Linear Algebra homework3.3

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1 执行 **3.3A** 的 6 个步骤，写出 A 的列空间和零空间，并求出 $A\boldsymbol{x}=\boldsymbol{b}$ 的所有解:

$$A=\begin{bmatrix}2&4&6&4\\2&5&7&6\\2&3&5&2\end{bmatrix}\quad \boldsymbol{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}4\\3\\5\end{bmatrix}$$

3 将完整解写成 $\boldsymbol{x}_p$ 加上零空间中 s 的任意倍；

$$\begin{aligned}&x+3y+3z=1\\&2x+6y+9z=5\\&-x-3y+3z=5\end{aligned}$$

5 b_1,b_2,b_3 之间满足什么条件可使得这个方程组有解? 在消元法中把 $\boldsymbol{b}$ 作为第四列。找到该条件下的所有解:

$$\begin{aligned}&x+2y-2z=b_1\\&2x+5y-4z=b_2\\&4x+9y-8z=b_3\end{aligned}$$

7 如果 $b_3-2b_2+4b_1=0$, 通过消元法证明 (b_1,b_2,b_3) 在 (A 的) 列空间中.

$$A=\begin{bmatrix}1&3&1\\3&8&2\\2&4&0\end{bmatrix}$$

A 的行之间怎样的组合得到一个全 0 行?

11 为什么一个 1×3 的方程组的解不为 $\boldsymbol{x}_p=(2,4,0)$ 以及 $\boldsymbol{x}_n=(1,1,1)$ 的任意倍数?

13 解释为什么这些都是错误的:
(a) 完整的解是 $\boldsymbol{x}_p$ 以及 $\boldsymbol{x}_n$ 的任意线性组合.
(b) 方程组 $A\boldsymbol{x}=\boldsymbol{b}$ 至多有一个特解.
(c) 所有自由变量为 0 的解 $\boldsymbol{x}_p$ 是最短的解 ($||\boldsymbol{x}||$ 的长度最短). 找出一个反例.
(d) 如果 A 是可逆的那么在零空间中就没有解 $\boldsymbol{x}_n$.

15 假设 U 的 3 行都没有轴元，那么 U 的行是 ____. 方程 $U\boldsymbol{x}=\mathbf{c}$ 仅仅在 ____ 时有解. 方程 $A\boldsymbol{x}=\mathbf{c}$ 有解、无解、还是可能无解？

16 3×5 的矩阵的秩最大可能为 ____. 那么 U 和 R 中的每一 ____ 都有一个轴元.$A\boldsymbol{x}=\mathbf{c}$ 的解是否存在且惟一？A 的列空间是 ____. 给出一个例子 $A=$ ____.

18 通过消元法得到 A 的秩以及 A^T 的秩.

$$(a)\begin{bmatrix}1&4&0\\2&11&5\\-1&2&10\end{bmatrix}\quad (b)\begin{bmatrix}1&0&1\\1&1&2\\1&1&q\end{bmatrix}$$

21 求出下列满秩方程组的完整解，解的形式为 $\boldsymbol{x}_p+\boldsymbol{x}_n$.
(a) $\quad x+y+z=4$

(b) $$\begin{aligned} x+y+z&=4 \\ x-y+z&=4. \end{aligned}$$

22 如果 $A\boldsymbol{x}=\boldsymbol{b}$ 有无穷多个解，为什么 $A\boldsymbol{x}=\mathbf{B}$(新的右边) 不可能只有一个解?$A\boldsymbol{x}=\mathbf{B}$ 可能会无解吗?

25 写出所有已知的 r,m,n 之间的关系式,（$m\times n$ 矩阵，秩是 r）如果 $A\boldsymbol{x}=\boldsymbol{b}$ 满足：

(a) 对于某些 $\boldsymbol{b}$ 没有解

(b) 每个 $\boldsymbol{b}$ 有无穷多个解

(c) 某些 $\boldsymbol{b}$ 只有一个解，其它的 $\boldsymbol{b}$ 没有解.

(d) 每个 $\boldsymbol{b}$ 都有一个解.

27 如果 A 是一个三角形矩阵，什么时候 (简化标准形)$R=rref(A)=I$?

29 应用高斯-约当消元法将 $R\boldsymbol{x}=0$ 和 $R\boldsymbol{x}=\mathbf{d}$ 简化为:

$$\begin{bmatrix} U & 0 \end{bmatrix}=\begin{bmatrix} 3&0&6&0 \\ 0&0&2&0 \\ 0&0&0&0 \end{bmatrix} \qquad \begin{bmatrix} U & \mathbf{c} \end{bmatrix}=\begin{bmatrix} 3&0&6&9 \\ 0&0&2&4 \\ 0&0&0&5 \end{bmatrix}$$

解 $U\boldsymbol{x}=0$ 或者 $R\boldsymbol{x}=0$ 找出 $\boldsymbol{x}_n$(自由变量=1).$R\boldsymbol{x}=\mathbf{d}$ 的解是什么？

30 化简 $U\boldsymbol{x}=c$(高斯消元法) 然后化简 $R\boldsymbol{x}=\mathbf{d}$(高斯-约当)：

$$A\boldsymbol{x}=\begin{bmatrix} 1&0&2&3 \\ 1&3&2&0 \\ 2&0&4&9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}=\boldsymbol{b}$$

求特解 $\boldsymbol{x}_p$ 和所有齐次解 $\boldsymbol{x}_n$.

31 找出满足性质的 A 和 B，如果找不出，说明原因.

(a) $A\boldsymbol{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 的唯一解是 $\boldsymbol{x}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b)$B\boldsymbol{x}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 的唯一解是 $\boldsymbol{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

33 $A\boldsymbol{x}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 的完整解为 $\boldsymbol{x}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. 求出 A.

$R\boldsymbol{x} = \boldsymbol{d}$ shows that the particular solution with free variables $= 0$ is $\boldsymbol{x_p} = (7, 0, -3, 0)$.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -4 & \mathbf{7} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}.$$

For the nullspace part $\boldsymbol{x_n}$ with $\boldsymbol{b} = \mathbf{0}$, set the free variables x_2, x_4 to $1, 0$ and also $0, 1$:

Special solutions $\quad \boldsymbol{s}_1 = (-2, \; 1, \; 0, \; 0)$ **and** $\boldsymbol{s}_2 = (4, \; 0, \; -4, \; 1)$

Then the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$ (and $R\boldsymbol{x} = \boldsymbol{d}$) is $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x_p} + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

The rows of A produced the zero row from $2(\text{row } 1) + (\text{row } 2) - (\text{row } 3) = (0, 0, 0, 0)$. Thus $\boldsymbol{y} = (2, 1, -1)$. The same combination for $\boldsymbol{b} = (4, 2, 10)$ gives $2(4) + (2) - (10) = 0$.

If a combination of the rows (on the left side) gives the zero row, then the same combination must give zero on the right side. Of course! *Otherwise no solution.*

Later we will say this again in different words: If every column of A is perpendicular to $\boldsymbol{y} = (2, 1, -1)$, then any combination $\boldsymbol{b}$ of those columns must also be perpendicular to $\boldsymbol{y}$. Otherwise $\boldsymbol{b}$ is not in the column space and $A\boldsymbol{x} = \boldsymbol{b}$ is not solvable.

And again: If $\boldsymbol{y}$ is in the nullspace of A^{T} then $\boldsymbol{y}$ must be perpendicular to every $\boldsymbol{b}$ in the column space of A. Just looking ahead...

Problem Set 3.3

1 (Recommended) Execute the six steps of Worked Example **3.3 A** to describe the column space and nullspace of A and the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $A\boldsymbol{x} = \boldsymbol{b}$ to be solvable. Together these two conditions put $\boldsymbol{b}$ into the _____ space (two planes give a line):

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

Questions 3–15 are about the solution of $A\boldsymbol{x} = \boldsymbol{b}$. Follow the steps in the text to $\boldsymbol{x}_p$ and $\boldsymbol{x}_n$. Start from the augmented matrix with last column $\boldsymbol{b}$.

3 Write the complete solution as $\boldsymbol{x}_p$ plus any multiple of $\boldsymbol{s}$ in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include $\boldsymbol{b}$ as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find $\boldsymbol{x}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination of the rows of A gives the zero row?

8 Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a)}\ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b)}\ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

9 (a) The Worked Example **3.3 A** reached $[\, U \ \ \boldsymbol{c} \,]$ from $[\, A \ \ \boldsymbol{b} \,]$. Put the multipliers into L and verify that LU equals A and $L\boldsymbol{c}$ equals $\boldsymbol{b}$.

(b) Combine the pivot columns of A with the numbers -9 and 3 in the particular solution $\boldsymbol{x}_p$. What is that linear combination and why?

10 Construct a 2 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$ with particular solution $\boldsymbol{x}_p = (2, 4, 0)$ and homogeneous solution $\boldsymbol{x}_n =$ any multiple of $(1, 1, 1)$.

11 Why can't a 1 by 3 system have $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_n =$ any multiple of $(1, 1, 1)$?

12 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has two solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, find two solutions to $A\boldsymbol{x} = \mathbf{0}$.

(b) Then find another solution to $A\boldsymbol{x} = \mathbf{0}$ and another solution to $A\boldsymbol{x} = \boldsymbol{b}$.

13 Explain why these are all false:

(a) The complete solution is any linear combination of $\boldsymbol{x}_p$ and $\boldsymbol{x}_n$.

(b) A system $A\boldsymbol{x} = \boldsymbol{b}$ has at most one particular solution.

(c) The solution $\boldsymbol{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\boldsymbol{x}\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution $\boldsymbol{x}_n$ in the nullspace.

14 Suppose column 5 of U has no pivot. Then x_5 is a _____ variable. The zero vector (is) (is not) the only solution to $A\boldsymbol{x} = \mathbf{0}$. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has _____ solutions.

15 Suppose row 3 of U has no pivot. Then that row is _____. The equation $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided _____. The equation $A\boldsymbol{x} = \boldsymbol{b}$ (*is*) (*is not*) (*might not be*) solvable.

Questions 16–20 are about matrices of "full rank" $r = m$ or $r = n$.

16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ (*always exists*) (*is unique*). The column space of A is _____. An example is $A =$ _____.

17 The largest possible rank of a 6 by 4 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ (*always exists*) (*is unique*). The nullspace of A is _____. An example is $A =$ _____.

18 Find by elimination the rank of A and also the rank of A^{T}:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \ \text{(rank depends on } q\text{)}.$$

19 Find the rank of A and also of $A^{\mathrm{T}}A$ and also of AA^{T}:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

20 Reduce A to its echelon form U. Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

21 Find the complete solution in the form $x_p + x_n$ to these full rank systems:

(a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$

22 If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right side) to have only one solution? Could $Ax = B$ have no solution?

23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

24 Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b
(b) ∞, regardless of b
(c) 0 or ∞, depending on b
(d) 1, regardless of b.

25 Write down all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b
(b) infinitely many solutions for every b
(c) exactly one solution for some b, no solution for other b
(d) exactly one solution for every b.

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix R.

26 Continue elimination from U to R. Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R:

$$U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

27 If A is a triangular matrix, when is $R = \mathsf{rref}(A)$ equal to I?

28 Apply Gauss-Jordan elimination to $Ux = \mathbf{0}$ and $Ux = c$. Reach $Rx = \mathbf{0}$ and $Rx = d$:

$$\begin{bmatrix} U & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $Rx = \mathbf{0}$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

29 Apply Gauss-Jordan elimination to reduce to $R\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \boldsymbol{d}$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $U\boldsymbol{x} = \mathbf{0}$ or $R\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}_n$ (free variable $= 1$). What are the solutions to $R\boldsymbol{x} = \boldsymbol{d}$?

30 Reduce to $U\boldsymbol{x} = \boldsymbol{c}$ (Gaussian elimination) and then $R\boldsymbol{x} = \boldsymbol{d}$ (Gauss-Jordan):

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \boldsymbol{b}.$$

Find a particular solution $\boldsymbol{x}_p$ and all homogeneous solutions $\boldsymbol{x}_n$.

31 Find matrices A and B with the given property or explain why you can't:

(a) The only solution of $A\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$.

(b) The only solution of $B\boldsymbol{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$.

32 Find the LU factorization of A and the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

33 The complete solution to $A\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + c \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$. Find A.