Artificial Intelligence 348 (2025) 104406

Contents lists available at ScienceDirect

Artificial Intelligence

journal homepage: www.elsevier.com/locate/artint

# Towards optimal subsidy bounds for envy-freeable allocations [☆,☆☆]

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ARTICLE INFO

*Keywords:*
Mechanism design with money
Envy-freeness
Algorithmic game theory

ABSTRACT

We study the fair division of indivisible items with subsidies among $n$ agents, where the absolute marginal valuation of each item is at most one. Under monotone nondecreasing valuations (where each item is a good), Brustle et al. [9] demonstrated that a maximum subsidy of $2(n-1)$ and a total subsidy of $2(n-1)^2$ are sufficient to guarantee the existence of an envy-freeable allocation. In this paper, we improve upon these bounds, even in a wider model. Namely, we show that, given an EF1 allocation, we can compute in polynomial time an envy-free allocation with a subsidy of at most $n-1$ per agent and a total subsidy of at most $n(n-1)/2$. Moreover, when the valuations are monotone nondecreasing, we provide a polynomial-time algorithm that computes an envy-free allocation with a subsidy of at most $n-1.5$ per agent and a total subsidy of at most $(n^2-n-1)/2$.

## 1. Introduction

We consider the problem of *fairly* dividing items among agents. The notion of fairness that has been extensively studied in the literature is *envy-freeness* [14]. It requires that no agent wants to swap her bundle with another agent's. When the items to be allocated are divisible, the classical result ensures the existence of an envy-free allocation [28]. In contrast, when the items are indivisible, envy-freeness is not a reasonable goal. For instance, consider $n$ agents with $n \geq 2$ and a single item valued at 1 by each agent. Allocating the only item to an agent results in envy from the other agents, as they get nothing. Thus, envy-free allocations may not exist when the items are indivisible.

One way to circumvent this issue is to relax the fairness requirement. For example, *envy-freeness up to one item* (EF1) requires that when agent $i$ envies agent $j$, the envy can be eliminated by either (i) removing one item from agent $j$'s bundle, or (ii) removing one item from agent $i$'s bundle. It is known that an EF1 allocation is guaranteed to exist if each item is either a good (whose

☆ A preliminary version of this paper [17] appeared in "Proceedings of the 38th Annual AAAI Conference on Artificial Intelligence (AAAI-24)".

☆☆ This work was partially supported by the joint project of Kyoto University and Toyota Motor Corporation, titled "Advanced Mathematical Science for Mobility Society", JST ERATO Grant Number JPMJER2301, JST PRESTO Grant Number JPMJPR2122, and JSPS KAKENHI Grant Numbers JP17K12646, JP19K22841, JP20H00609, JP20H05967, JP20K19739, JP20H00609, JP21H04979, JP21K17708, and JP21H03397.

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https://doi.org/10.1016/j.artint.2025.104406
Received 6 May 2024; Received in revised form 2 April 2025; Accepted 18 August 2025

Available online 28 August 2025

marginal valuation is non-negative) or a chore (whose marginal valuation is non-positive) for any agent, i.e., doubly monotone valuations [19,8]. Moreover, for general valuations, the existence of an EF1 allocation is assured when there are only 2 agents [11].

Another way to circumvent this issue is monetary compensation (subsidy). Since money is divisible, it can be a powerful tool to achieve envy-freeness. However, since the subsidy payments must be provided by an external agent (e.g., a government or a funding agency), it is desirable that the total subsidy amount is bounded. Thus, in this paper, we study the fair division of indivisible items with limited subsidies.

Most of the existing works on the fair division of indivisible items with limited subsidies focus on some special cases. For example, Maskin [21] and Klijn [18] consider the case that the number of agents and the number of items are equal and each agent can be allocated at most one item. Halpern and Shah [16] consider an extended model where the number of agents and number of items may differ, and each agent can be allocated more than one item, assuming the valuation of each agent is additive. Goko et al. [15] consider the case that the valuation of each agent is matroidal (which is not necessarily additive). Barman et al. [5] examine a broader class of valuations in which the marginal valuation of each item is dichotomous. As far as the authors are aware, the most general model considered so far is monotone nondecreasing valuations [9], where the marginal contribution of each item is non-negative.

In this paper, we study the fair division of indivisible items with limited subsidies when the valuations are not restricted to be monotone nondecreasing. We assume that the valuations are normalized so that the absolute marginal value of each item is at most 1 (i.e., between $-1$ and 1).

For monotone nondecreasing valuations, which are special cases of our model, Brustle et al. [9] show that envy-free allocation always exists with a subsidy of amount at most $2(n-1)$ per agent, and the total amount is $2(n-1)^2$. However, the only known lower bound on the total subsidy is $n-1$ (which can be obtained using the case with $n$ agents and one item described at the beginning of this section). A central open question asks whether total subsidies can be bounded as $O(n^{2-\epsilon})$ for some $\epsilon > 0$ under monotone nondecreasing valuations [20, Open Question 9].

*Our results*

In this paper, we present improved upper bounds for the subsidies necessary to achieve envy-freeness. We demonstrate that, given an EF1 allocation, an envy-free allocation with a subsidy can be constructed in polynomial time where each agent receives a subsidy of at most $n-1$ and $n(n-1)/2$ in total (Theorem 1). To prove these improved bounds, we reveal that the structure of the minimum subsidy vectors satisfies: (i) the minimum subsidy vector remains unchanged irrespective of the maximum weight matching (Lemma 2), and (ii) how the subsidies alter when the weights are changed (Lemma 4). When valuation functions are doubly monotone or there are only two agents, such envy-free allocations with limited subsidies can be computed in polynomial time. Notably, this improves upon the necessary subsidy amount for the existing case of monotone nondecreasing valuations, as monotone nondecreasing valuations are also doubly monotone. Furthermore, when $n = 2$, our obtained bounds are best possible since a subsidy of 1 is indispensable. We also show that from an EF$k$ allocation (i.e., pairwise envy can be eliminated by removing at most $k$ items), we can construct an envy-free allocation with a subsidy of at most $k \cdot (n-1)$ per agent and a total subsidy of $k \cdot n(n-1)/2$.

It is worth mentioning that our upper bounds of $n-1$ per agent and $n(n-1)/2$ in total cannot be improved when considering an arbitrary EF1 allocation (Example 1). We overcome this impossibility by slightly modifying the bundles. To be exact, for three or more agents with monotone nondecreasing valuations, we improve the bounds further to $n-1.5$ per agent and $(n^2-n-1)/2$ in total (Theorem 2).

Finally, for the general valuation case, we show that we can eliminate envies by at most $2\lceil m/n \rceil$ units of subsidy per agent, where $m$ is the number of items (Theorem 3). This bound is effective when the number of items is relatively small compared to the number of agents.

*Related work*

The concept of compensating an indivisible resource allocation with money has been prevalent in classical economics literature [1, 21,18,22,25–27]. Much of the classical work has focused on the unit-demand case in which each agent is allocated at most one good. Examples include the famous rent-division problem of assigning rooms to several housemates and dividing the rent among them [24]. It is known that, for a sufficient amount of subsidies, an envy-free allocation exists [21] and can be computed in polynomial time [2,18].

Most classical literature, however, has not considered a situation in which the number of items to be allocated exceeds the number of agents, in contrast to the rich body of recent literature on the multi-demand fair division problem. Halpern and Shah [16] recently extended the model to the multi-demand setting wherein multiple items can be allocated to one agent. Despite the existence of numerous related papers, Halpern and Shah [16] is the first work to study the asymptotic bounds on the amount of subsidy required to achieve envy-freeness. They showed that an allocation is envy-freeable with a subsidy if and only if the agents cannot increase the utilitarian social welfare by permuting bundles. This characterization implies that an allocation that can be made envy-free with a subsidy needs to satisfy some efficiency condition; hence, an approximately fair allocation, such as an EF1 allocation [10], may not be an envy-freeable allocation. It was conjectured in Halpern and Shah [16] that, for additive valuations in which the value of each item is at most 1, giving at most 1 to each agent is sufficient to eliminate envies. Brustle et al. [9] affirmatively settled this conjecture by developing an algorithm that iteratively computes a maximum-matching. Barman et al. [5] examined a model with dichotomous marginals and obtained the same bounds.

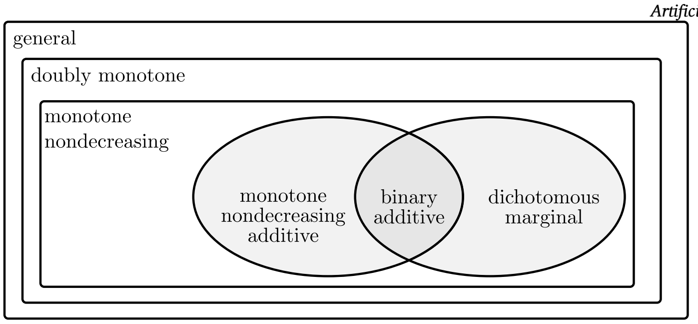


**Fig. 1.** Relationships of classes of valuation functions.

Babaioff et al. [4] and Benabbou et al. [7] studied the fair allocation of indivisible items with matroidal valuations. The prioritized egalitarian mechanism proposed by Babaioff et al. [4] returns an allocation that maximizes the Nash welfare and achieves envy-freeness up to any good and utilitarian optimality. Benabbou et al. [7] focused more on the balance between efficiency and fairness. They demonstrated that when agents have matroidal valuations, leximin allocations are equivalent to MNW allocations. For matroidal valuations, Goko et al. [15] developed *subsidized egalitarian* mechanism that is strategy-proof, utilitarian optimal, polynomial-time implementable, and envy-free with a subsidy amounting to at most 1 per agent and $n-1$ in total. They also provide a strategy-proof, utilitarian optimal, and envy-free mechanism with subsidies for more general valuations. Recently, Wu et al. [29] examined the upper bound on the total subsidy required to ensure *proportionality*.

Caragiannis and Ioannidis [12] studied the computational complexity of approximating the minimum amount of subsidies required to achieve envy-freeness. Aziz [3] considered another fairness requirement, the so-called *equitability*, in conjunction with envy-freeness and characterized an allocation that can be made both equitable and envy-free with a subsidy. Narayan et al. [23] studied a related but different setting with transfer payments; they analyzed the impact of introducing some amount of transfers on the Nash welfare and utilitarian welfare while achieving envy-freeness.

## 2. Preliminaries

We model fair division with a subsidy as follows. For each natural number $k \in \mathbb{N}$, we denote $[k] = \{1, \dots, k\}$. Let $[n]$ be the set of $n$ agents and let $M = \{e_1, \dots, e_m\}$ be the set of $m$ indivisible items. Each agent $i$ has a *valuation function*, denoted as $v_i : 2^M \to \mathbb{R}$, where $\mathbb{R}$ represents the set of real numbers. We assume that the functions $v_i$'s are given as value oracles. In addition, we assume that the valuation of the empty set is 0 (i.e., $v_i(\emptyset) = 0$ for any $i \in [n]$) and the maximum marginal contribution of each item is at most one (i.e., $|v_i(X \cup \{e\}) - v_i(X)| \le 1$ for any $i \in [n]$, $e \in M$, and $X \subseteq M \setminus \{e\}$).

We define an item $e \in M$ as a *good* for agent $i \in [n]$ if $v_i(X \cup \{e\}) \ge v_i(X)$ for every $X \subseteq M \setminus \{e\}$. Additionally, we define an item $e \in M$ as a *chore* for agent $i \in [n]$ if $v_i(X \cup \{e\}) \le v_i(X)$ for every $X \subseteq M \setminus \{e\}$, with at least one of these inequalities being strict. An instance is said to be *monotone nondecreasing* if every $e \in M$ is a good for any agent $i \in [n]$. Furthermore, an instance is said to be *doubly monotone* if every item $e \in M$ is either a good or a chore for any agent $i \in [n]$. An instance is *additive* if $v_i(X) = \sum_{e \in X} v_i(\{e\})$ for all $i \in [n]$ and $X \subseteq M$ and *dichotomous marginal* if $v_i(X \cup \{e\}) - v_i(X) \in \{0, 1\}$ for all $i \in [n]$, $X \subseteq M$, and $e \in M \setminus X$. Fig. 1 depicts the relationship among classes of valuation functions.

An *allocation* is an ordered partition $\boldsymbol{A} = (A_1, \dots, A_n)$ of $M$, i.e., $\bigcup_{i \in [n]} A_i = M$ and $A_i \cap A_j = \emptyset$ for any distinct $i, j \in [n]$. In allocation $\boldsymbol{A}$, each agent $i$ receives the items of bundle $A_i$. A *subsidy* is a non-negative real vector $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+$, where $p_i$ is the amount of money given to agent $i \in [n]$. In an allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$, the utility of each agent $i$ is $v_i(A_i) + p_i$. An allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ is *envy-free* if $v_i(A_i) + p_i \ge v_i(A_j) + p_j$ for any pair of agents $i, j \in [n]$. Our goal is to find an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that the total subsidy $\sum_{i \in [n]} p_i$ (or maximum subsidy $\max_{i \in [n]} p_i$) is minimized.

An allocation $\boldsymbol{A}$ is called *envy-free up to one item (EF1)* if, for all $i, j \in [n]$, it holds that $v_i(A_i \setminus X) \ge v_i(A_j \setminus X)$ for some $X \subseteq A_i \cup A_j$ with $|X| \le 1$. Similarly, an allocation $\boldsymbol{A}$ is called *envy-free up to $k$ items (EF$k$)* if, for all $i, j \in [n]$, it holds that $v_i(A_i \setminus X) \ge v_i(A_j \setminus X)$ for some $X \subseteq A_i \cup A_j$ with $|X| \le k$. It is known that an EF1 allocation always exists and can be found in polynomial time if the valuations are doubly monotone [19,8] or $n = 2$ [11].

### 2.1. Envy-free structure of subsidies

An allocation $\boldsymbol{A}$ is called *envy-freeable* if there exists a subsidy vector $\boldsymbol{p}$ such that $(\boldsymbol{A}, \boldsymbol{p})$ is envy-free. We call such a subsidy vector *envy-eliminating*. In this subsection, we describe the structure of envy-eliminating subsidies.

We fix an allocation $\boldsymbol{A} = (A_1, \dots, A_n)$. Let $w \in \mathbb{R}^{[n] \times [n]}$ be a weight matrix such that $w_{i,j} = v_i(A_j)$ for each $i, j \in [n]$. For a permutation $\sigma$ of $[n]$, let $\boldsymbol{A}^\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)})$ and let $P^\sigma$ be the set of envy-eliminating subsidy vectors:

$$P^\sigma = \left\{ \boldsymbol{p} \in \mathbb{R}^n_+ \;\middle|\; w_{i,\sigma(i)} + p_{\sigma(i)} \ge w_{i,j} + p_j \ (\forall i, j \in [n]) \right\}.$$

We call a permutation $\sigma$ *maximum weight permutation* for $w$ if it maximizes $\sum_{i=1}^n w_{i,\sigma(i)}$ among permutations. Halpern and Shah [16] proved that an allocation $\boldsymbol{A}$ is envy-freeable if and only if $\sum_{i=1}^n w_{i,i} \ge \sum_{i=1}^n w_{i,\sigma(i)}$ for any permutation $\sigma$ of $[n]$. From this result, the polyhedron $P^\sigma$ is non-empty if $\sigma$ is a maximum weight permutation. The polyhedron $P^\sigma$ contains a unique minimal element (i.e., a vector $\boldsymbol{p}$ such that $\boldsymbol{p} \le \boldsymbol{p}'$ for any $\boldsymbol{p}' \in P^\sigma$) because it is lower bounded by non-negativity and $\boldsymbol{p}, \boldsymbol{p}' \in P^\sigma$ implies that $(\min\{p_i, p'_i\})_{i\in[n]}$ is also in $P^\sigma$. The unique minimal vector is called the *minimum subsidy vector* for $w$ with respect to $\sigma$. Note that, for $\boldsymbol{p} \in P^\sigma$, the subsidy $p_i$ is associated with the bundle $A_i$, not the agent $i$.

For the pair $(w, \sigma)$ of a weight $w$ and a permutation $\sigma$ of $[n]$, we define the *envy graph* $G^{w,\sigma} = (V, E; \gamma)$ as a weighted directed complete graph in which each agent is a vertex (i.e., $V = [n]$), and each edge $(i, j) \in E$ $(= \{(i', j') \mid i', j' \in [n], i' \neq j'\})$ has a weight $\gamma_{i,j} = w_{i,\sigma(j)} - w_{i,\sigma(i)}$. Note that $\gamma_{i,j}$ represents the envy from $i$ towards $j$ in allocation $\boldsymbol{A}^\sigma$. The minimum subsidy vector can be characterized by using this envy graph.

**Lemma 1** *(Halpern and Shah [16, Theorem 2][1]). For any maximum weight permutation $\sigma$, the minimum subsidy $p_{\sigma(i)}$ is the maximum length of any path in $G^{w,\sigma}$ starting at $i$.*

It should be noted that the envy graph $G^{w,\sigma}$ does not contain any positive-weight directed cycle if $\sigma$ is a maximum weight permutation. Hence, we only need to consider simple paths. Although there may exist several maximum weight permutations, the following lemma states that the corresponding minimum subsidy vectors are identical.

**Lemma 2.** *Let $\sigma$ and $\sigma'$ be maximum weight permutations for $w$. Also, let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be the minimum subsidy vectors for $w$ with respect to $\sigma$ and $\sigma'$, respectively. Then, $\boldsymbol{p} = \boldsymbol{p}'$.*

**Proof.** It is sufficient to prove that $\boldsymbol{p} \in P^{\sigma'}$ and $\boldsymbol{p}' \in P^\sigma$. In addition, by symmetry, it is sufficient to show only the former.

Define a vector $\boldsymbol{q} \in \mathbb{R}^n$ as $q_i = w_{i,\sigma(i)} + p_{\sigma(i)}$ for each $i \in [n]$ and a weight $w' \in \mathbb{R}^{[n]\times[n]}$ as $w'_{i,j} = w_{i,j} + p_j - q_i$ for each $i, j \in [n]$. By definition of $\boldsymbol{p}$, we have $w'_{i,j} = (w_{i,j} + p_j) - (w_{i,\sigma(i)} + p_{\sigma(i)}) \le 0$ $(\forall i, j \in [n])$ and $w'_{i,\sigma(i)} = 0$ $(\forall i \in [n])$.

For any permutation $\pi$ of $[n]$, the difference between the total weights of $w$ and $w'$ is

$$\sum_{i\in[n]} w_{i,\pi(i)} - \sum_{i\in[n]} w'_{i,\pi(i)} = \sum_{i\in[n]} p_i - \sum_{i\in[n]} q_i,$$

which is a constant independent of $\pi$. Thus, $\sigma$ and $\sigma'$ are maximum weight permutations for $w'$, and hence the total weight of $\sigma'$ for $w'$ is $\sum_{i\in[n]} w'_{i,\sigma'(i)} = \sum_{i\in[n]} w'_{i,\sigma(i)} = 0$. As $w'$ is nonpositive, it holds that $w'_{i,\sigma'(i)} = 0$ for every $i \in [n]$. Thus, for any $i, j \in [n]$, we have

$$w_{i,\sigma'(i)} + p_{\sigma'(i)} = w'_{i,\sigma'(i)} + q_i = q_i \ge w'_{i,j} + q_i = w_{i,j} + p_j,$$

where the inequality holds by $w'_{i,j} \le 0$. Hence, $\boldsymbol{p}$ must be in $P^{\sigma'}$. $\square$

From this lemma, the minimum subsidy vector is determined for a weight matrix $w$ without specifying a maximum weight permutation. In the following, we simply call it the *minimum subsidy vector* for $w$.

It should be noted that the subsidy vector $\boldsymbol{p}$ and the vector $\boldsymbol{q} \in \mathbb{R}^n$ defined in the proof of Lemma 2 can be viewed as dual variables of an assignment problem. Here, the primal of the assignment problem is

$$\begin{array}{lll} \max & \sum_{i=1}^n \sum_{j=1}^n w_{i,j} x_{i,j} & \\ \text{s.t.} & \sum_{j=1}^n x_{i,j} = 1 & (\forall i \in [n]), \\ & \sum_{i=1}^n x_{i,j} \ge 1 & (\forall j \in [n]), \\ & x_{i,j} \ge 0 & (\forall i, j \in [n]), \end{array}$$

and its dual is

$$\begin{array}{lll} \min & \sum_{i=1}^n q_i - \sum_{j=1}^n p_j & \\ \text{s.t.} & w_{i,j} + p_j - q_i \le 0 & (\forall i, j \in [n]), \\ & p_j \ge 0 & (\forall j \in [n]). \end{array}$$

By the complementary slackness, we have $x_{i,j}(q_i - w_{i,j} - p_j) = 0$ $(\forall i, j \in [n])$ if $\boldsymbol{x}$ and $(\boldsymbol{p}, \boldsymbol{q})$ are optimal solutions.

The maximum weight permutation can be computed in polynomial time by a maximum-weight bipartite perfect matching algorithm. The minimum subsidy vector for $w$ can be computed in polynomial time by a shortest path algorithm (e.g., the Floyd–Warshall algorithm).

## 3. Bounding subsidy based on EF1 allocations

In this section, we prove the following key lemma.

[1] Note that the original work of Halpern and Shah [16] focused on the goods only instances. However, their proof can be extended to non-monotone settings because it relies only on the relative differences in agents' utilities, not on their absolute values.

**Lemma 3.** *Let $w \in \mathbb{R}^{[n]\times[n]}$ be the weight matrix for an allocation. We denote the sequence of numbers in descending order of $(\max_{j\in[n]}(w_{i,j} - w_{i,i}))_{i\in[n]}$ as $(\beta_1, \beta_2, \dots, \beta_n)$, i.e., $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. Let $\boldsymbol{p}^*$ be the minimum subsidy vector for $w$. Then, the $r$th largest value among $p_1^*, p_2^*, \dots, p_n^*$ is at most $\sum_{\ell=1}^{n-r} \beta_\ell$ for $r = 1, 2, \dots, n$.*

This result requires careful analysis because the identity permutation id might not be a maximum weight permutation for the allocation in Lemma 3. Consequently, this lemma cannot be directly derived from Lemma 1 because $G^{w,\mathrm{id}}$ can contain a positive directed cycle that results in an infinitely long path. This lemma becomes particularly valuable since EF1 allocations cannot always be realized through maximum weight permutations. We recall from Section 2.1 that minimum subsidies are defined independently of any specific maximum weight permutation.

For an EF1 allocation $\boldsymbol{A} = (A_1, \dots, A_n)$, if we set the weight matrix as $w = (v_i(A_j))_{i,j\in[n]}$, then the numbers $\beta_i$ ($i \in [n]$) in Lemma 3 are nonnegative and at most 1 because $v_i(A_i) \geq v_i(A_j) - 1$ for all $i, j \in [n]$. Thus, the $r$th largest value in the minimum subsidy vector is at most $n - r$. Here, note that $\boldsymbol{A}$ itself may not be envy-freeable.

**Theorem 1.** *Suppose that $|v_i(X_i \cup \{e\}) - v_i(X)| \leq 1$ for any $i \in [n]$, $e \in M$, and $X \subseteq M \setminus \{e\}$. Given any complete EF1 allocation $\boldsymbol{A}$, we can compute a complete envy-free allocation with a subsidy $(\boldsymbol{A}^\sigma, \boldsymbol{p})$ such that $\max_{i=1}^n p_i \leq n - 1$ and $\sum_{i=1}^n p_i \leq \sum_{\ell=1}^n (n - \ell) = n(n-1)/2$ in polynomial time.*

Recall that an EF1 allocation $\boldsymbol{A}$ can be found in polynomial time if the valuations are doubly monotone [8] or $n = 2$ [11]. Additionally, a maximum weight permutation $\sigma$ and the minimum subsidy vector can be computed in polynomial time. Thus, we can obtain the following corollaries.

**Corollary 1.** *If the valuations are doubly monotone, we can compute an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i=1}^n p_i \leq n - 1$ and $\sum_{i=1}^n p_i \leq \sum_{\ell=1}^n (n - \ell) = n(n-1)/2$ in polynomial time.*

**Corollary 2.** *If $n = 2$, we can compute an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\min\{p_1, p_2\} = 0$ and $\max\{p_1, p_2\} \leq 1$ in polynomial time.*

Lemma 3 also implies that, even when the valuations are general and we only have an EF$k$ allocation, we can still derive an envy-free allocation with a subsidy of at most $k \cdot (n - 1)$ per agent and $k \cdot n(n-1)/2$ in total. [6] recently established the existence of EF3 allocations for *non-negative* valuations (i.e., $v_i(M) \geq 0$ for all $i \in [n]$). This yields the following corollary.

**Corollary 3.** *If the valuations are non-negative, there exists an envy-free allocation with subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i=1}^n p_i \leq 3(n-1)$ and $\sum_{i=1}^n p_i \leq \sum_{\ell=1}^n (n - \ell) = \frac{3}{2} \cdot n(n-1)$.*

We first observe that the bound of Lemma 3 is tight even when the valuations are monotone nondecreasing and additive.

**Example 1.** Consider an instance with $M = \{e_{i,j} \mid i \in [n],\ j \in [n+1]\}$. The valuation of agent $i \in [n]$ for an item $e_{i',j}$ ($i' \in [n]$, $j \in [n+1]$) is

$$v_i(\{e_{i',j}\}) = \begin{cases} 1 & \text{if } i' = i - 1, \\ n/(n+1) & \text{if } i' = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_i = \{e_{i,1}, \dots, e_{i,n+1}\}$ for each $i \in [n]$. Then, $\boldsymbol{A} = (A_1, \dots, A_n)$ is an envy-freeable EF1 allocation.[2] Indeed, the weight matrix $w = (v_i(A_j))_{i,j\in[n]}$ is

$$\begin{array}{cc} & \begin{array}{cccccccc} A_1 & A_2 & \cdots & A_{i-1} & A_i & \cdots\cdots & A_{n-1} & A_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n-1 \\ n \end{array} & \begin{pmatrix} n & & & & & & & \\ n+1 & n & & & & & & \\ & n+1 & \ddots & & & & & \\ & & & n+1 & n & & & \\ & & & & \ddots & \ddots & & \\ & & & & & n+1 & n & \\ & & & & & & n+1 & n \end{pmatrix} \end{array}.$$

[2] This allocation also satisfies a stronger version of EF1 defined in Conitzer et al. [13]: for each $j \in [n]$ with $A_j \neq \emptyset$, there exists a good $g_j \in A_j$ such that for all $i \in [n]$, we have $v_i(A_i) \geq v_i(A_j - g_j)$. Thus, Theorem 1 is tight even for this stronger version of EF1.

It is not difficult to see that the identity permutation id is a maximum weight one, and a path that visits $n, n-1, \dots, 2, 1$ in this order is the longest one in $G^{w,\mathrm{id}}$. Thus, the minimum subsidy vector for $\boldsymbol{A}$ is $\boldsymbol{p} = (0, 1, \dots, n-1)$. Hence, $\max_{i\in[n]} p_i = n-1$ and $\sum_{i\in[n]} p_i = n(n-1)/2$ hold.

In what follows, we prove Lemma 3. Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an allocation and let $\boldsymbol{p}^*$ be the minimum subsidy vector for the weight matrix $w = (v_i(A_j))_{i,j\in[n]}$. Let $\sigma^*$ be a maximum weight permutation for $w$. Note that the envy graph $G^{w,\sigma^*}$ may have an edge with a large weight. Thus, the proof is not straightforward in such a case.

To prove the lemma, we establish the following framework. For each $i \in [n]$, we denote $q_i = \max_{j\in[n]}(v_i(A_j) + p_j^*)$, i.e., the maximum valuation for agent $i$ over bundles including subsidies, and $r_i = -(v_i(A_i) + p_i^* - q_i)$, i.e., the maximum envy for agent $i$. Also, let us define

$$\hat{w}_{i,j} = \begin{cases} v_i(A_i) + r_i & \text{if } i = j, \\ v_i(A_j) & \text{if } i \neq j \end{cases} \tag{1}$$

for each $i, j \in [n]$. Note that $r_i \geq 0$ and $\hat{w}_{i,j} \geq w_{i,j}$ for any $i, j \in [n]$. We verify that with this weight modification, the identity permutation becomes a maximum-weight permutation. Let $\hat{\boldsymbol{p}}$ denote the minimum subsidy vector for $\hat{w}$. We demonstrate the equality $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$. These results enable a direct application of Lemma 3 to the original allocation, yielding the subsidy bounds.

**Lemma 4.** *The identity permutation is a maximum weight permutation for $\hat{w}$. Moreover, the minimum subsidy vectors $\boldsymbol{p}^*$ and $\hat{\boldsymbol{p}}$ are the same.*

**Proof.** Since $(\boldsymbol{A}^{\sigma^*}, \boldsymbol{p}^*)$ is envy-free by Lemma 2, we have $q_i = \max_{j\in[n]}(v_i(A_j) + p_j^*) = v_i(A_{\sigma^*(i)}) + p^*_{\sigma^*(i)}$. Thus,

$$w_{i,j} + p_j^* - q_i = v_i(A_j) + p_j^* - q_i \leq 0 \tag{2}$$

for all $i, j \in [n]$, and

$$w_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = v_i(A_{\sigma^*(i)}) + p^*_{\sigma^*(i)} - q_i = 0 \tag{3}$$

for all $i \in [n]$. By the definition of $\hat{w}$, we have

$$\hat{w}_{i,j} + p_j^* - q_i = w_{i,j} + p_j^* - q_i \leq 0 \tag{4}$$

for all $i, j \in [n]$ with $i \neq j$ and

$$\hat{w}_{i,i} + p_i^* - q_i = w_{i,i} + p_i^* - q_i + r_i = 0, \tag{5}$$

for all $i \in [n]$, where the inequality in (4) holds by (2). In addition, for any $i \in [n]$ with $i \neq \sigma^*(i)$, we have

$$\hat{w}_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = w_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = 0 \tag{6}$$

by (3). Thus, for each $i \in [n]$, we obtain

$$\hat{w}_{i,\sigma^*(i)} = q_i - p^*_{\sigma^*(i)} = w_{i,\sigma^*(i)} \tag{7}$$

since the first equality holds by (5) and (6) and the second equality holds by (3).

By (4) and (5), the total weight $\sum_{i=1}^n \hat{w}_{i,\sigma(i)}$ is at most $\sum_{i\in[n]} q_i - \sum_{j\in[n]} p_j^*$ for any permutation $\sigma$. Thus, $\sigma^*$ and the identical permutation id are maximum weight permutations for $\hat{w}$ since the total weight of $\sigma^*$ and id for $\hat{w}$ are $\sum_{i\in[n]} q_i - \sum_{j\in[n]} p_i^*$ by (5) and (6). Moreover, $\boldsymbol{p}^*$ is an envy-eliminating subsidy for $\hat{w}$ (with respect to $\sigma^*$) since $\hat{w}_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} = q_i \geq \hat{w}_{i,j} + p_j^*$ for any $i, j \in [n]$ by (4), (5), and (6).

As $\boldsymbol{p}^*$ is an envy-eliminating subsidy for $\hat{w}$, we have $\hat{\boldsymbol{p}} \leq \boldsymbol{p}^*$. To prove that $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$, what is left is to show that $\hat{\boldsymbol{p}}$ is an envy-eliminating subsidy vector for $w$.

Define $\hat{q}_i = \max_{j\in[n]}(\hat{w}_{i,j} + \hat{p}_j)$ for each $i \in [n]$. Since $\hat{\boldsymbol{p}}$ is the minimum subsidy vector for $\hat{w}$ with respect to $\sigma^*$ by Lemma 2, we have

$$\hat{w}_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} - \hat{q}_i = 0 \quad (\forall i \in [n]). \tag{8}$$

Thus, for each $i \in [n]$, we have

$$w_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} = \hat{w}_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} = \hat{q}_i = \max_{j\in[n]}(\hat{w}_{i,j} + \hat{p}_j) \geq \max_{j\in[n]}(w_{i,j} + \hat{p}_j),$$

where the first equality holds by (7), the second equality holds by (8), and the last inequality holds by $\hat{w}_{i,j} \geq w_{i,j}$ for any $i, j \in [n]$. Therefore, $\hat{\boldsymbol{p}}$ is an envy-eliminating subsidy vector for $w$, which completes the proof. □

By definition, the weight of each edge in $G^{\hat{w},\mathrm{id}}$ is at most that of the corresponding edge in $G^{w,\mathrm{id}}$ because

$$\hat{w}_{i,j} - \hat{w}_{i,i} = w_{i,j} - (w_{i,i} + r_i) \leq w_{i,j} - w_{i,i}$$

for any $i, j \in [n]$ with $i \neq j$. By combining Lemma 4 with Lemma 1, we prove Lemma 3.

**Proof of Lemma 3.** Recall that $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$ and id is a maximum weight permutation by Lemma 4. For each $i \in [n]$, let $P_i \subseteq E$ be a longest path in $G^{\hat{w},\mathrm{id}}$ starting from $i$. By Lemma 1, $\hat{p}_i$ ($= p_i^*$) is the length of $P_i$ in $G^{\hat{w},\mathrm{id}}$. Note that $P_i$ is a simple path. As the weight of each edge in $G^{\hat{w},\mathrm{id}}$ is at most that of the corresponding edge in $G^{w,\mathrm{id}}$, we have

$$p_i^* = \sum_{(s,t)\in P_i} (\hat{w}_{s,t} - \hat{w}_{s,s}) \le \sum_{(s,t)\in P_i} (w_{s,t} - w_{s,s}) \le \sum_{\ell=1}^{|P_i|} \beta_\ell \tag{9}$$

for each $i \in [n]$.

What is left is to provide upper bounds on the numbers of edges in the longest paths $P_1, \dots, P_n$. Let $S = \bigcup_{i\in[n]} P_i$. Without loss of generality, we may assume that, if two paths $P_i$ and $P_j$ share a common vertex, all of the edges that follow the vertex in these two paths are identical. Then, $S$ is a directed forest and $|S| \le n-1$. Since $S$ is acyclic, we can relabel the vertices in the order of a topological sort of $S$. Then, the number of edges in $P_i$ is at most $n-i$ for $i = 1, 2, \dots, n$, because the vertices in $P_i$ form an increasing subsequence of $i, i+1, \dots, n$.

Therefore, for $r = 1, 2, \dots, n$, the $r$th largest value in $p^*$ is at most $\sum_{\ell=1}^{|P_r|} \beta_\ell \le \sum_{\ell=1}^{n-r} \beta_\ell$ by (9). $\square$

## 4. Improved bounds for monotone nondecreasing valuations

In this section, we provide an improved upper bound of subsidy when the valuations are monotone nondecreasing. As observed in Example 1, a maximum subsidy of $n-1$ is required to guarantee envy-freeness for an EF1 allocation. We demonstrate that the upper bound can be improved by slightly modifying a given EF1 allocation. Here, Lemma 3 plays a central role again. To reduce large subsidies, we attempt to shorten long paths in the envy graph by moving an item. Formally, we present the following theorem.

**Theorem 2.** *Suppose that $n \ge 3$ and $0 \le v_i(X_i \cup \{e\}) - v_i(X) \le 1$ for any $i \in [n]$, $e \in M$, and $X \subseteq M \setminus \{e\}$. Then, there exists a complete envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i\in[n]} p_i \le n - 1.5$ and $\sum_{i\in[n]} p_i \le (n^2 - n - 1)/2$. Moreover, such an envy-free allocation with a subsidy can be computed in polynomial time.*

Note that if $n = 2$, Theorem 1 implies that there exists an envy-free allocation with a subsidy where only one agent receives a subsidy of at most 1. This bound cannot be improved even when there is one item with a value of 1 for each agent. In what follows, we assume that $n \ge 3$.

We describe that we can obtain in polynomial time an EF1 allocation $\boldsymbol{A}$ satisfying $\sum_{i\in[n]} v_i(A_i) \ge \sum_{i\in[n]} v_i(A_{\sigma(i)})$ for any permutation $\sigma$ such that $\boldsymbol{A}^\sigma$ is EF1. We first compute an EF1 allocation $\boldsymbol{X}$ in polynomial time using the envy-cycles algorithm [19]. Next, we permute $\boldsymbol{X}$ as follows. Construct a bipartite graph $([n], [n]; E)$ where an edge $(i, j) \in E$ exists if and only if the EF1 criterion still holds for agent $i$ when we swap bundles of agents $i$ and $j$, i.e., $v_i(X_j) \ge \min_{Y\subseteq X_k : |Y|\le 1} v_i(X_k \setminus Y)$ for all $k \in [n]$. We assign the weight of an edge $(i, j) \in E$ as $v_i(X_j)$. Then we find the maximum weight perfect matching on the bipartite graph. By permuting bundles according to the matching, we can obtain the desired allocation $\boldsymbol{A}$.

Define $w = (v_i(A_j))_{i,j\in[n]}$. Let $\boldsymbol{p}^*$ be the minimum subsidy vector for $w$. In addition, let $q_i = \max_j (v_i(A_j) + p_j^*)$ and $r_i = -(v_i(A_i) + p_i^* - q_i)$ for each $i \in [n]$. Let $\hat{w}$ be the weights defined as (1).

A main task in the proof of Theorem 2 is to show that there exists an allocation with a subsidy $(\boldsymbol{A}'', \boldsymbol{p}'')$ such that $\max_{i\in[n]} p_i'' \le n - 1.5$ by modifying $\boldsymbol{A}$ and $\boldsymbol{p}^*$. We assume that $\max_{i\in[n]} p_i^* > n - 1.5$ since otherwise (i.e., $\max_{i\in[n]} p_i^* \le n - 1.5$) we have $\sum_{i\in[n]} p_i^* \le \sum_{k=1}^{n-1} \min\{k, n-1.5\} = (n^2-n-1)/2$ by Lemma 3. Here, $\beta_i \le 1$ ($\forall i \in [n]$) in the lemma since $\boldsymbol{A}$ is EF1. By Lemma 4, $\boldsymbol{p}^*$ is the minimum subsidy vector for $\hat{w}$, and id is a maximum weight permutation for $\hat{w}$. Since $r_i \ge 0$ ($\forall i \in [n]$), the weight of edge $(i, j)$ in $G^{\hat{w},\mathrm{id}}$ is

$$\hat{w}_{i,j} - \hat{w}_{i,i} \le v_i(A_j) - v_i(A_i) \le 1. \tag{10}$$

By Lemma 1, the length of a longest path in $G^{\hat{w},\mathrm{id}}$ is $\max_{i\in[n]} p_i^*$ ($> n - 1.5$). Since the weight of each edge is at most 1 by (10), the longest path must contain all the vertices and $n-1$ positive weight edges. Without loss of generality, we may assume that $(n, n-1, \dots, 1)$ is the longest path (see Fig. 2).

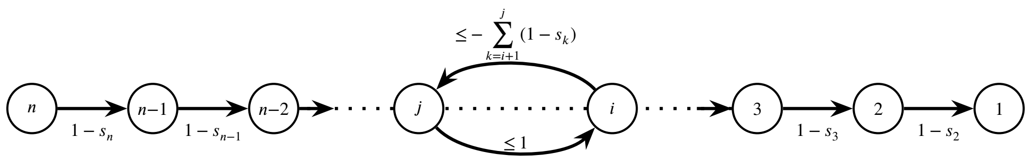


**Fig. 2.** The envy graph $G^{\hat{w},\mathrm{id}}$.

Let $s_i = 1 + r_i + v_i(A_i) - v_i(A_{i-1})$ for $i = 2, 3, \dots, n$. Then, for each $i \in \{2, 3, \dots, n\}$, $1 - s_i$ is the positive weight of $(i, i-1)$, and

$$0 \le r_i \le s_i \le 1, \tag{11}$$

since $1 - s_i = v_i(A_{i-1}) - (v_i(A_i) + r_i) = (v_i(A_{i-1}) - v_i(A_i)) - r_i \leq 1 - r_i$. For each $i \in [n]$, the path $(i, i-1, \dots, 1)$ must be a longest path starting at $i$ in $G^{\hat{w},\mathrm{id}}$. This is because, if there was a longer path starting at $i$, we could replace the subpath of the longest path $(n, n-1, \dots, 1)$ starting from $i$ with this longer path, thereby creating a longer path, contradicting the assumption that $(n, n-1, \dots, 1)$ is a longest path. Thus, for each $i \in [n]$, it holds that

$$p_i^* = \sum_{j=2}^{i}(\hat{w}_{j,j-1} - \hat{w}_{j,j}) = \sum_{j=2}^{i}(v_j(A_{j-1}) - v_j(A_j) - r_j) = \sum_{j=2}^{i}(1 - s_j).$$

In addition, since $(n, n-1, \dots, 1)$ is a longest path, $\max_{i\in[n]} p_i^*$ is achieved by $i = n$. Then since $\sum_{i=2}^{n}(1 - s_i) = \max_{i\in[n]} p_i^* > n - 1.5$, we have

$$0 \leq \sum_{i=2}^{n} r_i \leq \sum_{i=2}^{n} s_i < 0.5. \tag{12}$$

Next, we observe that the weight of each edge, from a vertex with a lower index to a higher index, is small. This observation will be used to evaluate modified allocations.

**Lemma 5.** *For $i, j \in [n]$ with $i < j$, it holds that*

$$\hat{w}_{i,j} - \hat{w}_{i,i} = v_i(A_j) - v_i(A_i) - r_i \leq -\sum_{k=i+1}^{j}(1 - s_k).$$

**Proof.** As the identity permutation is a maximum weight permutation for $\hat{w}$ by Lemma 4, envy graph $G^{\hat{w},\mathrm{id}}$ contains no positive-weight directed cycle. Hence, for $i, j \in [n]$ with $i < j$, we have

$$(\hat{w}_{i,j} - \hat{w}_{i,i}) + \sum_{k=i+1}^{j}(\hat{w}_{k,k-1} - \hat{w}_{k,k}) \leq 0.$$

Since $\hat{w}_{k,k-1} - \hat{w}_{k,k} = 1 - s_k$, we obtain

$$v_i(A_j) - v_i(A_i) - r_i = \hat{w}_{i,j} - \hat{w}_{i,i} \leq -\sum_{k=i+1}^{j}(\hat{w}_{k,k-1} - \hat{w}_{k,k}) = -\sum_{k=i+1}^{j}(1 - s_k). \qquad \square$$

From this lemma and (12), we have

$$v_i(A_j) - v_i(A_i) \leq r_i - \sum_{k=i+1}^{j}(1 - s_k) \leq r_i - (1 - s_{i+1}) \leq -1 + s_i + s_{i+1} < -0.5$$

for any $i, j \in [n]$ with $2 \leq i < j$. However, due to the lack of an upper bound for $r_1$, the evaluation $v_1(A_j) - v_1(A_1) - r_1 \leq -\sum_{k=2}^{j}(1 - s_k)$ is not useful. Therefore, we provide alternative inequalities that do not use $r_1$.

**Lemma 6.** *For every $j \in \{2, 3, \dots, n\}$, we have*

$$v_1(A_j) \leq \max\left\{ v_1(A_1) - (1 - s_2),\ \min_{e\in A_1} v_1(A_1 \setminus \{e\}) \right\}.$$

**Proof.** Let $j^* \in \arg\max_{j\in\{2,3,\dots,n\}} v_1(A_j)$ and

$$\boldsymbol{A}^{(j^*)} = (A_{j^*}, A_1, A_2, \dots, A_{j^*-1}, A_{j^*+1}, \dots, A_n).$$

It suffices to prove that $v_1(A_{j^*}) \leq v_1(A_1) - (1 - s_2)$ under the assumption that $v_1(A_{j^*}) > \min_{e\in A_1} v_1(A_1 \setminus \{e\})$.

In the allocation $\boldsymbol{A}^{(j^*)}$, each agent $j \in \{2, 3, \dots, n\}$ does not get worse than $\boldsymbol{A}$ because $v_j(A_{j-1}) - v_j(A_j) = 1 - s_j \geq 0$, and thus the EF1 criterion is still satisfied for agent $j$. By the choice of $j^*$, we have $v_1(A_{j^*}) \geq v_1(A_j)$ for all $j \in \{2, 3, \dots, n\}$. Hence, the allocation $\boldsymbol{A}^{(j^*)}$ is EF1 if $v_1(A_{j^*}) > \min_{e\in A_1} v_1(A_1 \setminus \{e\})$.

Since $\boldsymbol{A}^{(j^*)}$ is an EF1 allocation, we have $\sum_{j\in[n]} v_j(A_j) \geq \sum_{j\in[n]} v_j(A_j^{(j^*)})$ by the definition of $\boldsymbol{A}$. This implies that

$$\begin{aligned} v_1(A_1) - v_1(A_{j^*}) &\geq \sum_{j=2}^{j^*}(v_j(A_{j-1}) - v_j(A_j)) \\ &= \sum_{j=2}^{j^*}(1 - s_j + r_j) \geq 1 - s_2 + r_2 \geq 1 - s_2, \end{aligned}$$

by (11). Consequently, we obtain that $v_1(A_j) \leq v_1(A_{j^*}) \leq v_1(A_1) - (1 - s_2)$ for every $j \in \{2, 3, \dots, n\}$. $\square$

As $v_2(A_1) = v_2(A_2) + r_2 + (1 - s_2) > 0$, we have $A_1 \neq \emptyset$. We choose $e^* \in A_1$ such that $v_2(A_2) \geq v_2(A_1 \setminus \{e^*\})$ and define

$$\boldsymbol{A}' = (A_1 \setminus \{e^*\}, A_2, A_3, \dots, A_{n-1}, A_n \cup \{e^*\}).$$

Note that such an item $e^*$ must exist since $\boldsymbol{A}$ is an EF1 allocation. Intuitively, this modification is expected to shorten paths in the envy graph. However, new envy from agent 1 to agent $n$ may arise, necessitating careful analysis.

Let $w'$ be the weights such that $w'_{i,j} = v_i(A'_j)$ for all $i, j \in [n]$. We will demonstrate that the minimum subsidy vector $\boldsymbol{p}'$ for $w'$ satisfies the conditions that $\max_{i \in [n]} p'_i \leq n - 1.5$.

Before we proceed to the proof, we observe the effect of this modification to the minimum subsidy vector for the instance in Example 1.

**Example 2.** Consider the instance observed in Example 1. Then, $\boldsymbol{A}$ in the example is an EF1 allocation such that $\sum_{i \in [n]} v_i(A_i) = n^2 \geq \sum_{i \in [n]} v_i(A_{\sigma(i)})$ for any permutation $\sigma$. Let $e^* = e_{1,1}$ and consider

$$\boldsymbol{A}' = (A_1 \setminus \{e^*\}, A_2, A_3, \dots, A_{n-1}, A_n \cup \{e^*\}).$$

Then the valuations for the bundles are

$$(v_i(A'_j))_{i,j \in [n]} = \begin{array}{cc} & \begin{array}{ccccc} A'_1 & A'_2 & \cdots & A'_{n-1} & A'_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \cdot \\ n-1 \\ n \end{array} & \begin{pmatrix} \frac{n^2}{n+1} & & & & \frac{n}{n+1} \\ n & n & & & \\ & n+1 & \ddots & & \\ & & \ddots & n & \\ & & & n+1 & n \end{pmatrix} \end{array}$$

and the minimum subsidy vector for this allocation is $(0, 0, 1, \dots, n-2)$.

Let us now proceed with the proof. To provide upper bounds with Lemma 3, we analyze the structure of $G^{w',\mathtt{id}}$.

**Lemma 7.** *For each $i, j \in [n]$, the weight of edge $(i, j)$ in $G^{w',\mathtt{id}}$ is*

$$w'_{i,j} - w'_{i,i} = v_i(A'_j) - v_i(A'_i) \leq \begin{cases} \max\{0.5,\, v_1(A'_n) - v_1(A'_1)\} & \text{if } i = 1, \\ 0.5 & \text{if } i = 2, \\ 1 & \text{if } i \geq 3. \end{cases}$$

**Proof.** Let $i, j \in [n]$. If $i = j$, the claim clearly holds as $w'_{i,j} - w'_{i,i} = 0$. Hence, we assume that $i \neq j$.

**Case 1:** $i = 1$. If $2 \leq j < n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_1(A_j) - v_1(A'_1) \\ &\leq \max\left\{ \begin{array}{c} v_1(A_1) - (1 - s_2), \\ \min_{e \in A_1} v_1(A_1 \setminus \{e\}) \end{array} \right\} - v_1(A_1 \setminus \{e^*\}) && \text{(by Lemma 6)} \\ &\leq \max\{s_2, 0\} = s_2 < 0.5 && \text{(by (12)).} \end{aligned}$$

If $j = n$, we have

$$w'_{i,j} - w'_{i,i} = v_1(A'_n) - v_1(A'_1) \leq \max\{0.5,\, v_1(A'_n) - v_1(A'_1)\}.$$

**Case 2:** $i = 2$. If $j = 1$, we have $w'_{i,j} - w'_{i,i} = v_2(A_1 \setminus \{e^*\}) - v_2(A_2) \leq 0$ by the choice of $e^*$ and the definition of $A'_1 = A_1 \setminus \{e^*\}$. If $2 < j < n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_2(A_j) - v_2(A_2) \\ &\leq -\sum_{k=3}^{j} (1 - s_k) + r_2 && \text{(by Lemma 5)} \\ &\leq -(1 - s_3) + s_2 = -1 + s_3 + s_2 \leq 0 && \text{(by (11) and (12)).} \end{aligned}$$

If $j = n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_2(A'_n) - v_2(A_2) \leq v_2(A_n) - v_2(A_2) + 1 \\ &\leq -\sum_{k=3}^{n} (1 - s_k) + 1 + r_2 && \text{(by Lemma 5)} \\ &\leq s_2 + s_3 < 0.5 && \text{(by (12) and } n \geq 3\text{).} \end{aligned}$$

**Case 3:** $i \geq 3$. If $j < n$, we have $w'_{i,j} - w'_{i,i} = v_i(A'_j) - v_i(A'_i) \leq v_i(A_j) - v_i(A_i) \leq 1$ by $A'_i \supseteq A_i$, $A'_j \subseteq A_j$, and (10). If $j = n$ (and hence $i \neq n$), we have

$$\begin{aligned}
w'_{i,j} - w'_{i,i} &= v_i(A'_n) - v_i(A_i) = v_i(A_n \cup \{e^*\}) - v_i(A_i) \\
&\leq v_i(A_n) - v_i(A_i) + 1 \\
&\leq -\sum_{k=i+1}^{n} (1 - s_k) + 1 + r_i && \text{(by Lemma 5)} \\
&\leq -(1 - s_n) + 1 + s_i = s_i + s_n && \text{(by (11) and } i \neq n\text{)} \\
&< 0.5 && \text{(by (12)).} \quad \square
\end{aligned}$$

Lemma 7 implies that all edges except $(1, n)$ have small weights. When the weight of $(1, n)$ is large, we show that

$$\boldsymbol{A}'' = (A'_n, A'_1, A'_2, \ldots, A'_{n-1})$$

induces small edge weights. Let $w''$ be the weights such that $w''_{i,j} = v_i(A''_j)$ $(i, j \in [n])$. Note that the minimum subsidy vector for $w''$ is $\boldsymbol{p}'' = (p'_n, p'_1, p'_2, \ldots, p'_{n-1})$ by Lemma 2.

**Lemma 8.** *For each $i, j \in [n]$, the weight of edge $(i, j)$ in $G^{w'',\mathtt{id}}$ is*

$$w''_{i,j} - w''_{i,i} = v_i(A''_j) - v_i(A''_i) \leq \begin{cases} 0.5 + \max\{0,\ v_1(A'_1) - v_1(A'_n)\} & \text{if } i = 1, \\ s_2 + s_3 & \text{if } i = 2, \\ s_i & \text{if } i \geq 3. \end{cases}$$

**Proof.** Let $i, j \in [n]$. The proof is clear when $i = j$, and thus we assume that $i \neq j$.

**Case 1:** $i = 1$. If $j = 2$, we have

$$w''_{i,j} - w''_{i,i} = v_1(A'_1) - v_1(A'_n) \leq 0.5 + \max\{0,\ v_1(A'_1) - v_1(A'_n)\}.$$

If $j > 2$, we have

$$\begin{aligned}
w''_{i,j} - w''_{i,i} &\leq v_1(A_{j-1}) - v_1(A'_n) \\
&\leq \max\left\{ \begin{array}{c} v_1(A_1) - (1 - s_2), \\ \min_{e \in A_1} v_1(A_1 \setminus \{e\}) \end{array} \right\} - v_1(A'_n) && \text{(by Lemma 6)} \\
&\leq \max\{v_1(A'_1) + s_2,\ v_1(A'_1)\} - v_1(A'_n) && \text{(by } A'_1 = A_1 \setminus \{e^*\}\text{)} \\
&\leq 0.5 + v_1(A'_1) - v_1(A'_n) && \text{(by (11) and (12))} \\
&\leq 0.5 + \max\{0,\ v_1(A'_1) - v_1(A'_n)\}.
\end{aligned}$$

**Case 2:** $i = 2$. Since $s_2 = 1 + r_2 + v_2(A_2) - v_2(A_1)$, we have

$$w''_{i,i} = v_2(A''_2) = v_2(A'_1) = v_2(A_1 \setminus \{e^*\}) \geq v_2(A_1) - 1 = v_2(A_2) - s_2 + r_2.$$

If $j = 1$, since $A''_1 = A'_n = A_n \cup \{e^*\}$, we have

$$\begin{aligned}
w''_{i,j} - w''_{i,i} &\leq v_2(A''_1) - v_2(A_2) + s_2 - r_2 \\
&= v_2(A_n \cup \{e^*\}) - v_2(A_2) + s_2 - r_2 \\
&\leq 1 + v_2(A_n) - v_2(A_2) + s_2 - r_2 \\
&\leq -\sum_{k=3}^{n}(1 - s_k) + 1 + s_2 && \text{(by Lemma 5)} \\
&\leq -(1 - s_3) + 1 + s_2 = s_2 + s_3 && \text{(by (11) and } n \geq 3\text{).}
\end{aligned}$$

If $j \geq 3$, we have

$$\begin{aligned}
w''_{i,j} - w''_{i,i} &\leq v_2(A'_{j-1}) - v_2(A_2) + s_2 - r_2 \\
&= v_2(A_{j-1}) - v_2(A_2) + s_2 - r_2
\end{aligned}$$

$$\le -\sum_{k=3}^{j-1}(1-s_k)+s_2 \qquad \text{(by Lemma 5)}$$
$$\le s_2 \le s_2+s_3 \qquad \text{(by (11))}.$$

**Case 3:** $i \ge 3$. By $s_i = 1 + r_i + v_i(A_i) - v_i(A_{i-1})$, we have

$$w''_{i,i} = v_i(A''_i) = v_i(A'_{i-1}) = v_i(A_{i-1}) = v_i(A_i) + (1-s_i) + r_i.$$

If $j = 1$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &= v_i(A_n \cup \{e^*\}) - v_i(A_i) - (1-s_i) - r_i \\ &\le 1 + v_i(A_n) - v_i(A_i) - (1-s_i) - r_i \\ &\le -\sum_{k=i+1}^{n}(1-s_k) + s_i && \text{(by Lemma 5)} \\ &\le -(1-s_n) + s_i = -1 + s_n + s_i \le 0 && \text{(by (11) and (12))}. \end{aligned}$$

If $j \ge 2$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &= v_i(A''_j) - v_i(A_i) - (1-s_i) - r_i \\ &= v_i(A'_{j-1}) - v_i(A_i) - (1-s_i) - r_i \\ &\le v_i(A_{j-1}) - v_i(A_i) - (1-s_i) && \text{(by } A'_{j-1} \subseteq A_{j-1} \text{ and } r_i \ge 0) \\ &\le 1 - (1-s_i) = s_i && \text{(by (10))}. \quad \square \end{aligned}$$

**Algorithm 1:** Envy-free allocation with maximum subsidy $n - 1.5$ for monotone nondecreasing valuations.

**Input:** Monotone nondecreasing valuations $(v_i)_{i\in[n]}$ where $n \ge 3$
**Output:** $(\boldsymbol{A}, \boldsymbol{p})$ with $\max_{i\in[n]} p_i \le n - 1.5$
Compute an EF1 allocation $\boldsymbol{X}$ using the envy-cycles algorithm;
Construct a bipartite graph $([n],[n];E)$ where $E = \{(i,j) \in [n]\times[n] \mid v_i(X_j) \ge \max_{k\in[n]} \min_{Y\subseteq X_k : |Y|\le 1} v_i(X_k \setminus Y)\}$;
Assign weight $v_i(X_j)$ for each edge $(i,j) \in E$;
Let $\boldsymbol{A}$ be a permutation of $\boldsymbol{X}$ according to the maximum weight perfect matching on the weighted bipartite graph;
Compute minimum subsidy vector $\boldsymbol{p}^*$ for $w = (v_i(A_j))_{i,j\in[n]}$;
**if** $\max_{i\in[n]} p^*_i \le n - 1.5$ **then**
    Find maximum weight permutation $\tau$ for $w$;
    **return** $((A_{\tau(1)}, \dots, A_{\tau(n)}), (p^*_{\tau(1)}, \dots, p^*_{\tau(n)}))$;
**else**
    Let $\hat{w}$ be the weights defined as (1);
    Relabel the indices so that $p^*_1 \le p^*_2 \le \cdots \le p^*_n$;
    Select $e^* \in A_1$ satisfying $v_2(A_2) \ge v_2(A_1 \setminus \{e^*\})$;
    Define $\boldsymbol{A}' = (A_1 \setminus \{e^*\}, A_2, A_3, \dots, A_n \cup \{e^*\})$;
    Find maximum weight permutation $\tau^*$ for $w' = (v_i(A'_j))_{i,j\in[n]}$;
    **return** $((A'_{\tau^*(1)}, \dots, A'_{\tau^*(n)}), (p'_{\tau^*(1)}, \dots, p'_{\tau^*(n)}))$;

Now, we are ready to prove Theorem 2. Our algorithm can be summarized as shown in Algorithm 1.

**Proof of Theorem 2.** In what follows, suppose that $n \ge 3$. If $\max_{i\in[n]} p^*_i \le n - 1.5$, the total subsidy $\sum_{i\in[n]} p^*_i$ is at most $\sum_{k=1}^{n-1} \min\{k, n-1.5\} = (n^2 - n - 1)/2$ by Lemma 3. Therefore, we assume that $\max_{i\in[n]} p^*_i > n - 1.5$. We show that in this case, $\boldsymbol{p}'$ and $\boldsymbol{p}''$ defined before are desired ones.

For $\boldsymbol{p}'$, by Lemmas 3 and 7, we have

$$\begin{aligned} \max_{i\in[n]} p'_i &\le n - 2 + \max\{0.5, v_1(A'_n) - v_1(A'_1)\} \\ &\le (n-1.5) + \max\{0, v_1(A'_n) - v_1(A'_1)\} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \sum_{i\in[n]} p'_i &\le \sum_{i=1}^{n-2}(n-i)\cdot 1 + 1\cdot \max\{0.5, v_1(A'_n) - v_1(A'_1)\} \\ &\le \frac{n^2-n-1}{2} + \max\{0, v_1(A'_n) - v_1(A'_1)\}. \end{aligned} \tag{14}$$

For $\boldsymbol{p}''$, by Lemmas 3 and 8, it holds that

$$\begin{aligned}\max_{i\in[n]} p''_i &\le s_3 + \sum_{i=2}^{n} s_i + 0.5 + \max\{0, v_1(A'_1) - v_1(A'_n)\}\\ &\le 1.5 + \max\{0, v_1(A'_1) - v_1(A'_n)\}\\ &\le (n-1.5) + \max\{0, v_1(A'_1) - v_1(A'_n)\} \end{aligned} \tag{15}$$

and

$$\begin{aligned}\sum_{i\in[n]} p''_i &\le \sum_{i=1}^{n-2}(n-i)\cdot\frac{1}{2} + \frac{1}{2} + \max\{0, v_1(A'_n) - v_1(A'_1)\}\\ &\le \frac{n^2-n-1}{2} + \max\{0, v_1(A'_1) - v_1(A'_n)\}. \end{aligned} \tag{16}$$

We observe from these bounds that if $v_1(A'_n) \le v_1(A'_1)$, then $\boldsymbol{p}'$ satisfies the requirements of this theorem, i.e., $\max_{i\in[n]} p'_i \le n-1.5$ by (13) and $\sum_{i\in[n]} p'_i \le (n^2-n-1)/2$ by (14); otherwise, $\boldsymbol{p}''$ satisfies the requirements by (15) and (16). However, recall that $\boldsymbol{A}''$ and $\boldsymbol{p}''$ are rearrangements of $\boldsymbol{A}'$ and $\boldsymbol{p}'$, respectively. Thus, both $\boldsymbol{p}'$ and $\boldsymbol{p}''$ satisfy the requirements. Hence, $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{p}}) = ((A'_{\tau^*(1)}, \dots, A'_{\tau^*(n)}), (p'_{\tau^*(1)}, \dots, p'_{\tau^*(n)}))$ is a desired envy-free allocation with a subsidy, where $\tau^*$ is a maximum weight permutation for $w'$. In addition, $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{p}})$ can be computed in polynomial time via computing $\tau^*$. □

## 5. Small number of items

In this section, we explore cases where the number of items $m$ is relatively small compared to the number of agents $n$. If we allocate entire items to an agent $i^*$ who maximizes $v_i(M)$ over all $i \in [n]$, then the required subsidy per agent is bounded by $|v_{i^*}(M)| \le m$. We propose a mechanism with a superior subsidy bound.

1. Divide the set of items $M$ into subsets of nearly equal size, denoted by $\mathbf{X} = (X_1, \dots, X_n)$, where the size of each subset $|X_i|$ is either $\lfloor m/n \rfloor$ or $\lceil m/n \rceil$.
2. Define $w_{i,j} = v_i(X_j)$ for each agent $i$ and subset $X_j$, and compute a maximum weight permutation $\sigma$ for $w$.
3. Output the allocation $\mathbf{A} = (X_{\sigma(1)}, \dots, X_{\sigma(n)})$ and determine the minimum subsidy vector for this allocation.

We establish the following upper bounds for the subsidies required by this mechanism:

**Theorem 3.** *Suppose that $|v_i(X_i \cup \{e\}) - v_i(X)| \le 1$ for any $i \in [n]$, $e \in M$, and $X \subseteq M \setminus \{e\}$. Then, there exists a complete envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i\in[n]} p_i \le 2\lceil m/n \rceil$. Furthermore, if the valuations are monotone nondecreasing, the maximum subsidy required per agent $\max_{i\in[n]} p_i$ reduces to at most $\lceil m/n \rceil$.*

**Proof.** Let $w^*_{i,j} = v_i(A_j)$ for each $i, j \in [n]$ and consider the envy graph $G^{w^*,\mathtt{id}}$, where the weight of each edge $(i,j) \in [n]\times[n]$ is $\gamma_{i,j} = w^*_{i,j} - w^*_{i,i}$. Note that, as $\mathtt{id}$ is a maximum weight permutation, the envy graph does not contain any positive-weight directed cycles. By the assumption that the marginal contribution of each item is at most one, we have $|v_i(A_i)| \le |A_i| \le \lceil m/n \rceil$. Hence, we get $-2\lceil m/n \rceil \le \gamma_{i,j} \le 2\lceil m/n \rceil$.

By Lemma 1, the minimum subsidy $p_i$ for agent $i \in [n]$ is the maximum length of any path in $G^{w^*,\mathtt{id}}$ starting at $i$. Suppose to the contrary that there exists a path of length longer than $2\lceil m/n \rceil$ from $i$ to some $j \in [n]$. Then, the cycle consisting of the path and the edge $(j,i)$ is a positive weight directed cycle, and this is a contradiction. Thus, $p_i$ must be at most $2\lceil m/n \rceil$ for any $i \in [n]$.

If the valuations are monotone nondecreasing, then we have $0 \le v_i(A_i) \le \lceil m/n \rceil$ and consequently $-\lceil m/n \rceil \le \gamma_{i,j} \le \lceil m/n \rceil$ for any $i, j \in [n]$. Thus, under monotone nondecreasing valuations, the maximum subsidy required per agent is indeed at most $\lceil m/n \rceil$. □

Specifically, if the valuations are monotone nondecreasing and the number of items is at most the number of agents, the required subsidy is at most one per agent.

## 6. Conclusion

In this paper, we have studied the fair division of indivisible items with limited subsidies, focusing on improving existing subsidy bounds required to achieve envy-freeness. Our contributions can be summarized as follows:

1. For general valuations (where each item's absolute marginal valuation is at most one), we demonstrated that given an EF1 allocation, it is possible to compute in polynomial time an envy-free allocation with a subsidy of at most $n-1$ per agent and a total subsidy of at most $n(n-1)/2$. To obtain these bounds, we established the relationship between dual variables of the assignment problem and subsidies, and introduced a novel technique of transforming the valuation function without altering the

minimum subsidy vector. We also demonstrated that these subsidy bounds are best possible without recombining the bundles of a given EF1 allocation.

2. For monotone nondecreasing valuations, we further improved these subsidy bounds. We provided a polynomial time algorithm that finds an envy-free allocation with a subsidy that is at most $n-1.5$ per agent and $(n^2-n-1)/2$ in total, by slightly modifying the initial EF1 allocation. This result breaks through the limitations achievable by arbitrary EF1 allocations.
3. For general valuations, we provided a polynomial time algorithm that finds an envy-free allocation with a subsidy of at most $O(m/n)$ per agent and a total subsidy of at most $O(m)$. This result is particularly useful when resources are scarce relative to the number of participants. Additionally, by combining the subsidy bound of $n-1$ per agent, we can also obtain that a subsidy of at most $O(\sqrt{m})$ per agent is sufficient to achieve envy-freeness.

These results not only improve upon existing bounds but also provide efficient algorithms for computing envy-free allocations with limited subsidies in various settings.

Despite our progress, several open questions remain. Notably, even for monotone nondecreasing valuations, it is still unresolved whether there exists an envy-free allocation with a total subsidy of $O(n^{2-\epsilon})$ for some $\epsilon > 0$. In addition to exploring tighter bounds for envy-free allocations with subsidies, it would be interesting to investigate subsidy bounds for other fairness concepts, such as proportionality or equitability. We hope that our techniques introduced in this paper will play a role in addressing these questions and further advancing the study of fair division with subsidies.

## CRediT authorship contribution statement

**Yasushi Kawase:** Project administration, Conceptualization, Writing – original draft, Methodology, Writing – review & editing, Formal analysis. **Kazuhisa Makino:** Formal analysis, Methodology. **Hanna Sumita:** Methodology, Formal analysis, Writing – review & editing. **Akihisa Tamura:** Methodology, Formal analysis, Writing – review & editing. **Makoto Yokoo:** Methodology, Writing – review & editing, Writing – original draft, Formal analysis.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yasushi Kawase, Kazuhisa Makino, Hanna Sumita, Akihisa Tamura, Makoto Yokoo reports financial support was provided by Toyota Motor Corporation. Yasushi Kawase, Kazuhisa Makino, Hanna Sumita, Akihisa Tamura, Makoto Yokoo reports financial support was provided by Japan Science and Technology Agency. Yasushi Kawase, Kazuhisa Makino, Hanna Sumita, Akihisa Tamura, Makoto Yokoo reports financial support was provided by Japan Society for the Promotion of Science. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## References

[1] A. Alkan, G. Demange, D. Gale, Fair allocation of indivisible goods and criteria of justice, Econometrica 59 (1991) 1023–1039.
[2] E. Aragones, A derivation of the money Rawlsian solution, Soc. Choice Welf. 12 (1995) 267–276.
[3] H. Aziz, Achieving envy-freeness and equitability with monetary transfers, in: Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI), 2021, pp. 5102–5109.
[4] M. Babaioff, T. Ezra, U. Feige, Fair and truthful mechanisms for dichotomous valuations, in: Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI), 2021, pp. 5119–5126.
[5] S. Barman, A. Krishna, Y. Narahari, S. Sadhukhan, Achieving envy-freeness with limited subsidies under dichotomous valuations, in: Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI), 2022, pp. 60–66.
[6] S. Barman, P. Verma, Fair division beyond monotone valuations, arXiv:2501.14609, 2025.
[7] N. Benabbou, A. Igarashi, M. Chakraborty, Y. Zick, Finding fair and efficient allocations for matroid rank valuations, ACM Trans. Econ. Comput. 9 (2021) 1–41.
[8] U. Bhaskar, A.R. Sricharan, R. Vaish, On approximate envy-freeness for indivisible chores and mixed resources, in: Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021), 2021, pp. 1:1–1:23.
[9] J. Brustle, J. Dippel, V.V. Narayan, M. Suzuki, A. Vetta, One dollar each eliminates envy, in: Proceedings of the 21st ACM Conference on Economics and Computation (EC), 2020, pp. 23–39.
[10] E. Budish, The combinatorial assignment problem: approximate competitive equilibrium from equal incomes, J. Polit. Econ. 119 (2011) 1061–1103.
[11] K. Bérczi, E.R. Bérczi-Kovács, E. Boros, F.T. Gedefa, N. Kamiyama, T. Kavitha, Y. Kobayashi, K. Makino, Envy-free relaxations for goods, chores, and mixed items, arXiv:2006.04428, 2020.
[12] I. Caragiannis, S. Ioannidis, Computing envy-freeable allocations with limited subsidies, in: Proceedings of the 18th Conference on Web and Internet Economics (WINE), 2022, pp. 522–539.
[13] V. Conitzer, R. Freeman, N. Shah, J.W. Vaughan, Group fairness for the allocation of indivisible goods, in: Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI), 2019, pp. 1853–1860.
[14] D.K. Foley, Resource allocation and the public sector, Yale Econ. Essays 7 (1967) 45–98.
[15] H. Goko, A. Igarashi, Y. Kawase, K. Makino, H. Sumita, A. Tamura, Y. Yokoi, M. Yokoo, A fair and truthful mechanism with limited subsidy, Games Econ. Behav. 144 (2024) 49–70.

[16] D. Halpern, N. Shah, Fair division with subsidy, in: Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT), 2019, pp. 374–389.
[17] Y. Kawase, K. Makino, H. Sumita, A. Tamura, M. Yokoo, Towards optimal subsidy bounds for envy-freeable allocations, in: Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI), 2024, pp. 9824–9831.
[18] F. Klijn, An algorithm for envy-free allocations in an economy with indivisible objects and money, Soc. Choice Welf. 17 (2000) 201–215.
[19] R.J. Lipton, E. Markakis, E. Mossel, A. Saberi, On approximately fair allocations of indivisible goods, in: Proceedings of the 5th ACM Conference on Electronic Commerce (ACM-EC), 2004, pp. 125–131.
[20] S. Liu, X. Lu, M. Suzuki, T. Walsh, Mixed fair division: a survey, J. Artif. Intell. Res. 80 (2024) 1373–1406.
[21] E.S. Maskin, On the Fair Allocation of Indivisible Goods, Palgrave Macmillan UK, London, 1987, pp. 341–349.
[22] H. Moulin, Fair Division and Collective Welfare, MIT Press, 2004.
[23] V.V. Narayan, M. Suzuki, A. Vetta, Two birds with one stone: fairness and welfare via transfers, in: Proceedings of the 14th International Symposium on Algorithmic Game Theory (SAGT), 2021, pp. 376–390.
[24] F.E. Su, Rental harmony: Sperner's lemma in fair division, Am. Math. Mon. 106 (1999) 930–942.
[25] N. Sun, Z. Yang, A general strategy proof fair allocation mechanism, Econ. Lett. 81 (2003) 73–79.
[26] L.G. Svensson, Large indivisibles: an analysis with respect to price equilibrium and fairness, Econometrica 51 (1983) 939–954.
[27] K. Tadenuma, W. Thomson, The fair allocation of an indivisible good when monetary compensations are possible, Math. Soc. Sci. 25 (1993) 117–132.
[28] H.R. Varian, Equity, envy and efficiency, J. Econ. Theory 9 (1974) 63–91.
[29] X. Wu, C. Zhang, S. Zhou, One quarter each (on average) ensures proportionality, in: Proceedings of the 19th Conference on Web and Internet Economics (WINE), 2023, pp. 582–599.