Linear Algebra homework2.6

本次作业参见书 104-107 页。

Problems 1-14 compute the factorization A = LU (and also A = LDU).

1 (Important) Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \mathbf{c}$:

$$\begin{matrix} x + \ \ y = \mathbf{5} \\ x + 2y = \mathbf{7} \end{matrix} \longrightarrow \begin{matrix} x + y = \mathbf{5} \\ y = \mathbf{2}, \end{matrix} \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 1 & 2 & \mathbf{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 0 & 1 & \mathbf{2} \end{bmatrix}$$

That step subtracted ℓ_{21} = _____ times row 1 from row 2. The reverse step adds ℓ_{21} times row 1 to row 2. The matrix for that reverse step is L = _____. Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_1 = \begin{bmatrix} \mathbf{5} \\ \mathbf{2} \end{bmatrix}$ to get _____=_____.

In letters, L multiplies $U\boldsymbol{x} = \mathbf{c}$ to give _____.

2 Write down the 2 by 2 triangular systems $L\mathbf{c} = \boldsymbol{b}$ and $U\boldsymbol{x} = \mathbf{c}$ from Problem 1. Check that $\mathbf{c} = (5, 2)$ solves the first one. Find $\boldsymbol{x}$ that solves the second one.

3 (Move to 3 by 3) Forward elimination changes $A\boldsymbol{x} = \boldsymbol{b}$ to a triangular $U\boldsymbol{x} = \mathbf{c}$:

$$\begin{matrix} x + \ \ y + \ \ z = 1 \\ x + 2y + 3z = 7 \\ x + 3y + 6z = 11 \end{matrix} \qquad \begin{matrix} x + y + z = 5 \\ y + 2z = 2 \\ 2y + 5z = 6 \end{matrix} \qquad \begin{matrix} x + y + z = 5 \\ y + 2z = 2 \\ z = 2 \end{matrix}$$

The equation $z = 2$ in $U\boldsymbol{x} = \mathbf{c}$ comes from the original $x+3y+6z = 11$ in $A\boldsymbol{x} = \boldsymbol{b}$ by subtracting ℓ_{31} = _____ times equation 1 and ℓ_{32} = _____ times the final equation 2. Reverse that to recover [1 3 6 11] in the last row of A and $\boldsymbol{b}$ from the final [1 1 1 5] and [0 1 2 2] and [0 0 1 2] in U and $\mathbf{c}$:

$$\text{Row 3 of } [A \quad \boldsymbol{b}] = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 \\ + 1 \text{ Row } 3) \text{ of } [U \quad \mathbf{c}].$$

In matrix notation this is multiplication by L. So A = LU and $\boldsymbol{b} = L\mathbf{c}$.

4 What are the 3 by 3 triangular systems $L\mathbf{c} = \boldsymbol{b}$. and $U\boldsymbol{x} = \mathbf{c}$ from Problem 3? Check that $\mathbf{c} = (5, 2, 2)$ solves the first one. Which $\boldsymbol{x}$ solves the second one?

5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

7 What three elimination matrices E_{21},E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and

E_{21}^{-1} to factor A into L times U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

8 What are L and D (the diagonal **pivot matrix**) for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$\textbf{Already triangular} \quad \mathbf{A} = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

9 A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$\textbf{Symmetric} \quad \mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \textbf{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

10 (Recommended) Compute L and U for the symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Problems 12-13 use L and U (without needing A) to solve Ax = b.

12 Solve the triangular system $L\mathbf{c} = \boldsymbol{b}$ to find $\mathbf{c}$. Then solve $U\boldsymbol{x} = \mathbf{c}$ to find $\boldsymbol{x}$:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

For safety multiply LU and solve $A\boldsymbol{x} = \boldsymbol{b}$ as usual. Circle $\mathbf{c}$ when you see it.

14 (a) When you apply the usual elimination steps to L, what matrix do you reach?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I, what matrix do you get?
(c) When you apply the same steps to LU, what matrix do you get?

15 If $A = LDU$ and also $A = L_1 D_1 U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. "The three factors are unique."
Derive the equation $L_1^{-1} L D = D_1 U_1 U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal l's). Then $D = D_1$.

16 Tridiagonal matrices have zero entries except on the main diagonal and the two adjacent diago-

nals. Factor these into $A = LU$ and $A = LDL^T$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

18 If A and B have nonzeros in the positions marked by x, which zeros (marked by 0) stay zero in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

19 Suppose you eliminate upwards (almost unheard of). Use the last row to produce zeros in the last column (the pivot is 1). Then use the second row to produce zero above the second pivot. Find the factors in the unusual order $A = UL$.

$$\textbf{Upper} \quad \textbf{times} \quad \textbf{lower} \qquad \mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

20 *Easy but important.* If A has pivots 5, 9, 3 with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix A_2 (without row 3 and column 3)?

中文翻译参考

1 对 $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$ 采用前向消元得到 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \mathbf{c}$

$$\begin{aligned} x + \ \ y &= \mathbf{5} \\ x + 2y &= \mathbf{7} \end{aligned} \longrightarrow \begin{aligned} x + y &= \mathbf{5} \\ y &= \mathbf{2}, \end{aligned} \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 1 & 2 & \mathbf{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 0 & 1 & \mathbf{2} \end{bmatrix}$$

上面的步骤是从第二行中减去 $\ell_{21} =$ ____ 乘以第一行. 反向步骤是将 ℓ_{21} 乘以第一行加到第二行. 反向步骤对应的下三角矩阵 $L =$ ____ L 乘以三角方程组 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x_1} = \begin{bmatrix} \mathbf{5} \\ \mathbf{2} \end{bmatrix}$ 得到 ____ $=$ ____. 用字符表示为: L 乘以 $U\boldsymbol{x} = \mathbf{c}$ 得到 ____.

2 写出第一题中的 2 乘 2 的三角方程组 $L\mathbf{c} = \boldsymbol{b}$ 和 $U\boldsymbol{x} = \mathbf{c}$，验证 $\mathbf{c} = (5, 2)$ 是第一个方程组的解。求出第二个方程组的解 $\boldsymbol{x}$.

3 对 $A\boldsymbol{x} = \boldsymbol{b}$ 采用前向消元得到三角方程组 $U\boldsymbol{x} = \mathbf{c}$:

$$\begin{array}{ccc} x + y + z = 1 & x + y + z = 5 & x + y + z = 5 \\ x + 2y + 3z = 7 & y + 2z = 2 & y + 2z = 2 \\ x + 3y + 6z = 11 & 2y + 5z = 6 & z = 2 \end{array}$$

$U\boldsymbol{x} = \mathbf{c}$ 中的等式 $z = 2$ 是由 $A\boldsymbol{x} = \boldsymbol{b}$ 中的 $x + 3y + 6z = 11$ 减去 $\ell_{31} =$ ____ 乘以方程组第一行再减去 $\ell_{32} =$ ____ 乘以第三个方程组的第二行得到的. 写出从 $\begin{bmatrix} U & \mathbf{c} \end{bmatrix}$ 中恢复 $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ 的第 3 行的过程，即：

$$\begin{bmatrix} 1 & 3 & 6 & 11 \end{bmatrix} = \ell_{31} \times \begin{bmatrix} 1 & 1 & 1 & 5 \end{bmatrix} + \ell_{32} \times \begin{bmatrix} 0 & 1 & 2 & 2 \end{bmatrix} + 1 \times \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$$

（用矩阵表示，即为：$A = LU, \boldsymbol{b} = L\mathbf{c}$）。

4 问题 3 中的 3×3 的三角方程组 $L\mathbf{c} = \boldsymbol{b}, U\boldsymbol{x} = \mathbf{c}$ 分别是什么？检查 $\mathbf{c} = (5, 2\ 2)$ 是不是第一个方程组的解. 求出第二个方程组的解 $\boldsymbol{x}$.

5 求矩阵 E，计算得 $EA = U$，U 是上三角矩阵。左乘 $L(L = E^{-1})$, 将 A 分解为 LU $(A = LU)$，求 L:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

7 求三个消元矩阵 E_{21},E_{31}, E_{32}，将 A 变为上三角形式 U (即 $E_{32}E_{31}E_{21}A = U$)。将 A 分解为 L 乘以 U，求 L。

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \quad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

8 矩阵分解 $A = LDU$ 中矩阵 L 和 D（对角轴元矩阵）分别是什么？$A = LU$ 中的 U 是什么？$A = LDU$ 中的 U 是什么？

$$\text{已经是上三角矩阵} \quad A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

9 A 和 B 关于主对角线对称 (因为 $4 = 4$)。求 A 和 B 的三角分解 LDU，即求 L, D 和 U，说明 L 和 U 之间有何关联？

$$\text{对称矩阵} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \text{和} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

10 计算对称矩阵 A 的分解形式 $A = LU$（计算 L 和 U）：

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

找出 a, b, c, d 的需要满足的四个条件，使得 $A = LU$ 有 4 个轴元.

12 求三角方程组 $L\mathbf{c} = \boldsymbol{b}$ 的解 $\mathbf{c}$，和方程组 $U\boldsymbol{x} = \mathbf{c}$ 的解 $\boldsymbol{x}$.

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \text{和} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \text{和} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

保险起见，通常通过计算 $A = LU$ 来解方程组 $A\boldsymbol{x} = \boldsymbol{b}$，试用消元法得到 $U\boldsymbol{x} = \mathbf{c}$ 的形式。

14 (a) 当你按照正常的步骤对 L 进行消元时，你会得到什么矩阵？

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

(b) 按照相同的步骤处理单位阵 I（即乘子相同，计算过程相同，如在第 (a) 问中，E 的第 2 行减去 E 的第 1 行乘 ℓ_{21}，这里变成 I 的第 2 行减去

I 的第 1 行乘 ℓ_{21}），你将得到什么矩阵？(c) 按照相同的步骤处理矩阵 LU，又将得到什么矩阵？

15 如果 $A = LDU$, $A = L_1D_1U_1$, 其中的每个矩阵都是可逆的,那么 $L = L_1, D = D_1, U = U_1$(三个分解矩阵是唯一的). 证明 $L_1^{-1}LD = D_1U_1U^{-1}$. 等号两边的矩阵是三角形矩阵还是对角矩阵? 推导：如果 $L = L_1, U = U_1$(对角线元素都是 1), 那么 $D = D_1$.

16 除了主对角线和主对角线的两个相邻对角线外，其他位置的元素为零，这种矩阵称为三对角矩阵。如下面两个三对角矩阵分解为 $A = LU$ 和 $A = LDL^T$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ 和 } A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

18 如果 A 和 B 中的 x 不为 0，分别求它们三角分解 LU，问矩阵 L 和 U 中原本为 0 的位置哪些仍然为 0?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \text{ 和 } B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

19 假设你对 A 向上进行消元 (几乎没听过). 保留最后一行，使最后一列除最后一个元素外都为零 (第一个轴元是 1)，然后在第二行 (也是倒数第二行) 的基础上获得第二个轴元（轴元上方的元素为 0）。求该消元法情况下的 $A = UL$ 的分解矩阵 U, L。

$$\text{上三角乘下三角} \qquad A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

20 如果 A 有轴元 5，9，3 无需做行交换，那么左上的 2×2 的子矩阵 A_2 的轴元是什么 (没有第 3 行和第 3 列)?