Optimal bailouts and strategic debt forgiveness in financial networks ☆

Panagiotis Kanellopoulos [a], [ID], Maria Kyropoulou [a], [ID], Hao Zhou [b], [ID], *

[a] *School of Computer Science and Electronic Engineering, University of Essex, CO4 3SQ, United Kingdom*
[b] *School of Software, Nanchang University, 330047, China*

A B S T R A C T

A financial system is represented by a network, where nodes correspond to banks, and directed labeled edges correspond to debt contracts between banks. Once a payment schedule has been defined, the liquidity of the system is defined as the sum of total payments made in the network. Maximizing systemic liquidity is a natural objective of any financial authority, so, we study the setting where the financial authority offers bailout money to some bank(s) or forgives the debts of others in order to help them avoid costs related to default, and, hence, maximize liquidity. We investigate the approximation ratio provided by the greedy bailout policy compared to the optimal one, and we study the computational hardness of finding the optimal debt-removal and budget-constrained optimal bailout policy, respectively.

We also study financial systems from a game-theoretic standpoint. We observe that the removal of some incoming debt might be in the best interest of a bank, if that helps one of its borrowers remain solvent and avoid costs related to default. Assuming that a bank's well-being (i.e., utility) is aligned with the incoming payments they receive from the network, we define and analyze a game among banks who want to maximize their utility by strategically giving up some incoming payments. In addition, we extend the previous game by considering bailout payments. After formally defining the above games, we prove results about the existence and quality of pure Nash equilibria, as well as the computational complexity of finding such equilibria.

1. Introduction

A financial system comprises a set of institutions, such as banks, that engage in financial transactions. The interconnections showing the liabilities (financial obligations or debts) among the banks can be represented by a network, where the nodes correspond to banks and the edges correspond to liability relations. Each bank has a fixed amount of external assets (not affected by the network) which are measured in the same currency as the liabilities. A bank's total assets comprise its external assets and its incoming payments, and may be used for (outgoing) payments to its lenders. If a bank's assets are not enough to cover its liabilities, that bank will be in default and the value of its assets will be decreased (e.g., by liquidation); the extent of this decrease is captured by *default costs* and essentially implies that the corresponding bank will have only a part of its total assets available for making payments.

On the liquidation day (also known as clearing), each bank in the system has to pay its debts in accordance with the following three principles of bankruptcy law (see, e.g., [13]): i) *absolute priority*, i.e., banks with sufficient assets pay their liabilities in full, ii) *limited liability*, i.e., banks with insufficient assets to pay their liabilities are in default and pay all of their assets to lenders, subject to default costs, and iii) *proportionality*, i.e., in case of default, payments to lenders are made in proportion to the respective liability. Payments that satisfy the above properties are called *clearing payments* and (perhaps surprisingly) these payments are not uniquely defined for a given financial system. However, maximal clearing payments, i.e., ones that point-wise maximize all corresponding payments, are known to exist and can be efficiently computed [36].

Importantly, banks that are in default after clearing cannot fully repay their liabilities, which can trigger a cascading effect on the network and lead to a collapse of the financial system. Previous work has considered several systemic risk measures (see, e.g., [8,29,14]), in an attempt to quantify the risk of default contagion. In this paper, we consider the total liquidity of a financial system, i.e. the sum of payments made at clearing, as a natural metric for the well-being of the system [30,10]. Financial authorities, e.g., governments or other regulators, wish to keep the systemic liquidity as high as possible and they might interfere, if their involvement is necessary and would considerably benefit the system. For example, in the not so far past, the Greek government (among others) took loans in order to bailout banks that were in danger of defaulting, to avert collapse. In this work, we study the possibility of a financial regulating authority performing cash injections (i.e., bailouts) to selected bank(s) and/or forgiving debts selectively, with the aim of maximizing the total liquidity of the system (total money flow). Similarly to cash injections, it is a fact that debt removal can have a positive effect on systemic liquidity. Indeed, the existence of default costs can lead to the counter-intuitive phenomenon whereby removing a debt/edge from the financial network might result in increased money flow, e.g., if the corresponding borrower avoids default costs because of the removal.

Even more surprising than the increase of liquidity by the removal of debts, is the fact that the removal of an edge from borrower *b* to lender *l* might result in *l* receiving more incoming payments, e.g., if *b* avoids default costs and there is an alternative path in the network where money can flow from *b* to *l*. This motivates the definition of an edge-removal game on financial networks, where banks act as strategic agents who wish to maximize their total assets and might intentionally give-up a part of their due incoming payments towards this goal. As implied earlier, removing an incoming debt could rescue the borrower from financial default, thereby avoiding the activation of default costs, and potentially increasing the lender's utility (total assets). This strategic consideration is meaningful both in the context where a financial authority performs cash injections or not. We consider the existence, quality, and computation of equilibria that arise in such games.

1.1. Our contribution

We consider computational problems related to maximizing systemic liquidity, when a financial authority can modify the network by appropriately removing debt, or by injecting cash into selected banks. We also consider financial network games where agents (corresponding to banks) can choose to forgive incoming debts.

We show how to compute the optimal cash injection policy in polynomial time when there are no default costs, by solving a linear program; the problem is **NP**-hard when non-trivial default costs apply. As our LP-based algorithm requires knowledge of the available budget and leads to non-monotone payments, we study the approximation ratio of a greedy cash injection policy that is budget-agnostic and guarantees monotone payments. Regarding debt removal, we prove that the problem of finding the set of liabilities whose removal maximizes systemic liquidity is **NP**-hard, and so are relevant optimization problems.

Regarding edge-removal games, with or without bailout, we study the existence and the quality of Nash equilibria, while also addressing computational complexity questions. Apart from arguing about well-established notions, such as the Price of Anarchy and the Price of Stability, we introduce the notion of the Effect of Anarchy (Stability, respectively) as a new measure on the quality of equilibria in this setting.

A preliminary version of this paper appeared in [24]. Besides providing full and polished proofs of all results, more extensive discussion of related work, and formal definitions of the computational problems we consider, this paper extends [24] by including a new result (Theorem 2) on the computational complexity of the greedy algorithm and a refinement of the definitions of the Effect of Anarchy/Stability. In particular, we distinguish between positive and adverse impact of strategic behavior, compared to the initial network, and define the Positive and Negative, respectively, Effect of Anarchy and Stability.

1.2. Related work

The interconnectedness of the modern financial system is widely regarded as a key factor contributing to the recent financial crisis. Systemic risk refers to the risk of failure that the financial system is prone to, due to its inherent characteristics. Various systemic risk measures have been discussed in the literature (see in, e.g., [8,29,14]). Moreover, a large body of work has emerged on default contagion in various financial network models, see in, e.g., [13,17,16,1]. Our model is based on the seminal work of Eisenberg and Noe [13] who introduced a widely adopted model for financial networks, assuming debt-only contracts and proportional payments. This was later extended by Rogers and Veraart [36] to allow for default costs; Rogers and Veraart prove the existence of maximal clearing payments when non-trivial default costs apply, and present an efficient algorithm to compute them. Schuldenzucker et al. [38] show the complexity of finding clearing payments with credit default swaps. Additional features have been since introduced, see e.g., [39] and [33] for models including credit default swaps. Ioannidis et al. [20] examine clearing solutions' irrationality and approximation strength, while Ioannidis et al. [21] explore the intractability of clearing problems in networks with derivatives and payment priorities.

When the financial regulator has available funds to bailout each bank of the network, Jackson and Pernoud [23] consider the minimum bailout budget to ensure systemic solvency and prove that computing it is an **NP**-hard problem. When the financial authority has limited bailout budget, Demange [9] proposes the threat index as a means to determine which banks should receive cash during a default episode and suggests a greedy algorithm for this process. Calafiore et al. [7] introduce a new concept termed default resilience margin that measures the maximum amplitude of assets price fluctuations without generating additional insolvencies. Egressy and Wattenhofer [12] study how central banks should decide which insolvent banks to bailout and formulate corresponding optimization problems, while considering the market value as a metric for the welfare of the system. They derive several hardness results for various natural objectives, most of which assume non-trivial default costs. Dong et al. [10] introduce an efficient greedy-based clearing algorithm for an extension of the Eisenberg-Noe model, while also studying bailout policies when banks in default have no assets to distribute. Note that the problem of injecting cash (as subsidies) in financial networks has been studied (in a different context) in microfinance markets [22]. Klages-Mundt and Minca [27] investigate the optimal bailout policy within a financial network model featuring cross-holdings and portfolio overlaps. Most relevant to our setting is the paper by Papachristou and Kleinberg [32] who consider a setting where there is a fixed amount that can be offered to each bank and prove that finding the optimal *binary*[1] bailout scheme is an **NP**-hard problem; they design and analyze approximation algorithms, and prove hardness-of-approximation results. Furthermore, in [31], they extend the static Eisenberg-Noe model in a dynamic manner, formulate the optimal cash injection problem as a Markov Decision Process, present how to efficiently compute optimal policies when amounts can come from a range and provide approximation algorithms for the setting where there is a single fixed amount that can be offered to each bank.

Relevant to debt forgiveness (or removal) is the concept of network/portfolio compression [11], which can be seen as the removal of a sequence of debts that form a cycle. Schuldenzucker and Seuken [37] analyze the impact of compression on social welfare and banks' incentives to accept compression proposals. Veraart [42] derives necessary structural conditions under which portfolio compression could be detrimental to the health of the financial system, while Amini and Feinstein [4] introduce a formulation of the optimal network compression problem for financial systems and demonstrate that it is computationally intractable for different compression models.

When taking the banks' incentives also into account, a growing body of recent work considers strategic aspects in financial networks. Departing from the principle of proportionality [13], Bertschinger et al. [6] frame payments in the event of bankruptcy as a decision-making process. They propose two distinctive payment schemes, namely unit-ranking (or, coin-ranking), and edge-ranking, respectively, and examine the properties of equilibria in games that emerge in each payment scheme. Subsequent research by Kanellopoulos et al. [26] delves into priority-proportional payment schemes. Papp and Wattenhofer [33] study the banks' incentives for redefining liabilities and donating external assets. In a follow-up study [35], they consider the reversible network model where banks may return from insolvency, and point out that early defaulting can be an advantageous strategy for banks in some cases, while Allouch et al. [3] investigate the behavior of strategic default in a two-period financial network, and show the existence of Nash equilibria of this default game and develop an algorithm to find them. Additionally, Papp and Wattenhofer [34] study the effects of debt swaps in risk mitigation, while Froese et al. [15] examine the algorithmic properties of debt swaps with ranking-based clearing. In recent contributions, Bertschinger et al. [5] study the existence and structure of equilibria in a game-theoretic model of fire sales, while Kanellopoulos et al. [25] examine debt transfer games, where banks strategically decide whether to transfer their debt claims. This theoretical study is complemented by empirical game analysis. Hoefer et al. [19] study claims trading in financial networks and analyze the structural properties and computational aspects across various forms of claims trading. Zhou et al. [43] analyze prepayment games in the existence, inefficiency, and computation of resulting equilibria. Tong et al. [40] focus on strategic donations from solvent banks to distressed banks, demonstrating how such donations can potentially mitigate losses and prevent default cascades. Meanwhile, in [41], the same group of authors explore the impact of selfish debt cancellations on systemic risk. To the best of our knowledge, our work is the first to investigate a model for financial networks where agents consider forgiving debts as strategic actions.

2. Preliminaries

A *financial network* $G = (V, E)$ consists of a set $V = \{v_1, \ldots, v_n\}$ of n banks, where each bank v_i initially has some non-negative *external assets* $e_i \in \mathbb{R}_{\geq 0}$ corresponding to income received from entities outside the financial system. Banks have payment obligations, i.e., *liabilities*, among themselves. In particular, a *debt contract* creates a liability l_{ij} of bank v_i (the borrower) to bank v_j (the lender); we assume that $l_{ij} \geq 0$ and $l_{ii} = 0$. Note that $l_{ij} > 0$ and $l_{ji} > 0$ may both hold simultaneously. Also, let $L_i = \sum_j l_{ij}$ be the total liabilities of bank v_i. Banks with sufficient funds to pay their obligations in full are called *solvent* banks, while ones that cannot are *in default*. Then, the relative liability matrix $\Pi \in \mathbb{R}^{n \times n}$ is defined by

$$\pi_{ij} = \begin{cases} l_{ij}/L_i, & \text{if } L_i > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let p_{ij} denote the actual payment[2] from v_i to v_j; we assume that $p_{ii} = 0$. These payments define a payment matrix $\mathbf{P} = (p_{ij})$ with $i, j \in [n]$, where by $[n]$ we denote the set of integers $\{1, \ldots, n\}$. We denote by $p_i = \sum_{j \in [n]} p_{ij}$ the total outgoing payments of bank v_i.

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[1] We remark that Papachristou and Kleinberg [32] refer to such cash injection policies, where a bank is either allocated a given fixed amount or not, as *discrete* and these are different to the integer payments considered in Theorem 4a of our paper.

[2] Note that the actual payment need not equal the liability, i.e., the payment obligation.

A bank in default may need to liquidate its external assets or make payments to entities outside the financial system (e.g., to pay wages). This is modeled using *default costs* defined by values $\alpha, \beta \in [0,1]$. A bank in default can only use an α fraction of its external assets and a β fraction of its incoming payments (the case without default costs is captured by $\alpha = \beta = 1$). The absolute priority and limited liability regulatory principles, discussed in the introduction, imply that a solvent bank must repay all its obligations to all its lenders, while a bank in default must repay as much of its debt as possible, taking default costs also into account; the requirement for *proportional payments* dictates how payments happen in the latter case. Summarizing, it holds that $p_{ij} \leq l_{ij}$ and, in particular, $\mathbf{P} = \Phi(\mathbf{P})$, where

$$\Phi(\mathbf{x})_{ij} = \begin{cases} l_{ij}, & \text{if } L_i \leq e_i + \sum_{j=1}^{n} x_{ji} \\ (\alpha e_i + \beta \sum_{j=1}^{n} x_{ji}) \cdot \pi_{ij}, & \text{otherwise.} \end{cases}$$

Proportional payments $\mathbf{P}$ that satisfy these constraints are called *clearing payments* and have been frequently studied in the financial literature (e.g., in [13,36,9]).

Given clearing payments $\mathbf{P}$, the *total assets* $a_i(\mathbf{P})$ of bank v_i are defined as the sum of external assets plus incoming payments, i.e.,

$$a_i(\mathbf{P}) = e_i + \sum_{j \in [n]} p_{ji}.$$

Maximal clearing payments, i.e., ones that point-wise maximize all corresponding payments[3] (and hence total assets), are known to exist [13,36] and can be computed in polynomial time. As is standard practice, we focus on these payments throughout the paper, also to avoid ambiguity.

We measure the total *liquidity* of the system (also referred to as systemic liquidity) $\mathcal{F}(\mathbf{P})$ as the sum of payments traversing through the network, i.e.,

$$\mathcal{F}(\mathbf{P}) = \sum_{i \in [n]} \sum_{j \in [n]} p_{ji}.$$

We sometimes omit dependence on $\mathbf{P}$ for clarity of exposition when the payments are clear from the context.

We assume that there exists a financial authority (a *regulator*) who aims to maximize the systemic liquidity. In particular, the regulator can decide to remove certain debts (edges) from the network or inject cash to some bank(s). In the latter case, we assume the regulator has a total *budget M* available in order to perform *cash injections* to individual banks. We sometimes refer to the total increased liquidity, $\Delta \mathcal{F}$, (as opposed to total liquidity) which measures the difference in the systemic liquidity before and after the cash injections. A *cash injection policy* is a sequence of pairs of banks and associated transfers $((i_1, t_1), (i_2, t_2), \ldots (i_L, t_L)) \in (V \times \mathbb{R})^L$, such that the regulator gives capital t_1 to bank i_1, t_2 to bank i_2, etc. We note that t_k can be a fractional number unless stated otherwise. These actions naturally define two corresponding optimization problems on the total (increased) liquidity, i.e., *optimal cash injection* and *optimal debt removal*.

Optimal Cash Injection *(OPT-CASH-INJECTION)*. Given a financial network G and a total budget M, compute the cash injection policy that maximizes the total increased liquidity $\Delta \mathcal{F}$.

Optimal Debt Removal *(OPT-DEBT-REMOVAL)*. Given a financial network G, compute a collection of edges whose removal maximizes the total liquidity $\mathcal{F}$.

We will also find useful the notion of the *threat index*,[4] μ_i, of bank v_i, which captures how many units of total increased liquidity will be realized if the financial authority injects one unit of cash into bank v_i's external assets [9]; a unit of cash represents a small enough amount of money so that the set of banks in default would not change after the cash injection. Naturally, the threat index of solvent banks is 0, while the threat index of banks in default will be at least 1. Formally, the threat index is defined as

$$\mu_i = \begin{cases} 1 + \sum_{j \in D} \pi_{ij} \mu_j, & \text{if } a_i(\mathbf{P}) < L_i \\ 0, & \text{otherwise,} \end{cases}$$

where $D = \{j \mid a_j(\mathbf{P}) < L_j\}$ is the set of banks who are in default.[5] We remark that for the maximum total increased liquidity it holds $\Delta \mathcal{F} \leq M \cdot \mu_{\max}$, where $\mu_{\max}$ is the maximum threat index.

An example. Fig. 1 provides an example of a financial network, inspired by an example in [9]. The clearing payments are as follows: $p_{21} = 4.4$, $p_{32} = 3.2$, and $p_{43} = p_{45} = 1$, implying that banks v_2, v_3 and v_4 are in default. We assume that there are no default costs,

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[3] In other words, a clearing payments matrix $\mathbf{P}$ is maximal if, for any other clearing payments matrix $\mathbf{P}'$, it holds $p_{ij} \geq p'_{ij}, \forall i, j \in [n]$.

[4] The term threat index aims to capture the "threat" posed to the network by a decrease in a bank's cash flow or even the bank's default; this index can be thought of as counting all the defaulting creditors that would follow a potential default of the said bank.

[5] In matrix form, the threat index for banks who are in default can be computed by $(\mathbb{I}_{D \times D} - \pi_{D \times D}) \mu_D = \mathbf{1}_D$. This is a homogeneous linear equation system where $\pi_{D \times D}$ is the relative liability matrix only involving in the banks in set D; $\mathbb{I}_{D \times D}$ and $\mathbf{1}_D$ represent the $|D| \times |D|$ dimension identity matrix and the $|D|$ dimension identity vector, respectively, while μ_D is the vector of threat indices for banks in default.
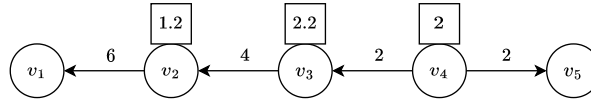
Fig. 1. A simple financial network. Nodes correspond to banks, edges are labeled with the respective liabilities, while external assets are in a rectangle above the relevant banks.

i.e., $\alpha = \beta = 1$. The threat indexes are computed as follows: $\mu_1 = \mu_5 = 0$, $\mu_2 = 1 + \mu_1$, $\mu_3 = 1 + \mu_2$, and $\mu_4 = 1 + \frac{1}{2}\mu_3 + \frac{1}{2}\mu_5$, implying that $\mu_3 = \mu_4 = 2$, $\mu_2 = 1$, while $\mu_1 = \mu_5 = 0$.

Additional definitions are deferred to the corresponding sections they are used in, for ease of exposition. Most importantly, we introduce (in Section 4) two novel notions, namely the Effect of Anarchy/Stability, that measure the effect strategic behavior has on the original network.

3. Computing optimal outcomes

In this section we present algorithmic and complexity results regarding the problems of computing optimal cash injection (see Section 3.1) and debt removal (Section 3.2) policies. Note that we omit referring to default costs in our statements for those results that hold when $\alpha = \beta = 1$.

3.1. Optimal cash injections

We begin with a positive result about computing the optimal cash injection policy when default costs do not apply. The proof utilizes a linear program which is a straightforward extension of the corresponding program for calculating clearing payments in the basic model without cash injections, see e.g. [13]. This has been also observed by Demange [9] and Ahn and Kim [2] without an explicit focus on the computational complexity aspect.

Theorem 1. *OPT-CASH-INJECTION can be solved in polynomial time.*

Proof. The proof follows by solving a linear program that computes the optimal cash injections and accompanying payments.

We denote by x_i the cash injection to bank i and by p_{ij} the payment from i to j. We aim to maximize the total liquidity, i.e., the total payments, subject to satisfying the limited liability and absolute priority principles. Recall that M is the budget, l_{ij} is the liability of i to j, e_i is the external assets of bank i, and L_i is the total liabilities of i.

$$
\begin{aligned}
\text{maximize} \quad & \sum_i \sum_j p_{ij} \\
\text{subject to} \quad & \sum_i x_i \leq M, && \forall i, j \\
& p_{ij} \leq l_{ij}, && \forall i, j \\
& p_{ij} \leq (x_i + e_i + \sum_k p_{ki}) \cdot \frac{l_{ij}}{L_i}, && \forall i, j \\
& x_i \geq 0, && \forall i \\
& p_{ij} \geq 0, && \forall i, j
\end{aligned}
$$

The first constraint corresponds to the budget constraint, while the second and third sets of constraints guarantee that no bank pays more than her total assets or more than a given liability; hence, the limited liability principle is satisfied. It remains to argue about the absolute priority principle, i.e., a bank can pay strictly less than her total assets only if she fully repays all outstanding liabilities.

Consider the optimal solution corresponding to a vector of cash injections and payments p_{ij}; we will show that this solution satisfies the absolute priority principle as well. We distinguish between two cases depending on whether a bank is solvent or in default. In the first case, consider a solvent bank i, i.e., $x_i + e_i + \sum_j p_{ji} \geq L_i$, for which $p_{ik} < l_{ik}$ for some bank k. By replacing p_{ik} with $p'_{ik} = l_{ik}$, we obtain another feasible solution that strictly increases the objective function; a contradiction to the optimality of the starting solution. Similarly, consider a bank i with $x_i + e_i + \sum_j p_{ji} < L_i$ for which $\sum_j p_{ij} < x_i + e_i + \sum_j p_{ji}$. Then, there necessarily exists a bank k for which $p_{ik} < (x_i + e_i + \sum_j p_{ji}) \cdot \frac{l_{ik}}{L_i}$ and it suffices to replace p_{ik} with $p'_{ik} = (x_i + e_i + \sum_j p_{ji}) \cdot \frac{l_{ik}}{L_i}$ to obtain another feasible solution that, again, strictly increases the objective function. Hence, we have proven that the optimal solution to the linear program satisfies the absolute priority principle and the claim follows by providing each bank i a cash injection of x_i. $\quad\square$

Note that the optimal policy does not necessarily satisfy certain desirable properties. In particular, as observed in [9], cash injections are not monotone with respect to the budget. To see that, consider the financial network in Fig. 1 and note that when $M = 0.5$, the optimal policy would give all available budget to bank v_3, while under an increased budget of 1.6, the entire budget would be allocated to v_4, hence v_3 would get nothing. Non-monotonicity in cash injections underscores the inefficiency of performing repeated

optimal allocations in a sequential setting where the bailout budget is released gradually over multiple periods. Additionally, non-monotonicity may create a perceived sense of unfairness among banks, thereby undermining the credibility and practical acceptability of the bailout mechanism. Furthermore, our LP-based algorithm crucially relies on knowledge of the available budget. In an attempt to alleviate these undesirable properties, we turn our attention to efficiently approximating the optimal cash injection policy by a natural and intuitive greedy algorithm, who only uses information on the available budget as a stopping criterion, and we compute its approximation ratio under a limited budget, when we care about the total *increased* liquidity.

Algorithm 1 The GREEDY cash injection policy.

Require: $M > 0$ ▷ Initial budget
1: $k = 0$ ▷ Let Round 0 be the initial network
2: **while** $M > 0$ and at least one bank in default **do**
3:      $k \leftarrow k + 1$;
4:      i_k: the bank with the highest threat index;
5:      t_k: the minimum amount that, if transferred to i_k, would lead to some previously defaulting bank to become solvent;
6:      Inject $\min\{M, t_k\}$ to i_k;
7:      $M \leftarrow M - \min\{M, t_k\}$
8: **end while**

Definition 1 (GREEDY's approximation ratio). The *approximation ratio* of GREEDY is defined as

$$\mathcal{R}_{\text{GREEDY}} = \min_{G,M} \frac{\Delta \mathcal{F}_{\text{GREEDY}}}{\Delta \mathcal{F}_{OPT}},$$

where the minimum is taken over all possible networks and budgets.

This ratio measures how much smaller the total *increased* liquidity (or money flow) achieved by GREEDY can be compared to the optimal total increased liquidity.

Let us revisit the example in Fig. 1, assuming a budget $M = 1.6$. Initially banks v_3 and v_4 have the highest threat index of $\mu_3 = \mu_4 = 2$ compared to $\mu_1 = \mu_5 = 0$, and $\mu_2 = 1$. We can assume[6] that bank v_3 would receive the first cash injection ($i_1 = v_3$) and in fact this will be equal to $t_1 = 0.8$. Indeed, a cash injection of 0.8 to v_3 will result in v_3 becoming solvent (notice that v_3 receives 1 from v_4), while a smaller cash injection would not impose any change on the threat index vector. At this stage, the threat index of each bank is as follows $\mu_1' = \mu_3' = \mu_5' = 0$ and $\mu_2' = \mu_4' = 1$. At this round, $i_2 = v_2$ would receive the remaining budget of $t_2 = 0.8$. Hence, the *total increased liquidity* achieved by GREEDY at this instance is $\Delta \mathcal{F}_{\text{GREEDY}} = 2.4$ (t_1 will traverse edges (v_3, v_2) and (v_2, v_1), while t_2 will traverse edge (v_2, v_1)). However, the optimal cash injection policy is to inject the entire budget $M = 1.6$ to bank v_4 resulting in $\Delta \mathcal{F}_{OPT} = 3.2$. Therefore, this instance reveals $\mathcal{R}_{\text{GREEDY}} \leq \frac{2.4}{3.2} = 3/4$.

Theorem 2. *Algorithm* GREEDY *runs in polynomial time.*

Proof. We note that in each round of GREEDY at least one (previously defaulted) bank will become irrevocably solvent, so GREEDY will terminate in at most n rounds.

We now show that any single round k of GREEDY takes polynomial time; clearly, this suffices for the result to follow. For a given network G^k at the beginning of the k-th round of GREEDY, we can run the polynomial-time Greatest Clearing Vector algorithm proposed in [36], see Theorem 3.7 and Definition 3.6 therein,[7] to compute the clearing payments. Based on these, we can determine the set of defaulted banks D^k and the total unpaid liabilities $\hat{L}_{j_k} = \sum_{i \in N}(l_{j_k,i} - p_{j_k,i})$ for each defaulted bank $j_k \in D^k$. In addition, the threat index for each bank in G^k can be computed in polynomial time by solving a system of homogeneous linear equations [9]. In particular, having calculated the set of defaulted banks D^k in the previous step, their threat index is given by the unique solution to $\mu_i = 1 + \sum_{j \in D_k} \pi_{ij} \mu_j$, $\forall i \in D^k$, otherwise $\mu_i = 0$. The bank i_k, which has the highest threat index, can then be straightforwardly selected. Furthermore, we can always find a sufficiently small value $\hat{t}_k$ of cash injection to bank i_k, such that the set of defaulted banks D^k remains unchanged after the injection. Indeed, a value $\hat{t}_k < \frac{\min_{j_k \in D^k} \hat{L}_{j_k}}{\max_i \mu_i}$ suffices as the additional payments cannot change D^k. Let δ_{i_k, j_k} represent the increase in incoming payments that bank $j_k \in D^k$ receives after this injection, which can be computed by rerunning the polynomial-time clearing algorithm. The ratio $\frac{\delta_{i_k, j_k}}{\hat{t}_k}$ measures the marginal increase in incoming payments that bank j_k receives due to the cash injection to bank i_k. Therefore, the cash injection t_k in GREEDY can be computed as

$$t_k = \min_{j_k \in D^k} \frac{\hat{L}_{j_k} \cdot \hat{t}_k}{\delta_{i_k, j_k}},$$

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[6] This is consistent to our tie breaking assumption that favors the least index.

[7] Besides [36], we refer the interested reader to [26] for a detailed exposition of (a generalized version of) the Greatest Clearing Vector algorithm and a proof of its computational complexity (see Algorithm 2 and Theorem 3 therein).
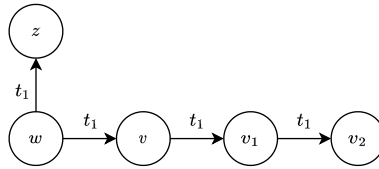
Fig. 2. An example network used in the lower bound of the approximation ratio of GREEDY for $\mu_v = 2$ and $\mu_w = 1 + \frac{1}{2} \cdot 2 = 2$. The claim in the proof is that for any arbitrary network such that the first cash injection made by GREEDY is t_1, and the highest threat index is an integer, e.g. 2 in this case, the network in this figure achieves at most the same approximation ratio, while satisfying properties (P1) and (P2).
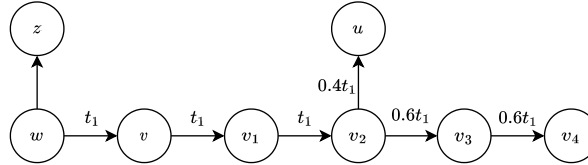

Fig. 3. Similarly to Fig. 2, for the case where the highest threat index is not an integer, e.g. $\mu_v = 3.6 = 1 + (1 + (1 + \frac{0.6t_1}{t_1} \cdot 1))$ and $\mu_w = 1 + \frac{t_1}{t_1 + t_1/2.6} \cdot \mu_v = 1 + \frac{2.6}{3.6} \cdot 3.6 = 3.6$.

where $\frac{\hat{L}_{j_k} \cdot \hat{i}_k}{\delta i_k, j_k}$ represents the minimum value of cash injection required for bank i_k such that the increased incoming payments to bank j_k can cover its not-yet-paid liabilities, making it solvent and thus changing the set of defaulted banks D^k. Clearly, computing t_k for any given round k takes polynomial time; this concludes the proof. $\square$

Theorem 3. *GREEDY's approximation ratio is at most* 3/4. *For inputs satisfying* $M \le t_1 \frac{\mu_v}{\mu_v - 1}$, *this ratio is tight.*

Proof. The upper bound follows from the instance in Fig. 1 and the discussion above; in the following, we argue about the lower bound when the total budget M satisfies the condition in the statement. We consider the following properties and claim that the worst approximation ratio of GREEDY is achieved at networks that satisfy both properties. To prove this claim we will show that starting from an arbitrary network G on which GREEDY approximates the optimal total increased liquidity by a factor of r, we can create a new network that satisfies properties (P1) and (P2), such that GREEDY approximates the optimal total increased liquidity in the new network by a factor of at most r. We can then bound the approximation ratio of GREEDY on the set of networks that satisfy these properties.

(P1) The total increased liquidity achieved by GREEDY is exactly $t_1 \mu_{i_1} + (M - t_1)$.
(P2) The optimal total increased liquidity is exactly $\mu_{i_1} M$.

Let the bank with the highest threat index in G be v and let μ_v be that threat index. If $\mu_v = 1$ or $M \le t_1$ then the claim is true (GREEDY trivially achieves an approximation ratio of 1), so we henceforth assume that $\mu_v > 1$ and $M > t_1$. We create G' (see Figs. 2 and 3) as follows. To formally describe our construction it is convenient to express the highest threat index as $\mu_v = x + \frac{a}{a+b}$ for some $a, b > 0$ that satisfy $a + b = t_1$ and x a fixed integer. If μ_v is an integer, then we set $x = \mu_v - 1$ (which implies $a = t_1$ and $b = 0$), while if μ_v is not an integer, we set $x = \lfloor \mu_v \rfloor$ and $a = (\mu_v - \lfloor \mu_v \rfloor)t_1$. Our network G' comprises $\lceil \mu_v \rceil + 4$ nodes, i.e., u, v, w, z and v_i, for $i = 1, \ldots, \lceil \mu_v \rceil$. There is a directed path of length $\lceil \mu_v \rceil$, including the following nodes in sequence v and v_i, for $i = 1, \ldots, \lceil \mu_v \rceil$, where the first $\lfloor \mu_v \rfloor - 1$ edges have liability t_1. The remaining edge(s) on that path has/have liability a and there is an edge $(v_{\lceil \mu_v \rceil - 1}, u)$ with liability b. Moreover, there are edges (w, v) and (w, z) such that $l_{wv} = t_1$ and $l_{wz} = \frac{t_1}{(\mu_v - 1)}$. Note that the liabilities of the two outgoing edges of w are selected so that w and v both have the highest threat index in G'; indeed, the threat index of v in G' is $\mu'_v = \mu_v$, and the threat index of w in G' is $\mu'_w = 1 + \frac{l_{wv}}{L_w} \mu_v = \mu_v$, where $L_w = l_{wv} + l_{wz}$. Immediate consequences of our construction are (i) $\mu_v = \frac{L_w}{l_{wz}}$ and (ii) $\frac{l_{wv}}{L_w} = \frac{\mu_v - 1}{\mu_v}$ which will be useful later.

To see that G' satisfies property (P1), it suffices to consider that in the first step, GREEDY offers a cash injection of t_1 to v in G'; w is then the only node in default and has threat index equal to 1. The total increased liquidity achieved by GREEDY in G' is exactly $t_1 \mu_v + (M - t_1)$, while GREEDY achieves at least that total increased liquidity in G since each node in default has threat index at least equal to 1. Assume now that the regulator offers the entire budget to node w in G'; this would result to total increased liquidity of $\mu'_w M$ as required by (P2), and since, by construction $\mu'_w = \mu_v$ is the maximum total increased liquidity in G, it holds that $\mu'_w M$ is an upper bound on the optimal total increased liquidity in the original network too. We can conclude that it is without loss of generality to restrict attention to networks that satisfy properties (P1) and (P2) when proving a lower bound on the approximation ratio of GREEDY.

Overall, it holds that the approximation ratio of GREEDY is lower bounded by

$$\mathcal{R}_{\text{GREEDY}} \ge \min_{\mathcal{G}^*, M} \frac{\Delta \mathcal{F}_{\text{GREEDY}}}{\Delta \mathcal{F}_{OPT}}$$

$$\geq \min_{\mathcal{G}^*, M} \frac{t_1 \mu_v + (M - t_1)}{M \mu_v},$$

where the minimum ranges over the set of networks $\mathcal{G}^*$ satisfying properties (P1) and (P2). Straightforward calculations after substituting $\mu_v = \frac{L_w}{l_{wz}}$ which holds by construction, leads to

$$\mathcal{R}_{\text{GREEDY}} \geq \min_{\mathcal{G}^*, M} \left\{ 1 + \left(\frac{t_1}{M} - 1 \right) \frac{l_{wv}}{L_w} \right\}$$
$$\geq \min_{\mathcal{G}^*} \left\{ 1 + \left(\frac{\mu_v - 1}{\mu_v} - 1 \right) \frac{\mu_v - 1}{\mu_v} \right\}$$
$$\geq 1 - 1/4$$
$$= 3/4,$$

where the second inequality holds by assumption that $M \leq t_1 \frac{\mu_v}{\mu_v - 1}$ and since, by construction $\frac{l_{wv}}{L_w} = \frac{\mu_v - 1}{\mu_v}$, while the third inequality holds since $-1/4$ is a global minimum of function $f(x) = (x - 1)x$. $\quad \square$

We note that the approximation ratio in Theorem 3 remains valid and serves as a lower bound when we consider total liquidity itself, i.e., $\frac{\mathcal{F}_{\text{GREEDY}}}{\mathcal{F}_{OPT}} = \frac{\mathcal{F}^i + \Delta \mathcal{F}_{\text{GREEDY}}}{\mathcal{F}^i + \Delta \mathcal{F}_{OPT}} \geq \frac{\Delta \mathcal{F}_{\text{GREEDY}}}{\Delta \mathcal{F}_{OPT}}$, where $\mathcal{F}^i$ is non-negative and represents the total liquidity without any cash injections. Next, we conclude this section with some computational complexity results. We note the contrast to similar results in the literature, which apply to slightly different models, e.g. "binary" bailout decisions in [32], or ensuring systemic solvency, i.e., all agents become solvent, in [23].

Theorem 4. *The following problems are* **NP**-*hard:*

  a) *OPT-CASH-INJECTION under the constraint of integer payments.*
  b) *OPT-CASH-INJECTION with default costs $\alpha \in [0, 1)$ and $\beta \in [0, 1]$.*
  c) *Compute the minimum budget so that a given agent becomes solvent, with default costs $\alpha \in [0, 1/2)$ and $\beta \in [0, 1]$.*

Proof. We begin with the case where all proportional payments need to be integers, and then prove the cases where default costs apply.

Hardness of computing the optimal cash injection policy under integer payments. We warm-up with a rather simple reduction from EXACT COVER BY 3-SETS (X3C), a well-known **NP**-complete problem. An instance of X3C consists of a set X of $3k$ elements together with a collection C of size-3 subsets of X. The question is whether there exists a subset $C' \subseteq C$ of size k such that each element in X appears exactly once in C'.

We build an instance of our problem as follows. We add an agent u_i for each $c_i \in C$ and an agent t_i for each element $x_i \in X$; we also add agent T. There are no external assets and the liabilities are as follows. Each agent u_i, corresponding to $c_i = \{a, b, c\}$ where a, b, c are elements in X, has liability 1 to each of the three agents t_a, t_b, t_c. Furthermore, each agent t_i has liability of 1 to agent T, while we assume that the budget equals $3k$.

Due to the integrality constraint and the fact that payments are proportional, a yes-instance for X3C, admitting a solution C', leads to a solution of liquidity $6k$, by injecting a payment of 3 to each u_i corresponding to the set $c_i \in C'$.

On the other direction, we argue that any solution with liquidity at least $6k$ leads to a solution for the instance of X3C. Indeed, any such solution must necessarily lead to liquidity of at least $3k$ to the edges from the u_i agents to the t_i agents. Since the budget equals $3k$ this implies that exactly k of the u_i agents must receive a payment of 3 and these agents should cover the entire set of the t_i agents.

Hardness of computing the optimal cash injection policy with default costs. Our proof follows by a reduction from the PARTITION problem, a well-known **NP**-complete problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \ldots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i \in X'} x_i = \sum_{i \notin X'} x_i = \frac{1}{2} \sum_{i \in X} x_i$.

The reduction works as follows. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ by adding an agent v_i for each element $x_i \in X$ and allocating an external asset of $e_i = x_i$ to v_i; we also include three additional agents S, T and G. Each agent v_i has liability equal to $\frac{4e_i}{3}$ to S and equal to $\frac{2e_i}{3}$ to T, while S has liability $\frac{2+\alpha}{3} \sum_i e_i$ to G; see also Fig. 4. We assume the presence of default costs $\alpha \in [0, 1)$, and $\beta \in [0, 1]$, while the budget is $M = \frac{1}{2} \sum_i e_i$; clearly, the reduction requires polynomial-time.

We first show that if $\mathcal{I}$ is a yes-instance for PARTITION, then the total liquidity is $\mathcal{F} = \frac{5\alpha + 10}{6} \sum_i e_i$. Indeed, consider a solution X' for instance $\mathcal{I}$ satisfying $\sum_{i \in X'} x_i = \frac{1}{2} \sum_{i \in X} x_i$, and let the set B contain agents v_i where $x_i \in X'$. Then, since $M = \frac{1}{2} \sum_i e_i = \sum_{i \in B} e_i$, we choose to inject an amount of e_i to any agent $v_i \in B$. The total assets of agent S are then

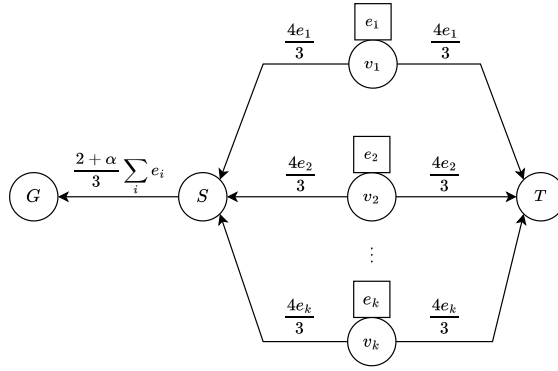$$a_S = \frac{4}{3} \sum_{i \in B} e_i + \frac{2\alpha}{3} \sum_{i \notin B} e_i$$

Fig. 4. The reduction used to show hardness of computing the optimal cash injection policy when $\alpha < 1$.

$$= \frac{2+\alpha}{3} \sum_i e_i,$$

and, hence, S is solvent. The total liquidity in this case is

$$\mathcal{F} = 2 \sum_{i \in B} e_i + \alpha \sum_{i \notin B} e_i + \frac{2+\alpha}{3} \sum_i e_i$$

$$= \frac{5\alpha + 10}{6} \sum_i e_i,$$

as desired.

We now show that any cash injection policy that leads to a total liquidity of at least $\frac{5\alpha+10}{6} \sum_i e_i$ leads to a solution for instance $\mathcal{I}$ of PARTITION. Assume any such cash injection policy and let B be the set of v_i agents that become solvent by it. Denote by $\mathcal{F}_v$ the liquidity arising solely from the payments made by the agents v_i, for $i = 1, \ldots, k$, to their direct neighbors, and we denote by t_S the amount of cash injected to S, we get that

$$\mathcal{F}_v = 2 \sum_{i \in B} e_i + \left(\sum_{i \notin B} e_i + \frac{\sum_i e_i}{2} - \sum_{i \in B} e_i - t_S \right) \cdot \alpha$$

$$= 2 \sum_{i \in B} e_i + \left(\frac{3}{2} \sum_i e_i - 2 \sum_{i \in B} e_i - t_S \right) \cdot \alpha$$

$$\leq 2(1-\alpha) \sum_{i \in B} e_i + \frac{3\alpha}{2} \sum_i e_i. \qquad (1)$$

It also holds that

$$\mathcal{F}_v \geq \left(1 + \frac{\alpha}{2} \right) \sum_i e_i \qquad (2)$$

since, by assumption, the policy under consideration leads to a total liquidity of at least $\frac{5\alpha+10}{6} \sum_i e_i$ and the payment from S to G can be at most $\frac{2+\alpha}{3} \sum_i e_i$.

Combining Inequalities (1) and (2), we get that

$$2(1-\alpha) \sum_{i \in B} e_i + \frac{3\alpha}{2} \sum_i e_i \geq \left(1 + \frac{\alpha}{2} \right) \sum_i e_i, \qquad \Longleftrightarrow \qquad 2(1-\alpha) \sum_{i \in B} e_i \geq (1-\alpha) \sum_i e_i,$$

which is equivalent to $\sum_{i \in B} e_i \geq \frac{1}{2} \sum_i e_i$ for $a < 1$. Moreover, note that each agent $v_i \in B$ needs (at least) an extra e_i to become solvent and, hence, it must be $\sum_{i \in B} e_i \leq \frac{1}{2} \sum_i e_i$, due to the budget constraint. We can conclude that $\sum_{i \in B} e_i = \frac{1}{2} \sum_i e_i$ and, hence, we can obtain a solution to instance $\mathcal{I}$ of PARTITION, as desired.

Hardness of computing the minimum budget that makes an agent solvent. The proof follows by the same reduction as in the previous case. We will prove that computing the minimum budget necessary to make agent S solvent corresponds to solving an instance from PARTITION. As before, whenever instance $\mathcal{I}$ admits a solution X', we inject an amount of e_i to each agent v_i such that $x_i \in X'$, and, as in the previous case, we obtain that a budget of $\frac{1}{2} \sum_i e_i$ suffices to make S solvent. We now argue that any cash injection policy with a budget of $\frac{1}{2} \sum_i e_i$ that can make agent S solvent leads to a solution for instance $\mathcal{I}$ when the default costs are $\alpha \in [0, 1/2), \beta \in [0, 1]$.

Let t_i be the cash injected at agent v_i and let t_S be the cash injected directly at agent S; clearly, $t_S + \sum_i t_i \leq \frac{\sum_i e_i}{2}$. As before, let B be the set of v_i agents that become solvent by the cash injection policy. Clearly, if $\sum_{i \in B} e_i = \frac{1}{2}\sum_i e_i$, we immediately obtain a solution to the PARTITION instance. Otherwise, $\sum_{i \in B} e_i < \frac{1}{2}\sum_i e_i$ and the total assets of S are

$$
\begin{aligned}
a_S &= \frac{4}{3}\sum_{i \in B} e_i + \frac{2\alpha}{3}\sum_{i \notin B}(e_i + t_i) + t_S \\
&\leq \frac{4}{3}\sum_{i \in B} e_i + \frac{2\alpha}{3}\sum_{i \notin B} e_i + t_S + \sum_{i \notin B} t_i \\
&= \frac{4}{3}\sum_{i \in B} e_i + \frac{2\alpha}{3}\sum_{i \notin B} e_i + \frac{\sum_i e_i}{2} - \sum_{i \in B} e_i \\
&= \frac{1}{3}\sum_{i \in B} e_i + \frac{1}{2}\sum_i e_i + \frac{2\alpha}{3}\sum_{i \notin B} e_i \\
&= \frac{1}{3}\sum_{i \in B} e_i + \frac{1}{2}\sum_i e_i + \frac{2\alpha}{3}\left(\sum_i e_i - \sum_{i \in B} e_i\right) \\
&= \left(\frac{1}{2} + \frac{2\alpha}{3}\right)\sum_i e_i + \frac{1 - 2\alpha}{3}\sum_{i \in B} e_i \\
&< \left(\frac{1}{2} + \frac{2\alpha}{3}\right)\sum_i e_i + \frac{1 - 2\alpha}{3}\left(\frac{\sum_i e_i}{2}\right) \\
&= \frac{2}{3}\sum_i e_i + \frac{\alpha}{3}\sum_i e_i,
\end{aligned}
$$

where the second equality holds due to the budget constraint and the strict inequality holds since $\alpha < 1/2$ and $\sum_{i \in B} e_i < \frac{1}{2}\sum_i e_i$; the claim follows. $\quad\square$

3.2. Optimal debt removal

In this section, we focus on maximizing systemic liquidity by appropriately removing edges/debts. As an example, consider again Fig. 1, where the central authority can increase systemic liquidity by removing the edge between v_4 and v_5.

Theorem 5. *OPT-DEBT-REMOVAL is* **NP***-hard.*

Proof. The proof relies on a reduction from the **NP**-complete problem RXC3 [18], a variant of EXACT COVER BY 3-SETS (X3C). In RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once. Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. We add an agent t_i for each element i of X and an agent s_i for each subset i in C, as well as two additional agents, S and T. Each s_i, corresponding to set $(x, y, z) \in C$, has external assets $e_i = 4$ and liability 1 to the three agents t_x, t_y, t_z corresponding to the three elements $x, y, z \in X$. Furthermore, each s_i has liability Z to agent S, where Z is a large integer. Finally, each agent t_i has liability 1 to agent T; see also Fig. 5. Note that this construction requires polynomial time.

When instance $\mathcal{I}$ is a yes-instance with a solution C', we claim that $\mathcal{I}'$ admits a solution with systemic liquidity $14k$. Indeed, it suffices to remove all edges from the s_i agents, with $i \in C'$, towards S. Then, the liquidity due to agents s_i, with $i \in C'$, equals $6k$, while each of the $2k$ agents whose edge towards S was preserved generates a liquidity of 4.

It suffices to show that any solution that generates liquidity at least $14k$ can lead to a solution for instance $\mathcal{I}$. First, observe that it is never strictly better for the financial authority to remove an edge from some agent s_i towards an agent t_j. Let S_k and S_r be the subsets of agents s_i whose edges towards S are kept and removed, respectively, and let $\chi = |S_r|$. The liquidity traveling from agents in S_k towards their direct neighbors is exactly $4(3k - \chi)$, while the liquidity traveling from agents in S_r towards their direct neighbors is exactly 3χ. The maximum liquidity traveling from agents t_j towards T is at most $\min\{3k, 3\chi + (3k - \chi)\frac{12}{Z+3}\}$. We conclude that the maximum liquidity is bounded by $12k - \chi + \min\{3k, 3\chi + (3k - \chi)\frac{12}{Z+3}\}$.

Note that, whenever $\chi > k$, then the maximum liquidity is at bounded by $15k - \chi < 14k$. Similarly, whenever $\chi < k$ and since Z is arbitrarily large, the maximum liquidity is bounded by $12k + 2\chi + (3k - \chi)\frac{12}{Z+3} < 14k$. It remains to show that whenever $\chi = k$, a liquidity of at least $14k$ necessarily leads to a solution in $\mathcal{I}$. Indeed, by the discussion above, any such solution must have liquidity equal to $3k$ traveling from agents t_j towards T, i.e., all these liabilities are fully repaid. This, in turn, can only happen if each of the t_j agents receives a payment of at least 1 from the s_i agents. Using the assumptions that i) $\chi = k$, ii) Z is arbitrarily large, iii) payments are proportional to liabilities, and iv) each t_j has exactly three neighboring s_i agents, this property holds only when the neighbors of the χ agents in S_r are disjoint. This directly translates to a solution for instance $\mathcal{I}$ and the RXC3 problem. $\quad\square$
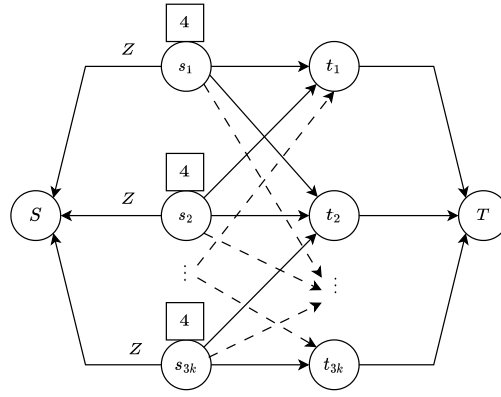
Fig. 5. The reduction used to show hardness of computing an edge-removal policy that maximizes systemic liquidity. All edges with missing labels correspond to liability 1.
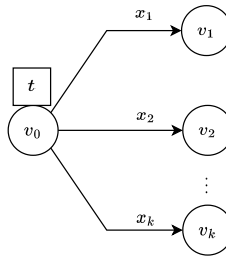

Fig. 6. An example of the reduction in the proof of Theorem 6.

We note that the objective of systemic solvency, i.e., guaranteeing that all agents are solvent, can be trivially achieved by removing all edges. However, adding a liquidity target, makes this problem more challenging.

Theorem 6. *In networks with default costs, the following problems are* **NP**-*hard:*

a) *OPT-DEBT-REMOVAL when required to ensure systemic solvency*
b) *Compute an edge set whose removal ensures systemic solvency and minimizes the amount of deleted liabilities.*
c) *Compute an edge set whose removal guarantees that a given agent is no longer in default and minimizes the amount of deleted liabilities.*

Proof. The proof for all these claims relies on a reduction from the SUBSET SUM problem, where the input consists of a set X of integers $\{x_1, x_2, \ldots, x_k\}$ and a target t and the question is whether there exists a subset $X' \subseteq X$ such that $\sum_{i \in X} x_i = t$. Given an instance $\mathcal{I}$ of SUBSET SUM, we construct an instance $\mathcal{I}'$ by adding an agent v_i for each integer x_i, adding an extra agent v_0 having $e_0 = t$ and setting the liability of v_0 to v_i to be equal to x_i; see also Fig. 6.

Since v_0 is in default, the goal becomes to remove an edge set so that the remaining liability of v_0 is at most t. Whenever instance $\mathcal{I}$ is a yes-instance for SUBSET SUM admitting a solution X', we remove edges from v_0 to agents corresponding to integers not in X'. Then, v_0 is solvent and the systemic liquidity equals t. Otherwise, if $\mathcal{I}$ is a no-instance, no edge set removal that leaves v_0 solvent can lead to systemic liquidity of (at least) t. □

4. Forgiving debt as a strategic action

In this section, we consider the case of strategic agents who have the option to forgive debt. We begin with some definitions.

Edge-removal games. Consider a financial network G of n banks who act strategically. Each bank v_i has the option to select a set of borrowers and forgive the corresponding debts they owe. More formally, the strategy set of a bank is the power set of its incoming edges and a strategy, s_i, denotes which of its incoming edges that bank will remove, thus erasing the corresponding debt owed to itself. A *pure strategy profile*, denoted by $\mathbf{s} = (s_1, \ldots, s_n)$ specifies a pure strategy for each bank; the pure strategy profile for all banks except v_i is denoted by $\mathbf{s}_{-i}$. The edge-removal game can be defined with and without cash-injections. A given strategy profile will result in realized payments through maximal clearing payments (as discussed in Section 2), including possible cash injections through a predetermined cash injection policy. Our results hold for both the optimal cash injection policy and GREEDY.

A bank is assumed to strategize over its incoming edges in order to maximize its utility, i.e., its total assets, where we remark that a possible cash injection can be seen as increasing one's external assets. The objective of the financial authority is to maximize

the total liquidity of the system, i.e., the social welfare is the sum of money flows that traverse the network. We will now define the notion of *Nash equilibrium* in an edge-removal game. Given a strategy profile $\mathbf{s}$ and the corresponding maximal clearing payments $\mathbf{P}$, the utility, i.e., the total assets, of bank v_i is $u_i(\mathbf{P}) = e_i + \sum_{j \in [n]} p_{ji}$. We say $\mathbf{s}$ is a pure Nash equilibrium if for each bank v_i with $i \in [n]$ and any alternative strategy s_i' of v_i, it holds that $u_i(\mathbf{P}) \geq u_i(\mathbf{P}')$, where $\mathbf{P}'$ denotes the maximal clearing payments under the profile $\mathbf{s} = (s_i', \mathbf{s}_{-i})$, where player v_i chooses strategy s_i while all other players keep using their equilibrium strategy. We denote the set of all Nash equilibria in a given edge-removal game by $\mathbf{s}_{eq}$. Let $\mathcal{F}_\mathbf{s}$ and $\mathcal{F}_\mathbf{G}$ be the systemic liquidity for a given strategy profile $\mathbf{s}$, and that on the original network (i.e., no edge removal), respectively.

The Price of Anarchy/Stability. The inefficiency of Nash equilibria in terms of liquidity is measured by the *Price of Anarchy* (*Stability*, respectively) [28], which is equal to the optimal systemic liquidity over that of the worst (best, respectively) pure Nash equilibrium. Note that the optimal systemic liquidity, $\mathcal{F}_{OPT}$, corresponds to the maximal one when the financial authority can dictate everyone's actions (edge-removals).

$$\text{PoA} = \max_G \max_{\mathbf{s} \in s_{eq}} \frac{\mathcal{F}_{OPT}}{\mathcal{F}_\mathbf{s}} \qquad \text{PoS} = \max_G \min_{\mathbf{s} \in s_{eq}} \frac{\mathcal{F}_{OPT}}{\mathcal{F}_\mathbf{s}}.$$

The Effect of Anarchy/Stability. We introduce a new notion that we use to measure the discrepancy between the systemic liquidity of the original network (no edge removal) and that of worst (best) Nash equilibrium. We call this the *Effect of Anarchy, (Stability, respectively)* and use it as an indication of the (positive or negative) effect that strategic behavior can have on the original network, in contrast to the "Price of" measure which indicates the extent to which the individual objectives of the banks and the objective of the regulator are (not) aligned. The Effect of Anarchy, (Stability, respectively) is defined as follows. For a given network G that admits at least one Nash equilibrium, we have

$$\text{EoA}(G) = \max_{\mathbf{s} \in s_{eq}} \frac{\mathcal{F}_G}{\mathcal{F}_\mathbf{s}} \qquad \text{EoS}(G) = \min_{\mathbf{s} \in s_{eq}} \frac{\mathcal{F}_G}{\mathcal{F}_\mathbf{s}}.$$

The Effect of Anarchy (Stability, respectively) of a given class[8] of networks is determined by the Effect of Anarchy (Stability, respectively) of the network in that class that demonstrates the maximum multiplicative distance of the worst (best, respectively) equilibrium from the original network in terms of liquidity. The Effect of Anarchy (Stability, respectively) of a given class of networks (is only defined when at least one network in the class admits some equilibrium and) can be positive or negative depending on whether the respective equilibria admit better or worse liquidity than the original network.

The Positive Effect of Anarchy and the Positive Effect of Stability of a given class of networks are defined as: Positive EoA = $\max_G \min_{\mathbf{s} \in g_{eq}} \frac{\mathcal{F}_\mathbf{s}}{\mathcal{F}_G}$, and Positive EoS = $\max_G \max_{\mathbf{s} \in g_{eq}} \frac{\mathcal{F}_\mathbf{s}}{\mathcal{F}_G}$, respectively, where $\mathbf{g}_{eq}$ denotes the set of all Nash equilibria in a given edge-removal game that are at least as good as the original network in terms of liquidity; the notions are well-defined only in the case where $\mathbf{g}_{eq}$ is non-empty in at least one network of the given class. For example, if a network in a given class has liquidity equal to 1 with no edge-removals, and the worst (best, respectively) equilibrium admits total liquidity equal to 2, we say that that class of networks has a Positive Effect of Anarchy (Stability, respectively) of at least 2.

The Negative Effect of Anarchy and the Negative Effect of Stability of a given class of networks are defined as: Negative EoA = $\max_G \max_{\mathbf{s} \in b_{eq}} \frac{\mathcal{F}_G}{\mathcal{F}_\mathbf{s}}$, and Negative EoS = $\max_G \min_{\mathbf{s} \in b_{eq}} \frac{\mathcal{F}_G}{\mathcal{F}_\mathbf{s}}$, respectively, where $\mathbf{b}_{eq}$ denotes the set of all Nash equilibria in a given edge-removal game that are worse than the original network in terms of liquidity; the notions are well-defined only in the case where $\mathbf{b}_{eq}$ is non-empty in at least one network of the given class. For example, if a network in a given class has liquidity equal to 1 with no edge-removals, and the worst (best, respectively) equilibrium admits total liquidity equal to $1/2$, we say that that class of networks has a Negative Effect of Anarchy (Stability, respectively) of at least 2.

We investigate properties of Nash equilibria in the edge-removal game with respect to their existence and quality, while we also address computational complexity questions under different assumptions on whether default costs and/or cash injections apply. Our results on the Effect of Anarchy of edge removal games imply that, rather surprisingly, in the presence of default costs even the worst Nash equilibrium can be arbitrarily better than the original network in terms of liquidity. However, the situation is reversed in the absence of default costs, where we observe that the original network can be considerably better in terms of liquidity than the worst equilibrium; in line with similar Price of Anarchy results. We begin with some results for the basic case, that is, without default costs; recall that we do not refer to default costs in the statements for results holding for $\alpha = \beta = 1$.

Our first result exploits the fact that for edge-removal games without cash injections, the strategy profile where all edges are preserved is a (not necessarily unique) Nash equilibrium.

Theorem 7. *Edge-removal games without cash injections always admit Nash equilibria.*

Proof. We claim that it is a Nash equilibrium if no bank removes an incoming edge. Indeed, consider a financial network and an arbitrary debt relation in that network represented by an edge e from v_i to v_j. Since there are no default costs, removing e and keeping everything else unchanged, will result in v_i instantly having p_{ij} additional assets to pay to its lenders (excluding v_j). This, however can lead to at most p_{ij} additional assets to reach v_j through indirect paths starting from v_i so it can not lead to more total

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[8] A class of networks is a set of networks having a common property. Common classes of networks are ones whose underlying graph is acyclic, or has a particular topology like a cycle or a tree.
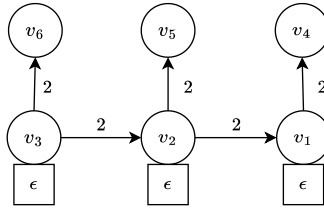
Fig. 7. An edge-removal game without Nash equilibria in the case without default costs and with budget $M = 2 - 3\epsilon$, where ϵ is an arbitrarily small constant.

payments from v_i to v_j.[9] Hence, v_j is better off not removing e. Since v_j is an arbitrary bank and e represents an arbitrary edge, our proof is complete. $\quad\square$

Theorem 8. *In edge-removal games without cash injections, the negative Effect of Anarchy is unbounded and the negative Effect of Stability is* 1.

Proof. The result on the Effect of Stability follows by the proof of Theorem 7, since the original network with no debt-removal is proved to be a Nash equilibrium. For the Effect of Anarchy, consider a simple network with two banks, each having a liability of 1 to the other. In the original network, the total liquidity is 2, while it is not hard to see that the network with both edges removed is a Nash equilibrium with a total liquidity of zero. $\quad\square$

Our next result shows that Nash equilibria may not exist once we allow for cash injections.

Theorem 9. *There is an edge-removal game with cash injections that does not admit Nash equilibria.*

Proof. Consider the network shown in Fig. 7, and a budget equals to $M = 2 - 3\epsilon$, where ϵ is an arbitrarily small constant.
We begin by observing that v_4, v_5 and v_6 will never remove their incoming edges, since each of these agents has a single incoming edge, originating from some agent with positive externals. Hence, it suffices to consider the strategic actions of banks v_1 and v_2 regarding the possible removal of edges (v_2, v_1) and (v_3, v_2), respectively. There are four possible cases:

A) *both edges are present.* In this case, v_3 has the highest threat index, i.e., $7/4$ and, thus receives the entire budget; it is not hard to verify that this is the optimal policy as well. The payments are, then, $p_{32} = 1 - \epsilon, p_{21} = 1/2$, resulting in total assets $a_1 = 1/2 + \epsilon, a_2 = 1$.
B) (v_2, v_1) *is present,* (v_3, v_2) *is removed.* In this case, v_2 receives the budget and the following payments are realized $p_{32} = 0, p_{21} = 1 - \epsilon$. This leads to total assets $a_1 = 1$ and $a_2 = 2 - 2\epsilon$.
C) *both edges are removed.* In this case, there is a tie on the maximum threat index (v_1, v_2 and v_3 have threat index 1), so, assuming that the banks with lower index are prioritized, v_1 receives all the budget; again, it is not hard to verify that this is an optimal policy as well. This results to payments $p_{32} = 0$ and $p_{21} = 0$ with $a_1 = 2 - 2\epsilon, a_2 = \epsilon$.
D) (v_2, v_1) *is removed,* (v_3, v_2) *is present.* In this case, v_3 receives the budget. The payments are $p_{32} = 1 - \epsilon$ and $p_{21} = 0$ (due to the removal), which implies $a_1 = \epsilon$ and $a_2 = 1$ respectively.

One can now easily check that the best response dynamics cycle, as starting from Case A, v_2 has an incentive to remove its incoming edge (v_3, v_2) and we reach Case B. Then, v_1 has an incentive to remove its incoming edge (v_2, v_1) (we are now in Case C), which leads v_2 to have an incentive to reinstate its incoming edge (v_3, v_2) (thus, reaching Case D). Finally, in Case D, v_1 has an incentive to reinstate (v_2, v_1), leading to Case A again. Therefore, the best-response dynamics cycles in an orbit of the form A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ A, repeatedly traversing these four exhaustive cases. $\quad\square$

Theorem 10. *The Price of Stability in edge-removal games (with or without cash injections) is unbounded.*

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of limited budget regardless of who receives it. Moreover, note that the proof does not assume default costs but the result immediately applies to that (more general) case as well, i.e., $\alpha = \beta = 1$ is a special case of default costs.
Consider the network shown in Fig. 8, where Z is an arbitrary large constant. Since each bank has exactly one incoming edge, no edge removals occur at the unique Nash equilibrium. By the proportionality principle it holds that the three payments have to be equal (to 1), that is $p_{13} = p_{12} = p_{21} = 1$, which results in the systemic liquidity of 3. However, systemic liquidity of $2Z$ can be achieved when v_3 removes its incoming edge. Indeed, then each remaining payment can be equal to Z. We conclude that PoS $\geq 2Z/3$, so the price of stability can be arbitrarily large for appropriately large values of Z, as desired. $\quad\square$

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[9] This would not be true if additional money was inserted to the network in the form of a cash injection.
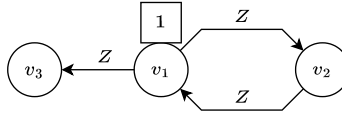
Fig. 8. A financial network with no default costs that admits unbounded price of stability.
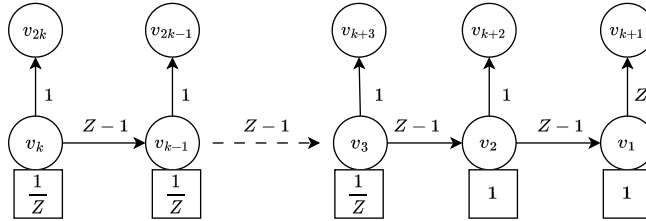

Fig. 9. A network that yields the negative Effect of Stability of $\Omega(n)$ for budget $M = 1$, $n = 2k$, and arbitrarily large Z.
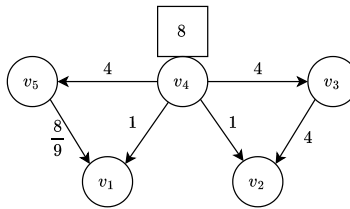

Fig. 10. An instance of an edge-removal game without Nash equilibria, when $\alpha = \beta = 1/4$.

Theorem 11. *The negative Effect of Stability in edge-removal games with cash injections is $\Omega(n)$.*

Proof. Consider the network in Fig. 9 where the budget $M = 1$, $k = n/2$ and Z is arbitrarily larger than k. We start by noticing that $\mu_1 = 1$, while for $i = 2, \ldots, k$, it holds that $\mu_i = 1 + \frac{Z-1}{Z}\mu_{i-1} \approx 1 + \mu_{i-1}$, for sufficiently large Z; all other banks are solvent. Hence, the total liquidity of the original network, which is in fact the optimal one, is achieved when v_k receives the entire budget of $M = 1$ as a cash injection, and is roughly $kM = n/2$, when Z is sufficiently large.

We now claim that under any Nash equilibrium, v_2 will receive the budget and all edges (v_i, v_{i-1}) for $i \in \{3, \ldots, k\}$ are removed. This would complete the proof, as the total liquidity would be at most $\frac{1}{Z}(k-2) + 2 + 2 \leq 5$. We now prove this claim. Consider any equilibrium and observe that v_i, for $i = k+1, \ldots, 2k$, must have their unique incoming edge present. Now, assume for a contradiction that some bank v_i with $i \in \{3, \ldots, k\}$ gets a cash injection; this implies that the edge (v_i, v_{i-1}) is present as, otherwise, the result holds trivially. Then, bank v_{i-1} has total assets $1 - 1/Z^2 + e_{i-1}$, but can increase them to $1 + e_{i-1}$ by strategically removing its incoming edge. So, under any Nash equilibrium, either v_2 or v_1 receives a cash injection. In the former case, where edge (v_2, v_1) is present, $a_1 = 3 - 2/Z$, while the assets of v_1 would be 2 if it removed its incoming edge and received the cash injection.

So far, we have proven that v_2 gets the cash injection and it remains to show no other edge (v_i, v_{i-1}) for $i \in \{3, \ldots, k\}$ exists in a Nash equilibrium. Now, observe that if such an edge exists, then neighboring edges on the horizontal path cannot exist as that would contradict that v_2 gets the cash injection. Then, when $i > 4$, bank v_{i-2} would have an incentive to add edge (v_{i-1}, v_{i-2}), thus, making bank v_i the recipient of the budget (for both optimal and greedy) and strictly increase its own total assets. The cases $i \in \{3, 4\}$ can be easily ruled out as well. Our proof is complete. $\square$

We now present a series of results for the case where default costs exist, but cash injections are not allowed. Contrary to the case with neither default costs nor cash injections, we show that a Nash equilibrium is no longer guaranteed to exist; the next result is complementary to Theorem 9.

Theorem 12. *There is an edge-removal game without cash injections but with default costs that does not admit Nash equilibria.*

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of limited budget regardless of who receives it, as a sufficiently small budget will not alter the players incentives.

Consider the financial network in Fig. 10 and assume that default costs $\alpha = \beta = 1/4$ are applied. We begin by claiming that v_5 (and, by symmetry v_3) will never remove its incoming edge. Indeed, since v_4 has positive external assets, then v_5 will receive a positive payment from v_4 if the edge between them remains, however v_5 will have zero incoming payments if the corresponding edge is removed. Similarly, v_1 (and, by symmetry v_2) will never remove the edge from v_5 (respectively v_3). Therefore, it suffices to consider the possible removal of edges (v_4, v_1) and (v_4, v_2). We will prove that none of the following states are at equilibrium: A) no edge is removed, B) (v_4, v_1) is removed but (v_4, v_2) remains, C) both edges are removed, and D) (v_4, v_1) remains but (v_4, v_2) is removed.
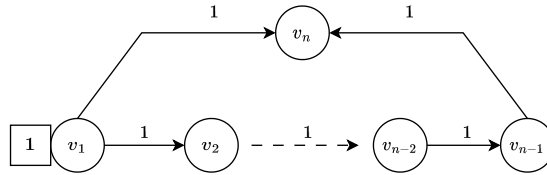
Fig. 11. A financial network with $\alpha = \beta = \epsilon$, for an arbitrarily small positive number ϵ, that admits a unbounded positive Effect of Anarchy.

Note that in Case A v_4 is in default, hence, due to the default costs, its payments are broken down as $\mathbf{p}_4 = (1/5, 1/5, 4/5, 0, 4/5)$. But then, v_3 and v_5 are also in default, and $p_{51} = p_{32} = 1/5$ in this case. So, the utility of both v_1 and v_2 is $2/5$. By removing (v_4, v_1) in Case A, and moving to Case B, v_1 would increase its utility to $8/9$. Indeed, in this case, $\mathbf{p}_4 = (0, 2/9, 8/9, 0, 8/9)$, while $p_{51} = 8/9$ and $p_{32} = 2/9$, so the utility of v_1 is $8/9$ and the utility of v_2 is $4/9$. But, then, it is beneficial for v_2 to remove (v_4, v_2). In this case (Case C), v_4 is now solvent and all existing debts are paid for, thus giving utility $8/9$ to v_1 and utility 4 to v_2. However, if v_1 then decides to reinstate its incoming edge (v_4, v_1), its utility in this case (Case D) is increased to $10/9$, while v_2's utility is $2/9$, since $\mathbf{p}_4 = (2/9, 0, 8/9, 0, 8/9)$, $p_{51} = 8/9$ and $p_{32} = 2/9$. Finally, v_2 prefers its utility in Case A to its utility in Case D so will decide to reinstate (v_4, v_2) when in Case D, thus defining a cycle between the four possible states. □

For some restricted topologies, however, the existence of Nash equilibria is guaranteed; in particular, keeping all edges is a Nash equilibrium.

Theorem 13. *Edge-removal games without cash injections but with default costs always admit Nash equilibria if the financial network is a tree or a cycle.*

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of limited budget regardless of who receives it.

Consider a financial network that is a tree (similar argument works in case of a directed cycle) and an arbitrary debt relation in that network represented by a directed edge (v_i, v_j). By definition of the tree structure, it holds that there is no other path in the network from v_i to v_j. Hence, v_j cannot benefit by removing that edge, even if the removal affects (increases) v_i's available assets. □

The following result demonstrates that the positive impact of (individually benefiting) edge removals dominates the negative impact of reducing the number of edges through which money can flow, hence, edge removals are in line with the regulator's best interest too.

Lemma 1. *Edge-removal games with default costs but no cash injections satisfy the following: given any network and any strategy profile, any unilateral removal of any edge(s) that weakly improves the total assets of the corresponding bank, also weakly improves the total assets of every other bank in the network. Consequently, the total liquidity of the system is increased.*

Proof. Consider a network $G = (V, E)$ and a strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$, under which banks have total assets according to $\mathbf{a} = (a_1, \ldots, a_n)$. Fix a bank i and let $\mathbf{s}' = (s_i', \mathbf{s}_{-i})$ be the strategy profile that is derived by $\mathbf{s}$ if bank i changes its strategy from s_i to s_i', where s_i' is derived by s_i by the removal of an edge $e = (v_j, v_i)$ (the argument can be applied repeatedly to prove the claim for more than one edge removals). By assumption, the total assets of bank i under $\mathbf{s}'$, a_i', satisfy $a_i' \geq a_i$. It holds that any bank reachable by i or j (the two endpoints of the edge that was removed) through a directed path will have at least the same total assets under $\mathbf{s}'$ than with $\mathbf{s}$, since there will be at least the same amount of money available to leave i and j and traverse these paths. The assets of banks not reachable by i or j will, clearly, not be affected by the removal of e. Hence, the assets of each bank in G are weakly higher under $\mathbf{s}'$ than under $\mathbf{s}$. The increase in the total liquidity follows since the total assets, by definition, equal external assets plus payments. □

In fact, the systemic liquidity of even the worst Nash equilibrium can be arbitrarily higher than at the original network. To see this, consider the network in the proof of Theorem 14, which admits a unique Nash equilibrium with arbitrarily higher total liquidity than that of the original network.

Theorem 14. *The positive Effect of Anarchy in edge-removal games with default costs and without cash injections is unbounded.*

Proof. Consider the network in Fig. 11 where default costs are $\alpha = \beta = \epsilon$ for some arbitrary small positive number ϵ. If no edge is removed, then all banks except v_n are in default and the following payments are realized: $p_{12} = p_{1n} = \epsilon/2$ and $p_{i,i+1} = \epsilon \cdot p_{i-1,i} = \epsilon^i/2$. The systemic liquidity is then $\mathcal{F}_G = \epsilon/2 + \sum_{i=1}^{n-1} \frac{\epsilon^i}{2} < \frac{\epsilon}{1-\epsilon}$. On the other hand, the unique Nash equilibrium is achieved when v_n removes the edge pointing from v_1 to itself. The systemic liquidity in this case is $n - 1$, and the proof follows. □

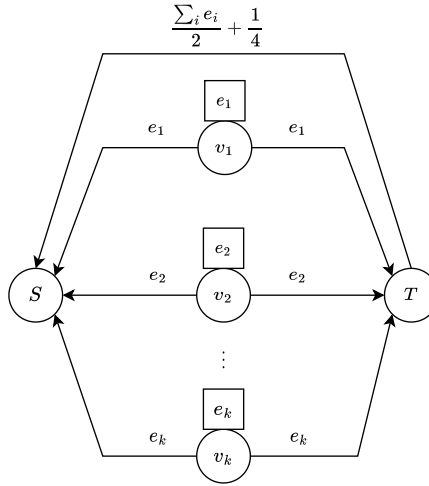We conclude with our results on computational complexity for the setting with default costs.

Fig. 12. The instance arising from the reduction used in Theorem 15b.

Theorem 15. *In edge-removal games with default costs, the following problems are* **NP***-hard:*

  a) *Decide whether a Nash equilibrium exists.*
  b) *Verify if a given strategy profile is a pure Nash equilibrium when pure equilibria are guaranteed to exist.*
  c) *Compute a best-response strategy.*
  d) *Compute a strategy profile that maximizes systemic liquidity.*

Proof. We begin by proving that the problem of verifying whether a strategy profile is a Nash equilibrium (even when its existence is guaranteed) is **NP**-hard.

Hardness of verifying a Nash equilibrium. Our proof follows by a reduction from the PARTITION problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \ldots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i \in X'} x_i = \sum_{i \notin X'} x_i = \frac{\sum_{i \in X} x_i}{2}$.

The reduction works as follows. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ by adding an agent v_i for each element $x_i \in X$ and allocating an external asset of $e_i = x_i$ to v_i; we also include two additional agents S and T. Each agent v_i has liability equal to e_i to each of S and T, while T has liability $\frac{\sum_i e_i}{2} + \frac{1}{4}$ to S; see also Fig. 12. We furthermore set default costs $\alpha = \beta = 1/\sum_i e_i$; clearly, the reduction requires polynomial-time.

Observe that, in any Nash equilibrium, agent T keeps all its incoming edges. Indeed, removing an edge from agent v_i will decrease T's total assets, as there is no alternative path for payments originating at v_i to reach T. Similarly, agent S keeps its incoming edge from T at any Nash equilibrium, as deleting it will reduce S's total assets. Therefore, the only strategic choice in this financial networks is by agent S about which edges from agents v_i to keep and which to remove. We denote by S_k and S_r the set of agents whose edges towards S are kept and removed, respectively, and observe that agents in S_k are in default while agents in S_r are not. Clearly, as S is essentially the only strategic agent, any best-response strategy by S forms a Nash equilibrium. This guarantees the existence of a Nash equilibrium in $\mathcal{I}'$. We will show that in instance $\mathcal{I}'$, agent S can compute her best response, and hence we can compute a Nash equilibrium, if and only if instance $\mathcal{I}$ of PARTITION is a yes-instance.

We first show that if $\mathcal{I}$ is a yes-instance for PARTITION, then agent S has total assets $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$. Indeed, consider the subset X' in $\mathcal{I}$ with $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$ and let S_k contain agents v_i where $x_i \in X'$, while S_r agents v_i with $x_i \notin X'$. Note that T obtains total assets

$$a_T = \sum_{i \in S_r} e_i + \frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2}$$
$$= \frac{\sum_i e_i}{2} + \frac{1}{4},$$

and is therefore solvent, while S obtains

$$a_S = \frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} + \frac{\sum_i e_i}{2} + \frac{1}{4}$$
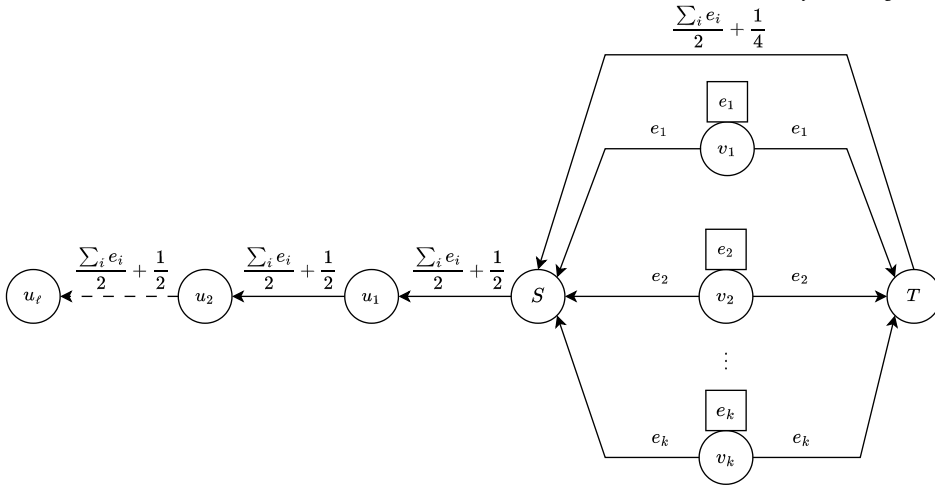$$= \frac{\sum_i e_i}{2} + \frac{1}{2}.$$

Fig. 13. The modified instance in the proof of Theorem 15d.

We will now show that if $\mathcal{I}$ is a no-instance, then S's total assets in $\mathcal{I}'$ are strictly less than $\frac{\sum_i e_i}{2} + \frac{1}{2}$; this suffices to prove the claim. Consider any subset $X' \subseteq X$ in $\mathcal{I}$ and the corresponding strategy profile in $\mathcal{I}'$ where S keeps incoming edges from agents in S_k while removes edges from agents in S_r. Let $\chi = \sum_{i \in S_r} e_i$ and observe that, as elements in X are integers, it holds either $\chi \leq \frac{\sum_i e_i}{2} - \frac{1}{2}$ or $\chi \geq \frac{\sum_i e_i}{2} + \frac{1}{2}$.

In the first case, when $\chi \leq \frac{\sum_i e_i}{2} - \frac{1}{2}$, we claim that T is in default. Indeed, T collects a total payment of χ from the agents in S_r and a total payment of $\frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} = \frac{1}{\sum_i e_i} \frac{\sum_i e_i - \chi}{2} = \frac{1}{2} - \frac{\chi}{2 \sum_i e_i}$ from the agents in S_k. So,

$$
\begin{aligned}
a_T &= \chi + \frac{1}{2} - \frac{\chi}{2 \sum_i e_i} \\
&= \chi \left(1 - \frac{1}{2 \sum_i e_i} \right) + \frac{1}{2} \\
&\leq \left(\frac{\sum_i e_i}{2} - \frac{1}{2} \right) \left(1 - \frac{1}{2 \sum_i e_i} \right) + \frac{1}{2} \\
&< \frac{\sum_i e_i}{2},
\end{aligned}
$$

i.e., less than T's liability to S; the first inequality follows by the assumption on χ. Therefore, S can collect $\frac{1}{\sum_i e_i} \frac{\sum_i e_i - \chi}{2}$ from the agents in S_k and strictly less than $\frac{1}{2}$ from T. We conclude that $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$ in this case.

In the second case, when $\chi \geq \frac{\sum_i e_i}{2} + \frac{1}{2}$, S obtains a total payment of $\frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} = \frac{1}{2 \sum_i e_i} (\sum_i e_i - \chi) \leq \frac{1}{2 \sum_i e_i} (\frac{\sum_i e_i}{2} - \frac{1}{2}) = \frac{1}{4} - \frac{1}{4 \sum_i e_i}$ from agents in S_k and a payment of at most $\frac{\sum_i e_i}{2} + \frac{1}{4}$ from T, i.e., $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$ again.

Since in any case, a no-instance $\mathcal{I}$ for PARTITION leads to an instance $\mathcal{I}'$ where $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$, the claim follows, as we cannot compute a best-response strategy for S, and, by the discussion above, a Nash equilibrium.

Hardness of computing a best-response strategy. The proof was given in the previous case.

Hardness of maximizing systemic liquidity. The proof follows by the reduction from PARTITION described in the previous case by adding a path of ℓ agents as shown in Fig. 13. By the discussion above, starting from a yes-instance in PARTITION, we have $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$, and S as well as any agent u_i are solvent. On the contrary, starting from a no-instance for PARTITION, agent S is in default and the payments traveling to the u_i agents get reduced by a factor of $\sum_i e_i$ at each edge. By selecting ℓ to be large enough, the claim follows.

Hardness of deciding the existence of Nash equilibria. Again, the proof follows by the reduction from PARTITION used in Theorem 15b by adding five agents, ℓ, m, r, x and y with liabilities and external assets are show in Fig. 14. Recall that the default costs are $\alpha = \beta = \frac{1}{\sum_i e_i}$ and, without loss of generality, we assume that $\sum_i e_i \geq 3$.

As argued earlier, T always keeps all its incoming edges, while S always keeps the edge from T. Similarly, agents ℓ and r always keep their incoming edges, as by removing any incoming edge their total assets strictly decrease. As agents v_i and m do not have
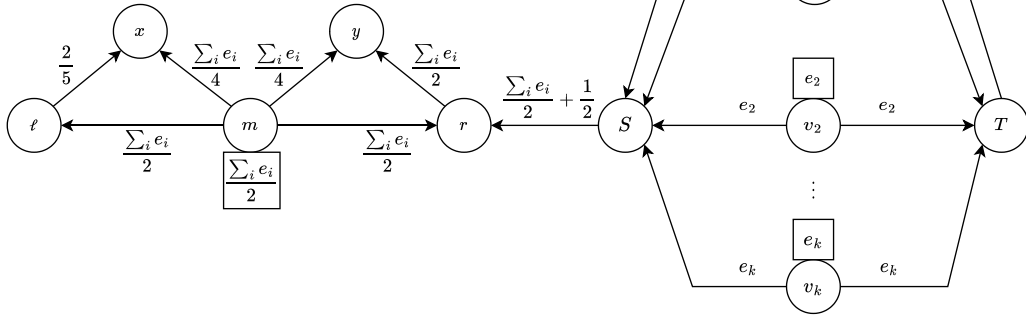
Fig. 14. The instance used in the proof of Theorem 15a.

incoming edges, the only strategic agents are x, y (with respect to the edges originating from m) and S (with respect to edges from the v_i agents). We first show that, if instance $\mathcal{I}$ of PARTITION is a yes-instance, then there is a Nash equilibrium. Indeed, we argued earlier that in this case $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$ and, hence, S is solvent. So, r is solvent as well and, therefore, y always keeps the edge from m as the liability from r is always fully paid. This implies that m is necessarily in default and agent x should keep the edge from m as well. To conclude, the strategy profile where x, y keep all edges is a Nash equilibrium.

On the other hand, when instance $\mathcal{I}$ is a no-instance, we have shown that S is in default. Then, r collects a payment of $\chi \leq \frac{1}{2} + \frac{1}{2\sum_i e_i}$ from S and is, hence, necessarily in default. When both x and y keep their edges from m, their total assets are $a_x = \frac{1}{3\sum_i e_i} + \frac{1}{6}$ and $a_y = \frac{1}{6} + (\frac{1}{3} + \chi)\frac{1}{\sum_i e_i}$. When x removes the edge and y keeps it, it is $a_x = \frac{2}{5}$ and $a_y = \frac{1}{5} + (\frac{2}{5} + \chi)\frac{1}{\sum_i e_i}$, i.e., x improves compared to the previous case. When they both remove these edges, we have $a_x = \frac{2}{5}$ and $a_y = \frac{\sum_i e_i}{2}$, i.e., y improves compared to the previous case. Similarly, when x keeps the edge and y removes it, the total assets are $a_x = \frac{3}{5}$ and $a_y = (\frac{2}{5} + \chi)\frac{1}{\sum_i e_i}$, i.e., x improves compared to the previous case. The claim follows by observing that, in the last case, y improves by keeping the edge. $\square$

5. Conclusions

We considered problems arising in financial networks, when a financial authority wishes to maximize the total liquidity either by injecting cash or by removing debt. We also studied the setting where banks are rational strategic agents that might prefer to forgive some debt if this leads to greater utility, and we analyzed the corresponding games with respect to properties of Nash equilibria. In that context, we also introduced the notion of the Effect of Anarchy (Stability, respectively) that compares the liquidity in the initial network to that of the worst (best, respectively) Nash equilibrium, while distinguishing between the cases where the equilibria (stable network structures) are better or worse than the original network.

Our work leaves some interesting problems unresolved. Given the computational hardness of some of the optimization problems, it makes sense to consider approximation algorithms. From the game-theoretic point of view, one can also consider the problems from a mechanism design angle, i.e., to design incentive-compatible policies where banks weakly prefer to keep all incoming liabilities. It would also be interesting to identify classes of games where all equilibria are better (or all equilibria are worse, respectively) than the original network, thus demonstrating only a positive (or only a negative, respectively) Effect of Anarchy/Stability. Finally, for the setting with cash injections where pure Nash equilibria are not guaranteed to exist (see Theorem 9), it is interesting to study properties of approximate equilibria.

CRediT authorship contribution statement

Panagiotis Kanellopoulos: Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization. **Maria Kyropoulou:** Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization. **Hao Zhou:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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