

MATH For ML



Linear Algebra

Week 1.

1. Motivations.

① Solve simultaneous problems,

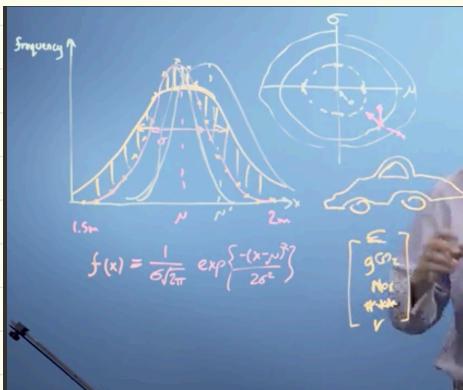
② the optimization problem of fitting some data with an equation with some fitting parameters,

exercises.

$$\begin{array}{l} 9x - 7y = -20 \\ 13x + 7y = -94 \end{array} \Rightarrow \begin{array}{l} x = 11 \\ y = 7 \end{array}$$

2. Vectors.

① get handle in vectors.



② operations

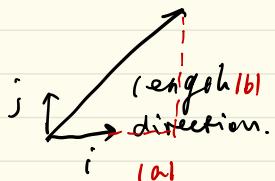
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \times 2 = \begin{bmatrix} 1 \times 2 \\ 2 \times 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

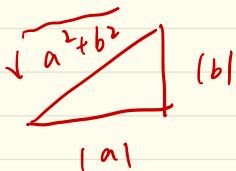
Week 2

- finding the size of vector, its size, and projection.

① Modulus



$$r = ai + bj = \begin{bmatrix} a \\ b \end{bmatrix} \quad |r| = \sqrt{a^2 + b^2}$$



② inner product.

$$S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad r = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad r \cdot S = r_i \cdot s_i + r_j \cdot s_j$$
$$= 3 \times (-1) + 2 \times 2 = 1$$

Property.

④ commutative.

$$r \cdot s = s \cdot r$$

⑤ distributive

$$r \cdot (s+t) = r \cdot s + r \cdot t$$

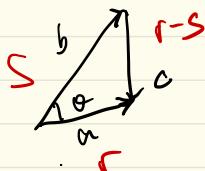
⑥ associative over scalar.

$$r \cdot (\alpha s) = \alpha \cdot (r \cdot s)$$

↓
scalar

⑦ $r \cdot r = |r|^2$ ~~内积与其 size²~~

⑧ cosine rule



$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

$$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta.$$

$$\begin{aligned} (r-s) \cdot (r-s) &= r \cdot r - s \cdot r - s \cdot r + s \cdot s \\ &= |r|^2 - 2s \cdot r + |s|^2 \end{aligned}$$

$$\Rightarrow -2s \cdot r = -2|r||s|\cos\theta$$

$$\Rightarrow r \cdot s = |r||s|\cos\theta.$$

⑥ projection.



$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\text{adj}}{|s|}$$

$$r \cdot s = |r| |s| \cos \theta.$$

$\underbrace{\text{adj}}$ = projection.

$$= |r| \times S_{\text{projection}}$$

a. Scalar projection: $\frac{r \cdot s}{|r|} = |s| \cos \theta$

b. Vector projection: $r \cdot \frac{r \cdot s}{|r| |r|}$

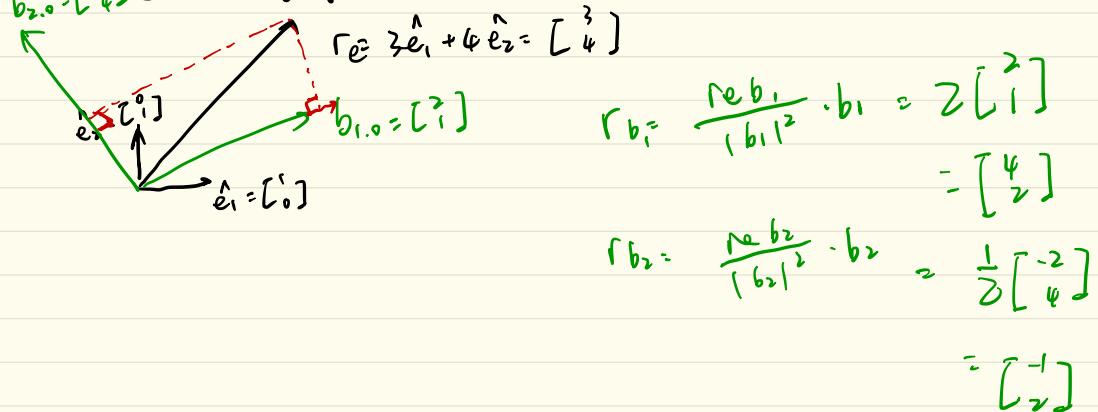
$r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $s = \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}$

$$Sp = \frac{r \cdot s}{|r|} = \frac{30 - 20}{\sqrt{9 + 16}} = \frac{10}{5} = 2$$

$r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $s = \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}$

$$Up = \frac{r \cdot s}{|r|} \cdot \frac{r}{|r|} = \frac{30 - 20}{5 \times 5} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}$$

(5) changing basis.



$$\# a. V = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \quad b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V_{b_1} = \frac{V \cdot b_1}{|b_1|} \cdot \frac{b_1}{|b_1|} = \frac{5+1-1}{(\sqrt{2})^2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 b_1$$

$$V_{b_2} = \frac{V \cdot b_2}{|b_2|} \cdot \frac{b_2}{|b_2|} = \frac{5+1+1}{(\sqrt{2})^2} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3 b_2$$

$$\therefore V_b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$b_1 \quad \frac{10 \times 3 - 20}{5} = 2 \quad \frac{10 \times 1 + 15}{5} = 11$$

$$c. \quad \frac{-6+2}{10} = -\frac{2}{5} \quad \frac{2+6}{10} = \frac{4}{5}$$

(6)

linear independent

Basis is a set of n vectors that:

- (i) are not linear combinations of each other (linearly independent)
- (ii) span the space
- The space is then n-dimensional

$$b_3 = a_1 b_1 + a_2 b_2, \quad a_1, a_2 \neq 0$$

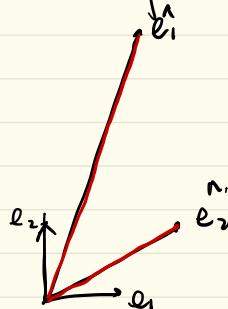
Week 3.

1. Intro.

$$\begin{matrix} 2a + 3b = 8 \\ 10a + 1b = 13 \end{matrix} \Rightarrow \underbrace{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}}_{\text{Matrices transform vector}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



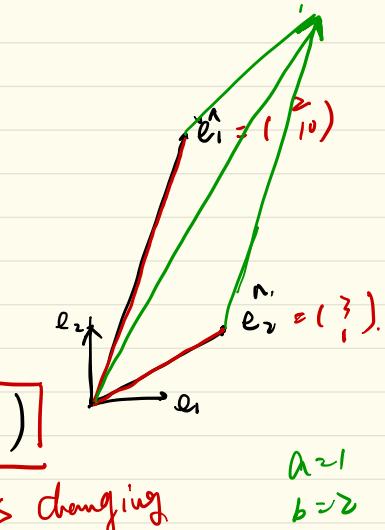
2. How Matrices transform space.

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} (a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

$$= a \boxed{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + b \boxed{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$= \underbrace{a \cdot \hat{e}_1}_{\text{basis changing}} + \underbrace{b \cdot \hat{e}_2}_{\text{basis changing}}$$



The matrix just tells us where the basis vectors go.
that's the transformation it does.

$$\# \quad \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -6 \\ 12 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5 \begin{pmatrix} 7 \\ 12 \end{pmatrix} + 6 \begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 \\ 108 \end{pmatrix}$$

3. Inverse.

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

A r S

$$A \cdot r = S.$$

$$A^{-1}A = E$$

若求得 A^{-1} 而 $r = A^{-1}S$. 解得 r .

求解矩阵的逆

$$A^{-1}A = E.$$

$$A^{-1}E = A^{-1}$$

A 逆 $\rightarrow A$ 没有逆 \rightarrow 通过 P 等效变换而求出

$$P_1 P_2 P_3 \cdots P_n A = E$$

$$P_1 P_2 P_3 \cdots P_n E = A^{-1}$$

$$\therefore (A : E) \rightarrow (E : A^{-1})$$

解法.

1. You go to the shops on Monday and buy 1 apple, 1 banana, and 1 carrot; the whole transaction totals €15. On Tuesday you buy 3 apples, 2 bananas, 1 carrot, all for €28. Then on Wednesday 2 apples, 1 banana, 2 carrots, for €23.

Construct a matrix and vector for this linear algebra system. That is, for

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \$\text{Mon} \\ \$\text{Tue} \\ \$\text{Wed} \end{bmatrix}$$

Where a, b, c , are the prices of apples, bananas, and carrots. And each s is the total for that day.

Fill in the components of A and s .

```
1 # Replace A and s with the correct values below:  
2 A = [[1, 1, 1],  
3      [3, 2, 1],  
4      [2, 1, 2]]  
5  
6 s = [15, 28, 23]  
7
```

运行

重置

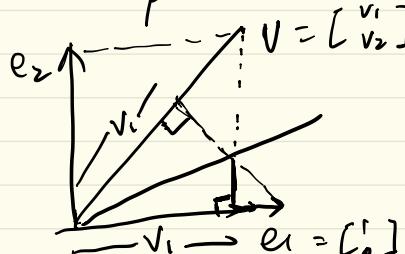
```
>> inv([1 1 1; 3 2 1; 2 1 2]) * [15; 28; 23]
```

ans =

3.0000
7.0000
5.0000

(Meek 4.

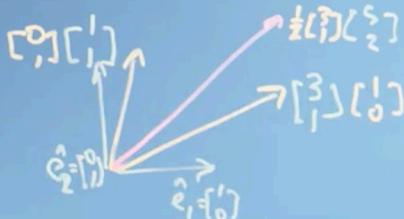
1. dot product.



2. 標

$$\begin{aligned} v_1 \cdot e_1 &= |v| \cdot |e_1| \cos \theta \\ &= |v| \cos \theta = V_1 \end{aligned}$$

2. Matrices Changing basis



Bear's basis vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in my frame.

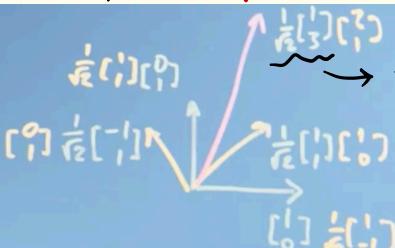
$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Bear's basis
in my coord.
vector

Bear's
vector

B

$$B = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ my basis in Bear's world}$$



B

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

B^{-1}

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Projections

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} 4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

我的坐标系在
熊的坐标系中
投影 = 原坐标
与熊坐标系的
积。

$$\left(\frac{1}{\sqrt{2}}[1], \frac{1}{\sqrt{2}}[-1] \right)$$

是单位正交基。

(加和) = 投影。

从另一个坐标系中观察向量变化.

从我的坐标系中观察
转回去的向量 → 从我的坐标系
转回去的向量 \rightarrow 将 (y) 转化为我自己的坐标系中
的向量 \rightarrow 向量 y .

$$\frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

B^{-1} R B
 $\xrightarrow{45^\circ}$ Bear's basis
 in my world.

vector, rotated, $\frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$
 in my basis

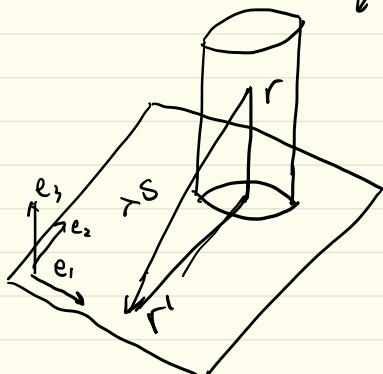
vector, rotation $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 5 & 3 \end{pmatrix}$
 in Bear's basis

$\boxed{B^{-1} R B}$

$$r' = \underline{A} r$$

$B^{-1} R B$

从 \vec{m} 看



$$r' = r + xS$$

$$r' \cdot e_3 = r \cdot e_3 + xS \cdot e_3$$

$$\Rightarrow r = \frac{-r \cdot e_3}{S \cdot e_3} \quad (S \cdot e_3 = S_3).$$

$$\Rightarrow r' = r - \underline{\frac{S \cdot (r \cdot e_3)}{S_3}}$$

4. Orthogonal matrices.

$$A^T A = I \quad |A| = 1$$

$$A^T = A^{-1}$$

$A_{ij}^T = A_{ji}$

$A^T A = I$

composed of vectors that are normal to each other and have unit length

向量组的正交化

设 $\alpha_1, \alpha_2, \dots, \alpha_m$ 为线性无关向量组，令

$$\beta_1 = \alpha_1,$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1,$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2,$$

.....

$$\beta_m = \alpha_m - \frac{(\alpha_m, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_m, \beta_2)}{(\beta_2, \beta_2)} \beta_2 - \dots - \frac{(\alpha_m, \beta_{m-1})}{(\beta_{m-1}, \beta_{m-1})} \beta_{m-1}.$$

(i) $\alpha_1, \alpha_2, \dots, \alpha_m$ 与 $\beta_1, \beta_2, \dots, \beta_m$ 等价; (ii) $\beta_1, \beta_2, \dots, \beta_m$ 为正交组。

再将 $\beta_1, \beta_2, \dots, \beta_m$ 为单位化，即得到单位正交向量组。

看出规律
来了吗？

先归化，再单位化

Gram-Schmidt

$$V = \{v_1, v_2, \dots, v_n\}$$

$$v_1$$

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = (v_2 \cdot e_1) \frac{e_1}{\|e_1\|} + u_2$$

$$u_2 = v_2 - (v_2 \cdot e_1) e_1 \quad \frac{u_2}{\|u_2\|} = e_2$$

$$v_3 = v_3 - (v_3 \cdot e_1) e_1 - (v_3 \cdot e_2) e_2 \quad \frac{u_3}{\|u_3\|} = e_3$$



6. Reflecting on a plane.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad E^{-1} \rightarrow \boxed{E} \rightarrow \boxed{T_E} \rightarrow \boxed{TE^{-1}v_3} \rightarrow r = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$E = [(\mathbf{e}_1) \ (\mathbf{e}_2) \ (\mathbf{e}_3)] = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right)$$

$$r' = \underbrace{E}_{\text{原点} \rightarrow \text{法线}} \underbrace{T_E}_{\text{法线} \rightarrow \text{法线}} \underbrace{\epsilon^T r}_{\text{法线} \rightarrow \text{原点}}$$

$$E^T = E^{-1}$$

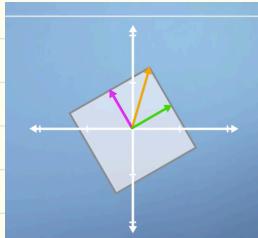
$$T_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Week 5

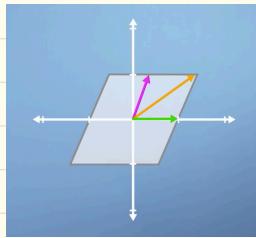
1. eigenvector and eigenvalue

$A\vec{x} = \lambda\vec{x}$

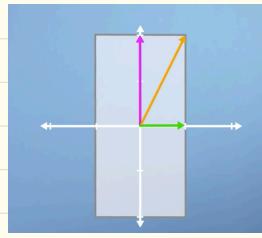
A로 변환된
2개의 베クト르의 선형 조합



1つ目



↑



2つ目

An eigenvector is a vector which applying the transformation stays in the same span.

예제 1: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$$

$$(A - \lambda I)x = 0 \quad = (1-\lambda)(2-\lambda) = 0$$

$$\text{@}\lambda=1: \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$$

$$\text{@}\lambda=2: \begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$$

$$\text{@}\lambda=1: x = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad \text{@}\lambda=2: x = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

$$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

$$T = CDC^{-1}$$

$$T^2 = CDC^{-1}CDC^{-1} = CDDC^{-1} = CD^2C^{-1}$$

$$T^n = CD^nC^{-1}$$

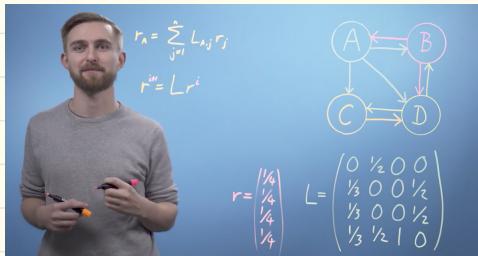
对角矩阵同作 C . 例如，先对 C 作相似变换得

$$\text{得 } C = PDP^{-1}$$

$$\text{R.1 } C^n = PDP^{-1}PDP^{-1}\cdots PDP^{-1}$$

$$= PD^nP^{-1}$$

Page Rank



	A	B	C	D
rank =	0.1250	0.2083	0.2083	0.4583
0.1318	0.2118	0.2118	0.4610	
0.1384	0.2151	0.2151	0.3594	
0.1084	0.2544	0.2544	0.3791	
0.1249	0.2311	0.2311	0.4129	
0.1227	0.2350	0.2350	0.4072	
0.1186	0.2425	0.2425	0.3963	
0.1313	0.2312	0.2412	0.3883	
0.1203	0.2395	0.2395	0.4007	
0.1201	0.2398	0.2398	0.4003	
0.1200	0.2401	0.2401	0.3999	
0.1200	0.2400	0.2400	0.4000	
0.1200	0.2400	0.2400	0.4000	
0.1200	0.2400	0.2400	0.4000	
0.1200	0.2400	0.2400	0.4000	
0.1200	0.2400	0.2400	0.4000	

$T = Lr$ 通过迭代，使得 r 与 T 不再改变。

所以 r 是 eigenvector, eigenvalue 是 1

