W = \text{span}\{1, t\}$.

Notice $\langle 1, t\rangle = \int_{-1}^{1} t\, dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0$

Thus $\{1, t\}$ is an orthogonal basis for $W$.

$$\text{Proj}_W(10t^3-5) = \frac{\langle 10t^3-5, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle 10t^3-5, t\rangle}{\langle t, t\rangle} t$$

and

- $\langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = t\Big|_{-1}^{1} = 2$
- $\langle 10t^3-5, 1\rangle = \int_{-1}^{1} (10t^3-5)\, dt = \left(\frac{10}{4}t^4 - 5t\right)\Big|_{-1}^{1}$

  $= -5\cdot 1 - (-5)\times(-1) = -10$
- $\langle t, t\rangle = \int_{-1}^{1} t^2 dt = \frac{1}{3}t^3\Big|_{-1}^{1} = \frac{2}{3}$
- $\langle 10t^3-5, t\rangle = \int_{-1}^{1} (10t^4 - 5t)\, dt = \left(\frac{10}{5}t^5 - \frac{5}{2}t^2\right)\Big|_{-1}^{1}$

  $= 2\cdot 1 - 2(-1) = 4$

$$\operatorname{Proj}_W(10t^3-5) = \frac{\langle 10t^3-5,\ 1\rangle}{\langle 1,\ 1\rangle}1 + \frac{\langle 10t^3-5,\ t\rangle}{\langle t,t\rangle}t$$

$$= \frac{-10}{2}\times 1 + \frac{4}{\frac{2}{3}}t$$

$$= -5 + 6t$$

5. Fall 2018 #16

**Keywords:** **Diagonalizable Matrices**

**Related questions:** Fall 2018 #24, Fall 2017 #22

Which of the following matrices are diagonalizable?

(i) $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$ ? yes, see below.

(ii) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ symmetric, yes

(iii) $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ distinct eigenvalues, yes

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$ symmetric, yes

A. (i) and (iii) only

B. (iii) and (iv) only

C. (ii) and (iii) only

D. (i), (ii) and (iv) only

E. (i), (ii), (iii) and (iv)

Check (i)

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda-1)(\lambda+2)-4$$

$$= \underbrace{\lambda^2+\lambda-6}_{\substack{\parallel \\ \text{Note, it is} \\ \lambda^2 - tr(A)\lambda + det(A)}} = (\lambda+3)(\lambda-2) = 0$$

Thus (i) has distinct eigenvalues. so it is diagonalizable.

6. Spring 2017 #17

**Keywords:** **Orthogonal Complement**

**Related questions:** Spring 2018 #5, Spring 2017 #17

Let $A$ be an $m \times n$ matrix. If $W$ is the column space of $A^T$, then $W^\perp$ (the orthogonal complement of $W$ ) is

A. the null space of $A$
B. the column space of $A$
C. the null space of $A^T$
D. the column space of $A^T$
E. none of the above

As $W = Col(A^T)$, then $W = Row(A)$

Then any $\vec{u} \in W^\perp$, $\vec{u}$ is orthogonal to $W = Row(A)$.

This means $\vec{u}$ inner product with each row of $A$ is 0.

So we have

$$A\vec{u} = \vec{0}$$

(In fact, $A\vec{u} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \vec{u} = \begin{bmatrix} \vec{a}_1 \cdot \vec{u} \\ \vec{a}_2 \cdot \vec{u} \\ \vec{a}_3 \cdot \vec{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ )

Thus $W^\perp$ is the same as $Nul\, A$.

Or we can apply the Thm in §6.1:

**Theorem 3** Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the null space of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

$$(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A \quad \text{and} \quad (\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$$

7. Fall 2015 #14

**Keywords: Symmetric Matrix**

**Related questions:** Spring 2017 #24.

Let $A$ be an $n \times n$ matrix, which of the following statements is FALSE?
A. If $A$ is a symmetric matrix, then $A^T$ is also symmetric.
B. The product $AA^T$ is always symmetric.
C. If $A$ is skew-symmetric, then $A^3$ is symmetric.
D. If $A$ is symmetric, then $A + A^2$ is symmetric.
E. The sum $A + A^T$ is always symmetric.

A. True $\quad (A^T)^T = A^T$

B. True. $\quad (AA^T)^T = (A^T)^T A^T = AA^T$

C. False. $\quad$ Note $A$ is skew-symmetric if $A^T = -A$.

won't be tested

If $A^T = -A$, then

$(A^3)^T = (AAA)^T = A^T A^T A^T = (A^T)^3 = (-A)^3$

$= (-A)(-A)(-A) = -A^3$

D. True. $\quad (A+A^2)^T = A^T + (A^2)^T = A + A^2$

E. True. $\quad (A+A^T)^T = A^T + (A^T)^T = A^T + A$

8. Midterm 2 #8

**Keywords:** **Linear Transformation, Finding Basis for Subspaces**

Let $\mathbb{P}_2$ denote the vector space of all polynomials of degree at most 2 in the variable $t$, and let $\mathbb{M}_{2\times 2}$ denote the vector space of all $2 \times 2$ matrices. Consider a linear transformation:

$$T : \mathbb{P}_2 \to \mathbb{M}_{2\times 2} \quad \text{given by } T(p(t)) = \begin{bmatrix} p(0) & p'(0) \\ p(1) & p'(1) \end{bmatrix}$$

(1) Find $T\left(at^2 + bt + c\right)$.

(2) Find a polynomial $p(t)$ in $\mathbb{P}_2$ such that $T(p(t)) = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$.

(3) Find a basis for the range of $T$.

(1) $p'(t) = 2at + b$

$$T(at^2+bt+c) = \begin{bmatrix} c & b \\ a+b+c & 2a+b \end{bmatrix}$$

(2) We need to find $a, b, c$ such that

$$T(at^2+bt+c) = \begin{bmatrix} c & b \\ a+b+c & 2a+b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c = 1 \\ b = 2 \\ a+b+c = 4 \\ 2a+b = 4 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c = 1 \end{cases}$$

Thus $p(t) = t^2 + 2t + 1$

B) Any element in the range of $T$ has the form

$$T(at^2+bt+c) = \begin{bmatrix} c & b \\ a+b+c & 2a+b \end{bmatrix}$$

$$= c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

which is a linear combination of

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}.$$

Note it is also a linearly independent set

Thus it is a basis for Range $T$.

Remark: To find $\ker(T)$, we need to find $a, b, c$ such that

$$T(p(t)) = \begin{bmatrix} c & b \\ a+b+c & 2a+b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{zero vector in } M_{2\times 2}}.$$

This implies $a = b = c = 0$.

Thus the only $p(t) \in \mathbb{P}_2$ that is sent to $0$ in $M_{2\times 2}$ is the zero polynomial.

So $\ker T = \{0\}$