Review of Linear Algebra Midterm 2

Additional Notes Summarized by Yourself

You can fill in this empty block to summarize the course contents that are not listed in this file.

Vector Spaces

Definition: A *vector space* is a non-empty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Examples:

1. The spaces $\mathbb{R}^n$, where $n \geq 1$.
2. The set $\mathbb{P}_n$ of polynomials of degree at most n, where $n \geq 0$.
3. The set $M_{m\times n}$ of all $m \times n$ matrices with real entries, where m and n are positive integers.
4. The set of all real-valued functions defined on a set $\mathbb{D}$.

Subspaces

Definition: A *subspace* of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Examples:

1. In every vector space V, the subsets $\{\mathbf{0}\}$ and V are subspaces.
2. A line through the origin in $\mathbb{R}^2$ or $\mathbb{R}^3$.
3. A plane through the origin in $\mathbb{R}^3$. For example, the solutions to the homogeneous equation $3x+4y+5z=0$ is a plane through the origin in $\mathbb{R}^3$.
4. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(a)=0$ for some fixed $a \in \mathbb{R}$ and positive integer n.
5. The set of all 3×3 symmetric matrices. Note we say an $n \times n$ matrix A is said to be symmetric if $A^T = A$. (Exercise 7 in the Lecture Notes §4.1).

Subspaces (continued)

Non-Examples:
1. A line in $\mathbb{R}^2$ or $\mathbb{R}^3$ not containing the origin.
2. A plane in $\mathbb{R}^3$ not containing the origin. For example, the solutions to the non-homogeneous equation $3x+4y+5z=6$ is a plane not containing the origin in $\mathbb{R}^3$. It is not a subspace of $\mathbb{R}^3$.
3. The first quadrant in $\mathbb{R}^2$.
4. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(a)=3$ for some fixed $a \in \mathbb{R}$ and positive integer n.

Basis, Dimension

Basis: A *basis* for a subspace H of a vector space V is a linearly independent set in H that spans H.

Dimension: If a vector space V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V.

Col A, Nul A, Row A

Col A:
- The *column space* of a matrix A is the set $\operatorname{Col} A$ of all linear combinations of the columns of A.
- Col A is a subspace of $\mathbb{R}^m$ if A is $m \times n$.
- The pivot columns of a matrix A form a basis for the column space of A.
- $\operatorname{rank} A = \dim \operatorname{Col} A$

Nul A:
- The *null space* of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x}=\mathbf{0}$.
- To test whether a given vector $\mathbf{v}$ is in Nul A, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector.
- To find a basis for $\operatorname{Nul} A$, we solve the equation $A\mathbf{x}=\mathbf{0}$ and write the solution for $\mathbf{x}$ in parametric vector form. The vectors in the parametric form give us a basis for $\operatorname{Nul} A$.
- The nullity of a matrix A is the dimension of its Nul$A$.

Row A:
- The set of all linear combinations of the row vectors of A is called the *row space* of A, and is denoted by Row A.
- Row A is a subspace of $\mathbb{R}^n$ if A is $m \times n$.
- If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.
- Row $A = \operatorname{Col} A^T$
- dim Row $A = \dim \operatorname{Col} A = \operatorname{rank} A$

Rank Thm: $\operatorname{rank} A +$ nullity $A =$ number of columns in A

Linearly Independent Sets

Definition: A set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ of vectors in a vector space V is said to be *linearly independent* if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_p = 0$. Otherwise the vectors are called *linearly dependant* (which also means that at least one of them can be written as a linear combination of the others).

Eigenvalues and Eigenvectors

Definition: A scalar λ is called an *eigenvalue* of A if $|\mathbf{A}-\lambda\mathbf{I}|=0$ (*characteristic equation*).
An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$.

Eigenspace: Given a particular eigenvalue λ of the n by n matrix A, the set $E=\{\mathbf{v} : (A-\lambda I)\mathbf{v}=\mathbf{0}\}$ is called the *eigenspace* of A associated with λ.

Properties:
- If $A\mathbf{x}=\lambda\mathbf{x}$, then $A^k\mathbf{x}=\lambda^k\mathbf{x}$ for any positive integer k. So λ^k is an eigenvalue for A^k. Check Practice Problems # 2 on Page 279. Solutions are on Page 282.
- If $A\mathbf{x}=\lambda\mathbf{x}$, then $s\lambda$ is an eigenvalue of sA for any real number s.
- The eigenvalues of a triangular matrix are the entries on its main diagonal.
- If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Similarity

Definition: If A and B are $n \times n$ matrices, then A *is similar to* B if there is an invertible matrix P such that $P^{-1}AP=B$, or, equivalently, $A=PBP^{-1}$.

Properties:
- Any square matrix A is similar to itself. (Reflexivity)
- A is similar to B if and only if B is similar to A. (Symmetry)
- If A is similar to B and B is similar to C, then A is similar to C. (Transitivity)
- If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).
- If A and B are similar, then $\det A = \det B$ (Example 4 in §5.2).
- Similar matrices have the same rank.

Warnings:
- It is not true that if two matrices have the same eigenvalues implies they are similar. Check the Lecture Notes §5.2 for an example.
- Similarity is not the same as row equivalence. Row operations on a matrix usually change its eigenvalues.

Diagonalization

Definition: A square matrix A is said to be *diagonalizable* if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Properties:
- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalizing A: Check Example 3 in Lecture Notes §5.3 as an exercise.
Step 1. Find the eigenvalues of A
Step 2. Find n linearly independent eigenvectors of A if A is $n \times n$. (A is not diagonalizable if this step fails.
Step 3. Construct P with the eigenvectors found in Step 2.
Step 4. Construct the diagonal matrix D with the the corresponding eigenvalues from columns of P.

Warnings:
- When A has fewer than n distinct eigenvalues, it is still possible to diagonalize A. (Example 3 in Lecture Notes §5.3)
- Diagonalizable $\not\Rightarrow$ Invertible. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable but not invertible.
- Invertible $\not\Rightarrow$ Diagonalizable. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.
- Diagonalizable $\not\Rightarrow$ no zero eigenvalues. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable and 0 is an eigenvalue.

Applications to Differential Equations

Constant Coeff. Homogeneous: $\mathbf{x}' = A\mathbf{x}$

Solution: $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots$,
where $\mathbf{x}_i$ are fundamental solutions from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for $\mathbf{x}' = A\mathbf{x}$ in §5.7:

We consider A to be 2×2, then the general solution is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$, with the fundamental solutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to review the method*)

  If $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$, $\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$.

Trajectories for the System $\mathbf{x}' = A\mathbf{x}$:

- **attractor**: A has distinct negative real eigenvalues.
- **repeller**: A has distinct positive real eigenvalues.
- **saddle point**: A has real eigenvalues of opposite sign.
- **spiral point**: A has complex conjugate eigenvalues with nonzero real parts.
- **center**: A has purely imaginary eigenvalues.

Suggested Concepts from Midterm 1 Material

Below is a list of topics from Midterm 1 that we suggest you to be familiar with for preparing Midterm 2. You can find the summary of the formula sheet via Brightspace:

- Existence and Uniqueness Theorem
- Row Reduction Method
- Matrix Equation $A\mathbf{x} = \mathbf{b}$
- Linear Combination and Span
- Transformation, Domain, Codomain, Image and Range
- Linear Transformation
- Determinant and its properties

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. Note the item 19 is new after Midterm 1.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. $\operatorname{Col} A = \mathbb{R}^n$.
15. $\operatorname{rank} A = n$.
16. $\dim \operatorname{Nul} A = 0$, i.e., nullity $A = 0$
17. $\operatorname{Nul} A = \{\mathbf{0}\}$.
18. $\det A \neq 0$.
19. The number 0 is not an eigenvalue of A.