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Identifying roles of formulas in inconsistency under Priest's minimally inconsistent logic of paradox

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ABSTRACT

It has been increasingly recognized that identifying roles of formulas of a knowledge base in the inconsistency of that base can help us better look inside the inconsistency. However, there are few approaches to identifying such roles of formulas from a perspective of models in some paraconsistent logic, one of typical tools used to characterize inconsistency in semantics. In this paper, we characterize the role of each formula in the inconsistency arising in a knowledge base from informational as well as causal aspects in the framework of Priest's minimally inconsistent logic of paradox. At first, we identify the causal responsibility of a formula for the inconsistency based on the counterfactual dependence of the inconsistency on the formula under some contingency in semantics. Then we incorporate the change on semantic information in the framework of causal responsibility to develop the informational responsibility of a formula for the inconsistency to capture the contribution made by the formula for the inconsistent information. This incorporation makes the informational responsibility interpretable from the point of view of causality, and capable of catching the role of a formula in inconsistent information concisely. In addition, we propose notions of naive and quasi naive responsibilities as two auxiliaries to describe special relations between inconsistency and formulas in semantic sense. Some intuitive and interesting properties of the two kinds of responsibilities are also discussed.

1. Introduction

Techniques for handling inconsistency in knowledge-based systems have been paid much attention in the community of knowledge representation and reasoning. In particular, it has been increasingly recognized that measuring inconsistency can provide a promising starting point to facilitating inconsistency handling in many applications [1,2], including requirements engineering [3,4], network security and intrusion detection [5], medical expert systems [6], and databases [7,8]. A variety of inconsistency measures have been proposed so far, and a more detailed survey of inconsistency measures has been given in [9].

The propositional logic has been considered one of the underlying logical frameworks appropriate for exemplifying inconsistency measuring in many cases because of its good balance between computational advantage and expressive power. A number of inconsistency measures built upon the framework of propositional logic have been tailored to many practical applications such as requirements engineering [3,4] and databases [7,8] by focusing on their own respective scenarios characterized by ground formulas. On the other hand, some works such as general information space presented in [10] aim to transform any general information space to an inconsistency equivalent propositional knowledge base to find the inconsistency of the general information space by

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applying the propositional inconsistency measures. Here the concept of general information space encompasses various types of databases and scenarios in AI systems [10]. Allowing for these, in this paper, we also focus on inconsistency measuring for propositional knowledge bases. Here a propositional knowledge base (knowledge base for short) is referred to as a finite set of propositional formulas.

Most inconsistency measures for a knowledge base focus on how to assess the degree of inconsistency of the whole knowledge base from their own respective perspectives. Such measures are categorized as base-level ones in [11]. In contrast, measures of the other type, called the formula-level measures in [11], aim to characterize the role of each formula in inconsistency of that knowledge base. As one of the earlier representative formula-level measures, the Shapley inconsistency value presented in [11] uses the Shapley value [12], a well known cooperation game model, to distribute a base-level measure for the whole knowledge base to each formula. Roughly speaking, it considers each formula of a knowledge base and each subset of that base as a player and a coalition in a coalitional game, respectively. Then the base-level inconsistency measure for each subset of that base is considered as the total payoffs the players (formulas) of that subset can obtain by cooperation, and the Shapley inconsistency value of each formula is exactly the amount of payoffs the formula gets according to the Shapley value mechanism. Certainly, the Shapley inconsistency value depends on the base-level inconsistency measure used as payoffs of coalitions.

In particular, when the base-level measure for a knowledge base is the number of minimal inconsistent subsets of that base, the corresponding Shapley inconsistency value can be characterized in a more concise way [13,11]. Here a minimal inconsistent subset refers to an inconsistent subset without an inconsistent proper subset. It may be considered as a natural characterization of inconsistency in a knowledge base from a syntactic perspective. Actually, minimal inconsistent subsets are attractive to developing formula-level measures. For example, the blame measure presented in [14] combines the principle of proportionality and minimal inconsistent subsets to describe the blame of each formula for the inconsistency in a knowledge base. The responsibility measure presented in [15] uses the minimal inconsistent subsets to characterize the responsibility of each formula for the inconsistency from a perspective of causality. The tacit culpability measure presented in [16] uses the minimal inconsistent subsets to identify the explicit role of a formula in causing the inconsistency. The DIM measure presented in [17] aims to capture the responsibility of each formula for the inconsistency based on the MUS-graph, which describes overlap relations of minimal inconsistent subsets.

Multi-valued semantics for an inconsistent knowledge base in some paraconsistent logic provides another important perspective to characterize inconsistency [18,11,19–21]. Particularly, Priest's minimally inconsistent logic of paradox (LP for short) [22] is considered appropriate for exemplifying inconsistency characterization based on paraconsistent models since it is simple enough but agrees with classical logic whenever the knowledge base is consistent [18].

In contrast to the case of minimal inconsistent subsets, there are few formula-level inconsistency measures built upon paraconsistent models. A representative work is that the minimal number of variables assigned to the non-classical truth value in Priest's minimal inconsistent models is used as the base-level measure to derive a special Shapley inconsistency value [11].

Notice that any inconsistent knowledge base has no classical model, and the principle of explosion tells us that anything can be deduced from it. Then for classical reasoning, any inconsistent knowledge base may be considered useless in providing information. In contrast, paraconsistent models of an inconsistent knowledge base capture both "good" (or "unpolluted") and "bad" (or "polluted") information conveyed by the knowledge base in semantics. Then compared to the case of minimal inconsistent subsets, it is natural and feasible to associate the identification of the role of a formula with changes of semantic information due to the absence of the formula under some circumstances. This means that we need to consider the contribution made by each formula to the inconsistent information in semantics as well as the role of each formula in causing inconsistency.

Interpretability is desirable in developing formula-level measures as well as base-level measures, because it allows us to easily understand the nature of inconsistency, and what exactly measures capture, and in what ways measures meet our hopes. In particular, an intuitive interpretation for formula-level measures can help stakeholders clearly recognize the intrinsic relation between each formula and the inconsistency in order to make a rational decision on resolving the inconsistency. It has been argued that the causal relation between formulas and the inconsistency of a knowledge base can help us characterize and interpret the role of each formula in the inconsistency in a concise way [15,23]. Moreover, the counterfactual dependence of the inconsistency on an individual formula (under some contingency) plays an essential role in identifying the responsibility of a formula for the inconsistency from a syntactic perspective [15,23] within the framework of Halpern and Pearl's causal model and Chockler and Halpern's notion of responsibility [24]. Here we say that A counterfactually depends on B if it is the case that if B had not happened, then A would not have happened. Then it is advisable to take into account the counterfactual dependence of the inconsistency on a formula in the context of paraconsistent semantics to characterize the role of the formula in the inconsistency from a semantic perspective.

In this paper, we consider a series of interpretable characterizations of the role of a formula in the inconsistency in a knowledge base from different aspects in the framework of Priest's minimally inconsistent logic of paradox. At first, we characterize the causal role of each formula in the inconsistency by identifying the counterfactual dependence of the inconsistency on a formula from a semantic perspective. Recall that the intention of paraconsistent semantics is to allow that a knowledge base has models whether it is consistent or not, by introducing some designated non-classical truth values. Then the inconsistency of a knowledge base can be characterized by its minimal inconsistent models. Intuitively, we may identify the counterfactual dependence of the inconsistency of a knowledge base K on a given formula α in semantic sense by checking that there exists an interpretation that falsifies α but is a classical model of all the formulas except for α. That is, if there is no need to introduce non-classical truth value in every minimal inconsistent model of $K - \{\alpha\}$ as long as we allow α to be assigned the false value, then we consider the inconsistency counterfactually depends on α in semantics. To bear this in mind, we consider a more general case where the inconsistency counterfactually depends on a given formula under some contingency, and then use such a counterfactual dependence to characterize the causal role of a formula.

Notice that the causal role of a formula is more concerned about the contribution made by the formula to making the inconsistency arise. That is, it is more interested in whether the inconsistency arises rather than how much semantic information on inconsistency arises. Then we further incorporate the semantic information in the framework of identifying the causal role of the given formula to capture the contribution made by the formula to the semantic information about the inconsistency. We use a notion of informational responsibility of a formula for the inconsistency to describe this aspect of the role of the formula in the inconsistency. Thanks to the framework of identifying causal role, the informational responsibility also has a good interpretability from a point of view of causality.

Instead of using the set of atoms assigned to the non-classical truth value by minimal inconsistent models, we consider using the probability distribution on the truth value assignments of each variable induced by minimal inconsistent models of a knowledge base to represent the semantic information on the inconsistency. Notice that each minimal inconsistent model of a given knowledge base provides a possible assignment of truth values to atoms, especially a possible set of atoms that have to be assigned to the non-classical truth value by the model in the presence of inconsistency. Then the uncertainty of assignments of truth values to atoms given a knowledge base is implied by the set of minimal inconsistent models of the base. Notice that the probability distribution on language has been used to measure the inconsistency for knowledge bases [25,26]. On the other hand, from the point of view of the uncertainty, the distribution with the maximum entropy is the one that makes fewest assumptions about the true distribution over the minimal inconsistent models. It considers that all the minimal inconsistent models are equally likely, that is, each minimal inconsistent model has the same probability. Actually, a similar idea has been considered by the principle of indifference about random worlds that is used to induce a probabilistic representation for beliefs from knowledge bases [27,28]. Then it is advisable and feasible to use probability distributions over assignments of truth values to atoms to represent the semantic information on inconsistency described by minimal inconsistent models. Allowing for this, in this paper we use the probability distribution with the maximum entropy over minimal inconsistent models to derive a sequence of probability distributions on assignments of truth values to atoms, and then use this sequence of probability distributions to represent the semantic information on the inconsistency.

The informational responsibility of a formula for the inconsistency is tightly associated with the causal role of the formula in the inconsistency. This implies that if a formula plays no role in making the inconsistency disappear, the formula has null informational responsibility for the inconsistency. Here we need to point out that even if a formula is not involved in any falsification making the inconsistency disappear, it does not necessarily imply that the formula has nothing to do with the inconsistency in semantics. That is, some formula with null informational responsibility may have some local contribution to the inconsistency. To illustrate this, consider a knowledge base consisting of two formulas $a \wedge \neg a \wedge b$ and $\neg b$. Evidently, falsifying $a \wedge \neg a \wedge b$ rather than $\neg b$ can make the inconsistency disappear, then we may consider that the inconsistency does not counterfactually depend on $\neg b$, and then $\neg b$ has neither informational responsibility nor causal responsibility for the inconsistency. But $\neg a \wedge \neg a \wedge b$, together with $\neg b$, determine the assignment of the non-classical truth value to the atom b in semantics. Then the absence of $\neg b$ from the knowledge base will change the assignment of the truth value to the atom b from the non-classical truth value to *true*, although it cannot bring any change on the assignment of the non-classical truth value to the atom a. There also exists the case that removing a formula with positive informational responsibility alone from a knowledge base does not change the semantics of the knowledge base, if the semantic information conveyed by the formula is also conveyed by some other formulas. That is, such a formula alone bears no local responsibility for the inconsistency.

Most formula-level inconsistency measures built upon minimal inconsistent subsets stem from the fact that removing a formula alone brings no change on the inconsistency if and only if the formula bears null responsibility for the inconsistency [11,15,23]. The two aforementioned scenarios make the semantic case different from the syntactic one. Allowing for these, we also propose two auxiliary notions of naive and quasi naive responsibilities of a formula to describe the aforementioned specific roles of the formula in the inconsistency from the semantic perspective. Some interesting properties of the four types of responsibilities are also discussed. These properties can help us better understand necessities of these notions in catching distinct behaviors of formulas within different contexts.

The rest of this paper is organized as follows. In Section 2, we give a brief introduction to some necessary notions on inconsistency. In Section 3, we consider the causal responsibility of a formula for the inconsistency. Then we consider the informational responsibility of a formula for the inconsistency in Section 4. In Section 5, we consider some interesting properties of the two kinds of responsibilities. In Section 6, we consider the two auxiliary notions of naive and quasi naive responsibilities of a formula for the inconsistency. In Section 7, we consider the computational complexity issues about these responsibilities. In Section 8, we use an example to illustrate how to apply the responsibilities of formulas for inconsistency to inconsistency handling in requirements engineering. In Section 9, we compare these responsibilities and some closely related works, and discuss some potential extensions. Finally, we conclude this paper in Section 10.

2. Preliminaries

Throughout this paper we use a finite propositional language $\mathcal{L}$ built from a finite set $\mathcal{A}$ of propositional atoms (or variables) and two propositional constants $\top$ (true) and $\bot$ (false) under connectives $\{\neg, \wedge, \vee\}$. Note that we also write $\mathcal{A}$ a sequence $\langle t_1, t_2, \cdots, t_n \rangle$ of all atoms if necessary, where n is the number of atoms in $\mathcal{A}$. We use a, b, c, $\cdots$, to denote propositional atoms, and α, β, γ, $\cdots$, to denote propositional formulas.

A *knowledge base* K is a finite set of propositional formulas built upon $\mathcal{A}$. We use $At(K)$ to denote the set of atoms appearing in formulas of K. K is *inconsistent* if there is a formula α such that $K \vdash \alpha$ and $K \vdash \neg\alpha$, where $\vdash$ is the classical consequence relation. We use $K \vdash \bot$ (resp. $K \nvdash \bot$) to denote that a knowledge base K is inconsistent (resp. consistent).

An inconsistent subset K' of K is called a *minimal inconsistent subset* of K if no proper subset of K' is inconsistent. We use $\mathcal{MI}(K)$ to denote the set of all the minimal inconsistent subsets of K. A formula in K is called a *free formula* if this formula does not belong to any minimal inconsistent subset of K [1]. We use $\mathcal{FF}(K)$ to denote the set of free formulas of K. Then $K = \left(\bigcup \mathcal{MI}(K)\right) \cup \mathcal{FF}(K)$, where $\bigcup \mathcal{MI}(K) = \cup_{M \in \mathcal{MI}(K)} M$. A consistent formula $\alpha \in K$ is called a safe formula of K if $At(K \setminus \{\alpha\}) \cap At(\{\alpha\}) = \emptyset$ [11,20]. Note that the safe formula is a special kind of free formula. We use $\mathcal{SF}(K)$ to denote the set of all the safe formulas of K. In addition, we use $\alpha \equiv \top$ to denote that α is a tautology.

The notion of minimal correction subset is tightly associated with the minimal inconsistent subset [29]. Given an inconsistent knowledge base K, a subset R of K is called a minimal correction subset of K if $K \setminus R \nvdash \perp$ and for any proper subset R' of R, $K \setminus R' \vdash \perp$. We use $\mathcal{MC}(K)$ to denote the set of minimal correction subsets of K.

Minimal inconsistent subsets of a knowledge base have been used to characterize inconsistency of that base by a number of methods of measuring inconsistency from a syntactic perspective. In contrast, non-classical models in some paraconsistent logic for an inconsistent knowledge base provide a semantic perspective to characterizing inconsistency. Priest's minimally inconsistent LP (Logic of Paradox) $\mathrm{LP_m}$ [22] is considered appropriate for exemplifying such a semantic characterization since it is simple enough but agrees with classical logic whenever the knowledge base is consistent [18].

The $\mathrm{LP_m}$ model of a knowledge base [22] is given in the framework of Priest's Logic of Paradox (Priest's LP for short) [30]. Roughly speaking, Priest's LP provides three-valued models for classically inconsistent knowledge bases by expanding the classical truth values $\{\mathsf{T}, \mathsf{F}\}$ to the set $\{\mathsf{T}, \mathsf{F}, \{\mathsf{T}, \mathsf{F}\}\}$, in which the third truth value $\{\mathsf{T}, \mathsf{F}\}$ (also abbreviated as B in [11,18]) is considered intuitively as both true and false [22]. Here we use the following notations and the concepts about the $\mathrm{LP_m}$ model used by [11]. An interpretation $\omega : \mathcal{A} \cup \{\top, \perp\} \to \{\mathsf{T}, \mathsf{F}, \mathsf{B}\}$ for $\mathrm{LP_m}$ models maps each propositional variable to one of the three truth values T, F, B such that

- $\omega(\top) = \mathsf{T}$, $\omega(\perp) = \mathsf{F}$,
- $\omega(\neg\alpha) = \mathsf{B}$ if and only if $\omega(\alpha) = \mathsf{B}$,
- $\omega(\neg\alpha) = \mathsf{T}$ if and only if $\omega(\alpha) = \mathsf{F}$,
- $\omega(\alpha \wedge \beta) = \min_{\leq_t}\{\omega(\alpha), \omega(\beta)\}$,
- $\omega(\alpha \vee \beta) = \max_{\leq_t}\{\omega(\alpha), \omega(\beta)\}$,

where $\mathsf{F} <_t \mathsf{B} <_t \mathsf{T}$. Then *the set of models of a formula* α is defined as $\mathcal{M}(\alpha) = \{\omega | \omega(\alpha) \in \{\mathsf{T}, \mathsf{B}\}\}$. Further, *the set of models of a knowledge base* K is defined as

$$\mathcal{M}(K) = \{\omega | \omega \in \mathcal{M}(\alpha) \text{ for all } \alpha \in K\}.$$

Let K be a knowledge base and ω be a model of K, then we use $\omega!(K)$ to denote the set of propositional variables of K assigned to B by ω, i.e., $\omega!(K) = \{x \in At(K) | \omega(x) = \mathsf{B}\}$.

Based on $\omega!(K)$, we call a model ω of K a *minimal (inconsistent) model* of K if there is no $\omega' \in \mathcal{M}(K)$ such that $\omega'!(K) \subset \omega!(K)$. We use $\mathcal{M}_{\min}(K)$ to denote the set of minimal models of K. Notice that each minimal model $\omega \in \mathcal{M}_{\min}(K)$ is one of the "most classical" models of K, and $\omega!(K)$ describes a minimal set of atoms that have to be assigned to B by ω. We call $\omega!(K)$ the B-atoms of the minimal model ω, and use $\mathsf{BA}(K)$ to denote the set of all B-atoms of minimal models of K [21]. Evidently, if K is consistent, then the set of minimal models of K is exactly the set of classical models of K, that is, $\mathsf{BA}(K) = \emptyset$.

Essentially, each minimal inconsistent model of a given inconsistent knowledge base describes a possible world in which some atoms have to be assigned to B. Instead of considering which atoms are assigned to B by minimal models, we focus on the probability that an individual atom has to be assigned to B by minimal models. We start with the probability distribution over the set of minimal models of a knowledge base.

Definition 1. Let K be a knowledge base and $\mathcal{M}_{\min}(K)$ the set of minimal models of K. We say that a non-negative function $Pr : 2^{\mathcal{M}_{\min}(K)} \mapsto [0, 1]$ is the probability function if it satisfies the following conditions.

- $Pr(\mathcal{M}_{\min}(K)) = 1$;
- If $\{A_i\} \subseteq 2^{\mathcal{M}_{\min}(K)}$ is a countable collection of pairwise disjoint sets, then $Pr\left(\bigcup_i A_i\right) = \sum_i Pr(A_i)$.

Given an atom $t \in \mathcal{A}$, we use X_t to denote the random variable describing the assignment of truth values to t by minimal models of K. Then the probability distribution of X_t is given as follows:

$$Pr(X_t = T | K) = \sum_{\omega \in \mathcal{M}_{\min}(K) \text{ s.t. } \omega(t) = T} Pr(\{\omega\}),$$

where $T \in \{\mathsf{B}, \mathsf{T}, \mathsf{F}\}$. Evidently, it holds that

$$Pr(X_t = \mathsf{T} | K) + Pr(X_t = \mathsf{F} | K) + Pr(X_t = \mathsf{B} | K) = Pr(\mathcal{M}_{\min}(K)) = 1.$$

It is well known that the distribution with the maximum entropy is the one that makes the fewest assumptions about the true distribution over the minimal models. Then in this paper we focus on the following maximum entropy distribution over the set of minimal models $\mathcal{M}_{\min}(K)$ of K:

$$\forall\omega \in \mathcal{M}_{\min}(K),\ Pr(\{\omega\}) = \frac{1}{|\mathcal{M}_{\min}(K)|},$$

where $|\mathcal{M}_{\min}(K)|$ is the cardinality of $\mathcal{M}_{\min}(K)$.

From now on, we use $Pr(X_t|K)$ to denote this truth value distribution of the atom t induced by the maximum entropy distribution over the set of minimal models of K, then the semantic information of the knowledge base K may be represented by the sequence

$$\langle Pr(X_{t_1}|K), Pr(X_{t_2}|K), \cdots, Pr(X_{t_n}|K)\rangle$$

of truth value distributions. We abbreviate the sequence as $\mathbf{Pr}(\mathcal{A}|K)$ if there is no confusion. Note that for any atom $t \notin At(K)$, $Pr(X_t|K)$ is the following fixed distribution:

$$Pr(X_t = \mathsf{B}|K) = 0,\ Pr(X_t = \mathsf{T}|K) = Pr(X_t = \mathsf{F}|K) = \frac{1}{2}.$$

Then just for simplicity, we use an alphabetic sequential expression of $At(K)$ instead of $\mathcal{A}$ in subsequent examples. For example, we may use

$$\langle Pr(X_a|K_0), Pr(X_b|K_0), Pr(X_c|K_0)\rangle$$

to represent the semantic information of $K_0 = \{a, \neg a \vee \neg b, b, c\}$ from a probabilistic perspective.

In particular, the probability $Pr(X_t = \mathsf{B}|K)$ for each atom t describes the uncertainty of t being assigned to the inconsistent truth value B by minimal models of K, then we use the sequence

$$\langle Pr(X_{t_1} = \mathsf{B}|K), Pr(X_{t_2} = \mathsf{B}|K), \cdots, Pr(X_{t_n} = \mathsf{B}|K)\rangle$$

to describe the inconsistency of K from a probabilistic perspective. We may abbreviate the sequence as $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K)$ if there is no confusion. Evidently, we have the following property about the sequence.

Proposition 1. $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) = \vec{0}$ *if and only if* K *is consistent, where* $\vec{0}$ *is the zero vector.*

Proof. $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) = \vec{0}$. $\Leftrightarrow$ It holds that $\omega!(K) = \emptyset$ for all $\omega \in \mathcal{M}_{\min}(K)$. $\Leftrightarrow$ For all $\omega \in \mathcal{M}_{\min}(K)$, ω is a classical model of K. $\Leftrightarrow$ K is consistent. □

This property shows that the sequence $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K)$ satisfies the property of *Consistency* presented in [11]. It means that the sequence is capable of distinguishing inconsistency from consistency, but cannot make a distinction between any two consistent knowledge bases.

Now we use the following example to illustrate these notions.

Example 1. Consider $K_0 = \{a, \neg a \vee \neg b, b, c\}$. Then

$$\mathcal{MI}(K_0) = \{M_1\},\ \mathcal{FF}(K_1) = \{c\},$$

where $M_1 = \{a, \neg a \vee \neg b, b\}$ is a unique minimal inconsistent subset of K_0.

Next we consider the set of minimal models of K_0:

$$\mathcal{M}_{\min}(K_0) = \{\omega_1, \omega_2\},$$

where

$$\omega_1(a) = \mathsf{B},\ \omega_1(b) = \mathsf{T},\ \omega_1(c) = \mathsf{T};$$

$$\omega_2(a) = \mathsf{T},\ \omega_2(b) = \mathsf{B},\ \omega_2(c) = \mathsf{T}.$$

Then the maximal entropy distribution is given as follows:

$$Pr(\{\omega_1\}) = Pr(\{\omega_2\}) = 0.5.$$

Furthermore, the sequence of probability distributions about truth values of atoms are given in Table 1, where each $Pr(X_t|K_0)$ is given by a column.

Obviously, $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K_0) = \langle 0.5, 0.5, 0\rangle$.

3. Causal responsibility

In this section, we identify the causal responsibility of a formula of a knowledge base for the inconsistency in that base from the semantic perspective. As mentioned earlier, characterizing the counterfactual dependence of the inconsistency on the formula under some contingency is central to identifying the causal responsibility of a formula for the inconsistency. Roughly speaking, the

Table 1
The sequence $\mathbf{Pr}(\mathcal{A}|K_0)$.

T	$Pr(X_a = T|K_0)$	$Pr(X_b = T|K_0)$	$Pr(X_c = T|K_0)$
B	0.5	0.5	0
T	0.5	0.5	1
F	0	0	0

assignment of the inconsistent truth value B in a minimal model may be considered as what we have to do if none of formulas is allowed to have the truth value F under the minimal model. Then we may identify the counterfactual dependence of inconsistency on a formula by falsifying the formula. We start with a notion of F-interpretation.

Definition 2. Let α be a formula and $\alpha \not\equiv \top$. An interpretation ω is called an F-interpretation of α, if $\omega(\alpha) = \mathsf{F}$.

Obviously, F-interpretations of α are exactly ones that falsify the formula in α. Notice that any formula except the tautology can be falsified by at least one 2-valued F-interpretation. This means that there is no need to introduce the inconsistent truth value B to falsify a formula. Then we only consider classical (2-valued) F-interpretations instead of 3-values ones in this paper. Generally, we call an interpretation ω an F-interpretation of a knowledge base K if $\omega(\alpha) = \mathsf{F}$ for all $\alpha \in K$.

Definition 3. Let K be an inconsistent knowledge base and $\alpha \in K$. The inconsistency in K counterfactually depends on α if there exists an F-interpretation ω of α such that $\omega \in \mathcal{M}_{\min}(K - \{\alpha\})$.

Given a knowledge base, notice that if there is a 2-valued minimal model of the base, then every minimal model of the base must be 2-valued, i.e., the knowledge base is consistent. Then it is evident that every minimal model of $K - \{\alpha\}$ is a 2-valued model, and is an F-interpretation of α in this case. Essentially, the counterfactual dependence of the inconsistency of K on α in paraconsistent semantics means that the B-atoms (characterization of inconsistency) would not have appeared in any minimal model of $K - \{\alpha\}$ if α had been falsified by one of these models. On the other hand, such a dependence implies that $\{\alpha\}$ is a minimal correction subset of K from the syntactic perspective. Now we use the following example to illustrate the counterfactual dependence.

Example 2. Consider $K_0 = \{a, \neg a \vee \neg b, b, c\}$ again. Let ω_3 be an interpretation such that

$$\omega_3(a) = \mathsf{F}, \quad \omega_3(b) = \mathsf{T}, \quad \omega_3(c) = \mathsf{T}.$$

Obviously, ω_3 is an F-interpretation of a, moreover, it is a classical model of $K - \{\alpha\}$. Then the inconsistency of K_0 counterfactually depends on a. Similarly, we can show that the inconsistency also counterfactually depends on either $\neg a \vee \neg b$ or b. But the inconsistency does not counterfactually depend on c.

The counterfactual dependence allows us to understand the relation between the inconsistency and an individual formula from a causal perspective intuitively. However, the counterfactual dependence under some contingency is a more general and important in identifying the causal relation in Halpern and Pearl's causal model [24]. Next we articulate the counterfactual dependence of the inconsistency on a formula under some contingency from the semantic perspective. We start with the following lemma.

Lemma 1. *Let K be an inconsistent knowledge base and $\Gamma \subseteq K$. If for any $\emptyset \subset \Gamma' \subseteq \Gamma$, any F-interpretation ω of Γ' is not a minimal model of $K - \Gamma'$, then $K - \Gamma \vdash \bot$.*

Proof. Assume, by way of contradiction, that $K - \Gamma \nvdash \bot$. Then there exists an interpretation ω such that $\omega(\alpha) = \mathsf{T}$ for all $\alpha \in K - \Gamma$. Moreover, ω is not an F-interpretation of Γ, then there exists $\Gamma_1 \subset \Gamma$ such that for all $\beta \in \Gamma_1$, $\omega(\beta) = \mathsf{F}$, and for all $\gamma \in \Gamma - \Gamma_1$, $\omega(\gamma) = \mathsf{T}$. This means that ω is an F-interpretation of Γ_1, moreover, it is also a model of $K - \Gamma_1$, which leads to the desired contradiction. $\square$

Definition 4. Given an inconsistent knowledge base K and $\alpha \in K$. Then the inconsistency of K counterfactually depends on α under a contingency Δ if the following conditions are satisfied.

(1) $\Delta \subseteq (K - \{\alpha\})$;
(2) There exists an F-interpretation ω of $\Delta \cup \{\alpha\}$ s.t. $\omega \in \mathcal{M}_{\min}(K - (\Delta \cup \{\alpha\}))$.
(3) For any $\emptyset \subset \Gamma \subset \Delta \cup \{\alpha\}$, any F-interpretation ω' of Γ, it holds that $\omega' \notin \mathcal{M}_{\min}(K - \Gamma)$.

Informally speaking, the counterfactual dependence of the inconsistency on α under a contingency Δ means that the B-atoms would not have appeared if all the formulas in $\Delta \cup \{\alpha\}$ had been falsified, moreover, the condition (3) implies that the falsification of each formula in $\Delta \cup \{\alpha\}$ is necessary to make the inconsistency disappear. The formulas in Δ are ones that have to be falsified in order to construct a scenario where the inconsistency counterfactually depends on α, i.e., they comprise the contingency. In particular, the counterfactual dependence of the inconsistency on a formula is exactly the case where $\Delta = \emptyset$.

Now we use the following example to illustrate this counterfactual dependence.

Example 3. Consider $K_1 = \{a, \neg a, \neg a \vee \neg b, b, c\}$. Note that ω_1 is the unique minimal model of K_1, where

$$\omega_1(a) = \mathsf{B},\ \omega_1(b) = \omega_1(c) = \mathsf{T}.$$

The inconsistency of K_1 counterfactually depends on the formula a, because ω_2, one of the F-interpretations of $\{a\}$, is a classical model of $K_1 - \{a\}$, where

$$\omega_2(a) = \mathsf{F},\ \omega_2(b) = \omega_2(c) = \mathsf{T}.$$

By contrast, the inconsistency of K_1 does not counterfactually depend on the formula b. To illustrate this, consider any F-interpretation ω of $\{b\}$ such that $\omega(b) = \mathsf{F}$. Neither $\omega(a) = \mathsf{T}$ nor $\omega(a) = \mathsf{F}$ makes both a and $\neg a$ true. That is, we have to introduce the inconsistent truth value to give a model to $K_1 - \{b\}$.

Similarly, we can also show that the inconsistency of K_1 does not counterfactually depend on the formula $\neg a$.

But the inconsistency counterfactually depends on the formula b under the contingency $\{\neg a\}$, because the F-interpretation ω_3 of $\{\neg a, b\}$ is a model of $K_1 - \{\neg a, b\}$, where

$$\omega_3(a) = \mathsf{T},\ \omega_3(b) = \mathsf{F},\ \omega_3(c) = \mathsf{T}.$$

This also implies that the inconsistency counterfactually depends on the formula $\neg a$ under the contingency $\{b\}$. Similarly, we can show that the inconsistency counterfactually depends on the formula $\neg a \vee \neg b$ under the contingency $\{\neg a\}$.

Lastly, neither counterfactual dependence on c nor counterfactual dependence on c under some contingency holds for the inconsistency, because for any $\Delta \subset (K - \{c\})$, $\Delta \cup \{c\}$ satisfies the condition (2) if and only if Δ satisfies it, that is, the condition (3) cannot be satisfied.

Definition 5. Given an inconsistent knowledge base K and $\alpha \in K$. Then $\Gamma \subseteq K - \{\alpha\}$ is called the smallest contingency for α if the following conditions are satisfied:

(1) The inconsistency of K counterfactually depends on α under the contingency Γ;
(2) For any $\Delta \subseteq K - \{\alpha\}$, if the inconsistency of K counterfactually depends on α under the contingency Δ, then $|\Gamma| \leq |\Delta|$.

We use $C_{\min}(\alpha|K)$ to denote the set of all the smallest contingencies for α. In particular, if the inconsistency counterfactually depends on α, then $C_{\min}(\alpha|K) = \{\emptyset\}$. Moreover, if there is no contingency under which the inconsistency counterfactually depends on α, then $C_{\min}(\alpha|K) = \emptyset$. Further, we use $\eta(\alpha)$ to denote the size of the smallest contingency for α if $C_{\min}(\alpha|K) \neq \emptyset$, that is, $|\Gamma| = \eta(\alpha)$ for all $\Gamma \in C_{\min}(\alpha|K)$.

Now we are ready to define the causal responsibility of a formula for the inconsistency from the semantic perspective.

Definition 6. Let K be an inconsistent knowledge base and $\alpha \in K$. Then the degree of causal responsibility of α for the inconsistency in K, denoted $dr_{\text{cau}}(\alpha|K)$, is 0 if there is no contingency under which the inconsistency counterfactually depends on α; it is $\frac{1}{\eta(\alpha)+1}$ if the inconsistency counterfactually depends on α under some contingency, and the smallest size of such contingencies is $\eta(\alpha)$.

Essentially, the degree of causal responsibility of a formula α is more interested in the minimum number $\eta(\alpha)$ of formulas that have to be falsified to construct a contingency where the inconsistency counterfactually depends on α, and then the formula shares the responsibility for the inconsistency with ones involved in such a contingency with the size $\eta(\alpha)$. Evidently, $dr_{\text{cau}}(\alpha|K) = 1$ if and only if the inconsistency of K counterfactually depends on α, that is, we may consider α bearing full responsibility if the inconsistency counterfactually depends on α.

Example 4. Consider K_1 again. From Example 3, we can get the following results:

$$\eta(a) = 0,\ \eta(\neg a) = \eta(\neg a \vee \neg b) = \eta(b) = 1.$$

Then the degree of causal responsibility of each formula is given as follows:

$$\begin{aligned}
&dr_{\text{cau}}(a|K_1) = 1,\\
&dr_{\text{cau}}(\neg a|K_1) = dr_{\text{cau}}(\neg a \vee \neg b|K_1) = dr_{\text{cau}}(b|K_1) = \frac{1}{2},\\
&dr_{\text{cau}}(c|K_1) = 0.
\end{aligned}$$

Proposition 2. *Let K be a knowledge base and $\alpha \in K$. Then the inconsistency of K counterfactually depends on α under a contingency Δ if and only if $\Delta \cup \{\alpha\}$ is a minimal correction subset of K.*

Proof. If the inconsistency of K counterfactually depends on α under a contingency Δ, then there exists an F-interpretation ω of $\Delta \cup \{\alpha\}$ s.t. $\omega \in \mathcal{M}_{\min}(K - (\Delta \cup \{\alpha\}))$. That is, $K - (\Delta \cup \{\alpha\}) \nvdash \perp$. So, $\Delta \cup \{\alpha\}$ is a correction subset of K. In addition, the condition (3) about the contingency ensures that $\Delta \cup \{\alpha\}$ is minimal w.r.t. $\subseteq$. Therefore, $\Delta \cup \{\alpha\}$ is a minimal correction subset of K.

If $\Delta \cup \{\alpha\}$ is a minimal correction subset of K, then for any classical model ω of $K - (\Delta \cup \{\alpha\})$, ω is an F-interpretation of $\Delta \cup \{\alpha\}$. Moreover, for any nonempty proper subset Γ of $\Delta \cup \{\alpha\}$, $K - \Gamma \vdash \perp$. This implies that any F-interpretation of Γ cannot be a model of $K - \Gamma$. So, the inconsistency of K counterfactually depends on α under a contingency Δ. □

This proposition implies that $\eta(\alpha) + 1$ is exactly the minimum size of minimal correction subsets that α belongs to. Then $dr_{\mathrm{cau}}(\alpha|K)$ can be considered as an identical counterpart in semantic case of $dr(\alpha, K)$ presented in our previous work [15], which is given in terms of the minimum size of minimal correction subsets including α from the syntactic perspective, and can be interpreted as the Chockler and Halpern's responsibility [31] within the Halpern and Pearl's causal model [24].

4. Informational responsibility

The causal responsibility of a formula for the inconsistency focuses on the role of the formula in bringing about the inconsistency. Identifying such a role is tightly associated with the falsification of some formulas that can make the inconsistency disappear. However, besides the disappearance of inconsistent truth values in models, the falsification brings some other changes on semantic information, which is not captured by the notion of causal responsibility. To illustrate this, let us consider K_1 in Example 3 again. Notice that $\{\neg a\}$ is the unique contingency under which the inconsistency counterfactually depends on b and $\neg a \vee \neg b$, respectively. But if we falsify $\{\neg a, b\}$, then the interpretation ω_3 captures the semantic information in this case, where $\omega_3(a) = \mathsf{T}$, $\omega_3(b) = \mathsf{F}$, and $\omega_3(c) = \mathsf{T}$. In contrast, if we falsify $\{\neg a, \neg a \vee \neg b\}$, then the interpretation ω_4 captures the semantic information in this case, where $\omega_4(a) = \mathsf{T}$, $\omega_4(b) = \mathsf{T}$, and $\omega_4(c) = \mathsf{T}$. Compared to the minimal model ω_1 of K_1, ω_3 changes both assignments of truth values to a and b, while ω_4 only changes the assignment of truth value to a. Such changes about semantic information have not yet been considered in the notion of causal responsibility.

In this section, we incorporate the semantic information change due to such falsifications in the framework of identifying the causal role to describe the role of each formula in semantic information on the inconsistency. Recall that we use a sequence $\mathbf{Pr}(\mathcal{A}|K)$ of truth value distributions to capture the semantic information for a knowledge base K, then it is natural to associate information change made by falsifying formulas of the smallest contingency Γ for the formula and the formula α with a comparison between $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\}))$. Hence we need to introduce some distance functions to compare $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\}))$.

At first, we introduce a general definition of distance function, and we further define the informational responsibility of the formula α for the inconsistency based on the distance between $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\}))$. Then we consider some appropriate specific distance functions and corresponding informational responsibilities. Finally, we show that the causal responsibility can be considered as a specific informational responsibility with a drastic distance function.

Definition 7. A distance function over a set X, denoted $dist$, is a function $dist : X \times X \to [0, +\infty)$ satisfying the following conditions for all x, x', $x'' \in X$:

(1) $dist(x, x') = 0$ if and only if $x = x'$.
(2) $dist(x, x') = dist(x', x)$.
(3) $dist(x, x') \leq dist(x, x'') + dist(x', x'')$.

Given an appropriate distance function $dist$, the distance $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - K'))$ between $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K - K')$ evaluates the difference between $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K - K')$. It can be considered as an assessment of semantic information change made by removing formulas of K' from K. Then we may use such a distance function to define the informational responsibility of a formula for the inconsistency as follows.

Definition 8. Let K be an inconsistent knowledge base and $\alpha \in K$. Then the degree of informational responsibility of α for the inconsistency in K w.r.t. a distance function $dist$, denoted $dr_{\mathrm{inf}}(\alpha|K, dist)$, is defined as follows:

$$dr_{\mathrm{inf}}(\alpha|K, dist) = \begin{cases} \min\limits_{\Gamma \in C_{\min}(\alpha|K)} \frac{dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\})))}{1 + |\Gamma|}, & C_{\min}(\alpha|K) \neq \emptyset \\ 0, & C_{\min}(\alpha|K) = \emptyset \end{cases}.$$

Essentially, given a formula, the minimal change of the semantic information made by falsifying the formula under the smallest contingency is equally shared between all the formulas falsified. Then the informational responsibility of the formula is exactly the share of the formula. On the other hand, the information change due to the falsification operation is described by the distance between the two corresponding sequences of probability distributions over truth value assignments of atoms. Evidently, the informational responsibility depends on the choice of distance functions. Now we consider some appropriate distance functions. Note that $\mathbf{Pr}(\mathcal{A}|K)$ is a sequence of truth value distributions of atoms, then we start with the following distances between two discrete probability distributions.

Given a finite set $X = \{x_1, x_2, \cdots, x_n\}$ and two probability measures P and Q defined on a measurable space $(X, \mathcal{F})$, where $\mathcal{F}$ is a σ-algebra on X. We use p_i and q_i as abbreviations of $P(\{x_i\})$ and $Q(\{x_i\})$ for $i = 1, 2, \cdots, n$, respectively. Then we use $\langle p_1, p_2, \cdots, p_n\rangle$ and $\langle q_1, q_2, \cdots, q_n\rangle$ to denote the probability measures P and Q, respectively. In addition, we use $\sqrt{P}$ and $\sqrt{Q}$ as abbreviations of vectors $\langle \sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_n}\rangle$ and $\langle \sqrt{q_1}, \sqrt{q_2}, \cdots, \sqrt{q_n}\rangle$, respectively.

Now we consider the following two well known distances, i.e., total variation distance [32] and Hellinger distance [33] between the two discrete probability distributions.

- The total variation distance between P and Q is defined as

$$d_T(P, Q) = \sup_{A \in \mathcal{F}} |P(A) - Q(A)|.$$

The total variation distance is exactly the largest possible difference between the probabilities that P and Q can assign to the same event. It is directly related to the L_1 norm $\| P - Q \|_1$ of the difference $P - Q$ between two vectors P and Q [32]:

$$d_T(P, Q) = \frac{1}{2} \| P - Q \|_1 = \frac{1}{2} \sum_{i=1}^{n} |p_i - q_i|.$$

- The Hellinger distance (also called Jeffreys distance) between P and Q is defined as

$$d_H(P, Q) = \frac{1}{\sqrt{2}} \| \sqrt{P} - \sqrt{Q} \|_2 = \frac{1}{\sqrt{2}} \left(\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \right)^{\frac{1}{2}}.$$

The Hellinger distance is tightly related to the L_2 norm $\| \sqrt{P} - \sqrt{Q} \|_2$ of the difference of the square root vectors.

In this paper, we focus on the distance $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K'))$ built upon the sequence $\langle d(Pr(X_t|K), Pr(X_t|K'))\rangle_{t\in\mathcal{A}}$, where $d(Pr(X_t|K), Pr(X_t|K'))$ is a distance between probability distributions $Pr(X_t|K)$ and $Pr(X_t|K')$. Here we give two instances.

Definition 9. Let $\mathcal{P}(\mathcal{A})$ be a set of sequences of truth value distributions and $d(\cdot,\cdot)$ a distance for probability distributions. Then a sum-distance function, denoted $dist_d^+$, is a function $dist_d^+ : \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}) \to \mathbf{R}_{\geq 0}$ such that for all K and K',

$$dis_d^+(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K')) = \sum_{t\in\mathcal{A}} d\left(Pr(X_t|K), Pr(X_t|K')\right).$$

We can further use the total variation distance d_T and the Hellinger distance d_H to define distance functions $dist_{d_T}^+$ and $dist_{d_H}^+$, respectively.

Similarly, we can define module-distance function $dist_d^m$ as follows: a module-distance function is a function $dist_d^m : \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}) \to \mathbf{R}_{\geq 0}$ such that for all K and K',

$$dis_d^m(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K')) = \left(\sum_{t\in\mathcal{A}} \left(d\left(Pr(X_t|K), Pr(X_t|K')\right)\right)^2 \right)^{\frac{1}{2}},$$

where $d(\cdot,\cdot)$ is a distance between probability distributions.

Essentially, the sum-distance function aggregates differences of truth value distributions of all atoms, while the module-distance function focuses on the magnitude of the vector consisting of differences of truth value distributions of all atoms.

Proposition 3. *If $d(\cdot,\cdot)$ is a distance function, then both $dist_d^+$ and $dist_d^m$ are distance functions.*

Proof. Note that for each $z \in \{+, m\}$ $dis_d^z(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K')) = 0$ if and only if $d\left(Pr(X_t|K), Pr(X_t|K')\right) = 0$ for all $t \in \mathcal{A}$. Notice that d is a distance function, then for all $t \in \mathcal{A}$, it is the case that $d\left(Pr(X_t|K), Pr(X_t|K')\right) = 0$ if and only if $Pr(X_t|K) = Pr(X_t|K')$, that is, $\mathbf{Pr}(\mathcal{A}|K) = \mathbf{Pr}(\mathcal{A}|K')$.

The symmetry property is obvious for both $dist_d^+$ and $dist_d^m$. Now we show that the two distances satisfy the triangle inequality. At first, we consider the distance $dist_d^+$. Let K, K', and K'' be any three knowledge bases, then

$$\begin{aligned}
&dis_d^+(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K')) = \sum_{t\in\mathcal{A}} d\left(Pr(X_t|K), Pr(X_t|K')\right) \\
&\leq \sum_{t\in\mathcal{A}} \left[d\left(Pr(X_t|K), Pr(X_t|K'')\right) + d\left(Pr(X_t|K''), Pr(X_t|K')\right) \right] \\
&= \sum_{t\in\mathcal{A}} d\left(Pr(X_t|K), Pr(X_t|K'')\right) + \sum_{t\in\mathcal{A}} d\left(Pr(X_t|K''), Pr(X_t|K')\right) \\
&= dis_d^+(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K'')) + dis_d^+(\mathbf{Pr}(\mathcal{A}|K''), \mathbf{Pr}(\mathcal{A}|K')).
\end{aligned}$$

For the distance $dist^m_d$, we have that

$$dis^m_d(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K'))=\left(\sum_{t\in\mathcal{A}}\left(d\left(Pr(X_t|K),Pr(X_t|K')\right)\right)^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{t\in\mathcal{A}}\left[d\left(Pr(X_t|K),Pr(X_t|K'')\right)+d\left(Pr(X_t|K''),Pr(X_t|K')\right)\right]^2\right)^{\frac{1}{2}}.$$

By applying Cauchy-Schwarz inequality to the right of the inequality above, we get that

$$\left(\sum_{t\in\mathcal{A}}\left[d\left(Pr(X_t|K),Pr(X_t|K'')\right)+d\left(Pr(X_t|K''),Pr(X_t|K')\right)\right]^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{t\in\mathcal{A}}\left(d\left(Pr(X_t|K),Pr(X_t|K'')\right)\right)^2\right)^{\frac{1}{2}}+\left(\sum_{t\in\mathcal{A}}\left(d\left(Pr(X_t|K''),Pr(X_t|K')\right)\right)^2\right)^{\frac{1}{2}}$$

$$=dis^m_d(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K''))+dis^m_d(\mathbf{Pr}(\mathcal{A}|K''),\mathbf{Pr}(\mathcal{A}|K')).\quad\square$$

We have the following relation between the informational responsibilities of a formula under different distances.

Proposition 4. *Let K be an inconsistent knowledge base and $\alpha\in K$. Then*

(1) $dr_{\inf}(\alpha|K,dist^+_{d_T})\leq\sqrt{2}dr_{\inf}(\alpha|K,dist^+_{d_H})$.
(2) $dr_{\inf}(\alpha|K,dist^m_{d_T})\leq\sqrt{2}dr_{\inf}(\alpha|K,dist^m_{d_H})$.
(3) $dr_{\inf}(\alpha|K,dist^m_{d_H})\leq\sqrt{dr_{\inf}(\alpha|K,dist^+_{d_T})}$.

Proof. Note that

$$d^2_H(P,Q)\leq d_T(P,Q)\leq\sqrt{2}d_H(P,Q)$$

holds for two probability distributions P and Q [33].

Then for any atom $t\in\mathcal{A}$, it holds that

$$d^2_H(Pr(X_t|K),Pr(X_t|K'))\leq d_T(Pr(X_t|K),Pr(X_t|K'))\leq\sqrt{2}d_H(Pr(X_t|K),Pr(X_t|K')).$$

Hence the three relations hold for the informational responsibilities. $\square$

Finally, we consider the relation between the causal and informational responsibilities. At first, we consider $dist_{0-1}$, a distance function given as follows:

$$dist_{0-1}(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K'))=\begin{cases}0, & \mathbf{Pr}(\mathcal{A}|K)=\mathbf{Pr}(\mathcal{A}|K')\\ 1, & otherwise\end{cases}.$$

This distance function just tells us whether two sequences of probability distribution are different from each other. The distance function of this kind is called the drastic distance measure in [34].

The following proposition shows that the causal responsibility is exactly a specific informational responsibility with the drastic distance function $dist_{0-1}$. This implies that the informational responsibility is more general than the notion of causal responsibility.

Proposition 5. *Let K be an inconsistent knowledge base and $\alpha\in K$. Then*

$$dr_{\mathrm{cau}}(\alpha|K)=dr_{\inf}(\alpha|K,dist_{0-1}).$$

Proof. If there is no contingency making the inconsistency of K counterfactually depend on α, then

$$dr_{\mathrm{cau}}(\alpha|K)=dr_{\inf}(\alpha|K,dist_{0-1})=0.$$

Otherwise, for any $\Gamma\in\mathcal{C}_{\min}(\alpha|K)$, it is the case that $\mathbf{Pr}(\mathcal{A}|K)\neq\mathbf{Pr}(\mathcal{A}|K-(\Gamma\cup\{\alpha\}))$, because there exists at least one atom t such that

$$Pr(X_t=\mathsf{B}|K)>0=Pr(X_t=\mathsf{B}|K-(\Gamma\cup\{\alpha\})).$$

Table 2
The sequences of $\mathbf{Pr}(\mathcal{A}|K_1)$ and $\mathbf{Pr}(\mathcal{A}|K_1 - (\Gamma \cup \{\alpha\}))$.

T	$Pr(X_t = T\|K)$											
	K_1			$K_1 - \{a\}$			$K_1 - \{\neg a, b\}$			$K_1 - \{\neg a, \neg a \vee \neg b\}$		
	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c
B	1	0	0	0	0	0	0	0	0	0	0	0
T	0	1	1	0	1	1	1	0	1	1	1	1
F	0	0	0	1	0	0	0	1	0	0	0	0

So,

$$dist_{0-1}(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\}))) = 1.$$

Therefore,

$$dr_{\inf}(\alpha|K, dist_{0-1}) = \frac{1}{1+|\Gamma|} = dr_{\text{cau}}(\alpha|K). \quad \square$$

Recall that the causal responsibility of a formula for the inconsistency just focuses on whether the inconsistency disappears when we falsify some formulas together with the formula. That is, it just focuses on whether the semantic information changes rather than how the information changes. This coincides with the sketchiness of the drastic distance function in this sense.

Now we use the following example to illustrate the informational role of formulas in the inconsistency.

Example 5. Consider $K_1 = \{a, \neg a, \neg a \vee \neg b, b, c\}$ again. We have mentioned that ω_1 is the unique minimal model of K_1, where

$$\omega_1(a) = \mathsf{B}, \omega_1(b) = \mathsf{T}, \omega_1(c) = \mathsf{T}.$$

Then the corresponding sequences of probability distributions about truth values of atoms are given in Table 2.

Then for the formula a,

$$d_T(Pr(X_a|K_1), Pr(X_a|K_1 - \{a\})) = 1,$$
$$d_T(Pr(X_b|K_1), Pr(X_b|K_1 - \{a\})) = 0,$$
$$d_T(Pr(X_c|K_1), Pr(X_c|K_1 - \{a\})) = 0.$$

So,

$$\begin{aligned} &dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{a\})) \\ =& \sum_{t \in \{a,b,c\}} d_T((Pr(X_t|K_1), Pr(X_t|K_1 - \{a\}) = 1. \end{aligned}$$

Therefore,

$$dr_{\inf}(a|K_1, dist^+_{d_T}) = \frac{dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{a\}))}{1 + |\emptyset|} = 1.$$

Note that for b, the inconsistency counterfactually depends on b under the contingency $\{\neg a\}$. Similarly, we can get that

$$dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{\neg a, b\})) = 2.$$

Therefore,

$$dr_{\inf}(b|K_1, dist^+_{d_T}) = \frac{dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{\neg a, b\}))}{1 + |\{\neg a\}|} = 1.$$

We have also shown that the inconsistency counterfactually depends on $\neg a \vee \neg b$ under the contingency $\{\neg a\}$. Then we get that

$$dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{\neg a, \neg a \vee \neg b\})) = 1.$$

Therefore,

$$dr_{\inf}(\neg a \vee \neg b|K_1, dist^+_{d_T}) = \frac{dis^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - \{\neg a, b\}))}{1 + |\{\neg a\}|} = \frac{1}{2}.$$

Table 3
The informational responsibility of each formula.

Formula α	$dr_{\mathrm{inf}}(\alpha\|K_1, dist)$				
	$dist_{0-1}$	$dist^+_{d_T}$	$dist^+_{d_H}$	$dist^m_{d_T}$	$dist^m_{d_H}$
a	1	1	1	1	1
$\neg a$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
b	$\frac{1}{2}$	1	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\neg a \vee \neg b$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
c	0	0	0	0	0

Recall that the inconsistency counterfactually depends on $\neg a$ under the contingency either $\{\neg a \vee \neg b\}$ or $\{b\}$. Then

$$\begin{aligned} & dr_{\mathrm{inf}}(\neg a|K_1, dist^+_{d_T}) \\ = & \min_{\Gamma \in \left\{\{b\},\{\neg a \vee \neg b\}\right\}} \frac{dist^+_{d_T}(\mathbf{Pr}(\mathcal{A}|K_1), \mathbf{Pr}(\mathcal{A}|K_1 - (\Gamma \cup \{\neg a\})))}{1+|\Gamma|} = \frac{1}{2}. \end{aligned}$$

For c, the inconsistency does not counterfactually depend on c under any contingency. So,

$$dr_{\mathrm{inf}}(c|K_1, dist^+_{d_T}) = 0.$$

Note that

$$dr_{\mathrm{cau}}(b|K_1) = dr_{\mathrm{cau}}(\neg a \vee \neg b|K_1) = \frac{1}{2},$$

but

$$dr_{\mathrm{inf}}(b|K_1, dist^+_{d_T}) = 1 > \frac{1}{2} = dr_{\mathrm{inf}}(\neg a \vee b|K_1, dist^+_{d_T}).$$

Both are intuitive from their own respective perspectives. Because we need to false one formula to make the inconsistency counterfactually depend on each of the two formulas, so they have the same causal responsibility for the inconsistency. But the two falsifications bring different changes on semantic information. In detail, falsifying $\{\neg a \vee \neg b, \neg a\}$ only changes the assignment of truth value to a, but falsifying $\{b, \neg a\}$ changes both the assignments of truth value to a and b. So, the formula b bears greater informational responsibility for the inconsistency.

If we use the module-distance function $dist^m_{d_T}$, we can get the following results:

$$\begin{aligned} & dr_{\mathrm{inf}}(a|K_1, dist^m_{d_T}) = 1, \quad dr_{\mathrm{inf}}(\neg a|K_1, dist^m_{d_T}) = \frac{1}{2}, \\ & dr_{\mathrm{inf}}(b|K_1, dist^m_{d_T}) = \frac{\sqrt{2}}{2}, \quad dr_{\mathrm{inf}}(\neg a \vee b|K_1, dist^m_{d_T}) = \frac{1}{2}, \\ & dr_{\mathrm{inf}}(c|K_1, dist^m_{d_T}) = 0. \end{aligned}$$

In addition, we may use the distance d_H in computing the informational responsibility for each formula. We summarize these results in Table 3.

5. Logical properties of informational responsibility

In this section, we consider logical properties of the informational responsibility. The most properties proposed so far are heavily oriented toward base-level measures. There are few specialized properties for characterizing the formula-level measure. Here we consider the properties of Minimality, Free Formula Independence, Monotonicity and Dominance used in [13,11,15]. All the last three properties are adapted from Hunter and Konieczny's set of properties for base-level measures [1,13,11].

Without loss of generality, we assume that a formula-level inconsistency measure dr assigns every formula α in K a nonnegative number $dr(\alpha|K)$ such that higher value implies more responsibility. Note that both dr_{cau} and dr_{inf} are in accord with this assumption. Under this assumption, the properties mentioned above are given as follows:

- *Minimality*: If $\alpha \in \mathcal{FF}(K)$, then $dr(\alpha|K) = 0$.
- *Free Formula Independence*: If $\beta \notin K$ and $\beta \in \mathcal{FF}(K \cup \{\beta\})$, then $dr(\alpha|K) = dr(\alpha|K \cup \{\beta\})$ for all $\alpha \in K$.
- *Monotonicity*: If $\beta \notin K$, then $dr(\alpha|K) \leq dr(\alpha|K \cup \{\beta\})$ for all $\alpha \in K$.
- *Dominance*: If $\alpha, \beta \notin K$, $\alpha \vdash \beta$ and $\alpha \nvdash \bot$, then $dr(\gamma|K \cup \{\alpha\}) \geq dr(\gamma|K \cup \{\beta\})$ for all $\gamma \in K$.

Roughly speaking, the property of Minimality states that free formulas bear no responsibility for the inconsistency [13], while the property of Free Formula Independence says that free formulas have no impact on the formula-level measure [13,15]. The property of Monotonicity requires that the formula-level measure is monotonic w.r.t. the extension of a knowledge base, whilst the property of Dominance states that the responsibility of a formula cannot decrease when we replace a formula with another logically stronger one. The last two properties have been used to characterize the formula-level measure $\mathrm{MIV}_{\mathrm{C}}$ value in [13] (Proposition 1 in [13]), which is built upon minimal inconsistent subsets.

The following two variations of Free Formula Independence are also considered, which are adapted from the Safe Formula Independence [11] and the Tautology Independence [35], respectively:

- *Safe Formula Independence*: If β is a safe formula of $K \cup \{\beta\}$, then $dr(\alpha|K) = dr(\alpha|K \cup \{\beta\})$ for all $\alpha \in K$.
- *Tautology Independence*: If $\beta \equiv \top$ and $\beta \notin K$, then $dr(\alpha|K) = dr(\alpha|K \cup \{\beta\})$ for all $\alpha \in K$.

In addition, the property of Fairness presented in [36] requires that any two formulas of minimal inconsistent knowledge base have the same degree of responsibility for the inconsistency. It is given as follows:

- *Fairness*: If K is minimally inconsistent, then $dr(\alpha|K) = dr(\beta|K)$ for all $\alpha, \beta \in K$.

Finally, we will consider some properties related to the modularity of the inconsistency as well. Now we start with the following proposition.

Proposition 6. *Let K be an inconsistent knowledge base and $\alpha \in K$. Then*

(1) $dr_{\mathrm{cau}}(\alpha|K) > 0$ if and only if $\alpha \in \bigcup \mathcal{MI}(K)$.
(2) $dr_{\mathrm{cau}}(\alpha|K) > 0$ if and only if $dr_{\mathrm{inf}}(\alpha|K, dist) > 0$.

Proof. Given an inconsistent knowledge base K and a formula $\alpha \in K$.

(1) Notice that $dr_{\mathrm{cau}}(\alpha|K) > 0$ if and only if the inconsistency of K counterfactually depends on α under some contingency Δ.
By Proposition 2, the inconsistency of K counterfactually depends on α under some contingency Δ if and only if $\{\alpha\} \cup \Delta$ is a minimal correction subset of K.
Allowing for the duality relation between minimal inconsistent subsets and the minimal correction subsets [29], then $\{\alpha\} \cup \Delta$ is a minimal correction subset of K if and only if $\{\alpha\} \cup \Delta \subseteq \bigcup \mathcal{MI}(K)$.
Therefore, $dr_{\mathrm{cau}}(\alpha|K) > 0$ if and only if $\alpha \in \bigcup \mathcal{MI}(K)$.
(2) Similar to the proof above, we know that $dr_{\mathrm{inf}}(\alpha|K, dist) > 0$ if and only if the inconsistency of K counterfactually depends on α under some contingency Δ, moreover, $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Delta \cup \{\alpha\})) > 0$.
Actually,

$$dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Delta \cup \{\alpha\})) > 0$$

if and only if

$$dist_0(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Delta \cup \{\alpha\})) = 1.$$

Therefore, $dr_{\mathrm{inf}}(\alpha|K, dist) > 0$ if and only if $dr_{\mathrm{inf}}(\alpha|K, dist_0) > 0$. By Proposition 5, we know that $dr_{\mathrm{cau}}(\alpha|K) = dr_{\mathrm{inf}}(\alpha|K, dist_0)$ for all $\alpha \in K$. Then it holds that $dr_{\mathrm{cau}}(\alpha|K) > 0$ if and only if $dr_{\mathrm{inf}}(\alpha|K, dist) > 0$. $\square$

This proposition shows that only the formulas involved in minimal inconsistent subsets need to bear informational responsibility as well as causal responsibility for the inconsistency in a knowledge base. The following direct consequence shows that a free formula bears neither the causal responsibility nor the informational responsibility for the inconsistency, i.e., the responsibilities satisfy the property of Minimality.

Corollary 1. *Let K be an inconsistent knowledge base and $\alpha \in K$. Then*

(1) $dr_{\mathrm{cau}}(\alpha|K) = 0$ if and only if $\alpha \in \mathcal{FF}(K)$.
(2) $dr_{\mathrm{cau}}(\alpha|K) = 0$ if and only if $dr_{\mathrm{inf}}(\alpha|K, dist) = 0$.

Proof. This is the direct consequence of Proposition 6. $\square$

Certainly, for tautologies and safe formulas, the two special kinds of free formulas, they need not bear any responsibility for the inconsistency according to the property of Minimality.

Corollary 2. *Let K be an inconsistent knowledge base and $\alpha \in K$. Then*

(1) $dr_{\mathrm{cau}}(\alpha|K) = dr_{\mathrm{inf}}(\alpha|K, dist) = 0$ if α is a tautology.

Table 4
The sequences of $\mathbf{Pr}(\mathcal{A}|K_2)$ and $\mathbf{Pr}(\mathcal{A}|K_2 - \{\alpha\})$.

T	$Pr(X_t = T\|K)$											
	K_2			$K_2 - \{a \wedge c\}$			$K_2 - \{\neg a \vee \neg b\}$			$K_2 - \{b\}$		
	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c
B	0.5	0.5	0	0	0	0	0	0	0	0	0	0
T	0.5	0.5	1	0	1	0.5	1	1	1	1	0	1
F	0	0	0	1	0	0.5	0	0	0	0	1	0

(2) $dr_{\text{cau}}(\alpha|K) = dr_{\text{inf}}(\alpha|K, dist) = 0$ *if* α *is a safe formula of* K.

It has been shown that the degree of responsibility satisfies the property of Fairness [15], i.e., any two formulas of a minimal inconsistent knowledge base have the same degree of responsibility for the inconsistency of that base. Then as the identical counterpart of the degree of responsibility, the causal responsibility also satisfies this property. But the informational responsibility may allow us to make a distinction between some formulas in a minimal inconsistent knowledge base. To illustrate this, consider the following example.

Example 6. Consider $K_2 = \{a \wedge c, \neg a \vee \neg b, b\}$. The set of minimal models of K_2 is given as

$$\mathcal{M}_{\min}(K_2) = \{\omega_1, \omega_2\},$$

where

$$\omega_1(a) = \mathsf{T},\ \omega_1(b) = \mathsf{B},\ \omega_1(c) = \mathsf{T};\ \omega_2(a) = \mathsf{B},\ \omega_2(b) = \omega_2(c) = \mathsf{T}.$$

Evidently, K_2 is a minimal inconsistent knowledge base, and the inconsistency of K_2 counterfactually depends on each of the three formulas. Moreover,

- $\mathcal{M}_{\min}(K - \{a \wedge c\}) = \{\omega_3, \omega_4\}$, where $\omega_3(a) = \mathsf{F}$, $\omega_3(b) = \mathsf{T}$, $\omega_3(c) = \mathsf{T}$; $\omega_4(a) = \mathsf{F}$, $\omega_4(b) = \mathsf{T}$, $\omega_4(c) = \mathsf{F}$.
- $\mathcal{M}_{\min}(K - \{\neg a \vee \neg b\}) = \{\omega_5\}$, where $\omega_5(a) = \omega_5(b) = \omega_5(c) = \mathsf{T}$.
- $\mathcal{M}_{\min}(K - \{b\}) = \{\omega_6\}$, where $\omega_6(a) = \omega_6(c) = \mathsf{T}$, and $\omega_6(b) = \mathsf{F}$.

The corresponding sequences of distributions over truth value assignments of atoms are given in Table 4.

Then for the causal responsibility, we have that

$$dr_{\text{cau}}(a \wedge c|K_2) = dr_{\text{cau}}(\neg a \vee \neg b|K_2) = dr_{\text{cau}}(b|K_2) = 1.$$

By contrast, the three formulas have different informational responsibilities:

$$dr_{\text{inf}}(a \wedge c|K_2, dist^+_{d_T}) = 2 > dr_{\text{inf}}(b|K_2, dist^+_{d_T}) = \frac{3}{2} > dr_{\text{inf}}(\neg a \vee \neg b|K_2, dist^+_{d_T}) = 1.$$

This distinction is rational if we consider different semantic information changes made by removing different formulas from K_2. This also implies that the property of Fairness is not appropriate for the case where we take into account the semantic information change. Instead, as shown later, the informational responsibility satisfies the property of *Reciprocity*, which is a more suitable counterpart of the property of Fairness for the semantic case.

It has been also shown that the degree of responsibility $dr(\alpha, K)$ satisfies the property of Free Formula Independence [15]. That is, it holds that $dr_{\text{cau}}(\alpha|K) = dr_{\text{cau}}(\alpha|K \cup \{\beta\})$ for all $\alpha \in K$ if a new formula $\beta \notin K$ is a free formula of $K \cup \{\beta\}$. However, the following example shows that the informational responsibility allows us to capture the semantic information change brought by free formulas, that is, adding free formulas may bring changes on the informational responsibility of some formula for the inconsistency.

Example 7. Consider $K_3 = \{a \wedge b, \neg a\}$. Evidently, $b \wedge c$ is a free formula of $K_3 \cup \{b \wedge c\}$. The set of minimal models of K_3 is given as follows: $\mathcal{M}_{\min}(K_3) = \{\omega_1\}$, where $\omega_1(a) = \mathsf{B}$ and $\omega_1(b) = \mathsf{T}$. Then

$$Pr(X_a = \mathsf{B}|K_3) = Pr(X_b = \mathsf{T}|K_3) = 1.$$

For the formula $a \wedge b$,

$$\mathcal{M}_{\min}(K_3 - \{a \wedge b\}) = \{\omega_2, \omega_3\},$$

where $\omega_2(a) = \omega_2(b) = \mathsf{F}$, and $\omega_3(a) = \mathsf{F}, \omega_3(b) = \mathsf{T}$. Then

$$Pr(X_a = \mathsf{F}|K_3 - \{a \wedge b\}) = 1,\ Pr(X_b = \mathsf{T}|K_3 - \{a \wedge b\}) = Pr(X_b = \mathsf{F}|K_3 - \{a \wedge b\}) = 0.5.$$

Then

$$dr_{\inf}(a \wedge b|K_3, dist^+_{d_T}) = 1.5.$$

Similarly, we can compute the informational responsibility of $\neg a$ as follows:

$$dr_{\inf}(\neg a|K_3, dist^+_{d_T}) = 1.$$

Now we consider the knowledge base $K_4 = K_3 \cup \{b \wedge c\}$. Then

$$\mathcal{M}_{\min}(K_4) = \{\omega_4\},$$

where $\omega_4(a) = \mathsf{B}, \omega_4(b) = \omega_4(c) = \mathsf{T}$. Then

$$Pr(X_a = \mathsf{B}|K_4) = Pr(X_b = \mathsf{T}|K_4) = Pr(X_c = \mathsf{T}|K_4) = 1.$$

Now we consider formulas $a \wedge b$ and $\neg a$ again.

$$\mathcal{M}_{\min}(K_4 - \{a \wedge b\}) = \{\omega_5\},$$

where $\omega_5(a) = \mathsf{F}, \omega_5(b) = \omega_5(c) = \mathsf{T}$. Then

$$Pr(X_a = \mathsf{F}|K_4 - \{a \wedge b\}) = 1$$

$$Pr(X_b = \mathsf{T}|K_4 - \{a \wedge b\}) = Pr(X_c = \mathsf{T}|K_4 - \{a \wedge b\}) = 1.$$

Then

$$dr_{\inf}(a \wedge b|K_4, dist^+_{d_T}) = 1 < 1.5 = dr_{\inf}(a \wedge b|K_3, dist^+_{d_T}).$$

This result of comparison is intuitive because the removal of $a \wedge b$ from K_3 brings the change on the semantic information about b, while the removal of $a \wedge b$ from K_4 makes the semantic information about b unchanged.

For the formula $\neg a$,

$$\mathcal{M}_{\min}(K_4 - \{\neg a\}) = \{\omega_6\},$$

where $\omega_6(a) = \omega_6(b) = \omega_6(c) = \mathsf{T}$. Evidently,

$$dr_{\inf}(\neg a|K_4, dist^+_{d_T}) = dr_{\inf}(\neg a|K_3, dist^+_{d_T}) = 1.$$

This is also rational because in the both two cases, the absence of the formula $\neg a$ only changes the truth value assignment of the atom a.

In summary, compared to the property of free formula independence, the informational responsibility characterizes the role of each formula in semantic information on the inconsistency realistically in the case where we add a free formula to a knowledge base.

On the other hand, the following proposition shows that the informational responsibility satisfies both the Safe Formula Independence and the Tautology Independence, which are considered as two weak versions of the property of Free Formula Independence.

Proposition 7. *Let K be an inconsistent knowledge base and $\alpha \in K$. Then*

(1) $dr_{\inf}(\alpha|K \cup \{\beta\}, dist) = dr_{\inf}(\alpha|K, dist)$ if β is a tautology.
(2) $dr_{\inf}(\alpha|K \cup \{\gamma\}, dist) = dr_{\inf}(\alpha|K, dist)$ if γ is a safe formula of $K \cup \{\gamma\}$.

Proof. Let K be an inconsistent knowledge base and $\alpha \in K$.

(1) Let β be a tautology, then $\mathcal{M}_{\min}(\Gamma \cup \{\beta\}) = \mathcal{M}_{\min}(\Gamma)$ for any $\Gamma \subseteq K$. Then for all $\alpha \in K$, if the inconsistency of K counterfactually depends on α under a contingency Δ, then the inconsistency of $K \cup \{\beta\}$ also counterfactually depends on α under a contingency Δ, moreover,

$$\mathcal{M}_{\min}(K \cup \{\beta\} - (\Delta \cup \{\alpha\})) = \mathcal{M}_{\min}(K - (\Delta \cup \{\alpha\})).$$

Therefore, $dr_{\inf}(\alpha|K \cup \{\beta\}, dist) = dr_{\inf}(\alpha|K, dist)$.

(2) Let γ be a safe formula of $K \cup \{\gamma\}$, then $At(\{\gamma\}) \cap At(K) = \emptyset$. Here we only need to consider the case where γ is not a tautology. For each atom $t \in At(\{\gamma\})$, it holds that

$$Pr(X_t = \mathsf{T}|\Gamma) = Pr(X_t = \mathsf{F}|\Gamma) = \frac{1}{2},$$

and

$$Pr(X_t|\Gamma \cup \{\gamma\}) = Pr(X_t|\{\gamma\})$$

for all $\emptyset \subset \Gamma \subseteq K$.
On the other hand, for each atom $s \in At(K)$, it holds that

$$Pr(X_s|\Gamma \cup \{\gamma\}) = Pr(X_s|\Gamma)$$

for all $\emptyset \subset \Gamma \subseteq K$. In particular,

$$Pr(X_s|K \cup \{\gamma\}) = Pr(X_s|K)$$

Moreover, given a formula $\alpha \in K$, the inconsistency in $K \cup \{\gamma\}$ counterfactually depends on α under a contingency Δ if and only if the inconsistency in K counterfactually depends on α under the contingency Δ. This implies that

$$dr_{\inf}(\alpha|K \cup \{\gamma\}, dist) = 0 \text{ if and only if } dr_{\inf}(\alpha|K, dist) = 0.$$

Hence we only need to consider the case that the inconsistency in K counterfactually depends on α under the contingency Δ. Then for $s \in At(K)$,

$$Pr(X_s|K - (\Delta \cup \{\alpha\})) = Pr(X_s|K \cup \{\gamma\} - (\Delta \cup \{\alpha\})).$$

And for each $t \in At(\{\gamma\})$,

$$Pr(X_t|K - (\Delta \cup \{\alpha\})) = Pr(X_t|K),$$
$$Pr(X_t|K \cup \{\gamma\} - (\Delta \cup \{\alpha\})) = Pr(X_t|K \cup \{\gamma\}) = Pr(X_t|\{\gamma\}).$$

This implies that for any distance function d,

$$\begin{aligned} & d(Pr(X_t|K), Pr(X_t|K - (\Delta \cup \{\alpha\}))) \\ = & d(Pr(X_t|K \cup \{\gamma\}), Pr(X_t|K \cup \{\gamma\} - (\Delta \cup \{\alpha\}))) \\ = & 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & d(Pr(X_s|K), Pr(X_s|K - (\Delta \cup \{\alpha\}))) \\ = & d(Pr(X_s|K \cup \{\gamma\}), Pr(X_s|K \cup \{\gamma\} - (\Delta \cup \{\alpha\}))) \end{aligned}$$

Hence,

$$\begin{aligned} & dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Delta \cup \{\alpha\}))) \\ = & dist(\mathbf{Pr}(\mathcal{A}|K \cup \{\gamma\}), \mathbf{Pr}(\mathcal{A}|K \cup \{\gamma\} - (\Delta \cup \{\alpha\}))). \end{aligned}$$

Therefore,

$$dr_{\inf}(\alpha|K \cup \{\gamma\}, dist) = dr_{\inf}(\alpha|K, dist). \quad \square$$

Next we consider two typical scenarios in exploring properties of inconsistency measures. The first scenario is that we add a new formula to a knowledge base, which is tightly related to the property of Monotonicity. In contrast, the second one is that we replace a formula in a knowledge base with a new logically stronger consistent formula, which is tightly related to the property of Dominance.

Now we consider the first scenario. Recall that we have shown that adding the free formula $b \wedge c$ to the knowledge base K_3 make the informational responsibility of $a \wedge b$ decrease in Example 7. Here we must point out that for some formulas in a knowledge base, as illustrated by the following example, adding a new formula may make their informational responsibilities for the inconsistency increase. This means that the informational responsibility does not satisfy the property of Monotonicity.

Example 8. Consider $K_3 = \{a \wedge b, \neg a\}$ again. We have obtained that

$$dr_{\inf}(a \wedge b|K_3, dist^{+}_{d_T}) = 1.5.$$

Now we consider $K_5 = K_3 \cup \{\neg b\}$. Then ω_1 is the unique minimal model of K_5, where $\omega_1(a) = \omega_1(b) = \mathsf{B}$. And ω_2 is the unique model of $K_5 - \{a \wedge b\} = \{\neg a, \neg b\}$, where $\omega_2(a) = \omega_2(b) = \mathsf{F}$. Then

$$dr_{\inf}(a \wedge b|K_5, dist^{+}_{d_T}) = 2.$$

Therefore,

$$dr_{\inf}(a \wedge b|K_5, dist^{+}_{d_T}) > dr_{\inf}(a \wedge b|K_3, dist^{+}_{d_T}).$$

Similarly,

$$dr_{\inf}(a \wedge b|K_5, dist^{m}_{d_T}) = \sqrt{2} > \frac{\sqrt{5}}{2} = dr_{\inf}(a \wedge b|K_3, dist^{m}_{d_T}).$$

If we use the Hellinger distance d_H instead of d_T, we obtain the same conclusion:

$$dr_{\inf}(a \wedge b|K_5, dist^{+}_{d_H}) = 2 > 1.5 = dr_{\inf}(a \wedge b|K_3, dist^{+}_{d_H});$$

$$dr_{\inf}(a \wedge b|K_5, dist^{m}_{d_H}) = \sqrt{2} > \frac{\sqrt{5}}{2} = dr_{\inf}(a \wedge b|K_3, dist^{m}_{d_H}).$$

If we summarize these scenarios about adding free formulas, safe formulas, and the formula $\neg b$, we may find that the change of semantic information of a knowledge base varies with new formulas added to the knowledge base, and the informational responsibility catches this variance.

As far as the second scenario is concerned, the following example illustrates that this kind of replacement does not have uniform impact on the informational responsibilities of formulas, that is, the informational responsibility does not satisfy the property of Dominance.

Example 9. Let $K_6 = \{\neg a, \neg b, \neg c\}$. Consider $K_7 = K_6 \cup \{a \vee b\}$ and $K_8 = K_6 \cup \{a \wedge b \wedge c\}$.

Obviously, the set of minimal models of K_7 is $\{\omega_1, \omega_2\}$, where $\omega_1(a) = \mathsf{B}, \omega_1(b) = \omega_1(c) = \mathsf{F}$ and $\omega_2(b) = \mathsf{B}, \omega_2(a) = \omega_2(c) = \mathsf{F}$. The inconsistency counterfactually depends on $\neg a$, and ω_3 is the unique minimal model of $K_7 - \{\neg a\}$, where $\omega_3(a) = \mathsf{T}$ and $\omega_3(b) = \omega_3(c) = \mathsf{F}$. Then

$$dr_{\inf}(\neg a|K_7, dist^{+}_{d_T}) = dr_{\inf}(\neg b|K_7, dist^{+}_{d_T}) = 1.5, \quad dr_{\inf}(\neg c|K_7, dist^{+}_{d_T}) = 0.$$

Evidently, ω_4 is the unique minimal model of K_8, where $\omega_4(a) = \omega_4(b) = \omega_4(c) = \mathsf{B}$. Moreover, the inconsistency in K_8 counterfactually depends on $\neg a$ under the contingency $\{\neg b, \neg c\}$. ω_5 is the unique minimal model of $K_8 - \{\neg a, \neg b, \neg c\}$, where $\omega_5(a) = \omega_5(b) = \omega_5(c) = \mathsf{T}$. Then

$$dr_{\inf}(\neg a|K_8, dist^{+}_{d_T}) = dr_{\inf}(\neg b|K_8, dist^{+}_{d_T}) = dr_{\inf}(\neg c|K_8, dist^{+}_{d_T}) = 1.$$

Notice that

$$dr_{\inf}(\neg a|K_7, dist^{+}_{d_T}) > dr_{\inf}(\neg a|K_8, dist^{+}_{d_T}),$$
$$dr_{\inf}(\neg b|K_7, dist^{+}_{d_T}) > dr_{\inf}(\neg b|K_8, dist^{+}_{d_T}),$$

but

$$dr_{\inf}(\neg c|K_7, dist^{+}_{d_T}) < dr_{\inf}(\neg c|K_8, dist^{+}_{d_T}).$$

The modularity of inconsistency in a knowledge base has been considered in characterizing inconsistency measures as well as measuring the inconsistency recently [15,37]. Here we consider some properties related to the modularity of inconsistency.

Definition 10. Given a knowledge base K, a subset $S \subseteq K$ is called a *block* of K if and only if

(1) $At(S) \cap At(K - S) = \emptyset$.
(2) For all $\emptyset \subset S' \subset S$, $At(S') \cap At(S - S') \neq \emptyset$.

Essentially, a block of a knowledge base is a minimal separate subset of the knowledge base with regard to the partition of atoms. For example, consider $K_1 = \{a, \neg a, \neg a \vee \neg b, b, c\}$ again, there are two blocks $S_1 = \{a, \neg a, \neg a \vee \neg b, b\}$ and $S_2 = \{c\}$. Evidently, it holds that $Pr(X_t|S) = Pr(X_t|K)$ for all $t \in At(S)$ if S is a block of K. From now on, we use $\mathcal{B}(K)$ to denote the set of blocks of K.

Given a block S, we use $\kappa(S)$ to denote the smallest size of minimal correction subsets of S, that is,

$$\kappa(S) = \min_{R \in \mathcal{MC}(S)} |R|.$$

Further, we use $\mathcal{MC}_{\min}(S)$ to denote the set of all minimal correction subsets with the smallest size, i.e.,

$$\mathcal{MC}_{\min}(S) = \{R \in \mathcal{MC}(S) \mid |R| \leq |R'| \text{ for all } R' \in \mathcal{MC}(S)\}.$$

Table 5
The sequences of $\mathbf{Pr}(\mathcal{A}|S)$ and $\mathbf{Pr}(\mathcal{A}|S - R_i)$.

T	$Pr(X_t = T\|K)$								
	$K = S$			$K = S - R_1$			$K = S - R_2$		
	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c
B	1	1	0	0	0	0	0	0	0
T	0	0	1	0	0.5	0.5	1	0.5	1
F	0	0	0	1	0.5	0.5	0	0.5	0

Obviously, $|R| = \kappa(S)$ for all $R \in \mathcal{MC}_{\min}(S)$.

Definition 11. Let S be a block, then the minimal semantic information change due to restoring the consistency, denoted by $\mathcal{I}(S|dist)$, is defined as

$$\mathcal{I}(S|dist) = \min_{R \in \mathcal{MC}_{\min}(S)} dist\big(\mathbf{Pr}(\mathcal{A}|S), \mathbf{Pr}(\mathcal{A}|S - R)\big).$$

Moreover, we call $R^*(S|dist)$ a minimal correction subset with the minimum information change if

$$R^*(S|dist) \in \operatorname*{argmin}_{R \in \mathcal{MC}_{\min}(S)} dist\big(\mathbf{Pr}(\mathcal{A}|S), \mathbf{Pr}(\mathcal{A}|S - R)\big).$$

The following proposition shows that any two formulas in a minimal correction subset with the minimum information change have the same informational responsibility for the inconsistency of an inconsistent block. We call this the property of *Reciprocity* because the inconsistency counterfactually depends on each formula under a contingency that exactly consists of the others. This property may be considered as a counterpart of that of Fairness in the semantic case.

Proposition 8. *Let S be an inconsistent block and $R^*(S|dist)$ a minimal correction subset with the minimum information change with $|R^*(S|dist)| \geq 2$. Then it holds that*

$$dr_{\inf}(\alpha|S, dist) = dr_{\inf}(\beta|S, dist) = \frac{\mathcal{I}(S|dist)}{|R^*(S|dist)|}$$

for all $\alpha, \beta \in R^*(S|dist)$.

Proof. Let S be a block and $R^*(S|dist) \subseteq S$ be a minimal correction subset with the minimum information change. Consider any two formulas $\alpha, \beta \in R^*(S|dist)$, then $\Delta \cup \{\beta\}$ (resp. $\Delta \cup \{\alpha\}$) is the contingency making the inconsistency of S counterfactually depend on α (resp. β), where $\Delta = R^*(S|dist) - \{\alpha, \beta\}$. Then

$$dr_{\inf}(\alpha|S, dist) = dr_{\inf}(\beta|S, dist) = \frac{\mathcal{I}(S|dist)}{|R^*(S|dist)|}. \quad \square$$

Example 10. Consider a block $S = \{a \wedge c, \neg a, c \wedge b \wedge \neg b\}$. Then $\mathcal{MC}_{\min}(S) = \{R_1, R_2\}$, where $R_1 = \{a \wedge c, c \wedge b \wedge \neg b\}$ and $R_2 = \{\neg a, c \wedge b \wedge \neg b\}$. Moreover, the corresponding sequences of probability distributions over the truth value assignment of atoms are given in Table 5. Evidently,

$$dist^{+}_{d_T}(\mathbf{Pr}(\mathcal{A}|S), \mathbf{Pr}(\mathcal{A}|S - R_2)) = 2 < 2.5 = dist^{+}_{d_T}(\mathbf{Pr}(\mathcal{A}|S), \mathbf{Pr}(\mathcal{A}|S - R_1)).$$

So, R_2 is the minimal correction subset with minimum information change of S w.r.t. $dist^{+}_{d_T}$, and $\neg a$ and $c \wedge b \wedge \neg b$ have the same informational responsibility as well as causal responsibility for the inconsistency in S.

Each block of a knowledge base can be considered as a module of the knowledge base. The following proposition shows that the informational responsibility of a formula can be represented in a modular way.

Proposition 9. *Let K be an inconsistent knowledge base and $\{S_i\}_{i=1}^n$ be the set of all the blocks of K. Let $\alpha \in S_1$ with $dr_{\mathrm{cau}}(\alpha|S_1) > 0$, then*

(1) $dr_{\mathrm{cau}}(\alpha|K) = \dfrac{1}{\frac{1}{dr_{\mathrm{cau}}(\alpha|S_1)} + \sum_{i=2}^{n} \kappa(S_i)}$.

(2) $dr_{\inf}(\alpha|K, dist^{+}_{d}) = dr_{\mathrm{cau}}(\alpha|K)\left(dr_{\inf}(\alpha|S_1, dist^{+}_{d}) + \sum_{i=2}^{n} \mathcal{I}(S_i|dist^{+}_{d})\right)$.

(3) $dr_{\inf}(\alpha|K, dist^{m}_{d}) = dr_{\mathrm{cau}}(\alpha|K)\left(dr^2_{\inf}(\alpha|S_1, dist^{m}_{d}) + \sum_{i=2}^{n} \mathcal{I}^2(S_i|dist^{m}_{d})\right)^{\frac{1}{2}}$.

Proof. If $dr_{\mathrm{cau}}(\alpha|S_1) > 0$, then the inconsistency of S_1 counterfactually depends on α under a contingency Δ, moreover, $dr_{\mathrm{cau}}(\alpha|S_1) = \frac{1}{1+|\Delta|}$.

Let $R^*(S_i|dist)$ be a minimal correction subset with the minimum information change of S_i with regard to $dist$, then $|R^*(S_i|dist)| = \kappa(S_i)$. And the inconsistency of K counterfactually depends on α under the contingency

$$\Gamma = \Delta \cup R^*(S_2|dist) \cup \cdots \cup R^*(S_n|dist).$$

Therefore,

$$dr_{\mathrm{cau}}(\alpha|K) = \frac{1}{|\Delta \cup \{\alpha\}| + \sum_{i=2}^{n} |R^*(S_i|dist)|} = \frac{1}{\frac{1}{dr_{\mathrm{cau}}(\alpha|S_1)} + \sum_{i=2}^{n} \kappa(S_i)}.$$

Furthermore, allowing for the definition of informational responsibility, it holds that

$$dr_{\inf}(\alpha|K, dist) = dr_{\mathrm{cau}}(\alpha|K) \times \big(\min_{\Gamma \in C_{\min}(\alpha|K)} dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \Gamma \cup \{\alpha\})) \big).$$

Notice that for each block S_i, $Pr(X_t|S_i) = Pr(X_t|K)$ and $Pr(X_t|S_i - K') = Pr(X_t|K - K')$ for $K' \subseteq K$. In particular,

$$\begin{aligned} & dist_d^+(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \Gamma \cup \{\alpha\})) \\ = & \sum_{i=1}^{n} \sum_{t \in At(S_i)} d(Pr(X_t|S_i), Pr(X_t|S_i - \Gamma \cup \{\alpha\})). \end{aligned}$$

Then

$$\begin{aligned} & \min_{\Gamma \in C_{\min}(\alpha|K)} dist_d^+(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \Gamma \cup \{\alpha\})) \\ = & \sum_{i=1}^{n} \min_{\Gamma \in C_{\min}(\alpha|K)} \left(\sum_{t \in At(S_i)} d(Pr(X_t|S_i), Pr(X_t|S_i - \Gamma \cup \{\alpha\})) \right). \\ = & dr_{\inf}(\alpha|S_1, dist_d^+) + \sum_{i=2}^{n} \min_{\Gamma \in C_{\min}(\alpha|K)} \left(\sum_{t \in At(S_i)} d(Pr(X_t|S_i), Pr(X_t|S_i - \Gamma \cap S_i)) \right) \\ = & dr_{\inf}(\alpha|S_1, dist_d^+) + \sum_{i=2}^{n} \mathcal{I}(S_i|dist_d^+). \end{aligned}$$

Therefore,

$$dr_{\inf}(\alpha|K, dist_d^+) = dr_{\mathrm{cau}}(\alpha|K) \left(dr_{\inf}(\alpha|S_1, dist_d^+) + \sum_{i=2}^{n} \mathcal{I}(S_i|dist_d^+) \right).$$

Notice that for the distance $dist_d^m$,

$$\begin{aligned} & \big(dist_d^m(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \Gamma \cup \{\alpha\}))\big)^2 \\ = & \sum_{i=1}^{n} \sum_{t \in At(S_i)} d^2(Pr(X_t|S_i), Pr(X_t|S_i - \Gamma \cup \{\alpha\})). \end{aligned}$$

Then we can get the following results in a similar way.

$$\begin{aligned} & \min_{\Gamma \in C_{\min}(\alpha|K)} \big(dist_d^m(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \Gamma \cup \{\alpha\}))\big)^2 \\ = & dr_{\inf}^2(\alpha|S_1, dist_d^m) + \sum_{i=2}^{n} \mathcal{I}^2(S_i|dist_d^m) \end{aligned}$$

So,

$$dr_{\inf}(\alpha|K, dist_d^m) = dr_{\mathrm{cau}}(\alpha|K) \left(dr_{\inf}^2(\alpha|S_1, dist_d^m) + \sum_{i=2}^{n} \mathcal{I}^2(S_i|dist_d^m) \right)^{\frac{1}{2}}. \quad \square$$

6. Naive responsibility

As mentioned above, given an inconsistent knowledge base K and a formula $\alpha \in K$, both the causal and informational responsibilities of α for the inconsistency of K is built upon the counterfactual dependence of the inconsistency on the formula under some contingency. This tight association makes the two kinds of responsibilities interpretable from the perspective of causality.

Except this causal perspective for characterizing the role of each formula in the inconsistency, we may find some interesting observations not covered by the two responsibilities. At first, we have shown that if a formula needs to bear the informational responsibility for the inconsistency, then the formula must be involved in a minimal inconsistent subset. To bear this in mind, if we remove such a formula alone from a knowledge base, the set of minimal inconsistent subsets must be changed because minimal inconsistent subsets including the formula are broken by the removal. That is, it certainly brings changes in the syntactic aspect. However, it does not necessarily bring changes in the semantic information. To illustrate this, consider the following example.

Example 11. Consider the knowledge base $K_1 = \{a, \neg a, \neg a \vee \neg b, b, c\}$ again. Recall that $dr_{\text{inf}}(\neg a \vee \neg b | K_1, dist) > 0$, i.e., the formula $\neg a \vee \neg b$ bears responsibility for the inconsistency arising in K_1. To be more specific, the inconsistency of K_1 counterfactually depends on $\neg a \vee \neg b$ given the contingency of falsifying the formula $\neg a$. This implies that we cannot make the inconsistency disappear just by falsifying $\neg a \vee \neg b$ alone.

Even if we know this, we wonder if the inconsistent information changes when we remove the formula $\neg a \vee \neg b$ alone from K_1. Actually, it is easy to check that

$$\mathcal{M}_{\min}(K_1 - \{\neg a \vee \neg b\}) = \mathcal{M}_{\min}(K_1) = \{\omega_1\},$$

where $\omega_1(a) = \mathsf{B}$ and $\omega_1(b) = \omega_1(c) = \mathsf{T}$. This signifies that removing the formula alone does not bring any change in the semantic information.

On the other hand, the free formula of a knowledge base can not be identified as a cause of the inconsistency in that knowledge base, and then it bears neither casual responsibility nor informational responsibility for the inconsistency. However, not all the free formulas are independent of the inconsistent information in semantics. To illustrate this, consider the following example.

Example 12. Consider $K_9 = \{a \wedge \neg a \wedge \neg b, b\}$. Evidently, b is a free formula of K_9, and

$$dr_{\text{cau}}(b|K_9) = dr_{\text{inf}}(b|K_9, dist) = 0.$$

Notice that ω_1 is the unique minimal model of K_9, where $\omega_1(a) = \omega_1(b) = \mathsf{B}$. That is, both the two atoms a and b are assigned to the inconsistent truth value by ω_1.

By contrast, for the case where b is absent from K_9, the minimal model ω_2 of $K_9 - \{b\}$ assigns B and F to atoms a and b, respectively. This implies that the formula b is tightly related to the semantic information of K_9. However, neither the causal responsibility nor the informative responsibility catches such a relation between the formula b and the semantics of K_9.

To address these issues, we propose two auxiliary notions to characterize the role of each formula in the inconsistency.

Definition 12. Let K be an inconsistent knowledge base and $\alpha \in K$. Then the primal naive responsibility of a for the inconsistency of K, denoted $dr_{\text{nai}_0}(\alpha|K, dist)$, is defined as

$$dr_{\text{nai}_0}(\alpha|K, dist) = dist\big(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \{\alpha\})\big).$$

Essentially, the primal naive responsibility of a formula of a knowledge base for the inconsistency aims to capture the direct impact of the absence of the formula on the semantic information of the knowledge base. In particular, the following proposition shows that if the inconsistency counterfactually depends on a formula, then the primal naive responsibility and the informational responsibility are identical to each other.

Proposition 10. *Let K be an inconsistent knowledge base and $\alpha \in K$. If $dr_{\text{cau}}(\alpha|K) = 1$, then*

$$dr_{\text{nai}_0}(\alpha|K, dist) = dr_{\text{inf}}(\alpha|K, dist) > 0.$$

Proof. If $dr_{\text{cau}}(\alpha|K) = 1$, then the inconsistency of K counterfactually depends on α under the contingency $\Delta = \emptyset$. So, $0 < dr_{\text{inf}}(\alpha|K, dist) = \frac{dist\left(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \{\alpha\})\right)}{1+0} = dr_{\text{nai}_0}(\alpha|K, dist)$. □

In addition, for the tautology, its primal naive responsibility is identical to its zero informational responsibility.

Proposition 11. *Let K be an inconsistent knowledge base and $\alpha \in K$. If α is a tautology, then*

$$dr_{\text{nai}_0}(\alpha|K, dist) = dr_{\text{inf}}(\alpha|K, dist) = 0.$$

Proof. If α is a tautology, then $\mathcal{M}_{\min}(K - \{\alpha\}) = \mathcal{M}_{\min}(K)$. So,

$$\mathbf{Pr}(\mathcal{A}|K) = \mathbf{Pr}(\mathcal{A}|K - \{\alpha\}).$$

Therefore,

$$dr_{\mathrm{nai}_0}(\alpha|K, dist) = dist\big(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \{\alpha\})\big) = 0 = dr_{\mathrm{inf}}(\alpha|K, dist). \quad \square$$

Recall that both the causal and the informational responsibilities take into account the change of inconsistency by introducing the counterfactual dependence of the inconsistency on a formula. That is, any non-zero responsibility is associated with a process from inconsistency to consistency. Notice that what captured by the primal naive responsibility is the change of semantic information. It is not necessarily the change on inconsistency. To focus on the inconsistency change, we consider the following indicator function:

$$I(\Gamma|K) = \begin{cases} 1, & if\ \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) \neq \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - \Gamma), \\ 0, & otherwise \end{cases}.$$

The value of this function is 0 if and only if for each atom, the probability of assigning inconsistent truth value to it remains unchanged when we remove formulas of Γ from K. Then we adapt the primal naive responsibility as follows:

$$dr_{\mathrm{nai}}(\alpha|K, dist) = I(\{\alpha\}|K) dr_{\mathrm{nai}_0}(\alpha|K, dist).$$

We call the adapted version $dr_{\mathrm{nai}}(\alpha|K, dist)$ the naive responsibility of α. Evidently, the naive responsibility also satisfies the two properties introduced in Proposition 10 and 11, respectively.

Proposition 12. *Let K be an inconsistent knowledge base and $\alpha \in K$. If α is a safe formula, then*

$$dr_{\mathrm{nai}}(\alpha|K, dist) = dr_{\mathrm{inf}}(\alpha|K, dist) = 0.$$

Proof. If α is a safe formula, then $\{\alpha\}$ is a block of K. Hence for each atom $t \notin At(\{\alpha\})$,

$$Pr(X_t = \mathsf{B}|K) = Pr(X_t = \mathsf{B}|K - \{\alpha\}),$$

and for each atom $s \in At(\{\alpha\})$,

$$Pr(X_s = \mathsf{B}|K) = Pr(X_s = \mathsf{B}|K - \{\alpha\}) = 0.$$

So, $I(\{\alpha\}|K) = 0$. Therefore,

$$dr_{\mathrm{nai}}(\alpha|K, dist) = 0 \times dist\big(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - \{\alpha\})\big) = 0 = dr_{\mathrm{inf}}(\alpha|K, dist). \quad \square$$

Note that $dr_{\mathrm{inf}}(\alpha|K, dist) = 0$ does not necessarily imply that $dr_{\mathrm{nai}}(\alpha|K, dist) = 0$, and vice versa. To illustrate this, consider the following example.

Example 13. Consider $K_1 = \{a, \neg a, \neg a \vee \neg b, b, c\}$ and $K_9 = \{a \wedge \neg a \wedge \neg b, b\}$ again. Note that

$$dr_{\mathrm{inf}}(\neg a \vee \neg b|K_1, dist^+_{d_T}) = \frac{1}{2},$$

but

$$dr_{\mathrm{nai}}(\neg a \vee \neg b|K_1, dist^+_{d_T}) = 0.$$

By contrast,

$$dr_{\mathrm{inf}}(\neg b|K_9, dist^+_{d_T}) = 0,$$

but

$$dr_{\mathrm{nai}}(b|K_9, dist^+_{d_T}) = 1.$$

All these results are intuitive from the semantic perspective as we analyzed in the two examples above.

Roughly speaking, given a formula α, if $dr_{\mathrm{inf}}(\alpha|K, dist) = 0$ but $dr_{\mathrm{nai}}(\alpha|K, dist) > 0$, then it means that although there is no contingency under which the inconsistency of K counterfactually depends on α from a causal perspective, i.e., there is no need to falsify the formula in order to make the inconsistency disappear, but the formula is really involved in the inconsistency in the semantics, and falsifying it really makes some change in the inconsistency from the semantic perspective.

By contrast, given a formula α, if $dr_{\text{inf}}(\alpha|K, dist) > 0$ but $dr_{\text{nai}}(\alpha|K, dist) = 0$, it means that although falsifying α together with its contingency can make the inconsistency disappear, falsifying it alone cannot bring any change on the inconsistency from the semantic perspective.

In addition, for the case where $dr_{\text{inf}}(\alpha|K, dist) = dr_{\text{nai}}(\alpha|K, dist) = 0$, we are more interested in whether the formula α is independent of any change on the inconsistency in semantics made by falsifying some formulas. We use the following notion of quasi naive responsibility to describe such cases.

Definition 13. Let K be an inconsistent knowledge base and $\alpha \in K$. We call a subset $\Gamma \subseteq K$ a context of α changing the inconsistency, if Γ satisfies the following conditions.

(1) $\alpha \notin \Gamma$ and $I(\Gamma \cup \{\alpha\}|K) = 1$.
(2) For any $\gamma \in \Gamma \cup \{\alpha\}$, if $I(\Gamma \cup \{\alpha\} - \{\gamma\}|K) = 1$, then

$$\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (\Gamma \cup \{\alpha\})) \neq \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (\Gamma \cup \{\alpha\} - \{\gamma\})).$$

(3) If there exists $\Gamma' \subseteq K$ satisfying both *(1)* and *(2)*, then $|\Gamma| \leq |\Gamma'|$.

The first condition states that removing $\Gamma \cup \{\alpha\}$ can bring some change on the inconsistency. The second ensures that each formula in $\Gamma \cup \{\alpha\}$ is necessary to make the change. Informally speaking, the context is a smallest set of formulas that have to be removed from the knowledge base, together with the formula α, to bring some change in the inconsistency.

We use $Cont(\alpha|K)$ to denote the set of all contexts of α changing the inconsistency. Evidently, if $Cont(\alpha|K) = \{\emptyset\}$, then removing α alone can change the inconsistent semantic information. So, $dr_{\text{nai}}(\alpha|K, dist) > 0$. If $Cont(\alpha|K) = \emptyset$, then we cannot change the inconsistent semantic information by removing α under the aforementioned conditions. So, $dr_{\text{nai}}(\alpha|K, dist) = 0$ in this case.

Definition 14. Let K be an inconsistent knowledge base and $\alpha \in K$. Then the degree of quasi naive responsibility of α for the inconsistency in K, denoted $dr_{\text{qna}}(\alpha|K, dist)$, is defined as

$$dr_{\text{qna}}(\alpha|K, dist) = \begin{cases} \min\limits_{\Gamma \in Cont(\alpha|K)} \frac{dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (\Gamma \cup \{\alpha\})))}{|\Gamma|+1}, & if\ Cont(\alpha|K) \neq \emptyset, \\ 0, & otherwise \end{cases}.$$

The quasi naive responsibility of a formula aims to describe the contribution made by the formula to a local change of the inconsistency by a way of removing formulas as few as possible.

Now we use the following example to illustrate how the notion of quasi naive responsibility is different from both the naive responsibility and the informational responsibility.

Example 14. Consider $K_{10} = \{a \wedge \neg a \wedge \neg b \wedge \neg c \wedge \neg d, b \wedge c, c \wedge d, d \wedge b\}$. Then ω_1 is the unique minimal model of K_{10}, where $\omega_1(a) = \omega_1(b) = \omega_1(c) = \omega_1(d) = \mathsf{B}$. Evidently, for each free formula $\alpha \in \{b \wedge c, c \wedge d, d \wedge b\}$, it holds that

$$dr_{\text{inf}}(\alpha|K_{10}, dist) = dr_{\text{nai}}(\alpha|K_{10}, dist) = 0.$$

That is, removing each formula alone has no impact on the inconsistency from the semantic perspective.

Evidently, ω_2 is the unique minimal model of $K_{10} - \{b \wedge c, c \wedge d\}$, where $\omega_2(a) = \omega_2(b) = \omega_2(d) = \mathsf{B}, \omega_2(c) = \mathsf{F}$. That is, removing both $b \wedge c$ and $c \wedge b$ from K_{10} can change the inconsistency information about c. In addition, ω_3 is the unique minimal model of $K_{10} - \{b \wedge c, b \wedge d\}$, where $\omega_3(a) = \omega_3(c) = \omega_3(d) = \mathsf{B}, \omega_3(b) = \mathsf{F}$. That is, removing $b \wedge c$ and $b \wedge d$ from K_{10} can change the inconsistency information about b.

Hence

$$dr_{\text{qna}}(b \wedge c|K_{10}, dist^+_{d_T}) = \frac{1}{2}.$$

Similarly, we obtain that

$$dr_{\text{qna}}(c \wedge d|K_{10}, dist^+_{d_T}) = dr_{\text{qna}}(d \wedge b|K_{10}, dist^+_{d_T}) = \frac{1}{2}.$$

The following proposition gives some relations between the three kinds of responsibilities of a formula.

Proposition 13. *Let K be an inconsistent knowledge base and $\alpha \in K$.*

(1) If $dr_{\text{nai}}(\alpha|K, dist) > 0$, then $dr_{\text{qna}}(\alpha|K, dist) = dr_{\text{nai}}(\alpha|K, dist)$.
(2) If $dr_{\text{nai}}(\alpha|K, dist) = 0$ and $dr_{\text{inf}}(\alpha|K, dist) > 0$, then $dr_{\text{qna}}(\alpha|K, dist) > 0$.

Proof. Let us consider a formula $\alpha \in K$.

Table 6
The four responsibilities of each formula.

Formula	Responsibilities			
	$dr_{\text{cau}}(\alpha\|K)$	$dr_{\text{inf}}(\alpha\|K,dist)$	$dr_{\text{nai}}(\alpha\|K,dist)$	$dr_{\text{qna}}(\alpha\|K,dist)$
$\alpha \in \bigcup \mathcal{MI}(K)$	>0	>0	≥ 0	>0
$\alpha \in \mathcal{FF}(K)$	$=0$	$=0$	≥ 0	≥ 0
$\alpha \equiv \top$	$=0$	$=0$	$=0$	$=0$
$\alpha \in \mathcal{SF}(K)$	$=0$	$=0$	$=0$	$=0$

(1) If $dr_{\text{nai}}(\alpha|K,dist)>0$, then $Cont(\alpha|K)=\{\emptyset\}$. So,

$$dr_{\text{qna}}(\alpha|K,dist)=\frac{dist(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K-(\{\alpha\})))}{|\emptyset|+1}=dr_{\text{nai}}(\alpha|K,dist).$$

(2) If $dr_{\text{nai}}(\alpha|K,dist)=0$ and $dr_{\text{inf}}(\alpha|K,dist)>0$, then for each contingency Δ of α, it holds that $\Delta\neq\emptyset$. Then there exist a contingency Δ and a subset $\emptyset\subset\Gamma\subseteq\Delta$ such that Γ is a context of α changing the inconsistency. Hence $dr_{\text{qna}}(\alpha|K,dist)>0$. □

Proposition 14. *Let K be an inconsistent knowledge base and $\alpha\in K$. Then*

(1) If α is a tautology, then $dr_*(\alpha|K,dist)=0$ for each $*\in\{\text{cau},\text{inf},\text{nai},\text{qna}\}$.
(2) If α is a safe formula, then $dr_*(\alpha|K,dist)=0$ for each $*\in\{\text{cau},\text{inf},\text{nai},\text{qna}\}$.
(3) If $dr_{\text{cau}}(\alpha|K)=1$, then $dr_*(\alpha|K,dist)=dr_{\text{inf}}(\alpha|K,dist)$ for each $*\in\{\text{nai},\text{qna}\}$.

Proof. Let us consider a formula α of an inconsistent knowledge base K. Notice that results (1) and (2) can be easily derived in the same way used in proofs of Proposition 11 and 12, respectively. Here we focus on the result (3).

If $dr_{\text{cau}}(\alpha|K)=1$, then $\mathbf{Pr}(\mathcal{A}|K)\neq\mathbf{Pr}(\mathcal{A}|K-\{\alpha\})$, and $Pr(X_t=\mathsf{B}|K-\{\alpha\})=0$ for all $t\in At(K)$. Then $I(\{\alpha\}|K)=1$, and

$$\begin{aligned}&dr_{\text{qna}}(\alpha|K,dist)=dr_{\text{inf}}(\alpha|K,dist)\\&=\frac{dist(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K-\{\alpha\}))}{1+|\emptyset|}\\&=1\times dist(\mathbf{Pr}(\mathcal{A}|K),\mathbf{Pr}(\mathcal{A}|K-\{\alpha\}))\\&=dr_{\text{nai}}(\alpha|K,dist).\end{aligned}$$ □

Now we summarize these results about the four kinds of responsibilities in Table 6.

Proposition 15. *Let K be an inconsistent knowledge base and $\alpha\in K$. If S be a block of K s.t. $\alpha\in S$, then*

$$dr_*(\alpha|S,dist)=dr_*(\alpha|K,dist)$$

for each $*\in\{\text{nai},\text{qna}\}$.

Proof. Given an inconsistent knowledge base K and a formula $\alpha\in K$. Let S' be a block not including α, then it holds that

$$Pr(X_t|K-\{\alpha\})=Pr(X_t|S')$$

for each $t\in At(S')$.

This further implies that if there exists $\Gamma\in Cont(\alpha|K)$, then $\Gamma\cap S'=\emptyset$. Hence,

$$Pr(X_t|K-\Gamma)=Pr(X_t|S')$$

for each $t\in At(S')$.

Let S be the block of K s.t. $\alpha\in S$, then $\Gamma\subseteq S$, and

$$Pr(X_t|K-\Gamma)=Pr(X_t|S-\Gamma)$$

for each $t\in At(S)$. Therefore,

$$dr_*(\alpha|S,dist)=dr_*(\alpha|K,dist)$$

for each $*\in\{\text{nai},\text{qna}\}$. □

This proposition shows that both the naive and quasi naive responsibilities of a formula in a knowledge base describe the impact of the formula on partial inconsistency of that knowledge base in some sense.

The following example illustrates that the two local responsibilities have similar behaviors in the case where a formula is replaced by a new logically stronger one.

Example 15. Consider K_7 and K_8 again. Note that

$$dr_{\text{nai}}(a|K_7, dist^+_{d_T}) = dr_{\text{qna}}(a|K_7, dist^+_{d_T}) = \frac{3}{2},$$
$$dr_{\text{nai}}(\neg c|K_7, dist^+_{d_T}) = dr_{\text{qna}}(\neg c|K_7, dist^+_{d_T}) = 0.$$
$$dr_{\text{nai}}(a|K_8, dist^+_{d_T}) = dr_{\text{qna}}(a|K_8, dist^+_{d_T}) = 1,$$
$$dr_{\text{nai}}(\neg c|K_8, dist^+_{d_T}) = dr_{\text{qna}}(\neg c|K_8, dist^+_{d_T}) = 1.$$

So, for each $* \in \{\text{nai}, \text{qna}\}$,

$$dr_*(a|K_7, dist^+_{d_T}) > dr_*(a|K_8, dist^+_{d_T}), \text{ but } dr_*(\neg c|K_7, dist^+_{d_T}) < dr_*(\neg c|K_8, dist^+_{d_T}).$$

For the case where adding a new formula to a knowledge base, the following example illustrates that the two local responsibilities have similar behaviors.

Example 16. Consider K_3, K_4 and K_5 again. Note that for $i \in \{3, 4, 5\}$,

$$dr_{\text{cau}}(a \wedge b|K_i) = 1.$$

Then according to Proposition 14, for each $* \in \{\text{nai}, \text{qna}\}$,

$$dr_*(a \wedge b|K_i, dist^+_{d_T}) = dr_{\text{inf}}(a \wedge b|K_i, dist^+_{d_T}).$$

So,

$$dr_*(a \wedge b|K_3, dist^+_{d_T}) < dr_*(a \wedge b|K_5, dist^+_{d_T}),$$

but

$$dr_*(a \wedge b|K_3, dist^+_{d_T}) > dr_*(a \wedge b|K_4, dist^+_{d_T})$$

for $* \in \{\text{nai}, \text{qna}\}$.

7. Computational complexity

Now we turn to computational complexity issues on the four kinds of responsibilities. We assume that the reader is familiar with the polynomial time hierarchy ($\Delta_0^p = \Sigma_0^p = \Pi_0^p = \text{P}$; and for all $i \geq 0$, $\Delta_{i+1}^p = \text{P}^{\Sigma_i^p}, \Sigma_{i+1}^p = \text{NP}^{\Sigma_i^p}, \Pi_{i+1}^p = \text{co}\Sigma_{i+1}^p$) as well as the classes of P, NP, and coNP [38]. We also use FC (e.g., FNP) to denote the corresponding class of function problems associated with languages in the complexity class C (resp. NP) of decision problems [38]. In addition, D_1^p is the complexity class of decision problems such that $L = L_1 \times L_2$, where L_1 is in NP and L_2 is in coNP, and the well known problem of SAT-UNSAT is one of the canonical D_1^p-complete problems [38].

Let Σ be a finite alphabet and $R \subseteq \Sigma^* \times \Sigma^*$ a polynomially bounded binary relation, for every string x, a string $y \in \Sigma^*$ is called a solution for x if $(x, y) \in R$. We use $R(x)$ to denote the set of all solutions for x, i.e., $R(x) = \{y \in \Sigma^* | (x, y) \in R\}$. Then the enumeration problem **Enum_R** associated with R is defined as follows, which aims to output the set $R(x)$ of solutions for x [39].

Enum_R:

- Instance: $x \in \Sigma^*$.
- Output: $R(x) = \{y \in \Sigma^* | (x, y) \in R\}$.

Let C be a decision complexity class, then we say that Enum_R $\in$ Inc$C$ if there is a random access machine (RAM for short) M with oracle L in C and a polynomial p, such that for any instance x, M enumerates $R(x)$ with delay $p(|x| + i)$ between the i-th and $(i + 1)$-th solutions for every $0 \leq i \leq |R(x)|$. Moreover, the size of every oracle call made between outputting the i-th and $(i + 1)$-th solutions is bounded by $p(|x| + i)$ for every $0 \leq i \leq |R(x)|$ [39].

We use $\text{FP}^{\text{Inc}C}$ to denote the complexity class of function problems that can be solved in polynomial time by a Turing machine equipped with an Inc$C$ oracle, where the Inc$C$ oracle solves whatever instance of a problem in Inc$C$ class in unit time.

Recall that the informational responsibility is built upon minimal models as well as minimal correction subsets. Next we consider some problems about computing minimal correction subsets and minimal models. We start with the following problem of finding a special minimal correction subset of a knowledge base. Let $k > 0$ be a fixed natural number, then we define the problem of finding a k-sized minimal correction subset of a knowledge base that contains a given formula as follows:

Find_A_Super_MCS_With_k_Size:

- Instance: a knowledge base K, a formula $\alpha \in K$.
- Output: $R \in \mathcal{MC}(K)$ such that $\{\alpha\} \subseteq R$ and $|R| = k$ or declare that no such R exists.

Evidently, for the case where $k = 1$, given a knowledge base K, a formula $\alpha \in K$, check that $K \vdash \bot$ and $K - \{\alpha\} \nvdash \bot$, if so, return $R = \{\alpha\}$, otherwise return No such R exists (NSRE for short). This implies that Find_A_Super_MCS_With_1_Size$\in \mathrm{F}D_1^p$. Next we consider the case where $k > 1$.

Lemma 2. *For $k > 1$, Find_A_Super_MCS_With_k_Size is in* $\mathrm{F}\Sigma_2^p$.

Proof. Given a knowledge base K and a formula $\alpha \in K$, if $|K| < k$, return NSRE. Otherwise, check that $K \vdash \bot$, if so, consider the following algorithm, otherwise return NSRE.

- If $|K| = k$, check that $\{\beta\} \vdash \bot$ for all $\beta \in K$, if so, return $R = K$, otherwise, return NSRE.
- If $|K| - 1 \geq k$, then
  - non-deterministically guess a $(k-1)$-sized subset H of $K - \{\alpha\}$.
  - check that $H \cup \{\alpha\}$ is a minimal correction subset of K.
  - if so, return $R = H \cup \{\alpha\}$.

Note that checking whether a knowledge base is inconsistent is coNP-complete. Then for the case where $|K| = k$, we need to make $|K|$ queries to an NP oracle. For the case where $|K| - 1 \geq k$, for a guess H, to check whether $H \cup \{\alpha\}$ is a minimal correction subset of K, we need to check that $K - (H \cup \{\alpha\}) \nvdash \bot$, $K - H \vdash \bot$ and $K - ((H - \{\beta\}) \cup \{\alpha\}) \vdash \bot$ for each $\beta \in H$, that is, we need to make $(k+1)$ queries to an NP oracle. Then Find_A_Super_MCS_With_k_Size is in $\mathrm{F}\Sigma_2^p$. □

Lemma 3. *Deciding whether a model of a knowledge base is minimal is* coNP*-complete.*

Proof. Membership: given a knowledge base K and a model ω of K, its minimality can be checked as follows:

- non-deterministically guess a proper subset S of $\omega!$ and guess a 2-valued interpretation ω' over $At(K) - S$ to compose an interpretation ω'' s.t. $\omega''! = S$ and $\omega''(t) = \omega'(t)$ for all $t \in At(K) - S$;
- check that $\omega'' \in \mathcal{M}(K)$;
- if so, return false.

Note that checking whether ω'' is a model of K can be done in polynomial time, then checking the minimality of a model is in coNP.

Hardness: given a formula ϕ, ϕ is unsatisfiable if and only if ω is a minimal model of $\{\neg\phi\}$, where $\omega(t) = \mathsf{T}$ for all $t \in At(\phi)$. Suppose that ω is not a minimal model of $\{\neg\phi\}$, then $\omega(\neg\phi) = \mathsf{F}$, i.e., $\omega(\phi) = \mathsf{T}$. That is, ϕ is satisfiable, and a contradiction arises. □

Proposition 16. *Finding a minimal model of a knowledge base is in* $\mathrm{F}\Sigma_2^p$.

Proof. Now we consider the following nondeterministic algorithm for finding a minimal model:

- guess an interpretation ω;
- check that ω is a model of K;
- if so, check its minimality;
- if so, return ω.

Note that checking whether ω is a model can be done in polynomial time. By Lemma 3, checking the minimality of a model is coNP-complete, i.e., we need to make one query to a NP oracle. Then finding a minimal model of a knowledge base is in $\mathrm{F}\Sigma_2^p$. □

The last three kinds of responsibilities are built upon the probability distribution over the set of minimal models. Notice that given $\mathcal{M}_{\min}(K)$, $\mathbf{Pr}(\mathcal{A}|K)$ can be computed in polynomial time ($O(|At(K)|)$). Moreover, we focus on the distance function $dist(x, y)$ that can be computed in polynomial time given x and y.

Computing the set of minimal models of a knowledge base is an enumeration problem in essence. Another enumeration problem is computing the set of all the k-sized minimal inconsistent subsets that contain a formula α, which plays an important role in identifying the informational responsibility of α if its causal responsibility is exactly $\frac{1}{k}$. Now we define the two enumeration problems as follows:

(1) **Enum_MinM**:
   - Input: a knowledge base K.
   - Output: $\mathcal{M}_{\min}(K)$.

(2) **Enum_Super_MCS_with_k_Size**:
- Input: a knowledge base K and a formula $\alpha \in K$.
- Output: $\mathcal{MC}(K|\alpha, k) = \{R \in \mathcal{MC}(K) | \{\alpha\} \subseteq R, \text{ and } |R| = k\}$.

Here $k > 0$ is a fixed natural number. Notice that if $k = 1$, then it holds that either $\mathcal{MC}(K|\alpha, 1) = \{\{\alpha\}\}$ or $\mathcal{MC}(K|\alpha, 1) = \emptyset$. So, we focus on the case where $k > 1$.

Lemma 4. *Enum_MinM is in* $\mathrm{Inc}\Sigma_2^p$.

Proof. Given a knowledge base K, we may enumerate the minimal models of K as follows:

- we start by copying K to the oracle registers, and get a very first minimal model ω by using a Σ_2^p oracle, according to Proposition 16;
- next we extend K in the oracle registers to $K' = K \cup \{\alpha_\omega\}$, and get a minimal model of K' by using a Σ_2^p oracle, where α_ω is referred to as the formula $\bigvee_{t \in At(K)} (t \neq \omega(t))$. Here for an interpretation ω', $\omega'(\alpha_\omega) = \mathsf{T}$ if $\exists t \in At(K)$ s.t. $\omega'(t) \neq \omega(t)$, otherwise $\omega'(\alpha_\omega) = \mathsf{F}$. Then every minimal model of K' must be a minimal model of K, which is certainly different from ω.
- we repeat such extensions and compute the rest of minimal models of K.

Repeatedly extending K and then computing a minimal model by using Σ_2^p oracle imply that Enum_MinM $\in \mathrm{Inc}\Sigma_2^p$. $\square$

Lemma 5. *For $k > 1$, Enum_Super_MCS_with_k_Size is in* $\mathrm{Inc}\Sigma_2^p$.

Proof. Given a knowledge base K and a formula $\alpha \in K$, if $|K| < k$, then return $\mathcal{MC}(K|\alpha, k) = \emptyset$; if $|K| = k$, check that $\{\beta\} \vdash \bot$ for all $\beta \in K$, if so, return $\mathcal{MC}(K|\alpha, k) = \{K\}$, otherwise, return $\mathcal{MC}(K|\alpha, k) = \emptyset$. Now we consider the case where $k < |K|$.

At first, we may express K as a sequence $(\beta_1, \beta_2, \cdots, \beta_{|K|})$ of formulas under some given order. Then for each subset H of K, we may use a unique 0-1 vector $\vec{v}(H) = (v^{(1)}, \cdots, v^{(|K|)})$ to represent H such that for all $1 \leq i \leq |K|$, $v^{(i)} = 1$ if $\beta_i \in H$, otherwise $v^{(i)} = 0$. Then we adapt the algorithm mentioned in the proof for Lemma 2 to find a k-sized minimal correction subset satisfying a constraint described by a set $\{\vec{v}_l\}_{l=0}^p$ of such vectors:

- non-deterministically guess a $(k-1)$-sized subset H of $K - \{\alpha\}$.
- check that $H \cup \{\alpha\}$ is a minimal correction subset of K.
- if so, check that $\min\limits_{0 \leq l \leq p} \| \vec{v}_l - \vec{v}(H) \|_1 > 0$.
- if so, return $R = H \cup \{\alpha\}$.

Here $\| \vec{v} - \vec{v'} \|_1 = \sum\limits_{m=1}^{|K|} |v^{(m)} - v'^{(m)}|$. Evidently, $\| \vec{v} - \vec{v'} \|_1 = 0$ if and only if $\vec{v} = \vec{v'}$. Then the satisfaction of condition $\min\limits_{0 \leq l \leq p} \| \vec{v}_l - \vec{v}(H) \|_1 > 0$ ensures that H is different from formulas represented by $\{\vec{v}_l\}_{l=0}^p$. Notice that checking whether $\min\limits_{0 \leq l \leq p} \| \vec{v}_l - \vec{v}(H) \|_1 > 0$ is true can be done in polynomial time.

Now we enumerate the k-sized minimal correction subsets containing α as follows:

- we start by copying a pair $(K, \{\vec{v}_0 = \vec{0}\})$ to the oracle registers, and get a very first k-sized minimal correction subset R s.t. $\alpha \in R$ by using a Σ_2^p-oracle, according to Lemma 2 and the adapted algorithm.
- next we extend $(K, \{\vec{v}_0 = \vec{0}\})$ in the oracle registers to $(K, \{\vec{v}_0, \vec{v}_1 = \vec{v}(R)\})$, and get a k-sized minimal correction subset different from R by using a Σ_2^p-oracle.
- we repeat such extensions and compute the rest of solutions.

Repeatedly Extending the constraints and making calls to a Σ_2^p-oracle imply that Enum_Super_MCS_with_k_Size is in $\mathrm{Inc}\Sigma_2^p$. $\square$

Now we are ready to discuss the computational complexity about computing these responsibilities.

We have shown that the causal responsibility $dr_{\mathrm{cau}}(\alpha|K)$ of a formula α for the inconsistency is exactly the identical counterpart of $dr(\alpha, K)$ defined in syntactic case [15]. It has been shown that computing $dr(\alpha, K)$ is in $\mathrm{FP}^{\Sigma_2^p[\log n]}$ (Proposition 5.8 in [15]). Allowing for this, here we give the following identical counterpart of Proposition 5.8 in [15].

Proposition 17. *Computing $dr_{\mathrm{cau}}(\alpha|K)$ for a knowledge base K and a formula $\alpha \in K$ is in* $\mathrm{FP}^{\Sigma_2^p[\log n]}$.

Roughly speaking, the complexity is due to the fact that

$$dr_{\mathrm{cau}}(\alpha|K) \in \{0, \frac{1}{|K|}, \frac{1}{|K|-1}, \cdots, \frac{1}{2}, 1\},$$

and the result that finding a minimal correction subset with a given size is in $\mathrm{F}\Sigma_2^p$ [15].

Proposition 18. *Computing the naive responsibility $dr_{\mathrm{nai}}(\alpha|K, dist)$ for a knowledge base K and a formula $\alpha \in K$ w.r.t. $dist$ is in* $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$.

Proof. By Lemma 4, we know that computing the set of minimal models of a knowledge base is in $\mathrm{Inc}\Sigma_2^p$. Then we need to make 2 calls to a $\mathrm{Inc}\Sigma_2^p$-oracle to compute $\mathcal{M}_{\min}(K)$ and $\mathcal{M}_{\min}(K-\{\alpha\})$. Given $\mathcal{M}_{\min}(K)$ and $\mathcal{M}_{\min}(K-\{\alpha\})$, $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K-\{\alpha\})$ can be computed in polynomial time, respectively. Further, $dist\big(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K-\{\alpha\})\big)$ and $I(\{\alpha\}|K)$ can be computed in polynomial time based on $\mathbf{Pr}(\mathcal{A}|K)$ and $\mathbf{Pr}(\mathcal{A}|K-\{\alpha\})$. So, computing the naive responsibility is in $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$. □

Recall that the informational responsibility is tightly associated with the causal one. Next we consider how to compute the informational responsibility when the causal responsibility is given.

Proposition 19. *Given the causal responsibility $dr_{\mathrm{cau}}(\alpha|K)$, computing the informational responsibility $dr_{\mathrm{inf}}(\alpha|K, dist)$ is in* $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$.

Proof. Given the causal responsibility $dr_{\mathrm{cau}}(\alpha|K)$,

- if $dr_{\mathrm{cau}}(\alpha|K) = 0$, then $dr_{\mathrm{inf}}(\alpha|K, dist) = 0$;
- if $dr_{\mathrm{cau}}(\alpha|K) = 1$, then $dr_{\mathrm{inf}}(\alpha|K, dist) = dr_{\mathrm{nai}}(\alpha|K, dist)$;
- if $dr_{\mathrm{cau}}(\alpha|K) = \frac{1}{k} < 1$, then consider the following algorithm:
  (1) compute $\mathcal{MC}(K|\alpha, k) = \{R \in \mathcal{MC}(K) | \{\alpha\} \subseteq R, \text{ and } |R| = k\}$;
  (2) compute $\mathcal{M}_{\min}(K)$ and $\mathbf{Pr}(\mathcal{A}|K)$;
  (3) for each $\Gamma \in \mathcal{MC}(K|\alpha, k)$,
    (a) compute $\mathcal{M}_{\min}(K-\Gamma)$ and $\mathbf{Pr}(\mathcal{A}|K-\Gamma)$,
    (b) compute $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K-\Gamma))$;
  (4) $d_{\min} = \min\limits_{\Gamma \in \mathcal{MC}(K|\alpha,k)} dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K-\Gamma))$;
  (5) $dr_{\mathrm{inf}}(\alpha|K, dist) = \frac{1}{k} \times d_{\min}$.

By proof for Proposition 18, we need to make 2 calls to an $\mathrm{Inc}\Sigma_2^p$-oracle to compute $dr_{\mathrm{nai}}(\alpha|K, dist)$. By Lemma 5, computing $\mathcal{MC}(K|\alpha, k)$ is in $\mathrm{Inc}\Sigma_2^p$. Computing the set of minimal models for a knowledge base is in $\mathrm{Inc}\Sigma_2^p$ according to Lemma 4. Note that $|\mathcal{MC}(K|\alpha, k)| \le \binom{|K|-1}{k-1}$. Then at Step (2) and Step (3), we need to make at most $1 + \binom{|K|-1}{k-1}$ calls to an $\mathrm{Inc}\Sigma_2^p$-oracle. So, computing the informational responsibility is in $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$. □

Lastly, we consider the quasi naive responsibility. Note that the quasi naive responsibility focuses on exploring potential change on inconsistency that can be made by a formula under some condition. Next we give an algorithmic framework for computing the quasi naive responsibility. However, it is difficult to get appropriate results on its computational complexity, as explained later.

Given a knowledge base K, a formula $\alpha \in K$, and a distance function $dist$, we may compute the quasi naive responsibility $dr_{\mathrm{nai}}(\alpha|K, dist)$ as follows:

1. compute the naive responsibility $dr_{\mathrm{nai}}(\alpha|K, dist)$.
2. If $dr_{\mathrm{nai}}(\alpha|K, dist) > 0$, then $dr_{\mathrm{qna}}(\alpha|K, dist) = dr_{\mathrm{nai}}(\alpha|K, dist)$.
3. If $dr_{\mathrm{nai}}(\alpha|K, dist) = 0$, then let $k_{\mathrm{qna}} = 0$ and $d_{\mathrm{qna}} = 0$, and
   (a) perform a sequential search on $\{1, 2, \cdots, |K|-1\}$ to find the minimum size k_{qna} of context of α changing the inconsistency by the following nondeterministic algorithm:
      i. guess a k-sized subset $H \subseteq K - \{\alpha\}$;
      ii. check that $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) \ne \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (H \cup \{\alpha\}))$;
      iii. if so, check that $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - ((H - \{\gamma\}) \cup \{\alpha\})) \ne \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (H \cup \{\alpha\}))$ for all $\gamma \in H$;
      iv. if so, $k_{\mathrm{qna}} = k$;
      v. $d_{\mathrm{qna}} = dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (H \cup \{\alpha\})))$;
      vi. return k_{qna} and d_{qna}.
   (b) if $k_{\mathrm{qna}} = 0$, then $dr_{\mathrm{qna}}(\alpha|K, dist) = 0$;
   (c) else for each k_{qna}-sized subset $H \subseteq K - \{\alpha\}$,
      i. check that $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) \ne \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (H \cup \{\alpha\}))$;
      ii. if so, check that $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - ((H - \{\gamma\}) \cup \{\alpha\})) \ne \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K - (H \cup \{\alpha\}))$ for all $\gamma \in H$;
      iii. if so, compute $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (H \cup \{\alpha\})))$;
      iv. if $d_{qna} > dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (H \cup \{\alpha\})))$, then

$$d_{qna} = dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K - (H \cup \{\alpha\}))).$$

(d) $dr_{\mathrm{qna}}(\alpha|K, dist) = \frac{d_{\mathrm{qna}}}{k_{\mathrm{qna}}+1}$.

Step 3(a) aims to find the minimum number k_{qna} of formulas that have to be removed together with α to make change on inconsistency information. Step 3(c) aims to determine the minimal change on information made by removing α and other k_{qna} formulas once k_{qna} is determined.

By Proposition 18, computing the naive responsibility $dr_{\mathrm{nai}}(\alpha|K, dist)$ is in $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$, moreover, from the proof of Proposition 18, both computing $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K-\Gamma))$ and deciding whether $\mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K) \neq \mathbf{Pr}(\mathcal{A} \leftarrow \mathsf{B}|K-\Gamma)$ for any $\Gamma \subseteq K$ are also in $\mathrm{FP}^{\mathrm{Inc}\Sigma_2^p}$, because we need make 2 calls to an $\mathrm{Inc}\Sigma_2^p$-oracle to enumerate corresponding minimal models for each of the two problems. However, at Step (3), for each k-size guess, we need to make $(k+1)$ calls to an $\mathrm{Inc}\Sigma_2^p$-oracle in this non-deterministic algorithm once k is given. To the best of our knowledge, it is difficult to provide appropriate complexity results on this scenario based on the current complexity classes for enumeration defined in [39]. This is an open question.

Notice that the most used base-level inconsistency measure built upon the paraconsistent models is I_{LP_m} measure presented in [11], which uses the ratio of the minimal number of atoms assigned to the truth value B in Priest's minimal models to the number of all atoms as an inconsistency assessment. It has been shown that computing either the I_{LP_m} measure or its tailored version in databases [8] is $\mathrm{FP}^{\mathrm{NP}[\log n]}$-complete [40,8]. By contrast, our responsibilities are formula-level measures built upon the probability distribution on minimal models, and then we guess that computing them is harder than computing the base-level measure I_{LP_m}.

8. An application in requirements engineering

In this section we consider using a small but explanatory example in requirements engineering designed in [15] (i.e., Example 7.1. in [15]) to illustrate the application of semantics-based responsibility for the inconsistency. Briefly, the example describes a scenario for eliciting requirements for updating an existing software. Three stakeholders, the seller of the new system, the user of the existing system, and the domain expert in requirements engineering provide demands from their own respective perspectives, and then negotiate on resolving inconsistencies if their demands conflict [15]. Here we adopt all the stakeholders and their demands in Example 7.1 in [15], and then we focus on illustrating how the informational responsibility can be used to help developers and stakeholders make a decision on revising the requirements.

Example 17. Consider the following scenario for eliciting requirements for updating an existing software given in [15]:

- The seller of the new system provides two demands:
  (a1) The user interface of the system-to-be should be in the modern idiom (i.e., fashionable).
  (a2) The system-to-be should be open, that is, the system-to-be could be extended easily.
- The user of the existing system provides three demands:
  (b1) The system-to-be should be developed based on the techniques used in the existing system.
  (b2) The user interface of the system-to-be should maintain the style of the existing system.
  (b3) The system-to-be should be secure.
- The domain expert in requirements engineering provides two constraints about security:
  (c1) To guarantee the security of the system-to-be, openness (or ease of extension) should not be considered.
  (c2) To improve the security of the system-to-be, the newest development techniques should be adopted.

These requirements are formulated by the following knowledge base in [15]:

$$K_R = \left\{ \begin{array}{ll} \text{Fash(int\_f)}, \ \text{Open(sys)}, & \neg\text{New(sys)}, \ \neg\text{Fash(int\_f)}, \ \text{Secu(sys)}, \\ \text{Secu(sys)} \rightarrow \text{New(sys)}, & \text{Secu(sys)} \rightarrow \neg\text{Open(sys)} \end{array} \right\},$$

where the ground predicates are explained as follows:

- Fash(int_f): the interface is fashionable;
- Open(sys): the system is open;
- New(sys): the system will be developed based on the newest techniques;
- Secu(sys): the system is secure.

For simplicity of discussion, we consider the following abbreviation:

$$K_R = \{a1, a2, b1, b2, b3, c1, c2\}.$$

It has been stated that K_R is inconsistent, moreover, the causal responsibility of each requirement for the inconsistency has been given as follows [15]:

$$dr_{\mathrm{cau}}(a1|K_R) = dr_{\mathrm{cau}}(b2|K_R) = dr_{\mathrm{cau}}(b3|K_R) = \frac{1}{2},$$

$$dr_{\mathrm{cau}}(a2|K_R) = dr_{\mathrm{cau}}(b1|K_R) = dr_{\mathrm{cau}}(c1|K_R) = dr_{\mathrm{cau}}(c2|K_R) = \frac{1}{3}.$$

The Example 7.1 in [15] has illustrated how stakeholders and developers negotiate on resolving inconsistency based on the (causal) responsibility of each formula for the inconsistency. Here we focus on how to facilitate such a negotiation by considering the informational responsibility as well as the casual responsibility.

We adopt the distance function $dist^+_{d_T}$ to compute the informational responsibilities of these requirements:

$$dr_{\inf}(a1|K_R, dist^+_{d_T}) = dr_{\inf}(b2|K_R, dist^+_{d_T}) = dr_{\inf}(b3|K_R, dist^+_{d_T}) = \frac{3}{2},$$
$$dr_{\inf}(a2|K_R, dist^+_{d_T}) = dr_{\inf}(b1|K_R, dist^+_{d_T}) = 1,$$
$$dr_{\inf}(c1|K_R, dist^+_{d_T}) = dr_{\inf}(c2|K_R, dist^+_{d_T}) = \frac{5}{6}.$$

According to the interpretation of the causal and the informational responsibilities, given a requirement α with a positive causal responsibility, then the inconsistency of K_R can be resolved if we abandon a $\frac{1}{dr_{\text{cau}}(\alpha|K_R)}$-sized minimal correction subset that contains α, and the semantic distance between K_R and the corresponding revision is evaluated by $\frac{dr_{\inf}(\alpha|K_R, dist^+_{d_T})}{dr_{\text{cau}}(\alpha|K_R)}$. This is the starting point of applying the two kinds of responsibilities to inconsistency resolving.

Now we consider the following scenario of negotiation on abandoning some requirements to restore the consistency of the set of requirements above.

The requirements with the highest degree of causal responsibility are often the preferred candidates to be changed at the beginning of negotiation, because changing requirements as few as possible is an unspoken consensus among developers and stakeholders. Then either $\{a1, b3\}$ or $\{b2, b3\}$ is the goal at the beginning of negotiation.

Note that the informational responsibility of each of the three requirements is $\frac{3}{2}$, then both the distance between K_R and $K_R - \{a1, b3\}$ and the distance between K_R and $K_R - \{b2, b3\}$ in semantics are 3. Then it is advisable to have no preference between $\{a1, b3\}$ and $\{b2, b3\}$.

Suppose that the user of the existing system refuses to change requirement $b3$. This implies that neither $\{a1, b3\}$ nor $\{b2, b3\}$ is acceptable. Developers have to consider 3-sized goals, i.e., developers need to consider either $a1$ or $b2$ together with two other requirements with the causal responsibility $\frac{1}{3}$.

All the rest requirements have the causal responsibility $\frac{1}{3}$, but $c1$ and $c2$ have the smaller informational responsibility $\frac{5}{6}$. Notice that both the distance between K_R and $K_R - \{a1, c1, c2\}$ and the distance between K_R and $K_R - \{b2, c1, c2\}$ in semantics are $\frac{5}{2}$. So, both $\{a1, c1, c2\}$ and $\{b2, c1, c2\}$ are preferred goals of the next round of negotiation from the point of view of semantic information change. Then developers recommend either $\{a1, c1, c2\}$ or $\{b2, c1, c2\}$ to stakeholders.

Suppose that after the second round of negotiation, the user agrees to abandon the requirement $b2$, but the domain expert insists to remain both the constraints $c1$ and $c2$ unchanged. The seller refuses to change the requirement $a1$. Instead, this stakeholder agrees to abandon the requirement $a2$ if the user agrees to withdraw the requirement $b1$.

Then developers recommend $\{a2, b1, b2\}$ as the goal of the third round of negotiation. Suppose that the three stakeholders agree with each other in this goal. Then all the three requirements $a2, b1, b2$ are chosen as ones to be abandoned. The revised set of requirements is given by the following consistent knowledge base

$$K^1_R = \{a1, b3, c1, c2\}.$$

Notice that the distance between K_R and K^1_R in semantics is 3. This implies that semantic difference between K_R and K^1_R is not greater than that between K_R and the revision $K_R - \{b2, b3\}$ with the minimum number of changes.

9. Comparison and discussion

The four kinds of responsibilities of a formula in a knowledge base for the inconsistency aim to characterize different aspects of the role of that formula in the inconsistency arising in that knowledge base from the perspective of semantics. Given a formula, the goal of the causal responsibility of the formula for the inconsistency is to catch the causality between the inconsistency and the formula. It is more interested in under what contingency the inconsistency counterfactually depends on the formula in semantics. We have shown that the causal responsibility built upon the falsification of formulas in semantics is identical to the notion of responsibility [15], which is built upon minimal inconsistent subsets and minimal correction subsets from the syntactic perspective. Taking the counterfactual dependence into account makes the related informational responsibility as well as the causal responsibility interpretable from the viewpoint of the causality. This interpretability is important to put the characterization of roles of formulas into applications as well as to better understand the inconsistency.

As far as the semantic information change is concerned, the causal responsibility is very sketchy. It tells us nothing other than whether a knowledge base becomes consistent by falsifying some formulas. By contrast, the informational responsibility allows us to incorporate proper evaluations of semantic information change in the framework of the causal responsibility. For a given formula, its informational role is characterized by considering the semantic information change due to the falsification of related formulas to identify the causal relation between the inconsistency and the formula.

As mentioned above, the causal responsibility may be considered as a specific informational responsibility with the drastic distance function. On the other hand, the causal responsibility is identical to the notion of responsibility [15], which is built from the syntactic perspective. In this sense, we may consider that the informational responsibility is in accord with the notion of responsibility in syntactic case. Then the detailed comparison given in [15] between the Shapley inconsistency value [1,13,11] and the notion of responsibility are still useful to understand the difference between informational responsibility and Shapley inconsistency values. Besides the Shapley inconsistency value, in [15] we compared the notion of responsibility with other formula-level measures, including a series of MinInc inconsistency values built upon minimal inconsistent subsets [13], the DIM measures built upon the MUS-graph [17], and the measure $I_{\mathcal{P}_m}$ built upon minimal proof [41]. These comparisons are also meaningful for the informational responsibility since it is built upon the framework of causal responsibility.

Both the causal and the informational responsibilities can make a distinction between formulas involved in the minimal inconsistent subsets and free formulas. Although it is intuitive that a free formula has zero informational responsibility as well as causal responsibility when we bear in mind that falsifying the free formula in any case cannot make the inconsistency (inconsistent truth value) disappear in semantics, the role of the free formula in semantics is more delicate than that in the syntactic case, and so does some formula involved in minimal inconsistent subsets. In the syntactic case, removing a formula involved in minimal inconsistent subsets must bring changes on the inconsistency, because it can break all the minimal inconsistent subsets that include the formula. On the other hand, removing a free formula cannot break any minimal inconsistent subset, then it cannot bring any change on the inconsistency in this sense. By contrast, some free formula may bring information tightly related to the inconsistency in semantics, then it is not surprised that removing such a free formula can bring some changes on the inconsistency from the semantic perspective. On the contrary, removing some formula with positive informational responsibility alone from a knowledge base may keep the inconsistency unchanged from the semantic perspective. This can be owed to the same semantic information conveyed by some other formulas. The naive responsibility, together with the quasi naive responsibility, may provide a subtle characterization for the role of a formula in such cases.

Both the causal and the informational responsibilities of a formula stem from the counterfactual dependence of the inconsistency on the formula under some contingency, then the two kinds of responsibilities are appropriate for one-step scenarios where inconsistencies are resolved by removing a minimal correction subset in practical applications. This has been illustrated in Section 8. By contrast, the naive responsibility and the quasi-naive responsibility are more suitable for stepwise deletion resolution [42]. In detail, the naive responsibility can help us choose the formula that can bring maximum semantic information change at each step, whilst the quasi-naive responsibility may help us evaluate the possible maximum semantic information change brought by subsequent steps given the selected formula at each step.

Recall that characterizing inconsistency from the semantic perspective is another important way to measuring inconsistency. The set of B-atoms of minimal models is the most used tool for characterizing and measuring inconsistency from this perspective. For example, the I_{LP_m} measure [11] and its adaptations for databases [7,8], the uniform framework for semantic inconsistency measures [43], and the notion of Bi-free formula [21] are built upon the set of B-atoms in minimal models. By contrast, we use the sequence $\mathbf{Pr}(\mathcal{A}|K)$ of probability distributions over the truth values for all atoms instead of B-atoms to describe the inconsistency of a knowledge base K from the semantic perspective. That is, we take into account the uncertainty of inconsistent truth value assignment. Moreover, the distance function between two such sequences is important to identify the last three kinds of responsibilities of formulas for the inconsistency from the perspective of semantic information change. The choice of distance function often depends on practical domains. Besides the two distance functions $dist_d^+$ and $dist_d^m$, we may define and use other suitable functions in identifying these responsibilities. For example, we may define $dist_d^{\max}$ as follows:

$$dist_d^{\max}(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K')) = \max_{t\in\mathcal{A}} d\left(Pr(X_t|K), Pr(X_t|K')\right).$$

This distance function catches the maximum difference between two distributions over the truth values of the same atom.

The tacit culpability measure presented in [16] considers the semantic relations between formulas involved in the inconsistency. For example, they argue that given a knowledge base $K_{11} = \{a \vee b, \neg a \vee b, \neg b, b\}$, b plays an explicit role in causing the inconsistency by participating in $\{b, \neg b\}$, moreover, b plays an implicit role in causing the inconsistency by the way of $\{a \vee b, \neg a \vee b, \neg b\}$, because b is entailed by $\{a \wedge b, \neg a \vee b\}$ [16]. However, such semantic relations have been also taken into account in the responsibilities. To illustrate this, consider a formula with positive informational (or causal) responsibility, if its naive responsibility is 0, then it means that the formula also plays an implicit role in the inconsistency, that is, there are some other formulas also conveying the information that conveyed by the formula. Notice that the causal and the informational responsibilities stem from the counterfactual dependence of the whole inconsistency on the formula under some contingency instead of accumulating the contributions to the parts of inconsistency. This is the main difference between the tacit culpability measure and the responsibilities.

Recall that the informational responsibility incorporates semantic information change in the framework of the causal responsibility. Notice that we have shown that the causal responsibility defined from the semantic perspective by falsification is identical to the responsibility presented in [15], which is defined in terms of minimal inconsistent subsets and minimal correction subsets from syntactic perspective. This implies that the informational responsibility may be considered a two-fold formula-level measure, i.e., it describes the role of a formula in semantic information explicitly, and captures the role of a formula in syntactic perspective implicitly. Then it is not surprising that the informational responsibility satisfies some properties for characterizing measures built upon minimal inconsistent subsets. However, the (causal) responsibility is identified according to counterfactual dependence of the inconsistency on a formula. This is the main difference between the causal responsibility and formula-level and base-level measures built upon minimal inconsistent subsets presented in [13,11]. On the other hand, decomposition of minimal inconsistent subsets has been taken into account in measuring inconsistency by minimal inconsistent subsets [13,11,36,17]. Recently, it has been generalized to the notion of modularity of inconsistency which is appropriate for both syntactic and semantic cases [37]. Then we consider the modularity instead of decomposition of minimal inconsistent subsets in characterizing the informational responsibility.

The falsification of formulas plays a central role in identifying the causal relation between the inconsistency and a given formula, which provides a foundation for both the informational responsibility and the causal responsibility. However, we have to confront constraints on resolving the inconsistency in many practical applications. In such cases, some falsifications of formulas may be prohibited, and then we need to take into account constraints in identifying the role of a formula for the inconsistency. Notice that in syntactic context, some special kinds of constraints have been taken into account in identifying the responsibility (i.e., the identical counterpart of the causal responsibility) of a formula by incorporating these constraints in the graphical representation of minimal inconsistent subsets [23]. However, as illustrated in [44], given a subset of a knowledge base, there is no explicit relation between minimal models of the knowledge base and that of the subset. This implies that the graphical representation of inconsistency used in [23] is not necessarily appropriate for the semantic case. How to extend such an incorporation in semantic context to identify the informational responsibility under constraints is an interesting topic for future research.

We use a distance function $dist(\mathbf{Pr}(\mathcal{A}|K), \mathbf{Pr}(\mathcal{A}|K'))$ to capture how the two knowledge bases K and K' are different from each other in semantics. However, such a probability-based distance function may be further adapted and applied to identify the closest revision to the original belief base in belief revision. In addition, if we further consider the conditional probability of $Pr(X_t|X_{t'}, K)$, it may lead us to a broader vision on the relations between assignments of truth values to atoms.

10. Conclusion

The problem of characterizing the role of a formula in the inconsistency in a knowledge base from a perspective of paraconsistent semantics has not yet been paid much attention.

In this paper, we have proposed an interpretable multi-aspect characterization of the role of a formula in the inconsistency in a knowledge base by taking into account paraconsistent semantics and causality.

At first, the causal responsibility of a formula for the inconsistency aims to describe the role of the formula in making the inconsistency arise by falsifying the formula, possibly together with some others (called a contingency), to make the inconsistency disappear. Taking the counterfactual dependence of the inconsistency on a formula under some contingency into account makes the causal responsibility interpretable from the perspective of causality.

By incorporating the semantic information change in the framework of causal responsibility, the informational responsibility of a formula catches the contribution made by the formula to the inconsistent information by using counterfactual dependence of inconsistency on the formula under some contingency. In particular, the informational responsibility can be reduced to the causal responsibility if we use a drastic distance function to evaluate the semantic information change. Notice that the causal responsibility is identical to the notion of responsibility [15] constructed in the syntactic case, then we may consider that the informational responsibility is in accord with the syntactic notion of responsibility in some sense.

Both the causal and informational responsibilities of a formula focus on describing the role of the formula in the global inconsistency rather than local inconsistency. By contrast, as two auxiliary notions, both the naive and the quasi naive responsibilities of a formula for the inconsistency are proposed to describe the role of the formula in making some local change on the inconsistency. They, together with the first two responsibilities provide a rich and multi-aspect characterization of the role of formulas in the inconsistency.

CRediT authorship contribution statement

Kedian Mu: Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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