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21. 令 $\begin{cases} u=2x+y \\ v=y-x \end{cases}$ $\therefore$ $u\in[4,7]$ $v\in[-2,1]$

$$\iint_D (2x^2-xy-y^2)\,dxdy = \iint_{D} -(2x+y)(y-x)\,dxdy = \iint_{D'} -uv|J|\,dudv$$

$|J| = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$ 原式化为 $-\frac{1}{3}\int_{-2}^{1} dv\int_4^7 uv\,du = \frac{33}{4}$

22. 令 $u=\sqrt{xy}$ $v=\sqrt{\frac{y}{x}}$ $\therefore$ $u\in[1,3]$ $v\in[1,2]$

原式化为 $\iint_{D'} (u+v)|J|\,dudv$ $|J| = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} & -\frac{1}{2}x^{-\frac{3}{2}}\sqrt{y} \\ \frac{1}{2}\sqrt{\frac{x}{y}} & \sqrt{\frac{1}{x}}\,\frac{1}{2}y^{-\frac{1}{2}} \end{vmatrix} \Rightarrow$

$|J| = \frac{2u}{v}$ 原式化为 $\iint_{D'}(u+v)\frac{2u}{v}\,dudv = \int_1^2 dv\int_1^3\left(\frac{2u^2}{v}+2u\right)du = 8+\frac{52}{3}\ln 2$

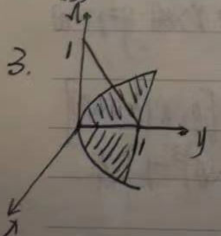
24. 令 $x=ar\cos\theta$ $y=br\sin\theta$ $|J| = \frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{vmatrix} = abr$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = r^2 \le 1$ $\therefore r\in[0,1]$ $\theta\in[0,2\pi]$

原式化为 $\iint_D\left[a^2r^2\cos^2\theta+b^2r^2\sin^2\theta\right]abr\,drd\theta = \int_0^{2\pi}(a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta\int_0^1 r^3\,dr = \frac{\pi ab}{4}(a^2+b^2)$

习题7.3

1. $\iiint (z+z^2)\,dV = \int_0^1 \pi(1-z^2)(z^2)\,dz = 2\pi\int_0^1(z^2-z^4)\,dz = \frac{4}{15}\pi$

3. 

$$\iiint x^2\sin x\,dxdydz = \iint_{D'} dydz\int x^2\sin x\,dx$$

$x^2\sin x$ 是关于x的奇函数

$\therefore \iiint x^2\sin x\,dxdydz = 0$

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5. $\iiint\limits_{\Omega}(x^2-y^2-z^2)dV \quad \Omega: x^2+y^2+z^2\le a^2$

$\iiint\limits_{\Omega}(x^2-y^2-z^2)dV = \iiint -z^2dV = \int_{-a}^{a} -z^2\pi(a^2-z^2)dz = -\frac{4}{15}\pi a^5$

7. $\iiint(y^2+z^2)dV \quad \Omega: 0\le a^2\le x^2+y^2+z^2\le b^2$

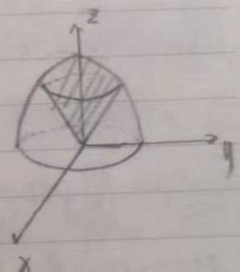
$\iiint(y^2+z^2)dV = \int_0^a 2z^2\cdot 2\cdot\pi(b^2-a^2)dz + \int_a^b 4\pi z^2(b^2-z^2)dz = \frac{8}{15}\pi(b^5-a^5)$

9. 令 $x=\rho\cos\theta \quad y=\rho\sin\theta \quad \therefore \rho^2\le R\rho\cos\theta$ 即 $\rho\le R\cos\theta$

$\rho\in[0,R\cos\theta] \quad \theta\in[-\frac{\pi}{2},\frac{\pi}{2}] \quad z\le\sqrt{R^2-r^2}$

原式化为 $\iiint z^2 r\,dr\,d\theta\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{R\cos\theta}dr\int_0^{\sqrt{R^2-r^2}}z^2 r\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{R\cos\theta}\frac{1}{6}(R^2-r^2)^{\frac{3}{2}}d(r^2)$

$\frac{R^5}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-[redacted]\sin^5\theta)d\theta = \frac{2\pi R^5}{15}-\frac{2R^5}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^5\theta\,d\theta = \frac{\pi R^5}{15}-\frac{32}{225}R^5$

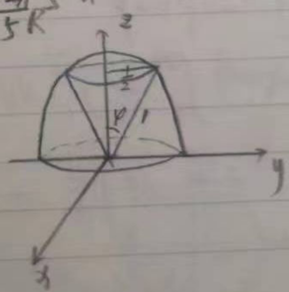


11. 令 $x=\rho\sin\varphi\cos\theta \quad y=\rho\sin\varphi\sin\theta \quad z=\rho\cos\varphi$

$\varphi\in[0,\frac{\pi}{4}] \quad \theta\in[0,2\pi] \quad \rho\in[0,R]$

原式化为 $\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^R\rho^2\sin^2\varphi\,\rho^2\sin\varphi\,d\rho$

$= 2\pi\times\frac{1}{5}R^5\int_0^{\frac{\pi}{4}}\sin^3\varphi\,d\varphi = \left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)\frac{2\pi}{5}R^5$



13. $\sqrt{3(x^2+y^2)}=\sqrt{1-y^2-x^2}$ 即 $x^2+y^2=(\frac{1}{2})^2 \quad \sin\varphi\in[0,\frac{1}{2}]$

$\varphi\in[0,\arcsin\frac{1}{2}]$ 令 $z=\rho\cos\varphi \quad \rho\in[0,[redacted]]$

原式化为 $\int_0^{2\pi}d\theta\int_0^{\arcsin\frac{1}{2}}d\varphi\int_0^1\rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho$

$= \frac{2\pi}{5}\int_0^{\arcsin\frac{1}{2}}\cos^2\varphi\sin\varphi\,d\varphi$

$= \frac{2}{5}\pi\left(-\frac{\sqrt{3}}{8}+\frac{1}{3}\right)$

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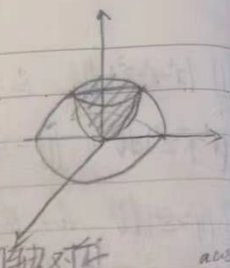
15. Ω $x^2+y^2+z^2=2a^2$, $az=x^2+y^2$ $\Rightarrow$ 



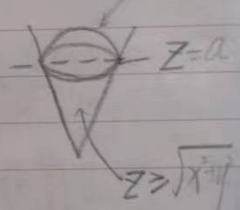
令 $x=r\cos\theta$ $y=r\sin\theta$ $\theta\in[0,2\pi]$ $z\in[a,\sqrt{2}a]$

$r\in[\sqrt{az},a]$ 原式化为 $\iiint \frac{xy+1}{x^2+y^2+z^2}dV$ 关于 x轴, y轴对称

$\therefore \iiint \frac{xy+1}{x^2+y^2+z^2}dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{\sqrt{2}a}\sin\varphi\, d\rho+\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\int_0^{\frac{a\cos\varphi}{\sin^2\varphi}}\rho$

$=2\sqrt{2}\pi a\left(\int_0^{\frac{\pi}{4}}\sin\varphi\, d\varphi+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\varphi}{\sin\varphi}d\varphi\right)=\sqrt{2}\,2\pi a\left[\left(\cos\varphi\Big|_{\frac{\pi}{4}}^{0}\right)+\ln\sin\varphi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$

$=2\pi a(\sqrt{2}-1+\ln\sqrt{2})$



17. $\iiint(x^3-\sin y+z)dV=\iiint z\,dV$

令 $x=\rho\sin\varphi\cos\theta$ $y=\rho\sin\theta\sin\varphi$ $z=\rho\cos\varphi$

原式化为 $\iiint \rho\cos\varphi\cdot\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$ $\theta\in[0,2\pi]$

$\rho^2\leq 2a\rho\cos\varphi$ $\rho\sin\varphi\leq\rho\cos\varphi$ $\varphi\in[0,\frac{\pi}{4}]$ $\rho\leq 2a\cos\varphi$

原式化为 $\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{2a\cos\varphi}\rho^3\sin\varphi\cos\varphi\, d\rho$

$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\frac{1}{4}\cdot(2a\cos\varphi)^4\sin\varphi\cos\varphi\, d\varphi$

$=2\pi\cdot 4a^4\int_0^{\frac{\pi}{4}}\cos^5\varphi\sin\varphi\, d\varphi=-8\pi a^4\int_0^{\frac{\pi}{4}}\cos^5\varphi\, d(\cos\varphi)$

$\overset{\cos\varphi=t}{=\!=}-\frac{8\pi a^4}{6}t^6\Big|_1^{\frac{\sqrt{2}}{2}}=\frac{4}{3}\pi a^4\times(1-\frac{1}{8})=\frac{7}{6}\pi a^4$

$u \in [1,2]$ $v \in [0,2]$ $w \in [0,3]$

18 令 $u = x$ $v = xy$ $w = 3z$ $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3x = 3u$

原式化为 $\iiint \frac{1}{3u}(uv + wv)\, d\sigma = \frac{1}{3}\int_1^2 du \int_0^2 dv \int_0^3 \left(v + \frac{wv}{u}\right) dw$

$\frac{1}{3}\int_1^2 du \int_0^2 \left(3v + \frac{9v}{2u}\right) dv = \frac{1}{3}\int_1^2 \left(6 + \frac{9}{u}\right) du$ $\left(2u + 3\ln u\right)\Big|_1^2 = 2 + 3\ln 2$

19 $\iiint (x+1)(y+1)\, dV = \iiint (xy + x + y + 1)\, dV = \iiint 1\, dV = V_\Omega$

令 $x = ar\sin\varphi\cos\theta$ $y = br\sin\varphi\sin\theta$ $z = cr\cos\varphi$ $\theta \in [0, 2\pi]$ $\varphi \in [0, \pi]$ $r \in [0, 1]$

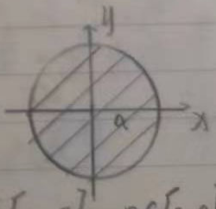
原式化为 $abc\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 r^2 \sin\varphi\, dr = \frac{4}{3}\pi abc$

21 ① 令 $x = r\cos\theta$ $y = r\sin\theta$ $\begin{cases} x^2+y^2+z^2 \le z \\ z = \sqrt{3x^2+3y^2} \end{cases}$ ∴ 相交 $x^2+y^2 = \frac{3}{16}$ $z = \frac{\sqrt{3}}{4}$ $r \in \left[0, \frac{\sqrt{3}}{4}\right]$
$\theta \in [0, 2\pi]$

∴ 原式化为 $\int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{4}} dr \int_{\sqrt{3}r}^{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}} f(\sqrt{r^2+z^2})\, r\, dz$

② 令 $x = \rho\sin\varphi\cos\theta$ $y = \rho\sin\varphi\sin\theta$ $z = \rho\cos\varphi$ 由①得 $\varphi \in \left[0, \frac{\pi}{6}\right]$ $\theta \in [0, 2\pi]$
$\rho \in [0, \cos\theta]$

原式化为 $\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} d\varphi \int_0^{\cos\varphi} f(\rho)\cdot \rho^2 \sin\varphi\, d\rho$



习题7.4 1. $\begin{cases} z = \sqrt{3a^2-x^2-y^2} \\ z = \dfrac{x^2+y^2}{2a} \end{cases}$ $\Rightarrow x^2+y^2 = 2a^2$ $z = a$

$z'_x = \frac{x}{a}$ $z'_y = \frac{y}{a}$

$z_x = \dfrac{-x}{\sqrt{3a^2-x^2-y^2}}$ $z_y = \dfrac{-y}{\sqrt{3a^2-x^2-y^2}}$

$S_1 = \iint_D \sqrt{1 + \frac{x^2+y^2}{a^2}}\, dx dy$ 令 $x = ar\cos\theta$ $y = ar\sin\theta$ $\theta \in [0, 2\pi]$ $r \in [0, a]$

∴ 原式化为 $\int_0^{2\pi} d\theta \int_0^{\sqrt{2}a} \frac{1}{a}\sqrt{a^2 + r^2}\, r\, dr = 2\pi \times \frac{1}{2} \times \frac{1}{a}\int_0^{a} \sqrt{a^2+r^2}\, d(r^2+a^2) = \frac{\pi}{a}\frac{2}{3}\left(a^2+r^2\right)^{\frac{3}{2}}\Big|_0^{\sqrt{2}a}$

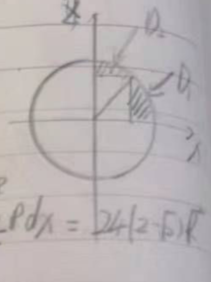
$= -\frac{2}{3}\pi a^2 + 2\sqrt{3}\pi a^2$

∴ $S = \frac{16}{3}\pi a^2$

$S_2 = \iint \frac{\sqrt{3}a}{\sqrt{3a^2-x^2-y^2}}\, dx dy = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}a} \frac{\sqrt{3}a}{\sqrt{3a^2-r^2}}\, r\, dr = 6\pi a^2 - 2\sqrt{3}\pi a^2$

3. $\begin{cases} x^2+y^2=R^2 \\ y^2+z^2=R^2 \\ x^2+z^2=R^2 \end{cases}$ 交线 $\begin{cases} x=y \\ x=z \\ y=z \end{cases}$

$x^2+y^2=R^2 \quad y_x=\frac{-2x}{\sqrt{R^2-x^2}} \quad y_z=0$



$S=2\times3\times8\iint\limits_{D_1}\sqrt{1+y_x^2+y_z^2}\,dx\,dz$

$=48\int_{\frac{R}{\sqrt2}}^{R}dx\int_0^{\sqrt{R^2-x^2}}\frac{R}{\sqrt{R^2-x^2}}dz=48\int_{\frac{R}{\sqrt2}}^{R}R\,dx=24(2-\sqrt2)R^2$

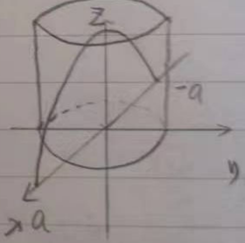
4. $V=8\iint\limits_D f(x,y)\,dx\,dy=8\left(\iint\limits_{D_1}\sqrt{R^2-x^2}\,dx\,dy+\iint\limits_{D_2}\sqrt{R^2-y^2}\,dx\,dy\right)$

$=\int_0^{\frac{\sqrt2}{2}R}dx\int_0^{x}\sqrt{R^2-x^2}\,dy+\int_{\frac{R}{\sqrt2}}^{R}dx\int_0^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2}\,dy=\frac{2-\sqrt2}{2}R^3$

同理 $\iint\limits_{D_2}\sqrt{R^2-y^2}\,dx\,dy=\frac{2-\sqrt2}{2}R^3$

$\therefore V=8(2-\sqrt2)R^3$

5 令 $x^2=r^2\cos^2\theta \quad y=r^2\sin^2\theta$

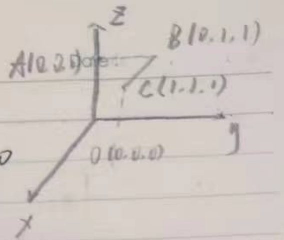


$\therefore r\in[0,\frac{a}{2}] \quad \theta\in[0,\pi] \qquad r\in[0,\frac{a}{2}]$

$V=2\iiint 1\,dx\,dy\,dz \qquad z\in[0,a-\frac{x^2}{a}]$

$=2\int_0^{\pi}d\theta\int_0^{\frac{a}{2}}dr\int_0^{a-\frac{x^2}{a}}r\,dz=2\int_0^{\pi}d\theta\int_0^{\frac{a}{2}}\left(a-\frac{x^2}{a}\right)r\,dr=2\int_0^{\pi}\frac{1}{8}a^3-\frac{1}{64}a^3\ldots$

$=2\left(\frac{\pi}{8}a^3-\frac{1}{64}a^3\int_0^{\pi}\frac{1+\cos2\theta}{2}d\theta\right)=2\left(\frac{\pi}{8}a^3-\frac{\pi}{128}a^3\right)=\frac{15}{64}\pi a^3$



1. $L = L_{OA} + L_{AB} + L_{BC}$

$\int_L (xy+yz+zx)\,ds = \int_{L_{OA}} (xy+yz+zx)\,ds = 0$ OA: $x=y=0$

$+\int_{L_{AB}} (xy+yz+zx)\,ds = \int_{L_{AB}} y\,ds$ AB: $x=0$ $z=1$

$+\int_{L_{BC}} (xy+yz+zx)\,ds = \int_{L_{BC}} (2x+1)\,ds$ BC: $y=1$ $z=1$

原式 $= \int_{L_{AB}} y\,ds + \int_{L_{BC}} (2x+1)\,ds = \int_0^1 y\,dy + \int_0^1 (2x+1)\,dx = \frac{5}{2}$

AB: $ds = \sqrt{0^2+0^2+1^2}\,dy = dy$ BC: $ds = dx$

4. $\int_L \frac{1}{x^2+y^2+z^2}\,ds = \int_0^{2\pi} \frac{1}{a^2+b^2t^2}\sqrt{a^2+b^2}\,dt = \frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{b}{a}t\Big|_0^{2\pi} = \frac{\sqrt{a^2+b^2}}{ab}\left(\arctan\frac{2\pi b}{a}\right)$

6. $x = a\cos t$ $y = b\sin t$ 原积分为 $\int_0^{\frac{1}{2}\pi} ab\sin t\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt \xRightarrow{x=\sin t} \int_0^1 abx\sqrt{(a^2-b^2)x^2+b^2}\,dx$

$= \frac{ab}{2(a^2-b^2)}\int_0^1 \left[(a^2-b^2)x^2+b^2\right]^{\frac{1}{2}} d\left[(a^2-b^2)x^2+b^2\right] \xrightarrow{(a^2-b^2)x^2+b^2=k} \frac{ab}{2(a^2-b^2)}\int_{b^2}^{a^2} k^{\frac{1}{2}}\,dk = \frac{ab(a^2+ab+b^2)}{3(a+b)}$

7. $x^2+y^2 = a^2(t^2+1)$ $dx/dt = -a\sin t + a\sin t + at\cos t = at\cos t$

代入可得: $dy/dt = a\cos t - a\cos t + at\sin t = at\sin t$

$\int_L \sqrt{x^2+y^2}\,ds = \int_0^{2\pi} a\sqrt{t^2+1}\sqrt{a^2t^2}\,dt = a^2\int_0^{2\pi}\sqrt{t^2+1}\,t\,dt = \frac{a^3}{3}(4\pi^2+1)^{\frac{3}{2}}$

8. $\int_L (x+\sqrt{y}-z^5)\,ds = \int_{L_1} (x+\sqrt{y}-z^5)\,ds + \int_{L_2} (x+\sqrt{y}-z^5)\,ds$

$= \int_0^1 2x\sqrt{1+4x^2}\,dx + \int_0^1 (2-z^5)\sqrt{1}\,dz$

$= \frac{1}{6}(5\sqrt{5}-1) + 2 - \frac{1}{6} = \frac{5\sqrt{5}}{6} + \frac{5}{3} = \frac{5\sqrt{5}+10}{6}$