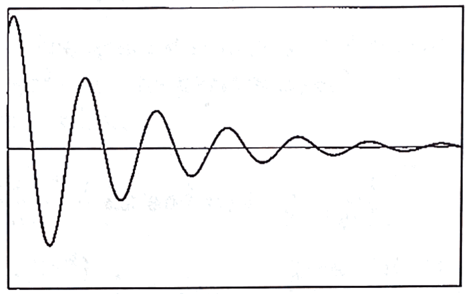
Green 1-7: CAB BEAD

Orange 1-7 BAD DACE

Alternate 1-7 ACED BAD

B **1.** For which values of the parameter α will the equation $y'' + \alpha y' + y = 0$ have solutions whose graphs are similar to the following graph?



A. $\alpha < 2$

B. $0 < \alpha < 2$

C. $\alpha = 0$

D. $\alpha > 2$

E. all α

① α is the damping constant c.

Thus $\alpha > 0$

② The char. eqn. has complex roots

$\alpha^2 - 4 < 0$

$\Rightarrow -2 < \alpha < 2$

By ① ②, $0 < \alpha < 2$

A **2.** *(10 pts.)* The solution to the initial value problem

$$y'' - y' - 2y = -6x + 5, \qquad y(0) = -4,\ y'(0) = 0 \text{ is}$$

A. $y = -e^{2x} + e^{-x} + 3x - 4$

B. $y = 15e^{2x} - 24e^{-x} - 6x + 5$

C. $y = e^{-2x} - e^{x} + 3x - 4$

D. $y = -5e^{-2x} - 4e^{x} - 6x + 5$

E. $y = \frac{1}{3}e^{2x} - \frac{1}{3}e^{-x} + 3x - 4$

D 3. *(10 pts.)* If the method of variation of parameters is used to find a particular solution to

$$x^2y'' - 3xy' + 4y = x^4$$

for $x > 0$, given that the general solution to the associated homogeneous equation is $c_1x^2 + c_2x^2 \ln x$, the particular solution obtained is $y = u_1x^2 + u_2x^2 \ln x$ where

A. $u_1 = \int \frac{-(x^2 \ln x)(x^2)}{2x^3 \ln x}\,dx$ and $u_2 = \int \frac{(x^2)(x^2)}{2x^3 \ln x}\,dx$

B. $u_1 = \int \frac{-(x^2 \ln x)(x^4)}{x^3}\,dx$ and $u_2 = \int \frac{-(x^2)(x^4)}{x^3}\,dx$

C. $u_1 = \int \frac{-(x^2 \ln x)(x^4)}{2x^3 \ln x}\,dx$ and $u_2 = \int \frac{(x^2)(x^4)}{2x^3 \ln x}\,dx$

D. $u_1 = \int \frac{-(x^2 \ln x)(x^2)}{x^3}\,dx$ and $u_2 = \int \frac{(x^2)(x^2)}{x^3}\,dx$

E. u_1 and u_2 cannot be determined because the equation is singular at $x = 0$.

D 4. *(10 pts.)* Find the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \mathbf{x}.$$

A. $\mathbf{x} = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

B. $\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

C. $\mathbf{x} = c_1 e^{2t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

D. $\mathbf{x} = c_1 e^{2t} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

E. $\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

A 5. *(10 pts.)* An undamped, free vibration modeled by $u'' + 64u = 0$ has initial conditions $u(0) = 3$, $u'(0) = -32$. The solution of this initial value problem can be written as $u = R\cos(\omega t - \alpha)$. What is the amplitude R of this solution?

A. $R = 5$

B. $R = 4$

C. $R = \sqrt{2}$

D. $R = \sqrt{5}$

E. $R = 3$

C 6. *(10 pts.)* Find the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix} \mathbf{x}.$$

A. $\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

B. $\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 5 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

C. $\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

D. $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

E. $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} -5 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E 7. *(10 pts.)* The correct form of a particular solution to use in the method of undetermined coefficients for the equation

$$y^{(4)} + 2y'' + y = t^2 \cos t \qquad \text{is}$$

A. $(At^2 + Bt + C)(D\cos t + E\sin t)$

B. $(At^4 + Bt^3 + Ct^2)(D\cos t + E\sin t)$

C. $(A_2t^2 + A_1t + A_0)\cos t + (B_2t^2 + B_1t + B_0)\sin t$

D. $(A_2t^3 + A_1t^2 + A_0t)\cos t + (B_2t^3 + B_1t^2 + B_0t)\sin t$

E. $(A_2t^4 + A_1t^3 + A_0t^2)\cos t + (B_2t^4 + B_1t^3 + B_0t^2)\sin t$

Note $r^4 + 2r^2 + 1 = 0$

$\Rightarrow (r^2+1)^2 = 0$

$\Rightarrow r = i, i, -i, -i$

8. *(10 pts.)* Rewrite the third order equation $y^{(3)} + 4y'' + 3y' + 2y = \sin(t)$ as a (3×3) system of first order equations. Suggestion: Let $u_1 = y$, etc.

$$\begin{cases} u_1' = u_2 \leftarrow 2\text{pts} \\ u_2' = u_3 \leftarrow 2\text{pts} \\ u_3' = -4u_3 - 3u_2 - 2u_1 + \sin t \end{cases}$$

$\uparrow$

5 pts

$\left.\begin{array}{l} u_1 = y \\ u_2 = y' \\ u_3 = y'' \end{array}\right\}$ 1pt

9. *(10 pts.)* Find the general solution to

$$y'' + 2y' + y = e^{-x} + e^{x}.$$

$r^2 + 2r + 1 = 0$

$\Rightarrow r = -1, -1.$

$y_c = C_1 e^{-x} + C_2 x e^{-x}$

$y_p = \underbrace{x^2 [Ae^{-x}]}_{3\text{pts}} + \underbrace{Be^{x}}_{2\text{pts}}$

$y_p' = 2Axe^{-x} - Ax^2 e^{-x} + Be^{x}$

$y_p'' = 2Ae^{-x} - 4Axe^{-x} + x^2 e^{-x} + Be^{x}$

$y_p'' + 2y_p' + y_p$

$= 2Ae^{-x} + 4Be^{x} = e^{-x} + e^{x}$

$\Rightarrow A = \frac{1}{2} \qquad B = \frac{1}{4}$

$y = \underbrace{C_1 e^{-x} + C_2 x e^{-x}}_{1\text{pt}} + \underbrace{\frac{1}{2} x^2 e^{-x}}_{2\text{pts}} + \underbrace{\frac{1}{4} e^{x}}_{2\text{pts}}$

10. *(10 pts.)* The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

is defective. It has only one eigenvalue $r = 2$, which has a one dimensional eigenspace spanned by the eigenvector $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, one solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x}_1 = \mathbf{a}e^{2t}$. Find a second linearly independent solution to the system.

ANS: Note $\vec{x}_2 = (\vec{v}_1 t + \vec{v}_2)\, e^{2t}$

Method 1. Find $\vec{v}_2$ first.

5pts

$(A-2I)^2 \vec{v}_2 = \vec{0}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}$ Let $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or any 2×1 vector.

Then $\vec{v}_1 = (A-2I)\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\leftarrow$ 4pts

Thus $\vec{x}_2 = \begin{bmatrix} -t+1 \\ t \end{bmatrix} e^{2t}$ $\leftarrow$ 1 pt

Method 2. Solve for $\vec{v}_2$ and let $\vec{v}_1 = \vec{a}$ given.

$(A-2I)\vec{v}_2 = \vec{v}_1 = \vec{a}$

$\Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\leftarrow$ 5pts

$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ or any vector satisfying the eqn. $\leftarrow$ 4pts

$\vec{x}_2 = \begin{bmatrix} -t+1 \\ t \end{bmatrix} e^{2t}$ $\leftarrow$ 1 pt