Review of Linear Algebra Midterm 1

Additional Notes Summarized by Yourself

You can fill in this empty block to summarize the course contents that are not listed in this file.

Reduced Row Echelon Form (RREF)

A matrix is in Reduced Row Echelon Form if it satisfies the following 4 conditions

1. All zero rows are at the bottom.
2. The first non-zero entry of every non-zero row is a 1 (leading one).
3. Leading ones go from left to right.
4. All entries above and below any leading one are zero.

If a matrix satisfies only the first 3 conditions above then we say it is in Row Echelon Form (REF).

The Row Reduction Algorithm

Step 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3: Use row replacement operations to create zeros in all positions below the pivot.

Step 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1 to 3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Step 5: Backward phase. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Steps 1-4 produce a matrix in row echelon form (REF). A fifth step produces a matrix in reduced row echelon form (RREF).

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, that is, if and only if an echelon form of the augmented matrix has no row of the form $\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$ with b nonzero.

If a linear system is consistent, then the solution set contains either
(i) a unique solution, when there are no free variables, or
(ii) infinitely many solutions, when there is at least one free variable.

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3 .

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

An equation in the form of $A\mathbf{x} = \mathbf{b}$ is called a *matrix equation.*

Theorem: If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

Theorem: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

3. The columns of A span $\mathbb{R}^m$.

4. A has a pivot position in every row.

5. $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$

Linear Combination and Span

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ is called a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

Homogeneous Linear Systems

Definition: A system of linear equations is said to be homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\iff$ the equation has at least one free variable

Parametric Vector Form

Summary: Writing a solution set (of a consistent system) in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.

2. Express each basic variable in terms of any free variables appearing in an equation.

3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Linear Independence

Definition: A set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ of vectors in $\mathbb{R}^n$ is said to be *linearly independent* if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_p = 0$. Otherwise the vectors are called *linearly dependant* (which also means that at least one of them can be written as a linear combination of the others).

Theorems:
1. A set containing only one vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v}$ is not the zero vector.
2. A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
3. An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
4. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
5. If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Sums and Scalar Multiples

Let A, B, and C be matrices of the same size, and let r and s be scalars.

- $A+B=B+A$
- $r(A+B)=rA+rB$
- $(A+B)+C=A+(B+C)$
- $(r+s)A=rA+sA$
- $A+\mathbf{0}=A$
- $r(sA)=(rs)A$

Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC)=(AB)C$ (associative law of multiplication)
- $A(B+C)=AB+AC$ (left distributive law)
- $(B+C)A=BA+CA$ (right distributive law)
- $r(AB)=(rA)B=A(rB)$ for any scalar r
- $I_m A=A=AI_n$ (identity for matrix multiplication)

Transpose of a Matrix

Definition: Given an $m \times n$ matrix A, the *transpose* of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Properties:

- $\left(A^T\right)^T=A$
- $(A+B)^T=A^T+B^T$
- For any scalar $r, (rA)^T=rA^T$
- $(AB)^T=B^T A^T$

Transformation, Domain, Codomain, Image and Range

A *transformation* (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
The set $\mathbb{R}^n$ is called the *domain* of T, and $\mathbb{R}^m$ is called the *codomain* of T.
For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the *image* of $\mathbf{x}$ (under the action of T).
The set of all images $T(\mathbf{x})$ is called the *range* of T.

Linear Transformations

Definition: A transformation (or mapping) T is *linear* if
(1) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
(2) $T(c\mathbf{u})=cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Properties: If T is a linear transformation, then
(1) $T(\mathbf{0})=\mathbf{0}$
(2) $T\left(c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p\right)=c_1 T\left(\mathbf{v}_1\right)+\cdots+c_p T\left(\mathbf{v}_p\right)$

Standard Matrix for the Linear Transformation

Let $T:\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. The *standard matrix for the linear transformation* T is

$$A=\left[\begin{array}{lll} T\left(\mathbf{e}_1\right) & \cdots & T\left(\mathbf{e}_n\right) \end{array}\right]$$

where $\mathbf{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$. A is the $m \times n$ matrix and

$$T(\mathbf{x})=A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Onto and One-to-One Linear Transformations

Onto:
- A mapping $T:\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$. This is an existence question.
- Let A be the standard matrix for T, then T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$ (if and only if A has a pivot position in every row).

One-to-One:
- A mapping $T:\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$. This is a uniqueness question.
- T is one-to-one if and only if the equation $T(\mathbf{x})=\mathbf{0}$ has only the trivial solution.
- Let A be the standard matrix for T, then T is one-to-one if and only if the columns of A are linearly independent.

Subspaces of $\mathbb{R}^n$

Definition: A *subspace* of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
(1) The zero vector is in H.
(2) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
(3) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Basis: A *basis* for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Coordinate Systems

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Col A, Nul A

Col A: - The *column space* of a matrix A is the set $\operatorname{Col} A$ of all linear combinations of the columns of A.
- The pivot columns of a matrix A form a basis for the column space of A.

Nul A: - The *null space* of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
- To test whether a given vector $\mathbf{v}$ is in Nul A, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector.
- To find a basis for $\operatorname{Nul} A$, we solve the equation $A\mathbf{x} = \mathbf{0}$ and write the solution for $\mathbf{x}$ in parametric vector form. The vectors in the parametric form give us a basis for $\operatorname{Nul} A$.
- The nullity of a matrix A is the dimension of its Nul$A$.

Dimension and Rank

Dimension: The *dimension* of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Rank: The *rank* of a matrix A, denoted by rank A, is the dimension of the column space of A.

The Rank Theorem: If a matrix A has n columns, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

The Inverse of a Matrix

Definition: Given a square matrix A its *inverse* (if it exists) is the matrix denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I$.

Find A^{-1}: (1) If the matrix is a 2×2 matrix, we use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(2) For a matrix of higher dimensions, row reduce the augmented matrix $[A \quad I]$ to get $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. If the matrix is not invertible, we will not get the identity on the left side after applying the row reduction process.

(3) We can also use the formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det A} \operatorname{adj} A,$$

where C_{ji} is a cofactor of A.

In particular, we have the (i, j)-entry of A^{-1} given by

$$(A^{-1})_{i,j} = \frac{1}{\det(A)} C_{j,i}.$$

Properties: If A is an invertible $n \times n$ matrix, then

- then A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$
- if B is $n \times n$ invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$
- The Invertible Matrix Theorem (next box).

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. $\operatorname{Col} A = \mathbb{R}^n$.
15. $\operatorname{rank} A = n$.
16. $\dim \operatorname{Nul} A = 0$.
17. $\operatorname{Nul} A = \{\mathbf{0}\}$.
18. $\det A \neq 0$.

Determinant

Minor: Given $A_{n\times n}$, the minor of entry ij is denoted by A_{ij} and is the determinant of the matrix obtained from A by removing row i and column j.

Cofactor: $C_{ij} = (-1)^{i+j} \det A_{ij}$

Determinant: Given an $n \times n$ matrix $A (n \geqslant 2)$

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

by expanding along the i^{th} row.

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

by expanding along the j^{th} column.

Properties: Given an $n \times n$ matrix A,

- if A has a zero row or zero column then $\det(A) = 0$.
- if we get matrix B by interchanging two rows of A then $\det(B) = -\det(A)$.
- if we get matrix B by multipying one row of A by $k \neq 0$ then $\det(B) = k \det(A)$.
- if we get matrix B by adding a multiple of a row to another of matrix A then $\det(B) = \det(A)$
- $\det(kA) = k^n \det(A)$
- $\det\left(A^T\right) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$

Cramer's Rule

For any invertible $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix}.$$

Then for any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n.$$

Area, Volume, and Linear Transformations

Theorem: If B is a 2×2 matrix, the area of the parallelogram determined by the columns of B is $|\det B|$.
If B is a 3×3 matrix, the volume of the parallelepiped determined by the columns of B is $|\det B|$.

Theorem: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A.
If S is a parallelogram in $\mathbb{R}^2$, then

$$\{ \text{ area of } T(S)\} = |\det A| \cdot \{ \text{ area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{ \text{ volume of } T(S)\} = |\det A| \cdot \{ \text{ volume of } S\}$$