

# Lecture Plan for Next Week

## Math 4B

 I will be traveling to Canada for a workshop next week (May 6th to May 10th). Here is the plan for next week's MATH 4B:

-  In-person Lectures will be replaced by online prerecorded videos.
  - Three prerecorded lectures will be uploaded and linked to Canvas before each lecture. The notes will also be uploaded.
- Your sections with Teaching Assistants will be same as usual.
- I will post information and practice for Midterm 2 by the end of Friday, May 10.
-  Next week's office hours will be moved to 2-5 pm Wednesday, May 15, two days before the Midterm 2.

# Lecture 13. Higher-order Linear Equations

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In this section, we will generalize the results we discussed in the previous lectures. Here is an outline:

## Lecture 13. Higher-order Linear Equations

1. General Solutions of Linear Equations
  - 1.1 Linearly Independent Solutions
    - Definition of linearly dependent/independent
    - Wronskian of  $n$  functions
  - 1.2.  $n$ -th order linear differential equation
    - Homogeneous linear equation
    - Higher-order Homogeneous Equations with Constant Coefficients

## 1. General Solutions of Linear Equations

### 1.1 Linearly Independent Solutions

#### Definition of linearly dependent/independent

The  $n$  functions  $f_1, f_2, \dots, f_n$  are said to be **linearly dependent** on the interval  $I$  if there exist constants  $c_1, c_2, \dots, c_n$  not all zero such that

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

for all  $x$  in  $I$ .

The  $n$  functions  $f_1, f_2, \dots, f_n$  are said to be **linearly independent** on the interval  $I$  if they are not linearly dependent. Equivalently, they are linearly independent on  $I$  if

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

holds on  $I$  only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

**Example 1** Show directly that the given functions are linearly dependent on the real line.

(1)  $f(x) = 3, g(x) = 2 \cos^2 x, h(x) = \cos 2x$

(2)  $f(x) = 5, g(x) = 2 - 3x^2, h(x) = 10 + 15x^2$

ANS: (1) By def, we need to find  $c_1, c_2, c_3$  not all zeros such that

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

$$\Rightarrow c_1 \cdot 3 + c_2 \cdot \frac{2 \cos^2 x}{\cos 2x + 1} + c_3 \cos 2x = 0$$

$$\Rightarrow c_1 \cdot 3 + c_2 (\cos 2x + 1) + c_3 \cos 2x = 0$$

Let  $c_2 = 1, c_3 = -1$  we have

$$c_1 \cdot 3 + \cancel{\cos 2x + 1} - \cancel{\cos 2x} = 0 \Rightarrow c_1 = -\frac{1}{3}$$

Thus  $-\frac{1}{3} \cdot 3 + 1 \cdot 2 \cos^2 x - \cos 2x = 0$

so  $f(x), g(x), h(x)$  are linearly dependent by def.

(2). We need to find  $c_1, c_2, c_3$  not all zeros such that

$$c_1 f + c_2 g + c_3 h = 0$$

$$\Rightarrow c_1 \cdot 5 + c_2 (2 - 3x^2) + c_3 (10 + 15x^2) = 0$$

Let  $c_2 = 5, c_3 = 1$ , then

$$c_1 \cdot 5 + 10 - 15x^2 + 10 + 15x^2 = 0 \Rightarrow c_1 \cdot 5 = -20$$

$$\Rightarrow c_1 = -4$$

$$\text{Thus } -4 \cdot 5 + 5(2 - 3x^2) + 1 \cdot (10 + 15x^2) = 0$$

## Wronskian of $n$ functions

Suppose that the  $n$  functions  $f_1, f_2, \dots, f_n$  are all  $n - 1$  times differentiable. Then their **Wronskian** is the  $n \times n$  determinant

$$W(x) = W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

- The Wronskian of  $n$  **linearly dependent** functions  $f_1, f_2, \dots, f_n$  is **identically zero**.

Idea of the proof:

- We show for the case  $n = 2$ . The case for general  $n$  is similar.
- If  $f_1$  and  $f_2$  are linearly dependent, then  $c_1 f_1 + c_2 f_2 = 0$  (\*) has nontrivial solutions for  $c_1$  and  $c_2$  ( $c_1$  and  $c_2$  are not all zeros).
- We also have  $c_1 f'_1 + c_2 f'_2 = 0$  from (\*).
- Thus we have the linear system of equations

$$\begin{aligned} c_1 f_1 + c_2 f_2 &= 0 \\ c_1 f'_1 + c_2 f'_2 &= 0 \end{aligned}$$

- By a theorem in linear algebra, the above system of equations has nontrivial solutions for  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  if and only if the determinant of the coefficient matrix is 0, that is,

$$\begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix} = 0$$

- So to show that the functions  $f_1, f_2, \dots, f_n$  are **linearly independent** on the interval  $I$ , it suffices to show that their Wronskian is **nonzero at just one point of  $I$** .

## 1.2. $n$ -th order linear differential equation

The general  **$n$ th-order linear** differential equation is of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

We assume that the coefficient functions  $P_i(x)$  and  $F(x)$  are continuous on some open interval  $I$ .

### Homogeneous linear equation

Similar to Lecture 3, we consider the **homogeneous linear equation**

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2)$$

### Theorem 1 Principle of Superposition for Homogeneous Equations

Let  $y_1, y_2, \dots, y_n$  be  $n$  solutions of the homogeneous linear equation (1) on the interval  $I$ . If  $c_1, c_2, \dots, c_n$  are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution of Eq. (1) on  $I$ .

### Theorem 4 General Solutions of Homogeneous Equations

Let  $y_1, y_2, \dots, y_n$  be  $n$  linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (1)$$

on an open interval  $I$  where the  $p_i$  are continuous. If  $Y$  is any solution of Eq. (1), then there exist numbers  $c_1, c_2, \dots, c_n$  such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all  $x$  in  $I$ .

**Example 2** Use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

$$f(x) = e^x, \quad g(x) = \cos x, \quad h(x) = \sin x; \quad \text{the real line}$$

**Remark:**  $3 \times 3$  matrix determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

**ANS:** By the previous page, we know it suffices to show that  $W(f, g, h) \neq 0$  at just one point on the real line.

$$W(f, g, h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix}$$

$$= e^x (\cancel{\sin^2 x + \cos^2 x}) - \cos x (-e^x \sin x - e^x \cos x) + \sin x (-e^x \cos x + e^x \sin x)$$

$$= e^x + e^x \cancel{\cos x \sin x} + \underline{e^x \cos^2 x} - \cancel{e^x \sin x \cos x} + \underline{e^x \sin^2 x}$$

$$= e^x + e^x (\cancel{\sin^2 x + \cos^2 x}) = 2e^x \neq 0 \quad \text{for all } x \in \mathbb{R}.$$

Thus  $f(x), g(x), h(x)$  are linearly independent.

## Higher-order Homogeneous Equations with Constant Coefficients

Recall in Lecture 10 and Lecture 11, we talked about 2nd-order homogeneous equations with constant coefficients of the following form

$$ay'' + by' + cy = 0$$

To solve for  $y$ , we first solve for  $r$  from the **characteristic equation**

$$ar^2 + br + c = 0,$$

which has roots  $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

**Case 1.**  $r_1, r_2$  are real and  $r_1 \neq r_2$  ( $b^2 - 4ac > 0$ ):

$$\text{General solution: } y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

**Case 2.**  $r_1, r_2$  are real and  $r_1 = r_2$  ( $b^2 - 4ac = 0$ ):

$$\text{General solution: } y = (c_1 + c_2 x) e^{r_1 x}$$

**Case 3.**  $r_1, r_2$  are complex numbers ( $b^2 - 4ac < 0$ ):

We can write  $r_{1,2} = A \pm Bi$ .

$$\text{General solution: } y = e^{Ax} (c_1 \cos Bx + c_2 \sin Bx)$$

In this lecture, we will discuss how to solve the general homogeneous equations with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad (1)$$

Similar to 2nd-order homogeneous equations, we look at the corresponding **characteristic equation**:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0 \quad (2)$$

We have 3 cases of the roots for Eq (2).

1. Distinct real roots
2. Repeated real roots
3. Complex roots
  - o distinct
  - o repeated

The results in Lecture 9 and 10 can be generalized in a natural way:

### Case 1. Distinct Real Roots

If the roots  $r_1, r_2, \dots, r_n$  of Eq(2) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

### Case 2. Repeated Real Roots

If Eq (2) has repeated root  $r$  with multiplicity  $k$ , then the part of a general solution of Eq(1) corresponds to  $r$  is

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{rx}$$

### Case 3. Complex Roots

**Unrepeated complex roots:** If  $r_{1,2} = A \pm Bi$  are roots of the characteristic equation, then the corresponding part to the general solution

$$y = e^{Ax} (c_1 \cos Bx + c_2 \sin Bx)$$

### Repeated complex roots

If the conjugate pair  $a \pm bi$  has multiplicity  $k$ , then the corresponding part of the general solution has the form

$$\begin{aligned} & (A_1 + A_2 x + \dots + A_k x^{k-1}) e^{(a+bi)x} + (B_1 + B_2 x + \dots + B_k x^{k-1}) e^{(a-bi)x} \\ &= \sum_{p=0}^{k-1} x^p e^{ax} (c_p \cos bx + d_p \sin bx) \end{aligned}$$

**Example 3** Find the general solution to the given differential equation.

$$y^{(3)} - 7y'' + 12y' = 0$$

ANS: The corresponding char. eqn . is

$$r^3 - 7r^2 + 12r = 0$$

$$\Rightarrow r(r^2 - 7r + 12) = 0$$

$$\Rightarrow r(r-3)(r-4) = 0$$

$$\Rightarrow r_1 = 0, r_2 = 3, r_3 = 4$$

Thus the general solution is  $y(x) = C_1 e^{0x} + C_2 e^{3x} + C_3 e^{4x}$   
 $= C_1 + C_2 e^{3x} + C_3 e^{4x}$

**Example 4.**

A 9th order, linear, homogeneous, constant coefficient differential equation has a characteristic equation which factors as follows.

$$9 \text{ linearly independent sols} \quad (r^2 + 4r + 8)^2 r^2 (r - 1)^3 = 0 \quad \otimes$$

Write the nine fundamental solutions to the differential equation as functions of the variable  $t$ .

ANS: We consider each factor appears in Eq  $\Theta$ .

- $r^2 = 0$  implies the corresponding solutions are

$$(C_1 + C_2 t) e^{0t} = C_1 + C_2 t$$

This gives the fundamental solution

$$y_1(t) = 1, \quad y_2(t) = t$$

- $(r-1)^3 = 0$  gives 3 repeated roots. Then the corresponding solutions are

$$(C_3 + C_4 t + C_5 t^2) e^{rt}$$

This gives  $y_3(t) = e^t$ ,  $y_4(t) = t e^t$ ,  $y_5(t) = t^2 e^t$

$$\bullet (r^2 + 4r + 8)^2 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i$$

each repeated twice. The corresponding solution is

$$(C_6 + C_7 t) e^{-2t} \cos 2t + (C_8 + C_9 t) e^{-2t} \sin 2t$$

Thus  $y_6(t) = e^{-2t} \cos 2t$ ,  $y_7(t) = t e^{-2t} \cos 2t$ ,

$$y_8(t) = e^{-2t} \sin 2t, \quad y_9(t) = t e^{-2t} \cos 2t.$$

**Example 5** Find a general solution the differential equation.

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0$$

ANS: The corresponding char. eqn. is

$$r^4 + 3r^3 + 3r^2 + r = 0$$

$$\Rightarrow r(r^3 + 3r^2 + 3r + 1) = 0 \quad \textcircled{*}$$

Notice  $r = -1$  is a solution.

$$\textcircled{*} \Rightarrow r(r - (-1))(\underbrace{r^2 + 2r + 1}_{\text{Long Division of Polynomials}}) = 0$$

Long Division of Polynomials.

$$\textcircled{*} \Rightarrow r(r+1)(r^2 + 2r + 1) = 0$$

$$\Rightarrow r(r+1)^3 = 0$$

$$\Rightarrow r_1 = 0, \quad r_2 = r_3 = r_4 = -1$$

$$\begin{array}{r} r^2 + 2r + 1 \\ \hline r+1 ) r^3 + 3r^2 + 3r + 1 \\ r^3 + r^2 \\ \hline 2r^2 + 3r + 1 \\ 2r^2 + 2r \\ \hline r + 1 \\ r + 1 \\ \hline 0 \end{array}$$

Thus the general solution is

$$y(x) = C_1 e^{r_1 x} + (C_2 + C_3 x + C_4 x^2) e^{r_2 x}$$

$$\Rightarrow y(x) = C_1 e^{-x} + (C_2 + C_3 x + C_4 x^2) e^{-x}$$

**Exercise 6.** Suppose that a fourth order differential equation has a solution  $y = 9e^{4x}x \cos(x)$ .

(a) Find such a differential equation, assuming it is homogeneous and has constant coefficients.

(b) Find the general solution to this differential equation.

**Solution.**

(a) We know a solution is of the form  $y = 9e^{4x}x \cos(x)$ , so we can figure out the corresponding root  $r$  to the characteristic equation.

Note the scalar 9 does not matter as the equation is homogeneous.

The  $x$  in  $e^{4x} \textcolor{red}{x} \cos(x)$  means the solution to the characteristic equation has repeated roots and repeated twice.

The part  $e^{4x} \cos(x)$  indicates there is a pair of complex root of type  $A \pm iB$ , where  $A = 4$  and  $B = 1$ . Thus we have  $r_{1,2} = 4 \pm i$  with multiplicity 2.

Therefore the characteristic equation is

$$(r - (4 + i))^2(r - (4 - i))^2 = r^4 - 16r^3 + 98r^2 - 272r + 289 = 0$$

Thus the answer is  $r^4 - 16r^3 + 98r^2 - 272r + 289 = 0$ .

(b) By the Equation (3) after case 3, we have the general solution as

$$y = (c_1 + c_2x)e^{4x} \sin(x) + (c_3 + c_4x)e^{4x} \cos(x)$$