MATH for ML

Linear Algebra

Linear Algebra

Week 1.

1. Motivations.

① Solve simultaneous problems.

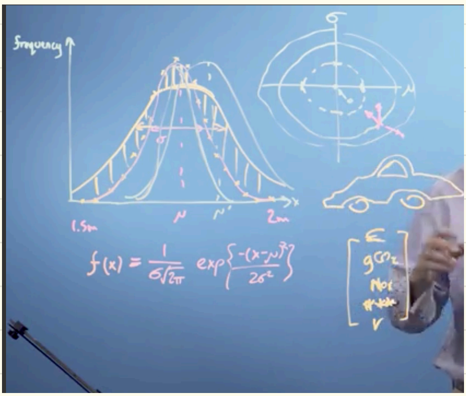
② the optimization problem of fitting some data with an equation with some fitting parameters.

exercises.

$$\begin{cases} 9x - 17y = -20 \\ -13x + 7y = -94 \end{cases} \Rightarrow \begin{cases} x = 11 \\ y = 7 \end{cases}$$

2. Vectors.

① get handle in vectors.



② operations

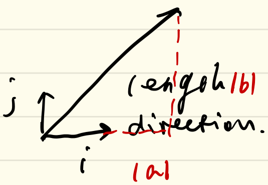
$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1+1\\2+2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix}\times 2=\begin{bmatrix}1\times 2\\2\times 2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

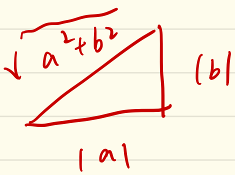
Week 2

1. Finding the size of vector. Its size. and projection.

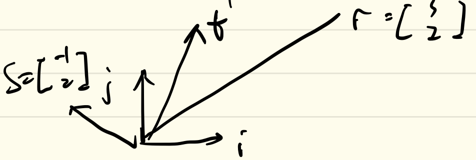
① Modulus



$r = ai + bj = \begin{bmatrix}a\\b\end{bmatrix}$ $\quad |r| = \sqrt{a^2+b^2}$



② inner product.



$r \cdot s = r_i \cdot s_i + r_j \cdot s_j$

$= 3\times(-1) + 2\times 2 = 1$

property.

① commutative. $r \cdot s = s \cdot r$

② distributive $r \cdot (s+t) = r \cdot s + r \cdot t$

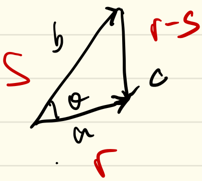
③ associative over scalar.

$$r \cdot (as) = a \cdot (r \cdot s)$$

a ↓ scalor

④ $r \cdot r = |r|^2$ 自身内积为其 size^2

⑤ cosine rule



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta.$$

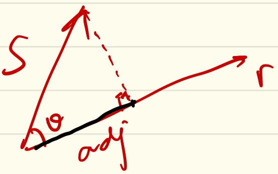
$$(r-s) \cdot (r-s) = r \cdot r - s \cdot r - s \cdot r + s \cdot s$$

$$= |r|^2 - 2 s \cdot r + |s|$$

$$\Rightarrow -2 s \cdot r = -2|r||s|\cos\theta$$

$$\Rightarrow r \cdot s = |r||s|\cos\theta.$$

④ projection.



$$\cos\theta = \frac{adj}{hyp} = \frac{adj}{|s|}$$

$$r \cdot s = |r|\underbrace{|s|\cos\theta}_{adj}. = \text{projection}.$$

$$= |r| \times S_{projection}$$

a. Scalar projection. $\frac{r \cdot s}{|r|} = |s|\cos\theta$

b. Vector projection. $r \cdot \frac{r \cdot s}{|r||r|}$

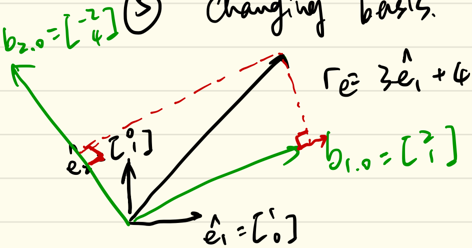
\# $r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $s = \begin{bmatrix} 10 \\ 5 \\ -6 \end{bmatrix}$

$$Sp = \frac{r \cdot s}{|r|} = \frac{30-20}{\sqrt{9+16}} = \frac{10}{5} = 2$$

\# $r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $s = \begin{bmatrix} 10 \\ 5 \\ -6 \end{bmatrix}$

$$Vp = \frac{r \cdot s}{|r|} \cdot \frac{r}{|r|} = \frac{30-20}{5 \times 5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{8}{5} \\ 0 \end{bmatrix}$$

(5) changing basis.



$r_e = 3\hat{e}_1 + 4\hat{e}_2 = \begin{bmatrix}3\\4\end{bmatrix}$

$$r_{b_1} = \frac{r_e b_1}{|b_1|^2}\cdot b_1 = 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

$$r_{b_2} = \frac{r_e b_2}{|b_2|^2}\cdot b_2 = \frac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

#a. $V=\begin{bmatrix}5\\-1\end{bmatrix}$ $b_1=\begin{bmatrix}1\\1\end{bmatrix}$ $b_2=\begin{bmatrix}1\\-1\end{bmatrix}$

$$V_{b_1} = \frac{V\cdot b_1}{|b_1|}\cdot\frac{b_1}{|b_1|} = \frac{5\times1-1}{(\sqrt{2})^2}\cdot\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2b_1$$

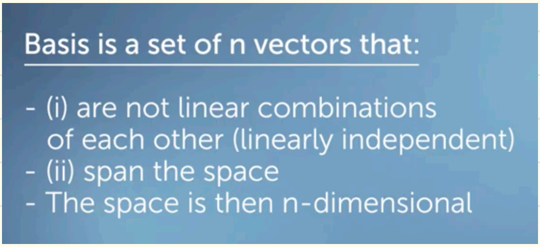
$$V_{b_2} = \frac{V\cdot b_2}{|b_2|}\cdot\frac{b_2}{|b_2|} = \frac{5\times1+1}{(\sqrt{2})^2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix} = 3b_2$$

$$\therefore V_b = \begin{bmatrix}2\\3\end{bmatrix}$$

b. $\frac{10\times3-20}{5} = 2$ $\frac{10\times4+15}{5} = 11$

c. $\frac{-6+2}{10} = -\frac{2}{5}$ $\frac{2+6}{10} = \frac{4}{5}$

⑥ linear independent



$b_3 = a_1 b_1 + a_2 b_2$ ， a、b 不存在

Week 3.

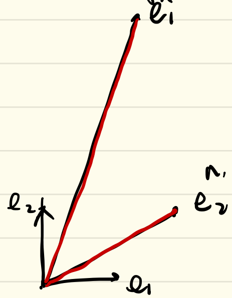
1. Intro.

$$\begin{aligned} 2a + 3b &= 8 \\ 10a + 1b &= 13 \end{aligned} \Rightarrow \underline{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

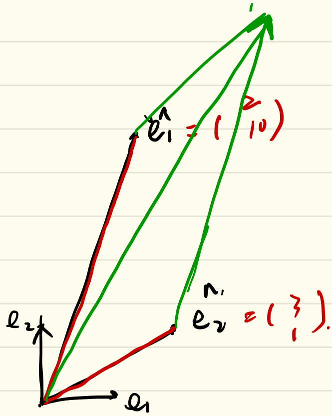
Matrices transform vector

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



2. How Matrices transform space.



$$\begin{pmatrix}2&3\\10&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}8\\12\end{pmatrix}$$

$$=\begin{pmatrix}2&3\\10&1\end{pmatrix}\left(a\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$=a\boxed{\begin{pmatrix}2&3\\10&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}+b\boxed{\begin{pmatrix}2&3\\10&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}$$

basis changing

$$=\underline{a}\cdot\hat{e}_1+\underline{b}\,\hat{e}_2$$

$a=1$
$b=2$

The matrix just tells us where the basis vectors go.
that's the transformation it does.

$\begin{pmatrix}5\\6\end{pmatrix}=5\begin{pmatrix}1\\0\end{pmatrix}+6\begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}7&-6\\12&8\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}=5\begin{pmatrix}7\\12\end{pmatrix}+6\begin{pmatrix}-6\\8\end{pmatrix}=\begin{pmatrix}-1\\108\end{pmatrix}$$

3. Inverse.

$$\begin{pmatrix}2&3\\10&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$$

$$A \qquad r \qquad s$$

$A \cdot r = S.$

$A^{-1}A = E$

若求得 A^{-1} 则 $r = A^{-1}S$. 解得 r.

求简单矩阵的逆

$A^{-}A = E.$

$A^{-}E = \cancel{A}\ A^{-}$

A 可逆 → A 满秩 → 一系列初等变换的乘积

$P_1P_2P_3 \cdots P_nA = E$

$P_1P_2P_3 \cdots P_nE = A^{-}$

$\therefore (A \vdots E) \longrightarrow (E \vdots A^{-})$

解法.

1。 You go to the shops on Monday and buy 1 apple, 1 banana, and 1 carrot; the whole transaction totals €15. On Tuesday you buy 3 apples, 2 bananas, 1 carrot, all for €28. Then on Wednesday 2 apples, 1 banana, 2 carrots, for €23.

Construct a matrix and vector for this linear algebra system. That is, for

$$A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} s_{\text{Mon}} \\ s_{\text{Tue}} \\ s_{\text{Wed}} \end{bmatrix}$$

Where a, b, c, are the prices of apples, bananas, and carrots. And each s is the total for that day.

Fill in the components of A and $\mathbf{s}$.

```
# Replace A and s with the correct values below:
A = [[1, 1, 1],
     [3, 2, 1],
     [2, 1, 2]]

s = [3, 7, 5]

```

运行

重置

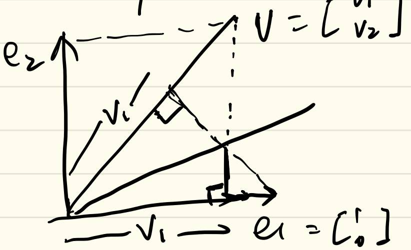
```
>> inv([1 1 1;3 2 1;2 1 2]) * [15; 28; 23]

ans =

    3.0000
    7.0000
    5.0000
```

Week4.

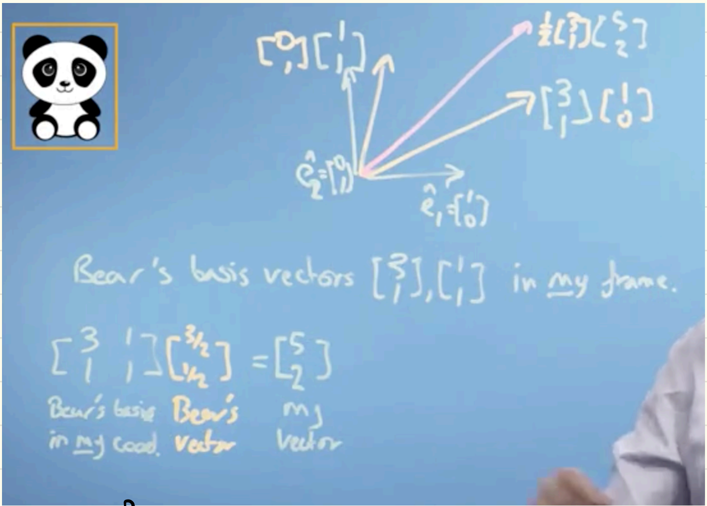
1. dot product.



对称

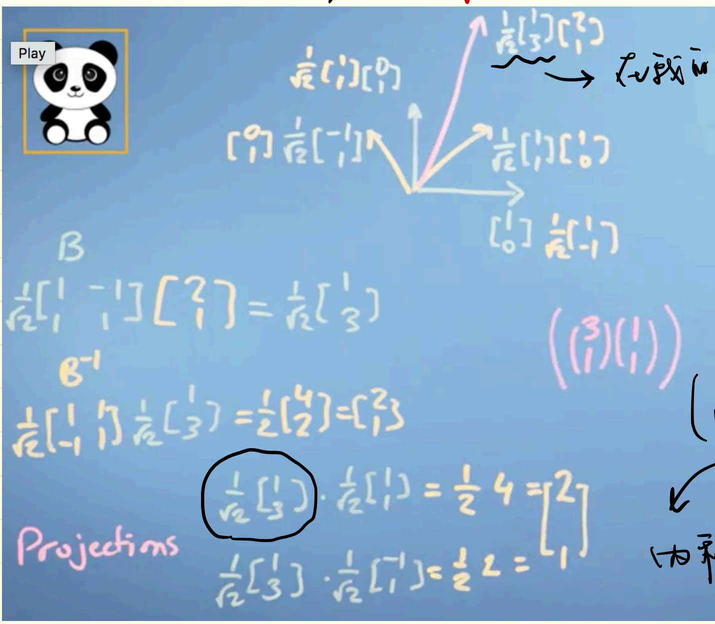
$$V \cdot e_1 = |V| \cdot |e_1| \cos\theta$$
$$= |V| \cos\theta = V_1$$

2. Matrices Changing basis



B

$B = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}$ My basis in Bear's world



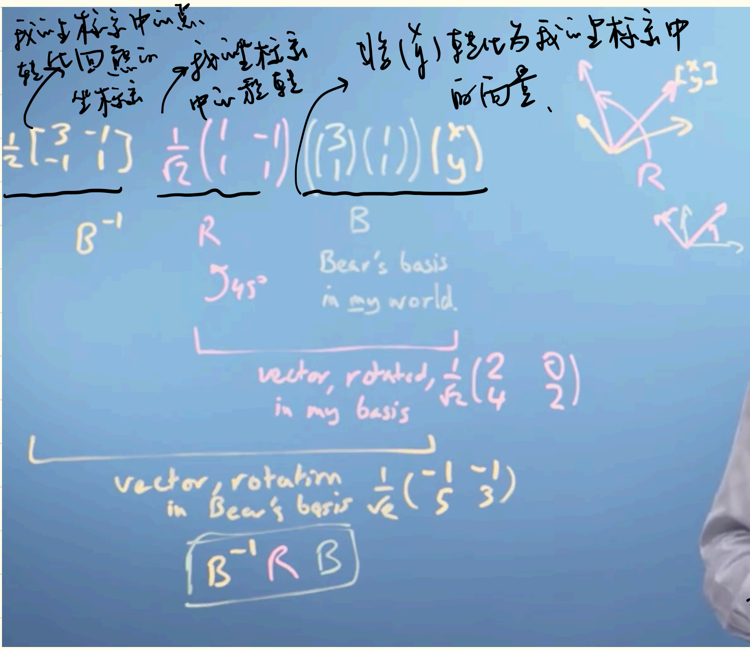
在我的坐标中

我的坐标在熊的坐标中的投影 = 我的坐标与熊坐标的内积

$\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)$ 是单位正交基.

内积 = 投影.

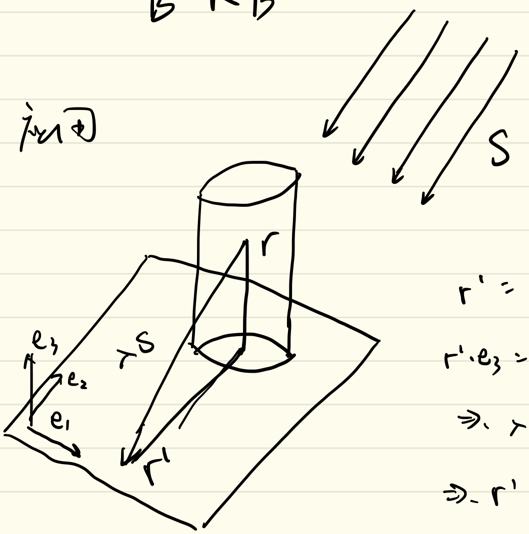
不同坐标系中旋转的变化.



$\Rightarrow \quad r' = \underline{A}\, r$

$B^{-1} R B$

2、应用



$r' = r + \lambda S$

$r' \cdot e_3 = r \cdot e_3 + \lambda S \cdot e_3$

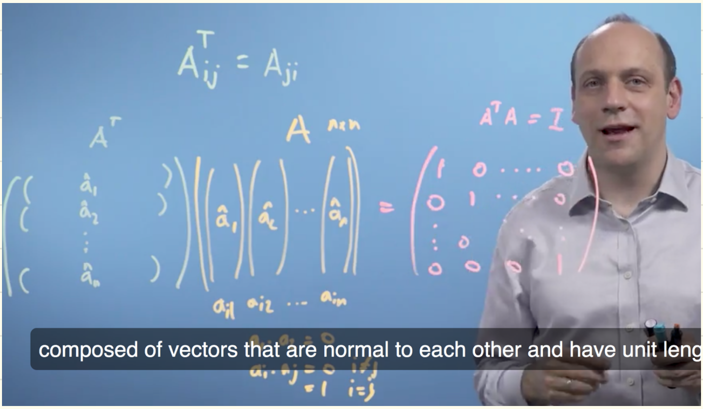
$\Rightarrow \lambda = \dfrac{-r \cdot e_3}{S \cdot e_3} \quad (S \cdot e_3 = S_3).$

$\Rightarrow r' = r - \dfrac{S (r \cdot e_3)}{S_3}$

4. Orthogonal matrices.

$A^TA = I \qquad |A| = 1$

$A^T = A^{-1}$



5. 向量组的正交规范化

设α_1，α_2，⋯，α_m为线性无关向量组，令

$$\beta_1 = \alpha_1,$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\ \beta_1)}{(\beta_1,\ \beta_1)}\beta_1,$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3,\ \beta_1)}{(\beta_1,\ \beta_1)}\beta_1 - \frac{(\alpha_3,\ \beta_2)}{(\beta_2,\ \beta_2)}\beta_2,$$

⋯⋯⋯⋯

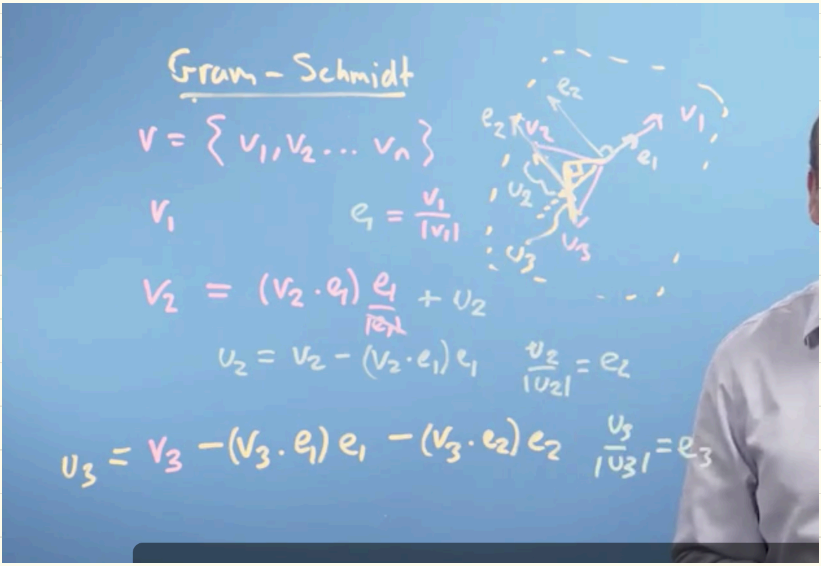
$$\beta_m = \alpha_m - \frac{(\alpha_m,\ \beta_1)}{(\beta_1,\ \beta_1)}\beta_1 - \frac{(\alpha_m,\ \beta_2)}{(\beta_2,\ \beta_2)}\beta_2 - \cdots - \frac{(\alpha_m,\ \beta_{m-1})}{(\beta_{m-1},\ \beta_{m-1})}\beta_{m-1}.$$

看出规律来了吗？

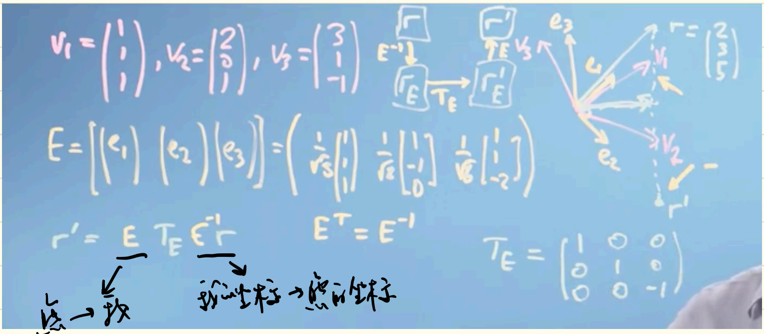
(i) $\alpha_1,\alpha_2,\cdots,\alpha_m$ 与 $\beta_1,\beta_2,\cdots,\beta_m$ 等价；(ii) $\beta_1,\beta_2,\cdots,\beta_m$ 为正交组。

再将β_1，β_2，⋯，β_m为单位化，即得到单位正交向量组。

先正交化，再单位化



6. Reflecting a plane.



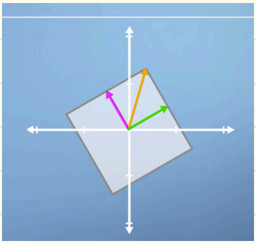
Week 5

1. eigenvector and eigenvalue

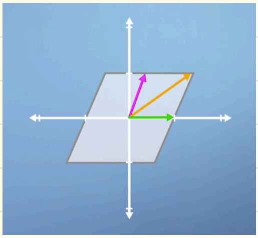
$A\alpha = \lambda\alpha$

A的特征向量

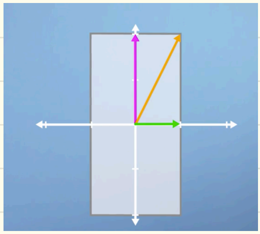
对应α的特征值



没有



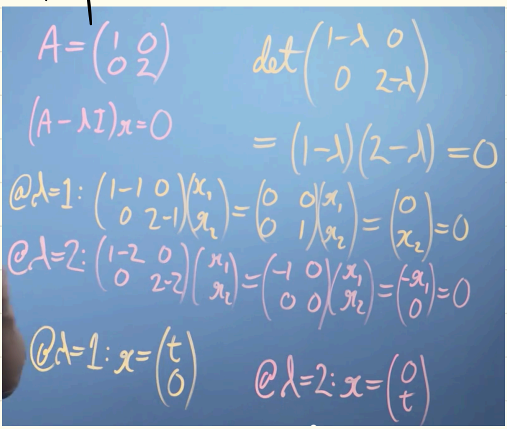
1个

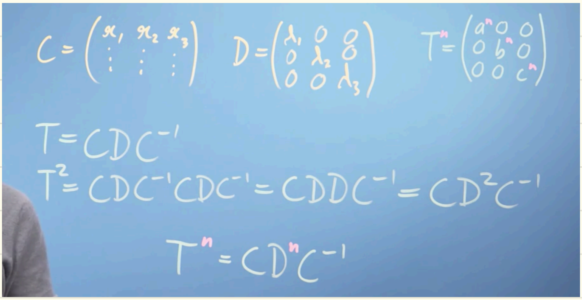


2个.

an eigenvector is a vector which applying the transformation stays in the same span.

特征值的计算





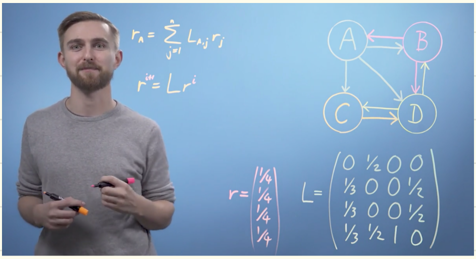
对任意方阵作变C. n次、先对C作相似对角化

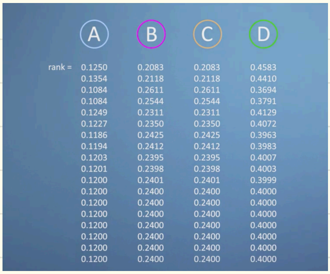
便得 $C = PDP^{-1}$

则 $C^n = P D \underline{P^{-1} P} D \underline{P^{-1}} \cdots P D P^{-1}$

$= P D^n P^{-1}$

Page Rank





	A	B	C	D
rank =	0.1250	0.2083	0.2083	0.4583
	0.1354	0.2118	0.2118	0.4410
	0.1084	0.2611	0.2611	0.3694
	0.1084	0.2544	0.2544	0.3791
	0.1249	0.2311	0.2311	0.4129
	0.1227	0.2350	0.2350	0.4072
	0.1186	0.2425	0.2425	0.3963
	0.1194	0.2412	0.2412	0.3983
	0.1203	0.2395	0.2395	0.4007
	0.1201	0.2398	0.2398	0.4003
	0.1200	0.2401	0.2401	0.3999
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000

$r = Lr$ 通过迭代、使得 r 的值不再改变、

此时、r 是 eigenvector、 eigenvalue 是 1

Multivariate Calculus.

Week2. Moving to multivariate

$$f(x,y,z) = \sin(x) e^{yz^2}$$

$$\frac{\partial f}{\partial x} = \cos(x) e^{yz^2}$$

$$\frac{\partial f}{\partial y} = \sin(x) e^{yz^2} \cdot z^2$$

$$\frac{\partial f}{\partial z} = \sin(x) \cdot e^{yz^2} \cdot 2yz.$$

total derivative

$$f(x,y,z) = \sin(x) e^{yz^2}$$

$$\begin{cases} x = t-1 \\ y = t^2 \\ z = \frac{1}{t} \end{cases} \Rightarrow \quad f(t) = \sin(t-1) e^{t^2 (\frac{1}{t})^2}$$

$$\boxed{f'(t) = \cos(t-1) e.}$$

$$\frac{d f(x,y,z)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

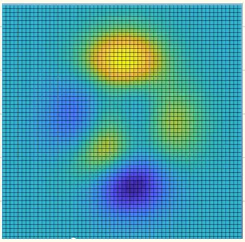
$$= \boxed{\cos(t-1) e}$$

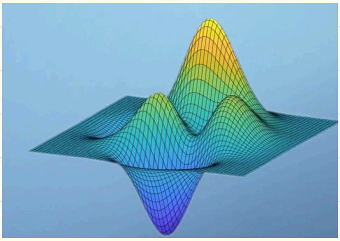
Jacobian.

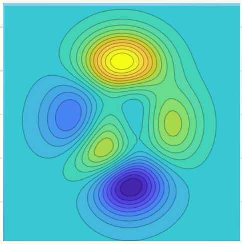
$f(x,y,z) = x^2y + 3z.$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2yx \\ \frac{\partial f}{\partial y} &= x^2 \\ \frac{\partial f}{\partial z} &= 3 \end{aligned}\right\} J = [2xy.\ x^2.\ 3]$$

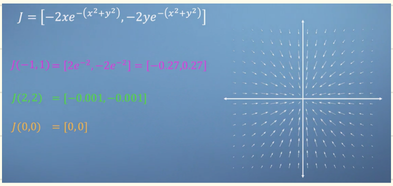
$J(0,0,0) = [0.\ 0.\ 3]$

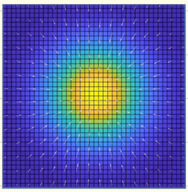


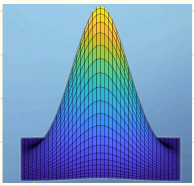




根据J，到原原方程

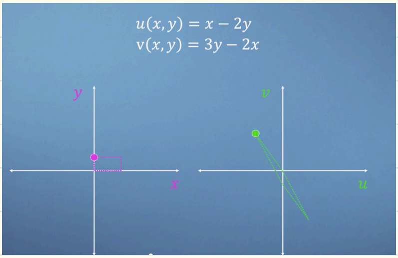






the maxmum and minmum value is $J = 0$

J Vector → J Matrices.



$$J_u = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \end{bmatrix}$$
$$J_v = \begin{bmatrix} \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$
$$J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

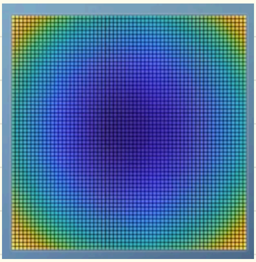
The Hessian.

$$\mathbf{H} = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 x_2 & \cdots & \partial^2 f/\partial x_1 x_n \\ \partial^2 f/\partial x_2 x_1 & \partial^2 f/\partial x_2^2 & \cdots & \partial^2 f/\partial x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 f/\partial x_n x_1 & \partial^2 f/\partial x_n x_2 & \cdots & \partial^2 f/\partial x_n^2 \end{bmatrix}$$

example. $f(x,y) = x^2 + y^2$

$J = [2x, 2y]$

$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



if $J=0$, How do we know it is maximum or minimum?

calculate Hessian,

① $|H| > 0$. We are dealing with either maximum or minimum.

② $H_{11} > 0$. Negtive. $H_{11} < 0$. Positive.

Mutivarible chain rule

$$f(\mathbf{x}) = f(x_1, x_2, x_3, \dots, x_n)$$

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix} \qquad \frac{d\mathbf{x}}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \\ \vdots \\ dx_n/dt \end{bmatrix} \qquad \frac{df}{dt} = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix} \cdot \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

$$f(\mathbf{x}(\mathbf{u}(t)))$$

$$f(\mathbf{x}) = f(x_1, x_2)$$

$$\mathbf{x}(\mathbf{u}) = \begin{bmatrix} x_1(u_1, u_2) \\ x_2(u_1, u_2) \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{d\mathbf{u}}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix}$$

eg 1. $f(x) = f(x_1, x_2) = x_1^2 x_2^2 + x_1 x_2$

$x_1(t) = 1 - t^2$

$x_2(t) = 1 + t^2$.

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = [2x_2^2 x_1 + x_2,\ 2x_1^2 x_2 + x_1] \begin{bmatrix} -2t \\ 2t \end{bmatrix}$$

2. $f(x) = f(x_1, x_2) = x_1^2 - x_2^2$

$x_1 = (u_1, u_2) = 2u_1 + 3u_2$

$x_2 = (u_1, u_2) = 2u_1 - 3u_2$

$u_1(t) = \cos(t/2)$

$u_2(t) = \sin(2t)$

$$\therefore \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial t} = [2x_1 \ \ -2x_2] \underbrace{\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}}_{J} \begin{bmatrix} -\frac{1}{2}\sin(\frac{t}{2}) \\ 2\cos(2t) \end{bmatrix}$$

Taylor Series.

if I know every thing at one point.

I also know every thing at everywhere.

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

$$g_0(x) = f(0)$$

$$g_1(x) = f(0) + f'(0)x$$

$$g_2(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2$$

$$g_3(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f^{(3)}(0)x^3$$

$$g_4(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f^{(3)}(0)x^3 + \frac{1}{24} f^{(4)}(0)x^4$$

$$g_4(x) = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$g_0(x) = f(p)$$

$$g_1(x) = f(p) + f'(p)(x-p)$$

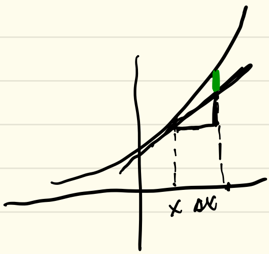
$$g_2(x) = f(p) + f'(p)(x-p) + \frac{1}{2} f''(p)(x-p)^2$$

Maclaurin Series

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Taylor Series

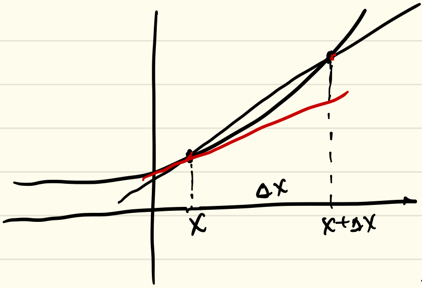
$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n$$



linearisation.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 + \frac{1}{6} f'''(x)\Delta x^3 \ldots\ldots$$

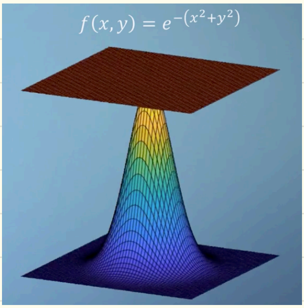
$$\therefore f(x+\Delta x) = f(x) + f'(x)\Delta x + O(\Delta x^2)$$

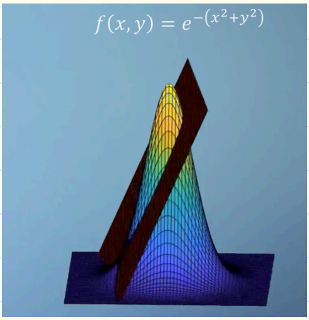


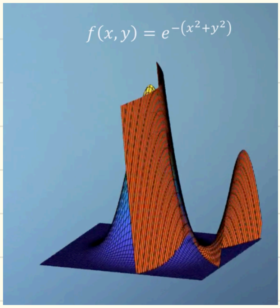
$$f'(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x} - f''(x)\Delta x - f'''(x)\Delta x^2 - \cdots$$

$$= \frac{f(x+\Delta x)-f(x)}{\Delta x} - O(\Delta x)$$

多变量的近似情况、







Mutivariable taylor series.

$$f(x+\Delta x, y+\Delta y) =$$

$$f(x,y)$$

$$+\left(\partial_x f(x,y)\,\Delta x + \partial_y f(x,y)\,\Delta y\right)$$

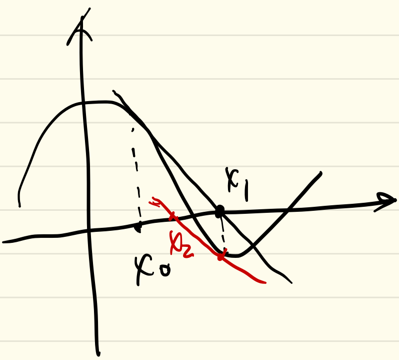
$$\Rightarrow \quad \left[\partial_x f(x,y), \partial_y f(x,y)\right]\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} \qquad \Rightarrow J_f \boldsymbol{\Delta x}$$

$$+\frac{1}{2}\left(\partial_{xx} f(x,y)\,\Delta x^2 + 2\partial_{xy} f(x,y)\Delta x\Delta y + \partial_{yy} f(x,y)\,\Delta y^2\right)$$

$$\Rightarrow \quad \frac{1}{2}[\Delta x, \Delta y]\begin{bmatrix}\partial_{xx} f(x,y) & \partial_{xy} f(x,y)\\ \partial_{yx} f(x,y) & \partial_{yy} f(x,y)\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} \quad \Rightarrow \frac{1}{2}\boldsymbol{\Delta x}^t \mathrm{H}_f \boldsymbol{\Delta x}$$

$$+ \ldots$$

Newton. method



$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0).$$

$$\Rightarrow \frac{f(x_0)}{f'(x_0)} = x_0 - x_1$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Grad.

$f(x,y) = x^2 y$

$$\frac{df}{dx} = 2xy \quad 求 \quad \underset{\text{Grad}}{Df} = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix}$$

$$\frac{df}{dy} = x^2$$

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ 方向的斜率

方向导数 $df = \frac{df}{dx}\cos\theta + \frac{df}{dy}\sin\theta$

$$= |A||1|\cos\alpha$$

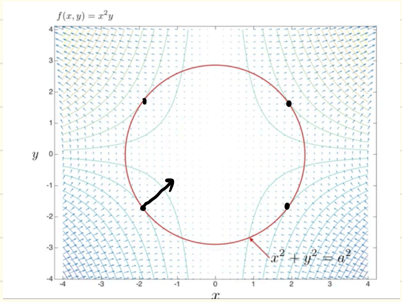
当 A 与 1 平行时，方向导数最大

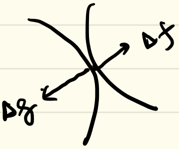
把 A 命名为梯度（函数变化最大的方向）

$$|A| = \sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}$$

Maximise $f(x,y) = x^2 y$

constraint $g(x,y) = x^2 + y^2 = a^2$





Solve $\Delta f = \lambda \Delta g$.

$$\Delta f = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \Delta g = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\Rightarrow$ ① $2xy = \lambda 2x \Rightarrow y = \lambda$

② $x^2 = \lambda 2y \Rightarrow x = \pm\sqrt{2}\, y$.

③ $x^2 + y^2 = a^2 \Rightarrow y = \pm \dfrac{a}{\sqrt{3}}$.

Solution. $\dfrac{a}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, $\dfrac{a}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$

$\dfrac{a}{\sqrt{3}}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ $\dfrac{a}{\sqrt{3}}\begin{bmatrix} -\sqrt{2} \\ -1 \end{bmatrix}$

$$f(x,y) = \frac{a^3}{3\sqrt{3}}\,2 \,,\ \frac{-2a^3}{3\sqrt{3}} \,,\ \frac{+2a^3}{3\sqrt{3}} .\ \frac{-2a^3}{3\sqrt{3}}$$