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Multi-rank smart reserves: A general framework for selection and matching diversity goals

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ABSTRACT

We study a problem where each school has flexible multi-ranked diversity goals, and each student may belong to multiple overlapping types, and consumes only one of the positions reserved for their types. We propose a novel choice function for a school to select students and show that it is the unique rule that satisfies three fundamental properties: maximal diversity, non-wastefulness, and justified envy-freeness. We provide a fast polynomial-time algorithm for our choice function that is based on the Dulmage Mendelsohn Decomposition Theorem as well as new insights into the combinatorial structure of constrained rank maximal matchings. Even for the case of minimum and maximum quotas for types (that capture two ranks), ours is the first known polynomial-time approach to compute an optimally diverse choice outcome. Finally, we prove that the choice function we design for schools, satisfies substitutability and hence can be directly embedded in the generalized deferred acceptance algorithm to achieve strategyproofness and stability. Our algorithms and results have immediate policy implications and directly apply to a variety of scenarios, such as where hiring positions or scarce medical resources need to be allocated while taking into account diversity concerns or ethical principles.

1. Introduction

Diversity goals are prevalent in many scenarios including student intake, hiring of employees, public housing, and rationing of scarce medical resources. While classical centralized matching algorithms have received widespread success, they do not directly apply to two-sided matching under diversity concerns such as those pertaining to proportionality goals or reaching minimum quotas for types. In this paper, we present a compelling solution to selection and matching problems under diversity concerns.

We first focus on the issue of how a single school should make selection decisions regarding students who may belong to multiple types such as being from disadvantaged groups. The school has diversity goals with respect to representation of the student types. The goal is to select the students in a way that takes into account the priority ordering of the school over students as well as the diversity goals. When there are multiple schools, students care about which school they are selected at. In that case, our goal is to make decisions about which student is assigned to which school while taking into account the preferences of the individual students, as well as the priorities and the diversity goals of the schools.

In both theory and practice, the diversity goals are typically achieved by setting minimum and maximum target representation of students of various types. Existing approaches either do not optimally achieve the diversity goals that are considered, or do so

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for restricted classes of diversity goals (such as those pertaining to a single rank of reserves, which we will explain later). In view of the limitations of existing approaches towards diverse matchings, there is a need for a more general model of controlled school choice that can capture flexible diversity goals, and meaningful methods that achieve these diversity goals while allocating resources to individuals.

In this paper, we propose a flexible framework for specifying a rich class of diversity goals. For the framework, we design a novel school choice function and a fast combinatorial algorithm for the function that draws connections from graph theory and theoretical computer science. We characterize the choice function as the only one satisfying three fundamental properties: maximal diversity, non-wastefulness, and justified envy-freeness. It also satisfies important consistency and incentive properties that make it suitable to be embedded in the broader problem of matching students to multiple schools (with their own priorities and diversity goals). Even when there is one school, the selection process can be framed as a matching problem whereby a student is either unmatched or is matched to some school seat that is reserved for a particular type satisfied by the student.

Our central contribution is the design of a choice function and a corresponding algorithm satisfying desirable properties in a two-sided matching problem with diversity goals. In addition to the school choice problems, our study has several other applications. When there is one school, the problem we study is equivalent to finding a diverse committee of candidates under a mutually agreed ranking over the candidates. If school seats are replaced by other scarce and sought-after resources such as spots for vaccine treatment, our model and results apply as well to them.

Below, we discuss our contributions in more detail.

- Our first contribution is designing a general model of matching under diversity goals that capture various diversity objectives including proportionality or egalitarian concerns. The key idea behind our approach is to partition the reserved seats for various types into different ranks. A student is only eligible for a seat, if the student satisfies a type corresponding to the seat. When selecting students at a school, maximally diversity is achieved if the maximum number of students are matched to eligible seats of rank 1, and conditional on that further ranked seats are filled up. We are interested in identifying subsets of students that give rise to matchings capturing a 'desirable' distribution of the reserved seats. We characterize our diversity objective via size-constrained rank maximal matchings of an underlying 'ranked reservation graph'. Our flexible diversity approach not only captures upper quota constraints, it also captures other diversity goals such as target proportions of types.
- We design a new choice function for each school that takes into account the priority ordering of the school over students as well as its diversity goals. The choice function is based on a key link between flexible diversity goals and the concept of rank maximality (that is well-studied in theoretical computer science). We demonstrate the desirability of the choice function by showing that it is characterized by three fundamental properties: maximal diversity, non-wastefulness, and justified envy-freeness. We also show that maximal diversity cannot be achieved by a natural adaptation of the horizontal envelope choice rule in [53] to the case of multi-ranks.
- We provide a fast polynomial-time algorithm for our choice function based on the Dulmage Mendelsohn Decomposition and combinatorial insights into constrained rank-maximal matchings that are required to match a certain subset of vertices. Our algorithms are the first known polynomial-time approaches to compute a maximally diverse choice outcome even when there are two ranks that capture minimum and maximum quotas for the number of admitted students of given types.
- We show that our new choice function generalizes existing choice functions (that handle more restricted forms of diversity). We prove via Berge's lemma that maximal diversity implies the utilization of the maximum number of reserved seats. Our choice function also incentivizes applicants to report all of their types.
- We prove that our choice function satisfies substitutability and can hence be directly embedded in the generalized deferred acceptance algorithm to achieve strategyproofness and stability.

Since our algorithm is the unique rule satisfying fundamental axiomatic properties, we make a compelling case for its adoption in many important matching problems where diversity is a central concern.

2. Related work

The interest in affirmative action concerns has a long history in matching and market design. The topic of school choice with diversity constraints has been referred to as *controlled school choice* [26]. In one of the seminal works on school choice, racial and gender balance concerns were already alluded to by Abdulkadiroğlu and Sönmez [2]. In another early work, Abdulkadiroğlu [1] focused on school choice with affirmative action in detail and imposed hard capacities or quotas on the number of admitted students of given types.

Typically, in controlled school choice, each school imposes a maximum quota and a minimum quota on each type [30,40,41]. Ehlers et al. [26] considered controlled school choice with lower and upper quotas for the types when students have exactly one type. They explored the implications of treating diversity quotas as hard bounds and soft bounds. For the case of soft quotas, they proposed a choice function of schools via dynamic priorities in which the students who belong to a type that is undersubscribed, are prioritized. Their dynamic priorities approach has inspired several followup works for more complex models. In a seminal work on choice functions for affirmative action, Echenique and Yenmez [25] examined the structure of choice functions that satisfy substitutability. They also assumed that each student has at most one type.

In reality, each student may satisfy multiple 'overlapping' types. Overlapping types have been considered in recent papers and deployed applications in the past few years including those in Brazil, Chile, Israel, and India (see, e.g., [6,16,22,43,28]).

Table 1
Literature on school choice with affirmative action.

	Types per student		Convention		Bounds		Diversity ranks			Algorithmic Approach	
	1	any	many-1	1-1	Hard	Soft	1	≤ 2	any	Dynamic Priorities	Smart Reserves
Huang [33], Biró et al. [19]		✓	✓		✓			✓			
Ehlers et al. [26]	✓			✓		✓		✓		✓	
Echenique and Yenmez [25]	✓			✓	✓	✓					
Abdulkadiroğlu [1]	✓			✓	✓		✓				
Kominers and Sönmez [41]	✓			✓		✓	✓				
Kurata et al. [43], Correa et al. [22], Aygün and Bó [4]		✓		✓		✓		✓		✓	
Aygün and Turhan [8], Aygun and Turhan [7]		✓		✓		✓		✓		✓	
Aygün and Bó [4]		✓		✓		✓		✓		✓	
Gonczarowski et al. [28], Aziz et al. [12]		✓	✓			✓		✓		✓	
Aziz [9], Aziz and Sun (2020)		✓	✓			✓			✓	✓	
Sönmez and Yenmez [52], Pathak et al. [48]		✓		✓		✓	✓				✓
Our paper		✓		✓		✓			✓		✓

When individuals have multiple types, there are two natural conventions, namely *one-for-all* and *one-for-one*, depending on how many reserved seats an individual takes up (to be consistent with the majority of text) or how reserved positions are accounted for [53]. Under the one-for-all convention, an individual takes the reserved seats of all types she satisfies [9,12,28]. For example, an aboriginal girl could take up one seat reserved for girls and one seat reserved for aboriginals. The one-for-all convention has been employed in algorithms for the Israeli Mechinot matching system [28]. Optimally meeting diversity requirements is NP-hard under the one-for-all convention [19]. Under the one-for-one convention, each individual takes up one of the reserved seats of only one of the types they belong to. This convention has the 'more widespread interpretation' [53]. Following our previous example, an Aboriginal girl could either take up a seat reserved for girls or a seat reserved for Aboriginal persons but not both. In this sense, the convention incorporates a stronger sense of affirmative action because it leaves open the utilization of the second seat.

The one-for-one convention has been considered in the case where students either have strict preferences over reserved seats of different types [6,43], or the indifferences are broken through fixed tie-breaking [16,22], or priorities are dynamically adjusted based on the count of matched students across different types [26].[1] In all of the above cases, the decision on which type a student should use, is made in a greedy or sequential manner and hence this greedy approach may not maximally satisfy diversity goals when students have overlapping types. A similar approach has been used by Kominers and Sönmez [41,42], Aygün and Bó [4], Aygün and Turhan [8] and Aygun and Turhan [7]. In contrast, our framework captures new diversity goals and our methodology optimally achieves these goals via a so called 'smart reserves approach'.

Sönmez and Yenmez [52] were the first to pioneer the approach of using '*smart reserves*' in the context of controlled school choice, where the actual decision of which type a student is going to use is used more flexibly. This allows us to maximally achieve the utilization of reserved seats for types. The underlying idea has also been applied to allocating medical resources to patients who satisfy various categories (Section 4, [48]). Aziz and Brandl [11] consider a generalized model in which the reserved seats heterogeneous priorities over the agents. They presented an algorithm with desirable efficiency, fairness, and strategic properties. Although the smart reserve approach of Sönmez and Yenmez [52] is compelling, their rule and framework have some limitations. In particular, their work is limited to a single rank of reserves in which types whose reserves are not filled can be viewed as under-subscribed. Their approach can be viewed as softly respecting minimum quotas but is unable to capture a standard requirement in many affirmative action models, where both a minimum and a maximum quota are present for each type. In school choice models where each type has a minimum and maximum quota, types that have not reached their minimum quota have a higher priority than the types which have not reached their maximum quota. Consequently, the model of Sönmez and Yenmez [52] does not generalize the model of Ehlers et al. [26], who considered one type per student and allowed upper quotas in addition to lower quotas for types. One rank of reserves is also unable to capture a wide umbrella of diversity goals such as enforcing ratios among types or expressing interleaving goals such as follows: "only care about the types t_3 and t_4 once the reserved seats of t_1 and t_2 are used up."

The literature on school choice with diversity concerns is naturally divided across several axes: (1) individual students having a single type versus students having multiple types; (2) two types versus more than two types (3) hard diversity constraints versus soft diversity constraints; (4) students types accounted under the one-for-all convention versus one-to-one convention; (5) methods based on dynamic priority versus a smart reserve approach. Table 1 classifies some of the main works on controlled school choice across these axes. The distinction between the one-for-all and one-to-one conventions was first explicitly made by Sönmez and Yenmez [52]. Smart reserves approaches are meaningful when students may have multiple types and the one-to-one convention is assumed.

We discuss some approaches using the one-for-all convention. Gonczarowski et al. [28] study the Israeli "Mechinot" gap-year problem in which each student has multiple types and schools impose soft minimum quotas and hard maximum quotas. They propose a greedy choice function for schools where students are chosen one by one based on priority orderings as long as the student has some type that helps to meet the diversity goals. Their model is different from ours as it assumes the one-for-all convention. In their setting, it is impossible to achieve fairness and strategyproofness properties simultaneously. Aziz et al. [12] proposed a strategyproof

[1] The concept of dynamic priority operates as follows: The highest precedence is assigned to types that have not yet met the minimum quotas, followed by types that have not reached the maximum quotas.

mechanism designed to attain both non-wastefulness and a form of weak fairness. This is achieved by eliminating justified envy solely among students who share the same set of types. Baswana et al. [16] designed and deployed an algorithm for Indian engineering colleges. They used a heuristic to deal with non-nested common quotas and their algorithm does not guarantee a fair outcome. Aziz [9] considers the one-for-all convention for diversity and proposed an algorithm for a choice rule that uses minimum quotas to specify diversity goals. Bredereck et al. [20] examined the complexity of multiwinner voting with diversity constraints under the one-for-all convention.

There has also been some recent work on matchings with constraints on the ratios of types [47]. The paper discusses how setting lower quotas on types may not be sufficient to achieve target ratios especially when school seats are not fully used. The paper assumes that each student/doctor has exactly one type. Our diversity goals are more flexible (capturing other objectives such as minimum and maximum quotas), and our algorithmic solution and axiomatic focus is different as well.

Sönmez and Yenmez [53] focus on vertical and horizontal reservations that are distinctive to affirmative action in Indian college admissions. Horizontal reservations are equivalent to a single rank of reserves in our framework, whilst vertical reservations are treated as set-aside seats for students who are not selected on ‘open merit’. Dur et al. [24] studied a class of mechanisms in which reserve seats are processed in a sequential manner. They highlighted the impact of the order in which reserves are processed.

Our model also bears some similarities with the hospital-resident matching problem with regional constraints [36,37], in which students are viewed as doctors, school seats are viewed as hospitals and schools are viewed as regions. However, in the hospital-resident matching problem, the distributional constraints are imposed on the number of doctors matched to different regions, rather than on the proportional composition of types of doctors. Aziz et al. [13] focus on the connection between matching with diversity constraints and matching with regional quotas where diversity constraints are set as hard bounds. However, they do not focus on the mechanism design of choice function of schools that maximize diversity goals.

Benabbou et al. [17] study the Singapore public housing program where ethnic quotas are imposed to each block: the percentage of each ethnic group cannot exceed a certain degree. Each agent has preferences over apartments within a block, whilst in our model, each student has preferences over schools rather than school seats within the school. Furthermore, they focus on the tradeoff between diversity and social welfare, while our attention is drawn to the maximization of diversity goals for each school. Ahmadi et al. [3] and Dickerson et al. [23] consider an optimisation-based approach to diverse team formation but do not take preferences and priorities into account.

There is a large body of AI literature that are related to matching problems under preferences. Goto et al. [29] focuses on mechanism design for school choice where schools are partitioned into regions and both regional minimum and maximum quotas are imposed as distributional constraints. Hamada et al. [31] considers a school choice model where some students are initially enrolled but prefer to be transferred, and minimum quotas are imposed on schools as distributional constraints. Suzuki et al. [56] further extends this model by generaling minimum quotas to M-convex distributional constraints and new strategy-proof algorithms are proposed. Liu et al. [44] considers a student-project-resource matching-allocation problem, where students have preferences over projects and the projects have preferences over students. Ismaili et al. [35] studies the computational complexity of deciding the existence of stable matchings under the budget constraints with respect to both coalitional stability and pairwise stability. Zhang et al. [57] proposes a fair and strategyproof mechanism under a union of symmetric M-convex constraints where non-wastefulness is incompatible with fairness in general. Sun et al. [55] studies the Japanese Residency Matching Program (JRMP) where hospitals form disjoint regions and both hospitals and regions are subject to quotas. They propose two algorithms and demonstrate their superiority over the currently deployed algorithm in JRMP, both theoretically and empirically. Sun et al. [54] considers a pairwise organ exchange problem among groups and study how to match a maximum number of pairwise compatible patient-donor pairs in a fair and individually rational way. Aziz and Sun [15] presents a general model of school choice with flexible diversity goals and specialized seats, as well as a polynomial-time algorithm that satisfies trategyproofness and stability under certain conditions. Cho et al. [21] introduces a new mechanism design problem over social networks where agents have incentives to invite their neighbours to the market. Aziz et al. [10] gives a survey on developments in the field of two-sided matching under various constraints, such as region, diversity, multi-dimensional capacities, and matroids. In two recent works [32,49], rank-maximal matching was used to elicit preferences of agents in the house allocation model (i.e., allocating indivisible objects among agents), which are different from our focus.

3. School choice with multi-rank diversity goals

An instance I of the school choice problem with soft diversity goals consists of a tuple $(S, C, q_C, T, \succ_S, \succ_C, \eta)$ where S and C denote the set of students and schools, respectively. The capacity vector $q_C = (q_c)_{c\in C}$ assigns each school c a capacity q_c. We denote by T the set of types. We overload the term to also capture the information about the types of each student. For each student s, let $T(s) \subseteq T$ denote the subset of types to which student s belongs. If $T(s) = \emptyset$, then student s does not have any privileged type. The vector $\eta = (\eta_1, \dots, \eta_{|C|})$ specifies the diversity goals of each school c. We will discuss diversity goals in detail later.

Let $\mathcal{X} = S \times C$ denote the set of possible student-school pairs. We also refer to these pairs as contracts. Given any $X \subseteq \mathcal{X}$, let X_s be the set of student-school pairs involving student s and let X_c be the set of student-school pairs involving school c.

Each student s has a strict preference ordering $\succ_s$ over $C \cup \{\emptyset\}$ where $\emptyset$ is a null school representing the option of being unmatched for student s. A school c is *acceptable* to student s if $c \succ_s \emptyset$ holds. Let $\succ_S = (\succ_{s_1}, \dots, \succ_{s_n})$ be the preference profile of all students S. Each school c has a strict priority ordering $\succ_c$ over $S \cup \{\emptyset\}$ where $\emptyset$ represents the option of leaving seats vacant for school c. Similarly, a student s is *acceptable* to school c if $s \succ_c \emptyset$ holds. Let $\succ_C = (\succ_{c_1}, \dots, \succ_{c_m})$ be the priority profile of all the schools. The priority ordering of the school could be based on the entrance exam scores, or in the case of automated hiring, on some objective measure that captures

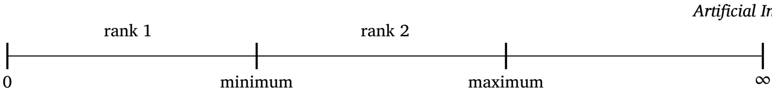

Fig. 1. An interpretation of minimum and maximum quotas.

the suitability of the applicants. Our results continue to hold if we assume weak preferences and priorities: we can artificially break the ties with fixed tie-breaking to induce strict orders.

An outcome (or a matching) X is a subset of $\mathcal{X}$. An outcome X is **feasible** (under soft bounds) for instance I if i) each student s is matched with at most one school, i.e., $|X_s| \leq 1$, and ii) the number of students matched to each school c does not exceed its capacity, i.e., $|X_c| \leq q_c$. A feasible outcome X is **individually rational** if each contract $(s, c) \in X$ is acceptable to both student s and school c. Please note that we treat diversity goals as soft constraints instead of hard feasibility constraints. For instance, it may be impossible to achieve all minimum quotas without violating individual rationality. Without loss of generality, we focus on acceptable and feasible matchings.

Except for Section 9, we will focus on how a single school makes decisions. A *choice function* of a school takes as input a given set of students $S' \subseteq S$ and selects a set of students $S^* \subseteq S'$ as output. In our setup, the choice function Ch_c of a school c will take into account its priority order $\succ_c$ and its diversity goals η_c to make the choice.

Diversity goals

In most of the literature on school choice with diversity constraints, schools typically impose minimum q_t^{min} and maximum quota q_t^{max} on each type t. We can interpret this type of diversity goal as using two ranks of quotas: for a given type t, rank 1 corresponds to the interval $[0, q_t^{min})$ and rank 2 corresponds to the interval $[q_t^{min}, q_t^{max})$. An implicit third rank is for types whose maximum quotas have already been met (Fig. 1).

In our model, we allow each school c to impose multiple ranks of quotas on each type. The parameter η_c specifies for each type t and rank j, the reserved quota $\eta_{c,t}^j$. We will denote by r the maximum number of ranks. A student can only take up a reserved seat of a type that she satisfies.[2] If some reserved seats remain unallocated, then in order to ensure maximal utilization of resources, any eligible student can take up a vacant seat. When a set of students is viewed as taking up reserved seats for the types they satisfy, the seats of earlier ranks are taken up before seats of later ranks are considered.[3]

Here, ranks are used to measure the importance of diversity goals [14]: the smaller the rank is, the more important the quota of a certain type is. For instance, consider two types t_1 and t_2 whose numbers of matched students at school c fall into rank i and j, respectively. If $i < j$, then the quota η_{c,t_1}^i of type t_1 and rank i is more important than the quota η_{c,t_2}^j of type t_2 and rank j to school c in terms of satisfying diversity goals. If $i = j$ holds, then both quotas η_{c,t_1}^i and η_{c,t_2}^j are tied.

Example 1 *(Example of a problem instance).* Consider the setting in which there are four students $S = \{s_1, s_2, s_3, s_4\}$ and two schools $C = \{c_1, c_2\}$. The type profile of the students is $T(s_1) = \{t_1, t_2\}$, $T(s_2) = \{t_1\}$, $T(s_3) = \emptyset$, $T(s_4) = \{t_3\}$. The capacity of c_1 is $q_{c_1} = 3$ and the capacity of c_2 is $q_{c_2} = 1$. School c_1 has diversity goals specified as follows: $\eta_{c_1,t_1}^1 = 1$, $\eta_{c_1,t_2}^1 = 1$, $\eta_{c_1,t_3}^1 = 0$, $\eta_{c_1,t_3}^2 = 1$. The term $\eta_{c_1,t_1}^1 = 1$ indicates that school c_1 has reserved 1 seat of rank 1 for someone satisfying t_1. The term $\eta_{c_1,t_2}^1 = 1$ indicates that school c_1 has reserved 1 seat of rank 1 for someone satisfying t_2. The term $\eta_{c_1,t_3}^2 = 1$ indicates that school c_1 has reserved 1 seat of rank 2 for someone satisfying t_3. On the other hand, school c_2 has no diversity goals. The preferences of the students are unanimous with c_1 preferred over c_2. The priorities of both schools are unanimous: $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4$ for $c \in C$.

The interpretation of the diversity goals of school c_1 is as follows: school c_1 wants to match as many students to slots of rank 1 as possible. One of the rank 1 slots is reserved for a student of type t_1 and one is reserved for a student of type t_2. Conditional on optimizing the number of students who can be matched to rank 1 slots, the school c_1 wants to next match some student of type t_3. Another interpretation of the diversity goals is in the form of setting quotas. The minimum and maximum quota for t_1 and t_2 is 1 whereas the minimum quota for t_3 is 0 and the maximum quota for t_3 is 1. ⋄

Below we further explain how our framework can capture diversity goals that the method of using a single layer of reserves used by Sönmez and Yenmez [52] cannot. Consider an example with both minimum and maximum quotas. Suppose that type t_1 has a soft minimum quota of 1 and a soft maximum quota of 2. On the other hand, t_2 has a soft minimum quota of 0 and a soft maximum quota of 3. When applying the approach suggested by Sönmez and Yenmez [52], the difficulty lies in establishing the ideal number of reserves for t_1 and t_2 due to the inherent two-layered nature of diversity goals.

Note that multiple ranks provide us with the ability to describe more complicated diversity goals. For instance, only using minimum and maximum quotas cannot capture the idea of "proportionality", a common diversity goal.

Example 2 *(Single rank reserves may not achieve proportionality).* Suppose each student is associated with one of three types t_1, t_2, t_3, and the percentages of students of each type are 30%, 30%, 40%, respectively. Consider one school c with capacity 100 which imposes

[2] This assumption is referred to as 'compliance with eligibility requirements' in the literature [48].

[3] This order of filling up seats is consistent with existing approaches in which minimum quotas are reached first before targeting the maximum quotas.

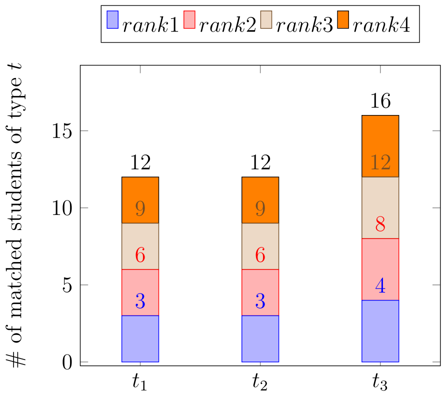

Fig. 2. Proportional goals in Example 2.

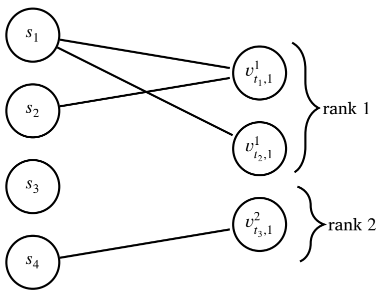

Fig. 3. The ranked reservation graph of school c_1 in Example 1.

minimum quotas 30, 30 and 40 on type t_1, t_2, t_3, respectively. Note that the numbers of students of each type who apply to school c do not match their percentages in the population. Say 15, 60, 60 students of type t_1, t_2, t_3 respectively apply for the school c, and the priority ordering of school c is consistent with students' types, say school c prefers students of type t_1, to students of type t_2, to students of type t_3.

Then one possible matching for school c with respect to minimum quotas could be 15, 45, 40, where each integer denotes the number of students of type t_1, t_2, t_3, respectively. However, another matching with 15, 37, 48 students of each type matched, seems better for achieving a diversity balance under the proportionality constraints, in which the numbers of matched students of type t_2 and t_3 are proportional in the ratio of 3:4. If we use our framework of defining diversity goals, we can try to achieve these. We illustrate how to set proportionality goals in Fig. 2.

Before we formally specify how the reserved seats information is used to define diversity goals, we give some intuition. Informally speaking, the diversity goals are achieved as follows. We want as many rank 1 seats to be filled by students who satisfy the corresponding types and conditional on that we want as many rank 2 seats filled, and so on. In the next section, we will give a formal description of how the reserved quotas $\eta^j_{c,t}$ give rise to the diversity goals discussed above. We will use the notion of a *ranked reservation graph*. This mathematical object is central to formally specifying diversity goals as well as other axiomatic properties.

4. A graph-theoretic view of diversity

Given a set of students $S' \subseteq S$ and a school c with reserved quotas η_c, a corresponding *ranked reservation graph* $G = (S' \cup V, E, \eta_c)$ is a bipartite graph whose vertices are partitioned into a set of students S' and a set of reserved seats V. Each reserved seat $v^j_{t,i} \in V$ has a rank j, a type t and an index i. For each rank j and each type t, we create $\eta^j_{c,t}$ reserved seats in G. The edge set E is specified as follows. There is an edge between a student s and a reserved seat $v^j_{t,i}$ if student s belongs to type t, i.e., $t \in T(s)$. Each edge $(s, v^j_{t,i})$ has a *rank* j corresponding to the rank j of the reserved seat $v^j_{t,i}$. We refer to all edges with rank j as j-ranked edges. The ranks of the edges lead to a natural partition of the edges: $E = E_1 \cup E_1 \cup \cdots \cup E_r$ where E_j denotes the set of the j-th ranked edges. Note that the ranked reservation graph is a generalization of the reservation graph used by Sönmez and Yenmez [52] in which all edges only have one rank. A ranked reservation graph G is a special ranked bipartite graph in which all edges incident to the same reserved seat $v^j_{t,i}$ have the same rank j.

Example 3 *(Example of a ranked reservation graph)*. Consider the problem instance in Example 1. We construct the corresponding reservation graph as shown in Fig. 3.

Before we proceed further, we formally specify important terms and concepts from matching theory.

4.1. Matching theory preliminaries

Given a graph G, a matching M in G is a set of pairwise non-adjacent edges such that no two edges have common vertices. Given a matching M, an **alternating path** is a path that begins with an unmatched vertex and whose edges belong alternately to the matching and not to the matching. An **augmenting path** is an alternating path that starts from and ends on unmatched vertices.

Let P denote an alternating path with respect to matching M. Then $M \oplus P = (M \setminus P) \cup (P \setminus M)$ denotes the symmetric difference of the two sets of edges, which is a new matching where the edges from $P \setminus M$ are matched while the edges from $M \cap P$ are not matched.

Consider a ranked bipartite graph $G = (A \cup B, E)$ in which each edge is assigned a rank. Suppose the edge set E is partitioned into r disjoint sets, i.e., $E = E_1 \cup E_2 \cup \cdots \cup E_r$ where E_i represents the set of edges of rank i. The **signature** $\rho(M) = \langle x_1, x_2, ..., x_r \rangle$ of a matching M in G is a tuple of integers where each element x_i represents the number of edges of rank i in M.

For a ranked bipartite graph, we compare the signatures of matchings in a lexicographical manner. A matching M' with $\rho(M') = \langle x_1, \cdots, x_r \rangle$ is **strictly better** than another matching M'' with $\rho(M'') = \langle y_1, \cdots, y_r \rangle$, if there exists an index $1 \le k \le r$ s.t. for $1 \le i < k$, $x_i = y_i$ and $x_k > y_k$. A matching M' is **weakly better** than another matching M'' if M'' does not provide strictly better diversity than M'. Let $M' \succ_{lex} M''$ denote that M' is strictly better than M'' and let $M' \succsim_{lex} M''$ denote that M' is weakly better than M''.

A matching M in a ranked bipartite graph G is **rank-maximal** if M is weakly better than any other matching M' in G. A rank maximal matching can be computed in polynomial time [34,45].

4.2. Maximally diverse matchings of a reservation graph

For a ranked reservation graph, a matching M' provides **strictly better diversity** than M'' if $M' \succ_{lex} M''$; and a matching M' provides **weakly better diversity** than M'' if $M' \succsim_{lex} M''$. A matching in a ranked reservation graph G is *maximally diverse* if it provides weakly better diversity than all other matchings of G. Alternatively, a matching in a ranked reservation graph G is *maximally diverse* if it is rank-maximal.

In this paper, one of our main goals is to formalize a way for a school to select a set of students while keeping diversity goals in mind. In the next section, we show how the concept of diverse matchings leads to a natural definition of a diverse set of students. Informally speaking, we will focus on selecting those sets of students that allow for diverse matchings in the underlying ranked reservation graph.

5. Desirable properties of a choice function

A choice function Ch_c of school c takes as input an $S' \subseteq S$, q_c, η_c, and $\succ_c$, and selects a set of students $S^* \subseteq S'$. We will assume that the choice function satisfies feasibility requirements: $|Ch_c(S', q_c, \eta_c, \succ_c)| \le q_c$.

Next, we extend the idea of comparing the signatures of two matchings in the reservation graph to that of comparing the diversity satisfaction of two subsets of students. The intuition is to check which subset of students leads to better utilization of the reserved seats in the ranked reservation graph.

Given a school c and two subsets of students $S_1, S_2 \subseteq S'$ with $|S_1|, |S_2| \le q_c$, we say that S_1 provides *strictly (weakly) better diversity* than S_2, if there exists a matching M in the ranked reservation graph $G = (S' \cup V, E, \eta_c)$ such that

1. the matched students in M are a subset of S_1;
2. M provides strictly (weakly) better diversity than any matching M' of G in which the matched students in M' are a subset of S_2.

The weakly better diversity relation is a total order on the set of sets of students. We use it to define a property called maximal diversity of a choice function.

Definition 1 *(Maximal Diversity).* A choice function Ch_c satisfies *maximal diversity* if for each instance $(S', q_c, \eta_c, \succ_c)$, it selects a feasible set of students $Ch_c(S', q_c, \eta_c, \succ_c)$ such that $Ch_c(S', q_c, \eta_c, \succ_c)$ provides weakly better diversity than all feasible subsets $S'' \subseteq S'$.

Non-wastefulness stipulates that either the school capacity should be saturated or all valid applicants must be accepted.[4]

Definition 2 *(Non-wastefulness).* A choice function satisfies non-wastefulness if a student is rejected only if the school capacity is reached.

[4] It can be assumed without loss of generality that we only consider those applicants that meet minimal acceptance requirements.

A student s has justified envy towards s' if i) s has a higher priority and ii) replacing s' with s does not compromise on the diversity goals.

Definition 3 *(Justified Envy-freeness).* For an instance $(S' \subseteq S, q_c, \eta_c, \succ_c)$, if a school c selects $Ch_c(S', q_c, \eta_c, \succ_c) = S^* \subseteq S'$, we say that a student $i \in S' \setminus S^*$ has justified envy towards another student $j \in S^*$ if $i \succ_c j$ and $S^* \cup \{i\} \setminus \{j\}$ provides weakly better diversity than S^*. The choice function Ch_c of a school c satisfies *justified envy-freeness,* if for each instance $(S', q_c, \eta_c, \succ_c)$, it selects a set of students $Ch_c(S', q_c, \eta_c, \succ_c) = S^* \subseteq S'$ s.t. there does not exist any student $i \notin S^*$ who has justified envy towards any student $j \in S^*$.

Note that whereas justified envy-freeness is referred to as a property of matchings in school choice, the definition above is a property of a choice function of an individual school. If an individual school does not satisfy the property in Definition 3, then the justified envy-freeness of matchings is also not satisfied. Note that a choice function Ch_c of school c that completely ignores diversity goals and selects the q_c highest priority students (or all of them when there are less than q_c applicants) satisfies justified envy-freeness.

Next, we introduce a property called *Maximal Reserves Utilization* that concerns whether the number of used reserved seats is maximal in the corresponding ranked reservation graph.

Given a school c and two subsets of students $S_1, S_2 \subseteq S$ with $|S_1|, |S_2| \leq q_c$, we say that S_1 *provides strictly (weakly) better reserves utilization* than S_2, if there exists a matching M in the ranked reservation graph $G = (S \cup V, E)$ such that

1. the matched students in M are a subset of S_1;
2. the size of M is strictly (weakly) larger than any matching M' of G in which the matched students in M' are a subset of S_2.

Definition 4 *(Maximal Reserves Utilization).* A choice function Ch_c satisfies *maximal reserves utilization* if for each instance $(S', q_c, \eta_c, \succ_c)$, it selects a feasible set of students $Ch_c(S', q_c, \eta_c, \succ_c) = S^*$ such that S^* provides weakly better reserves utilization than any feasible subsets $S'' \subseteq S'$.

Finally, we consider another property of choice functions. An algorithm is *privilege monotonic* if, under any circumstances, a student does not gain an advantage by reporting only a restricted subset of their true types. More precisely, there should be no scenario in which a student is selected when reporting a strictly smaller subset of their types, but is not selected when all of their types. The term has been used previously (see e.g. Aygün and Bó [4]).

6. A new choice function

Next, we design a new choice function for schools and show that it satisfies the compelling properties discussed above. The choice function relies on computing size-constrained *rank maximal* matchings of the corresponding reservation graph.

The choice function works as follows. We first compute the corresponding reservation graph G that includes the ranks of the edges. We then compute the signature of the rank-maximal matching in G constrained to the school's capacity. At this point we do not make a decision on which individuals are to be chosen. We simply require that the students selected should give rise to the same signature. We go down the priority list of students and check whether a given student s can be matched along with the previously selected students so that we still can get the same signature from some matching. If this is possible, we select s. Otherwise, we do not select s. The process continues until we have exhausted the priority list and we have a set of students who can all be matched in a rank-maximal matching of G. At this point, if the school's quota is not exceeded, we again go through students from the start of the priority list and add unselected students if the quota is still not exceeded. The algorithm is specified formally as Algorithm 1.

Input: $S' \subseteq S$, q_c, η_c, $\succ_c$.
Output: A set of students $S^* \subseteq S'$

1 Selected students $S^* \leftarrow \emptyset$
2 Construct the corresponding ranked reservation graph $G = (S' \cup V, E, \eta_c)$
3 **for** student $s \notin S^*$ down the list in $\succ_c$ **do**
4   **if** there exists a matching in G of size at most q_c that satisfies the following two conditions
    1.   it is rank maximal among all matchings in G of size at most q_c
    2.   it matches all students in $S^* \cup \{s\}$
     then
5   Add s to S^*
6 **for** student s down the list in $\succ_c$ **do**
7   **if** $|S^*| < q_c$ and $s \notin S^*$ **then**
8   Add s to S^*
9 **return** S^*

Algorithm 1: Choice function Ch_c of school c.

We next illustrate how Algorithm 1 works through the following Example. The details of how to check whether there exists a rank maximal matching including a certain set of students (i.e., line 4) is explained later.

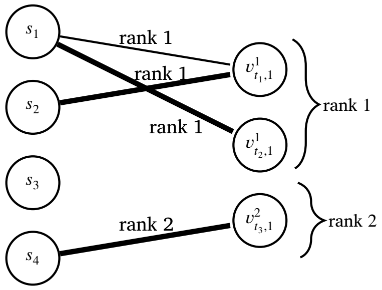

Fig. 4. The maximal diversity matching for ranked reservation graph of school c_1 in Example 1 is illustrated by thick bold edges.

Example 4. Consider the problem instance in Example 1 again. Suppose all the four students apply to school c_1. The corresponding maximal diversity matching is shown in Fig. 4, which is the unique rank maximal matching of the reservation graph. Then the school c_1 first selects s_1, then selects s_2, and finally chooses s_4. As the school capacity is fulfilled, student s_3 is rejected by school c_1.

7. Axiomatic properties of the new choice function

In this section, we analyze the fundamental properties satisfied by our choice function. We show that a choice function satisfies maximal diversity, justified envy-freeness, and non-wastefulness if and only if it selects the same set of students as the choice function in Algorithm 1 does. The following lemmata show that our choice function satisfies three important axioms.

Lemma 1. *The choice function in Algorithm 1 satisfies non-wastefulness.*

Proof. Non-wastefulness is ensured in the last for loop in Algorithm 1 where students keep getting selected until either no student is left or the school quota is reached. □

Lemma 2. *The choice function in Algorithm 1 satisfies maximal diversity.*

Proof. For any given set of students $S' \subseteq S$, Algorithm 1 iteratively selects a set of students $S^* \subseteq S'$ such that there exists a rank maximal matching of size at most q_c in the corresponding reservation graph that includes all students from S^*. By definition of rank maximality, we know it provides weakly better diversity than any other matchings of size at most q_c in the reservation graph. Thus students S^* provide weakly better diversity than all feasible subsets $S'' \subseteq S'$. □

Lemma 3. *The choice function in Algorithm 1 satisfies justified envy-freeness.*

Proof. Let the set of selected students be $x_1, \dots, x_m$ in descending order. We prove by induction on $k \in [m]$ that no student can have justified envy for x_k. For the base consider the first student x_1 that is selected. By the design of Algorithm 1 no unselected student y with priority higher than x_1 can be selected because then there is no rank maximal matching of size at most q_c for the corresponding ranked reservation graph in which y is present. We suppose that the induction hypothesis holds for $j = 1, \dots, k$.

We next show that it also holds for $j = k + 1$. By the induction hypothesis, no unselected student has justified envy for students in $x_1, \dots, x_k$. Suppose that some unselected student has higher priority than x_{k+1} and has justified envy for x_{k+1}. Let y be such a student with the highest priority. This means that there exists a rank maximal matching of size at most q_c that matches all the students in $(\{x_1, \dots, x_m\} \setminus \{x_{k+1}\}) \cup \{y\}$. In particular there exists a rank maximal matching of size at most q_c that matching all the students in $\{x_1, \dots, x_k\} \cup \{y\}$. But in that case, Algorithm 1 would have selected y before x_{k+1} which is a contradiction. Thus the assumption is wrong, and the induction still holds for $j = k + 1$. This completes the proof of Lemma 3. □

We have established that our choice functions satisfy maximal diversity, non-wastefulness, and justified envy-freeness. Interestingly, the converse holds as well.

Size-constrained rank-maximal matchings of the ranked reservation graph.

Lemma 4. *If a choice function satisfies maximal diversity, non-wastefulness, and justified envy-freeness, then it is equivalent to the choice function in Algorithm 1.*

Proof. Consider the set of students selected under the three conditions. We will show that the same set of students is selected under the choice function in Algorithm 1.

To begin with, we first show that the combination of maximal diversity and elimination of justified envy implies that the same set of students selected in the first for loop of Algorithm 1 must be selected. Suppose that S' is the minimal set of students selected due to the combination of maximal diversity and justified envy-freeness. Let the students in S' be $x_1, \dots, x_{|S'|}$ in decreasing order of priority of $\succ_c$. We will prove by induction on $k \in [|S'|]$ that each of the students $x_1 \ldots, x_k$ must be selected by the choice function in Algorithm 1 in the first for loop in the specified order.

For the base case $k = 1$, consider the first student s_1 who is selected by Algorithm 1. We know from the construction of Algorithm 1, that no student with priority higher than s_1 can be selected without compromising on size-constrained rank maximality. The selection of s_1 allows for a matching M of size at most q_c that is rank maximal in the corresponding reservation graph.

For the sake of contradiction, suppose there exists a selection of students not involving s_1 that allows for size-constrained rank maximality and justified envy-freeness. Then, there exists a matching satisfying maximal diversity and justified envy-freeness that does not match s_1. Let M' be such a matching that has minimal symmetric difference from M in its edge set among the size-constrained rank-maximal matchings of the ranked reservation graph. Consider $M \oplus M'$ the symmetric difference of M and M'. We consider this edge set from the perspective of matching M'. Such an edge set must not have an augmenting path with respect to M' or else it contradicts that M' satisfies size-constrained rank maximality. We have established that $M \oplus M'$ must consist of non-augmenting alternating paths. Next, we claim that the edge set $M \oplus M'$ consists of a single alternating path p with respect to M' that starts from unmatched vertex s_1. Suppose for contradiction that this is not true. Then either there exists another alternating path involving edges in E' that either does not include s_1 or does not start with s_1. In either case, it contradicts the assumption that M' is the closest to M (among the size-constrained rank-maximal matchings). In the alternating path p, s_1 is not matched in M'. Now consider the matching in M'' which is the same as M' except that the reserved seat $v = M(s_1)$ is not matched to $M'(v)$ and matched to s_1. The signature of the matching is not affected as the reserved seat matched in M'' is the same as the reserved seats matched in M'. However, we have replaced an agent $M'(v)$ with s_1 where $s_1 \succ_c M'(v)$. This shows that M' violates justified envy-freeness which is a contradiction.

Suppose that the inductive statement holds for $j = 1, \ldots, k$. By the induction hypothesis, we know that Algorithm 1 also chooses $x_1, \ldots, x_k$ in the same order. Let the next student selected by the Algorithm 1 be s_{k+1}. Such a student exists because otherwise Algorithm 1 can select x_{k+1} without compromising on size-constrained rank maximality. We know that there exists no student of priority higher than s_{k+1} and lower than x_k who can additionally be selected without violating maximal diversity. We show that $s_{k+1} = x_{k+1}$. We know from the induction hypothesis that $x_1, \ldots, x_k$ need to be selected. Suppose for contradiction that there exists a matching of size at most q_c that satisfies maximal diversity and justified envy-freeness that selects $x_1, \ldots, x_k$ but does not select s_{k+1}. Let M' be such a matching that has minimal difference from M in its edge set among the the size-constrained rank-maximal matchings in the reservation graph. The symmetric difference of M and M' must consist of a single alternating path p starting from s_{k+1}, since other components in the symmetric difference would either contradict with the maximality of M or with the assumption that M' is the closest to M (among the size-constrained rank-maximal matchings in the new graph). In the alternating path p, s_{k+1} is not matched in M'. The alternating path must contain some student y not in $x_1, \ldots, s_{k+1}$ or else the students matched in p can be matched using the same edges as M. Such an agent y must have less priority than s_{k+1}. Consider M'' that is the same as M' except that it removes the edges in M' that are present in p up till the edge that matches y. Instead, it selects the alternating edges in p (uptill one agent before y in p) that were not used in M'. The matching M'' selects the same agents as in M' except that y is now not matched but s_{k+1} is. The diversity of the matching is not affected as the reserved seats matched in M'' are the same as the reserved seats matched in M'. However, we have replaced a student y with s_{k+1}. This shows that M' violates justified envy-freeness which is a contradiction.

We have now established that maximal diversity and justified envy-freeness imply that Algorithm 1 selects S'. Next, non-wastefulness and justified envy-freeness imply that the same students are selected that Algorithm 1 selects in the second for loop. □

Based on the lemmata above, we obtain the following characterization of our choice function.

Theorem 1 *(Characterization of the Choice Function). A choice rule satisfies maximal diversity, justified envy-freeness, and non-wastefulness if and only if it is the choice function in Algorithm 1.*

Next, we discuss other axiomatic properties of our choice function. Maximal diversity puts special focus on the initial layers before focusing on the latter layers. In view of this focus, a question arises whether a maximally diverse matching uses the maximum number of reserved seats. In the next lemma, we prove that a matching that satisfies maximal diversity uses the maximum number of reserved seats by students who are eligible for the reserved seats. The statement contrasts with the fact that in general bipartite graphs, a rank maximal matching of a bipartite graph need not be a maximum size matching. This is because our ranked reservation graph is a special type of ranked bipartite graph where edges incident to the same reserved seat have the same rank.

Lemma 5. *A matching that satisfies maximal diversity uses the maximum number of reserved seats.*

Proof. For the sake of contradiction, suppose that the size-constrained rank maximal matching M of the ranked reservation graph is not a maximum size matching of the rank reservation graph. Let P denote all seat nodes matched in matching M. Then by Berge's lemma [18], M admits an augmenting path p if we ignore the weights, and only consider the edges. Let $M^* = M \oplus p$ denote a new matching. Now suppose we switch the matching M to M^* by removing the matched edges in the augmenting path p and selecting the complement of the edges in p. Each vertex in the augmenting path that was matched in M continues to be matched in M^*. These vertices include all the vertices in p that are from P and were matched in M. Let P^* denote the set of seat nodes that are matched in M^*. Additionally one vertex from P in path p is now matched in M^*. Observe that every vertex in P^* corresponds to some seat node for some given rank. Therefore, having it matched contributes equally (in M^* and M) to the count of seats of particular ranks being matched. Hence, M^* dominates M in terms of rank maximality which is a contradiction. Thus the assumption is wrong and any size-constrained rank maximal matching of the ranked reservation graph is also a maximum size matching of the rank reservation graph. □

The lemma above gives us the following theorem.

Theorem 2. *The choice function in Algorithm 1 satisfies maximal reserves utilization.*

The result also shows that our choice function generalizes the horizontal envelope choice rule of Sönmez and Yenmez [52] in another way by utilizing the maximum number of reserved seats.

Next, we prove that Algorithm 1 incentivizes the students to not hide their types.

Theorem 3. *Algorithm 1 is privilege monotonic.*

Proof. Let's assume for the sake of contradiction that Algorithm 1 is not type-strategyproof. This implies there exists an instance where a student s_j fails to be selected by school c but would be chosen if s_j reports $T' \subset T(s_j)$. Let's presume that when s_j gets selected by reporting T', the sequence of student selected by Algorithm 1 is $s_1, \ldots, s_j, \ldots$,.

We will now demonstrate that if s_j reports her true types $T(s_j)$, she will indeed be selected. Let G' denote a modified reservation graph when s_j reports $T(s_j)$, with additional edges incorporated. If the selection of students remains unchanged or changes only after s_j is selected, we encounter a contradiction. Therefore, let's assume s_i (where $i \leq j$) is not selected. If s_i is no longer selected, it must be due to the rank maximal matching in G' utilizing one of the additional edges instead of those incident to s_i. However, since all additional edges concern s_j, it follows that s_j should still be selected, contradicting our initial assumption. □

8. Combinatorial insights and a fast algorithm

In this section, we take a closer look at the choice function defined in Algorithm 1. Our first observation is that the outcome of Algorithm 1 can be computed in polynomial time via a reduction to maximum weight matching. We denote $\sum_{j=1}^{r} \sum_{t\in T} \eta_{c,t}^{j}$ by B where r denotes the max number of ranks. Please recall the terms and notations on matching theory in Section 4.

Theorem 4. *An outcome of Algorithm 1 can be computed in polynomial time $O(r|V|^3 \cdot |S|)$ where $|V| = (|S| + min(B, q_c))$, r is the max number of ranks, and B is the total number of reserved seats.*

Proof. The algorithm goes over the students in list $\succ_c$. The main step is to check the 'if' condition in Algorithm 1, which can be done as follows. We compute a rank maximal matching M of graph G that is constrained to the following requirements: the size of the matching is at most q_c and it matches all the vertices in $S^* \cup \{s\}$. The signature of such a matching can be compared with that of matching that is rank maximal among matchings of size at most q_c. Student s is added if and only if the signature is the same. Next we explain how to compute a matching that is size-constrained rank maximal among the set of matchings that match a certain set of students.

We use the reduction of computing a rank maximal matching to computing a maximum weight matching as follows. A rank maximal matching corresponds to the maximum weight matching in which the rank j edges get weight $2^{\lceil \log|V|\rceil(r-j)}$ where r is the number of ranks and V is the set of vertices in the graph. The idea is discussed by Michail [46]. He showed that a rank-maximal matching can be computed via this reduction in time $O(r|V|(|V|^2 + |V|\log(|V|))) = O(r|V|(|V|^2))$. Note that for the ranked reservation graph, $|V| = (|S| + min(B, q_c))$.

We tweak the standard reduction from computing a rank maximal matching to computing a maximum weight matching as follows. We add a large enough weight $2^{\lceil \log|V|\rceil(r+1)}$ to all edges pertaining to vertices in $S^* \cup \{s\}$ to obtain a weighted graph G'. This ensures that the maximum weight matching will match these vertices if possible. We claim that a maximum weight matching M of G finds a rank maximal matching among all matchings that maximizes the number of students in $S^* \cup \{s\}$ that are matched. Suppose for contradiction that there exists a matching M' of G' that also matches all the students in $S^* \cup \{s\}$ and has strictly better rank signature than M. Since the weight contribution from the additional large weight is the same in both M and M', it follows that M' also has a higher weight than M in G' which contradicts the definition of M.

As for the requirement that matching is of size at most q_c, this is handled by the Hungarian algorithm that builds a maximum weight matching incrementally: at stage j, it has a maximum weight matching of size j (Section 17.2 [51]). This completes the proof of Theorem 4. □

Next, we present a tailored algorithm that is faster. Apart from its computational advantage, the design of the algorithm is based on further combinatorial insights into the structure of our solutions. Our algorithm does not directly call existing fast algorithms for rank maximal matchings but carefully relies on some of the combinatorial insights underpinning previous algorithms. The running time of the algorithm is $O(rm\sqrt{n} + nm)$ where r, n and m denotes the number of ranks, the number of nodes, the number of edges in the reservation graph.

8.1. A detour to the algorithm of Irving et al. [34]

In order to set up the groundwork for our fast algorithm, we first present the classical algorithm of Irving et al. [34] for computing a rank maximal matching of a given ranked bipartite graph. It is based on the Dulmage Mendelsohn Decomposition which we describe below.

Let M denote a maximum matching for a bipartite graph G. Then we can partition all vertices into three categories:

- $\mathcal{E}$: vertices which are reachable via even length alternating paths from a free vertex with respect to M.
- $\mathcal{O}$: vertices which are reachable via odd length alternating paths from a free vertex with respect to M.
- $\mathcal{U}$: vertices that do not belong to $\mathcal{E}$ or $\mathcal{O}$.

Theorem 5 *(Irving et al. [34]). The Dulmage Mendelsohn Decomposition consists of the following statement:*

- *$\mathcal{E}, \mathcal{O}, \mathcal{U}$ are invariant among all maximum matchings.*
- *G does not contain an $\mathcal{E}\mathcal{U}$ / $\mathcal{E}\mathcal{E}$ edge.*
- *No maximum matching contains an $\mathcal{O}\mathcal{O}$ / $\mathcal{O}\mathcal{U}$ edge.*
- *Every vertex in $\mathcal{O}$ is matched (to some vertex in $\mathcal{E}$) and every vertex in $\mathcal{U}$ is matched (to some vertex in $\mathcal{U}$).*
- *The cardinality of any maximum matching is $|\mathcal{O}| + |\mathcal{U}|/2$.*

The following Algorithm 2 is currently the fastest algorithm for computing a rank maximal matching [34] based on the Dulmage Mendelsohn Decomposition. The main idea in Algorithm 2 is to convert the problem of computing a rank maximal matching into a problem of computing a maximum matching in a carefully chosen subgraph, in which all edges that never belong to any rank maximal matching are deleted, as shown in Theorem 6.

Input: a ranked bipartite graph $G = (S \cup V, E_1 \cup \cdots \cup E_L)$
Output: a rank maximal matching M_L
$G'_1 = G_1$ where G_1 denotes the induced subgraph of G with edges of rank 1 only and M_1 is an arbitrary maximum matching in G'_1.
for $i = 1$ to $L - 1$ **do**
  Partition all nodes into $\mathcal{E}_i, \mathcal{O}_i, \mathcal{U}_i$ in subgraph G'_i;
  Delete all edges incident to a node in $\mathcal{O}_i, \mathcal{U}_i$ from E_j for all $j > i$;
  Delete all edges $\mathcal{O}_i\mathcal{O}_i$ and $\mathcal{O}_i\mathcal{U}_i$ in G'_i;
  Obtain G'_{i+1} by adding edges E_{i+1} to G'_i;
  Compute a maximum matching M_{i+1} in G'_{i+1} by finding all augmenting paths w.r.t M_i.
return matching M_L

Algorithm 2: Algorithm of Irving et al. [34] for computing a rank maximal matching.

Theorem 6 *(Irving et al. [34]). For every $1 \leq i \leq L$, we have:*

- *Every rank maximal matching in G_i has all of its edges in G'_i.*
- *M_i is a rank maximal matching for G_i.*

Note that $G_i = (S \cup V, E_1 \cup \cdots \cup E_i)$ represents a subgraph of G with only edges of rank 1 to i and G'_i represents a pruned subgraph by removing some edges from G'_i that could not be part of any rank maximal matching. In Theorem 6, the matching M_i is obtained by augmenting M_{i-1}. Note that not every maximum matching M'_i in subgraph graph G'_i is a rank maximal matching.

Theorem 7 *(Irving et al. [34]). A rank-maximal matching can be computed in time $O(rm\sqrt{n})$ where r, n and m denotes the number of ranks, the number of nodes, the number of edges in the ranked bipartite graph.*

Although the algorithm of Irving et al. [34] provides an efficient way to compute a rank maximal matching and hence a diverse matching, it cannot be directly used to design a fast algorithm for our choice function. There are at least two aspects that need to be simultaneously handled: (1) our problem is subject to a size constraint on the matchings due to school capacity; (2) we need to match target subsets of students with respect to school priority. Next we address these issues by designing a new algorithm that takes inspiration from the combinatorial insights of Irving et al. [34].

8.2. Modified rank maximal matching

Since we take school capacity q_c into consideration, it is possible that the size of a rank maximal matching M in G is larger than school capacity. Given the school capacity q_c and a rank maximal matching M^* in G with the signature $\rho(M^*) = \langle x_1, \cdots, x_r \rangle$, let k denote the rank s.t. $\sum_{i=1}^{k-1} x_i \leq q_c < \sum_{i=1}^{k} x_i$. The **maximal clipped signature** of G with respect to school capacity q_c is denoted as $\hat{\rho}(G, q_c) = \langle y_1, \cdots, y_k, \cdots, y_r \rangle$ where

Input: a ranked reservation graph $G=(S\cup V, E_1\cup\cdots\cup E_r)$, q_c
Output: a pruned reservation graph G'_k and a rank maximal matching M_k in G'_k

$G'_1 = G_1$ where G_1 denotes the induced subgraph of G with edges of rank 1 only.
Compute a maximum matching M_1 in subgraph G'_1.
for $i=1$ to $r-1$ **do**
    Partition all nodes into $\mathcal{E}_i, \mathcal{O}_i, \mathcal{U}_i$ w.r.t M_i in G'_i.
    Delete all edges incident to some node in $\mathcal{O}_i, \mathcal{U}_i$ from E_j for all $j>i$.
    Delete all edges $\mathcal{O}_i\mathcal{O}_i$ and $\mathcal{O}_i\mathcal{U}_i$ in G'_i.
    if $|M_i|\ge q_c$ **then**
        $k\leftarrow i$
        return G'_k and M_k
    else
        Obtain G'_{i+1} by adding edges E_{i+1} to G'_i.
        Compute a maximum matching M_{i+1} in G'_{i+1} by finding augmenting paths w.r.t. M_i.
$k\leftarrow r$
return G'_k and M_k

Algorithm 3: Computing a modified rank maximal matching.

$$y_i = \begin{cases} x_i, & \text{if } i\le k-1,\\ q_c-\sum_{i=1}^{k-1} x_i & \text{if } i=k,\\ 0 & \text{if } i>k.\end{cases}$$

We reduce the element y_k to be $q_c-\sum_{i=1}^{k-1} x_i$ and set each element y_i to be 0 for $i>k$. It is safe to remove all edges of rank $k+1$ or above from graph G, since they cannot be matched in any size-constrainted rank maximal matching due to the school capacity.

Note that a matching with maximal clipped signature is rank maximal among all matchings in G of size at most q_c. By the definition of maximal diversity, a matching M is maximally diverse if it has the maximal clipped signature $\hat{\rho}(G,q_c)$.

The following Algorithm 3 is a modified version of Algorithm 2 that computes a matching with maximal clipped signature with respect to school capacity. Next, we give some intuition about Algorithm 3.

Algorithm 3 yields two objects: a **pruned reservation graph** G'_k and a rank maximal matching M_k in G'_k. Here k denotes some integer in the range $[1,r]$, depending on school capacity q_c.

Note that in subgraph G'_k, all edges of G that do not belong to any rank maximal matching are removed, while every rank maximal matching in G_k has all of its edges in G'_k by Theorem 6.

If the size of the returned matching M_k in G'_k is larger than q_c, then we can just remove $|M_k|-q_c$ edges of rank k from M_k to obtain a new matching M'_k of size q_c, which is still rank maximal among all matchings of size at most q_c in G'_k. However, we do not immediately make decisions about which edges of rank k should be deleted. We need to take school priorities into account to decide which students should be selected, as discussed in the next section.

8.3. A fast algorithm for our choice function

Next, we present a fast solution (Algorithm 4) that implements the choice function in Algorithm 1. There are two main differences from the previous solution in Theorem 4. First, we do not compute a maximum weight matching, which is computational inefficient. Second, we do not compute a new rank maximal matching containing a certain set of students S^* each time. Instead, we make full use of an existing rank maximal matching by computing alternating and augmenting paths.

Theorem 8. *Algorithm 1 and Algorithm 4 return the same set of students.*

Proof. In order to establish the equivalence between the outcomes of Algorithm 4 and Algorithm 1, we first prove the following lemma.

Lemma 6. *Consider a pruned reservation graph G'_k of a ranked reservation graph G and a matching M with the maximal clipped signature $\hat{\rho}(G,q_c)$ that matches all vertices in $S^*\subseteq S$, and does not match a given vertex $s\in S\setminus S^*$. Then, there exists a matching M' with the same clipped signature $\hat{\rho}(G,q_c)$ that matches all the vertices in $S^*\cup\{s\}$ if and only if one of the following two conditions holds:*

1. *there exists an alternating path P w.r.t. M that starts from s and end with some $s'\in S\setminus S^*$;*
2. *(a) there exists an augmenting path P w.r.t. M that starts from s and ends at some unmatched seat v of rank k and (b) there exists some student $s'\in S\setminus S^*$ who is matched to some seat v' of rank k in M.*

Proof. $\Leftarrow$ Supposes one of the two conditions holds. Case (1): There exists an alternating path P starting from s and ending at another student $s'\in S\setminus S^*$ with respect to M. Then the matching $M\oplus P$ is a matching including S^* and s by definition. Next we show that $M\oplus P$ has the maximal signature of G'_k. Let path P be a sequence of nodes $s, v_1, s_1, v_2, \cdots, v_w, s'$ where the edges (v_1,s_1), $(v_2,s_2)\cdots$ (v_w,s') are matched in matching M. While in the matching $M\oplus P$, edges (s,v_1), $(s_1,v_2)\cdots(s_{w-1},v_k)$ are matched. Note that in the reservation graph G, all edges incident to the same seat v have the same rank. Thus each edge in $M\setminus P$ is replaced by another edge in $P\setminus M$ with the same rank. In other words, the matching $M\oplus P$ contains students S^* and s with the maximal signature of G'_k.

Input: $S' \subseteq S$, q_c, $\succ_c$, η_c.
Output: A set of students $S^* \subseteq S'$

Construct a ranked reservation graph G.
Compute a pruned reservation graph G'_k as well as a rank maximal matching M in G'_k by running Algorithm 3 on input G, q_c.
if $|M| > q_c$ **then**
    Remove $|M| - q_c$ edges of rank k from M arbitrarily.
$S^* \leftarrow \emptyset$
for each student s in descending ordering of $\succ_c$ **do**
    if s is matched in M **then**
        $S^* \leftarrow S^* \cup \{s\}$
    else if there exists an *alternating path* P w.r.t M that starts from s and ends at $s' \in S \setminus S^*$ **then**
        $S^* \leftarrow S^* \cup \{s\}$
        $M \leftarrow M \oplus P$
    else if there exist i) an *augmenting path* P w.r.t M that starts from s and ends at some free seat v of rank k and ii) some student $s' \in S \setminus S^*$ who is matched to some seat $v^k_{t,i}$ of rank k in M **then**
        $S^* \leftarrow S^* \cup \{s\}$
        $M \leftarrow M \oplus P$
        $M \setminus \{(s', v^k_{t,i})\}$ % Remove student s' from the matching M
for each student s in descending ordering of $\succ_c$ **do**
    if $|S^*| < q_c$ and $s \notin S^*$ **then**
        $S^* \leftarrow S^* \cup \{s\}$
return S^*

Algorithm 4: A Fast Algorithm for the Choice Function.

Case (2): There exists an augmenting path P starting from student s and ending at some unmatched seat v of rank k with respect to M as well as some student $s' \in S \setminus S^*$ who is matched to some seat v' of rank k in the matching M. Then the matching $M \oplus P$ is a matching including S^* and s by definition. Let path P be a sequence of nodes $s, v_1, s_1, v_2, \cdots, v_w, s_w, v$ where the edges (v_1, s_1), $(v_2, s_2) \cdots (v_w, s_w)$ are matched in the matching M. While in the matching $M \oplus P$, edges (s, v_1), $(s_1, v_2) \cdots (s_{w-1}, v_w)$ and (s_w, v) are matched. In the reservation graph G, all edges incident to the same seat v have the same rank. Thus each edge in $M \setminus P$ is replaced by another edge in $P \setminus M$ with the same rank, and one more edge (s_w, v) of rank k is matched, which means the signature $\rho(M \oplus P)$ of $M \oplus P$ is larger than the maximal matching. Since there is one more student $s' \in S \setminus S^*$ who is matched to some seat v' of rank k in M. Then we can remove the edge (s', v') from $\rho(M \oplus P)$ to get a matching with maximal signature and the same number of matched children.

$\Rightarrow$ Suppose there exists a matching M' including S^* and s with the maximal signature of G'_k. Consider the subgraph G' by taking the symmetric difference of M and M': $(M \setminus M') \cup (M' \setminus M)$. For any graph, if each vertex has degree at most 2, then it only consists of isolated vertices, cycles and paths. By the assumption that student s is not matched in M and is matched in M', we know there exists a path P whose edges alternate between M' and M starting from student s. There are two cases where the path P ends with either a student or a seat corresponding to the two conditions in Lemma 6. For case (1), the path P ends at another student s' who is matched in M but not matched in M'. As we know all students S^* are matched in M', then student s' cannot belong to S^*. For case (2), the path P ends at an unmatched seat v. Note that seat v can only be incident to edges of rank k, otherwise it leads to a contradiction that both M and M' have the maximal clipped signature of G'_k. Since P is an augmenting path with respect to matching M which increases the number of matched edges of rank k by one, we have to remove one more edge of rank k from M'. Then there must exist another student $s' \in S \setminus S^*$ who is matched to some seat of rank k in M. This completes the proof of Lemma 6. □

For both Algorithm 1 and Algorithm 4, they always select the same student in the for loop by Lemma 6. This completes the proof of Theorem 8. □

Theorem 9. *Algorithm 4 runs in time $O(rm\sqrt{n} + nm)$ where r, n and m denotes the number of ranks, the number of nodes, the number of edges in the pruned reservation graph.*

Proof. Algorithm 4 first computes a rank maximal matching M in the pruned reservation graph G'_k by Algorithm 3, which takes time $O(rm\sqrt{n})$ by Theorem 7. Then we arbitrarily remove $|M| - q_c$ edges of rank k from M if $|M| > q_c$ in constant time. In the first for loop, for each student, we check whether there exists an alternating path or an augmenting path in time $O(m)$. Thus the total running time of the first for loop is $O(nm)$. The second for loop checks whether we can add some students without exceeding school capacity in time $O(n)$. The whole running time of Algorithm 4 is $O(rm\sqrt{n} + nm)$. □

Next, we present an illustrating example for Algorithm 4.

Example 5. Consider the following setting in which there are four students $S = \{s_1, s_2, s_3, s_4\}$ and one school $C = \{c\}$. The priority ordering of school c is $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4$ and all students consider school c acceptable. The type profile of the students is $T(s_1) = \{t_1, t_2\}$, $T(s_2) = \{t_1\}$, $T(s_3) = \{t_3, t_4\}$, $T(s_4) = \{t_4\}$. The school capacity of c is $q_c = 3$ and it has diversity goals specified as follows: $\eta^1_{c,t_2} = \eta^1_{c,t_3} = 0$, $\eta^1_{c,t_1} = \eta^2_{c,t_2} = \eta^2_{c,t_3} = \eta^1_{c,t_4} = 1$. The corresponding reservation graph is depicted in Fig. 5 where the superscript j of a seat $v^j_{t,i}$ represents its rank.

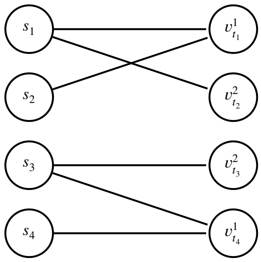

Fig. 5. The corresponding reservation graph of school c in Example 5.

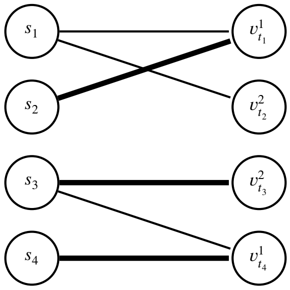

Fig. 6. Compute a rank maximal matching of size $q_c = 3$ and initialize S^* to be $\emptyset$.

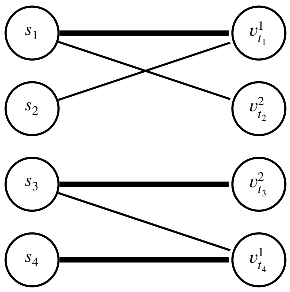

Fig. 7. Add s_1 to S^*, as we find a rank maximal matching of size 3 including s_1.

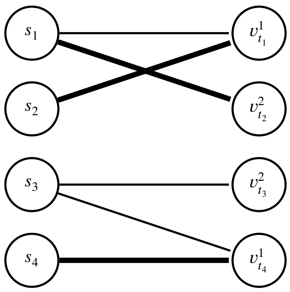

Fig. 8. Add s_2 to S^*, as we find a rank maximal matching of size 3 including s_1 and s_2.

The rank maximal matching yielded by Algorithm 3 is $\{(s_1, v^2_{t_2}), (s_2, v^1_{t_1}), (s_3, v^2_{t_3}), (s_4, v^1_{t_4})\}$. Since the school capacity is 3, we randomly delete one edge of rank 2, say $(s_1, v^2_{t_2})$, as shown in Fig. 6.

Next, we go through students one by one based on school priority and check whether there exists a matching of size 3 with maximal clipped signature $\langle 2, 1\rangle$. Initially set S^* to be empty.

- For student s_1, there exists an alternating path $s_1, v^1_{t_1,1}, s_2$. Thus we add s_1 to S^* and update M to be $\{(s_1, v^1_{t_1}), (s_3, v^2_{t_3}), (s_4, v^1_{t_4})\}$, as shown in Fig. 7.
- For student s_2, there exist an augmenting path $s_2, v^1_{t_1}, s_1, v^2_{t_2}$ and another student $s_3 \notin S^*$ who is matched to some seat of rank 2. Thus we add s_2 to S^* and update M to be $\{(s_1, v^2_{t_2}), (s_2, v^1_{t_1}), (s_4, v^1_{t_4})\}$, as shown in Fig. 8.

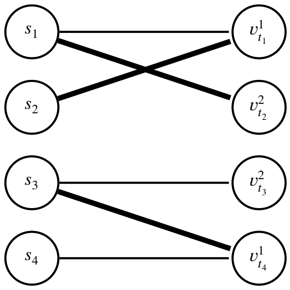

Fig. 9. Add s_3 to S^*, as a rank maximal matching of size 3 including s_1, s_2, s_3 exists.

- For student s_3, there exists an alternating path $s_3, v^1_{t_4}, s_4$. Thus we add s_3 to S^* and update M to be $\{(s_1, v^2_{t_2}), (s_2, v^1_{t_1}), (s_3, v^1_{t_4})\}$, as shown in Fig. 9.
- For student s_4, we cannot find either i) an alternating path starting from s_4 and ending at some $s' \notin S^*$ or ii) an augmenting path starting from s_4 and another student $s' \notin S^*$ who is matched to some seat of rank 2. Thus we cannot add s_4 to S^*.

9. Controlled school choice

In this section, we analyze the properties of Student Proposing Deferred Acceptance (SPDA) armed with new our choice function. Firstly, we compare with two previous approaches and show that our choice functions are equivalent in restricted domains and still achieve maximal diversity in general domains while previous ones do not. Secondly, we show that SPDA with our choice function satisfies stability and strategy-proofness for students. Furthermore, it returns a students-optimal stable outcome which is the one weakly preferred by all students among stable outcomes.

9.1. Student proposing deferred acceptance

In two-sided matching, the most important algorithm is the Deferred Acceptance (DA) algorithm [27]. We next explain how our proposed choice function can be embedded in Student Proposing Deferred Acceptance.

Input: Instance I, Ch_C
Output: A matching X
while no more student makes any proposal **do**
  (Proposal) All the unmatched students apply to their most preferred schools that have not rejected them yet.
  (Selection) For each school $c \in C$, let S_c denote the set of students who are matched to c or who now apply to c. Each school c selects a set of students $Ch_c(S_c)$ and rejects the rest $S_c \setminus Ch_c(S_c)$.
return Matching X that represents the current matches.

Algorithm 5: Student Proposing Deferred Acceptance (SPDA).

Algorithm 5 repeatedly adopt two procedures: unmatched students first propose to their favourite schools that have not rejected them yet; schools then choose a set of students from currently matched students and new applicants, and reject unchosen students. It terminates when each student does not make any proposal (i.e., either being matched or been rejected by all the schools). Note that Algorithm 5 still works in the same way as the classical Deferred Acceptance algorithm [27,50], except that schools choose students based on more complicated choice functions instead of priority orderings solely.

9.2. Generalization of previously proposed choice functions

Next, we show that our choice function coincides with two prominent choice functions in restricted domains.

We first show the choice function in Algorithm 1 is equivalent to choice function C_c^{SB} of Ehlers et al. [26]. We first recast in Algorithm 6, the definition of C_c^{SB} as an algorithm for the case of two ranks. The algorithm iteratively chooses the highest priority students whose type is undersubscribed with respect to the first rank. After that the algorithm iteratively chooses the highest priority students whose type is undersubscribed with respect to the second rank. After that it selects students until either the school capacity is reached or all the students are selected.

Theorem 10. *If there are exactly two ranks and each agent has exactly one type, then the choice function in Algorithm 1 is equivalent to choice function C_c^{SB} of Ehlers et al. [26].*

Proof. It is sufficient to prove that for two ranks and each agent having exactly one type, Algorithm 6 gives the same outcome as that of Algorithm 1 and that the students are selected in the same order. Suppose for contradiction that the order and selection is different. Then consider the first student s who is selected by Algorithm 6 but not by Algorithm 1. Suppose s is selected by Algorithm 6 in

Input: $S' \subseteq S$, q_c, η_c, $\succ_c$.
Output: A set of students $S^* \subseteq S'$

Selected students $S^* \leftarrow \emptyset$
for student s down the list in $\succ_c$ **do**
  if $s \notin S^*$ and $\exists t \in T(s)$ s.t. $|S_t^*| \le \eta_{c,t}^1$ **then**
    Add s to S^*
for student s down the list in $\succ_c$ **do**
  if $s \notin S^*$ and $\exists t \in T(s)$ s.t. $|S_t^*| \le \eta_{c,t}^1 + \eta_{c,t}^2$ **then**
    Add s to S^*
for student s down the list in $\succ_c$ **do**
  if $|S^*| < q_c$ and $s \notin S^*$ **then**
    Add s to S^*
return S^*

Algorithm 6: Choice function C_c^{SB} of Ehlers et al. [26] for school c when there are 2 ranks and no student has overlapping types.

the first loop. Denote by S' the set of students selected before s. Since students have exactly one type, S' corresponds to the exactly count of the types of reserved seats that are to be used. Then, it implies that selecting s as well as all the students in S' (students selected before s) leads to a matching that is not rank maximal. If s cannot be selected in the rank maximal matching, then it implies that any rank maximal matching will not fully use $\eta_{c,t}^1$ reserved seats. But in that case s must have been used in the rank maximal matching which is a contradiction. We have established that both algorithms select the same students and in the same order for rank one seats. The same argument is then repeated for rank two seats. Finally, the last for loop of both algorithms are identical. $\square$

We remark that the choice function C_c^{SB} of Ehlers et al. [26] does not require optimisation via maximum weight or maximum cardinality matching because Ehlers et al. [26] assume that each student has exactly one type.

Next we show that if there is one rank, then the choice function in Algorithm 1 is equivalent to the horizontal envelope choice rule of Sönmez and Yenmez [52]. In order to show the equivalence, we do not need to formally specify the horizontal envelope choice rule of Sönmez and Yenmez [52]. We show the equivalence by showing that in the case of one rank, the choice function in Algorithm 1 satisfies axiomatic properties (defined for the case of one rank) that characterize the rule of Sönmez and Yenmez [52].

Theorem 11. *If there is one rank, then the choice function in Algorithm 1 is equivalent to the horizontal envelope choice rule of Sönmez and Yenmez [52].*

Proof. If there is exactly one rank, our maximal diversity is equivalent to the 'maximally complies with reservations condition' studied by Sönmez and Yenmez [52]. Similarly, justified envy-freeness is a generalization of 'elimination of justified envy' in the context of a single rank as studied by Sönmez and Yenmez [52]. The notion is non-wastefuless is the same. Sönmez and Yenmez [52] proved that their horizontal envelope choice rule is the unique rule satisfying maximal compliance with reservations, non-wastefulness, and elimination of justified envy. Hence the theorem follows. $\square$

We next explain why our new choice function is more general and powerful to capture diversity goals. Firstly, unlike [52], we do not assume that the sum of all reserved quotas is no larger than school capcaity, i.e., $\sum_{j=1}^{r} \sum_{t \in T} \eta_{c,t}^j \le q_c$. Relaxing this assumption can be useful for more complex diversity goals.[5]

Secondly, for the case of multiple ranks of reserves, a naive approach could be to run the horizontal envelope choice rule of Sönmez and Yenmez [52] for the first rank to select a set of students and match them to reserved seats of the first rank only. These students as well as reserved seats already allocated are removed from consideration and the same process is repeated for the next rank. However such an approach might not guarantee justified envy-freeness, as shown in Example 6.

Example 6. Consider the following setting in which there are three students $S = \{s_1, s_2, s_3\}$ and one school $C = \{c\}$. The priority ordering of school c is $s_1 \succ_c s_2 \succ_c s_3$ and all students consider school c acceptable. The type profile of the students is $T(s_1) = \{t_1, t_2\}$, $T(s_2) = \{t_1\}$, $T(s_3) = \{t_3\}$. The school capacity of c is $q_c = 2$ and it has diversity goals specified as follows: $\eta_{c,t_2}^1 = \eta_{c,t_3}^1 = 0$, $\eta_{c,t_1}^1 = \eta_{c,t_2}^2 = \eta_{c,t_3}^2 = 1$. The corresponding reservation graph is depicted in Fig. 10 where the superscript j of a seat $v_{t,i}^j$ represents its rank.

The horizontal envelope choice rule of Sönmez and Yenmez [52] works as follows: 1) There is only one reserved seat of type t_1 with rank 1 and two students s_1, s_2 have type t_1. Since student s_1 has higher daycare priority than s_2, reserved seat $v_{t_1,1}^1$ is assigned to student s_1. 2) For the remaining students s_2 and s_3 as well as the remaining reserved seats $v_{t_2,2}^1, v_{t_3,2}^1$ of rank 2, student s_3 is assigned to seat $v_{t_3,2}^1$ while student s_2 cannot be matched to any reserved seat.

In contrast, our choice function finds a matching that satisfies maximal diversity in which student s_1 is matched to seat $v_{t_2,1}^2$, student s_2 is matched to seat $v_{t_1,1}^1$ and student s_3 is unmatched.

We argue that the latter outcome is better, as student s_2 has higher priority than s_3 and both outcomes satisfy maximal diversity.

[5] Even for the case of minimum and maximum quotas, it can be possible that the sum of upper quotas of types exceeds the school capacity.

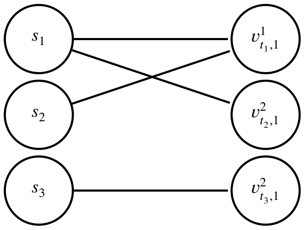

Fig. 10. School priority $s_1 \succ_c s_2 \succ_c s_3$.

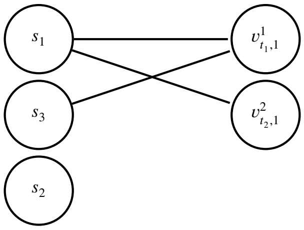

Fig. 11. School priority $s_1 \succ_c s_2 \succ_c s_3$.

The horizontal envelope choice rule of Sönmez and Yenmez [52] also does not guarantee rank maximality even if we apply it rank by rank, as shown in Example 7.

Example 7. Consider the following setting in which there are three students $S = \{s_1, s_2, s_3\}$ and one school $C = \{c\}$. The priority ordering of school c is $s_1 \succ_c s_2 \succ_c s_3$ and all students consider school c acceptable. The type profile of the students is $T(s_1) = \{t_1, t_2\}$, $T(s_2) = \{t_3\}$, $T(s_3) = \{t_2\}$. The school capacity of c is $q_c = 2$ and it has diversity goals specified as follows: $\eta^1_{c,t_2} = \eta^1_{c,t_3} = \eta^2_{c,t_3} = 0$, $\eta^1_{c,t_1} = \eta^2_{c,t_2} = 1$. The corresponding reservation graph is depicted in Fig. 11 where the superscript j of a seat $v^j_{t,i}$ represents its rank.

The horizontal envelope choice rule of Sönmez and Yenmez [52] works as follows: it first matches student s_1 to reserved seat $v^1_{t_1,1}$ and then selects student s_2 while leaving reserved seat $v^2_{t_2,1}$ vacant.

When Sönmez and Yenmez [52] presented their horizontal envelope choice rule for the case of a single rank of reserves, they showed that their rule maximizes the number of reserved seats used whereas other approaches that do not use smart-reserves may not reach diversity goals optimally. We say that an approach is *diversity-based greedy* if it fills up reserved seats in a greedy manner with students who satisfy the corresponding type. Note that the algorithms of [8] and Kurata et al. [43] are diversity-based greedy. We note here for problems with one rank of reserved seats, diversity-based greedy will reach at least reach half of the optimal reserve seat utilization.

Theorem 12. *For problems with only rank one reserves, any diversity-based greedy rule reaches a 2-approximation of the maximum possible utilization of reserved seats.*

Proof. This statement follows from viewing the corresponding reservation graph. A maximum size matching of the reservation graph provides an optimal reserve seat utilization. We note that a greedy-based diversity algorithm results in a matching of students to reserved seats such that the matching is a maximal matching of the reservation graph. We then use the well-known fact that a maximal matching provides a 2-approximation of a maximum size matching. On the other hand, there exist examples for which a non-smart approach may only utilize half of the reserved seats.

Example 8. Consider the setting in which there are three students s_1, s_2, s_3. The type profile of the students is $T(s_1) = \{t_1, t_2\}$, $T(s_2) = \emptyset$, and $T(s_3) = \{t_1\}$. The quota of c is $q_c = 2$. If $\eta^1_{c,t_1} = 1$ and $\eta^1_{c,t_2} = 1$, then a greedy approach that selects s_1 first and then s_2 will satisfy only one reserved seat where it is possible to satisfy both reserved seats by selecting s_1 and s_3.

This completes the proof of Theorem 12. □

9.3. Stability and strategy-proofness

The cornerstone result in the theory of matching markets is that the SPDA algorithm is strategyproof for students and returns a matching which satisfies a natural notion of stability. We say that an algorithm or rule is strategyproof for students if there exists no student who can misreport her preferences to get a better outcome. Stability is defined in Definition 5. Here we use a natural notion of choice function of a given student Ch_s as taking as input a set of schools and returning the most preferred school according to preference ordering $\succ_s$.

Definition 5 *(Stability).* A matching M is stable if it is

1. (Individual rationality) for each agent $i \in S \cup C$, $M(i) = Ch_i(M(i))$.
2. (No blocking pair) there exists no pair of student s and school c such that $c \succ_s M(s)$ and $s \in Ch_c(M(c) \cup \{s\})$.

Stability in Definition 5 requires that matching M needs to satisfy two conditions. First, for each agent i (being either a student or a school), the assignment to agent i equals the outcome of agent i's choice function given $M(i)$ as input. In other words, each student / school is matched to some acceptable school / a set of acceptable students. Second, there does not exist a pair of student and school who prefer to deviate from matching M and be matched with each other.

The results rely on the assumption that the choice function satisfies certain axioms we define below. We first formally define these axioms and then show that they are satisfied by our choice function.

Definition 6 *(Substitutability [38]).* A choice rule Ch satisfies substitutability if for every $S' \subseteq S$, $i \in Ch(S')$ and $j \neq i \implies i \in Ch(S' \setminus \{j\})$.

Definition 6 requires that given a set of applicants S', if student $i \in S'$ is selected, then student i is still selected when some student $j(\neq i) \in S'$ is absent. Note that the one-to-all convention does not satisfy substitutability.

Definition 7 *(Law of aggregate demand (LAD)).* A choice rule Ch satisfies the *law of aggregate demand (LAD)* if for $S'' \subseteq S' \subseteq S$, if $|Ch(S'')| \leq |Ch(S')|$.

Definition 7 requires that the number of selected students does not decrease when the set of applicants strictly increases (i.e., a superset).

Definition 8 *(Irrelevance of rejected individuals [5]).* A choice rule Ch satisfies the *irrelevance of rejected individuals* condition if for every $S' \subseteq S$, $i \in S \setminus Ch(S') \implies Ch(S' \setminus \{i\}) = Ch(S')$.

Definition 8 requires that given a set of applicants S', if $i \in S'$ is not selected, then removing student i from S' will not affect the set of selected students $Ch(S')$.

Lemma 7. *The choice function in Algorithm 1 satisfies substitutability.*

Proof. We next prove that for every $S' \subseteq S$, $i \in Ch(S')$ and $j \neq i \implies i \in Ch(S' \setminus \{j\})$. There are three possible cases for student j: i) student j was not selected; ii) student j was selected in the second for loop; iii) student j was selected in the first for loop.

If j was not selected, then either a) i contributes to rank maximality and was selected in the first for loops or b) i has higher priority than j and was selected in the second for loop. For both cases, removal of child j does not affect the selection of child i.

If j was selected in the second for loop, then either i was selected in the first for loop (for maximal diversity), or i was selected in the second for loop (for non-wastefulness). In the first case, exclusion of j does not affect the first for loop, so i will continue to be selected. In the second case, i will continue to be selected as well in the absence of j since the number of students who come before i are either the same or one less.

Now suppose j was selected in the first for loop and i was selected in the first for loop as well. Let M be the rank maximal matching computed when j is not removed. A rank maximal matching corresponds to the maximum weight matching in which the rank i edges get weight $2^{\lceil \log|V| \rceil (L-i)}$ where L is the number of ranks and V is the set of vertices in the graph. We will use this view of rank maximal matchings as maximum weight matchings of a weighted version of the reservation graph to prove our claim. Let j be the removed vertex and $M(j) = b$ its former pair in M. Take any maximum weight matching M' in the new graph, that has minimal difference from M in its edge set among the maximum weight matchings in the new graph. The symmetric difference of M and M' must consist of a single alternating path starting from b, since other components in the symmetric difference would either contradict with the maximality of M or with the assumption that M' is the closest to M (among the maximum weight matchings in the new graph). Hence i will continue to be in the rank maximal matching.

The final case is that j was selected in the first for loop and i was selected in the second for loop. In that case, the reservation graph will continue to select all other agents who were selected in the original rank maximal matching. Hence the size of the matching decreases by at most one. In that case i will be selected when j is removed. $\square$

Lemma 8. *The choice function in Algorithm 1 satisfies law of aggregate demand.*

Proof. Given two sets of students S' and S'' with $S' \subseteq S''$, we next show that $|Ch(S')| \leq |Ch(S'')|$. If $|Ch(S')| = q_c$, then we have $|Ch(S'')| = q_c$ as well, as the last for loop guarantees that at least q_c students are selected. If $|Ch(S')| < q_c$, then S'' will provide weakly better diversity than S' and thus the number of chosen students in the first for loop weakly increase. The second loop chooses students up to school capacity and thus the number of chosen students from S'' will not be smaller than the one for S'. □

Lemma 9. *The choice function in Algorithm 1 satisfies irrelevance of rejected individuals condition.*

Proof. We next show that given a set of students S', removal of any student i who was not selected by Ch does not affect the selection of $Ch(S')$. Suppose $S_1 \subseteq Ch(S')$ is the set of students who are selected in the first for loop and $S_2 \subseteq Ch(S')$ is the set of students who are selected in the second for loop. Let G denote the corresponding ranked reservation graph. We know that c cannot provide strictly better diversity than any student $s \in S_1$, otherwise c would be chosen in the first for loop. And removal of student i does not harm maximal diversity, since we can still choose students S_1 in the same order as i is part of the input. The second for loop is selected based on school priority only, removal of student i does not change the assignment of S_2. Thus for both for loops, the exactly same set of students is chosen. □

Theorem 13. *If the choice functions of the schools are of the class defined in Algorithm 1, then SPDA is strategyproof for students and returns a stable outcome.*

Proof. We have already shown that the choice function defined in Algorithm 1 satisfies the conditions of substitutability, law of aggregate demand, and irrelevance of rejected individuals. It follows from existing results (see, e.g., Aygün and Sönmez [5]) that SPDA with choice function of schools that satisfy these three properties is strategyproof for students and results in a stable outcome. □

Theorem 14. *SPDA with choice functions of the schools defined in Algorithm 1 returns a student-optimal stable outcome.*

Proof. We next show that for the outcome yielded by SPDA with choice functions of the schools defined in Algorithm 1, no student s is rejected by any school c if student s can be matched to school c in some stable outcome. We refer to such school c as a "potential" school for student s. Suppose during the process of SPDA, student s is the first student who is rejected by some potential school c. Let $X = \{x_1, \cdots, x_k\}$ denote the set of students who are matched to school c currently. By Algorithm 1, we know for each student $x \in X$, either x provides better diversity than s (i.e., selected in the first for loop) or x has higher priority than s (i.e., selected in the second for loop). Since we assume that student s is the first student who is rejected by a potential school c, then each student $x \in X$ is not rejected by any potential school. For the sake of contradiction, assume there exists another stable outcome in which s is matched to c. Then we know at least one student $x \in X$ needs to be matched to a worse school than c. However, in that case, x and c could form a blocking pair to block the outcome, a contradiction. Thus no student is rejected by any potential school during the process of SPDA and we know that it is the most preferred one by all students. □

10. Discussion

We consider a natural and general model of diversity goals in which the schools first want to fill in reserves for the first rank and then subsequent ranks. For this model, we designed a choice function that achieves diversity maximally. The choice function also serves as a useful decision-tool for committee or set selection problems (see e.g., [39]). In the latter problems, a set of candidates is selected after aggregating the preferences of the voters. If we have already aggregated the preferences to derive common (priority) ranking over the candidates, then our devised choice function can be used to additionally cater for diversity.

In our framework, a student can use a reserved seat if she satisfies the corresponding type of the seat. Our framework and all of our technical results immediately extend to scenarios where each reserved seat has a corresponding criterion based on types and a student can use a reserved seat if she satisfied the criterion. For example a criterion could indicate that 'any student of any type can use the seat.' This captures the seats that are under termed as 'open category seats.' Another criterion could be that the user 'satisfies type t_1 and t_2 but not t_3.' Note that it can be the case that one meta type is more constrained than another meta type. For example there could be some seats reserved for people with disabilities and separate seats reserved for people with a special disability.

Finally, we mention that the general approach for our choice function (Algorithm 1) can also be applied to other settings in which the priority list of applicants needs to be processed subject to various constraints. The approach deals with the applicants in decreasing order of priority and only adds an applicant if doing so will maintain the possibility of getting a feasible set of applicants.

CRediT authorship contribution statement

Haris Aziz: Writing – review & editing, Writing – original draft, Supervision. **Zhaohong Sun:** Writing – review & editing, Writing – original draft, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

References

[1] A. Abdulkadiroğlu, College admissions with affirmative action, Int. J. Game Theory 33 (2005) 535–549.
[2] A. Abdulkadiroğlu, T. Sönmez, School choice: a mechanism design approach, Am. Econ. Rev. 93 (2003) 729–747.
[3] S. Ahmadi, F. Ahmed, J.P. Dickerson, M. Fuge, S. Khuller, An algorithm for multi-attribute diverse matching, in: Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020, 2020, pp. 3–9.
[4] O. Aygün, I. Bó, College admission with multidimensional privileges: the Brazilian affirmative action case, Am. Econ. J. Microecon. 13 (2021) 1–28.
[5] O. Aygün, T. Sönmez, Matching with contracts: comment, Am. Econ. Rev. 103 (2013) 2050–2051.
[6] O. Aygün, B. Turhan, Dynamic reserves in matching markets: Theory and applications, Technical Report, 2016.
[7] O. Aygun, B. Turhan, Designing Direct Matching Mechanism for India with Comprehensive Affirmative Action, Papers, arXiv.org, arXiv:2004.13264, 2020.
[8] O. Aygün, B. Turhan, Dynamic reserves in matching markets: theory and applications, J. Econ. Theory 188 (2020) 105069.
[9] H. Aziz, A rule for committee selection with soft diversity constraints, Group Decis. Negot. (2019) 1–8.
[10] H. Aziz, P. Biró, M. Yokoo, in: Thirty-Sixth AAAI Conference on Artificial Intelligence, AAAI 2022, Thirty-Fourth Conference on Innovative Applications of Artificial Intelligence, IAAI 2022, the Twelveth Symposium on Educational Advances in Artificial Intelligence, EAAI 2022 Virtual Event, February 22 - March 1, 2022, AAAI Press, 2022, pp. 12308–12316.
[11] H. Aziz, F. Brandl, Efficient, fair, and incentive-compatible healthcare rationing, in: EC '21: The 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18–23, 2021, ACM, 2021, pp. 103–104.
[12] H. Aziz, S. Gaspers, Z. Sun, Mechanism design for school choice with soft diversity constraints, in: Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020, 2020, pp. 153–159.
[13] H. Aziz, S. Gaspers, Z. Sun, T. Walsh, From matching with diversity constraints to matching with regional quotas, in: Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, 2019, pp. 377–385.
[14] H. Aziz, S. Gaspers, Z. Sun, M. Yokoo, Multiple levels of importance in matching with distributional constraints, in: Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS), Extended Abstract, 2020, pp. 1759–1761.
[15] H. Aziz, Z. Sun, School choice with flexible diversity goals and specialized seats, in: Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021, 2021, pp. 17–23, ijcai.org.
[16] S. Baswana, P.P. Chakrabarti, S. Chandran, Y. Kanoria, U. Patange, Centralized admissions for engineering colleges in India, in: Proceedings of the 20th ACM Conference on Economics and Computation, 2019, pp. 323–324.
[17] N. Benabbou, M. Chakraborty, X. Ho, J. Sliwinski, Y. Zick, Diversity constraints in public housing allocation, in: Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2018, Stockholm, Sweden, July 10-15, 2018, 2018, pp. 973–981.
[18] C. Berge, Two theorems in graph theory, Proc. Natl. Acad. Sci. USA 43 (1957) 842–844.
[19] P. Biró, T. Fleiner, R.W. Irving, D.F. Manlove, The college admissions problem with lower and common quotas, Theor. Comput. Sci. 411 (2010) 3136–3153.
[20] R. Bredereck, P. Faliszewski, A. Igarashi, M. Lackner, P. Skowron, Multiwinner elections with diversity constraints, in: Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence, (AAAI-18), the 30th Innovative Applications of Artificial Intelligence (IAAI-18), and the 8th AAAI Symposium on Educational Advances in Artificial Intelligence (EAAI-18), New Orleans, Louisiana, USA, February 2–7, 2018, 2018, pp. 933–940.
[21] S. Cho, T. Todo, M. Yokoo, Two-sided matching over social networks, in: L.D. Raedt (Ed.), Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, Vienna, Austria, 23–29 July 2022, 2022, pp. 186–193, ijcai.org.
[22] J.R. Correa, R. Epstein, J. Escobar, I. Rios, B. Bahamondes, C. Bonet, N. Epstein, N. Aramayo, M. Castillo, A. Cristi, B. Epstein, School choice in Chile, in: Proceedings of the 2019 ACM Conference on Economics and Computation, EC 2019, Phoenix, AZ, USA, June 24-28, 2019, 2019, pp. 325–343.
[23] J.P. Dickerson, K.A. Sankararaman, A. Srinivasan, P. Xu, Balancing relevance and diversity in online bipartite matching via submodularity, in: The Thirty-Third AAAI Conference on Artificial Intelligence, AAAI 2019, the Thirty-First Innovative Applications of Artificial Intelligence Conference, IAAI 2019, the Ninth AAAI Symposium on Educational Advances in Artificial Intelligence, EAAI 2019, Honolulu, Hawaii, USA, January 27 - February 1, 2019, 2019, pp. 1877–1884.
[24] U. Dur, P.A. Pathak, T. Sönmez, Explicit vs. statistical targeting in affirmative action: theory and evidence from Chicago's exam schools, J. Econ. Theory 187 (2020) 104996.
[25] F. Echenique, M.B. Yenmez, How to control controlled school choice, Am. Econ. Rev. 105 (2015) 2679–2694.
[26] L. Ehlers, I.E. Hafalir, M.B. Yenmez, M.A. Yildirim, School choice with controlled choice constraints: hard bounds versus soft bounds, J. Econ. Theory 153 (2014) 648–683.
[27] D. Gale, L.S. Shapley, College admissions and the stability of marriage, Am. Math. Mon. 69 (1962) 9–15.
[28] Y.A. Gonczarowski, N. Nisan, L. Kovalio, A. Romm, Matching for the Israeli "mechinot" gap year: handling rich diversity requirements, in: Proceedings of the 20th ACM Conference on Economics and Computation, 2019, p. 321.
[29] M. Goto, A. Iwasaki, Y. Kawasaki, R. Kurata, Y. Yasuda, M. Yokoo, Strategyproof matching with regional minimum and maximum quotas, Artif. Intell. 235 (2016) 40–57.
[30] I.E. Hafalir, M.B. Yenmez, M. Yildirim, Effective affirmative action in school choice, Theor. Econ. 8 (2013) 325–363.
[31] N. Hamada, C. Hsu, R. Kurata, T. Suzuki, S. Ueda, M. Yokoo, Strategy-proof school choice mechanisms with minimum quotas and initial endowments, Artif. Intell. 249 (2017) 47–71.
[32] H. Hosseini, V. Menon, N. Shah, S. Sikdar, Necessarily optimal one-sided matchings, in: Thirty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2021, Thirty-Third Conference on Innovative Applications of Artificial Intelligence, IAAI 2021, the Eleventh Symposium on Educational Advances in Artificial Intelligence, EAAI 2021, Virtual Event, February 2-9, 2021, AAAI Press, 2021, pp. 5481–5488.
[33] C. Huang, Classified stable matching, in: Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2010, pp. 1235–1253.
[34] R.W. Irving, T. Kavitha, K. Mehlhorn, D. Michail, K.E. Paluch, Rank-maximal matchings, ACM Trans. Algorithms 2 (2006) 602–610.
[35] A. Ismaili, N. Hamada, Y. Zhang, T. Suzuki, M. Yokoo, Weighted matching markets with budget constraints, J. Artif. Intell. Res. 65 (2019) 393–421.
[36] Y. Kamada, F. Kojima, Efficient matching under distributional constraints: theory and applications, Am. Econ. Rev. 105 (2015) 67–99.
[37] Y. Kamada, F. Kojima, Recent developments in matching with constraints, Am. Econ. Rev. 107 (2017) 200–204.

[38] A.S. Kelso, V.P. Crawford, Job matching, coalition formation, and Gross substitutes, Econometrica 50 (1982) 1483–1504.
[39] D.M. Kilgour, E. Marshall, Approval balloting for fixed-size committees, in: Electoral Systems. Springer. Studies in Choice and Welfare, 2012, pp. 305–326, chapter 12.
[40] F. Kojima, School choice: impossibilities for affirmative action, Games Econ. Behav. 75 (2012) 685–693.
[41] S.D. Kominers, T. Sönmez, Designing for diversity in matching, in: Proceedings of the Fourteenth ACM Conference on Electronic Commerce, EC 2013, Philadelphia, PA, USA, June 16-20, 2013, 2013, pp. 603–604.
[42] S.D. Kominers, T. Sönmez, Matching with slot-specific priorities: theory, Theor. Econ. 11 (2016) 683–710.
[43] R. Kurata, N. Hamada, A. Iwasaki, M. Yokoo, Controlled school choice with soft bounds and overlapping types, J. Artif. Intell. Res. 58 (2017) 153–184.
[44] K. Liu, K. Yahiro, M. Yokoo, Strategyproof mechanism for two-sided matching with resource allocation, Artif. Intell. 316 (2023) 103855.
[45] D.F. Manlove, Algorithmics of Matching Under Preferences, World Scientific, 2013.
[46] D. Michail, Reducing rank-maximal to maximum weight matching, Theor. Comput. Sci. 389 (2007) 125–132.
[47] T. Nguyen, R. Vohra, Stable matching with proportionality constraints, Oper. Res. 67 (2019) 1503–1519.
[48] P.A. Pathak, T. Sönmez, M.U. Ünver, M.B. Yenmez, Fair allocation of vaccines, ventilators and antiviral treatments: leaving no ethical value behind in health care rationing, in: Proceedings of the 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18-23, EC 2021, ACM, 2021, pp. 785–786.
[49] J. Peters, Online elicitation of necessarily optimal matchings, in: Thirty-Sixth AAAI Conference on Artificial Intelligence, AAAI 2022, Thirty-Fourth Conference on Innovative Applications of Artificial Intelligence, IAAI 2022, the Twelveth Symposium on Educational Advances in Artificial Intelligence, EAAI 2022 Virtual Event, February 22 - March 1, 2022, AAAI Press, 2022, pp. 5164–5172.
[50] A.E. Roth, The college admissions problem is not equivalent to the marriage problem, J. Econ. Theory 36 (1985) 277–288.
[51] A. Schrijver, Combinatorial Optimization - Polyhedra and Efficiency, Springer, 2003.
[52] T. Sönmez, M.B. Yenmez, Affirmative action with overlapping reserves, http://fmwww.bc.edu/EC-P/wp990.pdf, 2020. Manuscript.
[53] T. Sönmez, M.B. Yenmez, Affirmative action in India via vertical and horizontal reservations, Econometrica 90 (2022) 1143–1176.
[54] Z. Sun, T. Todo, T. Walsh, in: Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021, 2021, pp. 419–425, ijcai.org.
[55] Z. Sun, T. Todo, M. Yokoo, in: Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19–27 August 2021, 2021, pp. 412–418.
[56] T. Suzuki, A. Tamura, K. Yahiro, M. Yokoo, Y. Zhang, Strategyproof allocation mechanisms with endowments and m-convex distributional constraints, Artif. Intell. 315 (2023) 103825, https://doi.org/10.1016/j.artint.2022.103825.
[57] Y. Zhang, K. Yahiro, N. Barrot, M. Yokoo, Strategyproof and fair matching mechanism for union of symmetric m-convex constraints, in: Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018, July 13-19, 2018, Stockholm, Sweden, 2018, pp. 590–596.