Midterm 1 Review of Common Ordinary Differential Equations

Separable Equations

Separable Equations: $\frac{dy}{dx} = g(x)k(y)$

Solution: $\int \frac{dy}{k(y)} = \int g(x)dx + C$

Also check if $k(y) = 0$ is a solution

Linear First-order Equations

Linear First-order Equations: $\frac{dy}{dx} + P(x)y = Q(x)$

Solution: $\rho y = \int \rho Q(x)dx$, where $\rho = e^{\int P(x)dx}$.

Applications: Mixture Problems: $\frac{dx}{dt} = r_i c_i - r_o c_o$,

where $c_o(t) = \frac{x(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_0)\, t$

Exact Equations

Exact Equations: $M(x,y)dx + N(x,y)dy = 0$, where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution: $F(x,y) = C$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

Homogeneous Equations

Homogeneous Equations: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

To identify: All $x^n y^m$ have total power $(n+m)$ the same (after rewriting).

Solution: Substitute $v = \frac{y}{x}$, then $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$
(This converts equation to a separable Diff. E.)

Bernoulli Equations

Bernoulli Equations: $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Rewrite: $y^{-n}y' + P(x)y^{1-n} = Q(x)$

Solution: $y^{1-n} = v$ and $y^{-n}y' = \frac{1}{1-n}v'$
(This converts equation to a linear Diff. E.)

Reducible Second-order Equations

Reducible Second-order Equations: $F(x, y, y'y'') = 0$

Case 1. y missing: Substitute: $p = y' = \frac{dy}{dx}$, $\quad y'' = \frac{dp}{dx}$.

Case 2. x missing: Substitute: $p = y' = \frac{dy}{dx}$, $\quad y'' = p\frac{dp}{dy}$.

Population Models

This topic was covered in Section 2.1. We talked about

- Solving the Logistic Equations.
- How solution curves behave near the equilibrium solutions

See illustrative examples from Lecture Notes Section 2.1.

Acceleration-Velocity Models

This topic was covered in Section 2.3. See the lecture notes and homework questions for examples.

Autonomous Equations and Equilibrium Solutions

Autonomous Equations: $\frac{dx}{dt} = f(x)$

Critical points: values of x such that $f(x) = 0$.
$f(x_0) = 0 \Rightarrow$ equilibrium solution at $x = x_0$
$f(x_0) < 0 \Rightarrow$ solutions go down at $x = x_0$
$f(x_0) > 0 \Rightarrow$ solutions go up at $x = x_0$

Stability of Critical Points: Phase diagram method
unstable = solutions go away (either side)
stable = solutions go towards (both sides)
semi-stable = solutions mixed

Euler's Method

Euler's Method: Consider $\dfrac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$

Euler's method with step size h: $\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot f(x_n, y_n) \end{cases}$

Existence and Uniqueness Theorem

First Order, General Initial Value Problem:

$$y' = f(x,y), \quad y(x_0) = y_0$$

- Solution exists and is unique if f and $\frac{\partial}{\partial y} f$ are continuous at (x_0, y_0).
- Solutions are defined somewhere inside the region containing (x_0, y_0), where f and $\frac{\partial}{\partial y} f$ are continuous.

Linearly Independent Functions

$f_1, \cdots, f_n$ are linearly independent if $c_1 f_1 + \cdots + c_n f_n = 0$ holds if and only if $c_1 = c_2 = \cdots = c_n = 0$.

Wronskian: $W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$.

The Wronskian of n linearly dependent functions $f_1, \cdots, f_n$ is **identically zero**.

2nd Order, Homogeneous Linear, Constant Coefficients

2nd Order, Homogeneous Linear, Constant Coefficients:

$$ay'' + by' + cy = 0$$

Characteristic Equation: $ar^2 + br + c = 0$

Solution depends on the type of roots:

- $r = r_1, r_2$(real, not repeated), $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
- $r = r_1 = r_1$ (repeated root), $y = (c_1 + c_2 x) e^{r_1 x}$.
- $r = r_{1,2} = A \pm Bi$(complex conjugates), $y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$

Higher Order, Homogeneous Linear, Constant Coefficients

Higher Order, Homogeneous Linear, Constant Coefficients: $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$

Characteristic Equation: $a_n r^n \cdots + a_1 r + a_0 = 0$

- Solution generalized from 2nd order case.
- Long division method can be used when solving char. eqn.

Reduction of Order

Consider

$$y'' + p(x)y' + q(x)y = 0,$$

with one solution $y = y_1(x)$ known.

Substitute:

$$y = v\cancel{y_1}$$
$$y' = v\cancel{y_1'} + v' y_1$$
$$y'' = v\cancel{y_1''} + 2v' y_1' + v'' y_1$$

Diff. E. becomes $(2v' y_1' + v'' y_1) + p v' y_1 = 0,$

which is separable: $\dfrac{1}{(v')}(v')' = -\left(p + \dfrac{2y_1'}{y_1}\right).$

Applications: Euler Equation: $ax^2 y'' + bxy' + cy = 0$

Additional Notes Summarized by Yourself

You can fill in this empty block to summarize the course contents that are not listed in this file.