Linear Algebra homework3.1

第一题到第八题是关于向量空间的. 在向量空间中，向量的加法 $\boldsymbol{x}+\boldsymbol{y}$ 和标量的乘法 $c\boldsymbol{x}$ 必须遵守以下的 8 个规则:

(1)$\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$

(2)$\boldsymbol{x}+(\boldsymbol{y}+\mathbf{z})=(\boldsymbol{x}+\boldsymbol{y})+\mathbf{z}$

(3)$\boldsymbol{x}+\mathbf{0}=\boldsymbol{x}$

(4)$\boldsymbol{x}+(-\boldsymbol{x})=0$

(5)$1\times\boldsymbol{x}=\boldsymbol{x}$

(6) $(c_1c_2)\boldsymbol{x}=c_1(c_2\boldsymbol{x})$

(7)$c(\boldsymbol{x}+\boldsymbol{y})=c\boldsymbol{x}+c\boldsymbol{y}$

(8)$(c_1+c_2)\boldsymbol{x}=c_1\boldsymbol{x}+c_2\boldsymbol{x}$.

1 假设 $(x_1,x_2)+(y_1+y_2)$ 定义为 (x_1+y_2,x_2+y_1). 满足乘法 $c\boldsymbol{x}=(cx_1,cx_2)$, 上面八个条件有哪些不满足？

2 假设在 $\mathbb{R}^2$ 中,$c\boldsymbol{x}$ 的结果 $(cx_1,0)$ 而不是 (cx_1,cx_2), 以上的 8 个条件满足吗？

3 (a) 如果我们仅仅使 $\mathbb{R}^1$ 中的 $x>0$, 哪个规则不满足?c 可取任意值. 射线不是子空间.

(b) 定义 $\boldsymbol{x}+\boldsymbol{y}=xy$，$c\boldsymbol{x}=x^c$, 也满足以上的 8 个条件. 当 $c=3,x=2,y=1$ 时，验证第 7 条规则是否正确. ($\boldsymbol{x}+\boldsymbol{y}=2$，$c\boldsymbol{x}=8$). 哪一个数字扮演 0 向量的角色？

4 矩阵 $A=\begin{bmatrix}2&-2\\2&-2\end{bmatrix}$ 是 2×2 矩阵空间 M 的一个“向量”. 写出这个空间中的 0 向量，向量 $\frac{1}{2}A$, 向量 $-A$. 哪些矩阵会在包含 A 的最小子空间中？

5 (a) 写出 M 的一个子空间，它包含 $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 但是不包含 $B=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$.

(b) 如果 M 的一个子空间包含 A 和 B, 它必须包含 I 吗？

(c) 写出 M 的一个子空间，它包含非零对角矩阵.

10 下面的 $\mathbb{R}^3$ 的子集有哪些是子空间?

(a) 向量 (b_1,b_2,b_3) 构成的平面，其中 $b_1=b_2$.

(b) 向量 (b_1,b_2,b_3) 构成的平面，其中 $b_1=1$.

(c) 向量 (b_1,b_2,b_3) 构成的平面，其中 $b_1b_2b_3=0$.

(d) $\mathbf{v}=(1,4,0)$ 和 $\mathbf{w}=(2,2,2)$ 的所有线性组合.

(e) 满足 $b_1+b_2+b_3=0$ 的所有向量.

(f) 满足 $b_1\le b_2\le b_3$ 的所有向量.

12 P 是在 $\mathbb{R}^3$ 中的一个平面，方程表达式为 $x+y-2z=4$. 原点 (0,0,0) 不在 P 中！在 P 中找出两个向量并验证它们的和不在 P 中.

14 $\mathbb{R}^3$ 的子空间是平面，线，$\mathbb{R}^3$ 本身, 或 Z 仅仅只包含 $(0,0,0)$.

(a) 写出 $\mathbb{R}^2$ 的三种类型的子空间.

(b) 写出 D 的所有子空间, D 是 2×2 的对角矩阵空间.

15 (a) 穿过 $(0,0,0)$ 的两个平面它们的交集可能是

$\mathbb{R}^3$ 中的 ____, 但也有可能是 ____. 不可能是 Z!
(b) 通过 $(0,0,0)$ 的平面和通过 $(0,0,0)$ 的直线它们的交点 (或交集) 可能是 ____, 但也有可能是 ____.
(c) 如果 $\mathbf{S}$ 和 $\mathbf{T}$ 是 $\mathbb{R}^5$ 的子空间，证明 $\mathbf{S}\cap\mathbf{T}$ 是 $\mathbb{R}^5$ 的一个子空间.$\mathbf{S}\cap\mathbf{T}$ 由位于两个子空间的向量组成. 如果 $\boldsymbol{x}$ 和 $\boldsymbol{y}$ 位于两个空间中，验证 $\boldsymbol{x}+\boldsymbol{y}$ 和 $c\boldsymbol{x}$ 属于 $\mathbf{S}\cap\mathbf{T}$.

17 (a) 证明 M 中可逆矩阵的集合不是子空间.
(b) 证明 M 中奇异矩阵的集合不是子空间.

19 描述下列特定矩阵的列空间 (线或者平面):

$$A=\begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix}\quad B=\begin{bmatrix}1&0\\0&2\\0&0\end{bmatrix}\quad C=\begin{bmatrix}1&0\\2&0\\0&0\end{bmatrix}$$

22 向量 (b_1,b_2,b_3) 取哪些值使得下列的方程组有解？
(a)

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

(b)

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

(c)

$$\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

23 如果我们对矩阵 A 再添加一列 $\boldsymbol{b}$, 列空间会变大除非 ____. 举一个列空间变大的例子和一个列空间不变的例子. 为什么当列空间没变大——A 与 $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix}$ 列空间一样时——$A\boldsymbol{x}=\boldsymbol{b}$ 才有解?

25 假设 $A\boldsymbol{x}=\boldsymbol{b}$ 以及 $A\boldsymbol{y}=\boldsymbol{b}^*$ 都有解，那么 $A\mathbf{z}=\boldsymbol{b}+\boldsymbol{b}^*$ 也是有解的.$\mathbf{z}$ 是什么？也可以表示为：如果 $\boldsymbol{b}$ 和 $\boldsymbol{b}^*$ 在列空间 $\mathbf{C}(A)$ 中，那么 $\boldsymbol{b}+\boldsymbol{b}^*$ 也在 $\mathbf{C}(A)$ 中.

27 判断对错 (如果错误给出反例)
(a) 不在列空间 $\mathbf{C}(A)$ 中的向量 $\boldsymbol{b}$ 构成了一个子空间.
(b) 如果 $\mathbf{C}(A)$ 只包含零向量，那么 A 就是零矩阵.
(c) $2A$ 的列空间等于 A 的列空间.
(d) $A-I$ 的列空间等于 A 的列空间.

29 如果 9×12 的方程组 $A\boldsymbol{x}=\boldsymbol{b}$ 对于任意的 $\boldsymbol{b}$ 都是有解的，那么 (A 的列空间)$\mathbf{C}(A)$=____.

Problem Set 3.1

The first problems 1–8 are about vector spaces in general. The vectors in those spaces are not necessarily column vectors. In the definition of a *vector space*, vector addition $x + y$ and scalar multiplication cx must obey the following eight rules:

(1) $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$

(2) $\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})=(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}$

(3) There is a unique "zero vector" such that $\boldsymbol{x}+\mathbf{0}=\boldsymbol{x}$ for all $\boldsymbol{x}$

(4) For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x}+(-\boldsymbol{x})=\mathbf{0}$

(5) 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$

(6) $(c_1c_2)\boldsymbol{x}=c_1(c_2\boldsymbol{x})$

(7) $c(\boldsymbol{x}+\boldsymbol{y})=c\boldsymbol{x}+c\boldsymbol{y}$

(8) $(c_1+c_2)\boldsymbol{x}=c_1\boldsymbol{x}+c_2\boldsymbol{x}$.

(1) to (4) about $\boldsymbol{x}+\boldsymbol{y}$

(5) to (6) about $c\boldsymbol{x}$

(7) to (8) connects them

1 Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $c\boldsymbol{x} = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

2 Suppose the multiplication $c\boldsymbol{x}$ is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2). With the usual addition in $\mathbf{R}^2$, are the eight conditions satisfied?

3 (a) Which rules are broken if we keep only the positive numbers $x > 0$ in $\mathbf{R}^1$? Every c must be allowed. The half-line is not a subspace.

(b) The positive numbers with $\boldsymbol{x}+\boldsymbol{y}$ and $c\boldsymbol{x}$ redefined to equal the usual xy and x^c *do* satisfy the eight rules. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $\boldsymbol{x}+\boldsymbol{y} = 2$ and $c\boldsymbol{x} = 8$.) Which number acts as the "zero vector"?

4 The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

5 (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of $\mathbf{M}$ does contain A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

6 The functions $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$ are "vectors" in $\mathbf{F}$. This is the vector space of all real functions. (The functions are defined for $-\infty < x < \infty$.) The combination $3\boldsymbol{f}(x) - 4\boldsymbol{g}(x)$ is the function $\boldsymbol{h}(x) =$ _____.

7 Which rule is broken if multiplying $\boldsymbol{f}(x)$ by c gives the function $f(cx)$? Keep the usual addition $\boldsymbol{f}(x) + \boldsymbol{g}(x)$.

8 If the sum of the "vectors" $\boldsymbol{f}(x)$ and $\boldsymbol{g}(x)$ is defined to be the function $f(g(x))$, then the "zero vector" is $\boldsymbol{g}(x) = x$. Keep the usual scalar multiplication $c\boldsymbol{f}(x)$ and find two rules that are broken.

Questions 9–18 are about the "subspace requirements": $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ (and then all linear combinations $c\boldsymbol{x} + d\boldsymbol{y}$) stay in the subspace.

9 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{x} + \boldsymbol{y}$ stays in the set but $\frac{1}{2}\boldsymbol{x}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{x}$ stays in the set but $\boldsymbol{x} + \boldsymbol{y}$ may be outside.

10 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces ?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

11 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12 Let P be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in P! Find two vectors in P and check that their sum is not in P.

13 Let $\mathbf{P}_0$ be the plane through $(0, 0, 0)$ parallel to the previous plane P. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

14 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

15 (a) The intersection of two planes through $(0,0,0)$ is probably a _____ in $\mathbf{R}^3$ but it could be a _____. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a _____ but it could be a _____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both subspaces. *Check that $\boldsymbol{x}+\boldsymbol{y}$ and $c\boldsymbol{x}$ are in $\mathbf{S} \cap \mathbf{T}$ if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both spaces.*

16 Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either _____ or _____.

17 (a) Show that the set of *invertible* matrices in $\mathbf{M}$ is not a subspace.

(b) Show that the set of *singular* matrices in $\mathbf{M}$ is not a subspace.

18 True or false (check addition in each case by an example):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

Questions 19–27 are about column spaces $C(A)$ and the equation $A\boldsymbol{x} = \boldsymbol{b}$.

19 Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

20 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

21 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of (B or C?) is also a combination of the columns of A. Which two matrices have the same column _____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

22 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{x} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$?

24 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

25 Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, *then* $\boldsymbol{b} + \boldsymbol{b}^*$ is in $\boldsymbol{C}(A)$.

26 If A is any 5 by 5 invertible matrix, then its column space is _____. Why?

27 True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

28 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

29 If the 9 by 12 system $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A) =$ _____.

Challenge Problems

30 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

(a) **Definition**: The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ or $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$. (Section 3.5 returns to this word "span".)

31 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

32 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$. Important point:

An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an _____ matrix.