Section 1.8 Introduction to Linear Transformations

A matrix equation $A\mathbf{x} = \mathbf{b}$ can be thought of the matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

For example, the equations

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{\vec{b}} \text{ and } \overbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}}^{\vec{u}} = \overbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}^{\vec{0}}$$

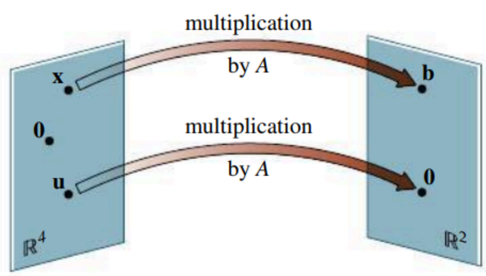

FIGURE 1 Transforming vectors via matrix multiplication.

Definition: transformation, domain, codomain, image, range

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.

The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the **range** of T.

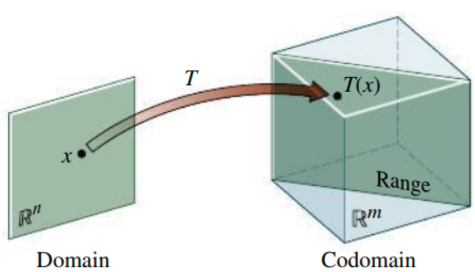

FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Matrix Transformations (special type of transformation/function/mapping)

The rest of this section focuses on mappings associated with matrix multiplication. For each $\mathbf{x}$ in $\mathbb{R}^n, T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.

Observe that

- the domain of T is $\mathbb{R}^n$ when A has n columns and
- the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.
- The range of T is the set of all linear combinations of the columns of A.

Example 1. Let A be a 4×3 matrix. What must a and b be in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

ANS: By discussion above, we know

- $a = 3$
- $b = 4$.

i.e. $\vec{x} \in \mathbb{R}^3$ and $A\vec{x} \in \mathbb{R}^4$

Example 2. Let T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$$

ANS: We need to find $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$.

The augmented matrix is

$$[A \quad \vec{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$ and the solution is unique.

Example 3.

(i) Find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

(ii) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

ANS: (i) We need to find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & \cancel{-3}\ ^{1} & \cancel{-6}\ ^{2} & \cancel{-6}\ ^{2} & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -9 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note: the pivot columns are 1, 2, 4.

$$\Rightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases} \qquad \vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

(2) The question is the same as asking is $[A \ \vec{b}]$ consistent.

$$[A \ \vec{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -18 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{array}\right]$$

Note the echelon form has the 4th row in the form of $[0 \ 0 \ 0 \ 0 \ 17]$, so the system is inconsistent (has no solution).

Thus $\vec{b}$ is not in the range of the transformation.

Every matrix transformation is linear (see Thm5 in §1.4). but not vice versa (Egs in Ch4 and Ch5).

Linear Transformations

Definition. A transformation (or mapping) T is **linear** if
(i) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
(ii) $T(c\mathbf{u})=cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

That is, linear transformations preserve the operations of vector addition and scalar multiplication.

Properties. If T is a linear transformation, then
(i) $T(\mathbf{0})=\mathbf{0}$

(ii) $T(c\mathbf{u}+d\mathbf{v})=cT(\mathbf{u})+dT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

(iii) $T\left(c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p\right)=c_1T\left(\mathbf{v}_1\right)+\cdots+c_pT\left(\mathbf{v}_p\right)$ (generalization of property (ii))

Example 4. Show that the transformation T defined by $T\left(x_1, x_2\right)=\left(x_1+3x_2, x_1+x_2+3, 7x_2\right)$ is not linear.

ANS: We can show that T does not map the zero vector into the zero vector, which does not satisfy the property (i) above. So T cannot be linear.
In fact, if $x_1=0$, $x_2=0$, $T(0,0)=(0,3,0)\neq$ a zero vector.

Example 5. Let $\mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}, \mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}, \mathbf{y}_1=\begin{bmatrix}3\\1\end{bmatrix}$, and $\mathbf{y}_2=\begin{bmatrix}-2\\3\end{bmatrix}$, and let $T:\mathbb{R}^2\rightarrow\mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix}-5\\2\end{bmatrix}$ and $\begin{bmatrix}x_1\\x_2\end{bmatrix}$.

ANS: We know $T(\vec{e}_1)=\vec{y}_1$, $T(\vec{e}_2)=\vec{y}_2$

General idea: Write the given vectors in terms of $\vec{e}_1$ and $\vec{e}_2$.
Then use property (ii).

- $\begin{bmatrix}-5\\2\end{bmatrix}=-5\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=-5\vec{e}_1+2\vec{e}_2$

That is, the given vector can be written as a linear combination of the (standard basis) $\vec{e}_1$ and $\vec{e}_2$.

By property (ii)

$$T\left(\begin{bmatrix}-5\\2\end{bmatrix}\right) = T\left(-5\vec{e}_1 + 2\vec{e}_2\right) \overset{\text{property (ii)}}{=} -5\,T(\vec{e}_1) + 2\,T(\vec{e}_2)$$

$$= -5\vec{y}_1 + 2\vec{y}_2$$

$$= -5\begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}-2\\3\end{bmatrix}$$

$$= \begin{bmatrix}-15-4\\-5+6\end{bmatrix} = \begin{bmatrix}-19\\1\end{bmatrix}$$

Similarly, $\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$

Thus $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2) \overset{\text{property (ii)}}{=} x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$

$$= x_1\vec{y}_1 + x_2\vec{y}_2 = x_1\begin{bmatrix}3\\1\end{bmatrix} + x_2\begin{bmatrix}-2\\3\end{bmatrix}$$

$$= \begin{bmatrix}3x_1 - 2x_2\\x_1 + 3x_2\end{bmatrix}$$