Midterm 2 Review

Note: A video will be uploaded to discuss these questions. We will meet on Nov. 9th Tuesday. I will talk about some comments on these questions and related subjects for around 30 minutes. The rest time for the lecture will be leaving for Q&A.

1. (Spring 2015 Spring Q13) §3.4

   An undamped forced oscillation $u'' + 4u = 0$ has initial conditions $u(0) = 4, u'(0) = 6$. The solution of the initial value problem can be written as $u = R\cos(\omega t - \delta)$. What are R and δ?

   A. $R = 5, \delta = \frac{\pi}{4}$

   B. $R = 4, \delta = \frac{\pi}{4}$

   C. $R = 2\sqrt{13}, \delta = \tan^{-1}\left(\frac{1}{2}\right)$

   D. $R = 5, \delta = \tan^{-1}\left(\frac{3}{4}\right)$

   E. $R = \sqrt{13}, \delta = \tan^{-1}\left(\frac{1}{2}\right)$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$u(t) = A\cos 2t + B\sin 2t$

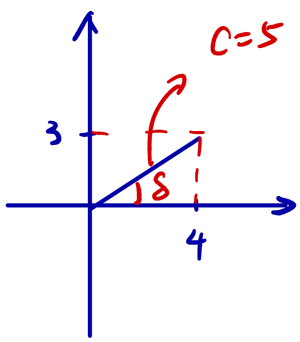
Since $u(0) = 4$, $A = 4$

Since $u'(0) = 6$, $u'(t) = -2A\sin 2t + 2B\cos 2t$

$u'(0) = 2B = 6 \Rightarrow B = 3$

Thus

$u(t) = 4\cos 2t + 3\sin 2t$



$R = \sqrt{4^2 + 3^2} = 5$

$\tan\delta = \frac{3}{4}$

δ lies in the 1st quadrant.

$\delta = \tan^{-1}\frac{3}{4}$

2. (2018 Spring Final Q14)

Note that $y_1(t) = \sqrt{t}$ and $y_2(t) = t^{-1}$ are solutions of the linear homogeneous differential equation

$$2t^2y'' + 3ty' - y = 0$$

Use variation of parameters to find the general solution of the nonhomogeneous differential equation

$$2t^2y'' + 3ty' - y = 4t^2 + 4t \quad ①$$

A. $C_1\sqrt{t} + C_2t^{-1} + \underline{\frac{4}{9}t^2} + \underline{2t}$

B. $C_1\sqrt{t} + C_2t^{-1} + \frac{4}{9}t^3 + 2t^2$

C. $C_1\sqrt{t} + C_2t^{-1} + \frac{8}{35}t^4 + \frac{2}{5}t^3$

D. $C_1\sqrt{t} + C_2t^{-1} + \frac{8}{35}t^2 + \frac{2}{5}t$

E. $C_1\sqrt{t} + C_2t^{-1} + \underline{\frac{4}{9}t^2} + \frac{2}{5}t$

Variation of Parameters:

$$y'' + P(x)y' + Q(x)y = f(x)$$

Note we need standard form like this!

homogeneous solution $y_c(x) = c_1y_1(x) + c_2y_2(x)$ known.

Then a particular solution is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

Wronskian: $W(x) = y_1y_2' - y_2y_1'$.

Remark: Let $u_1 = -\int \frac{y_2(x)f(x)}{W(x)}dx$ and $u_2 = \int \frac{y_1(x)f(x)}{W(x)}dx$, then the above equation becomes

$$y_p(x) = u_1y_1 + u_2y_2$$

· First we rewrite ① as the standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 2 + \frac{2}{t} = f(t)$$

· As $y_1 = \sqrt{t} = t^{\frac{1}{2}}$, $y_2 = t^{-1}$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = \underline{-\frac{3}{2}t^{-\frac{3}{2}}}$$

· Compute u_1

$$u_1 = -\int \frac{y_2 f}{W}dt = -\int \frac{t^{-1}\left(2 + \frac{2}{t}\right)}{-\frac{3}{2}t^{-\frac{3}{2}}}dt$$

$$= \frac{4}{3}\int \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)dt$$

$$= \frac{4}{3}\left(\frac{1}{1+\frac{1}{2}}t^{\frac{3}{2}} + \frac{1}{1-\frac{1}{2}}t^{\frac{1}{2}}\right)$$

$$\Rightarrow \quad u_1 = \frac{8}{9}t^{\frac{3}{2}} + \frac{8}{3}t^{\frac{1}{2}}$$

· Compute u_2

$$u_2 = \int \frac{y_1 f}{W}dt = \int \frac{t^{\frac{1}{2}}(2 + 2t^{-1})}{-\frac{3}{2}t^{-\frac{3}{2}}}dt$$

$$= -\frac{4}{3}\int (t^2 + t)dt$$

$$= -\frac{4}{3}\left(\frac{1}{3}t^3 + \frac{1}{2}t^2\right)$$

$$= -\frac{4}{9}t^3 - \frac{2}{3}t^2$$

Then $y_p = u_1 y_1 + u_2 y_2$.

$$= \left(\frac{8}{9}t^{\frac{3}{2}} + \frac{8}{3}t^{\frac{1}{2}}\right)t^{\frac{1}{2}} + \left(-\frac{4}{9}t^3 - \frac{2}{3}t^2\right)t^{-1}$$

$$= \frac{8}{9}t^2 + \frac{8}{3}t - \frac{4}{9}t^2 - \frac{2}{3}t$$

$$= \frac{4}{9}t^2 + 2t$$

So $y = y_c + y_p = c_1\sqrt{t} + c_2 t^{-1} + \frac{4}{9}t^2 + 2t$

§ 3.5

3. (2018 Spring Final Q9) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used on

poly of degree 1

$$y'' + y = t + t\sin(t)$$

A. $Y(t) = At + B + t(Ct + D)\cos(t) + t(Et + F)\sin(t)$

B. $Y(t) = At + B + (\cancel{Ct} + D)\cos(t) + (Et + F)\sin(t)$

C. $Y(t) = At + B + t(Ct \cancel{+ D})(\cos(t) + \sin(t))$

D. $Y(t) = t(\cancel{At} + B) + t(Ct + D)\cos(t) + t(Et + F)\sin(t)$

E. $Y(t) = \cancel{At} + t(Bt + C)\sin(t)$

Remark: Similar questions are 2017 Fall #13, 2018 Fall #12, Online and Handwritten HW, etc. Exercise in the book

ANS:

Undetermined Coefficients:

The general nonhomogeneous n th-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Find y_p by guessing a form and then plugging into DE (x^s is chosen so that y_i's are not terms of y_c)

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$

The char. eqn for $y'' + y = 0$ is

$r^2 + 1 = 0 \Rightarrow r = \pm i$. Then $y_c = C_1\cos t + C_2\sin t$

Thus

$$Y(t)\,(= y_p(t)) = At + B + t^1(Ct + D)\sin t + t^1(Et + F)\cos t$$

as sint cost appear once in y_c.

§5.2

4. (Spring 2019 Final Q17) Find the general solution of the following system

$$x' = \begin{pmatrix} 2 & 7 \\ 1 & -4 \end{pmatrix} x$$

A

(A) $\mathbf{x}(t) = \cancel{c_1 e^{2t} \begin{pmatrix} 7 \\ 1 \end{pmatrix}} + c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(B) $\mathbf{x}(t) = \cancel{c_1 e^{2t} \begin{pmatrix} 7 \\ 1 \end{pmatrix}} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(C) $\mathbf{x}(t) = \cancel{c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(D) $\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 7 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(E) $\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 7 \\ 1 & -4-\lambda \end{vmatrix}$$

$$= (\lambda+4)(\lambda-2) - 7$$

$$= \lambda^2 + 2\lambda - 8 - 7$$

$$= \lambda^2 + 2\lambda - 15$$

$$= (\lambda+5)(\lambda-3) = 0$$

$\lambda_1 = -5, \quad \lambda_2 = 3$

• $\lambda_1 = -5$, we solve

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 7 & 7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a = -b, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\vec{\mathbf{v}}$ such that $(\mathbf{A} - \lambda \mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

• $\lambda_2 = 3$. $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\Rightarrow \begin{bmatrix} -1 & 7 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -a + 7b = 0$$

Let $b = 1$, then $a = 7$ $\quad \vec{V}_2 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

5. (Fall 2009 Final Q21) §.5.2

The general solution of the system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$ is

A. $c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{-t}$

B. $\cancel{c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-4t}} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$

C. $\cancel{c_1 \begin{pmatrix} \cos t \\ 2\cos t - \sin t \end{pmatrix}} e^{t} + c_2 \begin{pmatrix} -\sin t \\ -2\sin t - \cos t \end{pmatrix} e^{t}$

D. $\cancel{c_1 \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{4t}} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

E. $c_1 \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix} e^{-t}$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix}$$

$$= (\lambda+3)(\lambda-1) + 5$$

$$= \lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm i$$

$\cdot\ \lambda = -1+i$, we find an eigenvector.

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (2-i)a - b = 0 & ① \\ 5a - (2+i)b = 0 & ② \end{cases}$$

Note $(2+i) \times ① = ②$

Assume $a = 1$, then $b = 2-i$.

So $\vec{v} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

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We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1 \vec{\mathbf{x}}_1(t) + c_2 \vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda \mathbf{I})^2 \vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda \mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)e^{\lambda t}$.

Then

$$\vec{v}e^{\lambda t} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{(-1+i)t}$$

Recall
$$e^{(A+Bi)t} = e^{At}(\cos At + i\sin At)$$

$$= \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{-t}(\cos t + i\sin t)$$

$$= e^{-t}\begin{bmatrix} \cos t + i\sin t \\ 2\cos t - i\cos t + 2i\sin t + \sin t \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix}}_{\vec{x}_1(t)} + i\cdot \underbrace{e^{-t}\begin{bmatrix} \sin t \\ -\cos t + 2\sin t \end{bmatrix}}_{\vec{x}_2(t)}$$

6. (Spring 2019 Final Q8)

Let $y(t)$ denote the unique solution to the initial value problem

$$y''' + 3y'' + 2y' = 0 \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = 1$$

What is the value of $y(1)$?

(A) $1 + 2e^{-1} + e^{-2}$

(B) $1 - e^{-2}$

(C) $1 + e^{-1}$

(D) $e + e^2$

(E) $2 + e^{-1} + 2e^{-2}$

$$r^3 + 3r^2 + 2r = 0$$

$$\Rightarrow r(r^2 + 3r + 2) = 0$$

$$\Rightarrow r(r+1)(r+2) = 0$$

$$\Rightarrow r = 0, \quad r = -1, \quad r = -2$$

$$y(t) = C_1 e^{0\cdot t} + C_2 e^{-t} + C_3 e^{-2t}$$

$$\Rightarrow y(t) = C_1 + C_2 e^{-t} + C_3 e^{-2t}$$

· As $y(0) = 2$, $\quad \underline{C_1 + C_2 + C_3 = 2}$

· As $y'(0) = -1$, $\quad y'(t) = -C_2 e^{-t} - 2C_3 e^{-2t}$

$$y'(0) = \underline{-C_2 - 2C_3 = -1}$$

· As $y''(0) = 1$, $\quad y''(t) = C_2 e^{-t} + 4C_3 e^{-2t}$

$$y''(0) = \underline{C_2 + 4C_3 = 1}$$

$\Rightarrow C_3 = 0$

· Then $C_3 = 0, \quad C_2 = 1, \quad C_1 = 1$

$$y(t) = 1 + e^{-t}$$

Then $y(1) = 1 + e^{-1}$

7. (Spring 2019 Final Q19) § 5.2 Eigenvalue Method

Given that

$$\vec{x} = e^{-t}\begin{pmatrix}\beta\\2\end{pmatrix} = \vec{v}e^{\lambda t} \Rightarrow \begin{cases}\lambda = -1 \text{ is an eigenvalue} \\ \begin{pmatrix}\beta\\2\end{pmatrix} \text{ is an eigenvector} \\ \text{corresponds to } \lambda = -1.\end{cases}$$

is a solution to

$$\mathbf{x}' = \underbrace{\begin{pmatrix}-3 & 5\\-2 & 4\end{pmatrix}}_{A}\mathbf{x}$$

Find β.

ANS: As $\vec{x} = e^{-t}\begin{pmatrix}\beta\\2\end{pmatrix}$ is a solution.

A. 8

We know

B. 5

- $\lambda = -1$ is an eigenvalue of A

C. 18

- $\begin{pmatrix}\beta\\2\end{pmatrix}$ satisfies. $(A-(-1)I)\cdot\begin{pmatrix}\beta\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ by def of eigenvector.

D. 2

E. 3

Thus we have

$$(A-(-1)I)\begin{pmatrix}\beta\\2\end{pmatrix} = \begin{bmatrix}-3+1 & 5\\-2 & 4+1\end{bmatrix}\begin{bmatrix}\beta\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-2 & 5\\-2 & 5\end{bmatrix}\begin{bmatrix}\beta\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow -2\beta + 10 = 0 \quad \Rightarrow \beta = 5$$

8. (2010 Spring Final Q19) From § 5.3. Note this section won't be tested. See the notes for a similar example.

Consider the system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$$

For what values of α is the equilibrium solution $\mathbf{x} = 0$ a saddle point?

9. (2018 Spring Final Q19)

Find the general solution to the system

$$\mathbf{x}' = \underbrace{\begin{pmatrix} -4 & 4 \\ -1 & -8 \end{pmatrix}}_{A}\mathbf{x}$$

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\vec{\mathbf{v}}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

ANS: $0 = |A - \lambda I| = \begin{vmatrix} -4-\lambda & 4 \\ -1 & -8-\lambda \end{vmatrix} = (\lambda+4)(\lambda+8) + 4 = \lambda^2 + 12\lambda + 36$

$= (\lambda+6)^2 = 0$

$\Rightarrow \lambda = -6$ with multiplicity 2.

Note $(A - \lambda I)\vec{v} = \begin{bmatrix} -4+6 & 4 \\ -1 & -8+6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus $\begin{cases} 2a + 4b = 0 & ① \\ -a - 2b = 0 & ② \end{cases}$ Note $-2 \times ② = ①$

So $a + 2b = 0$. If we assume $b = 1$, then $a = -2$.

Then $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponds to eigenvalue $\lambda = -6$.

This means $\vec{x}_1(t) = \vec{v}e^{\lambda t} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-6t}$ is a solution.

★ We need to find another solution $\vec{x}_2(t)$ that is linearly independent to $\vec{x}_1(t)$

We can use the algorithm in the notes: (Please check the discussion of the algorithm in the notes and the remarks)

- First we find $\vec{v}_2$ s.t.

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \vec{v}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ Note this means there is no restriction for a, b.

So we can choose $a=1$ $b=0$. $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then we compute.

$$\vec{v}_1 = (A-\lambda I)\,\vec{v}_2$$

$$= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector to $\lambda=-6$.

- Note this $\vec{v}_1$ is $-\vec{v}$ we found earlier. where $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponds to $\lambda = -6$.
- It doesn't matter because we are using the linear combination of $\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-6t}$ and $\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2)e^{\lambda t}$ as the general solution.

Thus the general solution is

Note you need $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ here

$$\vec{x}(t) = C_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-6t} + C_2 \begin{bmatrix} 2t + 1 \\ -t \end{bmatrix} e^{-6t}$$

10. (2019 Spring Final Q9)

Find the general solution to the second order differential equation

$$y'' - 2y' + 5y = 20\sin t$$

ANS: Recall the general solution has the form

$$y(t) = y_c(t) + y_p(t)$$

where y_c is the general solution to the corresponding homogeneous eqn and y_p is a particular solution

- $r = r_{1,2} = A \pm Bi$(complex conjugates), $y = e^{Ax}\left(c_1 \cos Bx + c_2 \sin Bx\right)$

- Find y_c

$$r^2 - 2r + 5 = 0$$

$$\Rightarrow r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Thus $y_c = e^t \left(C_1 \cos 2t + C_2 \sin 2t\right)$

- Find y_p

Assume $y_p = A\cos t + B\sin t$

Then $y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$$y_p'' - 2y_p' + 5y_p = \underline{-A\cos t} - B\sin t + 2A\sin t \underline{-2B\cos t}$$

$$\underline{+5A\cos t} + 5B\sin t$$

$$= (-2B + 4A)\cos t + (4B + 2A)\sin t$$

$$= 20\sin t$$

$$\Rightarrow \begin{cases} -2B + 4A = 0 \\ 4B + 2A = 20 \end{cases} \Rightarrow \begin{cases} -2B + 4A = 0 \\ 2B + A = 10 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 4 \end{cases}$$

So $y_p = 2\cos t + 4\sin t$

Thus

$$y(t) = y_c(t) + y_p(t)$$

$$= e^{t}(C_1 \cos 2t + C_2 \sin 2t) + 2\cos t + 4\sin t$$

is the general solution to the given equation.