INTRODUCTION

TO

LINEAR

ALGEBRA

Fourth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

3 This problem gives the diagonals $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v}=(3,3)$ and $\boldsymbol{w}=(2,-2)$.

4 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=($ add first answers$)=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination gives $(0,0,0)$. Stated another way: $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero. $c=3$ and $d=9$ give $(3,3,-6)$.

7 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=(0,1,2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Three possible parallelograms!

10 $\boldsymbol{i}-\boldsymbol{j}=(1,1,0)$ is in the base (x-y plane). $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=(1,1,1)$ is the opposite corner from $(0,0,0)$. Points in the cube have $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

11 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Centers of faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 A four-dimensional cube has $2^4=16$ corners and $2\cdot 4=8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum $=$ zero vector. Sum $=-2{:}00$ vector $=8{:}00$ vector. $2{:}00$ is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j}=(0,1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j}=(0,12)$.

15 The point $\dfrac{3}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u}=\dfrac{1}{2}\boldsymbol{v}+\dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V}=-\boldsymbol{v}+2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\ge 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\le c\le 1$ and $0\le d\le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square.

19 With $c\ge 0$ and $d\ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, then the cone is the whole quadrant $x\ge 0$, $y\ge 0$. *Question*: What if $\boldsymbol{w}=-\boldsymbol{v}$? The cone opens to a half-space.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1,2) + 4(3,1) = (14,8)$.

27 The combinations of $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1,1,0)$ fill the xy plane in xyz space.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4,5,6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2,5,8)$. Add to find $2\boldsymbol{v} = (6,10,14)$ so $\boldsymbol{v} = (3,5,7)$ and $\boldsymbol{w} = (1,0,-1)$.

29 Two combinations out of infinitely many that produce $\boldsymbol{b} = (0,1)$ are $-2\boldsymbol{u} + \boldsymbol{v}$ and $\frac{1}{2}\boldsymbol{w} - \frac{1}{2}\boldsymbol{v}$. **No**, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the x-y plane could fail to produce $\boldsymbol{b}$ if all three lie on a line that does not contain $\boldsymbol{b}$. *Yes*, if one combination produces $\boldsymbol{b}$ then two (and infinitely many) combinations will produce $\boldsymbol{b}$. This is true even if $\boldsymbol{u} = \mathbf{0}$; the combinations can have different $c\boldsymbol{u}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} 2c & -d & & = 1 \\ -c & +2d & -e & = 0 \\ & -d & +2e & = 0 \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 19

1 $\boldsymbol{u} \cdot \boldsymbol{v} = -1.8 + 3.2 = 1.4$, $\boldsymbol{u} \cdot \boldsymbol{w} = -4.8 + 4.8 = 0$, $\boldsymbol{v} \cdot \boldsymbol{w} = 24 + 24 = 48 = \boldsymbol{w} \cdot \boldsymbol{v}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 10$. Then $1.4 < (1)(5)$ and $48 < (5)(10)$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (3,1)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2,1,2)/3$. $\boldsymbol{U}_1 = (1,-3)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-1,3)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1,-2,0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a *line*.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60^\circ$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90^\circ$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60^\circ$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135^\circ$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1+v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular!

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v}\cdot\boldsymbol{w} = 0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ means angle $> 90^\circ$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5) - c(1,1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v}\cdot(\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c,d,-c)$. In that plane, $\boldsymbol{v} = (1,0,-1)$ and $\boldsymbol{w} = (0,1,0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u} = (1,-1,0,0)$, $\boldsymbol{v} = (0,0,1,-1)$, $\boldsymbol{w} = (1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in their 3D hyperplane."

15 $\frac{1}{2}(x+y) = (2+8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

16 $\|\boldsymbol{v}\|^2 = 1+1+\cdots+1 = 9$ so $\|\boldsymbol{v}\| = 3$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3},\ldots,\frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

18 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20 + 5$.

19 Start from the rules (1), (2), (3) for $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})$ and $(c\boldsymbol{v})\cdot\boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v} + (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w}$. By rule (1) this is $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w}) + \boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$! The main point is to be free to open up parentheses.

20 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. When $\theta < 90^\circ$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2+y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2+y^2) \le 1$.

26 The vectors $\boldsymbol{w} = (x, y)$ with $(1,2)\cdot\boldsymbol{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1,2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w} = (1,2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

27 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than 90° with each other: for example $(1,0), (-1,4), (-1,-4)$. Four vectors could *not* do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n+1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n+1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n+1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

29 For a specific example, pick $\boldsymbol{v} = (1,2,-3)$ and then $\boldsymbol{w} = (-3,1,2)$. In this example $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x + y + z = 0$:

$$\boldsymbol{v}\cdot\boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $\boldsymbol{v}\cdot\boldsymbol{w} = 0 - \dfrac{1}{2}\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \dfrac{1}{2}$.

30 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x+y+z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A) =$ same as A! So $a \ge g$ says that $A^3 \ge g^2A = x(y+z-A)A$. But $(y+z-A)A = (y-A)(A-z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1x_2\cdots x_n)^{1/n} \le A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

31 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

32 The commands $V = \mathbf{randn}\,(3,30)$; $D = \mathbf{sqrt}\,(\mathbf{diag}\,(V' * V))$; $U = V \backslash D$; will give 30 random unit vectors in the columns of U. Then $u' * U$ is a row matrix of 30 dot products whose average absolute value may be close to $2/\pi$.

Problem Set 1.3, page 29

1 $2s_1 + 3s_2 + 4s_3 = (2,5,9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2,3,4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1)\cdot \boldsymbol{x} \\ (\text{row } 2)\cdot \boldsymbol{x} \\ (\text{row } 2)\cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

3
$$\begin{aligned} y_1 \qquad\qquad &= B_1 \\ y_1 + y_2 \qquad &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= \qquad -B_2 + B_3 \end{aligned} \quad = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: independent columns in A and S!

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

6 $c = 3$ $\quad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & \mathbf{3} \end{bmatrix}$ has column 3 = 2 (column 1) + column 2

$c = -1$ $\quad \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column 3 = − column 1 + column 2

$c = 0$ $\quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column 3 = 3 (column 1) − column 2

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8
$$\begin{aligned} x_1 - 0 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned} \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

10 $\begin{aligned} z_2 - z_1 &= b_1 \\ z_3 - z_2 &= b_2 \\ 0 - z_3 &= b_3 \end{aligned} \qquad \begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= \quad\;\; -b_2 - b_3 \\ z_3 &= \qquad\quad\;\; -b_3 \end{aligned} \qquad = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

3 The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

5 If x, y, z satisfy the first two equations they also satisfy the third equation. The line **L** of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

6 Equation $1 +$ equation $2 -$ equation 3 is now $0 = -4$. Line misses plane; *no solution*.

7 Column $3 =$ Column 1 makes the matrix singular. Solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns).

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3$, $y + z + t = 3$, $z + t = 3$, $t = 2$.

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x}$. By rows or by columns: **9** separate multiplications for 3 by 3.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 $A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space, but the columns are in m-dimensional space.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

15 (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 $90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z). Q is the inverse of P.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$ and $E^{-1}E\boldsymbol{v}$ recovers $(3, 4, 5)$.

20 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ has $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by $45°$. The columns of R are the results from rotating $(1, 0)$ and $(0, 1)$!

22 The dot product $A\boldsymbol{x} = [1 \;\; 4 \;\; 5]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

24 $A * \boldsymbol{v} = [3 \;\; 4 \;\; 5]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

25 **ones**$(4, 4) *$ **ones**$(4, 1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

26 The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$.

27 The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a ***line***.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15,15,15)$; $M_4(1,1,1,1) = (34,34,34,34)$ because $1+2+\cdots+16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5,7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1,\ldots,1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1+\cdots+9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

1 Multiply by $\ell_{21} = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. (The two lines in the row picture are the same line, containing all solutions).

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or 3, that avoids a row operation.

13

$$\begin{aligned} 2x - 3y &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d - 10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

16 (a) **Example of 2 exchanges**

$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down**

$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(rows 1 and 3 are not consistent)

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

18 *Example* $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions.

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows 1+2 = row 3 on the left side but not the right side: $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 4$. No parallel planes but still no solution.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$ after elimination. Back substitution gives $t = 4, z = -3, y = 2, x = -1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If ordinary elimination leads to $x + y = 1$ and $2y = 3$, the original second equation could be $2y + \ell(x + y) = 3 + \ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y = 3$.

24 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$.

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

26 Solvable for $s = 10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2 + s$ on the right sides). The four equations for a, b, c, d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

27 Elimination leaves the diagonal matrix diag$(3, 2, 1)$ in $3x = 3, 2y = 2, z = 4$. Then $x = 1, y = 1, z = 4$.

28 $A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

29 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! *With row exchanges* in MATLAB's lu code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5, 5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1, \dots, j$ of A. If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

32 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **eye**(100). A better example has 99 random rows (or the numbers $1^i, \dots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 63

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}b = (1, -5, -35)$ but $E_{21}E_{32}b = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5, y = \frac{1}{2}, \boldsymbol{x} = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mathbf{7} & 0 & 1 \end{bmatrix}$ is $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}$.

8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. $\det M* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$!

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

12 The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB. A column that starts at zero will stay at zero. (b) E could add row 2 to row 3 to change a zero row to a nonzero row.

14 E_{21} has $-\ell_{21} = \frac{1}{2}$, E_{32} has $-\ell_{32} = \frac{2}{3}$, E_{43} has $-\ell_{43} = \frac{3}{4}$. Otherwise the E's match I.

15 $a_{ij} = 2i - 3j$: $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & \mathbf{0} & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -\mathbf{12} & -24 \end{bmatrix}$. The zero became -12, an example of *fill-in*. To remove that -12, choose $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

16 (a) The ages of X and Y are x and y: $x - 2y = 0$ and $x + y = 33$; $x = 22$ and $y = 11$ (b) The line $y = mx + c$ contains $x = 2$, $y = 5$ and $x = 3$, $y = 7$ when $2m + c = 5$ and $3m + c = 7$. Then $m = 2$ is the slope.

17 The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{aligned} a + \ \ b + \ \ c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$. Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1,1,1)$, $(1,2,3)$, $(1,4,9)$ is a "Vandermonde matrix."

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. In the opposite order, two row exchanges give $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, If M exchanges rows 2 and 3 then $M^2 = I$ (also $(-M)^2 = I$). There are many square roots of I: Any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

20 (a) Each column of EB is E times a column of B (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ \mathbf{2} & \mathbf{4} & \mathbf{8} \end{bmatrix}$. All rows of EB are *multiples* of $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.

21 **No.** $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ give $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ but $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

22 (a) $\sum a_{3j} x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j} x_j$.

23 $E(EA)$ subtracts 4 times row 1 from row 2 (EEA does the row operation twice). AE subtracts 2 times column 2 of A from column 1 (multiplication by E on the right side acts on **columns** instead of rows).

24 $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 4 & 1 & \mathbf{17} \end{bmatrix} \to \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 0 & -5 & \mathbf{15} \end{bmatrix}$. The triangular system is $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$ Back substitution gives $x_1 = 5$ and $x_2 = -3$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

26 (a) Add two columns $\boldsymbol{b}$ and $\boldsymbol{b}^*$ $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \to \boldsymbol{x} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ and $\boldsymbol{x}^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Many solutions if $d = 0 = c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

29 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ subtracts each row from the next row. The result $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ still has multipliers $= 1$ in a 3 by 3 Pascal matrix. The product M of all elimination matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page 88.

30 Given positive integers with $ad - bc = 1$. Certainly $c < a$ and $b < d$ would be impossible. Also $c > a$ and $b > d$ would be impossible with integers. This leaves row 1 $<$ row 2 OR row 2 $<$ row 1. An example is $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Multiply by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then multiply twice by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This shows that $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

31 $E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2/3 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$,

$$E_{43} E_{32} E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/3 & 2/3 & 1 & \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix}$$

Problem Set 2.4, page 75

1 If all entries of A, B, C, D are 1, then $BA = 3\,\mathbf{ones}(5)$ is 5 by 5; $AB = 5\,\mathbf{ones}(3)$ is 3 by 3; $ABD = 15\,\mathbf{ones}(3,1)$ is 3 by 1. DBA and $A(B+C)$ are not defined.

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

3 $AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 $A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from column 1 of AB and row 2 of C (multiply columns times rows).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False: usually $(AB)^2 \neq A^2B^2$.

8 The rows of DA are 3 (row 1 of A) and 5 (row 2 of A). Both rows of EA are row 2 of A. The columns of AD are 3 (column 1 of A) and 5 (column 2 of A). The first column of AE is zero, the second is column 1 of A + column 2 of A.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

12 $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \mathbf{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with B and C (and all other matrices) are multiples of I: $A = aI$.

13 $(A-B)^2 = (B-A)^2 = A(A-B) - B(A-B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A-B)^2$.

14 (a) True (A^2 is only defined when A is square) (b) False (if A is m by n and B is n by m, then AB is m by m and BA is n by n). (c) True (d) False (take $B = 0$).

15 (a) mn (use every entry of A) (b) $mnp = p \times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

17 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i, j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$ (this will be an example of a *rank one matrix*).

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

20 $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^4 =$ zero matrix for *strictly triangular* A.

Then $A\boldsymbol{v} = A \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^4\boldsymbol{v} = 0$.

21 $A = A^2 = A^3 = \cdots = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ but $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$ and $(AB)^2 =$ zero matrix!

22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

23 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. Note: Any matrix $A =$ column times row $= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ will have $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = 0$ if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but $A^3 = 0$; strictly triangular as in Problem 20.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

25 $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ h \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

26 **Columns of A times rows of B** $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} =$
$\begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix} = AB$.

27 (a) (row 3 of A) $\cdot$ (column 1 of B) and (row 3 of A) $\cdot$ (column 2 of B) are both zero.
(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: **both upper**.

28 A times B with cuts $A\begin{bmatrix} \mid & \mid & \mid \end{bmatrix}$, $\begin{bmatrix} \text{---} \end{bmatrix}B$, $\begin{bmatrix} \text{---} \end{bmatrix}\begin{bmatrix} \mid & \mid & \mid \end{bmatrix}$, $\begin{bmatrix} \mid & \mid \end{bmatrix}\begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$

29 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ produce zeros in the $2,1$ and $3,1$ entries.
Multiply E's to get $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. Then $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is the result of both E's since $(E_{31}E_{21})A = E_{31}(E_{21}A)$.

30 In **29**, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{c}\boldsymbol{b}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

31 $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ B\boldsymbol{x} + A\boldsymbol{y} \end{bmatrix}$ real part / imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.

32 A times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

33 $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

34 $A *$ **ones** $= \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ agrees with **ones** $*A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$ when $b = c$ and $a = d$

Then $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

35 $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$,

aba, ada	**cba, cda**	These show
bab, bcb	**dab, dcb**	16 2-step
abc, adc	**cbc, cdc**	paths in
bad, bcd	**dad, dcd**	the graph

36 Multiplying AB =(m by n)(n by p) needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC = (n$ by $p)(p$ by $q)$ needs npq and then $A(BC)$ needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7) + (2)(7)(10) =$ **196** with the larger number $(2)(4)(10) + (4)(7)(10) =$ **360**. So AB first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are N by 1, then $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ needs $2N$ multiplications but $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$ needs N^2 to find $\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}$ and N^2 more to multiply by the row vector $\boldsymbol{u}^{\mathrm{T}}$. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1} + n^{-1}$ with $p^{-1} + m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying $\boldsymbol{v}$ for $AB\boldsymbol{v}$, **don't multiply the matrices first**.

37 The proof of $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$ used the column rule for matrix multiplication—this rule is clearly linear, column by column.

Even for nonlinear transformations, $A(B(\boldsymbol{c}))$ would be the *"composition"* of A with B (applying B then A). This composition $A \circ B$ is just AB for matrices.

One of many uses for the associative law: The left-inverse B = right-inverse C from $B = B(AC) = (BA)C = C$.

Problem Set 2.5, page 89

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 A simple row exchange has $P^2 = I$ so $P^{-1} = P$. Here $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} = "transpose" of P, coming in Section 2.7.

3 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination.

4 The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.

5 An upper triangular U with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any a. And also $-U$.

6 (a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7 (a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \boldsymbol{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

9 If you exchange rows 1 and 2 of A to reach B, you exchange **columns** 1 and 2 of A^{-1} to reach B^{-1}. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this P.

10 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B).

11 (a) If $B = -A$ then certainly $A + B$ = zero matrix is not invertible. (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

13 $M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by C and the right by A : $B^{-1} = CM^{-1}A$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

15 If A has a column of zeros, so does BA. Then $BA = I$ is impossible. There is no A^{-1}.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

17 $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E.$

Reverse the order and change -1 to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} = L = E^{-1}$. Notice the 1's unchanged by multiplying in this order.

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

19 The $(1,1)$ entry requires $4a - 3b = 1$; the $(1,2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b = a$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

20 $A * \mathrm{ones}(4,1)$ is the zero vector so A cannot be invertible.

21 Six of the sixteen $0-1$ matrices are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$;
$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$.

23 $[\boldsymbol{A} \;\; \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to$
$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to$
$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] =$
$[\boldsymbol{I} \;\; \boldsymbol{A}^{-1}]$.

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

25 $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{\mathbf{-1}} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so B^{-1} does not exist.

26 $E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. $E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach $DE_{12}E_{21}A = I$. Then $A^{-1} = DE_{12}E_{21} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$.
This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$: row exchanges are certainly allowed in Gauss-Jordan.

29 (a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible) (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows) (c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

30 This A is not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

32 $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. When the triangular A alternates 1 and -1 on its diagonal, A^{-1} is *bidiagonal* with 1's on the diagonal and first superdiagonal.

33 $\boldsymbol{x} = (1, 1, \dots, 1)$ has $P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P - Q)\boldsymbol{x} = \boldsymbol{0}$.

34 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

35 A can be invertible with diagonal zeros. B is singular because each row adds to zero.

36 The equation $LDLD = I$ says that $LD = \mathsf{pascal}\,(4, 1)$ is its own inverse.

37 hilb(6) is not the exact Hilbert matrix because fractions are rounded off. So inv(hilb(6)) is not the exact either.

38 The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then $\mathsf{inv}(P) = \mathsf{inv}(L^{\mathrm{T}})\mathsf{inv}(L)$.

39 $Ax = b$ has many solutions when $A = \mathsf{ones}\,(4, 4) =$ singular matrix and $b = \mathsf{ones}\,(4, 1)$. $A\backslash \boldsymbol{b}$ in MATLAB will pick the shortest solution $\boldsymbol{x} = (1, 1, 1, 1)/4$. This is the only solution that is combination of the rows of A (later it comes from the "pseudoinverse" $A^{+} = \mathsf{pinv}(A)$ which replaces A^{-1} when A is singular). Any vector that solves $A\boldsymbol{x} = \boldsymbol{0}$ could be added to this particular solution $\boldsymbol{x}$.

40 The inverse of $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (This would be a good example for the cofactor formula $A^{-1} = C^{\mathrm{T}}/\det A$ in Section 5.3)

41 The product $\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix}$ that in this order the multipliers shows a, b, c, d, e, f are unchanged in the product (**important for $A = LU$ in Section 2.6**).

42 $MM^{-1} = (I_n - UV)\ (I_n + U(I_m - VU)^{-1}V)$ (this is testing formula **3**)
$= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$ (keep simplifying)
$= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$ (formulas **1**, **2**, **4** are similar)

43 4 by 4 still with $T_{11} = 1$ has pivots $1, 1, 1, 1$; reversing to $T^{*} = UL$ makes $T^{*}_{44} = 1$.

44 Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

45 The block pivots are A and $S = D - CA^{-1}B$ (and $d - cb/a$ is the correct second pivot of an ordinary 2 by 2 matrix). The example problem has $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix}\dfrac{1}{2}\begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -6 & -5 \end{bmatrix}$.

46 Inverting the identity $A(I+BA)=(I+AB)A$ gives $(I+BA)^{-1}A^{-1}=A^{-1}(I+AB)^{-1}$. So $I+BA$ and $I+AB$ are both invertible or both singular when A is invertible. (This remains true also when A is singular: Problem 6.6.19 will show that AB and BA have the same nonzero eigenvalues, and we are looking here at $\lambda=-1$.)

Problem Set 2.6, page 102

1 $\ell_{21}=1$ multiplied row 1; $L=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ times $\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}=\boldsymbol{c}$ is $A\boldsymbol{x}=b$: $\begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\7\end{bmatrix}$.

2 $L\boldsymbol{c}=\boldsymbol{b}$ is $\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}5\\7\end{bmatrix}$, solved by $\boldsymbol{c}=\begin{bmatrix}5\\2\end{bmatrix}$ as elimination goes forward.
$U\boldsymbol{x}=\boldsymbol{c}$ is $\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}$, solved by $\boldsymbol{x}=\begin{bmatrix}3\\2\end{bmatrix}$ in back substitution.

3 $\ell_{31}=1$ and $\ell_{32}=2$ (and $\ell_{33}=1$): reverse steps to get $A\boldsymbol{u}=\boldsymbol{b}$ from $U\boldsymbol{x}=\boldsymbol{c}$: 1 times $(x+y+z=5)+2$ times $(y+2z=2)+1$ times $(z=2)$ gives $x+3y+6z=11$.

4 $L\boldsymbol{c}=\begin{bmatrix}1&&\\1&1&\\1&2&1\end{bmatrix}\begin{bmatrix}5\\2\\2\end{bmatrix}=\begin{bmatrix}5\\7\\11\end{bmatrix}$; $\quad U\boldsymbol{x}=\begin{bmatrix}1&1&1\\&1&2\\&&1\end{bmatrix}\begin{bmatrix}\\\boldsymbol{x}\\\\\end{bmatrix}=\begin{bmatrix}5\\2\\2\end{bmatrix}$; $\boldsymbol{x}=\begin{bmatrix}5\\-2\\2\end{bmatrix}$.

5 $EA=\begin{bmatrix}1&&\\0&1&\\-3&0&1\end{bmatrix}\begin{bmatrix}2&1&0\\0&4&2\\6&3&5\end{bmatrix}=\begin{bmatrix}2&1&0\\0&4&2\\0&0&5\end{bmatrix}=U$. With E^{-1} as L, $A=LU=\begin{bmatrix}1&&\\0&1&\\3&0&1\end{bmatrix}U$.

6 $\begin{bmatrix}1&&\\0&1&\\0&-2&1\end{bmatrix}\begin{bmatrix}1&&\\-2&1&\\0&0&1\end{bmatrix}A=\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}=U$. Then $A=\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}U$ is the same as $E_{21}^{-1}E_{32}^{-1}U=LU$. The multipliers $\ell_{21},\ell_{32}=2$ fall into place in L.

7 $E_{32}E_{31}E_{21}\ A=\begin{bmatrix}1&&\\&1&\\&\mathbf{-2}&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\\mathbf{-3}&&1\end{bmatrix}\begin{bmatrix}1&&\\\mathbf{-2}&1&\\&&1\end{bmatrix}\begin{bmatrix}1&0&0\\2&2&2\\3&4&5\end{bmatrix}$. This is $\begin{bmatrix}1&0&1\\0&2&0\\0&0&2\end{bmatrix}=U$. Put those multipliers 2, 3, 2 into L. Then $A=\begin{bmatrix}1&0&0\\\mathbf{2}&1&0\\\mathbf{3}&\mathbf{2}&1\end{bmatrix}U=LU$.

8 $E=E_{32}E_{31}E_{21}=\begin{bmatrix}1&&\\&1&\\&-c&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\-b&&1\end{bmatrix}\begin{bmatrix}1&&\\-a&1&\\&&1\end{bmatrix}=\begin{bmatrix}1&&\\-a&1&\\ac-b&-c&1\end{bmatrix}$.
The multipliers are just a,b,c and the upper triangular U is I. In this case $A=L$ and its inverse is that matrix $E=L^{-1}$.

9 2 by 2: $d=0$ not allowed; $\begin{bmatrix}1&1&0\\1&1&2\\1&2&1\end{bmatrix}=\begin{bmatrix}1&&\\l&1&\\m&n&1\end{bmatrix}\begin{bmatrix}d&e&g\\&f&h\\&&i\end{bmatrix}$ $d=1,e=1$, then $l=1$
$f=0$ is not allowed
no pivot in row 2

10 $c = 2$ leads to zero in the second pivot position: exchange rows and not singular. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

11 $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$ has $L = I$ (A is already upper triangular) and $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$;
$A = LU$ has $U = A$; $A = LDU$ has $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ with 1's on the diagonal.

12 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; $\boldsymbol{U}$ **is** $\boldsymbol{L}^{\mathrm{T}}$

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

13 $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$. Need $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$. All of the multipliers are $\ell_{ij} = 1$ for this A

14 $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $a \neq 0$, $b \neq r$, $c \neq s$, $d \neq t$

15 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.
$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Forward to $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boldsymbol{c}$.

16 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}c = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ gives $c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}x = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ gives $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.
Those are the forward elimination and back substitution steps for
$A\boldsymbol{x} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

17 (a) L goes to I (b) I goes to L^{-1} (c) LU goes to U. Elimination multiply by L^{-1}!

18 (a) Multiply $LDU = L_1D_1U_1$ by inverses to get $L_1^{-1}LD = D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.
(b) L, U, L_1, U_1 have diagonal 1's so $D = D_1$. Then $L_1^{-1}L$ and U_1U^{-1} are both I.

19 $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU$; $\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = (\text{same } L)\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$ (same U). A tridiagonal matrix A has **bidiagonal factors** L and U.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find ℓ and then one for the new pivot!). $T =$ bidiagonal L times bidiagonal U.

21 For the first matrix A, L keeps the 3 lower zeros at the start of rows. But U may not have the upper zero where $A_{24} = 0$. For the second matrix B, L keeps the bottom left zero at the start of row 4. U keeps the upper right zero at the start of column 4. One zero in A and two zeros in B are filled in.

22 Eliminating *upwards*, $\begin{bmatrix}5&3&1\\3&3&1\\1&1&1\end{bmatrix} \to \begin{bmatrix}4&2&0\\2&2&0\\1&1&1\end{bmatrix} \to \begin{bmatrix}2&0&0\\2&2&0\\1&1&1\end{bmatrix} = L$. We reach a *lower* triangular L, and the multipliers are in an *upper* triangular U. $A = \boldsymbol{UL}$ with $U = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$.

23 The 2 by 2 upper submatrix A_2 has the first two pivots $5, 9$. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

24 The upper left blocks all factor at the same time as A: A_k is $L_k U_k$.

25 The i, j entry of L^{-1} is j/i for $i \geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

26 $(K^{-1})_{ij} = j(n-i+1)/(n+1)$ for $i \geq j$ (and symmetric): $(n+1)K^{-1}$ looks good.

Problem Set 2.7, page 115

1 $A = \begin{bmatrix}1&0\\9&3\end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix}1&9\\0&3\end{bmatrix}$, $A^{-1} = \begin{bmatrix}1&0\\-3&1/3\end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix}1&-3\\0&1/3\end{bmatrix}$;

$A = \begin{bmatrix}1&c\\c&0\end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \dfrac{1}{c^2}\begin{bmatrix}0&c\\c&-1\end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. Transpose that to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

3 (a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$. (b) If U is upper triangular, so is U^{-1}: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

4 $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

5 (a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&\mathbf{5}&6\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \mathbf{5}$ (b) $\boldsymbol{x}^{\mathrm{T}}A = \begin{bmatrix}4&5&6\end{bmatrix}$ (c) $A\boldsymbol{y} = \begin{bmatrix}2\\5\end{bmatrix}$.

6 $M^{\mathrm{T}} = \begin{bmatrix}A^{\mathrm{T}}&C^{\mathrm{T}}\\B^{\mathrm{T}}&D^{\mathrm{T}}\end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

7 (a) False: $\begin{bmatrix}0&A\\A&0\end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$. (b) False: The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$ when A and B are symmetric $\begin{bmatrix}0&A\\A&0\end{bmatrix}$ transposes to $\begin{bmatrix}0&A^{\mathrm{T}}\\A^{\mathrm{T}}&0\end{bmatrix}$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$. (c) True: Invertible symmetric matrices have symmetric in verses! Easiest proof is to transpose $AA^{-1} = I$. (d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices A, B, and C).

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices $\ldots$ ($n!$ overall).

9 $P_1P_2 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ but $P_2P_1 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$.
If P_3 and P_4 exchange *different* pairs of rows, $P_3P_4 = P_4P_3$ does both exchanges.

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. $(1,2,3,4)$, $(4,3,2,1)$ make 12 even P's.

11 $PA = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix} = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$ is upper triangular. Multiplying *on the right* by a permutation matrix P_2 exchanges the *columns*. To make this A lower triangular, we also need P_1 to exchange rows 2 and 3: $P_1AP_2 = \begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix} A \begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix} = \begin{bmatrix}6&0&0\\5&4&0\\3&2&1\end{bmatrix}$.

12 $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ since $P^{\mathrm{T}}P = I$. In general $P\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot P^{\mathrm{T}}\boldsymbol{y} \neq \boldsymbol{x}\cdot P\boldsymbol{y}$:
Non-equality where $P \neq P^{\mathrm{T}}$: $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\2\end{bmatrix} \neq \begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}$.

13 A cyclic $P = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ or its transpose will have $P^3 = I$: $(1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$. $\widehat{P} = \begin{bmatrix}1&0\\0&P\end{bmatrix}$ for the same P has $\widehat{P}^4 = \widehat{P} \neq I$.

14 The "reverse identity" P takes $(1,\ldots,n)$ into $(n,\ldots,1)$. When rows and also columns are reversed, $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$. In particular $(PAP)_{11}$ is A_{nn}.

15 (a) If P sends row 1 to row 4, then P^{T} sends row 4 to row 1 (b) $P = \begin{bmatrix}E&0\\0&E\end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

16 $A^2 - B^2$ (but not $(A+B)(A-B)$, this is different) and also ABA are symmetric if A and B are symmetric.

17 (a) $A = \begin{bmatrix}1&1\\1&1\end{bmatrix} = A^{\mathrm{T}}$ is not invertible (b) $A = \begin{bmatrix}0&1\\1&1\end{bmatrix}$ needs row exchange (c) $A = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ has $D = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

19 (a) The transpose of $R^{\mathrm{T}}AR$ is $R^{\mathrm{T}}A^{\mathrm{T}}R^{\mathrm{T}\,\mathrm{T}} = R^{\mathrm{T}}AR = n$ by n when $A^{\mathrm{T}} = A$ (any m by n matrix R) (b) $(R^{\mathrm{T}}R)_{jj} =$ (column j of R)$\cdot$(column j of R) $=$ (length squared of column j) ≥ 0.

20 $\begin{bmatrix}1&3\\3&2\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\0&-7\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}$; $\begin{bmatrix}1&b\\b&c\end{bmatrix}=\begin{bmatrix}1&0\\b&1\end{bmatrix}\begin{bmatrix}1&0\\0&c-b^2\end{bmatrix}\begin{bmatrix}1&b\\0&1\end{bmatrix}$

$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}=\begin{bmatrix}1&&\\-\frac{1}{2}&1&\\0&-\frac{2}{3}&1\end{bmatrix}\begin{bmatrix}2&&\\&\frac{3}{2}&\\&&\frac{4}{3}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}&0\\&1&-\frac{2}{3}\\&&1\end{bmatrix}=\boldsymbol{LDL}^{\mathrm{T}}.$

21 Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix. The examples $\begin{bmatrix}2&4&8\\4&3&9\\8&9&0\end{bmatrix}$ and $\begin{bmatrix}1&b&c\\b&d&e\\c&e&f\end{bmatrix}$ lead to $\begin{bmatrix}-5&-7\\-7&-32\end{bmatrix}$ and $\begin{bmatrix}d-b^2&e-bc\\e-bc&f-c^2\end{bmatrix}$.

22 $\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix}A=\begin{bmatrix}1&&\\0&1&\\2&3&1\end{bmatrix}\begin{bmatrix}1&0&1\\&1&1\\&&-1\end{bmatrix}$; $\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}A=\begin{bmatrix}1&&\\1&1&\\2&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\&-1&1\\&&1\end{bmatrix}$

23 $A=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=P$ and $L=U=I$. This cyclic P exchanges rows 1-2 then rows 2-3 then rows 3-4.

24 $PA=LU$ is $\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}\begin{bmatrix}0&1&2\\0&3&8\\2&1&1\end{bmatrix}=\begin{bmatrix}1&&\\0&1&\\0&1/3&1\end{bmatrix}\begin{bmatrix}2&1&1\\&3&8\\&&-2/3\end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A=L_1P_1U_1=\begin{bmatrix}1&&\\3&1&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&2\\0&0&2\end{bmatrix}$.

25 The **splu** code will not end when **abs**$(A(k,k))<$ tol line 4 of the **slu** code on page 100. Instead **splu** looks for a nonzero entry below the diagonal in the current column k, and executes a row exchange. The 4 lines to exchange row k with row r are at the end of Section 2.7 (page 113). To *find* that nonzero entry $A(r,k)$, follow **abs**$(A(k,k))<$ tol by locating the first nonzero (or the largest $A(r,k)$ out of $r=k+1,\ldots,n$).

26 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

27 (a) $E_{21}=\begin{bmatrix}1&&\\-3&1&\\&&1\end{bmatrix}$ puts 0 in the $2,1$ entry of $E_{21}A$. Then $E_{21}AE_{21}^{\mathrm{T}}=\begin{bmatrix}1&0&0\\0&2&4\\0&4&9\end{bmatrix}$ is still symmetric, with zero also in its $1,2$ entry. (b) Now use $E_{32}=\begin{bmatrix}1&&\\&1&\\&-4&1\end{bmatrix}$ to make the $3,2$ entry zero and $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}}=D$ also has zero in its $2,3$ entry. Key point: Elimination from both sides gives the symmetric LDL^{T} directly.

28 $A=\begin{bmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{bmatrix}=A^{\mathrm{T}}$ has $0,1,2,3$ in every row. (I don't know any rules for a symmetric construction like this)

29 Reordering the rows and/or the columns of $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ will move the entry **a**. So the result cannot be the transpose (which doesn't move **a**).

30 (a) Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

(b) Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$.

31 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x};\ A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck 1 plane

32 $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost* of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value* of outputs.

33 $P^3 = I$ so three rotations for 360°; P rotates around $(1, 1, 1)$ by 120°.

34 $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = EH =$ (elementory matrix) times (symmetric matrix).

35 $L(U^{\mathrm{T}})^{-1}$ is lower triangular times lower triangular, so lower triangular. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{T}\,\mathrm{T}} = U^{\mathrm{T}}DU$ again, so $U^{\mathrm{T}}DU$ is symmetric. The factorization multiplies lower triangular by symmetric to get LDU which is A.

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

37 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U =$ upper triangular.

38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

39 To split A into (symmetric B) + (anti-symmetric C), the only choice is $B = \frac{1}{2}(A + A^{\mathrm{T}})$ and $C = \frac{1}{2}(A - A^{\mathrm{T}})$.

40 Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for Q is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Problem Set 3.1, page 127

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$.
The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

6 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1 + c_2)\boldsymbol{f} = \boldsymbol{f}((c_1 + c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.

8 If $(\boldsymbol{f} + \boldsymbol{g})(x)$ is the usual $\boldsymbol{f}(\boldsymbol{g}(x))$ then $(\boldsymbol{g} + \boldsymbol{f})x$ is $\boldsymbol{g}(\boldsymbol{f}(x))$ which is different. In Rule 2 both sides are $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{h}(x)))$. Rule 4 is broken there might be no inverse function $\boldsymbol{f}^{-1}(x)$ such that $\boldsymbol{f}(\boldsymbol{f}^{-1}(x)) = x$. If the inverse function exists it will be the vector $-\boldsymbol{f}$.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

10 The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

12 For the plane $x + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

13 The parallel plane $\mathbf{P}_0$ has the equation $x + y - 2z = 0$. Pick two points, for example $(2, 0, 1)$ and $(0, 2, 1)$, and their sum $(2, 2, 2)$ is in $\mathbf{P}_0$.

14 (a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0, 0)$, and $(0, 0)$ by itself (b) The subspaces of $\mathbf{R}^4$ are $\mathbf{R}^4$ itself, three-dimensional planes $\boldsymbol{n} \cdot \boldsymbol{v} = 0$, two-dimensional subspaces $(\boldsymbol{n}_1 \cdot \boldsymbol{v} = 0$ and $\boldsymbol{n}_2 \cdot \boldsymbol{v} = 0)$, one-dimensional lines through $(0, 0, 0, 0)$, and $(0, 0, 0, 0)$ by itself.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$
(b) The plane and line probably intersect in the point $(0, 0, 0)$
(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

16 The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

17 (a) The invertible matrices do not include the zero matrix, so they are not a subspace
(b) The sum of singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular: not a subspace.

18 (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}} = -A$ do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.

19 The column space of A is the x-axis = all vectors $(x, 0, 0)$. The column space of B is the xy plane = all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + 2b_3$ in equation 3: Solution only if $b_3 = -b_1$.

21 A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space.

22 (a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.
$[\,A \quad \boldsymbol{b}\,] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{x} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ ($\boldsymbol{b}$ is in column space) ($A\boldsymbol{x} = \boldsymbol{b}$ has a solution)

24 The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B = 0$ and $A \neq 0$ is a case when $AB = 0$ has a smaller column space than A.

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

26 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

27 (a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$. (d) *False*: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

28 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $(1, 1, 1)$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line.

29 When $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\mathbf{R}^9$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

Problem Set 3.2, page 140

1 (a) $U=\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3 (b) $U=\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}$ Free x_3 Pivot x_1, x_2

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, $(0,0,-3,0,1)$ (b) Free variable x_3: solution $(1,-1,1)$. Special solution for each free variable.

3 The complete solution to $A\boldsymbol{x} = \mathbf{0}$ is $(-2x_2, x_2, -2x_4 - 3x_5, x_4, x_5)$ with x_2, x_4, x_5 free. The complete solution to $B\boldsymbol{x} = \mathbf{0}$ is $(2x_3, -x_3, x_3)$. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when there are no free variables.

4 $R = \begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$, $R = \begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$, R has the same nullspace as U and A.

5 $A = \begin{bmatrix}-1&3&5\\-2&6&10\end{bmatrix} = \begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}-1&3&5\\0&0&0\end{bmatrix}$; $B = \begin{bmatrix}-1&3&5\\-2&6&7\end{bmatrix} = \begin{bmatrix}1&0\\2&1\end{bmatrix}$ $\begin{bmatrix}-1&3&5\\0&0&-3\end{bmatrix} = LU$.

6 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. Total of pivot and free is n.

7 (a) The nullspace of A in Problem 5 is the plane $-x + 3y + 5z = 0$; it contains all the vectors $(3y + 5z, y, z) = y(3,1,0) + z(5,0,1) =$ combination of special solutions. (b) The *line* through $(3,1,0)$ has equations $-x+3y+5z = 0$ and $-2x+6y+7z = 0$. The special solution for the free variable x_2 is $(3,1,0)$.

8 $R = \begin{bmatrix}1&-3&-5\\0&0&0\end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix}1&-3&0\\0&0&1\end{bmatrix}$ with $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

9 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

10 (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I$, $R = I$.

11 $\begin{bmatrix}0&1&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\\0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1&1&1&1\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$

12 $\begin{bmatrix}\mathbf{1}&1&0&1&1&1&0&0\\0&0&\mathbf{1}&1&1&1&0&0\\0&0&0&0&0&0&\mathbf{1}&0\\0&0&0&0&0&0&0&\mathbf{1}\end{bmatrix}$, $\begin{bmatrix}0&\mathbf{1}&1&0&0&1&1&1\\0&0&0&\mathbf{1}&0&1&1&1\\0&0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&0&0&0&0\end{bmatrix}$. Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

13 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $\boldsymbol{x} = (0,0,0,1,0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

15 If a matrix has n columns and r pivots, there are $\boldsymbol{n - r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r = n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All important!

16 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

17 $A = [1 \ \ -3 \ \ -1]$ gives the plane $x - 3y - z = 0$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

18 Fill in **12** then **4** then **1** to get the complete solution to $x - 3y - z = 12$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{4} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{x}_{\text{particular}} + \boldsymbol{x}_{\text{nullspace}}$.

19 If $LU\boldsymbol{x} = \mathbf{0}$, multiply by L^{-1} to find $U\boldsymbol{x} = \mathbf{0}$. Then U and LU have the same nullspace.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

21 For special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables x_3, x_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

22 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through $(4, 3, 2, 1)$.

23 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $(1, 1, 5)$ and $(0, 3, 1)$ in $\boldsymbol{C}(A)$ and $(1, 1, 2)$ in $\boldsymbol{N}(A)$. *Which other A's?*

24 This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

25 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1, 1, 1)$ in $\boldsymbol{C}(A)$ and only the line (c, c, c, c) in $\boldsymbol{N}(A)$.

26
30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice $\text{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

27 If nullspace = column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

28 If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ *equals* $\boldsymbol{N}(A)$. (For $B = 0, \boldsymbol{C}(B)$ is smaller.)

29 For A = random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with fourth row of zeros. What about a random 3 by 4 matrix?

31 If $\boldsymbol{N}(A)$ = line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

32 Any zero rows come after these rows: $R = [\,1 \quad -2 \quad -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's !

34 One reason that R is the same for A and $-A$: They have the same nullspace. They also have the same column space, but that is not required for two matrices to share the same R. (R tells us the nullspace and row space.)

35 The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

37 *Currents*: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 3.3, page 151

1 (a) and (c) are correct; (b) is completely false; (d) is false because R might have 1's in nonpivot columns.

2 $A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$ has $R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The rank is $r = 1$;

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ has $R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The rank is $r = 2$;

$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ has $R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The rank is $r = 1$

3 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = [\,R_A \quad R_A\,]$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero rows go to the bottom

4 If all pivot variables come last then $R = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

5 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

6 A and A^{T} have the same rank $r =$ number of pivots. But *pivcol* (the column number) is 2 for this matrix A and 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

7 Special solutions in $N = [\,-2 \quad -4 \quad 1 \quad 0;\ -3 \quad -5 \quad 0 \quad 1\,]$ and $[\,1 \quad 0 \quad 0;\ 0 \quad -2 \quad 1\,]$.

8 The new entries keep rank 1: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{2} & 6 & \mathbf{-3} \\ 1 & \mathbf{3} & \mathbf{-3/2} \\ 2 & 6 & -3 \end{bmatrix}$,

$M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$.

9 If A has rank 1, the column space is a *line* in $\mathbf{R}^m$. The nullspace is a *plane* in $\mathbf{R}^n$ (given by one equation). The nullspace matrix N is n by $n-1$ (with $n-1$ special solutions in its columns). The column space of A^{T} is a *line* in $\mathbf{R}^n$.

10 $\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

11 A rank one matrix has one pivot. (That pivot is in row 1 after possible row exchange; it could come in any column.) The second row of U is zero.

12 Invertible r by r submatrices
Use pivot rows and columns $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

15 The product of rank one matrices has rank one or zero. These particular matrices have rank$(AB) = 1$; rank$(AM) = 1$ except $AM = 0$ if $c = -1/2$.

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

17 (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \leq$ rank (B). No new pivot columns! (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. It drops to zero for $A_2 = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$.

18 If we know that rank$(B^{\mathrm{T}}A^{\mathrm{T}}) \leq$ rank(A^{T}), then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have rank$(AB) \leq$ rank(A).

19 We are given $AB = I$ which has rank n. Then rank$(AB) \leq$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as the R they share.
(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 $A =$ (pivot columns)(nonzero rows of R) $= \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} +$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$ columns times rows $= \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

23 If $c = 1, R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free. If $c \neq 1, R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_3, x_4 free. Special solutions in $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for $c = 1$) and $N = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (for $c \neq 1$). If $c = 1, R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and x_1 free; if $c = 2, R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ and x_2 free; $R = I$ if $c \neq 1, 2$. Special solutions in $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(c = 1)$ or $N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $(c = 2)$ or $N = 2$ by 0 empty matrix.

24 $A = \begin{bmatrix} I & I \end{bmatrix}$ has $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$ has the same N; $C = \begin{bmatrix} I & I & I \end{bmatrix}$ has $N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$.

25 $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & 2 & 3 \\ \mathbf{0} & \mathbf{1} & 0 & 1 \end{bmatrix} =$ (pivot columns) times R.

26 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

27 $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; $\mathbf{rref}(R^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\mathbf{rref}(R^{\mathrm{T}}R) =$ same R

28 The *row-column reduced echelon form* is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is r by r.

Problem Set 3.4, page 163

1 $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - 2\mathbf{b}_1 \end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $s_1 = (-1, -1, 1, 0)$ and $s_2 = (2, -2, 0, 1)$; $x_{complete} = x_p + c_1 s_1 + c_2 s_2$;

$$\begin{bmatrix} R & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ gives the particular solution $x_p = (4, -1, 0, 0)$.

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix}$ Then $[\,R \quad \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) =$ line through $(2,6,4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 $\boldsymbol{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. The matrix is singular but the equations are still solvable; $\boldsymbol{b}$ is in the column space. Our particular solution has free variable $y = 0$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

5 $\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.
Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \mathbf{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$
(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[0 \;\; 0 \;\; 0 \;\; 0] =$ row 3 − 2 (row 2) + 4(row 1) **provided $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$**.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) We need $b_3 = 2b_2$, because $(\text{row }3) - 2(\text{row }2) = \mathbf{0}$.

9 $L\begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix}$
$= \begin{bmatrix} A & b \end{bmatrix}$; particular $\boldsymbol{x}_p = (-9, 0, 3, 0)$ means $-9(1,2,3) + 3(3,8,7) = (0,6,-6)$. This is $A\boldsymbol{x}_p = \boldsymbol{b}$.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$.

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $\boldsymbol{c}_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F\,]$ for any 3 by 2 matrix F.

17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The solution is *unique*. The nullspace contains only the zero *vector*. An example is $A = R = [\, I \quad F\,]$ for any 4 by 2 matrix F.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 Both matrices A have rank 2. Always $A^{\mathrm{T}}A$ and AA^{T} have **the same rank** as A.

20 $A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$; $A = LU \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathbf{2} & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$.

21 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution.

22 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

24 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}[\, x\,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b\,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

26 $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R = I$.

27 If U has n pivots, then R has n pivots **equal to 1**. Zeros above and below those pivots make $R = I$.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

29 $[\, R \ \ \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ leads to $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $[\, R \ \ \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$: no solution because of the 3rd equation

30 $\begin{bmatrix}1&0&2&3&\mathbf{2}\\1&3&2&0&\mathbf{5}\\2&0&4&9&\mathbf{10}\end{bmatrix}\to\begin{bmatrix}1&0&2&3&\mathbf{2}\\0&3&0&-3&\mathbf{3}\\0&0&0&3&\mathbf{6}\end{bmatrix}\to\begin{bmatrix}1&0&2&0&\mathbf{-4}\\0&1&0&0&\mathbf{3}\\0&0&0&1&\mathbf{2}\end{bmatrix};\begin{bmatrix}-4\\3\\0\\2\end{bmatrix};\boldsymbol{x}_n=x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}.$

31 For $A=\begin{bmatrix}1&1\\0&2\\0&3\end{bmatrix}$, the only solution to $A\boldsymbol{x}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ is $\boldsymbol{x}=\begin{bmatrix}0\\1\end{bmatrix}$. B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 $A=\begin{bmatrix}1&3&1\\1&2&3\\2&4&6\\1&1&5\end{bmatrix}$ factors into $LU=\begin{bmatrix}1&&&\\1&1&&\\2&2&1&\\1&2&0&1\end{bmatrix}\begin{bmatrix}1&3&1\\0&-1&2\\0&0&0\\0&0&0\end{bmatrix}$ and the rank is $r=2$. The special solution to $A\boldsymbol{x}=\boldsymbol{0}$ and $U\boldsymbol{x}=\boldsymbol{0}$ is $\boldsymbol{s}=(-7,2,1)$. Since $\boldsymbol{b}=(1,3,6,5)$ is also the last column of A, a particular solution to $A\boldsymbol{x}=\boldsymbol{b}$ is $(0,0,1)$ and the complete solution is $\boldsymbol{x}=(0,0,1)+c\boldsymbol{s}$. (Or use the particular solution $\boldsymbol{x}_p=(7,-2,0)$ with free variable $x_3=0$.)

For $\boldsymbol{b}=(1,0,0,0)$ elimination leads to $U\boldsymbol{x}=(1,-1,0,1)$ and the fourth equation is $0=1$. No solution for this $\boldsymbol{b}$.

33 If the complete solution to $A\boldsymbol{x}=\begin{bmatrix}1\\3\end{bmatrix}$ is $\boldsymbol{x}=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\c\end{bmatrix}$ then $A=\begin{bmatrix}1&0\\3&0\end{bmatrix}$.

34 (a) If $\boldsymbol{s}=(2,3,1,0)$ is the only special solution to $A\boldsymbol{x}=\boldsymbol{0}$, the complete solution is $\boldsymbol{x}=c\boldsymbol{s}$ (line of solution!). The rank of A must be $4-1=3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R must be $\begin{bmatrix}1&0&-2&0\\0&1&-3&0\\0&0&0&1\end{bmatrix}$.

(c) $A\boldsymbol{x}=\boldsymbol{b}$ can be solve for all $\boldsymbol{b}$, because A and R have *full row rank* $r=3$.

35 For the $-1,2,-1$ matrix K(9 by 9) and constant right side $\boldsymbol{b}=(10,\cdots,10)$, the solution $\boldsymbol{x}=K^{-1}\boldsymbol{b}=(45,80,105,120,125,120,105,80,45)$ rises and falls along the parabola $x_i=50i-5i^2$. (A formula for K^{-1} is later in the text.)

36 If $A\boldsymbol{x}=\boldsymbol{b}$ and $C\boldsymbol{x}=\boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b}=\boldsymbol{0}$). If $\boldsymbol{b}=$ column 1 of A, $\boldsymbol{x}=(1,0,\ldots,0)$ solves $A\boldsymbol{x}=\boldsymbol{b}$ so it solves $C\boldsymbol{x}=\boldsymbol{b}$. Then A and C share column 1. Other columns too: $A=C$!

Problem Set 3.5, page 178

1 $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=0$ gives $c_3=c_2=c_1=0$. So those 3 column vectors are independent. But $\begin{bmatrix}1&1&1&2\\0&1&1&3\\0&0&1&4\end{bmatrix}[\,\boldsymbol{c}\,]=\begin{bmatrix}0\\0\\0\end{bmatrix}$ is solved by $\boldsymbol{c}=(1,1,-4,1)$. Then $\boldsymbol{v}_1+\boldsymbol{v}_2-4\boldsymbol{v}_3+\boldsymbol{v}_4=\boldsymbol{0}$ (dependent).

2 $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1,1,1,1)\cdot\boldsymbol{v}=0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$. A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the difference are *dependent* and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

9 (a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\, \boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* "$\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$" —since $\boldsymbol{v}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{v}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{v}_1 + 3(0, 0, 0) = \mathbf{0}$.

10 The plane is the nullspace of $A = [\,1 \;\; 2 - 3 - 1\,]$. Three free variables give three solutions $(x, y, z, t) = (2, -1 - 0 - 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False*. The zero vector is always in the row space.

13 The column space and row space of A and U all have the same dimension = 2. *The row spaces of A and U are the same*, because the rows of U are combinations of the rows of A (and vice versa!).

14 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$. The two pairs *span* the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent*.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$.

16 These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $N(I) = \{\text{zero vector}\}$.

17 The column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row 1 + row 2) and (row 1 and − row 2) are bases for the row spaces of U.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

19 n-independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ *rank* $= m = n$. The rank counts the number of *independent* columns.

20 One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 (a) The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

23 Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces.

24 (a) *False* $A = [\,1 \ \ 1\,]$ has dependent columns, independent row (b) *False* column space $\neq$ row space for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) *True*: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) *False*, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

25 A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

26 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

27 I, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$; echelon matrices do *not* form a subspace; they ***span*** the upper triangular matrices (not every U is echelon).

28 $\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}$; $\begin{bmatrix}1&-1&0\\-1&1&0\end{bmatrix}$ and $\begin{bmatrix}1&0&-1\\-1&0&1\end{bmatrix}$.

29 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c) I by itself spans the space of all multiples cI.

30 $\begin{bmatrix}-1&2&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}-1&0&2\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&0\\-1&2&0\end{bmatrix}$, $\begin{bmatrix}0&0&0\\-1&0&2\end{bmatrix}$.

31 (a) $y(x) =$ constant C (b) $y(x) = 3x$ this is one basis for the 2 by 3 matrices with $(2, 1, 1)$ in their nullspace (4-dim subspace). (c) $y(x) = 3x + C = y_p + y_n$ solves $dy/dx = 3$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

33 (a) $y(x) = e^{2x}$ is a basis for, all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

35 Basis 1, x, x^2, x^3, for cubic polynomials; basis $x - 1$, $x^2 - 1$, $x^3 - 1$ for the subspace with $p(1) = \mathbf{0}$.

36 Basis for $\mathbf{S}$: $(1, 0, -1, 0)$, $(0, 1, 0, 0)$, $(1, 0, 0, -1)$; basis for $\mathbf{T}$: $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3, -3, 2, 1) =$ nullspace for 3 equation in $\mathbf{R}^4$ has dimension 1.

37 The subspace of matrices that have $AS = SA$ has dimension *three*.

38 (a) No, 2 vectors don't span $\mathbf{R}^3$ (b) No, 4 vectors in $\mathbf{R}^3$ are dependent (c) Yes, a basis (d) No, these three vectors are dependent

39 If the 5 by 5 matrix $[\,A\ \ \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A\ \ \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

40 (a) The functions $y = \sin x$, $y = \cos x$, $y = e^x$, $y = e^{-x}$ are a basis for solutions to $d^4y/dx^4 = y(x)$.

(b) A particular solution to $d^4y/dx^4 = y(x) + 1$ is $y(x) = -\mathbf{1}$. The complete solution is $y(x) = -1 + c, \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$ (or use another basis for the nullspace of the 4th derivative).

41 $\boldsymbol{I} = \begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix} - \begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix} + \begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix} + \begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix} - \begin{bmatrix}&&1\\1&&\\&1&\end{bmatrix}$. The six P's are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

42 The dimension of $\boldsymbol{S}$ spanned by all rearrangements of $\boldsymbol{x}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1,1,1,1)$ (c) three when $\boldsymbol{x} = (1,1,-1,-1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1,1,1,1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim $\boldsymbol{S} = 2$. I owe this nice problem to Mike Artin—the answers are the same in higher dimensions: $0, 1, n-1, n$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients = zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to an m by n matrix fill $\mathbf{R}^n$. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

45 If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

46 If A^2 = zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$, and we must have $r \le 10 - r$ and $r \le 5$.

Problem Set 3.6, page 190

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}})) = 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space basis = row 1 = $(1,2,4)$; nullspace $(-2,1,0)$ and $(-4,0,1)$; column space basis = column1 = $(1,2)$; left nullspace $(-2,1)$. B: Row space basis = both rows = $(1,2,4)$ and $(2,5,8)$; column space basis = two columns = $(1,2)$ and $(2,5)$; nullspace $(-4,0,1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$.

3 Row space basis = rows of $U = (0,1,2,3,4)$ and $(0,0,0,1,2)$; column space basis = pivot columns (of A not U) $= (1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1)$ = last row of E^{-1}!

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has those rows spanning its row space $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ has the same rows spanning its nullspace and $BA^{\mathrm{T}} = 0$.

6 A: dim $\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: dim $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix} = 3$ by 2 have *row space dimensions* $= 3, 3, 0 =$ column space dimensions; *nullspace dimensions* $2, 3, 2$; left *nullspace dimensions* $0, 2, 3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank$=3$, nullspace and left nullspace contain only $(0,0,0)$. For **rand** $(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here. (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \boldsymbol{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space (same dimension!) (b) *True*: A and $-A$ have the same four subspaces (c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

17 Row space $= yz$ plane; column space $= xy$ plane; nullspace $= x$ axis; left nullspace $= z$ axis. For $I + A$: Row space = column space $= \mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row $3 - 2$ row $2 +$ row $1 =$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x} = \boldsymbol{0}$ leads to $0 = b_3 - b_2 - b_1$ so $(-1,-1,1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3 - 2b_1 = 0$ and $b_4 + b_2 - 4b_1 = 0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace. *Why*? Those vectors multiply the matrix to give *zero rows*. Section 4.1 will show another approach: $A\boldsymbol{x} = \boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} & \boldsymbol{z}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$ has column space spanned by $\boldsymbol{u}$ and $\boldsymbol{w}$, row space spanned by $\boldsymbol{v}$ and $\boldsymbol{z}$.

23 As in Problem 22: Row space basis $(3,0,3),(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A = [1 \quad 0]$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A) = \boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}} = A$ or $A^{\mathrm{T}} = -A$ they are also the same for A^{T})

26 The rows of $C = AB$ are combinations of the rows of B. So rank $C \le$ rank B. Also rank $C \le$ rank A, because the columns of C are combinations of the columns of A.

27 Choose $d = bc/a$ to make $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ a rank-1 matrix. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$. Those two vectors are perpendicular !

28 B and C (checkers and chess) both have rank 2 if $p \ne 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero; $N(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $N(C)$ is a challenge.

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1.$

30 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \ne 0$.

31 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; dim(nullspace) $= 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank $= 1$).

2 The nullspace of a 3 by 2 matrix with rank 2 is $\mathbf{Z}$ (only zero vector) so $\boldsymbol{x}_n = \mathbf{0}$, and row space $= \mathbf{R}^2$. Column space $=$ plane perpendicular to left nullspace $=$ line in $\mathbf{R}^3$.

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2 = 0$; take $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

4 If $AB = 0$, the columns of B are in the *nullspace* of A. The rows of A are in the *left nullspace* of B. If rank $= 2$, those four subspaces would have dimension 2 which is impossible for 3 by 3.

5 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$. $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = 0$. (b) If $A^{\mathrm{T}}\boldsymbol{y} = (1,1,1)$ has a solution, $(1,1,1)$ is in the **row space** and is orthogonal to every $\boldsymbol{x}$ in the nullspace.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Equations add to $0 = 1$ so no solution: $\boldsymbol{y} = (1, 1, -1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

7 Multiply the 3 equations by $\boldsymbol{y} = (1, 1, -1)$. Then $x_1 - x_2 = 1$ plus $x_2 - x_3 = 1$ *minus* $x_1 - x_3 = 1$ is $0 = 1$. Key point: This $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is not orthogonal to $\boldsymbol{b} = (1, 1, 1)$ so $\boldsymbol{b}$ is not in the column space and $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \boldsymbol{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $\boldsymbol{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \boldsymbol{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

11 ***For A***: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is the perpendicular line through $(3, -1)$. ***For B***: The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for A.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1, -1) + (1, 1) = (2, 0)$. Notice $\boldsymbol{N}(A^{\mathrm{T}})$ is a plane $(1, 0) = (1, 1)/2 + (1, -1)/2 = \boldsymbol{x}_r + \boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W$ = zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $A\boldsymbol{x} = B\widehat{\boldsymbol{x}}$ means that $[\,A \quad B\,]\begin{bmatrix} \boldsymbol{x} \\ -\widehat{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $\boldsymbol{x} = (3, 1)$ and $\widehat{\boldsymbol{x}} = (1, 0)$ and $A\boldsymbol{x} = B\widehat{\boldsymbol{x}} = (5, 6, 5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

15 A p-dimensional and a q-dimensional subspace of $\mathbf{R}^n$ share at least a line if $\boldsymbol{p + q > n}$. (The $p + q$ basis vectors of $\boldsymbol{V}$ and $\boldsymbol{W}$ cannot be independent.)

16 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ leads to $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = 0$. Then $\boldsymbol{y} \perp A\boldsymbol{x}$ and $\boldsymbol{N}(A^{\mathrm{T}}) \perp \boldsymbol{C}(A)$.

17 If $\boldsymbol{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, then $\boldsymbol{S}^{\perp}$ is $\mathbf{R}^3$. If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, then $\boldsymbol{S}^{\perp}$ is the plane spanned by $(1, -1, 0)$ and $(1, 0, -1)$. If $\boldsymbol{S}$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, then $\boldsymbol{S}^{\perp}$ is the line spanned by $(0, 1, 0)$.

18 $\boldsymbol{S}^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $\boldsymbol{S}^{\perp}$ is a *subspace* even if $\boldsymbol{S}$ is not.

19 $\boldsymbol{L}^{\perp}$ is the 2-*dimensional subspace* (*a plane*) in $\mathbf{R}^3$ perpendicular to $\boldsymbol{L}$. Then $(\boldsymbol{L}^{\perp})^{\perp}$ is a 1-*dimensional subspace* (*a line*) perpendicular to $\boldsymbol{L}^{\perp}$. In fact $(\boldsymbol{L}^{\perp})^{\perp}$ is $\boldsymbol{L}$.

20 If $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$, then $\boldsymbol{V}^{\perp}$ contains only the *zero vector*. Then $(\boldsymbol{V}^{\perp})^{\perp} = \mathbf{R}^4 = \boldsymbol{V}$.

21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $\boldsymbol{S}^{\perp}$ = nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

22 $(1, 1, 1, 1)$ is a basis for $\boldsymbol{P}^{\perp}$. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has $\boldsymbol{P}$ as its nullspace and $\boldsymbol{P}^{\perp}$ as row space.

23 $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is perpendicular to any vector in $\boldsymbol{V}$. Since $\boldsymbol{V}$ contains all the vectors in $\boldsymbol{S}$, $\boldsymbol{x}$ is also perpendicular to any vector in $\boldsymbol{S}$. So every $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is also in $\boldsymbol{S}^{\perp}$.

24 $AA^{-1} = I$: Column 1 of A^{-1} is orthogonal to the space spanned by the 2nd, 3rd, ..., nth rows of A.

25 If the columns of A are unit vectors, all mutually perpendicular, then $A^{\mathrm{T}}A = I$.

26 $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, This example shows a matrix with perpendicular columns. $A^{\mathrm{T}}A = 9I$ is *diagonal*: $(A^{\mathrm{T}}A)_{ij} =$ (column i of A) $\cdot$ (column j of A). When the columns are *unit vectors*, then $A^{\mathrm{T}}A = I$.

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are **parallel**. They are the same line if $b_2 = 2b_1$. In that case (b_1, b_2) is perpendicular to $(-2, 1)$. The nullspace of the 2 by 2 matrix is the line $3x + y = 0$. One particular vector in the nullspace is $(-1, 3)$.

28 (a) $(1, -1, 0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

29 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$; A has $\boldsymbol{v} = (1, 2, 3)$ in row space and column space B has $\boldsymbol{v}$ in its column space and nullspace. $\boldsymbol{v}$ **can not** be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} \neq 0$.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $\boldsymbol{C}(B) \leq$ dimension of $\boldsymbol{N}(A)$. This means rank$(B) \leq 4 -$ rank(A).

31 null(N') produces a basis for the *row space* of A (perpendicular to $\boldsymbol{N}(A)$).

32 We need $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{n} = 0$ and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\ell} = 0$. All possible examples have the form $a\boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ with $a \neq 0$.

33 Both $\boldsymbol{r}$'s orthogonal to both $\boldsymbol{n}$'s, both $\boldsymbol{c}$'s orthogonal to both $\boldsymbol{\ell}$'s, each pair independent. All A's with these subspaces have the form $[\boldsymbol{c}_1\ \boldsymbol{c}_2]M[\boldsymbol{r}_1\ \boldsymbol{r}_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; $\boldsymbol{p} = 5\boldsymbol{a}/3$; $\boldsymbol{e} = (-2, 1, 1)/3$ (b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; $\boldsymbol{p} = \boldsymbol{a}$; $\boldsymbol{e} = \boldsymbol{0}$.

2 (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$ (b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1\boldsymbol{b} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

4 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. P_1 projects onto $(1, 0)$, P_2 projects onto $(1, -1)$ $P_1P_2 \neq 0$ and $P_1 + P_2$ is not a projection matrix.

5 $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. P_1 and P_2 are the projection matrices onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$ $P_1P_2 =$ *zero matrix because* $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

XXX Above solution does not fit in 3 lines.

6 $\boldsymbol{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = \boldsymbol{b}$.

7 $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$
We *can* add projections onto *orthogonal vectors*. This is important.

8 The projections of $(1,1)$ onto $(1,0)$ and $(1,2)$ are $\boldsymbol{p}_1 = (1,0)$ and $\boldsymbol{p}_2 = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project on all of $\mathbf{R}^2$.

10 $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}, P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P_1P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$. This is not $\boldsymbol{a}_1 = (1,0)$ *No,* $P_1P_2 \neq (P_1P_2)^2$.

11 (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2,3,0)$, $\boldsymbol{e} = (0,0,4)$, $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$ (b) $\boldsymbol{p} = (4,4,6)$, $\boldsymbol{e} = \mathbf{0}$.

12 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$ Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

13 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, P = square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

14 The projection of this $\boldsymbol{b}$ onto the column space of A is $\boldsymbol{b}$ itself when $\boldsymbol{b}$ is in that space.
But P is not necessarily I. $P = \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$ and $\boldsymbol{b} = P\boldsymbol{b} = \boldsymbol{p} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$.

15 $2A$ has the same column space as A. $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

17 If $P^2 = P$ then $\boldsymbol{(I-P)^2} = (I-P)(I-P) = I - PI - IP + P^2 = \boldsymbol{I - P}$. When P projects onto the column space, $I - P$ projects onto the *left nullspace*.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$
(b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

19 For any basis vectors in the plane $x - y - 2z = 0$, say $(1,1,0)$ and $(2,0,1)$, the matrix P is $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

20 $\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \frac{\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}}{\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$.

22 $P^{\mathrm{T}} = (A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}})^{\mathrm{T}} = A((A^{\mathrm{T}}A)^{-1})^{\mathrm{T}}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = P$. ($A^{\mathrm{T}}A$ is symmetric!)

23 If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $e = 0$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\mathbf{0}$.

25 **The column space of $\boldsymbol{P}$ will be $\boldsymbol{S}$.** Then r = dimension of $S = n$.

26 A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.

27 If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is in the nullspace of A^{T}. But $A\boldsymbol{x}$ is always in the column space of A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace*.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 $A = B^{\mathrm{T}}$ has independent columns, so $A^{\mathrm{T}}A$ (which is BB^{T}) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ (columns of A project to themselves) and $AP_R = A$. Then $P_C A P_R = A$!

31 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $\boldsymbol{C}(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

33 If $P_1P_2 = P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^{T} is invertible as in Problem 29. Then $(A^{\mathrm{T}}A)(BB^{\mathrm{T}})$ = product of r by r invertible matrices, so rank r. AB can't have rank $< r$, since A^{T} and B^{T} cannot increase the rank. *Conclusion*: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

$E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

2 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. This $A\boldsymbol{x} = \boldsymbol{b}$ is unsolvable. Change $\boldsymbol{b}$ to $\boldsymbol{p} = P\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$; $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ exactly solves $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$.

3 In Problem 2, $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (1,5,13,17)$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1,3,-5,3)$. $\boldsymbol{e}$ is perpendicular to both columns of A. This shortest distance $\|\boldsymbol{e}\|$ is $\sqrt{44}$.

4 $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$. These normal equations are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [\,1 \quad 1 \quad 1 \quad 1\,]$ and $A^{\mathrm{T}}A = [\,4\,]$. $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = 9 =$ best height C. Errors $\boldsymbol{e} = (-9, -1, -1, 11)$.

6 $\boldsymbol{a} = (1,1,1,1)$ and $\boldsymbol{b} = (0,8,8,20)$ give $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$ and the projection is $\widehat{x}\boldsymbol{a} = \boldsymbol{p} = (9,9,9,9)$. Then $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = (-9,-1,-1,11)^{\mathrm{T}}(1,1,1,1) = 0$ and $\|\boldsymbol{e}\| = \sqrt{204}$.

7 $A = [\,0 \quad 1 \quad 3 \quad 4\,]^{\mathrm{T}}$, $A^{\mathrm{T}}A = [\,26\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,112\,]$. Best $D = 112/26 = 56/13$.

8 $\widehat{\boldsymbol{x}} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

10 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Then $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$. Exact cubic so $\boldsymbol{p} = \boldsymbol{b}$, $\boldsymbol{e} = \boldsymbol{0}$. This Vandermonde matrix gives exact interpolation by a cubic at $0, 1, 3, 4$

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{\boldsymbol{b}} = 9$ when $\widehat{t} = 2$.
(b) The first equation $Cm + D\sum t_i = \sum b_i$ divided by m gives $C + D\widehat{t} = \widehat{\boldsymbol{b}}$.

12 (a) $\boldsymbol{a} = (1,\dots,1)$ has $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/m$ is the **mean** of the b's (b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$ $\boldsymbol{b} = (1,2,b)$ $\|\boldsymbol{e}\|^2 = \sum_{i=1}^{m}(b_i - \widehat{x})^2 =$ **variance**
(c) $\begin{matrix} \boldsymbol{p} = (3,3,3) \\ \boldsymbol{e} = (-2,-1,3) \end{matrix}$ $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. $P = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. When $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ averages to $\boldsymbol{0}$, so does $\widehat{\boldsymbol{x}} - \boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ is $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}$. When the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ will be the *output covariance matrix* $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\sigma^2 A(A^{\mathrm{T}}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm{T}}A)^{-1}$.

15 When A has 1 column of ones, Problem 14 gives the expected error $(\widehat{x} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} = \sigma^2/m$. By taking m measurements, the variance drops from σ^2 to σ^2/m.

16 $\dfrac{1}{10}b_{10} + \dfrac{\mathbf{9}}{\mathbf{10}}\widehat{x}_9 = \dfrac{1}{10}(b_1 + \cdots + b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

17 $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5,13,17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2,-6,4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

19 If $\boldsymbol{b} =$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \boldsymbol{0}$.

20 If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5,13,17)$ then $\widehat{\boldsymbol{x}} = (9,4)$ and $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A) = \{\boldsymbol{0}\} =$ zero vector only.

22 The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm{T}}A$

23 $\boldsymbol{e}$ is orthogonal to $\boldsymbol{p}$; then $\|\boldsymbol{e}\|^2 = e^{\mathrm{T}}(\boldsymbol{b} - \boldsymbol{p}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{p}$.

24 The derivatives of $\|A\boldsymbol{x} - b\|^2 = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} - 2\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$ (this term is constant) are zero when $2A^{\mathrm{T}}A\boldsymbol{x} = 2A^{\mathrm{T}}\boldsymbol{b}$, or $\boldsymbol{x} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

25 3 points on a line: *Equal slopes* $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$. Linear algebra: Orthogonal to $(1, 1, 1)$ and (t_1, t_2, t_3) is $\boldsymbol{y} = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$.

26 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$. At $x, y = 0, 0$ the best plane $2 - x - \frac{3}{2}y$ has height $C = \mathbf{2} =$ average of $0, 1, 3, 4$.

27 The shortest link connecting two lines in space is *perpendicular to those lines*.

28 Only 1 plane contains $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

29 There is exactly one hyperplane containing the n points $\mathbf{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ *When the $n-1$ vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are linearly independent*. (For $n = 3$, the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ must be independent. Then the three points $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ determine a plane.) The equation of the plane in $\mathbf{R}^n$ will be $\boldsymbol{a}_n^{\mathrm{T}}\boldsymbol{x} = 0$. Here $\boldsymbol{a}_n$ is any nonzero vector on the line (it is only a line!) perpendicular to $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$.

Problem Set 4.4, page 239

1 (a) *Independent* (b) *Independent* and *orthogonal* (c) *Independent* and *orthonormal*. For orthonormal vectors, (a) becomes $(1, 0)$, $(0, 1)$ and (b) is $(.6, .8)$, $(.8, -.6)$.

2 Divide by length 3 to get $\boldsymbol{q}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. $\boldsymbol{q}_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. $Q^{\mathrm{T}}Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $QQ^{\mathrm{T}} = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$.

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries 1, 4, 9.

4 (a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $QQ^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$. Any Q with $n < m$ has $QQ^{\mathrm{T}} \neq I$. (b) $(1, 0)$ and $(0, 0)$ are *orthogonal*, not *independent*. Nonzero orthogonal vectors *are* independent. (c) Starting from $\boldsymbol{q}_1 = (1, 1, 1)/\sqrt{3}$ my favorite is $\boldsymbol{q}_2 = (1, -1, 0)/\sqrt{2}$ and $\boldsymbol{q}_3 = (1, 1, -2)/\sqrt{b}$.

5 *Orthogonal* vectors are $(1, -1, 0)$ and $(1, 1, -1)$. *Orthonormal* are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm T}Q_1Q_2 = Q_2^{\mathrm T}Q_1^{\mathrm T}Q_1Q_2 = Q_2^{\mathrm T}Q_2 = I$.

7 When Gram-Schmidt gives Q with orthonormal columns, $Q^{\mathrm T}Q\widehat{\boldsymbol x} = Q^{\mathrm T}\boldsymbol b$ becomes $\widehat{\boldsymbol x} = Q^{\mathrm T}\boldsymbol b$.

8 If $\boldsymbol q_1$ and $\boldsymbol q_2$ are *orthonormal* vectors in $\mathbf R^5$ then $(\boldsymbol q_1^{\mathrm T}\boldsymbol b)\boldsymbol q_1 + (\boldsymbol q_2^{\mathrm T}\boldsymbol b)\boldsymbol q_2$ is closest to $\boldsymbol b$.

9 (a) $Q = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}$ has $P = QQ^{\mathrm T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(QQ^{\mathrm T})(QQ^{\mathrm T}) = Q(Q^{\mathrm T}Q)Q^{\mathrm T} = QQ^{\mathrm T}$.

10 (a) If $\boldsymbol q_1, \boldsymbol q_2, \boldsymbol q_3$ are *orthonormal* then the dot product of $\boldsymbol q_1$ with $c_1\boldsymbol q_1 + c_2\boldsymbol q_2 + c_3\boldsymbol q_3 = \mathbf 0$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. *Independent* q's (b) $Q\boldsymbol x = \mathbf 0 \Rightarrow Q^{\mathrm T}Q\boldsymbol x = \mathbf 0 \Rightarrow \boldsymbol x = \mathbf 0$.

11 (a) Two *orthonormal* vectors are $\boldsymbol q_1 = \frac{1}{10}(1,3,4,5,7)$ and $\boldsymbol q_2 = \frac{1}{10}(-7,3,4,-5,1)$
(b) Closest in the plane: *project* $QQ^{\mathrm T}(1,0,0,0,0) = (0.5, -0.18, -0.24, 0.4, 0)$.

12 (a) Orthonormal $\boldsymbol a$'s: $\boldsymbol a_1^{\mathrm T}\boldsymbol b = \boldsymbol a_1^{\mathrm T}(x_1\boldsymbol a_1 + x_2\boldsymbol a_2 + x_3\boldsymbol a_3) = x_1(\boldsymbol a_1^{\mathrm T}\boldsymbol a_1) = x_1$
(b) Orthogonal $\boldsymbol a$'s: $\boldsymbol a_1^{\mathrm T}\boldsymbol b = \boldsymbol a_1^{\mathrm T}(x_1\boldsymbol a_1 + x_2\boldsymbol a_2 + x_3\boldsymbol a_3) = x_1(\boldsymbol a_1^{\mathrm T}\boldsymbol a_1)$. Therefore $x_1 = \boldsymbol a_1^{\mathrm T}\boldsymbol b/\boldsymbol a_1^{\mathrm T}\boldsymbol a_1$
(c) x_1 is the first component of A^{-1} times $\boldsymbol b$.

13 The multiple to subtract is $\frac{\boldsymbol a^{\mathrm T}\boldsymbol b}{\boldsymbol a^{\mathrm T}\boldsymbol a}$. Then $\boldsymbol B = \boldsymbol b - \frac{\boldsymbol a^{\mathrm T}\boldsymbol b}{\boldsymbol a^{\mathrm T}\boldsymbol a}\boldsymbol a = (4,0) - 2\cdot(1,1) = (2,-2)$.

14 $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol q_1 & \boldsymbol q_2 \end{bmatrix}\begin{bmatrix} \|\boldsymbol a\| & \boldsymbol q_1^{\mathrm T}\boldsymbol b \\ 0 & \|\boldsymbol B\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt2 & 1/\sqrt2 \\ 1/\sqrt2 & -1/\sqrt2 \end{bmatrix}\begin{bmatrix} \sqrt2 & 2\sqrt2 \\ 0 & 2\sqrt2 \end{bmatrix} = QR.$

15 (a) $\boldsymbol q_1 = \frac13(1,2,-2)$, $\boldsymbol q_2 = \frac13(2,1,2)$, $\boldsymbol q_3 = \frac13(2,-2,-1)$ (b) The nullspace of $A^{\mathrm T}$ contains $\boldsymbol q_3$ (c) $\widehat{\boldsymbol x} = (A^{\mathrm T}A)^{-1}A^{\mathrm T}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol p = (\boldsymbol a^{\mathrm T}\boldsymbol b/\boldsymbol a^{\mathrm T}\boldsymbol a)\boldsymbol a = 14\boldsymbol a/49 = 2\boldsymbol a/7$ is closest to $\boldsymbol b$; $\boldsymbol q_1 = \boldsymbol a/\|\boldsymbol a\| = \boldsymbol a/7$ is $(4,5,2,2)/7$. $\boldsymbol B = \boldsymbol b - \boldsymbol p = (-1,4,-4,-4)/7$ has $\|\boldsymbol B\| = 1$ so $\boldsymbol q_2 = \boldsymbol B$.

17 $p = (\boldsymbol a^{\mathrm T}\boldsymbol b/\boldsymbol a^{\mathrm T}\boldsymbol a)\boldsymbol a = (3,3,3)$ and $\boldsymbol e = (-2,0,2)$. $\boldsymbol q_1 = (1,1,1)/\sqrt3$ and $\boldsymbol q_2 = (-1,0,1)/\sqrt2$.

18 $\boldsymbol A = \boldsymbol a = (1,-1,0,0)$; $\boldsymbol B = \boldsymbol b - \boldsymbol p = (\frac12, \frac12, -1, 0)$; $\boldsymbol C = \boldsymbol c - \boldsymbol p_A - \boldsymbol p_B = (\frac13, \frac13, \frac13, -1)$. Notice the pattern in those orthogonal $\boldsymbol A, \boldsymbol B, \boldsymbol C$. In $\mathbf R^5$, $\boldsymbol D$ would be $(\frac14, \frac14, \frac14, \frac14, -1)$.

19 If $A = QR$ then $A^{\mathrm T}A = R^{\mathrm T}Q^{\mathrm T}QR = R^{\mathrm T}R =$ *lower* triangular times *upper* triangular (this Cholesky factorization of $A^{\mathrm T}A$ uses the same R as Gram-Schmidt!). The example has $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac13\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$ and the same R appears in $A^{\mathrm T}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^{\mathrm T}R$.

20 (a) *True* (b) *True*. $Q\boldsymbol x = x_1\boldsymbol q_1 + x_2\boldsymbol q_2$. $\|Q\boldsymbol x\|^2 = x_1^2 + x_2^2$ because $\boldsymbol q_1\cdot\boldsymbol q_2 = 0$.

21 The orthonormal vectors are $\boldsymbol q_1 = (1,1,1,1)/2$ and $\boldsymbol q_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol b = (-4,-3,3,0)$ projects to $\boldsymbol p = (-7,-3,-1,3)/2$. And $\boldsymbol b - \boldsymbol p = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol q_1$ and $\boldsymbol q_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. These are not yet unit vectors.

23 You can see why $\boldsymbol{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \boldsymbol{q}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \boldsymbol{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&4\\0&3&6\\0&0&5\end{bmatrix} = QR$.

24 (a) One basis for the subspace $\boldsymbol{S}$ of solutions to $x_1 + x_2 + x_3 - x_4 = 0$ is $\boldsymbol{v}_1 = (1,-1,0,0)$, $\boldsymbol{v}_2 = (1,0,-1,0)$, $\boldsymbol{v}_3 = (1,0,0,1)$ (b) Since $\boldsymbol{S}$ contains solutions to $(1,1,1,-1)^{\mathrm{T}}\boldsymbol{x} = 0$, a basis for $\boldsymbol{S}^{\perp}$ is $(1,1,1,-1)$ (c) Split $(1,1,1,1) = \boldsymbol{b}_1 + \boldsymbol{b}_2$ by projection on $\boldsymbol{S}^{\perp}$ and $\boldsymbol{S}$: $\boldsymbol{b}_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $\boldsymbol{b}_1 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$.

25 This question shows 2 by 2 formulas for QR; breakdown $R_{22} = 0$ when A is singular. $\begin{bmatrix}2&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2&-1\\1&2\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}5&3\\0&1\end{bmatrix}$. Singular $\begin{bmatrix}1&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}2&2\\0&\mathbf{0}\end{bmatrix}$. The Gram-Schmidt process breaks down when $ad - bc = 0$.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

27 When a and b are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. We must use the orthogonal A and B (or orthonormal $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$) to be allowed to add 1D projections.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

29 $\boldsymbol{q}_1 = \frac{1}{3}(2,2,-1)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,-1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(1,-2,-2)$.

30 The columns of the wavelet matrix W are *orthonormal*. Then $W^{-1} = W^{\mathrm{T}}$. See Section 7.2 for more about wavelets: a useful orthonormal basis with many zeros.

31 (a) $c = \frac{1}{2}$ normalizes all the orthogonal columns to have unit length (b) The projection $(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a}$ of $\boldsymbol{b} = (1,1,1,1)$ onto the first column is $\boldsymbol{p}_1 = \frac{1}{2}(-1,1,1,1)$. (Check $\boldsymbol{e} = \mathbf{0}$.) To project onto the plane, add $\boldsymbol{p}_2 = \frac{1}{2}(1,-1,1,1)$ to get $(0,0,1,1)$.

32 $Q_1 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

34 (a) $Q\boldsymbol{u} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$. This is $-\boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ equals 1 (b) $Q\boldsymbol{v} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$.

35 Starting from $\boldsymbol{A} = (1,-1,0,0)$, the orthogonal (not orthonormal) vectors $\boldsymbol{B} = (1,1,-2,0)$ and $\boldsymbol{C} = (1,1,1,-3)$ and $\boldsymbol{D} = (1,1,1,1)$ are in the directions of $\boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$. The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal Q!) are $\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D}\end{bmatrix} = \begin{bmatrix}1&1&1&1\\-1&1&1&1\\0&-2&1&1\\0&0&-3&1\end{bmatrix}$ and

$$\begin{bmatrix}1&1&1&1&1\\-1&1&1&1&1\\0&-2&1&1&1\\0&0&-3&1&1\\0&0&0&-4&1\end{bmatrix}$$

36 $[Q, R] = \boldsymbol{qr}(A)$ produces from A (m by n of rank n) a "*full-size*" square $Q = [\,Q_1 \;\; Q_2\,]$ and $\begin{bmatrix} R \\ 0 \end{bmatrix}$. The columns of Q_1 are the orthonormal basis from Gram-Schmidt of the *column space* of A. The $m-n$ columns of Q_2 are an orthonormal basis for the *left nullspace* of A. Together the columns of $Q = [\,Q_1 \;\; Q_2\,]$ are an orthonormal basis for $\mathbf{R}^m$.

37 This question describes the next $\boldsymbol{q}_{n+1}$ in Gram-Schmidt using the matrix Q with the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ (instead of using those $\boldsymbol{q}$'s separately). Start from $\boldsymbol{a}$, *subtract its projection* $\boldsymbol{p} = Q^{\mathrm{T}}\boldsymbol{a}$ onto the earlier $\boldsymbol{q}$'s, *divide by the length* of $\boldsymbol{e} = \boldsymbol{a} - Q^{\mathrm{T}}\boldsymbol{a}$ to get $\boldsymbol{q}_{n+1} = \boldsymbol{e}/\|\boldsymbol{e}\|$.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$.

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (b) *True*: The product rule extends to ABC (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$ (d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

4 Exchange rows 1 and 3 to show $|J_3| = -1$. Exchange rows 1 and 4, then 2 and 3 to show $|J_4| = 1$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

6 To prove Rule 6, multiply the zero row by $t = 2$. The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So $2\det(A) = \det(A)$ and $\det(A) = 0$.

7 $\det(Q) = 1$ for rotation and $\det(Q) = -1$ for reflection $(1 - 2\sin^2\theta - 2\cos^2\theta = -1)$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so det can't blow up.

9 $\det A = 1$ from two row exchanges . $\det B = 2$ (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). $\det C = 0$ (equal rows) even though $C = A + B$!

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

12 $\det(A^{-1})$ divides twice by $ad - bc$ (once for each row). This gives $\frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$.

13 Pivots $1, 1, 1$ give determinant $= 1$; pivots $1, -2, -3/2$ give determinant $= 3$.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has $\det = 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

16 A singular rank one matrix has determinant $= 0$. The skew-symmetric K also $\det K = 0$ (see #**17**).

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$. So we must have $\det(K) = 0$ for 3 by 3.

18 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$ (to reach 2 by 2, eliminate a and a^2 in row 1 by column operations). Factor out $b-a$ and $c-a$ from the 2 by 2: $(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$.

19 For triangular matrices, just multiply the diagonal entries: $\det(U) = 6, \det(U^{-1}) = \frac{1}{6}$, and $\det(U^2) = 36$. 2 by 2 matrix: $\det(U) = ad, \det(U^2) = a^2d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

20 $\det\begin{bmatrix} a-Lc & b-Ld \\ c-\ell a & d-\ell b \end{bmatrix}$ reduces to $(ad-bc)(1-L\ell)$. The determinant changes if you do two row operations at once.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

22 $\det(A) = 3, \det(A^{-1}) = \frac{1}{3}, \det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. *Note to instructor*: If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify $\lambda = 1$ and $\lambda = 3$ as the eigenvalues of A.

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$. $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

24 Here $A = LU$ with $\det(L) = 1$ and $\det(U) = -6$ product of pivots, so also $\det(A) = -6$. $\det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A)$ and $\det(U^{-1}L^{-1}A)$ is $\det I = 1$.

25 When the i, j entry is ij, row 2 = 2 times row 1 so $\det A = 0$.

26 When the ij entry is $i + j$, row 3 − row 2 = row2 − row 1 so A is singular: $\det A = 0$.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b-a)(c-b)$ by doing elimination.

28 (a) *True*: $\det(AB) = \det(A)\det(B) = 0$ (b) *False*: A row exchange gives $-\det =$ product of pivots. (c) *False*: $A = 2I$ and $B = I$ have $A - B = I$ but the determinants have $2^n - 1 \neq 1$ (d) *True*: $\det(AB) = \det(A)\det(B) = \det(BA)$.

29 A is rectangular so $\det(A^{\mathrm{T}}A) \neq (\det A^{\mathrm{T}})(\det A)$: these determinants are not defined.

30 Derivatives of $f = \ln(ad - bc)$: $\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$.

31 The Hilbert determinants are $1, 8\times10^{-2}, 4.6\times10^{-4}, 1.6\times10^{-7}, 3.7\times10^{-12}, 5.4\times10^{-18}, 4.8\times10^{-25}, 2.7\times10^{-33}, 9.7\times10^{-43}, 2.2\times10^{-53}$. Pivots are ratios of determinants so the 10th pivot is near 10^{-10}. The Hilbert matrix is numerically difficult (*ill-conditioned*).

32 Typical determinants of rand(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. randn(n) with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

33 I now know that maximizing the determinant for $1, -1$ matrices is **Hadamard's problem** (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences (**research.att.com/~njas**) includes the solution for small n (and more references) when the problem is changed to $0, 1$ matrices. That sequence A003432 starts from $n = 0$ with 1, 1, 1, 2, 3, 5, 9. Then the $1, -1$ maximum for size n is 2^{n-1} times the $0, 1$ maximum for size $n-1$ (so $(32)(5) = \mathbf{160}$ for $n = 6$ in sequence **A003433**).

To reduce the $1, -1$ problem from 6 by 6 to the $0, 1$ problem for 5 by 5, multiply the six rows by ± 1 to put $+1$ in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix S of $-2, 0$ and divide S by -2.

Here is an advanced MATLAB code and a $1, -1$ matrix with largest $\det A = 48$ for $n = 5$:

```
n = 5; p = (n − 1)^2; A0 =ones(n); maxdet= 0;
for k = 0 : 2^p − 1
Asub = rem(floor(k. ∗ 2.^(−p + 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 − 2∗
reshape(Asub, n − 1, n − 1);
if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A;
end
end
```

Output: maxA = $\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{matrix}$ maxdet = 48.

34 Reduce B by row operations to [row 3; row 2; row 1]. Then $\det B = -6$ (odd permutation).

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1 $\det A = 1+18+12-9-4-6 = 12$, rows are independent; $\det B = 0$, row 1+row 2 = row 3; $\det C = -1$, independent rows ($\det C$ has one term, odd permutation)

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

3 All cofactors of row 1 are zero. A has rank ≤ 2. Each of the 6 terms in $\det A$ is zero. Column 2 has no pivot.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

5 Four zeros in the same row guarantee $\det = 0$. $A = I$ has 12 zeros (maximum with $\det \neq 0$).

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

7 $5!/2 = 60$ permutation matrices have det $= +1$. Move row 5 of I to the top; starting from $(5, 1, 2, 3, 4)$ elimination will do four row exchanges.

8 Some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows 1, 2, . . ., n into rows α, β, . . ., ω. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations, the matrix needs an *even* number of -1's. To get $+1$ for the odd P's, the matrix needs an *odd* number of -1's. So all six terms $= +1$ in the big formula and det $= 6$ are impossible: max(det) $=$ **4**.

10 The $4!/2 = 12$ even permutations are $(1, 2, 3, 4), (2, 1, 4, 3), (3, 1, 4, 2), (4, 3, 2, 1)$, and 8 P's with one number in place and even permutation of the other three numbers. $\det(I + P_{\text{even}}) = 16$ or 4 or 0 (16 comes from $I + I$).

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why*?

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

14 We must choose 1's from column 2 then column 1, column 4 then column 3,and so on. Therefore n must be even to have $\det A_n \neq 0$. The number of row exchanges is $n/2$ so $C_n = (-1)^{n/2}$.

15 The $1, 1$ cofactor of the n by n matrix is E_{n-1}. The $1, 2$ cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is 1, 0, -1, -1, 0, 1 and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1, 1$ cofactor of the n by n matrix is F_{n-1}. The $1, 2$ cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the $1, 2$ entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

17 $|B_4| = 2\det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & \\ & -1 & -1 \end{bmatrix} = 2|B_3| - \det\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$ $2|B_3| - |B_2|$. $|B_3|$ and $-|B_2|$ are cofactors of row 4 of B_4.

18 Rule 3 (linearity in row 1) gives $|B_n| = |A_n| - |A_{n-1}| = (n + 1) - n = 1$.

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x - a)(x - b)(x - c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(\boldsymbol{x}) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n - 1) =$ (product of the λ's).

21 $S_1 = 3, S_2 = 8, S_3 = 21$. The rule looks like every second number in Fibonacci's sequence $\ldots 3, 5, 8, 13, 21, 34, 55, \ldots$ so the guess is $S_4 = 55$. Following the solution to Problem 30 with 3's instead of 2's confirms $S_4 = 81+1-9-9-9 = 55$. Problem 33 directly proves $S_n = F_{2n+2}$.

22 Changing 3 to 2 in the corner reduces the determinant F_{2n+2} by 1 times the cofactor of that corner entry. This cofactor is the determinant of S_{n-1} (one size smaller) which is F_{2n}. Therefore changing 3 to 2 changes the determinant to $F_{2n+2} - F_{2n}$ which is F_{2n+1}.

23 (a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves entries from A times entries from D leading to $(\det A)(\det D)$
(b) and (c) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. See #25.

24 (a) All L's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, \frac{-1}{3}$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

26 If A is a row and B is a column then $\det M = \det AB =$ dot product of A and B. If A is a column and B is a row then AB has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$). This block matrix is invertible when AB is invertible which certainly requires $m \le n$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

28 Row 1 $-$ 2 row 2 $+$ row 3 $= 0$ so this matrix is singular.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+ (1,1)(2,2)(3,3)(4,4) + (1,2)(2,1)(3,4)(4,3) - (1,2)(2,1)(3,3)(4,4) - (1,1)(2,2)(3,4)(4,3) - (1,1)(2,3)(3,2)(4,4)$. Total -1.

30 The 5 products in solution 29 change to $16 + 1 - 4 - 4 - 4$ since A has 2's and -1's:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2).$$

31 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take $4, 1, 2, 3$ into $1, 2, 3, 4$; $\det(P^2) = (\det P)(\det P) = +1$ so $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is *not right*.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:
$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$.

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

35 Subtracting 1 from the n, n entry subtracts its cofactor C_{nn} from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

Problem Set 5.3, page 279

1 (a) $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$, $\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6$, $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4$, $|B_1| = 3$, $|B_2| = 2$, $|B_3| = 1$. Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad-bc)$ (b) $y = \det B_2/\det A = (fg-id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0}/\mathbf{0}$: *undetermined*.

4 (a) $x_1 = \det\big([\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,]\big)/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\,\boldsymbol{a}_2\,\boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\,\boldsymbol{a}_2\,\boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{a}_3|$ which is $x_1 \det A$.

5 If the first column in A is also the right side b then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\dfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

7 If all cofactors $= 0$ then A^{-1} would be the zero matrix if it existed; cannot exist. (And the cofactor formula gives $\det A = 0$.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1,3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

10 Take the determinant of $AC^{\mathrm{T}} = (\det A)I$. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

12 Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A$ must be 1 or -1.

13 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \dfrac{1}{5}C^{\mathrm{T}}$.

14 (a) Lower triangular L has cofactors $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}$, $C_{31} = C_{13}, C_{32} = C_{23}$ make S^{-1} symmetric. (c) Orthogonal Q has cofactor matrix $C = (\det Q)(Q^{-1})^{\mathrm{T}} = \pm Q$ *also orthogonal*. Note $\det Q = 1$ or -1.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

16 (a) Area $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

17 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces $=$ length of cross product $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{matrix} -2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k} \\ \text{length} = 6\sqrt{2} \end{matrix}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

19 $\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ because the transpose has the same determinant. See #22.

20 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$.)

21 The maximum volume $L_1L_2L_3L_4$ is reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved by ± 1.

22 This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for A to the parallelogram for A^{T}, without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

23 $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{array}{lcl} \det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2 \\ \det A & = & \pm\|a\|\|b\|\|c\| \end{array}$

24 The box has height 4 and volume $= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4$. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ and $(\boldsymbol{k} \cdot \boldsymbol{w}) = 4$.

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

27 $x = r\cos\theta$, $y = r\sin\theta$ give $J = r$. The columns are orthogonal and their lengths are 1 and r.

28 $J = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & \theta \end{vmatrix} = \rho^2\sin\varphi$. This Jacobian is needed for triple integrals inside spheres.

29 From x, y to r, θ: $\begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ (-\sin\theta)/r & (\cos\theta)/r \end{vmatrix}$ $= \dfrac{1}{r} = \dfrac{1}{\text{Jacobian in } \mathbf{27}}$.

30 The triangle with corners $(0,0), (6,0), (1,4)$ has area 24. Rotated by $\theta = 60°$ the area is *unchanged*. The determinant of the rotation matrix is $J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1$.

31 Base area 10, height 2, volume 20.

32 The volume of the box is $\det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20$.

33 $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$. This is $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

34 $(\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$: *Even permutation* of $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ keeps the same determinant. Odd permutations reverse the sign.

35 $S = (2,1,-1)$, area $\|PQ \times PS\| = \|(-2,-2,-1)\| = 3$. The other four corners can be $(0,0,0)$, $(0,0,2)$, $(1,2,2)$, $(1,1,0)$. The volume of the tilted box is $|\det| = 1$.

36 If $(1,1,0)$, $(1,2,1)$, (x,y,z) are in a plane the volume is $\det \begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$. The "box" with those edges is flattened to zero height.

37 $\det \begin{bmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 7x - 5y + z$ will be *zero* when (x,y,z) is a combination of $(2,3,1)$ and $(1,2,3)$. The plane containing those two vectors has equation $7x - 5y + z = 0$.

38 Doubling each row multiplies the volume by 2^n. Then $2 \det A = \det(2A)$ only if $n = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1} = 1/\det A$, construct A^{-1} using the cofactors. *Invert to find* A.

40 The cofactor formula adds 1 by 1 determinants (which are just entries) *times* their cofactors of size $n-1$. Jacobi discovered that this formula can be generalized. For $n = 5$, Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns $a < b$) times a 3 by 3 determinant from rows 3-5 (using the remaining columns $c < d < e$).

The key question is $+$ or $-$ sign (as for cofactors). The product is given a $+$ sign when a, b, c, d, e is an even permutation of 1, 2, 3, 4, 5. This gives the correct determinant $+1$ for that permutation matrix. More than that, all other P that permute a, b and separately c, d, e will come out with the correct sign when the 2 by 2 determinant for columns a, b multiplies the 3 by 3 determinant for columns c, d, e.

41 The Cauchy-Binet formula gives the determinant of a square matrix AB (and AA^{T} in particular) when the factors A, B are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from A and B (printed in boldface):

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & c \\ \boldsymbol{d} & \boldsymbol{e} & f \end{bmatrix}\begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ \boldsymbol{h} & \boldsymbol{k} \\ i & \ell \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{a} & b & \boldsymbol{c} \\ \boldsymbol{d} & e & \boldsymbol{f} \end{bmatrix}\begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ h & k \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix} \qquad \begin{bmatrix} a & \boldsymbol{b} & \boldsymbol{c} \\ d & \boldsymbol{e} & \boldsymbol{f} \end{bmatrix}\begin{bmatrix} g & j \\ \boldsymbol{h} & \boldsymbol{k} \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix}$$

Check $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \qquad AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$

Cauchy-Binet: $\quad (4-2)(4-2) + (7-3)(7-3) + (14-12)(14-12) = \mathbf{24}$
$(14)(66) - (30)(30) = \mathbf{24}$

Problem Set 6.1, page 293

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2,1)$ and $x_2 = (1,1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to $\mathbf{0}$ and $\mathbf{6}$. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1,1)$ and $\boldsymbol{x}_2 = (2,-1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $x_1 = (3,-2)$ and $x_2 = (1,1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3. $A+B$ has $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A+B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

7 The eigenvalues of U (on its diagonal) are the *pivots* of A. The eigenvalues of L (on its diagonal) are all 1's. The eigenvalues of A *are not* the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A-\lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$ (b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A+I)\boldsymbol{x} = (\boldsymbol{\lambda}+\mathbf{1})\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1,2)$ and $\boldsymbol{x}_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A-\lambda_1 I$ are in the nullspace of $A-\lambda_2 I$ because $M = (A-\lambda_2 I)(A-\lambda_1 I)$ $=$ zero matrix **[**this is the *Cayley-Hamilton Theorem* in Problem 6.2.32**]**. Notice that M has *zero eigenvalues* $(\lambda_1-\lambda_2)(\lambda_1-\lambda_1) = 0$ and $(\lambda_2-\lambda_2)(\lambda_2-\lambda_1) = 0$.

12 The projection matrix P has $\lambda = 1, 0, 1$ with eigenvectors $(1,2,0)$, $(2,-1,0)$, $(0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda = 1$. *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \boldsymbol{0}$ (c) $\boldsymbol{x}_1 = (-1,1,0,0)$, $\boldsymbol{x}_2 = (-3,0,1,0)$, $\boldsymbol{x}_3 = (-5,0,0,1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}$.

14 Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1 = (1,i)$ and $\boldsymbol{x}_2 = (1,-i)$ (more generally $c\boldsymbol{x}_1$, and $d\boldsymbol{x}_2$ with $cd \neq 0$).

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A-\lambda I) = (\lambda_1-\lambda)\dots(\lambda_n-\lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a+d+\sqrt{(a-d)^2+4bc})$ and $\lambda_2 = \frac{1}{2}(a+d-\sqrt{})$ add to $a+d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A-\lambda I) = (\lambda-3)(\lambda-4) = \lambda^2 - 7\lambda + 12$.

18 These 3 matrices have $\lambda = 4$ and 5, trace 9, det 20: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

19 (a) rank $= 2$ (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 $A = \begin{bmatrix} 0 & 1 \\ \mathbf{-28} & \mathbf{11} \end{bmatrix}$ has trace 11 and determinant 28, so $\lambda = 4$ and 7. Moving to a 3 by 3 companion matrix, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{6} & \mathbf{-11} & \mathbf{6} \end{bmatrix}$ has $\det(C-\lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = (1-\lambda)(2-\lambda)(3-\lambda)$. Notice the trace $6 = 1+2+3$, determinant $6 = (1)(2)(3)$, and also $\mathbf{11} = (1)(2) + (1)(3) + (2)(3)$.

21 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have *different eigenvectors*.

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

24 $\lambda = 0, 0, 6$ (*notice rank* 1 *and trace* 6) with $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

25 With the same n λ's and $\boldsymbol{x}$'s, $A\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ equals $B\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ for all vectors $\boldsymbol{x}$. So $A = B$.

26 The block matrix has $\lambda = 1, 2$ from B and $5, 7$ from D. All entries of C are multiplied by zeros in $\det(A - \lambda I)$, so C has no effect on the eigenvalues.

27 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

29 Triangular matrix: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; Rank-1 matrix: $\lambda(C) = 0, 0, 6$.

30 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (\boldsymbol{a} + \boldsymbol{b})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = \boldsymbol{d} - \boldsymbol{b}$ to produce the correct trace $(a + b) + (d - b) = a + d$.

31 Eigenvector $(1, 3, 4)$ for A with $\lambda = 11$ and eigenvector $(3, 1, 4)$ for PAP^{T}. Eigenvectors with $\lambda \neq 0$ must be in the column space since $A\boldsymbol{x}$ is always in the column space, and $\boldsymbol{x} = A\boldsymbol{x}/\lambda$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

33 If $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$ then $A^2 = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{v}^{\mathrm{T}}$ is the zero matrix and $\lambda^2 = 0, 0$ and $\lambda = 0, 0$ and trace $(A) = 0$. This zero trace also comes from adding the diagonal entries of $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix} \quad \text{has trace } u_1v_1 + u_2v_2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$$

34 $\det(P - \lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $\boldsymbol{x}_1 = (1, 1, 1, 1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1, -1, 1, -1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

35 3 by 3 permutation matrices: Since $P^{\mathrm{T}}P = I$ gives $(\det P)^2 = 1$, the determinant is 1 or -1. The pivots are always 1 (but there may be row exchanges). The trace of P can be 3 (for $P = I$) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and -1 and $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

37 (a) Since the columns of A add to 1, one eigenvalue is $\lambda = 1$ and the other is $c - .6$ (to give the correct trace $c + .4$).

(b) If $c = 1.6$ then both eigenvalues are 1, and all solutions to $(A - I)\,\boldsymbol{x} = \mathbf{0}$ are multiples of $\boldsymbol{x} = (1, -1)$.

(c) If $c = .8$, the eigenvectors for $\lambda = 1$ are multiples of (1, 3). Since all powers A^n also have column sums $= 1$, A^n will approach $\dfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ = rank-1 matrix A^∞ with eigenvalues $1, 0$ and correct eigenvectors. $(1, 3)$ and $(1, -1)$.

Problem Set 6.2, page 307

1 $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

2 Put the eigenvectors in S and eigenvalues in Λ. $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S. $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

5 With $S = I$, $A = S\Lambda S^{-1} = \Lambda$ is a diagonal matrix. If S is triangular, then S^{-1} is triangular, so $S\Lambda S^{-1}$ is also triangular.

6 The columns of S are nonzero multiples of (2,1) and (0,1): either order. Same for A^{-1}.

7 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}/2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for any a and b.

8 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $S\Lambda^k S^{-1} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 (a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1, 1)$, $\boldsymbol{x}_2 = (1, -2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, ...

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (A will have two different eigenvalues)

15 $A^k = S\Lambda^k S^{-1}$ approaches zero **if and only if every $|\boldsymbol{\lambda}| < 1$**; $A_1^k \to A_1^\infty$, $A_2^k \to 0$.

16 $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S\Lambda^k S^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Multiply those last three matrices to get $A^k = \frac{1}{2} \begin{bmatrix} 1 + 3^k & 1 - 3^k \\ 1 - 3^k & 1 + 3^k \end{bmatrix}$.

19 $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof works when A is *diagonalizable*.

21 trace $ST = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1} =$ trace of $(\Lambda S^{-1})S = \Lambda$: *sum of the* λ's.

22 $AB - BA = I$ is impossible since trace $AB -$ trace $BA = zero \neq$ trace I. $AB - BA = C$ is possible when trace $(C) = 0$, and $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has $EE^{\mathrm{T}} - E^{\mathrm{T}}E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

23 If $A = S\Lambda S^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$. So B has the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same S. When $S = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

25 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real. Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 $A^{\mathrm{T}} = A$ gives $\boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}(B\boldsymbol{x}) \le \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$ by the Schwarz inequality. $B^{\mathrm{T}} = -B$ gives $-\boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) \le \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$. Add to get Heisenberg's Uncertainty Principle when $AB - BA = I$. Position-momentum, also time-energy.

29 The factorizations of A and B into $S\Lambda S^{-1}$ are the same. So $A = B$. (This is the same as Problem 6.1.25, expressed in matrix form.)

30 $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from $S\Lambda_1 S^{-1}S\Lambda_2 S^{-1} = S\boldsymbol{\Lambda_1\Lambda_2}S^{-1} = S\boldsymbol{\Lambda_2\Lambda_1}S^{-1} = S\Lambda_2 S^{-1}S\Lambda_1 S^{-1} = BA$.

31 (a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

32 When $A = S\Lambda S^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = S(\Lambda - \lambda_j I)S^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. In the product $p(A) = (A - \lambda_1 I)\cdots(A - \lambda_n I)$, each inside S^{-1} cancels S. This leaves S times (*product of diagonal matrices* $\Lambda - \lambda_j I$) times S^{-1}. That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen this reasoning but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.3 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)\ \mathrm{cof}\ (A - \lambda I)^{\mathrm{T}} = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for λ. If other matrices B are substituted, does the identity remain true? If $AB \ne BA$, even the order of multiplication seems unclear . . .

33 $\lambda = 2, -1, 0$ are in $\boldsymbol{\Lambda}$ and the eigenvectors are in S (below). $A^k = S\boldsymbol{\Lambda}^k S^{-1}$ is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\boldsymbol{\Lambda^k}\frac{1}{6}\begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check $k = 4$. The $(2, 2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

34 If $AB = BA$, then B has the same eigenvectors $(1,0)$ and $(0,1)$ as A. So B is also diagonal $b = c = 0$. The nullspace for the following equation is 2-dimensional: $AB - BA = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} - \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & -b\\ c & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. The coefficient matrix has rank $4 - 2 = 2$.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

36 The eigenvalues of $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and $\det = 1$). Their eigenvectors are $(1,-i)$ and $(1,i)$:

$$A^n = S\Lambda^n S^{-1} = \begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}\begin{bmatrix}e^{in\theta} & \\ & e^{-in\theta}\end{bmatrix}\begin{bmatrix}i & -1\\ i & 1\end{bmatrix}/2i$$
$$= \begin{bmatrix}(e^{in\theta}+e^{-in\theta})/2 & \cdots\\ (e^{in\theta}-e^{-in\theta})/2i & \cdots\end{bmatrix} = \begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

38 Note that $\text{ones}(n) * \text{ones}(n) = n * \text{ones}(n)$. This leads to $C = 1/(n+1)$.

$$AA^{-1} = (\text{eye}(n) + \text{ones}(n)) * (\text{eye}(n) + C * \text{ones}(n))$$
$$= \text{eye}(n) + (1 + C + Cn) * \text{ones}(n) = \text{eye}(n).$$

Problem Set 6.3, page 325

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\ 0\end{bmatrix}$, $\boldsymbol{u}_2 = e^t\begin{bmatrix}1\\ -1\end{bmatrix}$. If $\boldsymbol{u}(0) = (5,-2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\ 0\end{bmatrix} + 2e^t\begin{bmatrix}1\\ -1\end{bmatrix}$.

2 $z(t) = 2e^t$; then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 (a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3,2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1,-1)$. Then the usual 3 steps:

1. Write $u(0) = \begin{bmatrix}4\\ 1\end{bmatrix}$ as $\begin{bmatrix}3\\ 2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2$
2. Follow those eigenvectors by $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{u}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3,2)$.

4 $d(v+w)/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\ \lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$; $\begin{matrix}v(1) = 20 + 10e^{-2}\\ w(1) = 20 - 10e^{-2}\end{matrix}$ $\begin{matrix}v(\infty) = 20\\ w(\infty) = 20\end{matrix}$

5 $\dfrac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has $\lambda = 0$ and $+2$: $v(t) = 20 + 10e^{2t} - \infty$ as $t \to \infty$.

6 $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$, and the solutions of $\boldsymbol{u}' = A\boldsymbol{u}$ approach zero. $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$, and all solutions of $\boldsymbol{v}' = B\boldsymbol{v}$ approach zero.

7 A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1,1)$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1,-1)$. For the solution to $\boldsymbol{u}' = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5$, $\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

9 (a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ i \end{bmatrix} + 2\begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $u(t) = 2e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

10 $\dfrac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

11 $e^{At} = I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{zeros} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$ $\begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$. This $y(t) = y(0) + y'(0)t$ solves the equation.

12 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1,3)$.

13 (a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{u}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

14 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal*.

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$; $\boldsymbol{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix} 1 \\ t \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

17 (a) $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ (b) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (c) $\begin{bmatrix}1&1\\-1&1\end{bmatrix}$. These show the unstable cases (a) $\lambda_1<0$ and $\lambda_2>0$ (b) $\lambda_1>0$ and $\lambda_2>0$ (c) $\lambda=a\pm ib$ with $a>0$

18 $d/dt(e^{At})=A+A^2t+\frac{1}{2}A^3t^2+\frac{1}{6}A^4t^3+\cdots=A(I+At+\frac{1}{2}A^2t^2+\frac{1}{6}A^3t^3+\cdots)$. This is exactly Ae^{At}, the derivative we expect.

19 $e^{Bt}=I+Bt$ (short series with $B^2=0$) $=\begin{bmatrix}\mathbf{1}&\mathbf{-4t}\\\mathbf{0}&\mathbf{1}\end{bmatrix}$. Derivative $=\begin{bmatrix}0&-4\\0&0\end{bmatrix}=B$.

20 The solution at time $t+T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1&4\\0&0\end{bmatrix}=\begin{bmatrix}1&4\\0&-1\end{bmatrix}\begin{bmatrix}\mathbf{1}&0\\0&\mathbf{0}\end{bmatrix}\begin{bmatrix}1&4\\0&-1\end{bmatrix}$; $\begin{bmatrix}1&4\\0&-1\end{bmatrix}\begin{bmatrix}\boldsymbol{e^t}&0\\0&\mathbf{1}\end{bmatrix}\begin{bmatrix}1&4\\0&-1\end{bmatrix}=\begin{bmatrix}e^t&4e^t-4\\0&1\end{bmatrix}$.

22 $A^2=A$ gives $e^{At}=I+At+\frac{1}{2}\boldsymbol{At^2}+\frac{1}{6}At^3+\cdots=I+(e^t-1)A=\begin{bmatrix}e^t&e^t-1\\0&1\end{bmatrix}$.

23 $e^A=\begin{bmatrix}e&4(e-1)\\0&1\end{bmatrix}$ from **21** and $e^B=\begin{bmatrix}1&-4\\0&1\end{bmatrix}$ from **19**. By direct multiplication $e^Ae^B\neq e^Be^A\neq e^{A+B}=\begin{bmatrix}e&0\\0&1\end{bmatrix}$.

24 $A=\begin{bmatrix}1&1\\0&3\end{bmatrix}=\begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}\mathbf{1}&0\\0&\mathbf{3}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}$. Then $e^{At}=\begin{bmatrix}\boldsymbol{e^t}&\frac{1}{2}(e^{3t}-e^t)\\0&\boldsymbol{e^{3t}}\end{bmatrix}$.

25 The matrix has $A^2=\begin{bmatrix}1&3\\0&0\end{bmatrix}^2=\begin{bmatrix}1&3\\0&0\end{bmatrix}=A$. *Then all* $A^n=A$. So $e^{At}=I+(t+t^2/2!+\cdots)A=I+(e^t-1)A=\begin{bmatrix}e^t&3(e^t-1)\\0&0\end{bmatrix}$ as in Problem 22.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x}=\lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x}=e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t}\neq 0$. To see $e^{At}\boldsymbol{x}$, write $(I+At+\frac{1}{2}A^2t^2+\cdots)\boldsymbol{x}=(1+\lambda t+\frac{1}{2}\lambda^2t^2+\cdots)\boldsymbol{x}=e^{\lambda t}\boldsymbol{x}$.

27 $(x,y)=(e^{4t},e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u}=(y,x)$ is $\begin{bmatrix}2&-2\\-4&0\end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $\boldsymbol{U}_{n+1}=\begin{bmatrix}1&\Delta t\\-\Delta t&1-(\Delta t)^2\end{bmatrix}\boldsymbol{U}_n$. At $\Delta t=1$, $\begin{bmatrix}1&1\\-1&0\end{bmatrix}$ has $\lambda=e^{i\pi/3}$ and $e^{-i\pi/3}$. Both eigenvalues have $\lambda^6=1$ so $A^6=I$. Therefore $\boldsymbol{U}_6=A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda=\pm i$ and $A^4=I$.
Second A has $\lambda=-1,-1$ and $A^n=(-1)^n\begin{bmatrix}1-2n&-2n\\2n&2n+1\end{bmatrix}$ Linear growth.

30 With $a=\Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1}=\dfrac{1}{1+a^2}\begin{bmatrix}1-a^2&2a\\-2a&1-a^2\end{bmatrix}\boldsymbol{U}_n$.

That matrix has orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\|=\|\boldsymbol{U}_n\|$

31 (a) $(\cos A)\boldsymbol{x}=(\cos\lambda)\boldsymbol{x}$ (b) $\lambda(A)=2\pi$ and 0 so $\cos\lambda=1,1$ and $\cos A=I$ (c) $\boldsymbol{u}(t)=3(\cos 2\pi t)(1,1)+1(\cos 0t)(1,-1)$ [$\boldsymbol{u}'=A\boldsymbol{u}$ has **exp**, $\boldsymbol{u}''=A\boldsymbol{u}$ has **cos**]

Problem Set 6.4, page 337

Note A way to complete the proof at the end of page 334, (perturbing the matrix to produce distinct eigenvalues) is now on the course website: "*Proofs of the Spectral Theorem*." **math.mit.edu/linearalgebra**.

1 $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$ $= \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$ $=$ **symmetric** $+$ **skew-symmetric**.

2 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C will be 6 by 6 and the triple product $A^{\mathrm{T}}CA$ is 3 by 3.

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$.

4 $\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

5 $Q = \dfrac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A
Each unit eigenvector could be multiplied by -1

6 $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has $\lambda = 0$ and 25 so the columns of Q are the two eigenvectors: $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or we can exchange columns or reverse the signs of any column.

7 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ's (c) trace $= \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues.

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is *symmetric* then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ requires $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

9 If λ is complex then $\overline{\lambda}$ is also an eigenvalue $(A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}})$. Always $\lambda + \overline{\lambda}$ is real. The trace is real so the third eigenvalue of a 3 by 3 real matrix must be real.

10 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

12 $[\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,]$ is an orthogonal matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,]\begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = I$; $P_1P_2 = \boldsymbol{x}_1(x_1^{\mathrm{T}}x_2)x_2^{\mathrm{T}} = 0$. Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

13 $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ has $\lambda = ib$ and $-ib$. The block matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are also skew-symmetric with $\lambda = ib$ (twice) and $\lambda = -ib$ (twice).

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

15 $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$. The good property for complex matrices is not $A^{\mathrm{T}} = A$ (symmetric) but $\overline{A}^{\mathrm{T}} = A$ (Hermitian with real eigenvalues and orthogonal eigenvectors: see Problem 20 and Section 10.2).

16 (a) If $A\boldsymbol{z} = \lambda\boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1, 0, -1, 0)$, $\boldsymbol{x}_2 = (0, 1, 0, -1)$, $\boldsymbol{x}_3 = (1, 0, 1, 0)$, $\boldsymbol{x}_4 = (0, 1, 0, 1)$.

17 The eigenvalues of $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ are $0, \sqrt{2}, -\sqrt{2}$ by Problem 16 with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$, $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}$.

18 **1.** $\boldsymbol{y}$ is in the nullspace of A and $\boldsymbol{x}$ is in the column space $=$ row space because $A = A^{\mathrm{T}}$. Those spaces are perpendicular so $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$.
2. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{y} = \beta\boldsymbol{y}$ then shift by β: $(A-\beta I)\boldsymbol{x} = (\lambda-\beta)\boldsymbol{x}$ and $(A-\beta I)\boldsymbol{y} = \boldsymbol{0}$ and again $\boldsymbol{x} \perp \boldsymbol{y}$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A
Not perpendicular for B
since $B^{\mathrm{T}} \neq B$

20 $A = \begin{bmatrix} 1 & 3+4i \\ 3-4i & 1 \end{bmatrix}$ is a *Hermitian matrix* ($\overline{A}^{\mathrm{T}} = A$). Its eigenvalues 6 and -4 are *real*. Adjust equations (1)–(2) in the text to prove that λ is always real when $\overline{A}^{\mathrm{T}} = A$:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \text{ leads to } \overline{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}. \text{ Transpose to } \overline{\boldsymbol{x}}^{\mathrm{T}}A = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda} \text{ using } \overline{A}^{\mathrm{T}} = A.$$

$$\text{Then } \overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x} \text{ and also } \overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda}\boldsymbol{x}. \text{ So } \lambda = \overline{\lambda} \text{ is real.}$$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$
(c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

26 Eigenvectors $(1, 0)$ and $(\mathbf{1}, \mathbf{1})$ give a 45° angle even with A^{T} very close to A.

27 The roots of $\lambda^2 + b\lambda + c = 0$ are $\frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$. Then $\lambda_1 - \lambda_2$ is $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

28 $A = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = \overline{A}^{\mathrm{T}}$ has real eigenvalues $\lambda = 5$ and -1 with trace $= 4$ and $\det = -5$. The solution to **20** proves that λ is real when $\overline{A}^{\mathrm{T}} = A$ is Hermitian; I did not intend to repeat this part.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The $1,1$ entries of $\overline{T}^{\mathrm{T}}T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $\begin{bmatrix} q_{11} \dots q_{1n} \end{bmatrix} \begin{bmatrix} \lambda_1\overline{q}_{11} \dots \lambda_n\overline{q}_{1n} \end{bmatrix}^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \cdots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

32 Since A is diagonalizable with eigenvalue matrix $\Lambda = 2I$, the matrix A itself has to be $S\Lambda S^{-1} = S(2I)S^{-1} = 2I$. (The unsymmetric matrix $[2\ 1\ ;\ 0\ 2]$ also has $\lambda = 2, 2$.)

Problem Set 6.5, page 350

1 Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$. (i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2 = \det = ac - b^2 > 0$. (ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

2 Only $A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues. $\boldsymbol{x}^{\mathrm{T}}A_1\boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms.

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x,y) = x^2 + 4xy + 9y^2 = (x+2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x+3y)^2$.

5 $x^2 + 4xy + 3y^2 = (x+2y)^2 - y^2 =$ *difference of squares* is negative at $x = 2$, $y = -1$, where the first square is zero.

6 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ produces $f(x,y) = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2xy$. A has $\lambda = 1$ and -1. Then A is an *indefinite matrix* and $f(x,y) = 2xy$ has a *saddle point*.

7 $R^{\mathrm{T}}R = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $R^{\mathrm{T}}R = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $R^{\mathrm{T}}R = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular (and positive semidefinite). The first two R's have independent columns. The 2 by 3 R cannot have full column rank 3, with only 2 rows.

8 $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots $3, 4$ outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 3(x+2y)^2 + 4y^2$

9 $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$, rank $A = 1$, eigenvalues are $24, 0, 0$, $\det A = 0$.

10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

11 Corner determinants $|A_1| = 2$, $|A_2| = 6$, $|A_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

13 $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ is an example with $a + c > 2b$ but $ac < b^2$, so not positive definite.

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}} A^{-1} \boldsymbol{x} = (A^{-1}\boldsymbol{x})^{\mathrm{T}} A (A^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$.

15 Since $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} > 0$ and $\boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ we have $\boldsymbol{x}^{\mathrm{T}}(A + B)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. Then $A + B$ is a positive definite matrix. The second proof uses the test $A = R^{\mathrm{T}} R$ (independent columns in R): If $A = R^{\mathrm{T}} R$ and $B = S^{\mathrm{T}} S$ pass this test, then $A + B = \begin{bmatrix} R & S \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R \\ S \end{bmatrix}$ also passes, and must be positive definite.

16 $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

17 If a_{jj} were smaller than all λ's, $A - a_{jj} I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj} I$ has a *zero* in the (j, j) position; impossible by Problem 16.

18 If $A\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} = \lambda \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$. If A is positive definite this leads to $\lambda = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} > 0$ (ratio of positive numbers). So positive energy $\Rightarrow$ positive eigenvalues.

19 All cross terms are $\boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues $\Rightarrow$ positive energy.

20 (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues (d) $A = -I$ has $\det = +1$ when n is even.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

23 $x^2/a^2 + y^2/b^2$ is $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ when $A = \operatorname{diag}(1/a^2, 1/b^2)$. Then $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = \frac{1}{3}$ and $b = \frac{1}{4}$. The points $(\frac{1}{3}, 0)$ and $(0, \frac{1}{4})$ are at the ends of the axes.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}} C = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

26 The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = A$.

27 Writing out $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$ gives $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$. So the LDL^{T} from elimination is exactly the same as *completing the square*. The example $2x^2 + 8xy + 10y^2 = 2(x + 2y)^2 + 2y^2$ with pivots $2, 2$ outside the squares and multiplier 2 inside.

28 $\det A = (1)(10)(1) = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive. So A is positive definite.

29 $H_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is semidefinite; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite at $(0, 1)$ where 1st derivatives $= 0$. This is a saddle point of the function $f_2(x, y)$.

30 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$) because the determinant $ac - b^2$ is *negative*.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero along the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n. Another subgroup: lower triangular elimination matrices E with diagonal 1's.

33 A product AB of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $AB = M^{-1}K$. (often we use *eig*(K, M) without actually inverting M.) All eigenvalues λ are positive:

$$AB\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (B\boldsymbol{x})^{\mathrm{T}}AB\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}B^{\mathrm{T}}AB\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} > 0.$$

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3},\ 2 - 1,\ 2,\ 2 + 1,\ 2 + \sqrt{3}$. The product of those eigenvalues is $6 = \det K$.

35 Put parentheses in $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \boldsymbol{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \boldsymbol{0}$ only happens when $\boldsymbol{x} = \boldsymbol{0}$. Thus $A^{\mathrm{T}}CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}}CA$ in a wide range of applications. I believe this is a unifying concept from linear algebra.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

2 $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3 $B=\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}=M^{-1}AM;$

$B=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}^{-1}\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix};$

$B=\begin{bmatrix}4&3\\2&1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}^{-1}\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}.$

4 A has no repeated λ so it can be diagonalized: $S^{-1}AS=\Lambda$ makes A similar to Λ.

5 $\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}$ are similar (they all have eigenvalues 1 and 0). $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is by itself and also $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ is by itself with eigenvalues 1 and -1.

6 *Eight families* of similar matrices: six matrices have $\lambda=0,1$ (one family); three matrices have $\lambda=1,1$ and three have $\lambda=0,0$ (two families each!); one has $\lambda=1,-1$; one has $\lambda=2,0$; two matrices have $\lambda=\frac{1}{2}(1\pm\sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}\boldsymbol{x})=M^{-1}(A\boldsymbol{x})=M^{-1}\mathbf{0}=\mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&2\\0&0\end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda=0,0$.

9 $A^2=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $A^3=\begin{bmatrix}1&3\\0&1\end{bmatrix}$, every $A^k=\begin{bmatrix}1&k\\0&1\end{bmatrix}$. $A^0=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$.

10 $J^2=\begin{bmatrix}c^2&2c\\0&c^2\end{bmatrix}$ and $J^k=\begin{bmatrix}c^k&kc^{k-1}\\0&c^k\end{bmatrix}$; $J^0=I$ and $J^{-1}=\begin{bmatrix}c^{-1}&-c^{-2}\\0&c^{-1}\end{bmatrix}$.

11 $\boldsymbol{u}(0)=\begin{bmatrix}5\\2\end{bmatrix}=\begin{bmatrix}v(0)\\w(0)\end{bmatrix}$. The equation $\dfrac{d\boldsymbol{u}}{dt}=\begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}\boldsymbol{u}$ has $\dfrac{dv}{dt}=\lambda v+w$ and $\dfrac{dw}{dt}=\lambda w$. Then $w(t)=2e^{\lambda t}$ and $v(t)$ must include $2te^{\lambda t}$ (this comes from the repeated λ). To match $v(0)=5$, the solution is $v(t)=2te^{\lambda t}+5e^{\lambda t}$.

12 If $M^{-1}JM=K$ then $JM=\begin{bmatrix}m_{21}&m_{22}&m_{23}&m_{24}\\0&\mathbf{0}&\mathbf{0}&0\\m_{41}&m_{42}&m_{43}&m_{44}\\0&0&0&0\end{bmatrix}=MK=\begin{bmatrix}\mathbf{0}&m_{12}&m_{13}&\mathbf{0}\\0&m_{22}&m_{23}&0\\0&m_{32}&m_{33}&0\\0&m_{42}&m_{43}&0\end{bmatrix}$.

That means $m_{21}=m_{22}=m_{23}=m_{24}=0$. M is not invertible, J not similar to K.

13 The five 4 by 4 Jordan forms with $\lambda=0,0,0,0$ are $J_1=$ zero matrix and

$$J_2=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad J_3=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$J_4=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\quad J_5=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Problem 12 showed that J_3 and J_4 are *not similar*, even with the same rank. Every matrix with all $\lambda = 0$ is "*nilpotent*" (its nth power is A^n = zero matrix). You see $J^4 = 0$ for these matrices. How many possible Jordan forms for $n = 5$ and all $\lambda = 0$?

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1} J_i M_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1} J M_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}} J^{\mathrm{T}} M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}} M_0^{-1} J M_0 M^{\mathrm{T}} = (M M_0 M^{\mathrm{T}})^{-1} A (M M_0 M^{\mathrm{T}})$, and A^{T} *is similar to* A.

15 $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda I M)$. This is $\det(M^{-1}(A - \lambda I)M)$. By the product rule, the determinants of M and M^{-1} cancel to leave $\det(A - \lambda I)$.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$; $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is similar to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. So two pairs of similar matrices but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: different eigenvalues!

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $AB\boldsymbol{x} = \lambda \boldsymbol{x}$ *then* $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

20 (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$. So A^2 is similar to B^2. (b) A^2 equals $(-A)^2$ but A may not be similar to $B = -A$ (it could be!). (c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because $\lambda_1 \neq \lambda_2$, so these matrices are similar. (d) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable (e) PAP^{T} is similar to A.

21 J^2 has three 1's down the *second* superdiagonal, and *two* independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note to professors: An interesting question: *Which matrices A have (complex) square roots $R^2 = A$?* If A is invertible, no problem. But any Jordan blocks for $\lambda = 0$ must have sizes $n_1 \geq n_2 \geq \ldots \geq n_k \geq n_{k+1} = 0$ that come in pairs like 3 and 2 in this example: $n_1 = (n_2$ or $n_2+1)$ and $n_3 = (n_4$ or $n_4+1)$ and so on.

A list of all 3 by 3 and 4 by 4 Jordan forms could be $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$,

$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ (for any numbers a, b, c) with $3, 2, 1$ eigenvectors; $\mathrm{diag}(a, b, c, d)$ and $\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix}$,

$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix}$ with $4, 3, 2, 1$ eigenvectors.

22 If all roots are $\lambda = 0$, this means that $\det(A - \lambda I)$ must be just λ^n. The Cayley-Hamilton Theorem in Problem 6.2.32 immediately says that A^n = zero matrix. The key example is a single n by n Jordan block (with $n-1$ ones above the diagonal): Check directly that J^n = zero matrix.

23 Certainly Q_1R_1 is similar to $R_1Q_1 = Q_1^{-1}(Q_1R_1)Q_1$. Then $A_1 = Q_1R_1 - cs^2I$ is similar to $A_2 = R_1Q_1 - cs^2I$.

24 A could have eigenvalues $\lambda = 2$ and $\lambda = \frac{1}{2}$ (A could be diagonal). Then A^{-1} has the same two eigenvalues (and is similar to A).

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

2 This $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a 2 by 2 matrix of rank 1. Its row space has basis $\boldsymbol{v}_1$, its nullspace has basis $\boldsymbol{v}_2$, its column space has basis $\boldsymbol{u}_1$, its left nullspace has basis $\boldsymbol{u}_2$:

$$\text{Row space } \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{Nullspace } \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{Column space } \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{N}(A^{\mathrm{T}}) \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

3 If A has rank 1 then so does $A^{\mathrm{T}}A$. The only nonzero eigenvalue of $A^{\mathrm{T}}A$ is its trace, which is the sum of all a_{ij}^2. (Each diagonal entry of $A^{\mathrm{T}}A$ is the sum of a_{ij}^2 down one column, so the trace is the sum down all columns.) Then σ_1 = square root of this sum, and σ_1^2 = this sum of all a_{ij}^2.

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}$, $\sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$. But A is indefinite
$\sigma_1 = (1+\sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5}-1)/2 = -\lambda_2(A)$; $\boldsymbol{u}_1 = \boldsymbol{v}_1$ but $\boldsymbol{u}_2 = -\boldsymbol{v}_2$.

5 A proof that ***eigshow*** finds the SVD. When $\boldsymbol{V}_1 = (1,0)$, $\boldsymbol{V}_2 = (0,1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at some angle θ. A 90° turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must produce $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ *at angle* $\pi/2$. Those orthogonal directions give $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 $AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$;

and $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\,\boldsymbol{u}_1 \;\; \boldsymbol{u}_2\,]\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[\,\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3\,]^{\mathrm{T}}$.

7 The matrix A in Problem 6 had $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ in Σ. The smallest change to rank 1 is **to make $\boldsymbol{\sigma_2 = 0}$**. In the factorization

$$A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\sigma_2\boldsymbol{v}_2^{\mathrm{T}}$$

this change $\sigma_2 \to 0$ will leave the closest rank–1 matrix as $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$. See Problem 14 for the general case of this problem.

8 The number $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$ is the same as $\sigma_{\max}(A)/\sigma_{\min}(A)$. This is certainly ≥ 1. It equals 1 if all σ's are equal, and $A = U\Sigma V^{\mathrm{T}}$ is a multiple of an orthogonal matrix. The ratio $\sigma_{\max}/\sigma_{\min}$ is the important **condition number** of A studied in Section 9.2.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

10 A rank–1 matrix with $A\boldsymbol{v} = 12\boldsymbol{u}$ would have $\boldsymbol{u}$ in its column space, so $A = \boldsymbol{u}\boldsymbol{w}^{\mathrm{T}}$ for some vector $\boldsymbol{w}$. I intended (but didn't say) that $\boldsymbol{w}$ is a multiple of the unit vector $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ in the problem. Then $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to get $A\boldsymbol{v} = 12\boldsymbol{u}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 1$.

11 If A has orthogonal columns $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^{\mathrm{T}}A$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the σ's are definitely the singular values of A (as expected). The eigenvalues of that diagonal matrix $A^{\mathrm{T}}A$ are the columns of I, so $V = I$ in the SVD. Then the $\boldsymbol{u}_i$ are $A\boldsymbol{v}_i/\sigma_i$ which is the unit vector $\boldsymbol{w}_i/\sigma_i$.

The SVD of this A with orthogonal columns is $A = U\Sigma V^{\mathrm{T}} = (A\Sigma^{-1})(\Sigma)(I)$.

12 Since $A^{\mathrm{T}} = A$ we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. But λ_2 is negative, so $\sigma_1 = 3$ and $\sigma_2 = 2$. The unit eigenvectors of A are the same $\boldsymbol{u}_1 = \boldsymbol{v}_1$ as for $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and $\boldsymbol{u}_2 = -\boldsymbol{v}_2$ (notice the sign change because $\sigma_2 = -\lambda_2$, as in Problem 4).

13 Suppose the SVD of R is $R = U\Sigma V^{\mathrm{T}}$. Then multiply by Q to get $A = QR$. So the SVD of this A is $(QU)\Sigma V^{\mathrm{T}}$. (Orthogonal Q times orthogonal U = orthogonal QU.)

14 The smallest change in A is to set its smallest singular value σ_2 to zero. See # **7**.

15 The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

16 This simulates the random walk used by *Google* on billions of sites to solve $A\boldsymbol{p} = \boldsymbol{p}$. It is like the power method of Section 9.3 except that it follows the links in one "walk" where the vector $p_k = A^k p_0$ averages over all walks.

17 $A = U\Sigma V^{\mathrm{T}} =$ [cosines including $\boldsymbol{u}_4$] **diag**(**sqrt**$(2-\sqrt{2}, 2, 2+\sqrt{2})$) [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

The SVD of the *derivative* on $[0,\pi]$ with $f(0) = 0$ has $\boldsymbol{u} = \sin nx$, $\sigma = n$, $\boldsymbol{v} = \cos nx$!

Problem Set 7.1, page 380

1 With $\boldsymbol{w} = \boldsymbol{0}$ linearity gives $T(\boldsymbol{v} + \boldsymbol{0}) = T(\boldsymbol{v}) + T(\boldsymbol{0})$. Thus $T(\boldsymbol{0}) = \boldsymbol{0}$. With $c = -1$ linearity gives $T(-\boldsymbol{0}) = -T(\boldsymbol{0})$. This is a second proof that $T(\boldsymbol{0}) = \boldsymbol{0}$.

2 Combining $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ and $T(d\boldsymbol{w}) = dT(\boldsymbol{w})$ with addition gives $T(c\boldsymbol{v} + d\boldsymbol{w}) = cT(\boldsymbol{v}) + dT(\boldsymbol{w})$. Then one more addition gives $cT(\boldsymbol{v}) + dT(\boldsymbol{w}) + eT(\boldsymbol{u})$.

3 (d) is not linear.

4 (a) $S(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (-1,0)$. Then $T(\boldsymbol{v}) + T(\boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w})$ but $T(\boldsymbol{v} + \boldsymbol{w}) = (0,0)$.

6 (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ does not satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ or $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ (b) and (c) are linear (d) satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$.

7 (a) $T(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(T(\boldsymbol{v})) = \boldsymbol{v} + (2,2)$ (c) $T(T(\boldsymbol{v})) = -\boldsymbol{v}$ (d) $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$.

8 (a) The range of $T(v_1, v_2) = (v_1 - v_2, 0)$ is the line of vectors $(c, 0)$. The nullspace is the line of vectors (c, c). (b) $T(v_1, v_2, v_3) = (v_1, v_2)$ has Range $\mathbf{R}^2$, kernel $\{(0, 0, \boldsymbol{v}_3)\}$ (c) $T(\boldsymbol{v}) = \mathbf{0}$ has Range $\{\mathbf{0}\}$, kernel $\mathbf{R}^2$ (d) $T(v_1, v_2) = (v_1, v_1)$ has Range = multiples of $(1,1)$, kernel = multiples of $(1,-1)$.

9 If $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$ then $T(T(\boldsymbol{v})) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2)$; $T^3(\boldsymbol{v}) = \boldsymbol{v}$; $T^{100}(\boldsymbol{v}) = T(\boldsymbol{v})$.

10 (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

11 For multiplication $T(\boldsymbol{v}) = A\boldsymbol{v}$: $\boldsymbol{V} = \mathbf{R}^n$, $\boldsymbol{W} = \mathbf{R}^m$; the outputs fill the column space; $\boldsymbol{v}$ is in the kernel if $A\boldsymbol{v} = \mathbf{0}$.

12 $T(\boldsymbol{v}) = (4,4); (2,2); (2,2)$; if $\boldsymbol{v} = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v}) = b(2,2) + (0,0)$.

13 The *distributive law* (page 69) gives $A(M_1 + M_2) = AM_1 + AM_2$. The *distributive law* over c's gives $A(cM) = c(AM)$.

14 This A is invertible. Multiply $AM = 0$ and $AM = B$ by A^{-1} to get $M = 0$ and $M = A^{-1}B$. The kernel contains only the zero matrix $M = 0$.

15 This A is *not* invertible. $AM = I$ is impossible. $A\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The range contains only matrices AM whose columns are multiples of $(1,3)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 For $T(M) = M^{\mathrm{T}}$ (a) $T^2 = I$ is True (b) True (c) True (d) False.

18 $T(I) = 0$ but $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$; these M's fill the range. Every $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is in the kernel. Notice that dim (range) + dim (kernel) $= 3 + 1 =$ dim (input space of 2 by 2 M's).

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

21 $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles the width of the house. $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$ *projects* the house (since $A^2 = A$ from trace $= 1$ and $\lambda = 0, 1$). The projection is onto the column space of $A =$ line through $(.7, .3)$. $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ will *shear* the house horizontally: The point at (x, y) moves over to $(x + y, y)$.

22 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ leaves the house AH sitting straight up (b) $A = 3I$ expands the house by 3 (c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the house.

23 $T(\boldsymbol{v}) = -\boldsymbol{v}$ rotates the house by $180°$ around the origin. Then the affine transformation $T(\boldsymbol{v}) = -\boldsymbol{v} + (1, 0)$ shifts the rotated house one unit to the right.

24 A code to add a chimney will be gratefully received!

25 This code needs a correction: add spaces between -10 10 -10 10

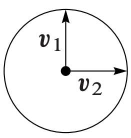
26 $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ compresses vertical distances by 10 to 1. $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects onto the $45°$ line. $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ rotates by $45°$ clockwise and contracts by a factor of $\sqrt{2}$ (the columns have length $1/\sqrt{2}$). $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant -1 so the house is "flipped and sheared." One way to see this is to factor the matrix as LDL^{T}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \text{(shear) (flip left-right) (shear).}$$

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

28 A code that adds two eyes and a smile will be included here with public credit given!

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0, 0)$.)

30 The circle 

transforms to the ellipse by rotating $30°$ and stretching the first axis by 2.

31 Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed $\boldsymbol{v}$) go to two parallel edges (edges differing by $T(\boldsymbol{v})$). So the output is a parallelogram.

Problem Set 7.2, page 395

1 For $S\boldsymbol{v} = d^2\boldsymbol{v}/dx^2$
$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 = 1, x, x^2, x^3$
$S\boldsymbol{v}_1 = S\boldsymbol{v}_2 = \boldsymbol{0}$, $S\boldsymbol{v}_3 = 2\boldsymbol{v}_1$, $S\boldsymbol{v}_4 = 6\boldsymbol{v}_2$;
The matrix for S is $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 $Sv = d^2v/dx^2 = 0$ for linear functions $\boldsymbol{v}(\boldsymbol{x}) = a + bx$. All $(a, b, 0, 0)$ are in the nullspace of the second derivative matrix B.

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

4 The third derivative matrix has **6** in the $(1,4)$ position; since the third derivative of x^3 is 6. This matrix also comes from AB. The fourth derivative of a cubic is zero, and B^2 is the zero matrix.

5 $T(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3) = 2\boldsymbol{w}_1+\boldsymbol{w}_2+2\boldsymbol{w}_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $\boldsymbol{v} = c(\boldsymbol{v}_2-\boldsymbol{v}_3)$ gives $T(\boldsymbol{v}) = \mathbf{0}$; nullspace is $(0,c,-c)$; solutions $(1,0,0)+(0,c,-c)$.

7 $(1,0,0)$ is not in the column space of the matrix A, and $\boldsymbol{w}_1$ is not in the range of the linear transformation T. Key point: *Column space* of matrix matches *range* of transformation.

8 We don't know $T(\boldsymbol{w})$ unless the $\boldsymbol{w}$'s are the same as the $\boldsymbol{v}$'s. In that case the matrix is A^2.

9 Rank of $A = 2 =$ dimension of the *range* of T. The outputs $A\boldsymbol{v}$ (column space) match the outputs $T(\boldsymbol{v})$ (the range of T). The "output space" $\boldsymbol{W}$ is like $\mathbf{R}^m$: it contains all outputs but may not be filled up.

10 The matrix for T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. For the output $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose input $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This means: For the output $\boldsymbol{w}_1$ choose the input $\boldsymbol{v}_1-\boldsymbol{v}_2$.

11 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ so $T^{-1}(\boldsymbol{w}_1) = \boldsymbol{v}_1-\boldsymbol{v}_2, T^{-1}(\boldsymbol{w}_2) = \boldsymbol{v}_2-\boldsymbol{v}_3, T^{-1}(\boldsymbol{w}_3) = \boldsymbol{v}_3$. The columns of A^{-1} describe T^{-1} from $\boldsymbol{W}$ back to $\boldsymbol{V}$. The only solution to $T(\boldsymbol{v}) = 0$ is $\boldsymbol{v} = 0$.

12 (c) $T^{-1}(T(\boldsymbol{w}_1)) = \boldsymbol{w}_1$ is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

13 (a) $T(\boldsymbol{v}_1) = \boldsymbol{v}_2, T(\boldsymbol{v}_2) = \boldsymbol{v}_1$ is its own inverse (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1, T(\boldsymbol{v}_2) = 0$ has $T^2 = T$ (c) If $T^2 = I$ for part (a) and $T^2 = T$ for part (b), then T must be I.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} =$ inverse of (a) (c) $A\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

15 (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} r \\ t \end{bmatrix}$ and $\begin{bmatrix} s \\ u \end{bmatrix}$; this is the "easy" direction. (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-\mathbf{1}}$ transforms in the inverse direction, back to the standard basis vectors. (c) $ad = bc$ will make the forward matrix singular and the inverse impossible.

16 $MW = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

17 Recording basis vectors is done by a *Permutation matrix.* Changing lengths is done by a *positive diagonal matrix.*

18 $(a,b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

19 $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$; $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ = first column of M^{-1} = coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

20 $\boldsymbol{w}_2(x) = 1 - x^2$; $\boldsymbol{w}_3(x) = \frac{1}{2}(x^2 - x)$; $\boldsymbol{y} = 4\boldsymbol{w}_1 + 5\boldsymbol{w}_2 + 6\boldsymbol{w}_3$.

21 $\boldsymbol{w}$'s to $\boldsymbol{v}$'s: $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$. $\boldsymbol{v}$'s to $\boldsymbol{w}$'s: inverse matrix $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. *The key idea*: The matrix multiplies the coordinates in the $\boldsymbol{v}$ basis to give the coordinates in the $\boldsymbol{w}$ basis.

22 The 3 equations to match 4, 5, 6 at $x = a, b, c$ are $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. This Vandermonde determinant equals $(b-a)(c-a)(c-b)$. So a, b, c must be distinct to have det $\neq 0$ and one solution A, B, C.

23 The matrix M with these nine entries must be invertible.

24 Start from $A = QR$. Column 2 is $\boldsymbol{a}_2 = r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2$. This gives $\boldsymbol{a}_2$ as a combination of the $\boldsymbol{q}$'s. So the change of basis matrix is $\boldsymbol{R}$.

25 Start from $A = LU$. Row 2 of A is ℓ_{21}(row 1 of U) + ℓ_{22} (row 2 of U). The change of basis matrix is always *invertible*, because basis goes to basis.

26 The matrix for $T(\boldsymbol{v}_i) = \lambda_i \boldsymbol{v}_i$ is $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

27 If T is not invertible, $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$ is not a basis. We couldn't choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$.

28 (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \mathbf{0}$ and $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_1$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_1$ (which combine into $T(\boldsymbol{v}_2) = \mathbf{0}$ by *linearity*).

29 $T(x, y) = (x, -y)$ is reflection across the x-axis. Then reflect across the y-axis to get $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

30 S takes (x, y) to $(-x, y)$. $S(T(\boldsymbol{v})) = (\mathbf{-1}, \mathbf{2})$. $S(\boldsymbol{v}) = (-2, 1)$ and $T(S(\boldsymbol{v})) = (\mathbf{1}, \mathbf{-2})$.

31 Multiply the two reflections to get $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ which is *rotation by* $2(\theta - \alpha)$. In words: $(1, 0)$ is reflected to have angle 2α, and that is reflected again to angle $2\theta - 2\alpha$.

32 False: We will not know $T(\boldsymbol{v})$ for *energy* $\boldsymbol{v}$ unless the n $\boldsymbol{v}$'s are linearly independent.

33 To find coordinates in the wavelet basis, multiply by $W^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Then $e = \frac{1}{4}w_1 + \frac{1}{4}w_2 + \frac{1}{2}w_3$ and $v = w_3 + w_4$. Notice again: W **tells us how the bases change**, W^{-1} **tells us how the coordinates change**.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, −2, −2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$, $(0,0,0,0,1,1,-1,-1)$ and 4 wavelets with a single pair $1,-1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplying by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\boldsymbol{v}_1 + c\boldsymbol{v}_3$. Similarly $T(\boldsymbol{v}_2) = a\boldsymbol{v}_2 + c\boldsymbol{v}_4$ and $T(\boldsymbol{v}_3) = b\boldsymbol{v}_1 + d\boldsymbol{v}_3$ and $T(\boldsymbol{v}_4) = b\boldsymbol{v}_2 + d\boldsymbol{v}_4$. The matrix for T in this basis is $\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$.

38 The matrix for T in this basis is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

2 Orthonormal bases: $\boldsymbol{v}_1$ for row space, $\boldsymbol{v}_2$ for nullspace, $\boldsymbol{u}_1$ for column space, $\boldsymbol{u}_2$ for $N(A^{\mathrm{T}})$. All matrices with those four subspaces are multiples cA, since the subspaces are just lines. Normally many more matrices share the same 4 subspaces. (For example, all n by n invertible matrices share $\mathbf{R}^n$.)

3 $A = QH = \dfrac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \dfrac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

5 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$ has $\lambda = 18$ and 2, $\boldsymbol{v}_1 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\sigma_1 = \sqrt{18}$ and $\sigma_2 = \sqrt{2}$.

6 $AA^{\mathrm{T}} = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$ has $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The same $\sqrt{18}$ and $\sqrt{2}$ go into Σ.

7 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

8 $A = U\Sigma V^{\mathrm{T}}$ splits into QK (polar): $Q = UV^{\mathrm{T}} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $K = V\Sigma V^{\mathrm{T}} = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

10 $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\lambda = 25, 0, 0$ and $\boldsymbol{v}_1 = \begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Here $A = [3 \;\; 4 \;\; 0]$ has rank 1 and $AA^{\mathrm{T}} = [25]$ and $\sigma_1 = 5$ is the only singular value in $\Sigma = [5 \;\; 0 \;\; 0]$.

11 $A=[\,1\,]\,[\,5\quad 0\quad 0\,]V^{\mathrm{T}}$ and $A^{+}=V\begin{bmatrix}.2\\0\\0\end{bmatrix}=\begin{bmatrix}.12\\.16\\0\end{bmatrix}$; $A^{+}A=\begin{bmatrix}.36&.48&0\\.48&.64&0\\0&0&0\end{bmatrix}$; $AA^{+}=[\,1\,]$

12 The zero matrix has no pivots or singular values. Then Σ = same 2 by 3 zero matrix and the pseudoinverse is the 3 by 2 zero matrix.

13 If $\det A=0$ then $\operatorname{rank}(A)<n$; thus $\operatorname{rank}(A^{+})<n$ and $\det A^{+}=0$.

14 A must be *symmetric and positive definite*, if $\Sigma=\Lambda$ and $U=V=$ eigenvector matrix Q is orthogonal.

15 (a) $A^{\mathrm{T}}A$ is singular (b) This $\boldsymbol{x}^{+}$ in the row space does give $A^{\mathrm{T}}A\boldsymbol{x}^{+}=A^{\mathrm{T}}\boldsymbol{b}$ (c) If $(1,-1)$ in the nullspace of A is added to $\boldsymbol{x}^{+}$, we get another solution to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$. But this $\widehat{\boldsymbol{x}}$ is longer than $\boldsymbol{x}^{+}$ because the added part is orthogonal to $\boldsymbol{x}^{+}$ in the row space.

16 $\boldsymbol{x}^{+}$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}}-\boldsymbol{x}^{+}$ in the nullspace of $A^{\mathrm{T}}A=$ nullspace of A. The right triangle has $c^2=a^2+b^2$.

17 $AA^{+}\boldsymbol{p}=\boldsymbol{p},\ AA^{+}\boldsymbol{e}=\boldsymbol{0},\ A^{+}A\boldsymbol{x}_r=\boldsymbol{x}_r,\ A^{+}A\boldsymbol{x}_n=\boldsymbol{0}$.

18 $A^{+}=V\Sigma^{+}U^{\mathrm{T}}$ is $\frac{1}{5}[\,.6\quad .8\,]=[\,.12\quad .16\,]$ and $A^{+}A=[\,1\,]$ and $AA^{+}=\begin{bmatrix}.36&.48\\.48&.64\end{bmatrix}=$ projection.

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

20 LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are determined by $\mathbf{1}+\mathbf{2}+\mathbf{0}$ numbers because A is *symmetric*.

21 Column times row multiplication gives $A=U\Sigma V^{\mathrm{T}}=\sum\sigma_i\boldsymbol{u}_i\boldsymbol{v}_i^{\mathrm{T}}$ and also $A^{+}=V\Sigma^{+}U^{\mathrm{T}}=\sum\sigma_i^{-1}\boldsymbol{v}_i\boldsymbol{u}_i^{\mathrm{T}}$. Multiplying $A^{+}A$ and using orthogonality of each $\boldsymbol{u}_i$ to all other $\boldsymbol{u}_j$ leaves the projection matrix $A^{+}A$: $A^{+}A=\sum 1\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}$. Similarly $AA^{+}=\sum 1\boldsymbol{u}_i\boldsymbol{u}_i^{\mathrm{T}}$ from $VV^{\mathrm{T}}=I$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A=U\Sigma V^{\mathrm{T}}$ has the required form $A=\widehat{U}M_1\Sigma_rM_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M=M_1\Sigma_rM_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix}0&A\\A^{\mathrm{T}}&0\end{bmatrix}\begin{bmatrix}\boldsymbol{u}\\\boldsymbol{v}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\\A^{\mathrm{T}}\boldsymbol{u}\end{bmatrix}=\sigma\begin{bmatrix}\boldsymbol{u}\\\boldsymbol{v}\end{bmatrix}$. The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

1 Det $A_0^{\mathrm{T}}C_0A_0=\begin{bmatrix}c_1+c_2&-c_2&0\\-c_2&c_2+c_3&-c_3\\0&-c_3&c_3+c_4\end{bmatrix}$ is by direct calculation. Set $c_4=0$ to find $\det A_1^{\mathrm{T}}C_1A_1=c_1c_2c_3$.

2 $(A_1^{\mathrm{T}}C_1A_1)^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}c_1^{-1}&&\\&c_2^{-1}&\\&&c_3^{-1}\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=$

$$\begin{bmatrix}c_1^{-1}&c_1^{-1}&c_1^{-1}\\c_1^{-1}&c_1^{-1}+c_2^{-1}&c_1^{-1}+c_2^{-1}\\c_1^{-1}&c_1^{-1}+c_2^{-1}&c_1^{-1}+c_2^{-1}+c_3^{-1}\end{bmatrix}.$$

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1+f_2+f_3=0$. $\boldsymbol{f}=(-1,0,1)$ gives $c_2u_1-c_2u_2=-1, c_3u_2-c_3u_3=-1, 0=0$. Then $\boldsymbol{u}_{\text{particular}}=(-c_2^{-1}-c_3^{-1},-c_3^{-1},0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}}=(1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx=-\left[c(x)\frac{du}{dx}\right]_0^1=0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx=0$.

5 $\displaystyle -\frac{dy}{dx}=f(x)$ gives $\displaystyle y(x)=C-\int_0^x f(t)dt$. Then $y(1)=0$ gives $\displaystyle C=\int_0^1 f(t)dt$ and $\displaystyle y(x)=\int_x^1 f(t)dt$. If the load is $f(x)=1$ then the displacement is $y(x)=1-x$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1=[1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

7 For 5 springs and 4 masses, the 5 by 4 A has two nonzero diagonals: all $a_{ii}=1$ and $a_{i+1,i}=-1$. With $C=\text{diag}(c_1,c_2,c_3,c_4,c_5)$ we get $K=A^{\mathrm{T}}CA$, symmetric tridiagonal with diagonal entries $K_{ii}=c_i+c_{i+1}$ and off-diagonals $K_{i+1,i}=-c_{i+1}$. With $C=I$ this K is the $-1,2,-1$ matrix and $K(2,3,3,2)=(1,1,1,1)$ solves $K\boldsymbol{u}=$ ones$(4,1)$. (K^{-1} will solve $K\boldsymbol{u}=$ ones(4).)

8 The solution to $-u''=1$ with $u(0)=u(1)=0$ is $u(x)=\frac{1}{2}(x-x^2)$. At $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ this gives $\boldsymbol{u}=2,3,3,2$ (discrete solution in Problem 7) times $(\Delta x)^2=1/25$.

9 $-u''=mg$ has complete solution $u(x)=A+Bx-\frac{1}{2}mgx^2$. From $u(0)=0$ we get $A=0$. From $u'(1)=0$ we get $B=mg$. Then $u(x)=\frac{1}{2}mg(2x-x^2)$ at $x=\frac{1}{3},\frac{2}{3},\frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is *not* proportional to the discrete $u=(3mg,5mg,6mg)$ at the meshpoints. This imperfection is because the discrete problem uses a 1-sided difference, less accurate at the free end. Perfect accuracy is recovered by a centered difference (discussed on page 21 of my CSE textbook).

10 (added in later printing, changing **10-11** below into **11-12**). The solution in this fixed-fixed case is $(2.25, 2.50, 1.75)$ so the second mass moves furthest.

11 The two graphs of 100 points are "discrete parabolas" starting at $(0,0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

12 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: $E=$ diag(ones$(6,1),1)$; $K=64*(2*$ eye$(7)-E-E')$; $D=80*(E-$ eye$(7))$; $(K+D)\backslash$ones$(7,1)$; % forward; $(K-D')\backslash$ones$(7,1)$; % backward; $(K+D/2-D'/2)\backslash$ones$(7,1)$; % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A=\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}$; nullspace contains $\begin{bmatrix}c\\c\\c\end{bmatrix}$; $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}$ for $\boldsymbol{y}=(1,-1,1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination on $b_1[\,A \quad \boldsymbol{b}\,] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$ leads to $[\,U \quad \boldsymbol{c}\,] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{bmatrix}$. The nonzero rows of U come from edges 1 and 3 in a tree. The zero row comes from the loop (all 3 edges).

4 For the matrix in Problem 3, $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for $\boldsymbol{b} = (1,1,0)$ and not solvable for $\boldsymbol{b} = (1,0,0)$. For solvable $\boldsymbol{b}$ (in the column space), $\boldsymbol{b}$ must be orthogonal to $\boldsymbol{y} = (1,-1,1)$; that combination of rows is the zero row, and $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1,-1,0)$ and not solvable for $\boldsymbol{f} = (1,0,0)$; $\boldsymbol{f}$ must be orthogonal to $(1,1,1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = 1, -1, 0$ and currents $-A\boldsymbol{x} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\boldsymbol{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\boldsymbol{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} +$ any $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = \frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\boldsymbol{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

9 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

10 The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The nonzero rows of U keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$
diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry $= -1$ if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

12 (a) The nullspace and rank of $A^{\mathrm{T}}A$ and A are always the same (b) $A^{\mathrm{T}}A$ is always positive semidefinite because $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \|A\boldsymbol{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1,1,1,1)$ is in the nullspace (c) Real eigenvalues all ≥ 0 because positive semidefinite.

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives four potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ I grounded $x_4 = 0$ and solved for $\boldsymbol{x}$ currents $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$

14 $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{0}$ for $\boldsymbol{x} = c(1,1,1,1) = (c,c,c,c)$. If $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ is solvable, then $\boldsymbol{f}$ in the column space (= row space by symmetry) must be orthogonal to $\boldsymbol{x}$ in the nullspace: $f_1 + f_2 + f_3 + f_4 = 0$.

15 The number of loops in this connected graph is $n - m + 1 = \mathbf{7 - 7 + 1} = \mathbf{1}$. What answer if the graph has two separate components (no edges between)?

16 Start from (4 nodes) $-$ (6 edges) $+$ (3 loops) $= 1$. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

18 A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \cdots + (n-1) = n(n-1)/2$. Tree has 5 edges.

Problem Set 8.3, page 437

1 Eigenvalues $\lambda = 1$ and $.75$; $(A - I)\boldsymbol{x} = 0$ gives the steady state $\boldsymbol{x} = (.6, .4)$ with $A\boldsymbol{x} = \boldsymbol{x}$.

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^{\infty} = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4 A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$, because each row of A^{T} adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A.)

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.

6 Add the components of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^{\infty}$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \le 1 \\ .4 + .6a \ge 0 \end{matrix}$

8 If $P =$ cyclic permutation and $\boldsymbol{u}_0 = (1, 0, 0, 0)$ then $\boldsymbol{u}_1 = (0, 0, 1, 0)$; $\boldsymbol{u}_2 = (0, 1, 0, 0)$; $\boldsymbol{u}_3 = (1, 0, 0, 0)$; $\boldsymbol{u}_4 = \boldsymbol{u}_0$. The eigenvalues $1, i, -1, -i$ are all *on the unit circle*. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue $\lambda = 1$.

9 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply on the right by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

11 Last row $.2, .3, .5$ makes $A = A^{\mathrm{T}}$; rows also add to 1 so $(1, \ldots, 1)$ is also an eigenvector of A.

12 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, \ .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

13 $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\boldsymbol{x} = (.9, .9, .9)$

14 $(I-A)(I+A+A^2+\cdots) = (I+A+A^2+\cdots)-(A+A^2+A^3+\cdots) = \boldsymbol{I}$. This says that $I + A + A^2 + \cdots$ is $(I-A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$ and the series adds to $\begin{bmatrix} 1+\frac{1}{2}+\cdots & \frac{1}{2}+\frac{1}{4}+\cdots \\ 1+\frac{1}{2}+\cdots & 1+\frac{1}{2}+\cdots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I-A)^{-1}$.

15 The first two A's have $\lambda_{\max} < 1$; $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I-A)^{-1}$ does not exist.

18 The Leslie matrix on page 435 has $\det(A-\lambda I) = \det \begin{bmatrix} F_1-\lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{bmatrix} = -\lambda^3 + F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2$. This is negative for large λ. It is positive at $\lambda = 1$ provided that $F_1 + F_2P_1 + F_3P_1P_2 > 1$. Under this key condition, $\det(A-\lambda I)$ must be zero at some λ between 1 and ∞. That eigenvalue means that the population grows (under this condition connecting F's and P's reproduction and survival rates).

19 Λ times $S^{-1}\Delta S$ has the same diagonal as $S^{-1}\Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $A\boldsymbol{x} = \lambda_{\max}(A)\boldsymbol{x} > 0$ then $B\boldsymbol{x} > \lambda_{\max}(A)\boldsymbol{x}$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment (6, 0) to (0, 3); minimum cost at (6, 0), maximum at (0, 3).

2 Feasible set has corners (0, 0), (6, 0), (2, 2), (0, 6). Minimum cost $2x - y$ at (6, 0).

3 Only two corners (4, 0, 0) and (0, 2, 0); let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From (0, 0, 2) move to $\boldsymbol{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 Cost = 20 at start (4, 0, 0); keeping $x_1+x_2+2x_3 = 4$ move to (3, 1, 0) with cost 18 and $r = -2$; or move to (2, 0, 1) with cost 17 and $r = -3$. Choose x_3 as entering variable and move to (0, 0, 2) with cost 14. Another step will reach (0, 4, 0) with minimum cost 12.

6 If we reduce the Ph.D. cost to \$1 or \$2 (below the student cost of \$3), the job will go to the Ph.D. with cost vector $\boldsymbol{c} = (2, 3, 8)$ the Ph.D. takes 4 hours $(x_1 + x_2 + 2x_3 = 4)$ and charges \$8.

The teacher in the dual problem now has $y \le 2, y \le 3, 2y \le 8$ as constraints $A^{\mathrm{T}}\boldsymbol{y} \le \boldsymbol{c}$ on the charge of y per problem. So the dual has maximum at $y = 2$. The dual cost is also \$8 for 4 problems and maximum = minimum.

7 $\boldsymbol{x} = (2, 2, 0)$ is a corner of the feasible set with $x_1 + x_2 + 2x_3 = 4$ and the new constraint $2x_1 + x_2 + x_3 = 6$. The cost of this corner is $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = (5, 3, 8) \cdot (2, 2, 0) = 16$. Is this the minimum cost?

Compute the reduced cost r if $x_3 = 1$ *enters* (x_3 was previously zero). The two constraint equations now require $x_1 = 3$ and $x_2 = -1$. With $\boldsymbol{x} = (3, -1, 1)$ the new cost is $3.5 - 1.3 + 1.8 = 20$. This is higher than 16, so the original $\boldsymbol{x} = (2, 2, 0)$ was optimal.

Note that $x_3 = 1$ led to $x_2 = -1$ and a negative x_2 is not allowed. If x_3 reduced the cost (it didn't) we would not have used as much as $x_3 = 1$.

8 $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \le \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} = (A^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} \le \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. The first inequality needed $\boldsymbol{y} \ge 0$ and $A\boldsymbol{x} - \boldsymbol{b} \ge 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$ Notice $j - k \ne 0$ in the denominator. If $j = k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

2 Three integral tests show that $1, x, x^2 - \frac{1}{3}$ are orthogonal on the interval $[-1,\ 1]$: $\int_{-1}^{1}(1)(x)\ \ dx = 0, \int_{-1}^{1}(1)(x^2 - \frac{1}{3})\ \ dx = 0, \int_{-1}^{1}(x)(x^2 - \frac{1}{3})\ \ dx = 0$. Then $2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$. Those coefficients $2, 0, \frac{2}{3}$ can come from integrating $f(x) = 2x^2$ times the 3 basis functions and dividing by their lengths squared—in other words using $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/a^{\mathrm{T}}a$ for functions (where $\boldsymbol{b}$ is $f(x)$ and $\boldsymbol{a}$ is 1 or x or $x^2 - \frac{1}{3}$) exactly as for vectors.

3 One example orthogonal to $\boldsymbol{v} = (1, \frac{1}{2}, \ldots)$ is $\boldsymbol{w} = (2, -1, 0, 0, \ldots)$ with $\|\boldsymbol{w}\| = \sqrt{5}$.

4 $\int_{-1}^{1}(1)(x^3 - cx)\,dx = 0$ and $\int_{-1}^{1}(x^2 - \frac{1}{3})(x^3 - cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3 - cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (*odd* k). The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$. That infinite series equals $\pi^2/8$.

7 The $-1, 1$ odd square wave is $f(x) = x/|x|$ for $0 < |x| < \pi$. Its Fourier series in equation (8) is $4/\pi$ times $[\sin x + (\sin 3x)/3 + (\sin 5x/5) + \cdots]$. The sum of the first N terms has an interesting shape, close to the square wave except where the wave jumps between -1 and 1. At those jumps, the Fourier sum spikes the wrong way to ± 1.09 (the *Gibbs phenomenon*) before it takes the jump with the true $f(x)$.

This happens for the Fourier sums of all functions with jumps. It makes shock waves hard to compute. You can see it clearly in a graph of the sum of 10 terms.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1 - a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi}(1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, 0, ... and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0, b_k = -2/k$.

10 The integral from $-\pi$ to π or from 0 to 2π (or from any a to $a + 2\pi$) is over one complete period of the function. If $f(x)$ is periodic this changes $\int_0^{2\pi} f(x)\,dx$ to $\int_0^{\pi} f(x)\,dx + \int_{-\pi}^{0} f(x)\,dx$. If $f(x)$ is **odd**, those integrals cancel to give $\int f(x)\,dx = 0$ over one period.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

12 $\dfrac{d}{dx}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x \\ \cos x \\ -2\sin 2x \\ 2\cos 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}.$ This shows the differentiation matrix.

13 The square pulse with $F(x) = 1/h$ for $-x \le h/2 \le x$ is an even function, so all sine coefficients b_k are zero. The average a_0 and the cosine coefficients a_k are

$$a_0 = \frac{1}{2\pi}\int_{-h/2}^{h/2} (1/h)dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-h/2}^{h/2} (1/h)\cos kx dx = \frac{2}{\pi kh}\left(\sin\frac{kh}{2}\right) \text{ which is } \frac{1}{\pi}\,\text{sinc}\left(\frac{kh}{2}\right)$$

(introducing the sinc function $(\sin x)/x$). As h approaches zero, the number $x = kh/2$ approaches zero, and $(\sin x)/x$ approaches 1. So all those a_k approach $1/\pi$.

The limiting "delta function" contains an equal amount of all cosines: a very irregular function.

Problem Set 8.6, page 458

1 The diagonal matrix $C = W^{\mathrm{T}}W$ is $\Sigma^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix}$ with no covariances (independent trials). Then solve $A^{\mathrm{T}}CA\widehat{\boldsymbol{x}} = A^{\mathrm{T}}C\boldsymbol{b}$ for this weighted least squares problem (notice $Ct + D$ instead of $C + Dt$):

$$A\boldsymbol{x} = \widehat{\boldsymbol{b}} \quad \text{is} \quad \begin{matrix} 0C + D = 1 \\ 1C + D = 2 \\ 2C + D = 4 \end{matrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

$$A^{\mathrm{T}}CA = \begin{bmatrix} 3 & 2 \\ 2 & 2.5 \end{bmatrix} \qquad A^{\mathrm{T}}C\boldsymbol{b} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \quad \widehat{\boldsymbol{x}} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10/7 \\ 6/7 \end{bmatrix}.$$

2 If the measurement b_3 is totally unreliable and $\sigma_3^2 = \infty$, then *the best line will not use* b_3. In this example, the system $A\boldsymbol{x} = \boldsymbol{b}$ becomes square (first two equations from Problem 1):

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ gives } \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \text{ The line } b = \boldsymbol{t+1} \text{ fits exactly.}$$

3 If $\sigma_3 = 0$ the third equation is exact. Then the best line has $Ct + D = b_3$ which is $2C + D = 4$. The errors $Ct + D - b$ in the measurements at $t = 0$ and 1 are $D - 1$ and $C + D - 2$. Since $D = 4 - 2C$ from the exact $b_3 = 4$, those two errors are $D - 1 = 3 - 2C$ and $C + D - 2 = 2 - C$. The sum of squares $(3-2C)^2 + (2-C)^2$ is a minimum at $8 = 5C$ (calculus or linear algebra in 1D). Then $C = 8/5$ and $D = 4 - 2C = 4/5$.

4 $0, 1, 2$ have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\boldsymbol{\Sigma} = \textbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \frac{1}{2}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$; $a = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$ recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\boldsymbol{\Sigma}L^{\mathrm{T}} = (A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1}A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}A(A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1} = P = (A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1}$.

9 The new grade matrix A has row 3 = − row 1 and row 4 = − row 2, so the rank is 7. The nullspace of A now includes $(1,-1,-1,1)$ as well as $(1,1,1,1)$. Compare to the grade matrix in Example 6 (not Example 5). The other two singular vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for Example 6 are still correct for this new $A(A\boldsymbol{v}_1$ is still orthogonal to $A\boldsymbol{v}_2)$:

$$A\begin{bmatrix} 2\boldsymbol{v}_1 & 2\boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 & -3 \\ -1 & 3 & -3 & 1 \\ -3 & 1 & -1 & -3 \\ 1 & -3 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -8 & -4 \\ -8 & 4 \\ 8 & 4 \end{bmatrix}.$$

Those last orthogonal columns are multiples of the orthonormal $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. This matrix A has $\sigma_1 = 8$ and $\sigma_2 = 4$ (only two singular values since the rank is 2). If you compute $A^{\mathrm{T}}A$ to find those singular vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ from scratch, notice that its trace is $\sigma_1^2 + \sigma_2^2 = 64 + 16 = 80$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 20 & -12 & -20 & 12 \\ -12 & 20 & 12 & -20 \\ -20 & 12 & 20 & -12 \\ 12 & -20 & -12 & 20 \end{bmatrix}.$$

Problem Set 8.7, page 463

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

2 For an affine transformation we also need T (origin), because $T(\mathbf{0})$ need not be $\mathbf{0}$ for affine T. Including this translation by $T(\mathbf{0})$, $(x, y, z, 1)$ is transformed to $xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) + T(\mathbf{0})$.

3 $TT_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix}$ is translation along $(1, 6, 8)$.

4 $S = \operatorname{diag}(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

6 $\begin{bmatrix} x & y & z & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} x & y & z & 1 \end{bmatrix}\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ -2 & -2 & -4 & 1 \end{bmatrix}$.

The first matrix translates by $(-1, -1, -2)$. The second matrix rescales by 2.

7 The three parts of Q in equation (1) are $(\cos\theta)I$ and $(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}$ and $-\sin\theta(\boldsymbol{a}\times)$. Then $Q\boldsymbol{a}=\boldsymbol{a}$ because $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}=\boldsymbol{a}$(unit vector) and $\boldsymbol{a}\times\boldsymbol{a}=\mathbf{0}$.

8 If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}=0$ and those three parts of Q (Problem 7) multiply $\boldsymbol{b}$, the results in $Q\boldsymbol{b}$ are $(\cos\theta)\boldsymbol{b}$ and $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}=\mathbf{0}$ and $(-\sin\theta)\boldsymbol{a}\times\boldsymbol{b}$. The component along $\boldsymbol{b}$ is $(\cos\theta)\boldsymbol{b}$.

9 $\boldsymbol{n}=\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ has $P=I-\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}=\frac{1}{9}\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}$. Notice $\|\boldsymbol{n}\|=1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_-PT_+=\frac{1}{9}\begin{bmatrix}5&-4&-2&0\\-4&5&-2&0\\-2&-2&8&0\\6&6&3&9\end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$. Row vectors!

12 The projection of a square onto a plane is a parallelogram (or a line segment). The sides of the square are perpendicular, but their projections may not be ($\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}=0$ but $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y})=\boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y}=\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{y}$ may be nonzero).

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I-2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\begin{bmatrix}1&-8&-4\\-8&1&-4\\-4&-4&7\end{bmatrix}=\left(-\frac{11}{3},-\frac{11}{3},-\frac{1}{3}\right)$.

15 $(3,3,3,1)\to(3,3,0,1)\to\left(-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1\right)\to\left(-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1\right)$.

16 Just subtracting vectors would give $\boldsymbol{v}=(x,y,z,0)$ ending in 0 (not 1). In homogeneous coordinates, add a **vector** to a point.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}|=|\text{entry/pivot}|\le 1$. $A=\begin{bmatrix}1&1&1\\0&1&-1\\-1&1&1\end{bmatrix}$.

2 The exact inverse of hilb(3) is $A^{-1}=\begin{bmatrix}9&-36&30\\-36&192&-180\\30&-180&180\end{bmatrix}$.

3 $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}11/6\\13/12\\47/60\end{bmatrix}=\begin{bmatrix}1.833\\1.083\\0.783\end{bmatrix}$ compares with $A\begin{bmatrix}0\\6\\-3.6\end{bmatrix}=\begin{bmatrix}1.80\\1.10\\0.78\end{bmatrix}$. $\|\Delta\boldsymbol{b}\|<.04$ but $\|\Delta\boldsymbol{x}\|>6$. The difference $(1,1,1)-(0,6,-3.6)$ is in a direction $\Delta\boldsymbol{x}$ that has $A\Delta\boldsymbol{x}$ near zero.

4 The largest $\|\boldsymbol{x}\|=\|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\|=1/\lambda_{\min}$ since $A^{\mathrm{T}}=A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $<wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1}=Q^{\mathrm{T}}$. So $QR\boldsymbol{x}=\boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x}=\boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

8 $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}$; $A \to \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}$.

9 $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has cofactors $C_{13} = C_{31} = C_{24} = C_{42} = 1$ and $C_{14} = C_{41} = -1$. A^{-1} is a full matrix!

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$.
(b) A has eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of Q: either $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$ **or** $Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$.

12 When A is multiplied by a plane rotation Q_{ij}, this changes the $2n$ (not n^2) entries in rows i and j. Then multiplying on the right by $(Q_{ij})^{-1} = (Q_{ij})^{\mathrm{T}}$ changes the $2n$ entries in columns i and j.

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

14 The $(2,1)$ entry of $Q_{21}A$ is $\frac{1}{3}(-\sin\theta + 2\cos\theta)$. This is zero if $\sin\theta = 2\cos\theta$ or $\tan\theta = 2$. Then the $2, 1, \sqrt{5}$ right triangle has $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$.

Every 3 by 3 rotation with $\det Q = +1$ is the product of 3 plane rotations.

15 This problem shows how elimination is more expensive (the nonzero multipliers are counted by **nnz**(L) and **nnz**(LL)) when we spoil the tridiagonal K by a random permutation.

If on the other hand we start with a poorly ordered matrix K, an improved ordering is found by the code **symamd** discussed in this section.

16 The "red-black ordering" puts rows and columns 1 to 10 in the odd-even order $1, 3, 5, 7, 9, 2, 4, 6, 8, 10$. When K is the $-1, 2, -1$ tridiagonal matrix, odd points are connected only to even points (and 2 stays on the diagonal, connecting every point to itself):

$$K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & 2 \end{bmatrix} \text{ and } PKP^{\mathrm{T}} = \begin{bmatrix} 2I & D \\ D^{\mathrm{T}} & 2I \end{bmatrix} \text{ with}$$

$$D = \begin{bmatrix} -1 & & & & & \\ -1 & -1 & & & & \\ 0 & -1 & -1 & & & \\ & & -1 & -1 & & \\ & & & -1 & -1 & \end{bmatrix} \begin{array}{l} 1 \text{ to } 2 \\ 3 \text{ to } 2,4 \\ 5 \text{ to } 4,6 \\ 7 \text{ to } 6,8 \\ 9 \text{ to } 8,10 \end{array}$$

17 Jeff Stuart's **Shake a Stick** activity has long sticks representing the graphs of two linear equations in the x-y plane. The matrix is nearly singular and Section 9.2 shows how to compute its condition number $c = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min} \approx 80,000$:

$$A = \begin{bmatrix} 1 & 1.0001 \\ 1 & 1.0000 \end{bmatrix} \|A\| \approx 2 \quad A^{-1} = 10000 \begin{bmatrix} -1 & 1.0001 \\ 1 & -1 \end{bmatrix} \quad \begin{array}{l} \|A^{-1}\| \approx 20000 \\ c \approx 40000. \end{array}$$

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2$, $c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1$, $c = 3$; $\|A\| = 2 + \sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

2 $\|A\| = 2$, $c = 1$; $\|A\| = \sqrt{2}$, $c =$ infinite (singular matrix); $A^{\mathrm{T}}A = 2I$, $\|A\| = \sqrt{2}$, $c = 1$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

4 $1 = \|I\| = \|AA^{-1}\| \le \|A\|\|A^{-1}\| = c(A)$.

5 If $\Lambda_{\max} = \Lambda_{\min} = 1$ then all $\Lambda_i = 1$ and $A = SIS^{-1} = I$. The only matrices with $\|A\| = \|A^{-1}\| = 1$ are *orthogonal matrices*.

6 All orthogonal matrices have norm 1, so $\|A\| \le \|Q\|\|R\| = \|R\|$ and in reverse $\|R\| \le \|Q^{-1}\|\|A\| = \|A\|$, then $\|A\| = \|R\|$. Inequality is usual in $\|A\| < \|L\|\|U\|$ when $A^{\mathrm{T}}A \ne AA^{\mathrm{T}}$. Use **norm** on a random A.

7 The triangle inequality gives $\|A\boldsymbol{x} + B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = |\lambda|$ for that particular vector $\boldsymbol{x}$. When we maximize the ratio over all vectors we get $\|A\| \ge |\lambda|$.

9 $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\rho(A) = 0$ and $\rho(B) = 0$ but $\rho(A+B) = 1$. The triangle inequality $\|A+B\| \le \|A\| + \|B\|$ fails for $\rho(A)$. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ also has $\rho(AB) = 1$; thus $\rho(A) = \max|\lambda(A)|$ = spectral radius is not a norm.

10 (a) The condition number of A^{-1} is $\|A^{-1}\|\|(A^{-1})^{-1}\|$ which is $\|A^{-1}\|\|A\| = c(A)$. (b) Since $A^{\mathrm{T}}A$ and AA^{T} have the same nonzero eigenvalues, A and A^{T} have the same norm.

11 Use the quadratic formula for $\lambda_{\max}/\lambda_{\min}$, which is $c = \sigma_{\max}/\sigma_{\min}$ since this $A = A^{\mathrm{T}}$ is positive definite:

$$c(A) = \left(1.00005 + \sqrt{(1.00005)^2 - .0001}\right) / \left(1.00005 - \sqrt{}\right) \approx 40,000.$$

12 $\det(2A)$ is not $2 \det A$; $\det(A + B)$ is not always less than $\det A + \det B$; taking $|\det A|$ does not help. The only reasonable property is $\det AB = (\det A)(\det B)$. The condition number should not change when A is multiplied by 10.

13 The residual $\boldsymbol{b} - A\boldsymbol{y} = (10^{-7}, 0)$ is much smaller than $\boldsymbol{b} - A\boldsymbol{z} = (.0013, .0016)$. But $\boldsymbol{z}$ is much closer to the solution than $\boldsymbol{y}$.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

15 $\boldsymbol{x} = (1, 1, 1, 1, 1)$ has $\|\boldsymbol{x}\| = \sqrt{5}$, $\|\boldsymbol{x}\|_1 = 5$, $\|\boldsymbol{x}\|_\infty = 1$. $\boldsymbol{x} = (.1, .7, .3, .4, .5)$ has $\|\boldsymbol{x}\| = 1$, $\|\boldsymbol{x}\|_1 = 2$ (sum) $\|\boldsymbol{x}\|_\infty = .7$ (largest).

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|\boldsymbol{x}\|_1^2$. $x_1^2 + \cdots + x_n^2 \le n \max(x_i^2)$ so $\|\boldsymbol{x}\| \le \sqrt{n}\|\boldsymbol{x}\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|\boldsymbol{x}\|_1 = \boldsymbol{x} \cdot \boldsymbol{y} \le \|\boldsymbol{x}\|\|\boldsymbol{y}\| = \sqrt{n}\|\boldsymbol{x}\|$. $\boldsymbol{x} = (1, \ldots, 1)$ has $\|\boldsymbol{x}\|_1 = \sqrt{n}\,\|\boldsymbol{x}\|$.

17 For the ℓ^∞ norm, the largest component of $\boldsymbol{x}$ plus the largest component of $\boldsymbol{y}$ is not less than $\|\boldsymbol{x} + \boldsymbol{y}\|_\infty$ = largest component of $\boldsymbol{x} + \boldsymbol{y}$.

For the ℓ^1 norm, each component has $|x_i + y_i| \le |x_i| + |y_i|$. Sum on $i = 1$ to n: $\|\boldsymbol{x} + \boldsymbol{y}\|_1 \le \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1$.

18 $|x_1| + 2|x_2|$ is a norm but $\min(|x_1|, |x_2|)$ is not a norm. $\|\boldsymbol{x}\| + \|\boldsymbol{x}\|_\infty$ is a norm; $\|A\boldsymbol{x}\|$ is a norm provided A is invertible (otherwise a nonzero vector has norm zero; for rectangular A we require independent columns to avoid $\|A\boldsymbol{x}\| = 0$).

19 $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + x_2y_2 + \cdots \le (\max|y_i|)(|x_1| + |x_2| + \cdots) = ||\boldsymbol{x}||_1\ ||\boldsymbol{y}||_\infty$.

20 With $\lambda_j = 2 - 2\cos(j\pi/n{+}1)$, the largest eigenvalue is $\lambda_n \approx 2 + 2 = 4$. The smallest is $\lambda_1 = 2 - 2\cos(\pi/n{+}1) \approx \left(\frac{\pi}{n+1}\right)^2$, using $2\cos\theta \approx 2 - \theta^2$. So the condition number is $c = \lambda_{\max}/\lambda_{\min} \approx (4/\pi^2)\, n^2$, growing with n.

21 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1.1 \end{bmatrix}$ has $A^n = \begin{bmatrix} 1 & q \\ 0 & (1.1)^n \end{bmatrix}$ with $q = 1 + 1.1 + \cdots + (1.1)^{n-1} = (1.1^n - 1)/(1.1 - 1) \approx 1.1^n/.1$. So the growing part of A^n is $1.1^n \begin{bmatrix} 0 & 10 \\ 0 & 1 \end{bmatrix}$ with $||A^n|| \approx \sqrt{101}$ times 1.1^n for larger n.

Problem Set 9.3, page 489

1 The iteration $\boldsymbol{x}_{k+1} = (I - A)\boldsymbol{x}_k + \boldsymbol{b}$ has $S = I$ and $T = I - A$ and $S^{-1}T = I - A$.

2 If $Ax = \lambda x$ then $(I-A)x = (1-\lambda)x$. Real eigenvalues of $B = I-A$ have $|1-\lambda| < 1$ provided λ is between 0 and 2.

3 This matrix A has $I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which has $|\lambda| = 2$. The iteration diverges.

4 Always $\|AB\| \le \|A\|\|B\|$. Choose $A = B$ to find $\|B^2\| \le \|B\|^2$. Then choose $A = B^2$ to find $\|B^3\| \le \|B^2\|\|B\| \le \|B\|^3$. Continue (or use induction) to find $\|B^k\| \le \|B\|^k$. Since $\|B\| \ge \max|\lambda(B)|$ it is no surprise that $\|B\| < 1$ gives convergence.

5 $Ax = \mathbf{0}$ gives $(S-T)x = \mathbf{0}$. Then $Sx = Tx$ and $S^{-1}Tx = x$. Then $\lambda = 1$ means that the errors do not approach zero. We can't expect convergence when A is singular and $Ax = b$ is unsolvable!

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^2$.

8 Jacobi has $S^{-1}T = \begin{bmatrix} a & \\ & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}$ with $|\lambda| = |bc/ad|^{1/2}$.

Gauss-Seidel has $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & -bc/ad \end{bmatrix}$ with $|\lambda| = |bc/ad|$.

So Gauss-Seidel is twice as fast to converge (or to explode if $|bc| > |ad|$).

9 Set the trace $2-2\omega+\frac{1}{4}\omega^2$ equal to $(\omega-1)+(\omega-1)$ to find $\omega_{\text{opt}} = 4(2-\sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

10 Gauss-Seidel will converge for the $-1, 2, -1$ matrix. $|\lambda|_{\max} = \cos^2(\pi/n+1)$ is given on page 485, with the improvement from successive over relaxation.

11 If the iteration gives all $x_i^{\text{new}} = x_i^{\text{old}}$ then the quantity in parentheses is zero, which means $Ax = b$. For Jacobi change x^{new} on the right side to x^{old}.

12 A lot of energy went into SOR in the 1950's! Now incomplete LU is simpler and preferred.

13 $u_k/\lambda_1^k = c_1x_1 + c_2x_2(\lambda_2/\lambda_1)^k + \cdots + c_nx_n(\lambda_n/\lambda_1)^k \to c_1x_1$ if all ratios $|\lambda_i/\lambda_1| < 1$. The largest ratio controls the rate of convergence (when k is large). $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $|\lambda_2| = |\lambda_1|$ and no convergence.

14 The eigenvectors of A and also A^{-1} are $x_1 = (.75, .25)$ and $x_2 = (1, -1)$. The inverse power method converges to a multiple of x_2, since $|1/\lambda_2| > |1/\lambda_1|$.

15 In the jth component of Ax_1, $\lambda_1 \sin\frac{j\pi}{n+1} = 2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.

16 $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ produces $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $u_3 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$. This is converging to the eigenvector direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with largest eigenvalue $\lambda = 3$. Divide u_k by $\|u_k\|$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $\boldsymbol{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}, \boldsymbol{u}_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to \boldsymbol{u}_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1+\sin^2\theta) & -\sin^3\theta \\ \boldsymbol{-\sin^3\theta} & -\cos\theta\sin^2\theta \end{bmatrix}$.

19 If A is orthogonal then $Q = A$ and $R = I$. Therefore $A_1 = RQ = A$ again, and the "QR method" doesn't move from A. But shift A slightly and the method goes quickly to Λ.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ by $\boldsymbol{q}_j^{\mathrm{T}}$ to find $\boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j = a_j$ (because the $\boldsymbol{q}$'s are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

22 Theoretically the $\boldsymbol{q}$'s are orthonormal. In reality this important algorithm is not very stable. We must stop every few steps to reorthogonalize—or find another more stable way to orthogonalize $\boldsymbol{q}, A\boldsymbol{q}, A^2q, \ldots$

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

24 The proof of $|\lambda| < 1$ when every absolute row sum < 1 uses $|\sum a_{ij}x_j| \le \sum |a_{ij}||x_i| < |x_i|$. (Here x_i is the largest component.) The application to the Gershgorin circle theorem (very useful) is printed after its statement in this problem.

25 For A and K, the maximum row sums give all $|\lambda| \le 1$ and all $|\lambda| \le 4$. The circles $|\lambda - .5| \le .5$ and $|\lambda - .4| \le .6$ around diagonal entries of A give tighter bounds. The circle $|\lambda - 2| \le 2$ for K contains the circle $|\lambda - 2| \le 1$ and all three eigenvalues $2 + \sqrt{2}$, 2, and $2 - \sqrt{2}$.

26 With diagonal dominance $a_{ii} > r_i$, the circles $|\lambda - a_{ii}| \le r_i$ don't include $\lambda = 0$ (*so A is invertible*!). Notice that the $-1, 2, -1$ matrix is also invertible even though its diagonals are only weakly dominant. They *equal* the off-diagonal row sums, $2 = 2$ except in the first and last rows, and more care is needed to prove invertibility.

27 From the last line of code, $\boldsymbol{q}_2$ is in the direction of $\boldsymbol{v} = A\boldsymbol{q}_1 - h_{11}\boldsymbol{q}_1 = A\boldsymbol{q}_1 - (\boldsymbol{q}_1^{\mathrm{T}}A\boldsymbol{q}_1)\boldsymbol{q}_1$. The dot product with $\boldsymbol{q}_1$ is zero. This is Gram-Schmidt with $A\boldsymbol{q}_1$ as the second input vector.

28 *Note* The five lines in Solutions to Selected Exercises prove two key properties of conjugate gradients—the residuals $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$ are orthogonal and the search directions are A-orthogonal ($\boldsymbol{p}_i^{\mathrm{T}}A\boldsymbol{p}_i = 0$). Then each new guess $\boldsymbol{x}_{k+1}$ is the **closest vector to $\boldsymbol{x}$** among all combinations of $\boldsymbol{b}$, $A\boldsymbol{b}$, $A^k\boldsymbol{b}$. Ordinary iteration $S\boldsymbol{x}_{k+1} = T\boldsymbol{x}_k + \boldsymbol{b}$ does not find this best possible combination $\boldsymbol{x}_{k+1}$.

The solution to Problem 28 in this Fourth Edition is straightforward and important. Since $H = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is symmetric if $A = A^{\mathrm{T}}$, and since H has only one lower diagonal by construction, then H has only one upper diagonal: H is tridiagonal and all the recursions in Arnoldi's method have only 3 terms (Problem 29).

29 $H = Q^{-1}AQ$ is *similar* to A, so H has the same eigenvalues as A (at the end of Arnoldi). When Arnoldi stops sooner because the matrix size is large, the eigenvalues of H_k (called *Ritz values*) are close to eigenvalues of A. This is an important way to compute approximations to λ for large matrices.

30 In principle the conjugate gradient method converges in 100 (or 99) steps to the exact solution $\boldsymbol{x}$. But it is slower than elimination and its all-important property is to give good approximations to $\boldsymbol{x}$ much sooner. (Stopping elimination part way leaves you nothing.) The problem asks how close $\boldsymbol{x}_{10}$ and $\boldsymbol{x}_{20}$ are to $\boldsymbol{x}_{100}$, which equals $\boldsymbol{x}$ except for roundoff errors.

Problem Set 10.1, page 498

1 (a)(b)(c) have sums $4, -2+2i, 2\cos\theta$ and products $5, -2i, 1$. Note $(e^{i\theta})(e^{-i\theta}) = 1$.

2 In polar form these are $\sqrt{5}e^{i\theta}, 5e^{2i\theta}, \frac{1}{\sqrt{5}}e^{-i\theta}, \sqrt{5}$.

3 The absolute values are $r = 10, 100, \frac{1}{10}$, and 100. The angles are $\theta, 2\theta, -\theta$ and -2θ.

4 $|z \times w| = 6,\ |z + w| \le 5,\ |z/w| = \frac{2}{3},\ |z - w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i;\ w^{12} = 1$.

6 $1/z$ has absolute value $1/r$ and angle $-\theta$; $(1/r)e^{-i\theta}$ times $re^{i\theta}$ equals 1.

7 $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}\begin{bmatrix} ac-bd \\ bc+ad \end{bmatrix}$ $\begin{matrix}\textbf{real part} \\ \textbf{imaginary part}\end{matrix}$ $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ is the matrix form of $(1+3i)(1-3i) = 10$.

8 $\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$ gives complex matrix = vector multiplication $(A_1 + iA_2)(\boldsymbol{x}_1 + i\boldsymbol{x}_2) = \boldsymbol{b}_1 + i\boldsymbol{b}_2$.

9 $2+i;\ (2+i)(1+i) = 1+3i;\ e^{-i\pi/2} = -i;\ e^{-i\pi} = -1;\ \frac{1-i}{1+i} = -i;\ (-i)^{103} = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

11 $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ includes aI (which just adds a to the eigenvalues and $b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So the eigenvectors are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The eigenvalues are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. We see $\overline{\boldsymbol{x}}_1 = \boldsymbol{x}_2$ and $\overline{\lambda}_1 = \lambda_2$ as expected for real matrices with complex eigenvalues.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2 - 4(ad-bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

15 The 6 by 6 cyclic shift P has $\det(P_6 - \lambda I) = \lambda^6 - 1 = 0$. Then $\lambda = 1, w, w^2, w^3, w^4, w^5$ with $w = e^{2\pi i/6}$. These are the six solutions to $\lambda^b = 1$ as in Figure 10.3 (The sixth roots of 1).

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

17 (a) $2e^{i\pi/3}, 4e^{2i\pi/3}$ (b) $e^{2i\theta}, e^{4i\theta}$ (c) $7e^{3\pi i/2}, 49e^{3\pi i}$ $(=-49)$ (d) $\sqrt{50}e^{-\pi i/4}$, $50e^{-\pi i/2}$.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

19 $a + ib = 1, i, -1, -i, \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. The root $\overline{w} = w^{-1} = e^{-2\pi i/8}$ is $1/\sqrt{2} - i/\sqrt{2}$.

20 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$ are cube roots of 1. The cube roots of -1 are -1, $e^{\pi i/3}$, $e^{-\pi i/3}$. Altogether six roots of $z^6 = 1$.

21 $\cos 3\theta = \text{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

22 If the conjugate $\overline{z} = 1/z$ then $|z|^2 = 1$ and z is any point $e^{i\theta}$ on the unit circle.

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2+2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

1 $\|\boldsymbol{u}\| = \sqrt{9} = 3$, $\|\boldsymbol{v}\| = \sqrt{3}$, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 3i + 2$, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = -3i + 2$ (this is the conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$).

2 $A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are Hermitian matrices. They share the eigenvalues 4 and 2.

3 $\boldsymbol{z} =$ multiple of $(1+i, 1+i, -2)$; $A\boldsymbol{z} = \boldsymbol{0}$ gives $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}} = \boldsymbol{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{H}})$, $\boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}A\boldsymbol{z} = \boldsymbol{0}$ then $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = 0$. This is $\|A\boldsymbol{z}\|^2 = 0$ so $A\boldsymbol{z} = \boldsymbol{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^{\mathrm{H}}$.

7 cA is still Hermitian *for real* c; $(iA)^{\mathrm{H}} = -iA^{\mathrm{H}} = -iA$ is skew-Hermitian.

8 This P is invertible and unitary. $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} -i & & \\ & -i & \\ & & -i \end{bmatrix} = -iI$. Then $P^{100} = (-i)^{33}P = -iP$. The eigenvalues of P are the roots of $\lambda^3 = -i$, which are i and $ie^{2\pi i/3}$ and $ie^{4\pi i/3}$.

9 One unit eigenvector is certainly $\boldsymbol{x}_1 = (1, 1, 1)$ with $\lambda_1 = i$. The other eigenvectors are $\boldsymbol{x}_2 = (1, w, w^2)$ and $\boldsymbol{x}_3 = (1, w^2, w^4)$ with $w = e^{2\pi i/3}$. *The eigenvector matrix is the Fourier matrix* F_3. The eigenvectors of any unitary matrix like P are orthogonal (using the correct complex form $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$ of the inner product).

10 $(1, 1, 1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 Not included in 4th edition $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2+5P+4P^2$ has $\begin{array}{l} \lambda = 2+5+4 = 11, \\ 2+5e^{2\pi i/3}+4e^{4\pi i/3}, \\ 2+5e^{4\pi i/3}+4e^{8\pi i/3}. \end{array}$

11 If $U^{\mathrm{H}}U = I$ then $U^{-1}(U^{\mathrm{H}})^{-1} = U^{-1}(U^{-1})^{\mathrm{H}} = I$ so U^{-1} is also unitary. Also $(UV)^{\mathrm{H}}(UV) = V^{\mathrm{H}}U^{\mathrm{H}}UV = V^{\mathrm{H}}V = I$ so UV is unitary.

12 Determinant = product of the eigenvalues *(all real)*. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

13 $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = \|A\boldsymbol{z}\|^2$ is positive unless $A\boldsymbol{z} = \boldsymbol{0}$. When A has independent columns this means $\boldsymbol{z} = \boldsymbol{0}$; so $A^{\mathrm{H}}A$ is positive definite.

14 $A = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

15 $K = (iA^{\mathrm{T}}$ in Problem 14$) = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$; λ's are imaginary.

16 $Q = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.

17 $V = \dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$. Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

18 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$: a typical orthonormal expansion.

19 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

20 $\boldsymbol{z} = (1, i, -2)$ completes an orthogonal basis for $\mathbf{C}^3$. So does any $e^{i\theta}\boldsymbol{z}$.

21 $R + iS = (R+iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

22 $\mathbf{C}^n$ has dimension n; the columns of any unitary matrix are a basis. For example use the columns of iI: $(i, 0, \ldots, 0), \ldots, (0, \ldots, 0, i)$

23 $[\,1\,]$ and $[-1]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

24 The eigenvalues of A^{H} are *complex conjugates* of the eigenvalues of A: $\det(A - \lambda I) = 0$ gives $\det(A^{\mathrm{H}} - \overline{\lambda}I) = 0$.

25 $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^{\mathrm{H}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ and also $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}} + 4\boldsymbol{u}(\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u})\boldsymbol{u}^{\mathrm{H}} = I$. The rank-1 matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ projects onto the line through $\boldsymbol{u}$.

26 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A + iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

27 We are given $A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. So that $\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$ is symmetric.

28 $AA^{-1} = I$ gives $(A^{-1})^{\mathrm{H}}A^{\mathrm{H}} = I$. Therefore $(A^{-1})^{\mathrm{H}}$ is $(A^{\mathrm{H}})^{-1} = A^{-1}$ and A^{-1} is Hermitian.

29 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\dfrac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

30 If U has (complex) orthonormal columns, then $U^{\mathrm{H}}U = I$ and U is *unitary*. If those columns are eigenvectors of A, then $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ is *normal*. The direct test for a normal matrix (which is $AA^{\mathrm{H}} = A^{\mathrm{H}}A$ because diagonals could be real!) and Λ^{H} surely commute:

$$AA^{\mathrm{H}}=(U\Lambda U^{\mathrm{H}})(U\Lambda^{\mathrm{H}}U^{\mathrm{H}})=U(\Lambda\Lambda^{\mathrm{H}})U^{\mathrm{H}}=U(\Lambda^{\mathrm{H}}\Lambda)U^{\mathrm{H}}=(U\Lambda^{\mathrm{H}}U^{\mathrm{H}})(U\Lambda U^{\mathrm{H}})=A^{\mathrm{H}}A.$$

An easy way to construct a normal matrix is $1+i$ times a symmetric matrix. Or take $A = S + iT$ where the real symmetric S and T commute (Then $A^{\mathrm{H}} = S - iT$ and $AA^{\mathrm{H}} = A^{\mathrm{H}}A$).

Problem Set 10.3, page 514

1 Equation (3) (the FFT) is correct using $i^2 = -1$ in the last two rows and three columns.

2 $F^{-1} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \dfrac{1}{2} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \dfrac{1}{2} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} = \dfrac{1}{4}F^{\mathrm{H}}.$

3 $F = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix}$ permutation last.

4 $D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix}$ (note 6 not 3) and $F_3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$.

5 $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} = \boldsymbol{w}/4$. Delta vector $\leftrightarrow$ all-ones vector.

6 $(F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$ and $(F_4)^4 = 16I$. Four transforms recover the signal!

7 $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = F\boldsymbol{c}$. Also $C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = FC$.
Adding $c + C$ gives $(1,1,1,1)$ to $(4,0,0,0) = 4$ (delta vector).

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8\boldsymbol{c}$.
$\boldsymbol{C} \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8\boldsymbol{C}$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

10 For every integer n, the nth roots of 1 add to zero. For even n, they cancel in pairs. For any n, use the geometric series formula $1 + w + \cdots + w^{n-1} = (w^n - 1)/(w - 1) = 0$. In particular for $n = 3$, $1 + (-1 + i\sqrt{3})/2 + (-1 - i\sqrt{3})/2 = 0$.

11 The eigenvalues of P are $1, i, i^2 = -1$, and $i^3 = -i$. Problem 11 displays the eigenvectors. And also $\det(P - \lambda I) = \lambda^4 - 1$.

12 $\Lambda = \text{diag}(1, i, i^2, i^3)$; $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and P^{T} lead to $\lambda^3 - 1 = 0$.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2} n \log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

16 The row $1, \overline{w}^k, \overline{w}^{2k}, \ldots$ in $\overline{F}$ is the same as the row $1, w^{N-k}, w^{N-2k}, \ldots$ in F because $w^{N-k} = e^{(2\pi i/N)(N-k)}$ is $e^{2\pi i} e^{-(2\pi i/N)k} = 1$ times $\overline{w}^k$. So F and $\overline{F}$ have the **same rows in reversed order** (except for row 0 which is all ones).