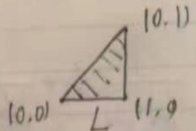
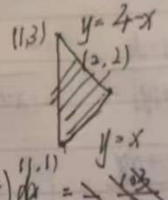
1(1) $P=-(x^3+x^2y)$ $Q=(xy^2+y^3)$ $\frac{\partial P}{\partial y}=-x^2$ $\frac{\partial Q}{\partial x}=y^2$

$\therefore$ 原式化为 $\iint_D (y^2+x^2)\,dxdy = a^2\iint_D 1\,dxdy = a^2\cdot\frac{1}{3}\pi a^2 = \frac{1}{3}\pi a^4$



(2) $P=y^2$ $Q=x^2$ $\frac{\partial P}{\partial y}=2y$ $\frac{\partial Q}{\partial x}=2x$

$\therefore$ 原式化为 $\iint_D (2x-2y)\,dxdy = \int_0^1 dx\int_0^{1-x} 2(x-y)\,dy = 0$



(3) $Q=y\left[xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right]$ $P=\sqrt{x^2+y^2}$

$\frac{\partial P}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$ $\frac{\partial Q}{\partial x}=y^2+\frac{y}{\sqrt{x^2+y^2}}$

原式化为 $\iint_D y^2\,dxdy = \int_1^2 dx\int_x^{4-x} y^2\,dy = \frac{1}{3}\int_1^2\left(-2x^3+12x^2-48x+64\right)dx = \frac{25}{6}$

(5) $P=e^x\sin y+\sin x-8y$ $Q=e^x\cos y-\sin y$ $\frac{\partial P}{\partial y}=e^x\cos y-8$ $\frac{\partial Q}{\partial x}=e^x\cos y$

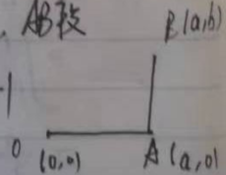
$-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=+8$ 原式化为 $\iint_D +8\,dxdy = -8S_D = -8\times\pi\times\left(\frac{a}{2}\right)^2\times\frac{1}{2} = -\pi a^2$

2(1) $S=-\frac{1}{2}\oint_L y\,dx-x\,dy = -\frac{1}{2}\int_0^{2\pi} a\sin t(1-\cos t)\,d\left[a(1-\cos t)\cos t\right]-a(1-\cos t)\cos t\,d\left[a(1-\cos t)\sin t\right]$

$=\frac{1}{2}\int_0^{2\pi} a^2(1-\cos t)^2\,dt = \frac{3}{2}\pi a^2$

4(3) $P(x,y)=e^x\cos y$ $Q(x,y)=-e^x\sin y$ $\frac{\partial P}{\partial y}=-e^x\sin y$ $\frac{\partial Q}{\partial x}=-e^x\sin y=\frac{\partial P}{\partial y}$

由于 P,Q 在原域内仍连续可导 $\therefore$ 积分值与路径无关，化为OA段、AB段



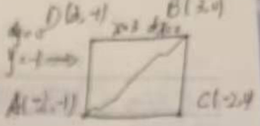
原式 $=\int_0^a e^x\,dx+\int_0^b\left(-e^a\sin y\right)dy = \cancel{e^a(\cos b-1)}\ e^a\cos b-1$

5. $P(x,y)=x^4+4xy^3$ $Q(x,y)=6x^2y^2-5y^4$ $\frac{\partial P}{\partial y}=12xy^2$ $\frac{\partial Q}{\partial x}=12xy^2$

$\therefore \frac{\partial P}{\partial y}\equiv\frac{\partial Q}{\partial x}$ $\therefore$ 积分值与路径无关

∴原积分为 $\int_{-2}^{3}3x^2dx+\int_{-1}^{0}3+6y^2-5y^4\,dy$

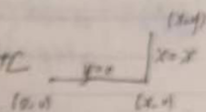
$=\frac{1}{3}x^3-2x\Big|_{-2}^{3}+(18y^3-y^5)\Big|_{-1}^{0}=45+17=62$



6.(1) $P(x,y)=x^2+2xy-y^2$ $Q(x,y)=x^2-2xy-y^2$ $\frac{\partial P}{\partial y}=2x-2y$ $\frac{\partial Q}{\partial x}=2x-2y=\frac{\partial P}{\partial y}$

取 $x_0,y_0=(0,0)$ 则有 $u(x,y)=\int_0^x x^2dx+\int_0^y(x^2-2xy-y^2)dy$

$=\frac{1}{3}x^3+(x^2y-xy^2-\frac{1}{3}y^3)+C=\frac{1}{3}x^3+x^2y-xy^2-\frac{1}{3}y^3+C$



∴ $u(x,y)=\frac{1}{3}x^3+x^2y-xy^2-\frac{1}{3}y^3+C$

(2) $P(x,y)=2x\cos y-y^2\sin x$ $Q(x,y)=2y\cos x-x^2\sin y$ $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ ∴满足条件

∴取 $(x_0,y_0)=(0,0)$ $u(x,y)=\int_0^x 0\,dx+\int_0^y 2y\cos x-x^2\sin y\,dy=y^2\cos x+x^2\cos y-x^2+C$

7 $P(x,y)=\frac{y^2+2xy+ax^2}{(x^2+y^2)^2}$ $Q(x,y)=-\frac{x^2+2xy+by^2}{(x^2+y^2)^2}$ 显然 $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ 时满足条件

取 $(x_0,y_0)=(0,0)$ 则有 $u(x,y)=\int_0^x\frac{y^2+2xy+ax^2}{(x^2+y^2)^2}dx+\int_0^y-\frac{x^2+2xy+by^2}{(x^2+y^2)^2}dy$

$=\int_0^x\frac{ax^2}{x^4}dx-\int_0^y\frac{x^2+2xy+by^2}{(x^2+y^2)^2}dy=\frac{x-y}{x^2+y^2}+C$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ 时 $a=b=-1$

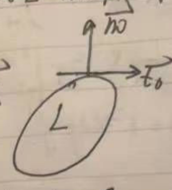
12.(1) $\iint_D v\Delta u\,d\sigma+\iint_D\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)d\sigma=\iint_D v\frac{\partial^2u}{\partial x^2}+v\frac{\partial^2u}{\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}d\sigma$

$=\iint_D\left(\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\right)d\sigma$ $\oint_{L^+}v\frac{\partial u}{\partial x}dy-v\frac{\partial u}{\partial y}dx$

而原式为 $\oint_{L^+}v\frac{\partial u}{\partial x}\cos\langle\vec{n},\vec{i}\rangle ds+v\frac{\partial u}{\partial y}\cos\langle\vec{n},\vec{j}\rangle ds=$ 右侧

原式成立

(2) 设 $\vec{t_0}$ 为 L^+ 的单位切向量，其方向余弦为 $\cos\alpha$, $\cos\beta$ $\vec{n_0}$ 为 L 的外法线方向单位向量



$\vec{n_0}=(a,b)$

$\therefore \begin{cases} \vec{n_0}\cdot\vec{t_0}=0 \\ \vec{n_0}\times\vec{t_0}=\vec{k} \end{cases}$ $\quad \vec{k}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & 0 \\ \cos\alpha & \cos\beta & 0 \end{vmatrix}=(a\cos\beta-b\cos\alpha)\vec{k}$

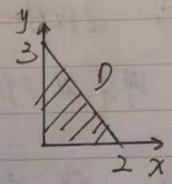
即表示为 $\begin{cases} a\cos\alpha+b\cos\beta=0 \\ a\cos\beta-b\cos\alpha=1 \end{cases}$ $\quad \begin{matrix} a=\cos\beta \\ b=-\cos\alpha \end{matrix}$ $\quad \vec{n_0}=(\cos\beta,-\cos\alpha)$

$$\oint_{L^+}\frac{\partial u}{\partial \vec{n}}ds=\oint_{L^+}\left(\frac{\partial u}{\partial x}\cos\beta-\frac{\partial u}{\partial y}\cos\alpha\right)ds=\oint_{L^+}\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx=\iint_D \Delta u\,dxdy$$

原式得证

8.4

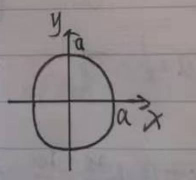
1. $Z=4\left(1-\frac{x}{2}-\frac{y}{3}\right)$ $\quad Z_x=-2$ $\quad Z_y=-\frac{4}{3}$



$\therefore \iint_S \left(2x+\frac{4}{3}y+Z\right)dS=\iint_D 4\sqrt{1+4+\frac{16}{9}}\,d\sigma$

$=\frac{4\sqrt{61}}{3}S_D=\frac{4\sqrt{61}}{3}\times\frac{1}{2}\times2\times3=4\sqrt{61}$

3. $Z=\sqrt{a^2-x^2-y^2}$ $\quad Z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}$ $\quad Z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$



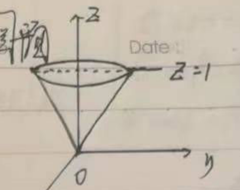
$\therefore \iint_S \left(x+y+\sqrt{a^2-x^2-y^2}\right)\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,d\sigma$

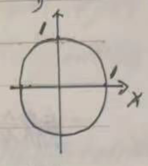
令 $x=r\cos\theta$ $\quad y=r\sin\theta$ $\quad \therefore$ 原式化为 $\int_0^{2\pi}d\theta\int_0^a\left[r(\sin\theta+\cos\theta)+\sqrt{a^2-r^2}\right]\sqrt{\frac{a^2}{a^2-r^2}}\,dr$

$=\int_0^{2\pi}d\theta\int_0^a r|a|\,dr+\int_0^{2\pi}d\theta\int_0^a |a|r(a^2-r^2)^{-\frac{1}{2}}(\sin\theta+\cos\theta)\,dr$

$=\pi a^3+\int_0^{2\pi}(\sin\theta+\cos\theta)d\theta\int_0^a |a|r^2(a^2-r^2)^{-\frac{1}{2}}\,dr=\cancel{2\pi a^2}\ \pi a^3$

D: 平行xoy的圆面和锥形平面 D_1: 锥平面 D_2: 圆面



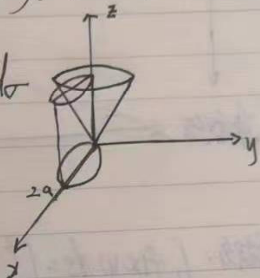


5. $Z=\sqrt{x^2+y^2}$ $Z_x=\frac{x}{\sqrt{x^2+y^2}}$ $Z_y=\frac{y}{\sqrt{x^2+y^2}}$

$\therefore \iint_S (x+y)^2 dS = \iint_D (x^2+2xy+y^2)\sqrt{1+Z_x^2+Z_y^2}\,dxdy$

$= \sqrt{2}\iint_{D_1}(x^2+2xy+y^2)\,dxdy + \iint_{D_2}(x^2+2xy+y^2)\,dxdy = (\sqrt{2}+1)\iint_D(x^2+2xy+y^2)\,dxdy$

$\xrightarrow[y=r\sin\theta]{x=r\cos\theta} (\sqrt{2}+1)\int_0^{2\pi}d\theta\int_0^1\left[r^2+2r^2\sin\theta\cos\theta\right]r\,dr = \frac{\sqrt{2}+1}{4}\int_0^{2\pi}(1+\sin 2\theta)\,d\theta = \frac{\sqrt{2}+1}{2}\pi$



6. $\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$ $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$ $dS=\sqrt{1+Z_x^2+Z_y^2}\,d\sigma=\sqrt{2}\,d\sigma$

原式化为 $\iint_D x(y+\sqrt{x^2+y^2})+\sqrt{x^2+y^2}(x+y)\sqrt{2}\,dxdy$

$=\sqrt{2}\iint_D xy+2x\sqrt{x^2+y^2}+y\sqrt{x^2+y^2}$

$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\left[r^2\sin\theta\cos\theta+2r\cos\theta\cdot r+r\sin\theta\cdot r\right]r\,dr$

$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\sin 2\theta}{2}+2\cos\theta+\sin\theta\right)d\theta\int_0^{2a\cos\theta}r^3\,dr = 4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin\theta\cos^5\theta+2\cos^5\theta+\sin\theta\cos^4\theta)\,d\theta$

$=2\times 8\sqrt{2}a^4\int_0^{\frac{\pi}{2}}\cos^5\theta\,d\theta = \frac{4\times 2}{5\times 3}\times\cancel{\frac{\pi}{2}}1\times 16\sqrt{2}a^4\cancel{\times 2} = \frac{128\sqrt{2}}{15}\cancel{\pi}a^4$

$$\int_0^{\frac{\pi}{2}}\sin^n\varphi\,d\varphi = \int_0^{\frac{\pi}{2}}\cos^n\varphi\,d\varphi = \begin{cases}\frac{(n-1)(n-3)\cdots\cdots 4\times 2}{n(n-2)\cdots\cdots 5\times 3}=\frac{(n-1)!!}{n!!} & n\text{为奇数}\\ \frac{(n-1)(n-3)\cdots\cdots 3\times 1}{n(n-2)\cdots\cdots 4\times 2}\times\frac{\pi}{2}=\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2} & n\text{为偶数}\end{cases}$$