§ 3.4 #35

We have

$$x'' + 2x' + x = 0 \quad , \ x(0) = 0, \ x'(0) = 1.$$

The charcteristic equation is

$$r^2 + 2r + 1 = 0 \Rightarrow \quad r_1 = r_2 = -1$$

Thus

$$x = (C_1 + C_2 t)e^{-t}$$

As $x(0) = 0$ and $x'(0) = 1$, we have

$$C_1 = 0 \text{ and } C_2 = 1.$$

Therefore

$$x(t) = te^{-t}$$

§ 3.5

#21. $y'' - 2y' + 2y = e^x \sin x$

The characteristic equation for the homogeneous equation $y'' - 2y' + 2y = 0$ is $r^2 - 2r + 2 = 0$

$$\Rightarrow r = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$$

Thus

$$y_c = e^x (C_1 \cos x + C_2 \sin x).$$

As $e^x \sin x$ is a solution to the homogeneous equation. we assume

$$y_p = x e^x (A \cos x + B \sin x)$$

#22 $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$. (1)

The characteristic equation for the corresponding homogeneous equation is

$$r^5 - r^3 = 0 \Rightarrow r^3(r^2 - 1) = 0$$

Thus $r_1 = r_2 = r_3 = 0$ and $r_4 = 1$, $r_5 = -1$

Then

$$y_c = (C_1 + C_2 x + C_3 x^2)\overset{1}{\cancel{e^{0x}}} + C_4 e^x + C_5 e^{-x}$$

We can rewrite Eq (1) as the following two equations:

$$y^{(5)} - y^{(3)} = e^x \qquad (2)$$

$$y^{(5)} - y^{(3)} = 2x^2 - 5 \qquad (3)$$

For Eq (2), since e^x is a solution to the corresponding homogeneous equation with multiplicity 1, we have

$$y_{p_1} = Axe^x$$

For Eq (3), since $e^{0x} = 1$ is a solution to the corresponding homogeneous equation with multiplicity 3, we have

$$y_{p_2} = x^3(B + Cx + Dx^2)$$

Therefore, we assume

$$y_p = y_{p_1} + y_{p_2} \Rightarrow y_p = Axe^x + Bx^3 + Cx^4 + Dx^5$$

#24

$$y^{(3)} - y'' - 12y' = x - 2xe^{-3x}$$

The corresponding homogeneous equation is

$$r^3 - r^2 - 12r = 0 \Rightarrow r(r^2 - r - 12) = r(r-4)(r+3) = 0$$

$\Rightarrow r_1 = 0,\ r_2 = -3,\ r_3 = 4.$

Thus

$$y_c = C_1 e^{0x} + C_2 e^{-3x} + C_3 e^{4x}$$
$$= C_1 + C_2 e^{-3x} + C_3 e^{4x}$$

We can break down the given eqn. as the following:

$$y^{(3)} - y'' - 12y' = x \quad ①$$
$$y^{(3)} - y'' - 12y' = -2xe^{-3x} \quad ②$$

Consider Eq ①, since $e^{0x} = 1$ is a solution to the homogeneous equation with multiplicity 1, we assume

$$y_{p_1} = x(Ax + B)$$

Consider Eq ②, since e^{-3x} is a solution to the homogeneous equation with multiplicity 1, we assume

$$y_{p_2} = x(Cx + D)e^{-3x}$$

Thus $y_p = y_{p_1} + y_{p_2} = Ax^2 + Bx + (Cx^2 + Dx)e^{-3x}$

#29 $(D-1)^3(D^2-4)y = xe^x + e^{2x} + e^{-2x}$

The characteristic equation for the homogeneous equation

$(D-)^3(D^2-4)y=0$ is $(r-1)^3(r^2-4)=0$.

Thus $r_1 = r_2 = r_3 = 1$ and $r_4 = 2$, $r_5 = -2$

Similarly to the previous two question, we can

assume

$$y_p = x^3(Ax+B)e^x + xCe^{2x} + xDe^{-2x}$$

#54

Use the method of variation of parameters to find a particular solution of the given differential equation

$$y'' + y = \csc^2 x \left(= \frac{1}{\sin^2 x}\right)$$

As $r^2+1=0 \Rightarrow r = \pm i$. Then $y_c = C_1 \cos x + C_2 \sin x$.

Thus $y_1 = \cos x$ and $y_2 = \sin x$.

Then

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

So $u_1(x) = -\int \frac{y_2(x)\ f(x)}{W(x)}\,dx$

$$= -\int \frac{\sin x \cdot \frac{1}{\sin^2 x}}{1}\, dx$$

$$= -\int \csc x \, dx = -\ln\left|\tan\frac{x}{2}\right| = \ln\left|\cot\frac{x}{2}\right|$$

$$u_2(x) = \int \frac{y_1(x) f(x)}{W(x)}\, dx = \int \frac{\cos x \cdot \frac{1}{\sin^2 x}}{1}\, dx$$

$$= \int \frac{1}{\sin^2 x}\, d\sin x = -\frac{1}{\sin x} = -\csc x$$

Thus $y_p = u_1 y_1 + u_2 y_2$

$$= \cos x \ln\left|\cot\frac{x}{2}\right| - \sin x \csc x$$

$$\Rightarrow y_p = \cos x \ln\left|\cot\frac{x}{2}\right| - 1$$

#61 $x^2 y'' + x y' + y = \ln x$; $y_c = C_1 \cos(\ln x) + C_2 \sin(\ln x)$

We have $y_1(x) = \cos(\ln x)$, $y_2(x) = \sin(\ln x)$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix}$$

$$= \frac{1}{x}\left(\cos^2(\ln x) + \sin^2(\ln x)\right) = \frac{1}{x}$$

Note the equation can be written as

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = \frac{1}{x^2} \ln x$$

Thus $f(x) = \frac{1}{x^2} \ln x$

We have

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)} dx = -\int \frac{\sin(\ln x) \cdot \frac{1}{x^2} \ln x}{\frac{1}{x}} dx$$

$$= -\int \frac{\ln x \sin(\ln x)}{x} dx \quad \left(\int u dv = uv - \int v du \right)$$

$$= + \int \ln x \, d\cos(\ln x) \quad (u = \ln x, \; v = \cos \ln x)$$

$$= \ln x \cdot \cos(\ln x) - \int \cos(\ln x) \, d\ln x$$

$$= \ln x \cdot \cos(\ln x) - \int \frac{\cos(\ln x)}{x} dx$$

$$= \ln x \cdot \cos(\ln x) - \sin(\ln x)$$

$$u_2 = \int \frac{y_1(x) f(x)}{W(x)} dx = \int \frac{\cos(\ln x) \cdot \frac{1}{x^2} \ln x}{\frac{1}{x}} dx$$

$$= \int \frac{\ln x \cos(\ln x)}{x} dx = \int \ln x \, d\sin(\ln x)$$

$$= \ln x \sin(\ln x) - \int \frac{\sin(\ln x)}{x} dx$$

$$= \ln x \sin(\ln x) + \cos(\ln x)$$

Therefore

$$y_p = u_1 y_1 + u_2 y_2$$

$$= [\ln x \cdot \cos(\ln x) - \sin(\ln x)] \cos(\ln x) + [\ln x \sin(\ln x) + \cos(\ln x)] \sin(\ln x)$$

$$= \cos^2(\ln x) \ln x + \sin^2(\ln x) \ln x$$

$$= \ln x$$

§ 3.6

#11

$$x'' + 4x' + 5x = 10\cos 3t, \quad x(0) = x'(0) = 0$$

ANS: As

$$r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i,$$

$$x_c = e^{-2t}(C_1 \cos t + C_2 \sin t)$$

Assume $x_p = A\cos 3t + B\sin 3t$, then by

$x_p'' + 4x_p' + 5x_p = 10\cos 3t$

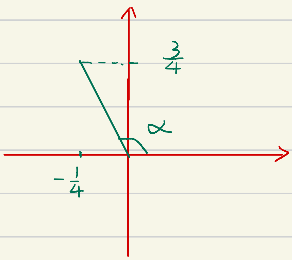
we have

$$-4A + 12B = 10 \quad \text{and} \quad 12A + 4B = 0$$

$$\Rightarrow A = -\frac{1}{4}, \quad B = \frac{3}{4}.$$

Thus $x_{sp}(t) = x_p = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t$

$$C = \sqrt{A^2 + B^2} = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{10}}{4}$$



$\alpha = \frac{\pi}{2} + \arctan\frac{1}{3} \approx 1.8925$

So $x_{sp}(t) \approx \frac{\sqrt{10}}{4}\cos(3t - 1.8925)$

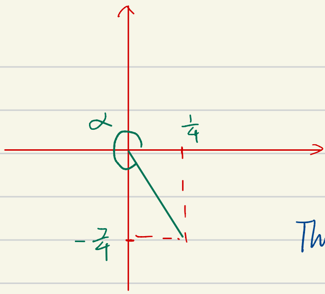
Now we have

$$x(t) = x_c + x_p = x_{tr} + x_p = e^{-2t}(C_1 \cos t + C_2 \sin t) + x_{sp}(t)$$

As $x(0) = x'(0) = 0$, we have $C_1 - \frac{1}{4} = 0$, $-2C_1 + C_2 + \frac{9}{4} = 0$

Thus $C_1 = \frac{1}{4}$, $C_2 = -\frac{7}{4}$.

Then $x_{tr} = e^{-2t}\left(\frac{1}{4}\cos t - \frac{7}{4}\sin t\right)$



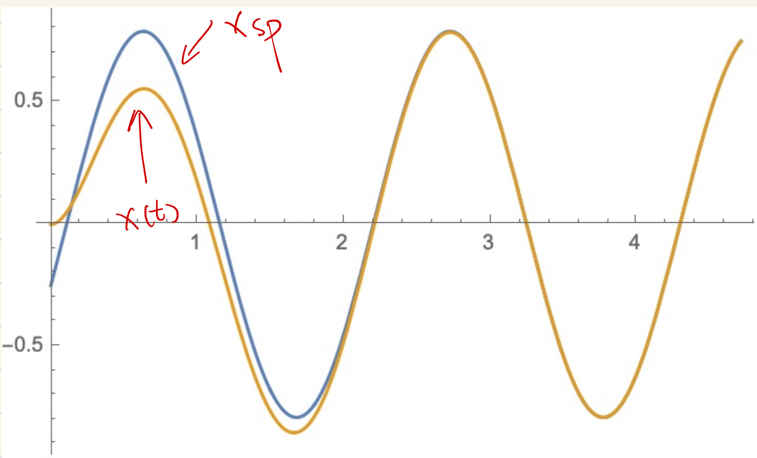
$C = \sqrt{(\frac{1}{4})^2 + (-\frac{7}{4})^2} = \frac{\sqrt{50}}{4} = \frac{5}{4}\sqrt{2}$

$\alpha = 2\pi - \arctan 7 \approx 4.8543$

Thus $x_{tr} \approx \frac{5}{4}\sqrt{2}\, e^{-2t} \cos(t - 4.8543)$

Thus

$x(t) = x_{tr}(t) + x_{sp}(t) = \frac{5}{4}\sqrt{2}\, e^{-2t}(t - 4.8543) + \frac{\sqrt{10}}{4}\cos(3t - 1.8925)$



12

$$x'' + 6x' + 13x = 10\sin 5t, \qquad x(0) = x'(0) = 0$$

ANS: As

$$r^2 + 6r + 13 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36 - 4\times 13}}{2} = -3 \pm 2i,$$

$$x_c = e^{-3t}(C_1 \cos 2t + C_2 \sin 2t)$$

Assume $x_p = A\cos 5t + B\sin 5t$, then

$$x_p' = -5A\sin 5t + 5B\cos 5t$$

$$x_p'' = -25A\cos 5t - 25B\sin 5t$$

Thus

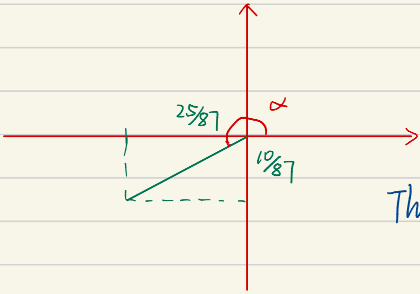
$$x_p'' + 6x_p' + 13x_p = 10\sin 5t$$

implies

$$\begin{cases} -12A + 30B = 0 \\ -30A - 12B = 10 \end{cases}$$

$$\Rightarrow \begin{cases} A = -\frac{25}{87} \\ B = -\frac{10}{87} \end{cases}$$

Thus $x_p = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$



$$C = \sqrt{\left(-\frac{25}{87}\right)^2 + \left(-\frac{10}{87}\right)^2} = \frac{5\sqrt{29}}{87}$$

$$\alpha = \pi + \arctan\frac{10}{25} \approx 3.5221$$

Thus $x_{sp}(t) = x_p(t) \approx \frac{5\sqrt{29}}{87}\cos(5t - 3.5221)$

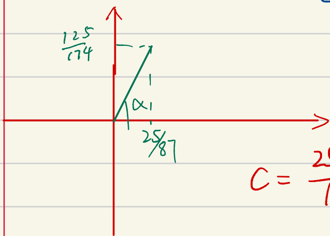
Thus we have

$$x(t) = x_c + x_p = e^{-3t}(C_1\cos 2t + C_2 \sin 2t) + x_p$$

As $x(0)=0$, $x'(0)=0$, we have

$$C_1 = \frac{25}{87}, \qquad C_2 = \frac{125}{174}$$

Thus $X_{tr} = e^{-3t}\left(\frac{25}{87}\cos 2t + \frac{125}{174}\sin 2t \right)$



$$C = \frac{25\sqrt{29}}{174}$$

$$\alpha = \arctan \frac{\frac{125}{174}}{\frac{25}{87}}$$

$$= \arctan \frac{5}{2}$$

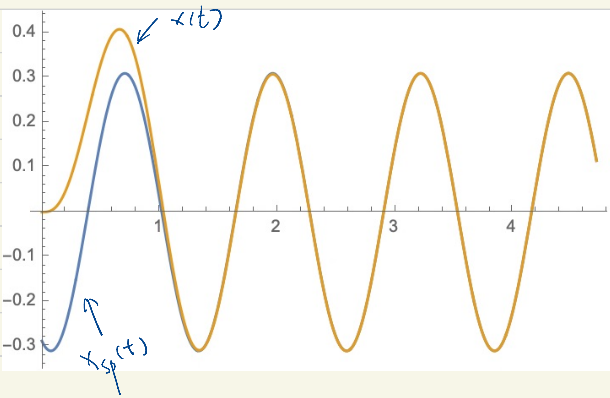
$$\approx 1.1903$$

So $X_{tr} = \frac{25\sqrt{29}}{174} e^{-3t} \cos(2t - 1.1903)$

Therefore

$$X(t) = X_{sp}(t) + X_{tr}(t)$$

$$\Rightarrow X(t) = \frac{5\sqrt{29}}{87}\cos(5t - 3.5221) + \frac{25\sqrt{29}}{174} e^{-3t}\cos(2t - 1.1903)$$



§4.1

\# 27

(a) $\begin{cases} x' = y \\ y' = -x. \end{cases}$

Then $y = x' \Rightarrow y' = x'' = -x$

$\Rightarrow x'' + x = 0 \qquad \Rightarrow x = C_1 \cos t + C_2 \sin t.$

Then $y = x' = -C_1 \sin t + C_2 \cos t$

So $x^2 + y^2 = C_1^2 \cos^2 t + C_2 \sin^2 t + 2C_1C_2 \cos t \sin t$

$\qquad + C_1^2 \sin^2 t + C_2 \cos^2 t - 2C_1C_2 \cos t + \sin t$

$\Rightarrow x^2 + y^2 = C_1^2 + C_2^2$

$\Rightarrow x^2 + y^2 = C^2$

Thus the trajectories of the system are circles

(b) $\begin{cases} x' = y \\ y' = x \end{cases}$

We have $y = x' \Rightarrow y' = x'' = x \Rightarrow x'' - x = 0.$

Thus $x = C_1 e^{x} + C_2 e^{-x}$

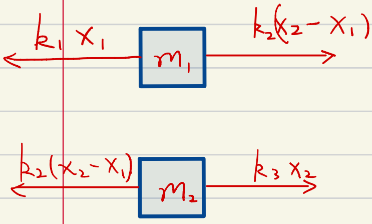
$y = x' = C_1 e^{x} - C_2 e^{-x}$

Then $x^2 - y^2 = C_1^2 e^{2x} + C_2^2 e^{-2x} + 2C_1C_2 - C_1^2 e^{2x} - C_2^2 e^{-2x} + 2C_1C_2$

$\Rightarrow x^2 - y^2 = 4C_1C_2 \Rightarrow x^2 - y^2 = C$

Thus the trajectories of the system are hyperbolas.

30



By Newton's law and Hooke's law.

$$m_1 a_1 = m_1 x_1'' = k_2(x_2 - x_1) - k_1 x_1$$

$$\Rightarrow m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x$$

$$m_2 a_2 = m_1 x_2'' = k_3 x_2 - k_2(x_2 - x_1)$$

$$\Rightarrow m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$$

HW #9

§5.1 #23

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}, \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

ANS:

$$\vec{x}_1' = e^{2t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{2t}$$

$$\vec{x}_2' = e^{-2t}\begin{bmatrix} -2 \\ -10 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ -10 \end{bmatrix} e^{-2t}$$

So $\vec{x}_1' = A\vec{x}_1$, $\vec{x}_2' = A\vec{x}_2$.

Thus $\vec{x}_1$, $\vec{x}_2$ are solutions of the given system.

$$W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 5 - 1 = 4 \neq 0$$

Thus $\vec{x}_1$ $\vec{x}_2$ are linearly independent.

Then the general solution of the given system is

$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t)$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + C_2\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1(t) = C_1 e^{2t} + C_2 e^{-2t} \\ x_2(t) = C_1 e^{2t} + 5C_2 e^{-2t} \end{cases}$$

#32 We have $x_1(0)=5$, $x_2(0)=-3$.

So

$$\begin{cases} 5 = C_1 + C_2 \\ -3 = C_1 + 5C_2 \end{cases} \Rightarrow \begin{cases} C_1 = 7 \\ C_2 = -2 \end{cases}$$

Thus

$$\begin{cases} x_1(t) = 7e^{2t} - 2e^{-2t} \\ x_2(t) = 7e^{2t} - 10e^{-2t} \end{cases}$$

Section [redacted] 5.2

#29

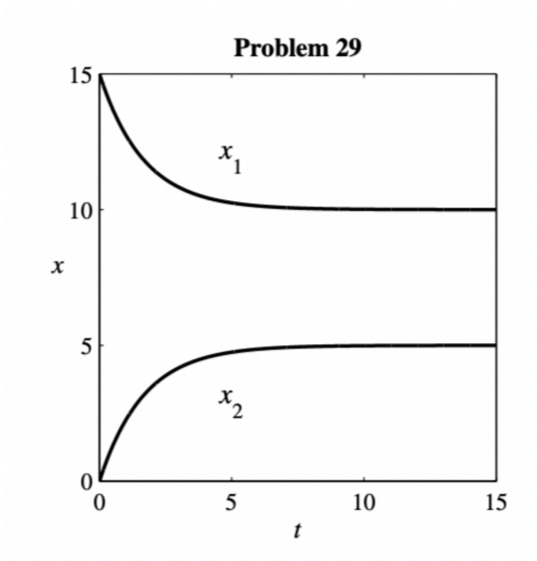
The coefficient matrix $\mathbf{A} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.6$, with eigenvectors $\mathbf{v}_1 = [2 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad -1]^{\mathrm{T}}$ that yield the general solution

$$x_1(t) = 2c_1 + c_2 e^{-0.6t}, \quad x_2(t) = c_1 - c_2 e^{-0.6t}.$$

The initial conditions $x_1(0) = 15, x_2(0) = 0$ give $c_1 = c_2 = 5$, so we get

$$x_1(t) = 10 + 5e^{-0.6t}, \quad x_2(t) = 5 - 5e^{-0.6t}.$$

The figure shows the graphs of $x_1(t)$ and $x_2(t)$.



#8

Characteristic equation $\lambda^2 + 4 = 0$ Eigenvalue $\lambda = 2i$ Eigenvector equation
$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = [5 \quad 1-2i]^{\mathrm{T}}$;

$$\mathbf{x}(t) = \mathbf{v}e^{2it} = \begin{bmatrix} 5\cos 2t + 5i \sin 2t \\ (\cos 2t + 2\sin 2t) + i(\sin 2t - 2\cos 2t) \end{bmatrix}$$

$$x_1(t) = 5c_1 \cos 2t + 5c_2 \sin 2t$$

$$x_2(t) = c_1(\cos 2t + 2\sin 2t) + c_2(\sin 2t - 2\cos 2t) = (c_1 - 2c_2)\cos 2t + (2c_1 + c_2)\sin 2t$$

#11

Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$; Eigenvalue $\lambda = 1 - 2i$.

$$\text{Eigenvector equation } \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

Eigenvector $\mathbf{v} = [1 \quad i]^{\mathrm{T}}$;
The real and imaginary parts of

$$\mathbf{x}(t) = [1 \quad i]^{\mathrm{T}} e^t(\cos 2t - i \sin 2t) = e^t[\cos 2t \quad \sin 2t]^{\mathrm{T}} + ie^t[-\sin 2t \quad \cos 2t]^{\mathrm{T}}$$

yield the general solution

$$x_1(t) = e^t\,(c_1 \cos 2t - c_2 \sin 2t)\,, \quad x_2(t) = e^t\,(c_1 \sin 2t + c_2 \cos 2t)$$

The particular solution with $x_1(0) = 0$ and $x_2(0) = 4$ is obtained with $c_1 = 0$ and $c_2 = 4$,
So

$$x_1(t) = -4e^t \sin 2t, \quad x_2(t) = 4e^t \cos 2t$$

#24

Characteristic equation $-\lambda^3 + \lambda^2 - 4\lambda + 4 = 0$;
Eigenvalues $\lambda = 1$ and $\lambda = \pm 2i$

With $\lambda = 1$ the eigenvector equation $\begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the eigenvector $\mathbf{v}_1 = [1 \quad -1 \quad 0]^{\mathrm{T}}$. To find an eigenvector $\mathbf{v} = [a \quad b \quad c]^{\mathrm{T}}$ associated with $\lambda = 2i$we must find a nontrivial solution of the equations
Subtraction of the first two equations yields

$$(6 - 2i)a + (4 + 2i)b = 0$$

so we take $a = 2 + i$ and $b = -3 + i$. Then the first equation gives $c = 3 - i$. Thus $\mathbf{v} = [2 + i \quad -3 + i \quad 3 - i]^{\mathrm{T}}$. Finally

$$\begin{aligned} (2 + i)e^{2it} &= (2\cos 2t - \sin 2t) + i(\cos 2t + 2\sin 2t) \\ (3 - i)e^{2it} &= (3\cos 2t + \sin 2t) + i(3\sin 2t - \cos 2t) \end{aligned}$$

so the solution is

$$\begin{aligned} &x_1(t) = c_1 e^t + c_2(2\cos 2t - \sin 2t) + c_3(\cos 2t + 2\sin 2t), \\ &x_2(t) = -c_1 e^t - c_2(3\cos 2t + \sin 2t) + c_3(\cos 2t - 3\sin 2t), \\ &x_3(t) = c_2(3\cos 2t + \sin 2t) + c_3(3\sin 2t - \cos 2t). \end{aligned}$$

Section 5.5

#2

Characteristic equation $\lambda^2 - 4\lambda + 4 = 0$; repeated eigenvalue $\lambda = 2$; Using the algorithm, we solve $(\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{v_2} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

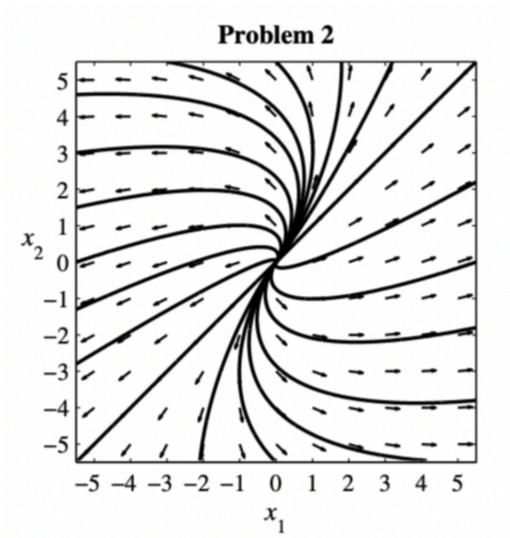
So we can assume $\mathbf{v}_2 = [1 \quad 0]^{\mathrm{T}}$. Then we compute

$$\mathbf{v_1} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v_2} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $\mathbf{x}_1 = \mathbf{v_1} e^{2t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ and $\mathbf{x}_2 = (\mathbf{v_1} t + \mathbf{v_2}) e^{2t} = \begin{bmatrix} t+1 \\ t \end{bmatrix} e^{2t}$.

Then $\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, which means we have $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t+1 \\ t \end{bmatrix}$

So we have $x_1(t) = (c_1 + c_2 + c_2 t)\, e^{2t}, \quad x_2(t) = (c_1 + c_2 t)\, e^{2t}$.



#3

Characteristic equation $\lambda^2 - 6\lambda + 9 = 0$; repeated eigenvalue $\lambda = 3$;

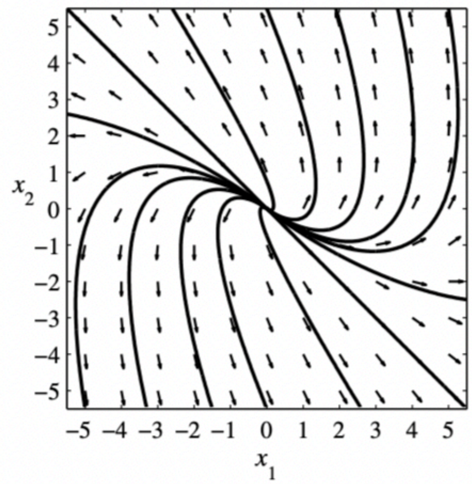
Using the algorithm, we solve $(\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{v_2} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\mathbf{v_1} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v_2} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Thus $\mathbf{x}_1 = \mathbf{v_1} e^{3t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}_2 = (\mathbf{v_1} t + \mathbf{v_2}) e^{3t} = \begin{bmatrix} -2t+1 \\ 2t \end{bmatrix} e^{3t}$. Then $\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

So we have $x_1(t) = (-2c_1 + c_2 - 2c_2 t)\, e^{3t}, \quad x_2(t) = (2c_1 + 2c_2 t)\, e^{3t}$.

Problem 3



#4

Characteristic equation $\lambda^2 - 8\lambda + 16 = 0$; repeated eigenvalue $\lambda = 4$;

Similar to questions 2 and 3, we have $\mathbf{v_2} = [1 \quad 0]^{\mathrm{T}}$

$$\mathbf{v_1} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{v_2} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus we have

$\mathbf{x}_1 = \mathbf{v_1} e^{4t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$ and $\mathbf{x}_2 = (\mathbf{v_1} t + \mathbf{v_2}) e^{4t} = \begin{bmatrix} -t+1 \\ t \end{bmatrix} e^{4t}$. Then $\mathbf{x}(t) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$.

Therefore $x_1(t) = (-c_1 + c_2 - c_2 t)\, e^{4t}, \quad x_2(t) = (c_1 + c_2 t)\, e^{4t}$

Problem 4