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习题10.4

2. (1) $\lim\limits_{n\to\infty} f_n(x)=0$ $\left|f_n(x)-f(x)\right|=\frac{1}{2^n+x^2}<\frac{1}{2^n}$ $\therefore f_n(x)$ 一致收敛

(3) $f_n(x)=\ln\left(1+\frac{x^2}{n^2}\right)$ $\lim\limits_{n\to\infty} f_n(x)=\frac{x^2}{n^2}=0$ $\left|f_n(x)-f(x)\right|=\ln\left(1+\frac{x^2}{n^2}\right)$

(a) $-l<x<l$ 原式 $<\ln\left(1+\frac{l^2}{n^2}\right)$ $\lim\limits_{n\to\infty}\ln\left(1+\frac{l^2}{n^2}\right)=\ln 1=0$ $f_n(x)$ 一致收敛

(b) $-\infty<x<+\infty$ 令 $x_n=n$ $\therefore \lim\limits_{n\to\infty}\ln\left(1+\frac{n^2}{n^2}\right)=\ln 2$ $f_n(x)$ 不一致收敛

(5) $\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}\frac{\sin x^n}{3+x^n}=\lim\limits_{n\to\infty}\frac{x^n}{3+x^n}=0$ $f_n(x)-f(x)=\frac{x^n}{3+x^n}$ (a) $0\le x\le 1-\delta$ (b) $0<x<1$

(a) $\left|f_n(x)-f(x)\right|=\frac{\sin x^n}{3+x^n}<\frac{x^n}{3+x^n}=\frac{1}{\frac{3}{x^n}+1}\le\frac{1}{\frac{3}{(1-\delta)^n}+1}=\frac{(1-\delta)^n}{3+(1-\delta)^n}$

$\lim\limits_{n\to\infty}\frac{(1-\delta)^n}{3+(1-\delta)^n}=0$ $\therefore f_n(x)$ 一致收敛

(b) 令 $x_n=\sqrt[n]{\frac{n-1}{n}}$ 原式化为: $\left|f_n(x_n)-f(x_n)\right|=\frac{\sin\left(1-\frac{1}{n}\right)}{3+1-\frac{1}{n}}=\frac{\sin\left(1-\frac{1}{n}\right)}{4-\frac{1}{n}}$

$\lim\limits_{n\to\infty}\frac{\sin\left(1-\frac{1}{n}\right)}{4-\frac{1}{n}}=\frac{\sin 1}{4}$ $f_n(x)$ 不一致收敛

3. (1) $u_n(x)=(-1)^n\frac{\sqrt{n}}{x^2+n^2}$ $\left|u_n(x)\right|=\frac{\sqrt{n}}{x^2+n^2}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{\frac{3}{2}}}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty}u_n(x)$ 一致收敛

(3) $u_n(x)=\frac{\sin nx}{\sqrt{(1+(x^2+n^2))^3}}\le\frac{1}{\sqrt{(x^2+n^2)^3}}\le\frac{1}{n^3}$ 收敛 $\therefore\sum\limits_{n=1}^{\infty}u_n(x)$ 一致收敛

(5) $u_n(x)=\frac{x^2}{(1+x)^n}\le\frac{x^2}{x^n}=\frac{1}{x^{n-2}}$ 一致收敛

4. 设 $u_n(x)=3^{-n}\sin 2^n x<\frac{2^n}{3^n}x$ 同时 $\left|u_n(x)\right|\le\frac{1}{3^n}$ $\therefore f(x)=\sum\limits_{n=1}^{\infty}f_n(x)$ 一致收敛

$f_n'(x)=\left(\frac{2}{3}\right)^n\cos 2^n x<\left(\frac{2}{3}\right)^n$ 收敛 $\therefore f_n'(x)$ 在R上一致收敛，且有连续的导函数

5. 令 $x_n=3^n$ $\therefore g(x)=\sum\limits_{n=1}^{\infty}2^n\sin 1$ $\lim\limits_{n\to\infty}f_n(x_n)=\lim\limits_{n\to\infty}2^n\sin 1\neq 0$

$\therefore g(x)$ 在 $(-\infty,+\infty)$ 上不一致收敛

$x\in[-M,M]$ 时 $\left|f_n(x)\right|=2^n\sin\frac{x}{3^n}\le\left|\frac{2^n}{3^n}x\right|\le\left(\frac{2}{3}\right)^n M$ 一致收敛

$f_n'(x)=\left(\frac{2}{3}\right)^n\cos x/3^n$ 在 $(-\infty,+\infty)$ 上一致收敛 $f_n'(x)$ 连续 $\therefore g(x)$ 在 $(-\infty,+\infty)$ 上有连续导数

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7. $|f_n(x)| = \left|\frac{\sin nx}{n^{p+2}}\right| \le \frac{1}{n^{p+2}}$ 收敛 $\therefore \sum_{n=1}^{\infty} f_n(x)$ 一致收敛. $f(x)$ 处处都收敛

$f_n'(x) = \frac{n\cos nx}{n^{p+2}} = \frac{\cos nx}{n^{p+1}} < \frac{1}{n^{p+1}}$ 收敛 $\sum_{n=1}^{\infty} f_n'(x)$ 一致收敛 $f_n'(x)$ 连续

$\therefore f(x)$ 在 $(-\infty,+\infty)$ 上有连续的导函数

习题五

1 (1) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n\cdot 2^n}} = \frac{1}{2}$ $R=2$ $\therefore$ 收敛半径 $=2$

(3) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}} = \lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)^{n+1}\left(1-\frac{1}{n+1}\right)^{-1} = \frac{1}{e}$ $\therefore R=e$ 收敛半径 $=e$

2 (1) 设 $u_n(x) = \frac{x^n}{\sqrt[3]{n}}$ $a_n = \frac{1}{\sqrt[3]{n}}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n+1}}{\sqrt[3]{n}} = \lim_{n\to\infty} \sqrt[3]{\frac{n+1}{n}} = 1$ $R=1$

当 $x=1$ 时 原级数发散 当 $x=-1$ 时, 原级数收敛

$\therefore$ 收敛区间 $(-1,1)$ 收敛域 $[-1,1)$

(3) 设 $a_n = (-1)^n \frac{1}{(2n+1)(2n+1)!}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (-1)\left(\frac{2n+2}{2n+1}\right)(2n+3)^2 = \infty$

$\therefore$ 收敛半径 $R=0$ 收敛区间 $x=0$

(5) $u_n(x) = x^{2n+1}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ 收敛半径 $R=1$

当 $x=1$ 时, 原级数发散. 当 $x=-1$ 时同理

综上, 收敛区间 $(-1,1)$ 收敛域 $(-1,1)$

(7) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1+e^{n+1})\,n}{(n+e^n)(n+1)e} = \frac{1}{e}$ 收敛半径 $R=e$

当 $x=e$ 时 $u_n = \left(\frac{1}{n}+e^{-n}\right)e^n = \frac{e^n}{n}+1$ 发散

当 $x=-e$ 时 $u_n = \frac{(-e)^n}{n}+(-1)^n$ 发散

$\therefore$ 收敛域 $(-e,e)$ 收敛区间 $(-e,e)$

3. (1) $a_n = n+1$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{n+2}{n+1} = 1$ $\therefore$ 收敛半径 $R=1$

$x\in(-1,1)$ 时 $\int_0^x S(t)\,dt = \int_0^x (n+1)t^n\,dt = \sum_{n=0}^{\infty} t^{n+1}\Big|_0^x = \cancel{t^{x+1}-t}$ $\sum_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x}$

两边求导得: $S(x) = \left(\frac{x}{1-x}\right)' = \frac{1-x-x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$ $(-1<x<1)$

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(3) $a_n=\frac{1}{n(n+1)}$ $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}}=1$ 收敛半径 $R=1$

$S(x)=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n(n+1)}$ $S'(x)=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ $S''(x)=\sum\limits_{n=1}^{\infty}(-1)^{n-1}x^{n-1}=\lim\limits_{n\to\infty}\frac{1-(-x)^n}{1-(-x)}$

$\therefore S'(x)=\int_0^x\frac{1}{1+t}dt=\ln(1+x)$ $S(x)=\int_0^x\ln(1+t)dt=t\ln(t+1)\Big|_0^x-\int_0^x t\frac{1}{t+1}dt=(x+1)\ln(1+x)-x$

$(-1<x<1)$

(5) $a_n=\cancel{\ }\frac{2n+1}{n!}$ $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{\frac{2n+3}{(n+1)!}}{\frac{2n+1}{n!}}=\lim\limits_{n\to\infty}\frac{2n+3}{(2n+1)(n+1)}=0$

当 $-\infty<x<+\infty$ $\int_0^x S(t)dt=\sum\limits_{n=1}^{\infty}\frac{1}{n!}\int_0^x(2n+1)t^{2n}dt=\sum\limits_{n=1}^{\infty}\frac{1}{n!}x^{2n+1}=xe^{x^2}$

$-\infty<x<+\infty$

求导 $S(x)=e^{x^2}+x\cdot 2xe^{x^2}=e^{x^2}(1+2x^2)$

(7) $a_n=\frac{1}{n(n+1)}$ $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=1$

当 $-1<x<1$ $S(x)=\sum\limits_{n=1}^{\infty}\left(\frac{x^n}{n}-\frac{x^n}{n+1}\right)=S(x_1)+S(x_2)$ $S(x_1)=\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$ $S(x_2)=\cdots$

$S'(x_1)=x^{n-1}$ $\sum\limits_{n=1}^{\infty}S'(x_1)=\lim\limits_{n\to\infty}\frac{1-x^n}{1-x}=\frac{1}{1-x}$ $S(x_1)=\int_0^x\frac{1}{1-t}dt=-\ln(1-x)$

同理 $\sum\limits_{n=1}^{\infty}\int_0^x S(t)dt=\sum\limits_{n=1}^{\infty}-x^{n+1}=-\frac{x(1-x^n)}{1-x}=-\frac{1}{1-x}$ 求导 $S(x_2)=\frac{0-1}{(x-1)^2}=\cdots$

$\therefore S(x)=S(x_1)+S(x_2)=\frac{1}{1-x}-\frac{1}{(1-x)^2}=\frac{1-x-1}{(1-x)^2}=\frac{-x}{(1-x)^2}$

习题 10.6

1 (4) $\ln\sqrt{\frac{1+x}{1-x}}=\frac{1}{2}\ln\frac{1+x}{1-x}=\frac{1}{2}\left[\ln(1+x)-\ln(1-x)\right]$

$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots\cdots+\frac{(-1)^{n-1}}{n}x^n$

$\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots\cdots-\frac{1}{n}x^n$

$\therefore$ 原式 $=\frac{1}{2}\left(2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots\cdots+\frac{2}{2n-1}x^{2n-1}\right)=\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}x^{2n-1}$

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=1$

$\therefore R=1$ 对端

$\therefore$ 收敛域 (

(5) $(1+x)e^{-x}$

$e^{-x}=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\cdots\cdots+\frac{(-1)^n}{n!}x^n$

$\therefore$ 原式 $=1-x+\frac{1}{2!}x^2+\cdots\cdots+\frac{(-1)^n}{n!}x^n+x-x^2+\frac{1}{2!}x^3+\cdots\cdots+\frac{(-1)^n}{n!}x^{n+1}$

$=1-\frac{1}{2!}x^2+\frac{1}{3}x^3-\frac{1}{8}x^4+\cdots\cdots+\frac{(-1)^n n+(-1)^n}{n!}x^n+\frac{(-1)^n}{n!}x^{n+1}$

$=\sum\limits_{n=1}^{\infty}\frac{(-1)^n(n+1)}{n!}x^n$ $\cancel{+\frac{(-1)^n}{n!}x^{n+1}}$

(7) sin

$\sin x=$

原式 $=$

(10)

2.

再

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(7) $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ 　 $\sin 3x = 3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 + \cdots\cdots + \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1}$

原式 $= \frac{1}{4}\left(3x - \frac{3}{3!}x^3 + \frac{3}{5!}x^5 + \cdots\cdots + \frac{3x(-1)^n}{(2n+1)!}x^{2n+1} - 3x + \frac{1}{3!}(3x)^3 - \frac{1}{5!}(3x)^5 + \cdots\cdots - \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1}\right)$

$= \frac{1}{4}\left(4x^3 - \frac{1}{5}x^5 + \cdots\cdots + \frac{-3x(-1)^n}{(2n+1)!}x^{2n+1}\right) = 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}$

(11) $\frac{5x-12}{x^2-5x-6} = \frac{6}{x-6} - \frac{1}{x+1}$

$\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$ 　　 $\therefore$ 原式 $= \sum_{n=0}^{\infty} x^n\left[(-1)^n + (6^{-n})\right]$

$\frac{6}{x-6} = \frac{1}{\frac{x}{6}-1} = -\sum_{n=0}^{\infty}\left(\frac{x}{6}\right)^n$

2. (2) 求微分 $y = \ln(x+\sqrt{1+x^2})$ 　 $y' = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots\cdots(-n+\frac{1}{2})}{n!}x^{2n}$

再求积分得: $y = x + \sum_{n=1}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots\cdots(-n+\frac{1}{2})}{n!\,(2n+1)}x^{2n+1}$

3. 证明: $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots\cdots + \frac{1}{n!}x^n$

$\therefore e^x - 1 = \sum_{n=1}^{\infty}\frac{1}{n!}x^n$ 　 $\therefore \frac{1}{x}(e^x-1) = \sum_{n=0}^{\infty}\frac{x^{n+1-1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}$

$\therefore \sum_{n=0}^{\infty}\frac{1}{(n+1)!} = e-1$ 　 $\sum_{n=1}^{\infty}\frac{n}{(n+1)!} = \sum_{n=0}^{\infty}\frac{n+1-1}{(n+1)!} = \sum_{n=0}^{\infty}\frac{1}{n!} - \sum_{n=0}^{\infty}\frac{1}{(n+1)!} = e-(e-1) = 1$

习题 11.1

1 (1) $\int_0^{+\infty} x e^{-x}dx = -(x+1)e^{-x}\Big|_0^{+\infty} = 0 + 1 = 1$ 　 $\therefore$ 原积分收敛, 其值为 1

(3) $\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 2\sqrt{2}\sigma\int_0^{+\infty} e^{-t^2}dt = 2\sqrt{2}\sigma \cdot \frac{\sqrt{\pi}}{2} = \sqrt{2\pi}\sigma$

$\therefore$ 原积分收敛 其值为 [illegible]