Linear Algebra homework3.2

中文翻译参考：

1 将 A, B 化简为它们的三角阶梯形 U。问哪些变量是自由的 (free)？

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

2 对于第 1 题的矩阵，求每一个自由变量的特殊解。（令一个自由变量为 1，其他自由变量都为 0）。

3 对第 1 问得到的 U 进行更深入的行操作 (row operation)，求它们的简化阶梯形 R。判断对错：R 的零空间等于 U 的零空间，说明原因。

5 判断对错 (如果正确说明原因，如果错误举例说明)：

(a) 方阵没有自由变量；

(b) 可逆矩阵没有自由变量；

(c) 一个 $m \times n$ 的矩阵的轴变量不超过 n 个；

(d) 一个 $m \times n$ 的矩阵的轴变量不超过 m 个；

6 在 4×7 的阶梯矩阵 U 中放置尽可能多的 1，使得 U 的轴元列是：(a)2，4，5。

7 在 4×8 的简化阶梯形 R 中放置尽可能多的 1，使得 R 的自由列是：(a)2，4，5，6。

8 假设一个 3×5 的矩阵的第 4 列都为零，那么 x_4 可以确定是一个 ____ 变量，该变量的特殊解是向量 $\boldsymbol{x} =$ ____。

9 假设一个 3×5 的矩阵的第 1 列和最后一列相等（不为零），那么 ____ 是自由变量，求出这个自由变量的特殊解。

10 假设一个 $m \times n$ 的矩阵有 r 个轴元，那么特殊解的个数为 ____，当 $r =$ ____ 时零空间仅包含 $\boldsymbol{x} = \mathbf{0}$，当 $r =$ ____ 时列空间是整个 $\mathbb{R}^m$。

13 (推荐) 平面 $x-3y-z=12$ 平行于 $x-3y-z=0$，该平面 $(x-3y-z=12)$ 上的一个特殊点是 $(12,0,0)$，所有平面上的点有以下形式：

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

填写上式留下的空。

15 构造一个矩阵，它的零空间 $\mathbf{N}(A) = (2,2,1,0)$ 和 $(3,1,0,1)$ 的所有线性组合。

18 构造一个矩阵，它的列空间包含 $(1,1,5)$ 和 $(0,3,1)$，零空间包含 $(1,0,1)$ 和 $(0,0,1)$。

20 创建一个 2×2 矩阵，它的零空间等于列空间。

21 为什么 3×3 矩阵的零空间不可能等于其列空间？

22 如果 $AB = 0$，那么 B 的列空间包含在 A 的 ____ 中，为什么？

24 举例证明下列三个说法一般是错误的：

(a) A 和 $A^\top$ 有相同的零空间；

(b) A 和 $A^\top$ 有相同的自由变量；

(c) 如果 R 是简化形式 $\mathbf{rref}(A)$ (即 $R = \mathbf{rref}(A)$)，那么 $R^\top = \mathbf{rref}(A^\top)$

26 如果 $R\boldsymbol{x} = \mathbf{0}$ 的特殊解在下列零空间矩阵 N 的列中，由特殊解反过来求简化矩阵 R 的非零行。

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{和} N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{和} N = \begin{bmatrix} \ \end{bmatrix} \text{空的 } 3 \times 1 \text{ 向量}$$

29 如果 4×4 矩阵 A 可逆，描述一下 4×8 矩阵 $B = \begin{bmatrix} A & A \end{bmatrix}$ 的零空间。

30 如果 $C = \begin{bmatrix} A \\ B \end{bmatrix}$，那么 C 的零空间 $\mathbf{N}(C)$ 与 A, B

的零空间 $\mathbf{N}(A), \mathbf{N}(B)$ 有何关系？

32 基尔霍夫电流定律 $A^\top \boldsymbol{x} = \mathbf{0}$ 说明在每个节点：流进的电流 = 流出的电流。比如在节点 1 有：$y_3 = y_1 + y_4$；写出四个节点的四个基尔霍夫方程 (箭头的方向表示每个电流 y 的方向)。把 $A^\top$ 简化成 R，然后在 $A^\top(4 \times 6$ 矩阵) 的零空间中找到三个特殊解。

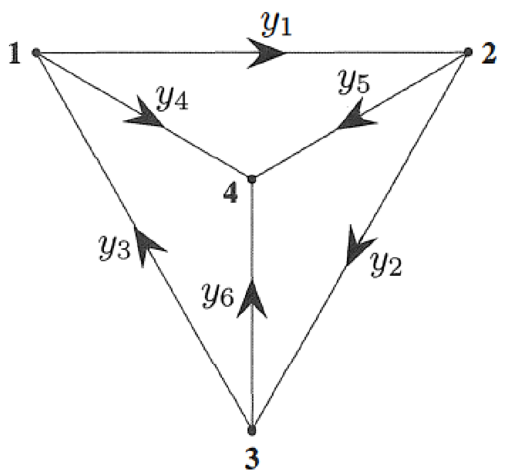


图 1. 第 32 题图

33 下面哪些定义给出了 A 的秩的正确定义：

(a) R(简化阶梯形) 的非零行的个数；

(b) 列的个数减去全部的行数；

(c) 列的个数减去自由列的个数；

(d) 矩阵 R 中 1 的个数。

35 假设轴元变量出现在最后而不是最前面，描述其简化阶梯形 R 的四个分块矩阵 (块矩阵 B 必须为 $r \times r$ 矩阵)。

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

问包含特殊解的零空间矩阵 N 是什么？

38 求 $R\boldsymbol{x} = \mathbf{0}$ 和 $\boldsymbol{y}^\top R = \mathbf{0}$ 的特殊解是什么？其中 R 为：

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{及 } R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

39 把下面的矩阵填完整使得它们的秩都是 1：

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \text{ 和 } B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \text{ 和 } M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

40 如果一个 $m \times n$ 矩阵 A 的秩是 $r = 1$，它的列都是某一列的倍数，它的行都是某一行的倍数。那么它的列空间是 $\mathbb{R}^m$ 中的 ____（线或平面），零空间是 $\mathbb{R}^n$ 中的 ____。零空间矩阵 N 的形状是 ____（几乘几矩阵)。

41 求向量 $\mathbf{u}$ 和 $\mathbf{v}$ 满足 $A = \mathbf{u}\mathbf{v}^\top$ = 列向量乘行向量：

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \text{ 和 } A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

矩阵 $A = \mathbf{u}\mathbf{v}^\top$ 是每一个秩为 1 的矩阵的自然形式 (natural form)。

44 假设 P 是一个 $m \times n$ 矩阵中只包含 r 个轴元列的子矩阵，解释为什么这个 $m \times r$ 子矩阵 P 的秩为 r？

45 对第 44 题中的 P 进行转置，找出 $r \times m$ 矩阵 $P^\top$ 的 r 个轴元列。将得到的结果（$r \times r$ 矩阵）转置回去，得到一个 P 和 A 的 $r \times r$ 可逆子矩阵 S。$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix}$，先求 3×2 矩阵 P，再求 2×2 可逆矩阵 S.

47 秩为 1 的矩阵 $\mathbf{u}\mathbf{v}^\top$ 乘秩为 1 的矩阵 $\mathbf{w}\mathbf{z}^\top$ 得到 $\mathbf{u}\mathbf{z}^\top$ 乘数字 ____。它们的积 $\mathbf{u}\mathbf{v}^\top\mathbf{w}\mathbf{z}^\top$ 的秩也是 1，除非 ____=0。

48 (a) 假设 B 的第 j 列是前面的列的线性组合，证明 AB 的第 j 列也是 AB 的前面列的相同的线性组合。那么 AB 不可能有新的轴元列，所以 $\operatorname{rank}(AB) \le \operatorname{rank}(B)$

(b) 已知 $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$，求矩阵 A_1, A_2，满足 $\operatorname{rank}(A_1B) = 1$ 以及 $\operatorname{rank}(A_2B) = 0$。

50 (重要) 假设 A, B 都是 $n \times n$ 矩阵，并且 $AB = I$。如果 $\operatorname{rank}(AB) \le \operatorname{rank}(A)$，证明：$A$ 的秩是 n。所以 A 不可逆，B 必须是它的双边逆矩阵 (Section 2.5)。所以 $BA = I$(这个不是显而易见的)。

52 假设 A, B 有相同的简化行阶梯形 R。

(a) 证明 A, B 有相同的零空间和相同的行空间；

(b) 若已知 $E_1A = R$ 和 $E_2B = R$，那么 A 等于一个 ____ 矩阵乘 B。

54 求 A, B 的简化行阶梯形 R 和秩 r(依赖于 c)。A 的轴元列是什么？特殊解是什么？

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \text{ 和 } B = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

56 简洁的事实。任意一个秩为 r 的 $m \times n$ 矩阵都可以简化为 $(m \times r)$ 矩阵乘 $(r \times n)$ 矩阵：

A=(A 的轴元列)(R 的前 r 行)= 列乘行。

假设 A 为 3×4 矩阵，所有的元素都为 1. 将 A 写成上述积的形式，“列”是由轴元列构成的 3×1 矩阵，“行”是由 R 得到的 1×4 矩阵。

3.2 C Find the row reduced form R and the rank r of A and B (*those depend on* c). Which are the pivot columns of A? What are the special solutions?

$$\textbf{Find special solutions} \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & c \\ c & c \end{bmatrix}.$$

Solution The matrix A has row $2 = 3$ (row 1). The rank of A is $r = 2$ ***except if*** $\boldsymbol{c = 4}$. Row $4 - 4$(row 1) ends in $c - 4$. The pivots are in columns 1 and 3. The second variable x_2 is free. Notice the form of R: Row 3 has moved up into row 2.

$$\boldsymbol{c \neq 4} \quad R = \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{c = 4} \quad R = \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two pivots leave one free variable x_2. But when $c = 4$, the only pivot is in column 1 (rank one). The second and third variables are free, producing two special solutions:

$$\boldsymbol{c \neq 4} \quad \text{Special solution } (-2, 1, 0) \qquad \boldsymbol{c = 4} \quad \text{Another special solution } (-1, 0, 1).$$

The 2 by 2 matrix $B = \left[\begin{smallmatrix} c & c \\ c & c \end{smallmatrix}\right]$ has rank $r = 1$ ***except if*** $\boldsymbol{c = 0}$, when the rank is zero!

$$\boldsymbol{c \neq 0} \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{c = 0} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and nullspace} = \mathbf{R}^2.$$

Problem Set 3.2

1 Reduce A and B to their triangular echelon forms U. Which variables are free?

$$\text{(a)} \ A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

3 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false with a reason*: The nullspace of R equals the nullspace of U.

4 For the same A and B, find the special solutions to $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is _____. This is the **Counting Theorem**: $\boldsymbol{r + (n - r) = n}$.

$$\text{(a)} \ A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \qquad \text{(b)} \quad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}.$$

Questions 5–14 are about free variables and pivot variables.

5 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

6 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) 2, 4, 5 (b) 1, 3, 6, 7 (c) 4 and 6.

7 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) 2, 4, 5, 6 (b) 1, 3, 6, 7, 8.

8 Suppose column 4 of a 3 by 5 matrix is all zero. Then x_4 is certainly a _____ variable. The special solution for this variable is the vector $\boldsymbol{x} =$ _____.

9 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.

10 Suppose an m by n matrix has r pivots. The number of special solutions is _____. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $r =$ _____. The column space is all of $\mathbf{R}^m$ when $r =$ _____.

11 The nullspace of a 5 by 5 matrix contains only $\boldsymbol{x} = \mathbf{0}$ when the matrix has _____ pivots. The column space is $\mathbf{R}^5$ when there are _____ pivots. Explain why.

12 The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which variables are free? The special solutions are _____ and _____.

13 (Recommended) The plane $x - 3y - z = 12$ is parallel to $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

14 Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column has no pivot? What is the special solution? Describe $\boldsymbol{N}(A)$.

Questions 15–22 ask for matrices (if possible) with specific properties.

15 Construct a matrix for which $\boldsymbol{N}(A) =$ all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

16 Construct A so that $\boldsymbol{N}(A) =$ all multiples of $(4, 3, 2, 1)$. Its rank is _____.

17 Construct a matrix whose column space contains $(1,1,5)$ and $(0,3,1)$ and whose nullspace contains $(1,1,2)$.

18 Construct a matrix whose column space contains $(1,1,0)$ and $(0,1,1)$ and whose nullspace contains $(1,0,1)$ and $(0,0,1)$.

19 Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

20 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

21 Why does no 3 by 3 matrix have a nullspace that equals its column space?

22 If $AB = 0$ then the column space of B is contained in the _____ of A. Why?

23 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What R is virtually certain if the random A is 4 by 3?

24 Show by example that these three statements are generally *false*:

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form $\mathbf{rref}(A)$ then R^{T} is $\mathbf{rref}(A^{\mathrm{T}})$.

25 If $\boldsymbol{N}(A) =$ all multiples of $\boldsymbol{x} = (2,1,0,1)$, what is R and what is its rank?

26 If the special solutions to $R\boldsymbol{x} = \mathbf{0}$ are in the columns of these nullspace matrices N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \ \text{(empty 3 by 1)}.$$

27 (a) What are the five 2 by 2 reduced matrices R whose entries are all 0's and 1's?

(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R?

28 Explain why A and $-A$ always have the same reduced echelon form R.

29 If A is 4 by 4 and invertible, describe the nullspace of the 4 by 8 matrix $B = [A \ \ A]$.

30 How is the nullspace $\boldsymbol{N}(C)$ related to the spaces $\boldsymbol{N}(A)$ and $\boldsymbol{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

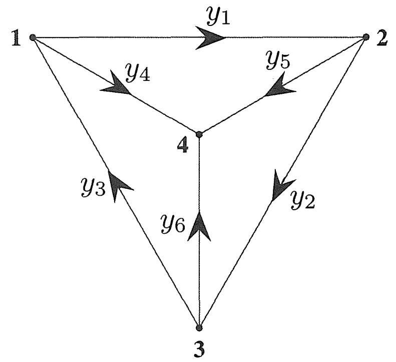
31 Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix with all entries equal to 4.

(b) The 3 by 4 matrix with $a_{ij} = i + j - 1$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

32 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ says that *current in = current out* at every node. At node 1 this is $y_3 = y_1 + y_4$. Write the four equations for Kirchhoff's Law at the four nodes (arrows show the positive direction of each y). Reduce A^{T} to R and find three special solutions in the nullspace of A^{T} (4 by 6 matrix).



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

33 Which of these rules gives a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.

(b) The number of columns minus the total number of rows.

(c) The number of columns minus the number of free columns.

(d) The number of 1's in the matrix R.

34 Find the reduced R for each of these (block) matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

35 Suppose all the pivot variables come *last* instead of first. Describe all four blocks in the reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N containing the special solutions?

36 (Silly problem) Describe all 2 by 3 matrices A_1 and A_2, with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is is true that $R_1 = A_1$ and $R_2 = A_2$ in this case? Does $R_1 - R_2$ equal $\mathbf{rref}(A_1 - A_2)$?

37 If A has r pivot columns, how do you know that A^{T} has r pivot columns? Give a 3 by 3 example with different column numbers in *pivcol* for A and A^{T}.

38 What are the special solutions to $R\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}^{\mathrm{T}}R = \mathbf{0}$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

39 Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

40 If A is an m by n matrix with $r = 1$, its columns are multiples of one column and its rows are multiples of one row. The column space is a _____ in $\mathbf{R}^m$. The nullspace is a _____ in $\mathbf{R}^n$. The nullspace matrix N has shape _____.

41 Choose vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

$A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is the natural form for every matrix that has rank $r = 1$.

42 If A is a rank one matrix, the second row of R is _____. Do an example.

Problems 43–45 are about r by r invertible matrices inside A.

43 *If A has rank r, then it has an r by r submatrix S that is invertible.* Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix S inside A, B, and C. You could keep the pivot rows and pivot columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

44 Suppose P contains only the r pivot columns of an m by n matrix. Explain why this m by r submatrix P has rank r.

45 Transpose P in Problem 44. Find the r pivot columns of P^{T} (which is r by m). Transposing back, **this produces an r by r invertible submatrix S inside P and A:**

$$\text{For } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} \text{ find } P \text{ (3 by 2) and then the invertible } S \text{ (2 by 2).}$$

Problems 46–51 show that $\operatorname{rank}(AB)$ is not greater than $\operatorname{rank}(A)$ or $\operatorname{rank}(B)$.

46 Find the ranks of AB and AC (rank one matrix times rank one matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

47 The rank one matrix $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ times the rank one matrix $\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is $\boldsymbol{u}\boldsymbol{z}^{\mathrm{T}}$ times the number _____. This product $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ also has rank one unless _____ $= 0$.

48 (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so $\mathbf{rank}(\boldsymbol{AB}) \leq \mathbf{rank}(\boldsymbol{B})$.

(b) Find A_1 and A_2 so that $\operatorname{rank}(A_1 B) = 1$ and $\operatorname{rank}(A_2 B) = 0$ for $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

49 Problem 48 proved that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. Then the same reasoning gives $\mathbf{rank}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}) \leq \operatorname{rank}(A^{\mathrm{T}})$. How do you deduce that $\mathbf{rank}(\boldsymbol{AB}) \leq \mathbf{rank}\,\boldsymbol{A}$?

50 *(Important)* Suppose A and B are n by n matrices, and $AB = I$. Prove from $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA = I$ *(which is not so obvious!).*

51 If A is 2 by 3 and B is 3 by 2 and $AB = I$, show from its rank that $BA \neq I$. Give an example of A and B with $AB = I$. For $m < n$, a right inverse is not a left inverse.

52 Suppose A and B have the *same* reduced row echelon form R.

(a) Show that A and B have the same nullspace and the same row space.

(b) We know $E_1 A = R$ and $E_2 B = R$. So A equals an _____ matrix times B.

53 Express A and then B as the sum of two rank one matrices:

$$\mathbf{rank} = \mathbf{2} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

54 Answer the same questions as in Worked Example **3.2 C** for

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

55 What is the nullspace matrix N (containing the special solutions) for A, B, C?

$$\text{Block matrices} \quad A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

56 *Neat fact* ***Every m by n matrix of rank r reduces to*** (m by r) ***times*** (r by n):

$$A = (\text{pivot columns of } A)\,(\text{first } r \text{ rows of } R) = (\textbf{COL})(\textbf{ROW}).$$

Write the 3 by 4 matrix A of all ones as the product of the 3 by 1 matrix from the pivot columns and the 1 by 4 matrix from R.