Below are the similar questions in the Exam 1 with detailed answers.

1. A stone is dropped from rest at an initial height $h = 25$ feet above the ground. Ignoring air resistance, assume that the acceleration due to gravity is $g = 32\mathrm{ft}/\mathrm{sec}^2$. How long does it take for the stone to hit the ground?

$$m \cdot \frac{d^2x}{dt^2} = -mg$$

$$\frac{dx}{dt} = -gt + \cancel{V_0} \quad 0$$

$$y = -\frac{1}{2}gt^2 + h = 0 \quad \text{when} \quad t^2 = \frac{2h}{g} = \frac{25}{16}$$

$$\Rightarrow t = 5/4 \text{ sec} = 1.25 \text{ sec}$$

2. A fish tank contains 20 gallons of a salt solution with a concentration of 5 grams of salt per gallon. A salt solution with a concentration of 10 grams/gallon is added to the tank at a rate of 2 gallons per minute. At the same time, well-mixed water is drained from the tank at a rate of 2 gallons per minute. How many grams of salt are in the tank after 10 minutes?

$\frac{dx}{dt} = 2\cdot 10 - 2\cdot\frac{x}{20}$ $\quad x(0) = 5\cdot 20 = 100$

$\Rightarrow \frac{dx}{dt} = 20 - \frac{x}{10} = \frac{200-x}{10}$

$\Rightarrow \frac{1}{200-x}\,dx = \frac{dt}{10}$

$\Rightarrow \frac{d(200-x)}{200-x} = -\frac{dt}{10}$

$\Rightarrow \ln(200-x) = -\frac{t}{10} + c$

As $x(0) = 100$

$\ln 100 = c$

$\ln(200-x) = -\frac{t}{10} + \ln 100$

$\Rightarrow \ln\left(2 - \frac{x}{100}\right) = -\frac{t}{10}$

$\Rightarrow 2 - \frac{x}{100} = e^{-\frac{t}{10}}$

$\Rightarrow x = 200 - 100e^{-\frac{t}{10}}$

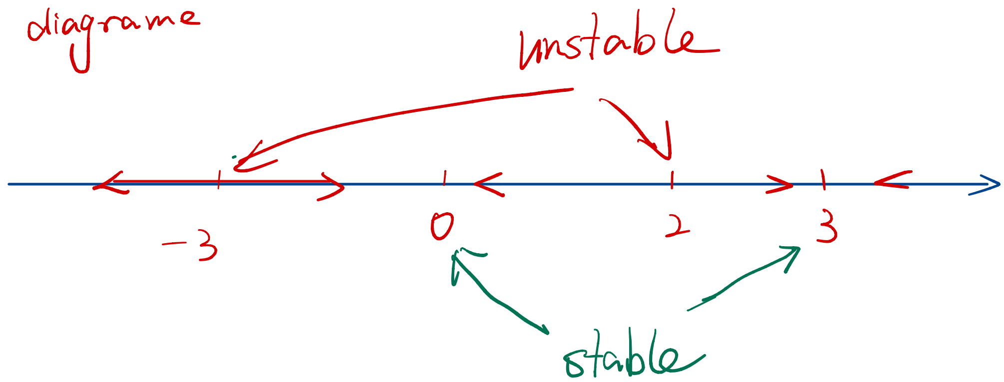
$\Rightarrow x(10) = 200 - 100e^{-1}$

3. Which one of the following statements is true about equilibrium solutions to

$$y' = y(2-y)\left(y^2 - 9\right)?$$

Critical pts are $y = 0, 2, -3, 3$

phase diagrame



4. Find the general solution to the differential equation

$$y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = 0$$

$$\Rightarrow (r+2)^2 = 0$$

$$\Rightarrow r = -2, -2$$

5. For the initial value problem $y' = t^2 + y^2, y(1) = 2$, use the Euler method with $h = 0.5$ to find an approximate value of $y(2)$.

$$y(1.5) \approx 2 + \frac{1}{2}(1^2 + 2^2) = \frac{9}{2}$$

$$y(2) \approx \frac{9}{2} + \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + \left(\frac{9}{2}\right)^2\right)$$

$$= \frac{9}{2} + \frac{1}{2}\left(\frac{9}{4} + \frac{81}{4}\right)$$

$$= \frac{9}{2} + \frac{90}{8}$$

$$= 15\frac{3}{4}$$

6. Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{xy^2}{x^2+1}, \quad y(0) = 3$$

$$\Rightarrow \int \frac{dy}{y^2} = \int \frac{x}{1+x^2}\,dx$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{2}\ln(1+x^2) + C$$

$$\xrightarrow{y(0)=3} -\frac{1}{3} = 0 + C \Rightarrow C = -\frac{1}{3}$$

$$y = \frac{-1}{\frac{1}{2}\ln(1+x^2) - \frac{1}{3}}$$

7. The change of variables $v = y/x^2$ transforms the equation

$$\frac{dy}{dx} = \sin\left(y/x^2\right) \quad \text{into}$$

$$v = y/x^2 \Rightarrow y = v x^2$$

$$\frac{dy}{dx} = 2x v + x^2 \frac{dv}{dx}$$

Thus $2xv + x^2 v' = \sin v$

8. Solve the initial value problem

$$2\frac{dy}{dx} + 3y = e^{2x}, \quad y(0) = 1$$

standard form: $y' + \frac{3}{2} y = \frac{1}{2} e^{2x}$

Integrating factor $\rho = e^{\int \frac{3}{2} dx} = e^{\frac{3}{2}x}$

$$e^{\frac{3}{2}x}\left(y' + \frac{3}{2}y\right) = \frac{1}{2} e^{2x} e^{\frac{3}{2}x}$$

$$\Rightarrow e^{\frac{3}{2}x} y = \frac{1}{2}\int e^{\frac{7}{2}x} dx = \frac{1}{7} e^{\frac{7}{2}x} + c$$

As, $x=0$, $y=1$, $1 \cdot 1 = \frac{1}{7} \cdot 1 + c$

$$\Rightarrow c = \frac{6}{7}$$

Thus $y = \frac{1}{7} e^{2x} + \frac{6}{7} e^{-\frac{3}{2}x}$

9. Explain why the following equation is exact and find an implicit formula for the solution to the initial value problem

$$\overset{M}{\left(\frac{y}{x}+6x\right)}+\overset{N}{(\ln x-2y)}\frac{dy}{dx}=0, \quad y(1)=2; \quad x>0$$

$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$. Thus exact.

$F(x,y) = \int M(x,y)\,dx = \int\left(\frac{y}{x}+6x\right)dx = y\ln x + 3x^2 + g(y)$

$\frac{\partial F}{\partial y} = \ln x + 0 + \frac{dg(y)}{dy} = N = \ln x - 2y$

$\Rightarrow \frac{dg(y)}{dy} = -2y$

$\Rightarrow g(y) = -y^2$

Thus general solution is $F(x,y) = y\ln x + 3x^2 - y^2 = C$

As $x=1$, $y=2$, $2\cdot 0 + 3 - 4 = C \Rightarrow C = -1$

Thus $y\ln x + 3x^2 - y^2 = -1$

10. Solve the initial value problem

$$y''-y'-2y=0, \quad y(0)=1, \quad y'(0)=1$$

$r^2 - r - 2 = 0$

$\Rightarrow (r-2)(r+1) = 0$

$\Rightarrow r = 2,\ r = -1$

$y = C_1 e^{-t} + C_2 e^{2t}$

As $y(0) = 1$, $\underline{C_1 + C_2 = 1}$

As $y'(0) = 1$, $y' = -C_1 e^{-t} + 2C_2 e^{2t}$

$\underline{-C_1 + 2C_2 = 1}$

$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + 2C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1}{3} \\ C_2 = \frac{2}{3} \end{cases}$ Thus $y = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t}$