Midterm 1 Review

We will talk about the following problems on Tuesday, Oct 5th, during the class.

1. (Spring 2017 Final Q2) Find the solution to the initial value problem

$$y' = \frac{3x^2+4x-4}{2y-4}, \quad y(1) = 3$$

A. $y^2 - 4y = x^3 + x^2 - x - 2$

B. $y^2 - 4y = x^3 + 2x^2 - 4x - 1$

C. $y^2 - 4y = x^3 + 2x^2 - 4x$

D. $y^2 - 4y - 2 = x^3 + 2x^2 - 4x$

E. $y^2 - 4y = x^3 + 2x^2 - 4x - 2$

separable? Yes

$$\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$$

$$\Rightarrow \int (2y-4)\,dy = \int (3x^2+4x-4)\,dx$$

$$\Rightarrow y^2 - 4y = x^3 + 2x^2 - 4x + c$$

As $y(1)=3$, we plug in $x=1$, $y=3$

$$-3 = 3^2 - 4\times 3 = 1 + 2 - 4 + c = -1 + c$$

$$\Rightarrow C = -3+1 = -2$$

2. (Spring 2019 Final Q1) Determine the interval where the solution guaranteed to exist for the following initial value problem

$$(t+2)y' + y = \frac{1}{t-1}, \quad y(0) = \frac{1}{2}$$

A. $(1, +\infty)$

B. $(-2, 1)$

C. $(-2, +\infty)$

D. $(-\infty, -2)$

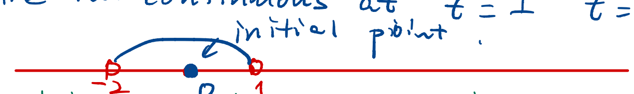
E. $(-\infty, 1)$

1. Write the given eqn in the form: $y' = f(x,y)$.

$$y' = \frac{\frac{1}{t-1} - y}{t+2}.$$

2. Find the intervals of x such that, f, $\frac{\partial}{\partial y} f$ are continuous

f, $\frac{\partial f}{\partial y}$ are not continuous at $t=1$ $t=-2$



So we have $(1,\infty)$ $(-2,1)$ $(-\infty,-2)$

3. Choose the interval such that contains the initial point (x_0, y_0)

The initial point is $(x_0, y_0) = (0, \frac{1}{2})$

Existence and Uniqueness Theorem

First Order, General Initial Value Problem:

$$y' = f(x,y), \quad y(x_0) = y_0$$

- Solution exists and is unique if f and $\frac{\partial}{\partial y} f$ are continuous at (x_0, y_0).
- Solutions are defined somewhere inside the region containing (x_0, y_0), where f and $\frac{\partial}{\partial y} f$ are continuous.

Fall $V_0 = 100$ $Q(0) = 10$

3. (~~Spring~~ 2018 Final Q2) Initially, a tank contains 100 L of water with 10 kg of sugar in solution. Water containing sugar flows into the tank at the rate of $2\ \mathrm{L/min}$ r_i, and the well-stirred mixture in the tank flows out at the rate of $5\ \mathrm{L/min}$ r_o. The concentration $c(t)$ of sugar in the incoming water varies as C_i $c(t) = 2 + \cos(3t)\ \mathrm{kg/L}$. Let $Q(t)$ be the amount of sugar (in kilograms) in the tank at time t (in minutes). Which differential equation does $Q(t)$ satisfy?

A. $\dfrac{dQ}{dt} = 2(2 + \sin(3t)) - \dfrac{5Q}{100 - 3t}$

B. $\dfrac{dQ}{dt} = 2(2 + \cos(3t)) - \dfrac{5Q}{100 - 3t}$

C. $\dfrac{dQ}{dt} = 2(2 + \sin(3t)) - \dfrac{5Q}{100}$

D. $\dfrac{dQ}{dt} = 2(2 + \cos(3t)) - \dfrac{5Q}{100}$

E. $\dfrac{dQ}{dt} = 2 + \cos(3t) - \dfrac{Q}{100 - t}$

Recall

$\dfrac{dQ}{dt} = r_i C_i - r_o C_o$, where

$C_o = \dfrac{Q(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_o)t$

We know

$V(t) = 100 + (2-5)t$

$= 100 - 3t$

So $\dfrac{dQ}{dt} = 2 \cdot (2 + \cos(3t)) - 5 \cdot \dfrac{Q}{100 - 3t}$

4. (Spring 2019 Final Q2) The general solution to $x^2 y' + 2xy = e^{3x}$ is

A. $y = \dfrac{3}{x^2} e^{3x} + c$

B. $y = ce^{3x}$

C. $y = \dfrac{1}{3x^2} e^{3x} + cx^{-2}$

D. $y = \dfrac{1}{2x^2} e^{3x}$

E. $y = \dfrac{1}{3x} e^{3x} + cx^{-2}$

Linear First-order Equations

Linear First-order Equations: $\dfrac{dy}{dx} + P(x)y = Q(x)$

Solution: $\rho y = \int \rho Q(x)dx$, where $\rho = e^{\int P(x)dx}$.

Applications: Mixture Problems: $\dfrac{dx}{dt} = r_i c_i - r_o c_o$,

where $c_o(t) = \dfrac{x(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_o)\,t$

$P(x)$ $Q(x)$

$y' + \dfrac{2}{x} y = \dfrac{e^{3x}}{x^2}$

- The integrating factor $\rho = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$
- Thus $x^2 y = \int x^2 \cdot \dfrac{e^{3x}}{x^2} dx = \int e^{3x} dx = \frac{1}{3}\int e^{3x} d3x = \frac{1}{3} e^{3x} + c$

$\Rightarrow \quad y = \frac{1}{3x^2} e^{3x} + Cx^{-2}$

5. (Spring 2018 Final Q2) Let $y(t)$ be exact solution of the following initial value problem

$$\begin{cases} ty' - 2y = t, & t > 0 \\ y(1) = 0 \end{cases}$$

We rewrite the give eqn in the following form

Let $y_{\text{app}}(t)$ be the approximate solution of this initial value problem by using Euler's method with step size $h = 1$. Then $y_{\text{app}}(3) =$

A. 6

B. -4

C. 3

D. -3

E. 4

Since $t_0 = 1$, $h = 1$ we need to take 2 steps

Euler's Method

Euler's Method:	Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$
Euler's method with step size h:	$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot f(x_n, y_n) \end{cases}$

We have $\frac{dy}{dt} = \frac{t + 2y}{t}$ $(= f(t, y))$, $y(1) = 0$ $(t_0 = 1, y_0 = 0)$

$h = 1$,

$t_0 = 1$	$y_0 = 0$
$t_1 = 2$	$y_1 = y_0 + h \cdot \frac{t_0 + 2y_0}{t_0} = 0 + 1 \times \frac{1 + 2 \cdot 0}{1} = 1$
$t_2 = 3$	$y_2 = y_1 + h \cdot \frac{t_1 + 2y_1}{t_1} = 1 + 1 \cdot \frac{2 + 2 \cdot 1}{2} = 1 + 2 = 3$

6. (Fall 2018 Final Q7) Which of the following is the set of unstable equilibrium solution(s) of

$$y' = y(y-1)(y+2)^2 = f(y)$$

A. $y(t) = 0$

B. $y(t) = 1$

C. $y(t) = -2, 1$

D. $y(t) = -2$

E. $y(t) = 0, 1$

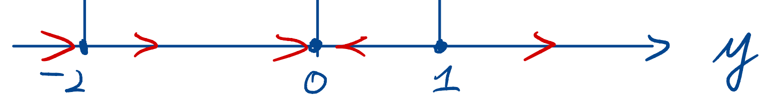
Autonomous Equations and Equilibrium Solutions

Autonomous Equations:	$\frac{dx}{dt} = f(x)$
Critical points:	values of x such that $f(x) = 0$. $f(x_0) = 0 \Rightarrow$ equilibrium solution at $x = x_0$ $f(x_0) < 0 \Rightarrow$ solutions go down at $x = x_0$ $f(x_0) > 0 \Rightarrow$ solutions go up at $x = x_0$
Stability of Critical Points:	Phase diagram method unstable = solutions go away (either side) stable = solutions go towards (both sides) semi-stable = solutions mixed

· Critical points: Let $y(y-1)(y+2)^2 = 0 \Rightarrow y = 0, 1, -2$

· Phase diagram:

$y' = y(y-1)(y+2)^2$ <0 <0 >0 >0 | $y' = y(y-1)(y+2)^2$ >0 | $y' = y(y-1)(y+2)^2$ <0 | $y' = y(y-1)(y+2)^2$ >0



Sec. 2.2

7. (Spring 2018 Final Q8) Consider the initial value problem,

$$y' = -y(y-1)^2(y-2), \quad y(0) = y_0$$

For which initial value y_0 is $\lim_{t\to\infty} y(t) = 1$?

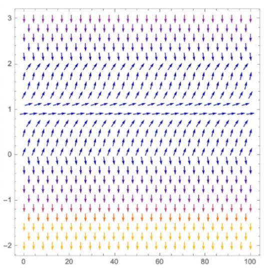
A. -0.001

B. 0.001

C. 1.001

D. 2.001

E. None of the above



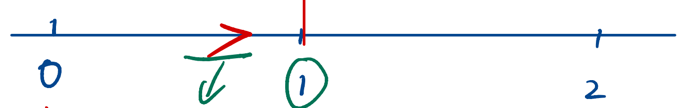
Let $-y(y-1)^2(y-2)=0$, we have critical pts $y=0, 1, 2$.

$0<y<1$

Phase diagram $y' = -y(y-1)^2(y-2)$

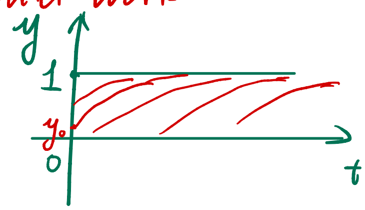
$<0 \quad >0 \quad <0$

>0



This means when $0<y_0<1$, $y(t)\uparrow$

Any y_0 such that $y_0 \in (0,1)$ would work



8. (Fall 2015 Final Q9) Find the general solution of the differential equation

$$y'' - 10y' + 27y = 0$$

2nd Order, Homogeneous Linear, Constant Coefficients

2nd Order, Homogeneous Linear, Constant Coefficients:

$$ay'' + by' + cy = 0$$

Characteristic Equation: $ar^2 + br + c = 0$

Solution depends on the type of roots:

- $r = r_1, r_2$ (real, not repeated), $y = c_1e^{r_1x} + c_2e^{r_2x}$.
- $r = r_1 = r_1$ (repeated root), $y = (c_1 + c_2x)e^{r_1x}$.
- $r = r_{1,2} = A \pm Bi$ (complex conjugates), $y = e^{Ax}(c_1\cos Bx + c_2 \sin Bx)$

$$r^2 - 10r + 27 = 0$$

$$r = \frac{10 \pm \sqrt{100 - 4\times 27}}{2}$$

$$= \frac{10 \pm \sqrt{100-108}}{2}$$

$$= \frac{10 \pm \cancel{\sqrt{-8}}\ 2\sqrt{2}i}{2}$$

$$= 5 \pm \sqrt{2}i \Rightarrow \begin{cases} A = 5 \\ B = \sqrt{2} \end{cases}$$

Thus

$$y = e^{5x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$$

9. (Spring 2018 Final Q5) Solve the initial value problem for the homogeneous equation

$$\frac{dy}{dx} = \frac{x^4+y^4}{xy^3}, \quad y(e)=e, \quad x>0$$

by using a substitution $v = y/x$.

ANS: We have $\frac{dy}{dx} = \frac{(x^4+y^4)/x^4}{(xy^3)/x^4} = \frac{1+\left(\frac{y}{x}\right)^4}{\left(\frac{y}{x}\right)^3}$

Let $v = \frac{y}{x}$, then $y = vx$, Taking diff. both sides

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{(product rule \& chain rule)}$$

Thus

$$v + x\frac{dv}{dx} = \frac{1+v^4}{v^3}$$

sep.

$$\Rightarrow \underline{x\frac{dv}{dx}} = \frac{1+v^4}{v^3} - v = \frac{1+v^4-v^4}{v^3} = \underline{\frac{1}{v^3}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v^3}$$

$$\Rightarrow \int v^3\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{4}v^4 = \ln x + C_1$$

$$\Rightarrow \frac{1}{4}\left(\frac{y}{x}\right)^4 = \ln x + C_1$$

As $y(e)=e$

$$\frac{1}{4} = \cancel{\ln e}^{\,1} + C_1 \Rightarrow C_1 = -\frac{3}{4}$$

So $\quad \frac{1}{4}\left(\frac{y}{x}\right)^4 = \ln x - \frac{3}{4}$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^4 = 4\ln x - 3$$

$$\Rightarrow \quad y^4 = x^4(4\ln x - 3)$$

or $\quad y = x\,(4\ln x - 3)^{\frac{1}{4}}$

10. (Fall 2015 Final Q6) Find the solution to the equation $2xy + 2xy^2 + 1 + \left(x^2 + 2x^2y + 2y\right)\frac{dy}{dx} = 0$ with $y(1) = 2$.

Exact Equations

Exact Equations: $M(x,y)dx + N(x,y)dy = 0$, where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution: $F(x,y) = C$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

Homogeneous ? No.

Exact ? Check $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$. ← scrach paper

ANS: We have $\underbrace{(2xy+2xy^2+1)}_{M(x,y)}dx + (\underbrace{x^2+2x^2y+2y}_{N(x,y)})dy = 0$

Let $M(x,y) = 2xy + 2xy^2 + 1$

$N(x,y) = x^2 + 2x^2y + 2y$

Since $\frac{\partial M}{\partial y} = 2x + 4xy = \frac{\partial N}{\partial x}$

the give equation is exact.

Thus there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M = 2xy + 2xy^2 + 1$$

$$\Rightarrow \quad F(x,y) = \int 2xy + 2xy^2 + 1\,dx$$

$$= x^2y + x^2y^2 + x + g(y)$$

Since $\frac{\partial F}{\partial y} = x^2 + 2x^2y + g'(y) = N(x,y) = x^2 + 2x^2y + 2y$

$\Rightarrow \quad g'(y) = 2y$

$\Rightarrow \quad g(y) = \int 2y\,dy = y^2$

Thus $F(x,y) = x^2y + x^2y^2 + x + y^2 = C$ is a general soluti

As $y(1) = 2$, we plug in $x=1, y=2$.

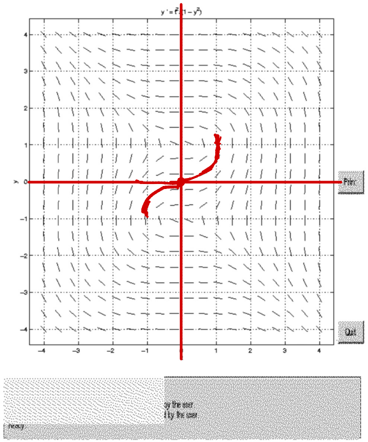
$$1^2 \cdot 2 + 1^2 \cdot 2^2 + 1 + 2^2 = C \Rightarrow C = 2 + 4 + 1 + 4 = 11$$

Thus the solution is $x^2y + x^2y^2 + x + y^2 = 11$.

2. What is the largest open interval for which a unique solution of the initial value problem $ty' + \dfrac{1}{t+1}y = \dfrac{t-2}{t-3}$, $y(1) = 0$ is guaranteed ?

A. $0 < t < 1$ **B.** $0 < t < 2$ **C.** $0 < t < 3$ **D.** $-1 < t < 3$ **E.** $-1 < t < 1$

3. Use the dfield plot below to estimate where the solution of $y' = \dfrac{t^2}{1-y^2}$, $y(0) = 0$ is defined:



A. $-1.2 < t < 1.2$ **B.** $-4 < t < 4$ **C.** $-1 < t < 2$ **D.** $-2 < t < 2$ **E.** $-4 < t < \infty$

MA266 Practice Problems

$\nearrow P(t)$ $\nearrow Q(t)$

1. If $y' + (1+\frac{1}{t})\,y = \frac{1}{t}$ and $y(1)=0$, then $y(\ln 2) =$

   A. $\ln 2 - \ln(\ln 2)$ **B.** $\ln(\ln 2)$ **C.** $\ln(\ln 2) + \dfrac{1}{2\ln 2}$ **D.** $\dfrac{1}{\ln 2}\,($

2. What is the largest open interval for which a unique solution c
   $ty' + \dfrac{1}{t+1}\,y = \dfrac{t-2}{t-3}$, $y(1) = 0$ is guaranteed ?

   A. $0<t<1$ **B.** $0<t<2$ **C.** $0<t<3$ **D.** $-1<t<3$

3. Use the dfield plot below to estimate where the solution of $y' =$

$$P = e^{\int 1+\frac{1}{t}\,dt} = e^{t+\ln t} = e^t \cdot t$$

$$te^t y = \int e^t\,dt = e^t + C$$