2.2 Equilibrium Solutions and Stability

Recall in Section 2.1, we talked about the logistic equations of the form $\dfrac{dP}{dt} = kP(M-P)$. For example,

$$\frac{dx}{dt} = 4x - x^2 \tag{1}$$

is a logistic equation.

Notice that $4x - x^2 = 0 \Rightarrow x(4-x) = 0 \Rightarrow x = 0$ or $x = 4$

$x(t) \equiv 0$, and $x(t) \equiv 4$ are solutions for (1)

To solve eqn, if $4x - x^2 \neq 0$,

$$\int \frac{dx}{x(4-x)} = \int dt$$

Let $\dfrac{1}{x(4-x)}$

$$= \frac{A}{x} + \frac{B}{4-x}$$

$$= \frac{4A - Ax + Bx}{x(4-x)}$$

$$= \frac{(B-A)x + 4A}{x(4-x)}$$

$$\Rightarrow \begin{cases} B - A = 0 \\ 4A = 1 \end{cases} \Rightarrow A = B = \frac{1}{4}$$

$$\Rightarrow \frac{1}{4}\int \left(\frac{1}{x} + \frac{1}{4-x}\right) dx = \int dt$$

$$\Rightarrow \ln|x| - \ln|4-x| = 4t + c$$

$$\Rightarrow \ln\left|\frac{x}{4-x}\right| = 4t + c$$

$$\Rightarrow \frac{x}{4-x} = Ce^{4t}$$

$$\Rightarrow x = (4-x)Ce^{4t}$$

$$\Rightarrow x(1 + Ce^{4t}) = 4Ce^{4t}$$

$$\Rightarrow x(t) = \frac{4Ce^{4t}}{1 + Ce^{4t}}$$

Critical points and equilibrium solutions

Let's consider the differential equation of the form

$$\frac{dx}{dt} = f(x) \qquad (1) \tag{1}$$

- We call it an **autonomous** first order equation--one in which the independent variable t does not appear explicitly.
- The solutions of the equation $f(x) = 0$ play an important role and are called **critical points** of the autonomous differential equation $\dfrac{dx}{dt} = f(x)$.
- If $x = c$ is a critical point of Eq. (1), then the differential equation has the constant solution $x(t) \equiv c$.
- A constant solution of a differentail equation is sometimes called an **equilibrium solution**.

For example,

$$\frac{dx}{dt} = 4x - x^2 \tag{1}$$

is an autonomous differntial equation with $f(x) = 4x - x^2$.

- Setting $f(x) = 0$, we have

$$4x - x^2 = x(4 - x) = 0 \implies x = 0 \text{ or } x = 4. \tag{2}$$

- So $x = 0$ and $x = 4$ are the critical points of the above equation.
- The constant functions $x(t) \equiv 0$ and $x(t) \equiv 4$ are the equilibrium solutions to Eq. (1).

Stability of Critial Points

Consider the differential equation of the form

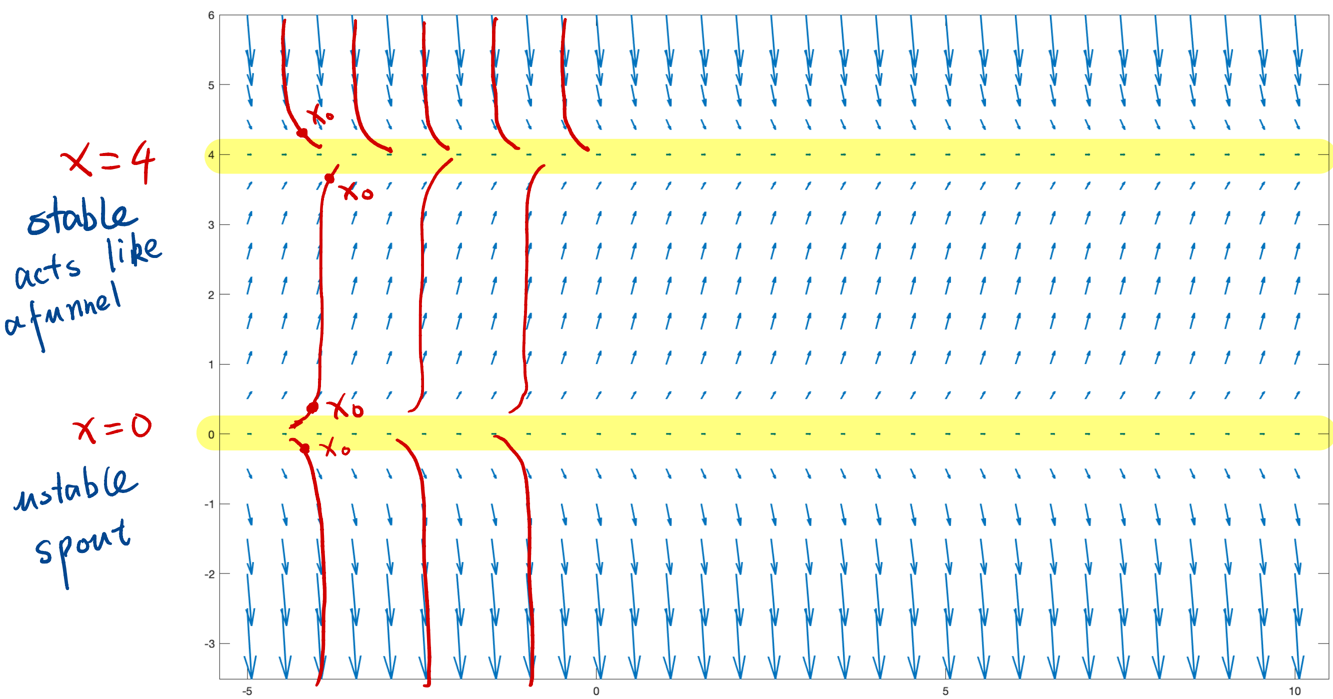
$$\frac{dx}{dt} = f(x) \qquad (1) \qquad (3)$$

- In general, a critical point $x = c$ of an autonomous first-order equation is said to be **stable** if the initial value x_0 is sufficiently close to c, then $x(t)$ remains close to c for all $t > 0$.
- The critical point $x = c$ is **unstable** if it is not stable.

Funnels and Spouts

For example, consider $\frac{dx}{dt} = 4x - x^2$. The vector field is as below.

Using the vector field, we can graph the solution curves:



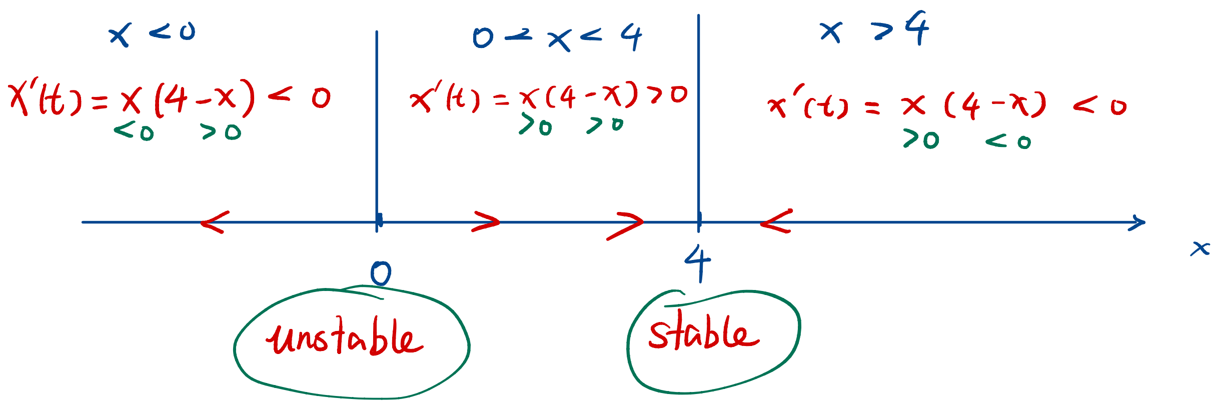
Matlab code for graphing the vector field of the equation $\dfrac{dx}{dt} = 4x - x^2$

```
[t, x]=meshgrid(-5:.5:10,-2.5:.5:6);
dx = 4*x - x.^2;
dt=ones(size(dx));
quiver(t,x,dt,dx,1.5,'LineWidth',1.5)
axis equal
```

Phase diagram

We can summarize the behavior of the solution by the phase diagram, which indicates the direction(phase) of change of x as a function of x itself.

$$(x'(t)) = \frac{dx}{dt} = 4x - x^2 = x(4-x)$$



In example 1 and 2,

- (1) first solve the equation $f(x) = 0$ to find the critical points of the given autonomous differential equation $\dfrac{dx}{dt} = f(x)$.
- (2) Then analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation.
- (3) Next, solve the differential equation explicitly for $x(t)$ in terms of t.
- Finally, (4) use a computer system or graphing calculator to plot a slope field and solution curves of the equation. Choose the correct graph below.

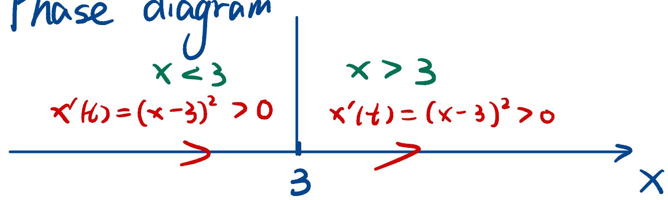
Example 1 $\dfrac{dx}{dt} = (x-3)^2$

ANS: (1) Let $f(x) = (x-3)^2 = 0$

$\Rightarrow$ $x = 3$ is the critical point

(2) & (4)

Phase diagram



$x = 3$ is semi-stable (unstable)

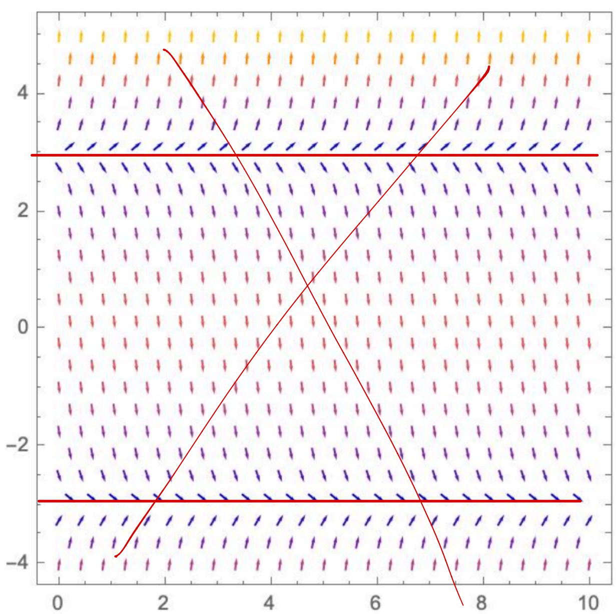
(3) If $(x-3) \neq 0$, we have

$$\int \frac{dx}{(x-3)^2} = \int dt$$

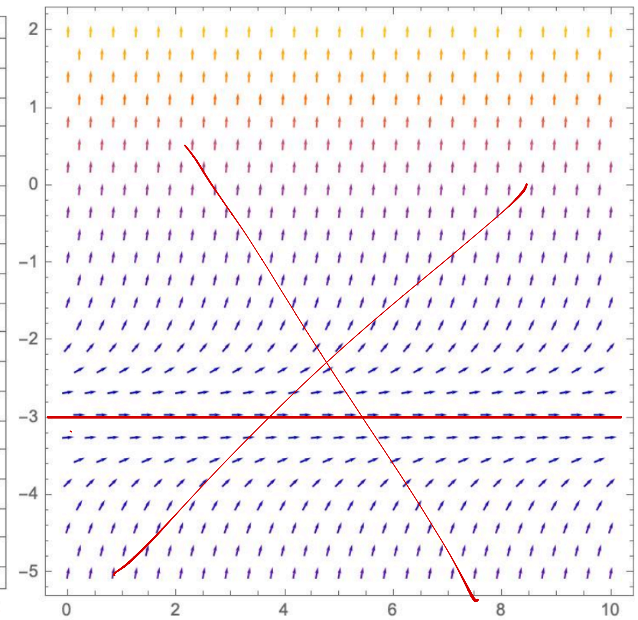
$$\Rightarrow \int (x-3)^{-2}\, d(x-3) = \int dt$$

$$\Rightarrow \frac{1}{x-3} = -t + C$$

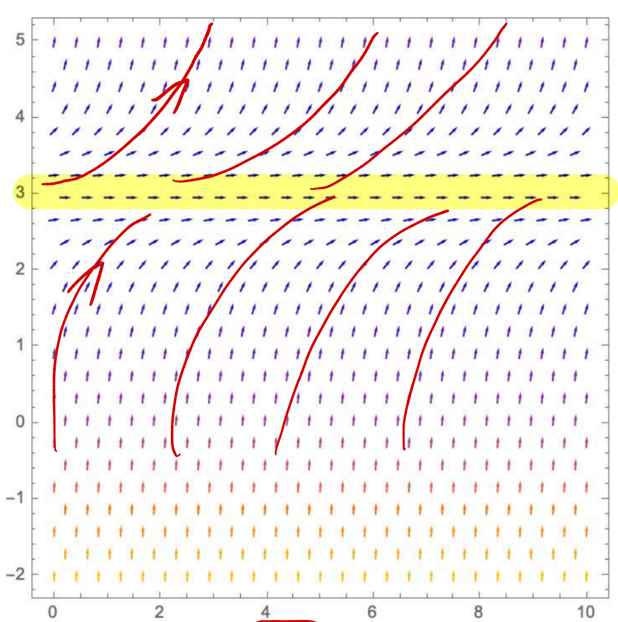
Choose the slope field of $\dfrac{dx}{dt} = (x-3)^2$ below.



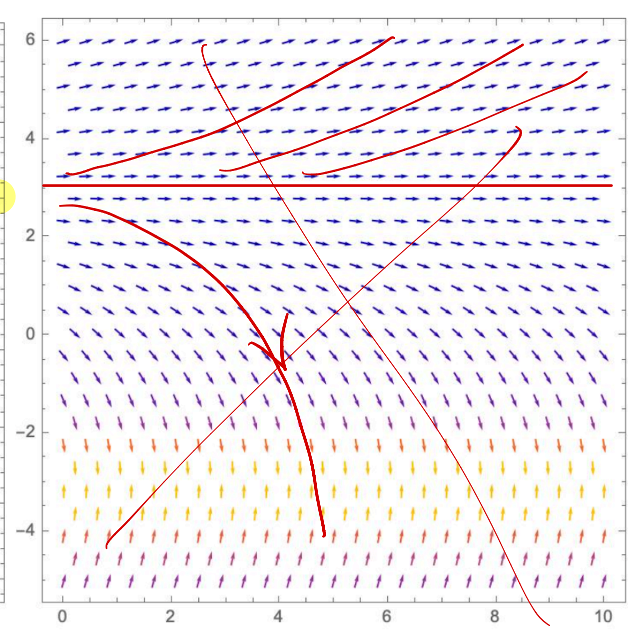
(A)



(B)



(C)



(D)

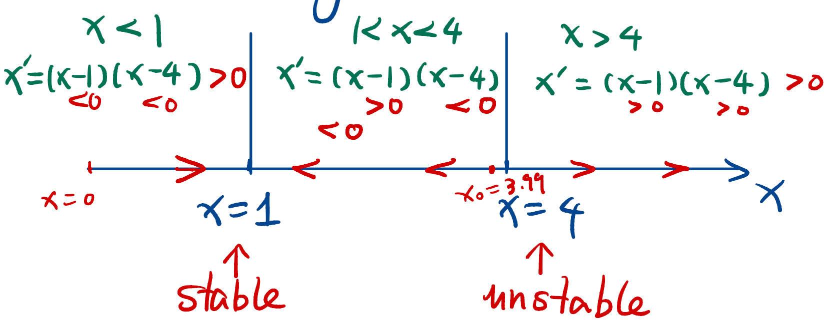
Example 2. $\overset{=x'}{\frac{dx}{dt}} = x^2 - 5x + 4$

ANS: (1) Let $f(x) = x^2 - 5x + 4$

$$= (x-1)(x-4) = 0$$

$\Rightarrow x = 1$ and $x = 4$ are critical points.

(2) Phase diagram



(3) If $(x-1)(x-4) \neq 0$.

$$\int \frac{dx}{(x-1)(x-4)} = \int dt$$

Let $\frac{1}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4}$

$$= \frac{Ax - 4A + Bx - B}{(x-1)(x-4)} = \frac{(A+B)x - 4A - B}{(x-1)(x-4)}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -4A - B = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = \frac{1}{3} \end{cases}$$

Thus

$$\frac{1}{3}\int \left(\frac{1}{x-4} - \frac{1}{x-1}\right) dx = \int dt$$

$$\Rightarrow \ln\left|\frac{x-4}{x-1}\right| = 3t + C$$

$$\Rightarrow \frac{x-4}{x-1} = ce^{3t}$$

Additional Questions:

Let $x' = x^2 - 5x + 4$,

1). If $x(0) = 0.001$, what is $\lim_{t\to\infty} x(t) = ?$ 1

2) If $x(0) = 3.99$, what is $\lim_{t\to\infty} x(t) = ?$ 1

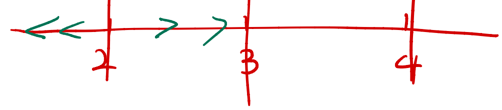
3) If $x(0) = 4.001$, what is $\lim_{t\to\infty} x(t) = ?$ ∞

See also #8 from Spring 2018 Final

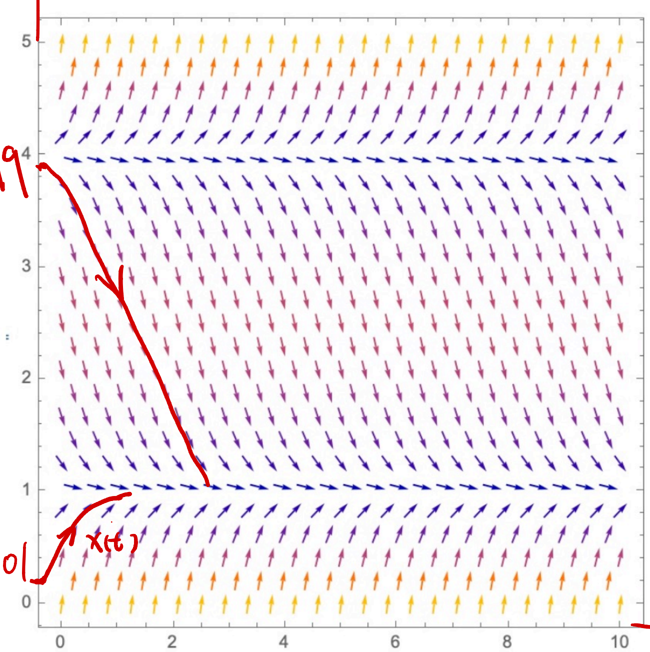
$\overset{>0}{} \quad \overset{<0}{} \quad \overset{<0}{}$

$\frac{dx}{dt} = (x-2)(x-3)(x-4) > 0$

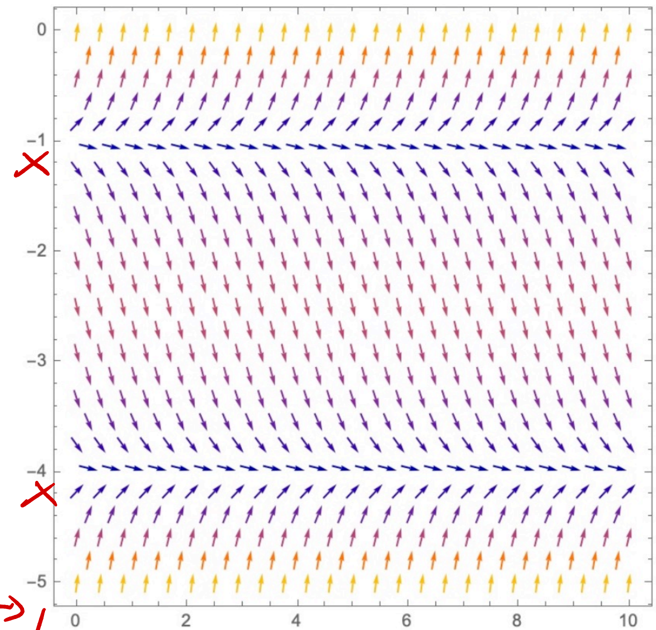
$x<2 \quad 2<x<3 \quad 3<x<4 \quad x>4$



Choose the slope field of $\dfrac{dx}{dt} = x^2 - 5x + 4$ below.



(A)



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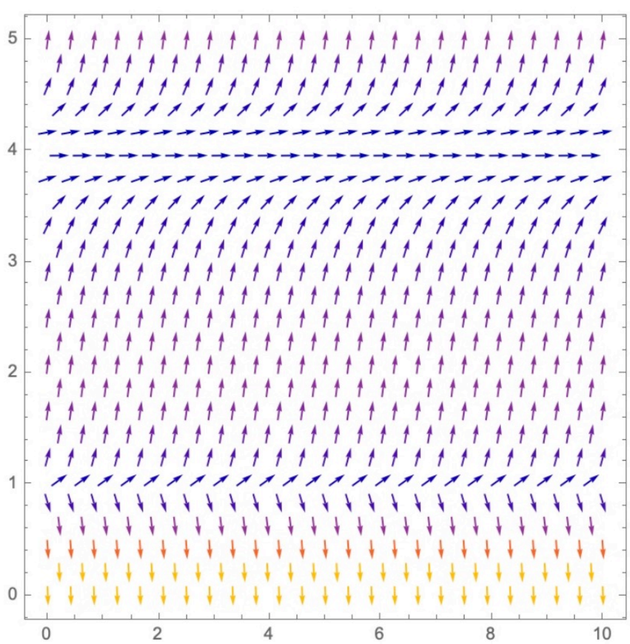

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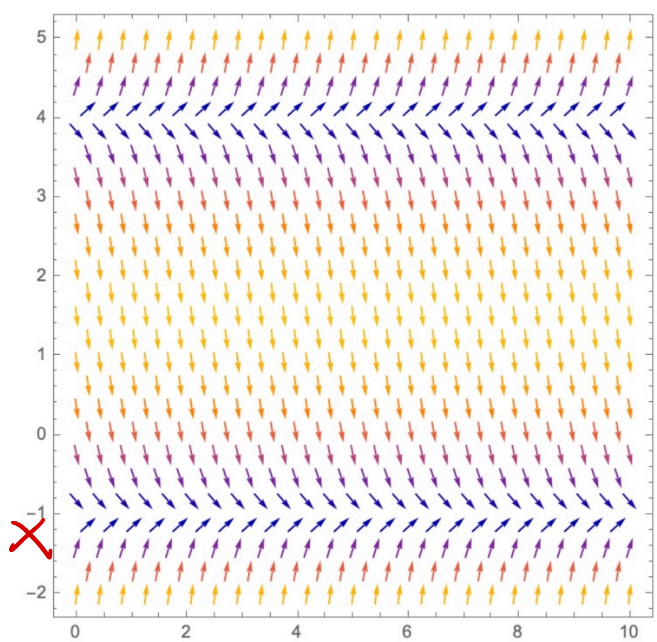
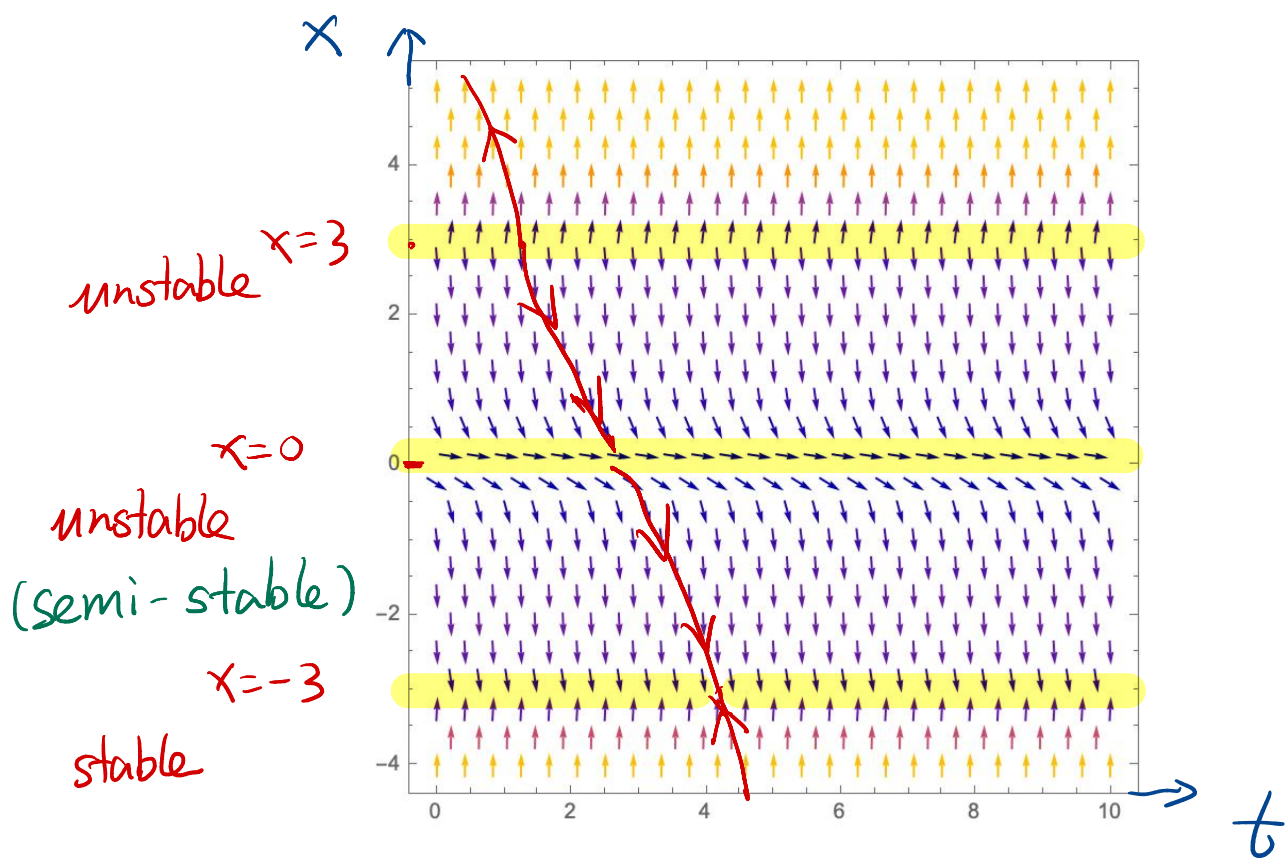


(D)

Example 3 Use a computer system or graphing calculator to plot a slope field and/or enough solution curves to indicate the stability or instability of each critical point of the given differential equation.

$$\frac{dx}{dt} = x^2(x^2 - 9) = x^2(x+3)(x-3)$$

critical points are $x=0$, $x=3$, $x=-3$



A biological/physical system modeled by a differential equation may depend on the values of certain coefficients. For example, the number of critical points of the equaiton may change as the value of a coefficient is changed.

Let's consider the following model.

Harvesting/Stocking a Logistic Population

The autonomous differential equation

$$\frac{dx}{dt} = ax - bx^2 \pm h \tag{4}$$

(with a, b, h positive) may be considered to describe a logistic polulation with harvesting/stocking.

Example 4 The differential equation $\frac{dx}{dt} = \frac{1}{100}x(x-5) + s$ ① models a population with stocking at rate s. Determine the dependence of the number of critical points c on the parameter s, and then construct the corresponding **bifurcation diagram** in the sc-plane.

ANS: To find the critical points, we set

$$f(x) = \frac{1}{100} x(x-5) + s = 0$$

$$\Rightarrow \quad x^2 - 5x + 100s = 0$$

$$\Rightarrow \quad x = \frac{5 \pm \sqrt{25 - 4 \times 100 s}}{2}$$

Recall

$ax^2 + bx + c = 0$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So the critical points are given by.

$$c = x = \frac{5 \pm 5\sqrt{1-16s}}{2}$$

- If. $1-16s < 0$, x has no (real-valued) solution.

  So if $s > \frac{1}{16}$, ① has no critical points.

- If $1-16s = 0$, x has only one solution $x = \frac{5}{2}$

  So if $s = \frac{1}{16}$, ① has one critical pt $c = \frac{5}{2}$.

- If $1-16 > 0$, x has two solutions.

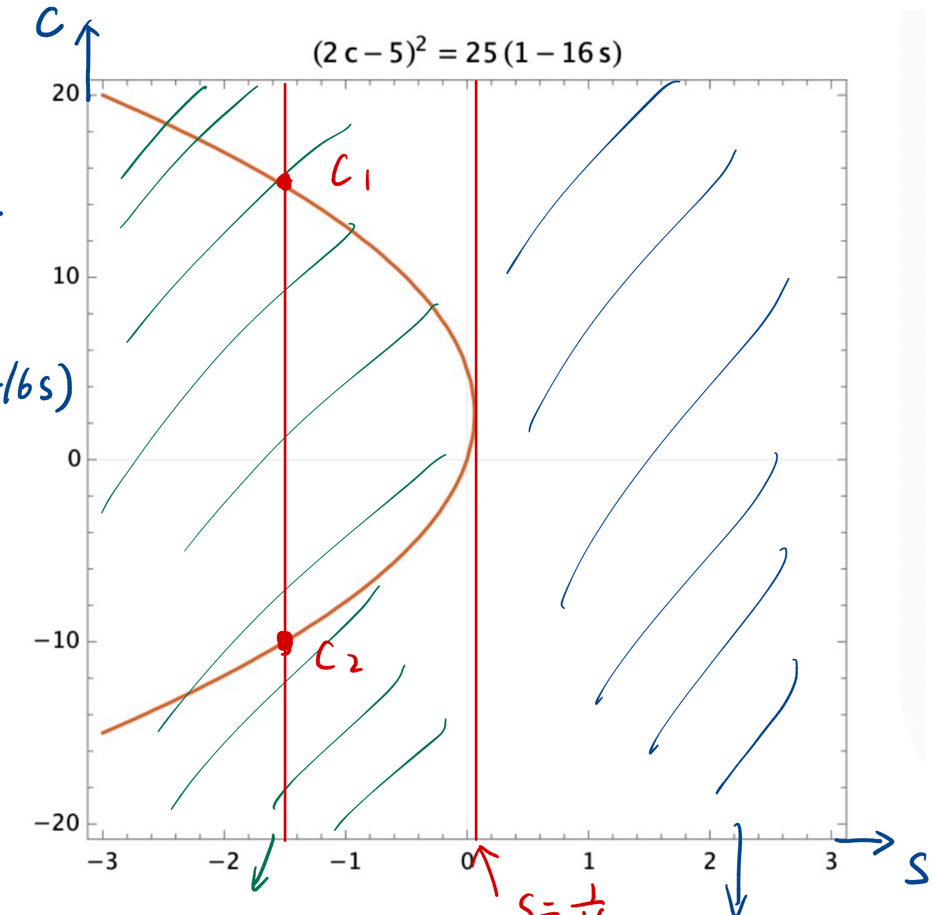
  So if $s < \frac{1}{16}$, ① has two critical pts.

A common way to visualize the bifurcation in the solution is to plot the **bifurcation diagram** with all points (s, c), where c is a critical point of the equation.

We need to graph.

$$c = \frac{5 \pm 5\sqrt{1-16s}}{2}$$

$$\Rightarrow (2c-5)^2 = 25(1-16s)$$



If $s < \frac{1}{16}$

two critical pts

only one critical pt.

If $s > \frac{1}{16}$

no critical pts.