Linear Algebra homework3.4

中文翻译参考：

1 证明 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ 线性独立，而 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ 线性不独立；

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

求解 $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ 或 $A\boldsymbol{x} = \mathbf{0}$。向量 $\mathbf{v}$ 构成矩阵 A 的四个列。

2 (推荐) 从下面向量中找出最大的可能线性独立的向量的个数：

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \; \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \; \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \; \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 证明如果 $a = 0$ 或 $d = 0$ 或 $f = 0$（三种情况），矩阵 U 的列线性不独立：

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 如果第 3 题中的 a, d, f 都是非零的，证明方程 $U\boldsymbol{x} = \mathbf{0}$ 的唯一解是 $\boldsymbol{x} = \mathbf{0}$，且上三角矩阵 U 的列线性独立。

7 如果 $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ 是线性独立的向量，证明向量 $\mathbf{v}_1 = \mathbf{w}_2 - \mathbf{w}_3$，$\mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3$，$\mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2$ 线性不独立。找出一个 $\mathbf{v}$ 的线性组合 $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$($c_1, c_2, c_3$ 不全为零)。求矩阵 A，使得 $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix} A$，试问 A 奇异吗？

8 如果 $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ 是线性独立的向量，证明向量 $\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3$，$\mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3$，$\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$ 也线性独立。（将 $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ 写成 $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ 的线性组合的形式，求解方程，证明方程的解为 $\mathbf{c} = \mathbf{0}$）。

10 在 $\mathbb{R}^4$ 空间的平面 $x + 2y - 3z - t = 0$ 上找到两个线性独立的向量，然后找出三个线性独立的向量，问：为什么不能找出四个线性独立的向量？这个平面是哪个矩阵的零空间？

11 描述由下列向量张成 (span) 的 $\mathbb{R}^3$ 的子空间 (是一条直线，一个平面，或者整个 $\mathbb{R}^3$)。

(a) 两个向量：$(1, 1, -2)$ 和 $(-1, -1, 1)$；

(b) 三个向量：$(0, 1, 1)$，$(1, 1, 0)$ 和 $(0, 0, 0)$；

(c) $\mathbb{R}^3$ 具有整数分量的所有向量 (分量是整数)；

(d) $\mathbb{R}^3$ 具有正的分量的所有向量 (分量为正数)。

12 当 ____ 有解时，向量 $\boldsymbol{b}$ 在由 A 的列张成的子空间中 (填一个方程组，如 $A\boldsymbol{x} = \boldsymbol{b}$)。当 ____ 有解，向量 $\mathbf{c}$ 在矩阵 A 的行空间中。判断对错：若零向量在行空间中，则行线性不独立。

13 求下面四个空间的维数。哪两个空间是一样的？

(a) A 的列空间；

(b) U 的列空间；

(c) A 的行空间；

(d) U 的行空间。

$$A=\begin{bmatrix}1&1&0\\1&3&1\\3&1&-1\end{bmatrix}\text{ 和 }U=\begin{bmatrix}1&1&0\\0&2&1\\0&0&0\end{bmatrix}.$$

15 如果向量 $\mathbf{v}_1,\cdots,\mathbf{v}_n$ 线性独立，则由它们张成的空间的维数是 ____。这些向量构成了该空间的一组 ____。如果这些向量是一个 $m\times n$ 矩阵的列，那么 m____n(比大小)。如果 $m=n$，那么矩阵是 ____(可逆？奇异？)。

16 求下列 $\mathbb{R}^4$ 的子空间的一组基。

(a) 所有分量相等的向量；（分量相等指向量的 4 个分量相等）

(b) 所有分量的和为 0 的向量；

(c) 所有垂直于 $(1,1,0,0),(1,0,1,1)$ 的向量；

(d) 单位矩阵 $I(4\times 4)$ 的列空间和零空间。

17 矩阵 $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$，求 U 的列空间的三组不同的基，求 U 的行空间的两组不同的基。

19 矩阵 A 的列是 n 个属于 $\mathbb{R}^m$ 的向量，如果这 n 个向量线性独立，那么 A 的秩是多少？如果这些向量张成 $\mathbb{R}^m$，A 的秩是多少？如果它们是 $\mathbb{R}^m$ 的基，A 的秩是多少？往前看：秩 r 是 ____ 列的个数。

20 求 $\mathbb{R}^3$ 空间平面 $x-2y+3z=0$ 的一组基。然后求该平面与 xOy 的平面的交线的一组基。然后求所有垂直于该平面的向量构成的子空间的一组基。

23 矩阵 U 等于矩阵 A 的第三行减去第一行得到的矩阵：

$$A=\begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix}\text{ 和 }U=\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}.$$

求两个矩阵的列空间的基，求两个矩阵的行空间的基，求两个矩阵的零空间的基。哪些空间在消元法过程中保持不变。

24 判断对错 (若正确证明，若错误举反例)：

(a) 如果矩阵的列线性不独立，则矩阵的行也线性不独立；

(b) 一个 2×2 矩阵的列空间和行空间是相同的；

(c) 一个 2×2 矩阵的列空间和行空间的维数是相同的；

(d) 矩阵的列是其列空间的一组基。

25 当 c 和 d 等于多少时，下面的矩阵的秩等于 2？

$$A=\begin{bmatrix}1&2&5&0&5\\0&0&c&2&2\\0&0&0&d&2\end{bmatrix}\text{ 和 }B=\begin{bmatrix}c&d\\d&c\end{bmatrix}.$$

26 求下面 3×3 矩阵的子空间的一组基和维数：

(a) 所有的对角矩阵；

(b) 所有的对称矩阵 ($A^\top=A$)；

(c) 所有的反对称矩阵 ($A^\top=-A$)。

29 由下列矩阵张成的 3×3 矩阵的子空间是什么？

(a) 可逆矩阵；

(b) 秩为 1 的矩阵；

(c) 单位矩阵。

30 2×3 矩阵的零空间包括 $(2,1,1)$，求此类矩阵构成的空间的一组基。

36 求由向量 (a,b,c,d)(其中 $a+c+d=0$) 构成的空间 $\mathbf{S}$ 的一组基；求由向量 (a,b,c,d)(其中 $a+b=0,c=2d$) 构成的空间 $\mathbf{T}$ 的一组基。问：空间 $\mathbf{S}$ 和 $\mathbf{T}$ 的交集 $\mathbf{S}\cap\mathbf{T}$ 的维数是多少？

37 如果对于平移矩阵 (shift matrix)S，有 $AS=SA$，证明 A 一定有下面的特殊形式：

$$\text{如果}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$$

$$\text{那么 }A=\begin{bmatrix}a&b&c\\0&a&b\\0&0&a\end{bmatrix}.$$

那么由该类矩阵 A 构成的矩阵空间的维数为 ____。

39 假设 A 是秩为 4 的 5×4 矩阵。当 5×5 矩阵 $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ 可逆时，证明方程组 $A\boldsymbol{x} = b$ 无解。证明当 $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ 奇异时，方程组 $A\boldsymbol{x} = \boldsymbol{b}$ 有解。

Problem Set 3.4

Questions 1–10 are about linear independence and linear dependence.

1 Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \mathbf{0}$ or $A\boldsymbol{x} = \mathbf{0}$. The $\boldsymbol{v}$'s go in the columns of A.

2 (Recommended) Find the largest possible number of independent vectors among

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $U\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. Then the upper triangular U has independent columns.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

6 Choose three independent columns of U. Then make two other choices. Do the same for A.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero. Which matrix A in $[\, \boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \boldsymbol{v}_3 \,] = [\, \boldsymbol{w}_1 \ \boldsymbol{w}_2 \ \boldsymbol{w}_3 \,]\, A$ is singular?

8 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the sums $\boldsymbol{v}_1 = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 + \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ are *independent*. (Write $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$ in terms of the $\boldsymbol{w}$'s. Find and solve equations for the c's, to show they are zero.)

9 Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because ____.

(b) The two vectors v_1 and v_2 will be dependent if ____.

(c) The vectors v_1 and $(0,0,0)$ are dependent because ____.

10 Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–14 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1,1,-1)$ and $(-1,-1,1)$

(b) the three vectors $(0,1,1)$ and $(1,1,0)$ and $(0,0,0)$

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

12 The vector b is in the subspace spanned by the columns of A when ____ has a solution. The vector c is in the row space of A when ____ has a solution.

True or false: If the zero vector is in the row space, the rows are dependent.

13 Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A, (b) column space of U, (c) row space of A, (d) row space of U:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

14 $v + w$ and $v - w$ are combinations of v and w. Write v and w as combinations of $v + w$ and $v - w$. The two pairs of vectors ____ the same space. When are they a basis for the same space?

Questions 15–25 are about the requirements for a basis.

15 If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension ____. These vectors are a ____ for that space. If the vectors are the columns of an m by n matrix, then m is ____ than n. If $m = n$, that matrix is ____.

16 Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space and the nullspace of I (4 by 4).

17 Find three different bases for the column space of $U = \left[\begin{smallmatrix}1&0&1&0&1\\0&1&0&1&0\end{smallmatrix}\right]$. Then find two different bases for the row space of U.

18 Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.

(b) Those vectors (are)(are not)(might be) linearly independent.

(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.

19 The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then? *Looking ahead*: The rank r counts the number of ____ columns.

20 Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

21 Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $A\boldsymbol{x} = \mathbf{0}$ has only the solution $\boldsymbol{x} = \mathbf{0}$ because ____ .

(b) If $\boldsymbol{b}$ is in $\mathbf{R}^5$ then $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because the basis vectors ____ $\mathbf{R}^5$.

Conclusion: A is invertible. Its rank is 5. Its rows are also a basis for $\mathbf{R}^5$.

22 Suppose $\mathbf{S}$ is a 5-dimensional subspace of $\mathbf{R}^6$. True or false (example if false):

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

23 U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

24 True or false (give a good reason):

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

25 For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Questions 26–30 are about spaces where the "vectors" are matrices.

26 Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All symmetric matrices ($A^{\mathrm{T}} = A$).

(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

27 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$.

28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

29 What subspace of 3 by 3 matrices is spanned (take all combinations) by

(a) the invertible matrices?

(b) the rank one matrices?

(c) the identity matrix?

30 Find a basis for the space of 2 by 3 matrices whose nullspace contains $(2, 1, 1)$.

Questions 31–35 are about spaces where the "vectors" are functions.

31 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

32 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace with $y(0) = 0$.

33 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$

(b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

34 Suppose $y_1(x), y_2(x), y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

35 Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

36 Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a + c + d = 0$ and also for the space $\mathbf{T}$ with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

37 If $AS = SA$ for the *shift matrix* S, show that A must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

"The subspace of matrices that commute with the shift S has dimension _____."

38 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$

(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$

(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$

(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$

39 Suppose A is 5 by 4 with rank 4. Show that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible. Show that $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $[\,A \;\; \boldsymbol{b}\,]$ is singular.

40 (a) Find a basis for all solutions to $d^4y/dx^4 = y(x)$.
(b) Find a particular solution to $d^4y/dx^4 = y(x) + 1$. Find the complete solution.

Challenge Problems

41 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 =$ zero matrix, and check entries to prove that c_1 to c_5 must all be zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

42 Choose $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including $\boldsymbol{x}$ itself, span a subspace $\mathbf{S}$. Find specific vectors $\boldsymbol{x}$ so that the dimension of $\mathbf{S}$ is: (a) zero, (b) one, (c) three, (d) four.

43 Intersections and sums have $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$. Start with a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ for the intersection $\mathbf{V} \cap \mathbf{W}$. Extend with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ to a basis for $\mathbf{V}$, and separately with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$ to a basis for $\mathbf{W}$. Prove that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s and $\boldsymbol{w}$'s together are ***independent***. The dimensions have $(r + s) + (r + t) = (r) + (r + s + t)$ as desired.

44 Mike Artin suggested a neat higher-level proof of that dimension formula in Problem 43. From all inputs $\boldsymbol{v}$ in $\mathbf{V}$ and $\boldsymbol{w}$ in $\mathbf{W}$, the "sum transformation" produces $\boldsymbol{v} + \boldsymbol{w}$. Those outputs fill the space $\mathbf{V} + \mathbf{W}$. The nullspace contains all pairs $\boldsymbol{v} = \boldsymbol{u}$, $\boldsymbol{w} = -\boldsymbol{u}$ for vectors $\boldsymbol{u}$ in $\mathbf{V} \cap \mathbf{W}$. (Then $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u} = \mathbf{0}$.) So $\dim(\mathbf{V} + \mathbf{W}) + \dim(\mathbf{V} \cap \mathbf{W})$ equals $\dim(\mathbf{V}) + \dim(\mathbf{W})$ (*input dimension from* $\mathbf{V}$ *and* $\mathbf{W}$) by the Counting Theorem.

dimension of outputs + dimension of nullspace = dimension of inputs.

Problem For an m by n matrix of rank r, what are those 3 dimensions? Outputs = column space. This question will be answered in Section 3.5, can you do it now?