5.3 A Gallery of Solution Curves of Linear Systems

Example 1

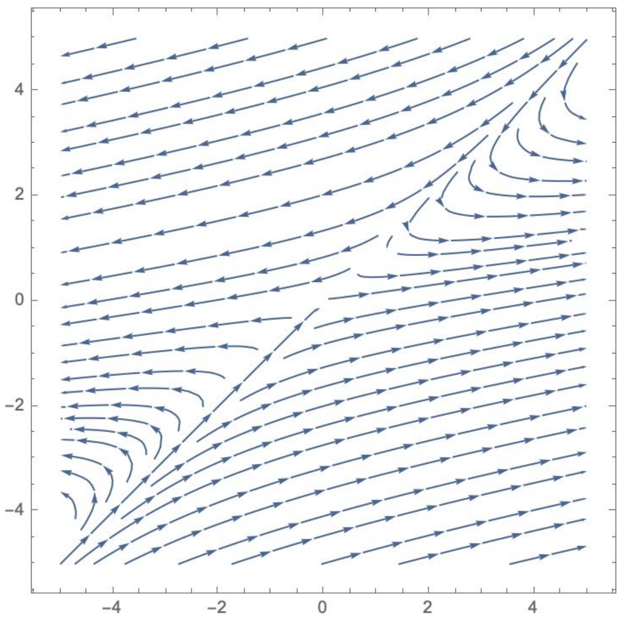
$$\begin{cases} x' = 6x - 7y = \frac{dx}{dt} \\ y' = x - 2y = \frac{dy}{dt} \end{cases} \tag{1}$$

We have $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}$.

Eigenvalues: $\lambda_1 = 5, \lambda_2 = -1$.

Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Direction field:



We have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{dy}{dx} = \frac{x-2y}{6x-7y}}$$

At some point (x, y), we can compute the slope of the line tangent to the solution curve

For example. if $(x,y) = (1,0)$, then the slope $\frac{dy}{dx} = \frac{1-2\cdot 0}{6\cdot 1 - 7\cdot 0} = \frac{1}{6}$

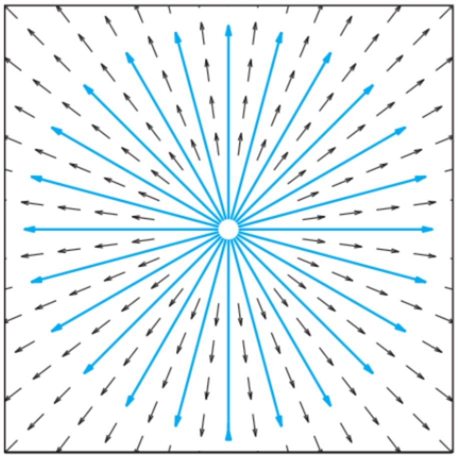
The general solution for this eqn is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 7 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

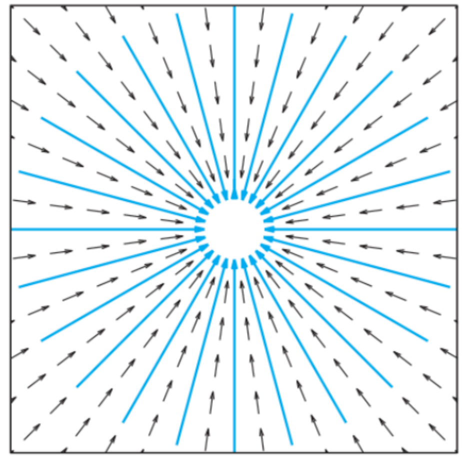
In Matlab, we use

```
[x,y] = meshgrid(-3:0.3:3,-3:0.3:3);
f1 = 6*x -7*y;
f2 = x -2*y;
quiver(x,y,f1,f2)
```

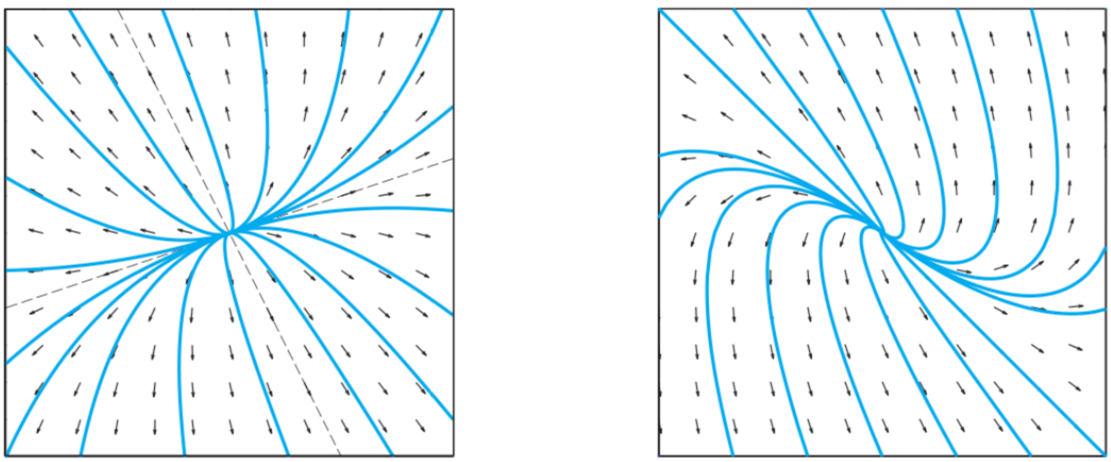
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Nodes



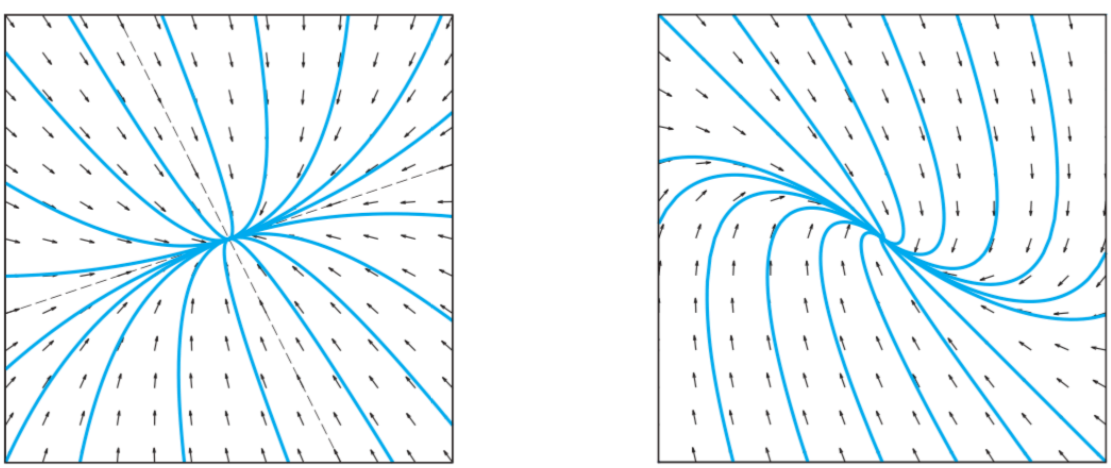
Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.



Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.

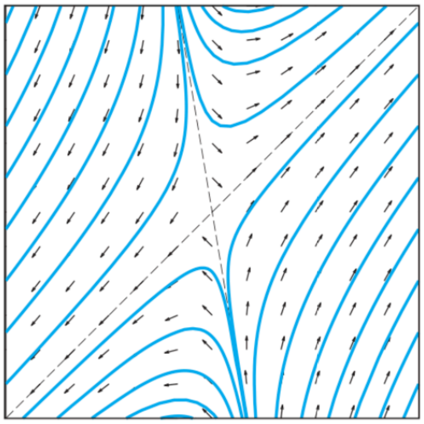


Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).

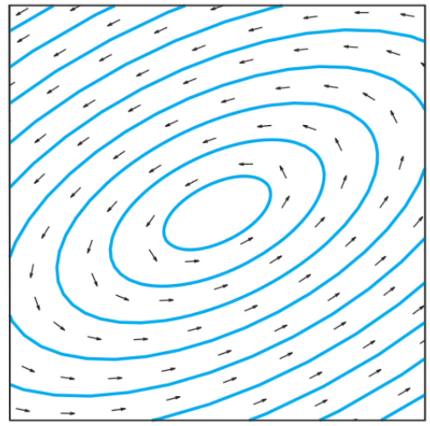


Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).

Gallery of Typical Phase Portraits for the System $\mathbf{x}' = A\mathbf{x}$: Saddles, Centers, Spirals, and Parallel Lines



Saddle Point: Real eigenvalues of opposite sign.



Center: Pure imaginary eigenvalues.

Consider

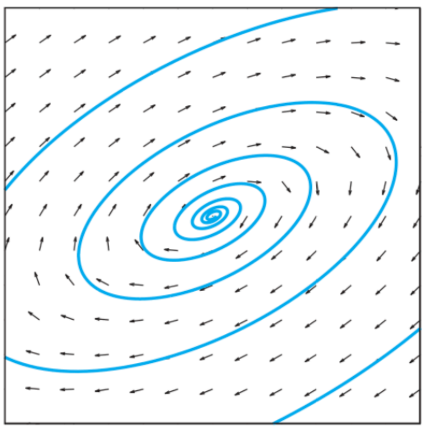
$$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$$

$$A = \begin{bmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{bmatrix}$$

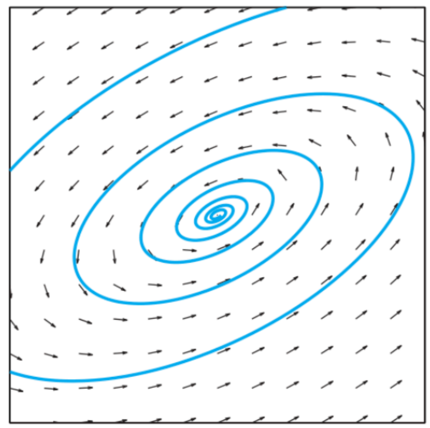
$$0 = |A - \lambda I| = \begin{vmatrix} \lambda & -2 \\ \frac{1}{2} & \lambda \end{vmatrix}$$

$$\Rightarrow \lambda^2 + 1 = 0$$

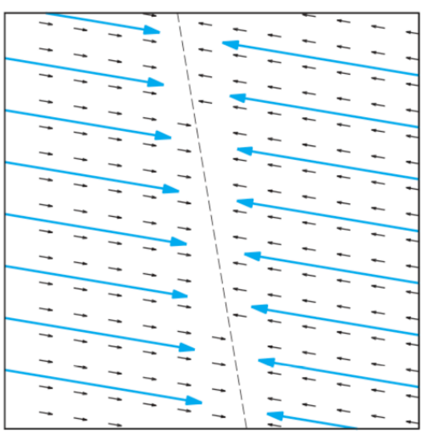
$$\lambda = \pm i$$



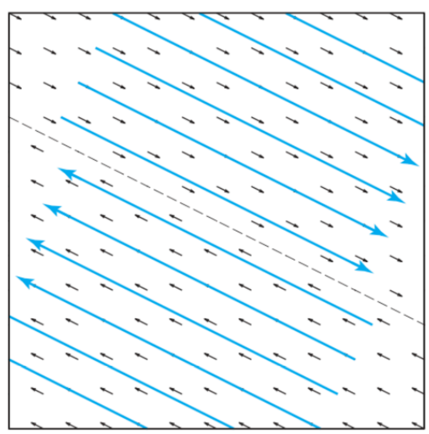
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow *away* from the dotted line.)



Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.

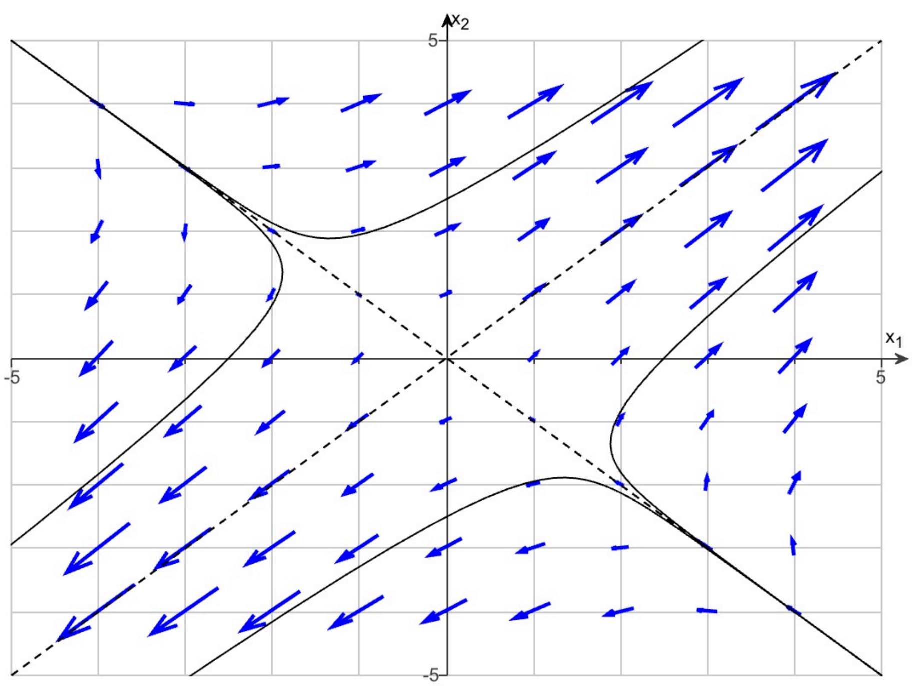
Example 2 Categorize the eigenvalues and eigenvectors of the coefficient matrix **A** according to the accompanying classifications and sketch the phase portrait of the system.

$$\mathbf{x}' = \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix} \mathbf{x} \tag{2}$$

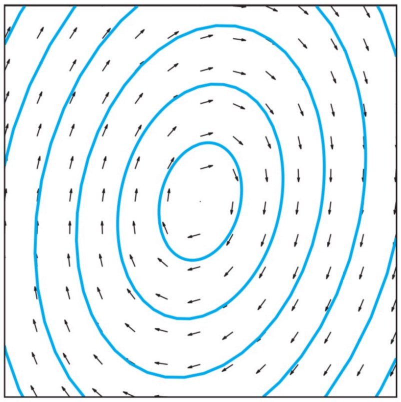
Eigenvalues: $\lambda_1 = -2, \lambda_2 = 12$

Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

The system shows a saddle point and its eigenvalues are distinct, opposite sign real.

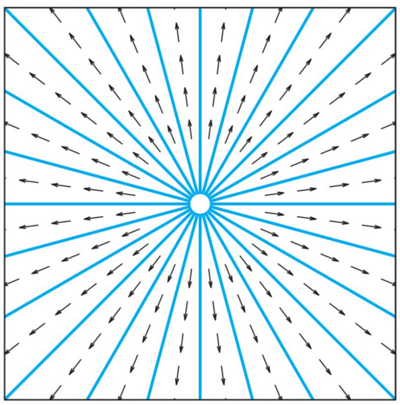


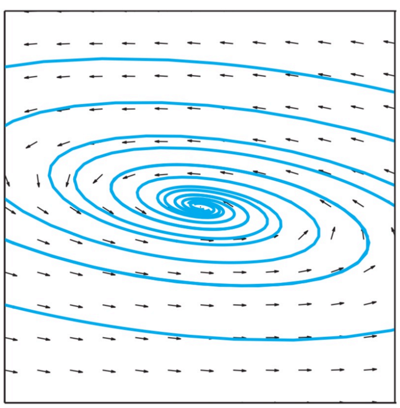
Example 3 The phase portraits the following problems correspond to linear systems of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in which the matrix $\mathbf{A}$ has two linearly independent eigenvectors. Determine the nature of the eigenvalues and eigenvectors of each system.



Center

pure imaginary eigenvalues





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Given

$$x' = \begin{pmatrix} 1 & \alpha \\ 3 & 1 \end{pmatrix} \mathbf{x} \tag{3}$$

what are the values of α if the origin is a saddle point in the phase plane?

(A) $\alpha > \frac{1}{3}$
(B) $\alpha < 0$
(C) $2 > \alpha > -2$
(D) $3 > \alpha > 1$
(E) $\alpha > \frac{1}{2}$

If the origin is a saddle point, then there are two real eigenvalues with opposite sign.

① $0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & \alpha \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 3\alpha = \lambda^2 - 2\lambda + (1-3\alpha) = 0$

$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$

Let λ_1 λ_2 be the eigenvalues of A. Then

- ① has two (distinct) real eigenvalue:

$\Delta = b^2 - 4ac = (-2) - 4(1-3\alpha) > 0$

$\Rightarrow 4 - 4 + 12\alpha > 0 \Rightarrow \alpha > 0$

- ① has two solutions with oppsite sign:

$\lambda_1 \lambda_2 = 1 - 3\alpha < 0 \Rightarrow 1 < 3\alpha \Rightarrow \alpha > \frac{1}{3}$

Thus we have $\left.\begin{matrix} \alpha > 0 \\ \alpha > \frac{1}{3} \end{matrix}\right\} \Rightarrow \alpha > \frac{1}{3}$