Formulae for Bonds Options

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Introduction

There are two or three or four versions of option models for bonds:

1. Bond Prices are log-normal
    - Bonds prices can go down to zero (so yields up to infinity)
    - Bond prices can go above sum(CF): 4yr 6.5% bond, sum of CF=$126. P=$130 ⇒ yld = -0.84%
    - Effectively, bond yields normal
2. Bond yields log-normal
    - 1.0%→1.1% same as 10%→11% same as 100%→110%
    - Yields cannot go negative
    - Maybe good, maybe bad
3. Bond yield normal
    - Commonly used now
4. Bond yield square-root process
    - One of my favorite, because mid-way between log-normal & normal

Option valuation with Equivalent Martingale Measures

In complete and frictionless markets we can derive simple arbitrage pricing relationships. We do this by choosing a suitable numeraire and using the equivalent martingale measure adapted to this numeraire. The resulting arbitrage pricing relationship is:

$$\frac{V(t)}{\mathcal{N}(t)} = E^Q\left[\frac{V(T)}{\mathcal{N}(T)}\right]$$

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(see `https://mfe.baruch.cuny.edu/wp-content/uploads/2019/12/IRC_Lecture4_2019.pdf` and `https://quant.stackexchange.com/questions/38530/change-of-numeraire-to-price-european-swaptions`)

This will hold for any and all securities traded. Most importantly for our purposes, it will apply for $V(t) = Bond$ and $V(t) = Option$, and other instruments that we may wish to value. For the case of a bond and option on the bond we know today's market price of the bond ($B(t)$) and for the bond we will use the pricing relationship $\frac{B(t)}{\mathcal{N}(t)} = E^Q\left[\frac{B(T)}{\mathcal{N}(T)}\right]$ to back out or calibrate the risk adjustment embedded in the equivalent martingale measure Q. This is similar conceptually to the way we back out a risk premium from the market price and yield of a risky bond (and also the corresponding UST) to uncover the market risky yield adjustment for risk-adjusted discounting. Here we are using equivalent martingale risk-adjusted valuation and using the market price of the bond to back out the risk adjustment embedded in the distribution. This valuation method is usually, but misleading, termed risk-neutral valuation. The term is misleading because the valuation is not risk-neutral in any way. It embeds the risk adjustment in the measure Q in a way that we can then take a simple expectation $E^Q\left[\frac{B(T)}{\mathcal{N}(T)}\right]$. This in no way assumes risk-neutrality, but rather builds the risk adjustment into the measure Q.

For each of the models (1)-(4), we will choose the appropriate underlying asset $\tilde{Y}(t)$, option payout $V(\tilde{Y}, T)$, numeraire $\mathcal{N}(t)$, and adapted equivalent martingale measure Q with expectation $E^Q[\cdot]$to ensure that we have a simple pricing relationship. This then means that our option model will be a simple expectation, and thus some form of Black-Scholes pricing formula.

Sample Option Pricing

Valuation Date 19-feb-2016

Expiry Date 1-sep-2018 (2.53 years)

Bond Maturity 1-sep-2045 (27 years from expiry)

Bond Coupon 4.7%, semi-annual frequency

Forward Bond Price 95.98, implies yield 4.972%

Volatility LNY 18.55%, LNP 13.486%, NormY 0.895%

Short Rate 4.97%sab $\Rightarrow$ $df = \frac{1}{(1+.0497/2)^{2\cdot 2.53}} = 0.88320$

1 Log-Normal Price

This is a forward-pricing model where the underlier $\tilde{Y}(t)$ is a forward bond price $\tilde{B}(t, T, m) = B_T(t)$ – the price at time t (say today) of a forward bond starting at T with m years maturity (from T). The call option payout at T is $C(T) = C(B_T(T)) = \max[B_T(T) - X, 0]$. The numeraire is the zero bond from t to T: *Z(t,T)*. Our arbitrage pricing relationship for the bond is:

$$\frac{B_T(t)}{Z(t,T)} = E^{QBZ}\left[\frac{B_T(T)}{Z(T,T)}\right] = E^Q\left[B_T(T)\right]$$

using the fact that $Z(T,T)=1$. I write the equivalent martingale measure as QBZ to emphasize that this is the measure over the bond price $\tilde{B}$, adapted to the zero-bond numeraire Z. We will choose our equivalent martingale measure to be log-normal with some variance σ^2 and mean μ_B. The variance we need to assume (unless we can back it out from some other options) but the mean we back out from the market price of the bond. The expectation of the bond distribution (over the whole distribution) will be

$$\frac{B_T(t)}{e^{-rT}} = \int_{B=0}^{B=\infty} \tilde{B}_T \cdot \varphi\left(\tilde{B}_T; \sigma, \mu_B\right) d\tilde{B}_T = \mu_B$$

This says that the mean of the distribution must be today's PV of the forward bond $B_T(t)$ inflated to forward value by e^{-rT} the PV of the zero bond – in other words the forward value of the forward bond.

We have now tied down the equivalent martingale measure Q (we have backed out the market's risk adjustment) and can apply the arbitrage pricing relationship to the call option:

$$\frac{C(t)}{Z(t,T)} = E^Q\left[\frac{C(T)}{Z(T,T)}\right] = E^Q\left[\max\left[B_T(T) - X, 0\right]\right]$$

again using $Z(T,T)=1$. This just gives us a Black-Scholes type formula for the call option:

$$Call = e^{-rT} \cdot E\left[(B_T - X)\,|B_T > X\right] = e^{-rT} \cdot \int_{B=X}^{B=\infty} (B_T - X)\,\varphi\left(B_T; \sigma, \mu = B_T(t)\right) dB_T$$

where $\varphi\left(B_T; \sigma, \mu = B_T(t)\right)$ is the log-normal density with *price* volatility σ and mean $\mu = B_T(t)$, today's forward bond price (in forward price dollars). This gives a Black-Scholes formula as in the picture below.

$$Call(0) = \left[N(d_1) \cdot B_T(0) - N(d_2) \cdot X\right] \cdot exp(-rT)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{B_T(0)}{X}\right) + \frac{\sigma^2 T}{2}\right]$$

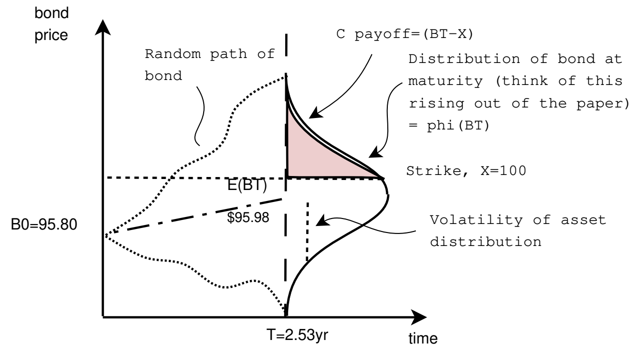
$$d_2 = d_1 - \sigma\sqrt{T}$$

The approximate relation between price and yield volatility is

$$\sigma_{price} = \frac{dp}{p} \approx \frac{y}{p} \cdot \frac{dy}{y} \cdot \frac{dp}{dy} = \frac{y}{p} \cdot \sigma_{yield} \cdot BPV = y \cdot ModDur \cdot \sigma_{yield}$$

The more accurate relationship is

$$\sigma_{price} = \sigma_{yield} \cdot \left[\frac{y_{fwd}}{P_{fwd}} \cdot BPV_{fwd\,yld} + \frac{c}{X} \cdot BPV_{strike\,yld}\right]/2$$



1.1 Pricing

Fwd Undlerlier $B_T(0) = 95.98$ (price of 27-year bond with 4.7% coupon at 4.972% yield)

Strike $X = 100$

σ Assume $\sigma_{yield} = 18.5\%$ which gives $\sigma_{price} = 0.185 \cdot \left[\frac{4.972}{95.98} \cdot 14.347 + \frac{4.70}{100} \cdot 15.207\right]/2 = 13.486\%$

d_1, $N(d_1)$ -0.0839, 0.4666

d_2, $N(d_2)$ -0.2985, 0.3827

Option $5.753, call on bond price

2 Log-Normal Yield

Here we are going to use the forward yield or swap coupon $\tilde{y}_T$ as the underlying stochastic variable. For a bond that pays a coupon c the PV of the bond is

$$PV(coup = c) = c \cdot PVAnn + 100 * DF(mat) = c \cdot A(\tilde{y}_T) + 100 \cdot df(\tilde{y}_T)$$

$PVAnn$ or $A(\tilde{y}_T)$ is the present value of a $1 annuity, paid at whatever frequency the bond coupons are paid. It is a function of the random forward yield $\tilde{y}_T$. The value of the bond is the (annual) coupon times this annuity, plus the PV of $100 at maturity (the discount factor at maturity). For a par bond,

$$100 = \tilde{y}_T \cdot PV(Ann) + 100 * DF(mat) = \tilde{y}_T \cdot A(\tilde{y}_T) + 100 \cdot df(\tilde{y}_T)$$

where $\tilde{y}T$ is the par bond yield-to-maturity (since for a par bond the coupon equals the yield-to-maturity).

Now we write down the call payout at T as

$$V(T) = C(B_T(T)) = \max\left[B_T(T) - 100, 0\right]$$

$$= \max\left[c \cdot A(\tilde{y}_T) + 100 \cdot df(\tilde{y}_T) - \tilde{y}(T) \cdot A(\tilde{y}_T) - 100 \cdot df(\tilde{y}_T), 0\right] = \max\left[(c - \tilde{y}(T)), 0\right] \cdot A(\tilde{y}_T)$$

The arbitrage relationship will hold for any numeraire:

$$\frac{C(t)}{\mathcal{N}(t)} = E^{Qy\mathcal{N}}\left[\frac{C(T)}{\mathcal{N}(t)}\right] = E^{Qy\mathcal{N}}\left[\frac{\max\left[\left(c-\tilde{y}(T)\right),0\right]\cdot A(\tilde{y}_T)}{\mathcal{N}(T)}\right]$$

where I write the measure as $Qy\mathcal{N}$ to emphasize that this is a measure over the random yield $\tilde{y}$ with some numeraire.

We have two difficulties with using this formula. First, numeraire. If we used the zero bond as our numeraire as above, we would have

$$\frac{C(t)}{Z(t,T)} = E^{QyZ}\left[\frac{C(T)}{Z(t,T)}\right] = E^{QyZ}\left[\max\left[\left(c-\tilde{y}(T)\right),0\right]\cdot A(\tilde{y}_T)\right]$$

The problem is that $A(\tilde{y}_T)$ is a convex function of $\tilde{y}$ and so the evaluation of the expectation (integral) is not trivial. This is not impossible, however, as there are decent numerical algorithms with which we can evaluate the expression, and at one time I devoted some effort to coding this up. But it is totally unnecessary. We have the freedom to choose a numeraire different from a zero bond Z. Why don't we choose the PV of the forward annuity, $A(t,T)$? In that case, the pricing expression is

$$\frac{C(t)}{A(t,T)} = E^{QyA}\left[\frac{\max\left[\left(c-\tilde{y}(T)\right),0\right]\cdot A(\tilde{y}_T)}{A(\tilde{y}_T)}\right] = E^{QyA}\left[\max\left[\left(c-\tilde{y}(T)\right),0\right]\right]$$

The annuity expression cancels and we are left with a very simple expression for the option.

The second difficulty is to back out the market risk-adjustment embedded in the martingale measure QyA. We cannot use the market price of the bond as above (in this case the forward par bond) because this would give

$$\frac{B(t)}{A(t,T)} = E^{QyA}\left[\frac{\tilde{y}_T\cdot A(\tilde{y}_T)+100\cdot df(\tilde{y}_T)}{A(\tilde{y}_T)}\right] = E^{QyA}\left[\tilde{y}_T+\frac{100\cdot df(\tilde{y}_T)}{A(\tilde{y}_T)}\right]$$

and involve the expectation of $\frac{100\cdot df(\tilde{y}_T)}{A(\tilde{y}_T)}$. Instead, we can use the market value of the forward annuity itself:

$$\frac{y_{fwd}\cdot A(t,T)}{A(t,T)} = y_{fwd} = E^{QyA}\left[\frac{\tilde{y}_T\cdot A(\tilde{y}_T)}{A(\tilde{y}_T)}\right] = E^{QyA}\left[\tilde{y}_T\right] = \mu_y$$

This simply says we set the mean of the yield distribution equal to the market forward yield.

So now we have

$$\frac{C(t)}{A(t,T)} = E^{QyA}\left[\frac{\max\left[\left(c-\tilde{y}(T)\right),0\right]\cdot \tilde{A}(T)}{\tilde{A}(T)}\right] = E^{QyA}\left[\max\left[\left(c-\tilde{y}(T)\right),0\right]\right]$$

or

$$\begin{aligned} Call\,on\,bond &= PV(forward\,Annuity)\cdot E^{QyA}\left[\left(c-\tilde{y}(T)\right)|\tilde{y}(T)<c\right] \\ &= PVFAnn\cdot\int_{y=0}^{y=c}\left(c-y_T\right)\varphi\left(y_T;\sigma,\mu=y_{fwd}\right)dy_T \end{aligned}$$

In other words, the call on the bond is a put on the forward yield. Now $\varphi\left(y_T;\sigma,\mu=y_T(t)\right)$ is the log-normal density for *rates*, with *log rate* volatility σ and mean $\mu=y_T(t)$, today's forward par bond yield. The formula for the call (using the Black-Scholes put on rates) is:

$$Call\,on\,bond = Put\,on\,rates = [N(d_2) \cdot c - N(d_1) \cdot y_T(0)] \cdot PVFAnn$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{y_T(0)}{c}\right) + \frac{\sigma^2 T}{2}\right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

2.1 Pricing

Fwd Undlerlier $y_T(0) = 4.972$ (95.98 price of 27-year bond with 4.7% coupon gives 4.972% yield)

Strike	$c = 4.700$
σ	Assume $\sigma_{yield} = 18.5\%$
FVFAnn	14.772 = FV (as of expiry date) of \$1 paid semi-annually for 27 years at 4.972% yield
PVFAnn	13.044 = FVFAnn discounted back for 2.53 years at 4.97%sab = $\frac{14.772}{(1+.0497/2)^{2 \cdot 2.53}}$
d_1, $N(d_1)$	0.3383, 0.6324
d_2, $N(d_2)$	0.0439, 0.5175
Option	\$5.741, call on bond price, put on rates

3 Normal Yield

This is the same as (2) except that Now $\varphi\,(y_T; \sigma, \mu = y_T(t))$ is the *normal* density for *level rates*, with *basis point* volatility σ and mean $\mu = y_T(t)$, today's forward par bond yield. This gives a formula similar to the Black-Scholes formula – the integral for a normal rather than log-normal density:

$$Put\,on\,bond = Call\,on\,rates = \sigma_{norm}\sqrt{T} \cdot [D \cdot N(D) + \phi(D)] \cdot PVFAnn$$

$$Call\,on\,bond = Put\,on\,rates = Call\,on\,rates - \sigma_{norm}\sqrt{T} \cdot D \cdot PVFAnn$$

$$= \sigma_{norm}\sqrt{T} \cdot [D \cdot (N(D) - 1) + \phi(D)] \cdot PVFAnn = \sigma_{norm}\sqrt{T} \cdot [-D \cdot N(-D) + \phi(D)] \cdot PVFAnn$$

$$D = \frac{y_T(0) - c}{\sigma_{norm}\sqrt{T}}$$

The approximate relation between log-normal and normal yield volatility is

$$\sigma_{norm} \approx \sigma_{lny} \cdot [y_T(0) + c]\,/2$$

3.1 Pricing

Fwd Undlerlier $y_T(0) = 4.972$ (95.98 price of 27-year bond with 4.7% coupon gives 4.972% yield)

Strike	$c = 4.700$
σ	Assume $\sigma_{lny} = 18.5\% \Rightarrow \sigma_{norm} = .185 \cdot (.04972 + .047)/2 = 0.895\%$
FVFAnn	14.772 = FV (as of expiry date) of \$1 paid semi-annually for 27 years at 4.972% yield
PVFAnn	13.044 = FVFAnn discounted back for 2.53 years at 4.97%sab $= \frac{14.772}{(1+.0497/2)^{2 \cdot 2.53}}$
D, $N(D)$, $\phi(D)$	0.1910, 0.5758, 0.3917
Option	\$5.770, call on bond price, put on rates