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Fair distribution of delivery orders

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A B S T R A C T

We initiate the study of fair distribution of delivery tasks among a set of agents wherein delivery jobs are placed along the vertices of a graph. Our goal is to fairly distribute delivery costs (distance traveled to complete the deliveries) among a fixed set of agents while satisfying some desirable notions of economic efficiency. We adopt well-established fairness concepts—such as *envy-freeness up to one item* (EF1) and *minimax share* (MMS)—to our setting and show that fairness is often incompatible with the efficiency notion of *social optimality*. We then characterize instances that admit fair and socially optimal solutions by exploiting graph structures. We further show that achieving fairness along with Pareto optimality is computationally intractable. We complement this by designing an XP algorithm (parameterized by the number of agents) for finding MMS and Pareto optimal solutions on every tree instance, and show that the same algorithm can be modified to find efficient solutions along with EF1, when such solutions exist. The latter crucially relies on an intriguing result that in our setting EF1 and Pareto optimality jointly imply MMS. We conclude by theoretically and experimentally analyzing the price of fairness.

1. Introduction

With the rise of digital marketplaces and the gig economy, package delivery services have become crucial components of e-commerce platforms like Amazon, AliExpress, and eBay. In addition to these novel platforms, traditional postal and courier services also require swift turnarounds for distributing packages. Prior work has extensively investigated the optimal partitioning of tasks among the delivery agents under the guise of *vehicle routing* problems (see [67] for an overview). However, these solutions are primarily focused on optimizing the efficiency (often measured by delivery time or distance traveled [47,57]), and do not consider fairness towards the delivery agents. This is particularly important in settings where agents do not receive monetary compensation, e.g., in volunteer-based social programs such as Meals on Wheels [56].

We consider fair distribution of delivery orders that are located on the vertices of a connected graph, containing a warehouse (the *hub*). Agents are tasked with picking up delivery packages (or *items*) from the fixed hub, delivering them to the vertices, and returning to the hub. In this setting, the cost incurred by an agent i is the total distance traveled, that is, the total number of the edges traversed by i in the graph. Let us illustrate this through an example.

Example 1. Consider seven delivery orders $\{a, b, \ldots, g\}$ and a hub (h) that are located on a graph as depicted in Fig. 1. An agent's cost depends on the graph structure and is *submodular*. For instance, the cost of delivering an order to vertex f is the distance from the hub h to f, which is 4[1]; but the cost of delivering to f and g is only 5 since they can both be serviced in the same trip.

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[1] Formally, there is also the cost of returning to h, but since, on trees, each edge must be traversed by an agent twice (once in each direction), we do not count the return cost for simplicity.

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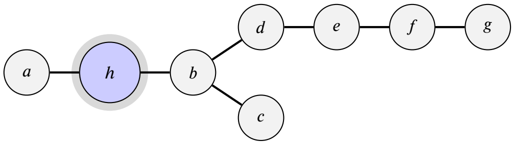


Fig. 1. An example graph with the hub, h, marked.

Let there be two (delivery) agents. If the objective were to simply minimize the total distance traveled (*social optimality*), then there are two solutions with the total cost of 7: either one agent delivers *all* the items or one agent services a while the other services the rest. However, these solutions do not distribute the delivery orders fairly among the agents.

One plausible fair solution may assign $\{a, b, f\}$ to the first agent and $\{c, d, e, g\}$ to the other, minimizing the cost discrepancy. However, as both agents benefit from exchanging f for c, this allocation is not efficient or, more precisely, it is not *Pareto optimal*. After the exchange, the first agent services $\{a, b, c\}$ and the second agent $\{d, e, f, g\}$, which in fact is a Pareto optimal allocation.

The above example captures the challenges in satisfying fairness in conjunction with efficiency, and consequently, motivates the study of fair distribution of delivery orders. The literature on fair division has long been concerned with the fair allocation of goods (or resources) [9,21,31,50], chores (or tasks) [25,29,38,43], and mixtures thereof [3,16,22,39].

It has resulted in a variety of fairness concepts and their relaxations. Most notably, *envy-freeness* and its relaxation—*envy-freeness up to one item* (EF1) [50]—have been widely studied in the context of fair division. Another well-studied fairness notion, minimax share (MMS) [21], requires that agents receive cost no more than what they would have received if they were to create (almost) equal partitions. A key question is how to adopt these fairness concepts to the delivery problems, and whether these fairness concepts are compatible with natural efficiency requirements.

1.1. Technical contributions

We initiate the study of fair distribution of delivery tasks among a set of agents. The tasks are placed on the vertices of an acyclic or tree graph. The cost of servicing a given set of tasks is the number of edges traversed to be able to reach each node. This cost function can be easily seen to be submodular. The primary objective is to find a fair partition of m delivery orders (represented by vertices of a graph), starting from a fixed hub, among n agents. We consider two well-established fairness concepts of EF1 and MMS and explore their existence and computation along with efficiency notions of social optimality (SO) and Pareto optimality (PO).

Existence of fair and efficient allocations. Table 1 summarizes our results on the coexistence of fairness and efficiency in arbitrary graphs. We first show that an EF1 allocation always exists and can be computed in polynomial time on trees (Proposition 3.1). In contrast, an MMS allocation is guaranteed to exist, but its computation remains NP-hard (Theorem 3.1). Out of the various combinations of fairness and efficiency, only MMS and PO allocations are guaranteed to exist. Finding a fair and efficient allocation remains NP-hard for each combination.

We note that our intractability results hold even for the restricted case of unweighted tree graphs. Consequently, in this paper we focus on the unweighted tree instances only. As we show, even this special case allows for a rich landscape of results. By doing a comprehensive study of the fair delivery problem on trees, we aim to establish a solid baseline for further work in the more general settings. Indeed, a popular technique to deal with computationally difficult problems on graphs is to provide algorithms with complexity parameterized by some measure of tree-resemblance [See for instance Chapter 7 in Cygan et al. [27]]. Trees can be also motivated in practice. Many suburban neighborhoods in the United States, among other countries, are designed with a cul-de-sac layout, with houses/mailboxes typically located at uniform distances. Then, a group of agents distributing a newspaper or a leaflet to every house in such a neighborhood, will face an unweighted tree instance, as in our problem. However, we note that many of our results already hold for the general case of cyclic and weighted graph, which we discuss in Appendix E.

In Section 4, we characterize the conditions for the existence of fair (EF1 and MMS) and efficient (PO and SO) allocations in tree instances. In particular, one of our most technically involved results provides a necessary condition for an allocation to be EF1 and PO (Proposition 4.1). In turn, this helps us to prove that such an allocation must be leximin optimal. As a result, we show that an EF1 and PO allocation *always satisfies* MMS as well (Theorem 4.1).

In Section 4.3, we present results on a stronger fairness notion of envy-freeness up to any item (EFX). We note that an EFX allocation always exists in this setting, however EFX combined with some efficiency requirement, for large classes of graphs, becomes as restrictive as envy-freeness (Theorem 4.3). Further determining whether an efficient EFX allocation exists is also NP-hard (Proposition 4.6).

Exact algorithm. We complement our existence results by designing an XP algorithm (Algorithm 1), parameterized by the number of agents, that finds the Pareto frontier of a given delivery instance (Theorem 5.1). This allows us to find an MMS and PO solution. Further, our characterization results in conjunction with Algorithm 1 enable us to check whether an instance admits an EF1 and PO allocation or a fair and SO allocation as well (Theorem 5.2). This algorithm also forms enables us to do extensive experiments on randomly generated instances.

Table 1
The summary of our results on EF1 and MMS in conjunction with efficiency requirements of PO and SO. ✓ denotes that the allocation always exists, and ✗ that it may not exist. We note that every NP-hard problem here can be solved with an XP algorithm parameterized by the number of agents (Theorem 5.2).

		–	PO	SO
EF1	existence	✓ (Proposition 3.1)	✗ (Proposition 4.1a)	✗ (Proposition 4.1a)
	computation	P (Proposition 3.1)	NP-h (Proposition 4.2)	NP-h (Proposition 4.3)
MMS	existence	✓ (Theorem 3.1)	✓ (Proposition 4.1b)	✗ (Theorem 4.2)
	computation	NP-h (Theorem 3.1)	NP-h (Proposition 4.2)	NP-h (Proposition 4.3)

Price of fairness and experiments. In our last set of theoretical results, we study the price of fairness of MMS and EF1 in our setting. Informally, price of fairness of a given instance is the ratio of the minimum possible sum of agent costs under a fair allocation to the minimum possible sum of agent costs under any allocation. To this end, we discuss the efficiency and computational complexity of finding minimum cost fair allocations Proposition 6.1 and Theorem 6.1.

We formally study price of fairness by establishing worst case upper bounds and typical values in a theoretical and experimental analysis, respectively. We present the theoretical bounds on the price of fairness in Section 6 and the experimental findings on price of fairness along with other experiments in Section 7. Our other experiments include checking how often EF1 and PO allocations and fair and SO allocations exist and studying the trajectories of the Pareto frontiers of randomly generated instances.

1.2. Related work

Fair division of indivisible items has garnered much attention in recent years. Several notions of fairness have been explored in this space, with EF1 [9,21,22,50] and MMS [5,32,37] being among the most prominent ones. An important result is from Caragiannis et al. [24] showing that an EF1 and PO allocation is guaranteed to exist for items with non-negative additive valuations. Some prior work has also looked at fair division on graphs [19,20,51,53,68], but in the settings that are very different from assigning delivery orders. The majority of these papers identified the vertices of a graph with goods and analyzed how to fairly partition the graph into contiguous pieces.

Some recent work has explored fairness in delivery settings [34,54,62,69,70] or ride-hailing platforms [30,61]. However, in these studies tasks cannot be combined to give lower aggregate cost than the sum of the individual costs. This is very different from our setting, where an agent delivering one order can deliver all orders on the way for no additional cost. Further, the prior work on these settings is largely experimental and does not provide any positive theoretical guarantees. While fairness has been studied in routing problems, the aim has been to balance the amount of traffic on each edge [47,57], which does not capture the type of delivery instances that we investigate in this paper. In Appendix A, we provide an extended review of the literature.

2. Our model

We denote a *delivery instance* by an ordered triple $I = \langle [n], G, h \rangle$, where $[n]$ is a set of agents, $G = (V, E)$ is an undirected acyclic graph (i.e., a tree) of delivery orders rooted in $h \in V$. The special vertex, i.e., the root, h is called the *hub*. By m, we denote the number of edges in the graph, which is also the number of non-hub vertices. We assume that $m \geq n$.

The goal is to assign each vertex in graph G, except for the hub, to a unique agent that will *service* it. Formally, an allocation $A = (A_1, \dots, A_n)$ is an n-partition of vertices in $V \setminus \{h\}$. We are only interested in *complete* allocations such that $\cup_{i \in [n]} A_i = V \setminus \{h\}$, and denote the set of all complete allocations by Π^n.

An agent's *cost* for servicing a vertex $v \in V$, denoted by $c(v)$, is the length of the shortest path from the hub h to v. An agent's cost for servicing a set of vertices $S \subseteq V \setminus \{h\}$ is equal to the minimum length of a walk that starts and ends in h and contains all vertices in S divided by two.[2] A walk to service vertices in S may pass through vertices in some superset of S, i.e., $S' \supseteq S$. Thus, the cost function c is *submodular* and belongs to the class of *coverage functions*. We use $G|_S$ to denote the minimal connected subgraph containing all vertices in $S \cup \{h\}$. Thus, we have $c(S) = |E(G|_S)|$. We say that an agent servicing S, *visits* all vertices in $G|_S$.

Fairness concepts. The most plausible fairness notion is *envy-freeness* (EF), which requires that no agent (strictly) prefers the allocation of another agent. An EF allocation may not exist; consider one delivery order and two agents. A prominent relaxation of EF is *envy-freeness up to one order* (EF1) [21,50], which requires that every pairwise envy can be eliminated by the removal of a single order served by the envious agent.

[2] On trees, in each such walk, each edge is traversed by an agent two times (once in both directions). For simplicity, we drop the return cost, hence the division by 2.

Definition 1 *(Envy-Freeness up to One Order (EF1)).* An allocation A is EF1 if for every pair $i, j \in [n]$, either $A_i = \emptyset$ or there exists $x \in A_i$ such that $c(A_i \setminus \{x\}) \le c(A_j)$.

Another well-studied notion is *minimax share* (MMS), which ensures that each agent gets at most as much cost as they would if they were to create an n-partition of the delivery orders but then receive their least preferred bundle. This notion is an adaptation of maximin share fairness—which was defined for positive valuations [21]—to settings with negative valuations and has been recently studied in fair allocation of chores (see e.g., Huang and Lu [43]).

Definition 2 *(Minimax Share (MMS)).* The minimax share cost of a given delivery instance I is given by

$$\mathrm{MMS}(I) = \min_{A \in \Pi^n} \max_{i \in [n]} c(A_i).$$

An allocation A is MMS if $c(A_i) \le \mathrm{MMS}_i(I)$ for all $i \in [n]$.

Under additive non-negative identical valuations, MMS and EF1 co-exist. For completeness, we give a formal proof in Appendix C. However, as our cost functions are submodular, neither EF1 nor MMS implies the other, even though the cost functions are identical (see Example 2).

Economic efficiency. Our first notion of efficiency is social optimality (SO). SO requires that the aggregate cost of all of the agents is minimum. That is, we want to minimize the number of agents that traverse any given edge. One simple way to do this would be to have a single agent traverse the entire graph. Here, the sum of all agents' costs is m i.e., equal to the number of edges in the graph. Clearly this is the minimum possible cost of any allocation. Thus, an allocation is socially optimal, if and only if, the sum of all agents' costs is m. Equivalently, SO is satisfied, if and only if, each vertex (except for the hub) is visited by exactly one agent.

Definition 3 *(Social Optimality (SO)).* An allocation A is *socially optimal* if $\sum_{i=1}^{n} c(A_i) = |E(G)|$. In other words, for every pair of agents $i \neq j \in [n]$, the only vertex they both visit is the hub, i.e., $V(G|_{A_i}) \cap V(G|_{A_j}) = \{h\}$.

Here, $V(\cdot)$ takes as an input a subgraph and returns the set of vertices in it. An allocation that assigns all vertices to a single agent is vacuously SO. However, as we discussed in Example 1 it may result in a very unfair distribution of orders. Therefore, we consider a weaker efficiency notion that allows for some overlap in vertices visited by the agents.

Definition 4 *(Pareto Optimality (PO)).* An allocation A *Pareto dominates* A' if $c(A_i) \le c(A'_i)$, for every agent $i \in [n]$, and there exists some agent $j \in [n]$ such that $c(A_j) < c(A'_j)$. An allocation is *Pareto optimal* if it is not Pareto dominated by any other allocation.

In other words, an allocation is PO if we cannot reduce the cost of one agent without increasing it for some other agent. Let us now follow up on Example 1 and analyze allocations satisfying our notions.

Example 2 *(Continuation of Example 1).* Consider the instance with 2 agents and the graph from Fig. 1. As previously noted, there are only two SO allocations and neither is EF1 or MMS.

PO allocation $(\{d, e, f, g\}, \{a, b, c\})$ satisfies MMS (vertex g must be serviced by some agent, hence the MMS cost cannot be smaller than 5), but it is not EF1. In fact, there is no EF1 and PO allocation in this instance, as an agent servicing g has to service f, e and d as well (otherwise giving them to this agent would be a Pareto improvement). But then, even when we assign the remaining vertices, a, b, c, to the second agent, the allocation would violate EF1.

Finally, observe that allocation $(\{a, b, f\}, \{c, d, e, g\})$ is EF1, but not MMS.

In order to study the (co-)existence of these fairness and efficiency notions, we use leximin optimality. Let us first set up some notation before defining it. Given an allocation A, we can *sort it in non-increasing cost order* to obtain allocation $B = \mathrm{sort}(A)$ such that $c(B_1) \ge c(B_2) \ge \cdots \ge c(B_n)$ and $B_i = A_{\pi(i)}$ for every agent $i \in [n]$ and some permutation of agents π.

Definition 5 *(Leximin Optimality).* An allocation A leximin dominates an allocation A' if there is agent $i \in [n]$ such that $c(B_i) < c(B'_i)$ and $c(B_j) = c(B'_j)$ for every $j \in [i-1]$, where $B = \mathrm{sort}(A)$ and $B' = \mathrm{sort}(A')$. An allocation is *leximin optimal* if it is not leximin dominated by any other allocation.

In other words, a leximin optimal allocation first minimizes the cost of the worst-off agent, then minimizes the cost of the second worst-off agent, and so on. We note that a leximin optimal allocation is always Pareto optimal as well.

3. Existence of fair allocations

In this section, we consider the existence and computation of EF1 and MMS allocations.

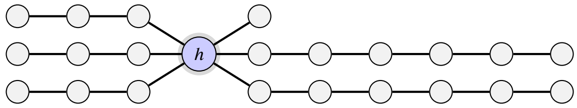


Fig. 2. An example of a spider (or starlike) graph.

Recall that our cost functions are submodular and *monotone*. Thus, an EF1 allocation can be obtained by adapting an *envy-graph* algorithm from Lipton et al. [50]: start with each agent having an empty set, pick an agent who currently has minimum cost (i.e., a sink of the envy-graph) and out of the unassigned vertices give them the one that results in a minimal increase of the cost.[3] Repeat this till all vertices in $V \setminus \{h\}$ are allocated. This algorithm always returns an EF1 solution as long as valuations are identical and monotone.

Proposition 3.1. *Given a delivery instance $I = \langle [n], G, h\rangle$, an* EF1 *allocation always exists and can be computed in polynomial time.*

We now shift our focus to allocations that satisfy MMS. An MMS allocation in our setting always exists. This follows from the fact that agents have identical cost functions (an allocation that minimizes the maximum cost will satisfy MMS). However, finding such an allocation is NP-hard. To establish this, we first show the hardness of finding the MMS cost.

Theorem 3.1. *Given a delivery instance $I = \langle [n], G, h\rangle$, an* MMS *allocation always exists. However, finding i) the* MMS *cost in* NP-hard *and ii) finding an MMS allocation is* NP-hard.

Proof. The existence of MMS allocations follows from the cost functions being identical. The same allocation will give the MMS threshold for all agents, hence would satisfy MMS for all.

For part i) we give a reduction from 3-Partition to a setting with unweighted trees. In the 3-Partition problem, we are given $3k$ positive integers $S = \{s_1, \dots, s_{3k}\}$ that sum up to kT for some $T \in \mathbb{N}$. The task is to decide if there is a partition of S into k pairwise disjoint subsets, $P = P_1, \dots, P_k \subseteq S$ such that the elements in each subset sum up to T. This problem is known to be NP-hard [46], even when the values of the integers are polynomial in k.

For each instance of 3-Partition let us construct a delivery instance with k agents. To this end, we construct a tree where for each integer $s_i \in S$, let us take s_i vertices $v_i^1, \dots, v_i^{s_i}$, which, with the hub, h, gives as a total of $3kT + 1$ vertices and all edges having a weight of 1. Next, for every $i \in [3k]$, let us connect all consecutive vertices in sequence $h, v_i^1, \dots, v_i^{s_i}$ with an edge to form a path. In this way, we obtain a graph that consists of the hub and $3k$ paths of different lengths outgoing from the hub (such graphs are known as *spider* or *starlike* graphs). See Fig. 2 for an illustration. As per our convention for trees, when defining edge costs, we count the traversal of each edge exactly once.

Let us show that minimax share cost in this instance is T, if and only if, there exists a desired partition in the original 3-Partition instance. If there is a partition P, consider allocation A obtained by assigning to every agent $j \in [k]$, all vertices corresponding to integers in P_j, i.e., $A_j = \cup_{s_i \in P_j} \{v_i^1, \dots, v_i^{s_i}\}$. In this allocation, the cost of every agent will be equal to T. Further, the maximum cost of an agent in any allocation cannot be smaller than T. If not, it would make the total cost smaller than the number of edges in the graph, which is not possible. Hence, the MMS cost is T.

Conversely, if we know that the MMS cost is T, we can show that a 3-Partition exists. Take an arbitrary MMS allocation A. Consider the leaves in the bundle of an arbitrary agent $j \in [k]$, i.e., vertices of the form $v_i^{s_i} \in A_j$ for some $i \in [3k]$. Observe that to service each such leaf, agent j has to traverse s_i edges and these costs are summed when the agent services multiple leaves. Now, we know that the total cost of j is at most T, i.e., $c(A_j) \le T$. Thus, the sum of integers corresponding to the leaves serviced by each agents is at most T. Hence, the leaves serviced by agents give us a desired partition P. Consequently, finding the MMS cost is NP-hard.

For part ii), observe that in the constructed instance from part i), in an MMS allocation A the cost of the agent that is the worst off, must be equal to MMS cost, i.e., $\max_{i\in[n]} c(A_i) = \text{MMS}_i(I)$. This is specifically true as the cost functions are identical. Hence, if we had a polynomial time algorithm for finding an MMS allocation, we would be able to find MMS cost by looking at the maximum cost of an agent. Consequently, we could decide a 3-partition exists. As a result, finding an MMS allocation is NP-hard. □

4. Characterizing fair and efficient solutions

The possible incompatibility of fairness and efficiency in our setting was established in Example 2. Recall that Example 2 gives an instance which does not admit an EF1 and PO allocation. In this section, we exploit the structure provided by trees to characterize the space of delivery instances for which fair and efficient allocations exist. We first discuss social optimality and then turn our attention to Pareto optimality.

[3] The original algorithm by Lipton et al. [50] also had an envy-cycle elimination step, which we do not need. Observe that as agents have identical cost functions, envy-cycles cannot arise here.

4.1. Pareto optimality

We first focus on Pareto optimality. We begin by noting that an MMS and PO allocation always exists, but the same is not true for an EF1 and PO allocation.

Proposition 4.1. *Given a delivery instance $I = \langle [n], G, h\rangle$,*

a. *an* EF1 *and* PO *allocation need not exist,*
b. *an* MMS *and* PO *allocation always exists.*

Proof. For *a*, the proof follows from the instance given in Example 2. As we have discussed earlier, in this example, no allocation is simultaneously EF1 and PO.

For *b*, note that as the cost functions are identical across the agents, a leximin optimal allocation will always be MMS and PO. As a result, an MMS and PO allocation always exists because a leximin optimal allocation always exists. □

The central result of this section is the proof that every EF1 and PO allocation will satisfy MMS as well. This comes in contrast to typical fair chore division settings, where under additive preferences EF1 and MMS are independent notions even in the presence of efficiency requirements. To this end, we first prove an insightful necessary condition for EF1 and PO allocations: in every such allocation, the pairwise difference in the costs of agents cannot be greater than 1.

Lemma 4.1. *Given a delivery instance $I = \langle [n], G, h\rangle$ and an* EF1 *allocation A, if $|c(A_i) - c(A_j)| > 1$ for some agents $i, j \in [n]$, then A is not* PO.

Proof. Given instance I, let A be an EF1 allocation. Without loss of generality, assume that A is sorted in non-increasing cost order, i.e., $c(A_1) \geq \cdots \geq c(A_n)$ (otherwise we can relabel the agents). We will show that if $c(A_n) < c(A_1) - 1$, then A is not PO, i.e., it is Pareto dominated by some allocation A' (not necessarily EF1).

For every vertex $x \in V \setminus \{h\}$, by $p(x)$ let us denote the *parent* of x in a tree G rooted in h. Also, for every agent $i \in [n]$, let $w(i)$ be the *worst* vertex in i's bundle, i.e., the vertex which on removal gives the largest decrease in cost (if there is more than one we take an arbitrary one). Formally,

$$w(i) = \arg\max_{x \in A_i} c(A_i) - c(A_i \setminus \{x\}).$$

As A is an EF1 allocation, for every agent i with maximal cost, i.e., such that $c(A_i) = c(A_1)$, we have that

$$c(A_i \setminus \{w(i)\}) \leq c(A_n) < c(A_i) - 1. \tag{1}$$

Observe that this is only possible if the parent of $w(i)$ is not serviced by i, i.e., $p(w(i)) \notin A_i$. To construct allocation A' which Pareto dominates A, we look at the agent servicing the parent of the worst vertex of agent 1, call this agent i_1. If i_1 incurs maximum cost, we look at the agent servicing the parent of the worst vertex of i_1. We continue in this manner and obtain a maximal sequence of pairwise disjoint agents $1 = i_0, i_1, \ldots, i_k$ such that $p(w(i_{s-1})) \in A_{i_s}$ and $c(A_{i_s}) = c(A_1)$ for every $s \in [k]$.

The cost incurred by the agent servicing the parent of the worst vertex of i_k, which we denote by i^* (i.e., $p(w(i_k)) \in A_{i^*}$) can create two cases. Either i^* does not incur maximum cost, i.e., $c(A_{i^*}) < c(A_1)$ (Case 1), or it already appears in the sequence, i.e., $i^* = i_j$ for some $j < k$ (Case 2).

<u>**Case 1**</u>. Consider the allocation A' which is obtained from A by exchanging the bundles of agent i_k and agent i^* with the exception of $w(i_k)$ (which continues to be serviced by i_k). See Fig. 3 for an illustration. Formally, $A'_{i^*} = A_{i_k} \setminus \{w(i_k)\}$, $A'_{i_k} = A_{i^*} \cup \{w(i_k)\}$, and $A'_t = A_t$, for every $t \in [n] \setminus \{i^*, i_k\}$. Since costs of agents in $[n] \setminus \{i^*, i_k\}$ are not affected, it suffices to prove that the cost of either i_k or i^* decreases without increasing the other's cost. To this end, observe that since parent of $w(i_k)$ belongs to A_{i^*}, adding this vertex to A_{i^*} increases the cost by 1, i.e.,

$$c(A'_{i_k}) = c(A_{i^*}) + 1. \tag{2}$$

Now, let us consider two subcases based on the original difference in costs of agents i_k and i^*.

<u>*Case 1a*</u>. If this difference is greater than one, i.e., $c(A_{i_k}) > c(A_{i^*}) + 1$, then from Equation (2) we get that

$$c(A'_{i_k}) = c(A_{i^*}) + 1 < c(A_{i_k}).$$

Hence, the cost of i_k decreases.

For i^*, from Equation (1), we have that

$$c(A'_{i^*}) = c(A_{i_k} \setminus \{w(i_k)\}) \leq c(A_n) \leq c(A_{i^*}),$$

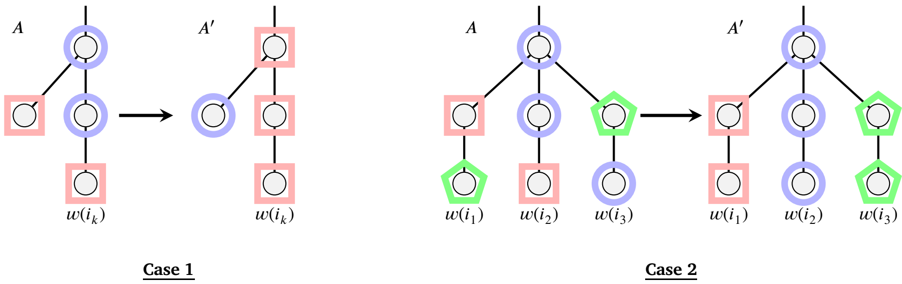


Fig. 3. Illustrating the proof of Proposition 4.1. In Case 1, i_k (red squares) exchanges its bundle with i^* (blue circles) except for $w(i_k)$. In Case 2, i_1, i_2, and i_3 (pentagons, squares, and circles, resp.) swap their worst vertices along the cycle. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

so agent i^* does not suffer from the exchange. Consequently, when $c(A_{i_k}) > c(A_{i^*}) + 1$, A' Pareto dominates A.

Case 1b. Otherwise, the difference in costs of i_k and i^* is exactly one, i.e., $c(A_{i_k}) = c(A_{i^*}) + 1$. Recall from Equation (2) that $c(A'_{i_k}) = c(A_{i^*} + 1) = c(A_{i_k})$. Thus, the cost of i_k stays the same in A'.

Now, in order to show that A' Pareto dominates A, we need that $c(A'_{i^*}) < c(A_{i^*})$. As, $c(A_n) < c(A_1) - 1$, and $c(A_1) = c(A_{i_k}) = c(A_{i^*}) + 1$ it must be that $c(A_n) < c(A_{i^*})$. Thus, from Equation (1) we have

$$c(A'_{i^*}) \le c(A_n) < c(A_{i^*}),$$

i.e., the cost of i^* decreases under A'. As a result, even when $c(A_{i_k}) = c(A_{i^*}) + 1$, A' Pareto dominates A.

Case 2. When $i^* = i_j$ for some $j < k$, we have a cycle of agents $i^* = i_j, i_{j+1}, \dots, i_k$ such that $c(A_{i_s}) = c(A_1)$ and $p(w(i_s)) \in A_{i_{s+1}}$ for every $s \in \{j, \dots, k\}$ (we denote i_j as i_{k+1} as well for notational convenience). Here, we consider two subcases, based on whether it happens somewhere in the cycle that the parent of the worst vertex of one agent is the worst vertex of the next agent, i.e., $p(w(i_s)) = w(i_{s+1})$ for some $s \in \{j, \dots, k\}$.

Case 2a. We first look at the subcase where there does exist $s \in \{j, \cdots, k\}$ s.t. $p(w(i_s)) = w(i_{s+1})$. Here, we consider allocation A' which is obtained from A by giving $w(i_{s+1})$ to agent i_s. Formally, $A'_{i_{s+1}} = A_{i_{s+1}} \setminus \{w(i_{s+1})\}$, $A'_{i_s} = A_{i_s} \cup \{w(i_{s+1})\}$, and $A'_t = A_t$, for every $t \in [n] \setminus \{i_s, i_{s+1}\}$.

Now, the cost of agent i_{s+1} decreases as it no longer services its worst vertex. Observe that agent i_s was already servicing a child of $w(i_{s+1})$ in A. Consequently, it was visiting $w(i_{s+1})$ on the way. Hence, the cost of i_s stays the same. As the bundles of the remaining agents did not change, A' Pareto dominates A.

Case 2b. Finally, we have the case where there is no agent in the cycle for which the parent of its worst vertex is the worst vertex of the next agent. In this case we consider the allocation A' obtained from A by swapping the worst vertices of the agents along the cycle (illustrated in Fig. 3). Formally, $A'_{i_s} = (A_{i_s} \setminus \{w(i_s)\}) \cup \{w(i_{s-1})\}$ for every $s \in \{j+1, \dots, k+1\}$ and $A'_i = A_i$ for every $i \in [n] \setminus \{i_{j+1}, \dots, i_{k+1}\}$.

As each i_s is servicing the parent of the worst vertex of the previous agent in the cycle, i.e., $p(w(i_{s-1})) \in A'_{i_s}$, servicing $w(i_{s-1})$ incurs an additional cost of 1. However, from Equation (1) giving away the worst vertex decreases the cost by more than 1. Hence, the cost of each agent in the cycle decreases. The other agents' costs stay the same, thus A' Pareto dominates A. □

We can now show that EF1 and PO imply MMS.

Theorem 4.1. *Given a delivery instance $I = \langle [n], G, h \rangle$, an* EF1 *and* PO *allocation exists if and only if for any leximin optimal allocation A, $\max_{i,j \in [n]} |c(A_i) - c(A_j)| \le 1$. Further, every* EF1 *and* PO *allocation satisfies* MMS.

Proof. We begin by proving that an EF1 and PO allocation A exists if and only if for any leximin optimal allocation A, we have that $\max_{i,j \in [n]} |c(A_i) - c(A_j)| \le 1$. Observe that by Lemma 4.1, we have that under a EF1 and PO allocation A, it must be that $\max_{i,j \in [n]} |c(A_i) - c(A_j)| \le 1$. We shall now that A is necessarily leximin optimal.

Without loss of generality, let A be s.t. $c(A_1) \ge \dots \ge c(A_n)$. For contradiction, assume that there exists A' that leximin dominates A. As the agents have identical costs and we are only comparing for leximin domination, we can assume without loss of generality that $c(A'_1) \ge \dots \ge c(A'_n)$. Thus, there exists $i \in [n]$ such that $c(A'_i) < c(A_i)$ and $c(A'_j) = c(A_j)$ for every $j \in [i-1]$.

Fix an agent $j \in [n]$ such that $j > i$ and thus $c(A_j) \le c(A_i)$. From Lemma 4.1 we know that it must be that either $c(A_j) = c(A_i)$ or $c(A_j) = c(A_i) - 1$. Recall that A' is also sorted and thus $c(A'_j) \le c(A'_i) < c(A_i)$. As a result, it must be that $c(A'_j) < c(A_i) - 1 \le c(A_j)$. Thus, A' also Pareto dominates A, which contradicts the assumption that A is PO.

Now for the converse, let A be a leximin optimal allocation such that $\max_{i,j\in[n]} |c(A_i)-c(A_j)| \le 1$. Clearly, A is PO. We shall now show that A is EF1. Fix $i, j \in [n]$, arbitrarily. We shall show that i is EF1 towards j. Firstly, if $c(A_i) \le c(A_j)$, clearly i does not envy j. From Lemma 4.1, the only remaining case is that $c(A_i) = c(A_j) + 1$. Choose a leaf x in the graph $G|A_i$. That is $x \in A_i$ is such that $c(A_i \setminus \{x\}) \le c(A_i) - 1 = c(A_j)$. Consequently, A is EF1.

As a result, an EF1 and PO allocation will always be leximin optimal. Hence, whenever an EF1 and PO allocation exists, it must also be MMS. □

Finally, by modifying the proof of Theorem 3.1 we show that deciding if there exists a PO allocation that is EF1 is also NP-hard. Also, we note that hardness for MMS and PO is directly implied by Theorem 3.1.

Proposition 4.2. *Given a delivery instance $I = \langle [n], G, h\rangle$, checking whether there exists an* EF1 *and* PO *or finding an* MMS *and* PO *allocation is* NP-hard.

Proof. We use the reduction from Theorem 3.1 to prove this result. Firstly, consider an MMS and PO allocation.

MMS and PO. Observe that the MMS allocations in the case when the original instance does admit a 3-Partition must also be PO. Hence, finding an MMS and PO allocation is NP-hard.

EF1 and PO. For the same reduction, we now show hardness of EF1 and PO allocations. Recall that from Lemma 4.1 for any EF1 and PO allocation, $A = (A_1, \dots, A_n)$, for any i, j, it must be that $|c(A_i) - c(A_j)| \le 1$. As $\sum_{i=1}^n c(A_i) = nT$, there exist i s.t. $c(A_i) < T$ if and only if there exists i' such that $c(A_{i'}) > T$. Thus, $c(A_{i'}) - c(A_i) \ge 2$ and for such an i, EF1 is violated. Consequently, an EF1 and PO allocation must give cost T to all agents.

Hence, an EF1 and PO allocation exists if any only if there exists a 3-Partition. Consequently, finding an EF1 and PO allocation when one exists is NP-hard. □

4.2. Social optimality

We now give more specific characterizations for fair and socially optimal allocations. Recall that in every socially optimal allocation, each edge is traversed by a unique agent, thus the sum of all agents' costs is m (the number of edges in a tree with $m+1$ vertices). This is a very demanding condition, hence it is not surprising that it may be impossible to satisfy it together with some fairness requirement. We show that checking if there exists a social optimal and fair allocation is computationally hard.

Proposition 4.3. *Given a delivery instance $I = \langle [n], G, h\rangle$, an* SO *allocation that satisfies* EF1 *or* MMS *need not exist. Moreover, checking whether an instance admits such an allocation is* NP-hard.

Proof. Example 2 shows that there can exist instances where no SO allocation satisfies MMS or EF1. Now recall the proof of Theorem 3.1 and Proposition 4.2. We find that the same reduction helps us prove the intractability of finding fair and SO allocations. We shall show that in the instance constructed in the reduction, a fair and SO allocation exists if and only if there exists a 3-Partition.

We have already shown that finding fair and PO allocations whenever they exist is NP-hard in Proposition 4.3. Using the same reduction (which is the one Theorem 3.1), we can show that in the instances constructed, we can show that any PO allocation must be SO. This will directly imply that finding a fair and SO allocation is NP-hard.

Let A be an allocation that is not SO. Thus, there must exist a vertex v_j^t that is visited by multiple agents. Clearly, v_j^t is not a leaf.

Recall that the reduction constructs a spider graph, where we create a branch of s_j vertices for each s_j in the given 3-Partition instance. The vertices on branch j are labeled $\{v_j^1, \cdots, v_j^{s_j}\}$ with $v_j^{s_j}$ being the leaf of the branch.

Let $v_j^{s_j}$ be serviced by agent i. Consider allocation A' which is identical to A with the exception that i services all of $v_j^1, \cdots, v_j^{s_j}$. Clearly, the cost of i is the same as that in A, but the costs of all other agents either decrease or remain the same. As multiple agents, including i, visit v_j^t, the cost of at least one agent reduces. Consequently, A' Pareto dominates A. Thus, every PO allocation in our constructed graph must be SO.

Hence, any polynomial time algorithm to find an MMS and SO allocation or find an EF1 and SO allocation when one exists, would also find the corresponding PO allocations and solve 3-Partition. Thus, we have the proposition. □

Despite the computational hardness result, we exploit the tree structure to characterize instances with such allocations.

Proposition 4.4. *Given a delivery instance $I = \langle [n], G, h\rangle$, an allocation A is SO, if and only if, every branch, B, outgoing from h, is fully contained in a bundle of some agent $i \in [n]$, i.e., $B \subseteq A_i$.*

Proof. Observe that the sum of the costs of all agents can never be smaller than the number of edges in the graph, i.e., $|E| = m$. Indeed, since every vertex has to be serviced by some agent, each edge must appear in $G|_{A_i}$ for some $i \in [n]$. Moreover, if every branch outgoing from the hub is fully contained in some bundle, every edge appears in $G|_{A_i}$ for exactly one $i \in [n]$. Hence, in every allocation satisfying the assumption, the total costs is equal to m and thus the allocation is SO.

Now, consider an allocation in which there exists a branch, B, and agents $i \neq j$ such that $B \cap A_i \neq \emptyset$ and $B \cap A_j \neq \emptyset$. Let u be a vertex in B connected to the hub, h. Then, edge (h, u) appears in both $G|_{A_i}$ and $G|_{A_j}$. Since every other edge appears in $G|_{A_k}$ for at least one $k \in [n]$, we get that the total cost is greater than m. Thus, such an allocation is not SO. □

Theorem 4.2. *Given a delivery instance $I = \langle [n], G, h \rangle$:*

a. *An* EF1 *and* SO *allocation exists if and only if there is a partition $(P_1, \dots, P_n)$ of branches out of h such that $\sum_{B \in P_i} |B| - \sum_{B \in P_j} |B| \leq 1$, for every $i, j \in [n]$,*
b. *An* MMS *and* SO *allocation exists if and only if there is a partition $(P_1, \dots, P_n)$ of branches outgoing from h such that $\sum_{B \in P_i} |B| \leq MMS_i(I)$ for every $i \in [n]$.*

Proof. From Proposition 4.4, we know that each SO allocation must be a partition of whole branches outgoing from h. Hence, *b* follows from the definition of MMS.

For *a*, observe that if an agent's bundle consists of a union of whole branches, then removing a vertex from its bundle reduces the cost of the agent by 1, if the vertex is a leaf, or by 0, otherwise. Hence, in order to achieve EF1 the costs of agents cannot differ by more than one. □

By Proposition 4.3, checking the conditions in the above theorem is NP-hard. However, we develop a polynomial-time verifiable necessary condition for EF1 and SO existence using the notion of the *center* of the graph (which was studied in computational social choice [63] and in theoretical computer science in general [33]).

Definition 6. The *center* of a tree $G = (V, E)$ is a set of vertices $C = \arg\min_{v \in V} \sum_{u \in V} \text{dist}(u, v)$, where $\text{dist}(x, y)$ is the length of a shortest path from x to y.

Proposition 4.5. *Given a delivery instance $I = \langle [n], G, h \rangle$, there exists an* EF1 *and* SO *allocation only if the hub is in the center of the tree.*

Proof. Let us show that if for some vertex v, there exists a partition of the branches outgoing from v such that the differences between the number of vertices in each bundle are equal or less than 1, then v has to be in the center. In conjunction with Theorem 4.2a, this will imply the thesis.

For contradiction, assume that there exists a tree $G = (V, E)$ and vertex v such that there is a partition of the branches outgoing from v such that the difference in the number of vertices in each pair of bundles is equal or less than 1, but at the same time there exists $u \in V$ such that

$$\sum_{w \in V} \text{dist}(w, v) > \sum_{w \in V} \text{dist}(w, u).$$

Let B, be the branch outgoing from v that contains vertex u. From the fact that the branches outgoing from v can be partitioned in such a way that the number of the vertices in each part is at most equal to one plus the number of vertices in each other part, we get that in particular $|V(B)| \leq |V \setminus V(B) \setminus \{v\}| + 1$. This implies that

$$|V(B)| \leq |V \setminus V(B)|. \tag{3}$$

Let $v = v_0, v_1, \dots, v_k = u$ be the path from vertex v to vertex u. Since sum of distances to v is greater than sum of distances to u, there has to be $i \in [k]$ such that

$$\sum_{w \in V} \text{dist}(w, v_{i-1}) > \sum_{w \in V} \text{dist}(w, v_i).$$

For arbitrary $w \in V$ consider the difference $\text{dist}(w, v_{i-1}) - \text{dist}(w, v_i)$. Since v_{i-1} and v_i are adjacent vertices, this difference has value 1, if w is closer to v_i than v_{i-1}, or -1, otherwise. Observe that every vertex in $V \setminus V(B)$ is closer to v_{i-1} than v_i. Thus, from Equation (3) we get that

$$\sum_{w \in V} \text{dist}(w, v_{i-1}) - \sum_{w \in V} \text{dist}(w, v_i) \leq |V(B)| - |V \setminus V(B)| \leq 0,$$

which is a contradiction. As a result, there cannot exist a vertex v that is not the center of the graph but there is a partition of branches outgoing from v such that the absolute difference in the number of vertices in each pair of bundles is at most one. □

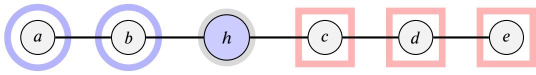


Fig. 4. An example graph with an EF1 and PO allocation marked. There is no EFX and PO allocation for this graph and two agents.

4.3. EFX allocations

We now consider a fairness notion that is stronger than EF1, but is still a relaxation of envy-freeness is *envy-freeness up to any item* (EFX). EFX has proven to be one of the most enigmatic fairness notions with the existence of EFX allocations being an open problem in most settings.

In our setting, allocation A is EFX if for every pair of agents $i, j \in [n]$ and every vertex $x \in A_i$, it holds that $c(A_i \setminus \{x\}) \le c(A_j)$. For chore division settings, Barman et al. [11] have shown that when allocating chores with identical monotone costs, an EFX allocation always exists and can be computed in pseudo-polynomial time. Observe that our setting is one where each vertex is a chore, and costs are identical submodular functions. Consequently, an EFX allocation must always exist. Further, in our case the costs of agents are polynomial with respect to the size of the input. Thus, the algorithm given by Barman et al. [11] runs in polynomial time.

While EFX allocations always exist in our setting, efficient EFX allocations need not. This can be seen clearly as an EFX allocation must also satisfy EF1 and efficient EF1 allocations need not exist. Although EFX is more restrictive than EF1, it seems less compatible with efficiency in delivery settings. To see this, observe that in an EFX allocation any envious agent, i, cannot posses a vertex, x, that is on a path to another of its vertices. Otherwise, removing x from the bundle of i would not decrease i's cost, which would contradict EFX. In fact, we show that for large classes of graphs, EFX combined with some efficiency requirement is as restrictive as envy-freeness.

Theorem 4.3. *Given an instance $I = \langle [n], G, h \rangle$ such that $m \ge n$ and an* EFX *allocation A, if*

a. *the hub is a parent of at most one leaf and A is* SO *or,*
b. *every vertex is a parent of at most one leaf and A is* PO*,*

then A satisfies EF.

Proof. Consider an arbitrary instance $I = \langle [n], G, h \rangle$ and an EFX allocation A. We shall establish both conditions a and b, separately.

Part a. For a, let us assume that A is both EFX and SO and I is such that h is the parent of at most one leaf. Now, assume for contradiction that there exists an agent, $i \in [n]$, that envies another agent. Observe that since $m \ge n$, we must have that $c(A_i) > 1$. Since the hub is the parent of at most one leaf, there must be a vertex in $x \in A_i$ that is not a leaf adjacent to h. As A is SO, then from Proposition 4.4 the whole branch containing x is given to i. Let y be a vertex of this branch adjacent to h. The branch has more then one vertex, so removing y does not decrease the cost of agent i, i.e., $c(A_i \setminus \{y\}) = c(A_i)$. But that contradicts the EFX.

Part b. For b, we assume that I is such that each vertex in G is the parent of at most one leaf and A is an EFX and PO allocation. We shall first show that all agents that envy others must only service leaves. Assume otherwise, i.e., i is envious and $x \in A_i$ is not a leaf. If there exists a vertex, y, that is a descendant of x in rooted tree (G, h), then removing x would not decrease the cost of agent i (as x is visited by i either way). But this violates EFX. On the other hand, if A_i does not contain any descendants of x, then let us denote arbitrary child of x by z and by j an agent that serves z, i.e., $z \in A_j$. Observe that removing vertex x from the bundle of i and giving it to j would decrease the cost of i, but not increase the cost of j (as it visits x either way) nor any other agent (as their bundles do not change). Hence, this constitutes a Pareto improvement, which violates PO.

Now, assume for contradiction that there exists an agent, $i \in [n]$, that envies another agent. From the previous paragraph, we know that A_i consists of leaves only. Let us take a leaf $x \in A_i$ that is the furthest from the hub h. Let y be the parent of x and j an agent that serves it, i.e., $y \in A_j$. From the previous paragraph we know that j is not envious of any other agent. Thus, $c(A_j) < c(A_i)$. As EFX implies EF1, by Proposition 3.1, this means that $c(A_j) = c(A_i) - 1$. Now, consider allocation A' obtained from A by giving j all of i's vertices except for x and to i all vertices of j and x. Formally, $A'_i = A_j \cup \{x\}$, $A'_j = A_i \setminus \{x\}$, and $A'_k = A_k$, for every $k \in [n] \setminus \{i, j\}$. Observe that adding vertex x to bundle A_j increases its cost by 1. Thus, $c(A'_i) = c(A_j) + 1 = c(A_i)$, which means that the cost for agent i did not increase. On the other hand, since A_i contains only leaves, x is the leaf in A_i that is the furthest from the hub, and y is not a parent of any other leaf, an agent serving $A_i \setminus \{x\}$ does not have to visit y. Hence, removing vertex x from A_i decreases the cost of at least 2, i.e., $c(A'_j) \le c(A_i) - 2 = c(A_j) - 1$. Hence, the cost of agent j decreased. Since the cost of the remaining agents is unchanged, A' Pareto dominates A—a contradiction. □

While EF1 is also not always compatible with PO it is much less restrictive then EFX. In particular, Fig. 4 shows an example of an instance in which there is no EFX and PO allocation but EF1 and PO allocation does exist. Indeed, for this instance there exist two different PO allocations (up to permutation of agents), i.e., the allocation shown in Fig. 4 and the allocation in which one agent serves all vertices. Neither of them is EFX.

Finally to conclude, the discussion of EFX, we observe that as any EFX allocation will be EF1. In particular, in the reduction in Proposition 4.2, an EFX and PO allocation will exist if and only if an EF1 and PO allocation exists. Consequently, we get the following result

Algorithm 1 FindParetoFrontier(n, G, h).

$\mathcal{F} \leftarrow [(\emptyset, \ldots, \emptyset)]$
for $u \in$ children of h **do**
  $T_u \leftarrow$ a subtree rooted in u
  $\mathcal{F}' \leftarrow$ FindParetoFrontier(n, T_u, u)
  for $A \in \mathcal{F}'$ **do** add u to A_1
  $\mathcal{F} \leftarrow$ maximal set of sorted combinations of allocations from $\mathcal{F}$ and $\mathcal{F}'$ such that none is weakly Pareto dominated by another within the set
end for
return $\mathcal{F}$

Proposition 4.6. *Given a delivery instance $I = \langle [n], G, h \rangle$, it is* NP-hard *to check whether there exists i) an* EFX *and* SO *allocation and ii) an* EFX *and* PO *allocation.*

Proof. We develop on the proof of Proposition 4.2 and Proposition 4.4. As any EFX allocation must satisfy EF1, analogous to the proof in Proposition 4.2 we have from Lemma 4.1 that an EFX and PO allocation to exists if and only if the given instance admits a 3-Partition. Thus, finding an EFX and PO allocation whenever it exists is NP-hard.

As shown in the proof of Proposition 4.4, in the constructed instance an allocation is PO if and only if it is socially optimal. As a result, finding an EFX and SO allocation, whenever it exists, is NP-hard. □

5. Computing fair and efficient solutions

We have shown that finding a fair and efficient allocation (or deciding if it exists) is computationally intractable. In this section, we develop a recursive algorithm for computing each combination of the fairness and efficiency notions we consider. This algorithm is XP with respect to the number of agents. That is, when the number of agents is bounded by a constant, the running time of our algorithm is polynomial.

Given a delivery instance, our algorithm (Algorithm 1) finds a set of PO allocations such that all possible cost distributions under a PO allocation are covered. We call this set, the Pareto frontier of the instance.

Definition 7. Given a delivery instance $I = \langle [n], G, h \rangle$, its *Pareto frontier* is a minimal set of allocations $\mathcal{F}$ such that for every PO allocation A there exists $B \in \mathcal{F}$ and permutation of agents π such that $c(A_i) = c(B_{\pi(i)})$, for every $i \in [n]$.

Clearly, a leximin optimal solution will be part of the Pareto frontier. Thus, we can use this algorithm to find an MMS and PO allocation. Further, we can use the tools built in Section 4 to identify if an allocation satisfying any other combination of fairness and efficiency exists.

Algorithm overview. Throughout the algorithm, we keep allocations in the list $\mathcal{F}$, with each allocation sorted in non-increasing cost order. That is, for each $A \in \mathcal{F}$, $c(A_1) \geq c(A_2) \geq \cdots c(A_n)$. The algorithm finds the Pareto frontier of a given tree rooted at h, by recursively finding the respective Pareto frontiers of the subtrees rooted at each of the children of h and combining them to generate PO allocations for the tree rooted at h. Specifically, this proceeds as follows:

First, the set $\mathcal{F}$ is initialized with just one empty allocation. Then, we look at all the vertices directly connected to the hub. For each, say u, we run our algorithm on a smaller instance where the graph is just the branch outgoing from h that u is on, and u is the hub. Let $\mathcal{F}_u$ be the Pareto frontier of this new tree, found recursively.

Now, we need to combine these Pareto frontiers to generate the Pareto frontier of the tree rooted at the hub. Before doing so, in each allocation in $\mathcal{F}_u$, we add u to the bundle of the first agent. This is a necessary step as none of the vertices connected to the hub had yet been allocated.

Finally, we combine these $\mathcal{F}_u$s, by looking at all possible sorted combinations of allocations in the lists. That is, for allocations $A \in \mathcal{F}_u$ and $A' \in \mathcal{F}_{u'}$ we consider allocations $A^\#$ where $A^\#_i = A_i \cup A'_j$ for $i, j \in [n]$. We only keep the ones that are not weakly Pareto dominated by any other (where an allocation weakly Pareto dominates another allocation if it Pareto dominates it, or all agents have equal costs in both allocations).

Example 3. We run our algorithm on the instance with 2 agents from Fig. 1. First, we run it on two smaller graphs: the vertex a as one and one on the branch containing vertices b, c, d, e, f and g. Vertex a is a leaf, so we get one allocation, i.e., $\mathcal{F} = \{(\{a\}, \emptyset)\}$. When b is the hub, we obtain two allocations: either one agent services g along with all its ancestors and the other agent services c, or one agent services everything. Thus, $\mathcal{F}' = \{(\{b,d,e,f,g\},\{c\}), (\{b,c,d,e,f,g\}, \emptyset)\}$. Finally, we combine $\mathcal{F}$ with $\mathcal{F}'$. We consider all four possible combinations. However, one of them, $(\{a,b,d,e,f,g\},\{c\})$ is Pareto dominated by another, $(\{b,c,d,e,f,g\},\{a\})$ (the cost of the first agent is the same, but for the second agent it decreases by 1). In conclusion, we return three allocations: $(\{b,d,e,f,g\},\{a,c\})$, $(\{b,c,d,e,f,g\},\{a\})$, and $(\{a,b,c,d,e,f,g\},\emptyset)$. We note that the first one is in fact MMS and PO allocation.

Now, we can prove the correctness of our algorithm. While we defer the full proof to Appendix D, we provide an outline here.

Table 2
The summary of our results on the price of EF1 and MMS. ✓ denotes that the min cost allocation is always PO, and ✗ that it may not be.

	PO	Computation	Upper Bound on PoF
Min Cost EF1	✗ (Proposition 6.1)	NP-h (Theorem 6.1)	$\frac{n(2m-n+1)}{2m}$ (Proposition 6.2)
Min Cost MMS	✗ (Proposition 6.1)	NP-h (Theorem 6.1)	$\frac{n(m-n+1)}{m}$ (Proposition 6.2)

Theorem 5.1. *Given a delivery instance $I = \langle [n], G, h\rangle$, Algorithm 1 computes its Pareto frontier and runs in time $O((n+2)!m^{3n+2})$, where $m = |E(G)|$.*

Proof (sketch). Here, we note two key observations.

For the first, consider an arbitrary instance $\langle [n], G, h\rangle$, and a smaller one obtained by taking a subtree rooted in some vertex $u \in V$, i.e., $\langle [n], T_u, u\rangle$. Then, every PO allocation A in the original instance, must be still PO when we cut it to the smaller instance (i.e., we remove vertices not in T_u). Otherwise, a Pareto improvement on the cut allocation, would also be a Pareto improvement for A. Hence, by looking at all combinations of PO allocations on the branches outgoing from h, we obtain all PO allocations in the original instance.

The second observation is that we do not need to keep two allocations that give the same cost for each agent. The maximum cost of each agent is m. Hence, we will never have more than $(m+1)^n$ different allocations in the list (in fact, since we keep all allocations sorted in non-increasing cost order, this number will be much smaller). Thus, we can combine two frontiers efficiently. □

We can use Algorithm 1 to obtain the desired allocations.

Theorem 5.2. *There exists an XP algorithm parameterized by n, that given a delivery instance $I = \langle [n], G, h\rangle$, computes an* MMS *and* PO *allocation, and decides whether there exist* MMS *and* SO*,* EF1 *and* PO*, and* EF1 *and* SO *allocations.*

Proof (sketch). Here, we focus on MMS and PO allocations (see the Appendix D for the other notions). By Theorem 5.1, we can obtain a Pareto frontier for every instance. Also, from the proof of Theorem 5.1, we know that a Pareto frontier contains at most $(m+1)^n$ allocations. Thus, we can search through them to find the leximin optimal one, which will be MMS and PO as well. □

We note that the proof for EF1 and PO here relies on Theorem 4.1.

6. Price of fairness

In this section, we study the *price of fairness*, i.e., the loss to the aggregate cost incurred by requiring allocations to be EF1 or MMS when delivery tasks are located on the vertices of a tree. Specifically, we show tight bounds for the ratio of the aggregate cost of a fair allocations to the cost of an SO allocation. The price of fairness concept has been well studied in fair division literature [10,17,65] and merits exploration in delivery settings as well.

Formally, price of fairness is defined as the ratio of the minimum total cost of an allocation satisfying the given notion to the minimum total cost of any allocation.

Definition 8 *(Price of Fairness).* Given a fairness concept F and an instance $I = \langle [n], G, h\rangle$, the price of F is given as:

$$\mathrm{PoF}(I) = \frac{\min_{A\in\Pi^n:\, A \text{ satisfies } F} \sum_{i\in[n]} c(A_i)}{\min_{A\in\Pi^n} \sum_{i\in[n]} c(A_i)}$$

Table 2 summarizes our results on the existence and computation of minimum cost fair allocations and the corresponding price of fairness. We first remark on the efficiency of minimum cost fair solutions.

Proposition 6.1. *Given a delivery instance $I = \langle [n], G, h\rangle$, we have that*

- *A minimum cost* MMS *allocation must be* PO.
- *A minimum cost* EF1 *allocation need not be* PO

Proof. We first consider minimum cost MMS allocations. Given a delivery instance $I = \langle [n], G, h\rangle$, let $A = (A_1, \cdots, A_n)$ be a minimum cost MMS allocation. We shall show that it must be PO.

For contradiction, let $B = (B_1, \cdots, B_n)$ be an allocation that Pareto dominates A. That is, for all $i \in [n]$, $c(B_i) \le c(A_i)$ and there exists $i^* \in [n]$ s.t. $c(B_{i^*}) < c(A_{i^*})$. As A is MMS, this implies that B must also be MMS. Further, due to Pareto domination, B must

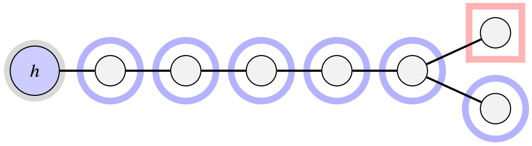


Fig. 5. An example with $m = 7$ and $n = 2$. A minimum cost MMS allocation is depicted. Here PoMMS $= \frac{12}{7} = \frac{2\cdot(7-2+1))}{7}$.

have strictly lower social cost than A. Clearly, this contradicts the fact that A is a minimum cost MMS allocation. Consequently, for any delivery instance, a minimum cost MMS allocation must be PO.

The same is not true for EF1 as an EF1 and PO allocation need not to exist (Proposition 4.1) but minimum cost EF1 allocations always exist. □

We now show that it is intractable to find minimum cost fair allocations.

Theorem 6.1. *Given a delivery instance $I = \langle [n], G, h\rangle$, computing a minimum cost* MMS *allocation or a minimum cost* EF1 *allocation is* NP-hard.

Proof. We first consider minimum cost MMS allocations. Clearly, whenever an MMS and SO allocation exists, it will also be a minimum cost MMS allocation. Thus, an algorithm that always finds a minimum cost MMS allocation would find an MMS and SO allocation, whenever it exists. Thus, finding a minimum cost MMS allocation is NP-hard.

For the case of minimum cost EF1 allocations, recall the reduction in Proposition 4.2. In the instance constructed, any EF1 and PO allocation will be SO. Consequently, it would be a minimum cost EF1 allocation. Hence, finding a minimum cost EF1 allocation is NP-hard. □

In the following proposition we identify the tight bounds for the price of EF1 and MMS.

Proposition 6.2. *Given a delivery instance $I = \langle [n], G, h\rangle$, it holds that* PoMMS$(I) \leq \frac{n(m-n+1)}{m}$ *and* PoEF1$(I) \leq \frac{n(2m-n+1)}{2m}$. *Both these bounds are tight.*

Proof. In delivery instances, the minimum total cost of any allocation, fair or otherwise, will be achieved by socially optimal allocations that have social cost m. Thus, for price of fairness in delivery instances, the denominator of the fraction will always be m. We first consider upper bounds on the price of MMS.

Price of MMS. Let $I = \langle [n], G, h\rangle$ be an arbitrary delivery instance and A a minimum cost MMS allocation. In any MMS allocation, each agent has cost at most the MMS threshold. In the worst case, the number of vertices that are visited by multiple agents are maximized. We have from Proposition 6.1 that a minimum cost MMS allocation must be PO. In a PO allocation, each agent must service at least one leaf. Otherwise, all of the vertices serviced by it can be given to the agents serving the descendants of these vertices decreasing the cost for this agent to 0 and incurring no additional cost on any other agent, which would be a Pareto improvement.

In the worst case, all non leaf vertices are visited by all agents and each agent additionally services a single leaf. This would mean that there are n leaves, and each agent visits the $m-n$ internal vertices and 1 leaf. This gives a social cost of $n(m-n+1)$. Thus, we have that

$$\mathrm{PoMMS}(I) \leq \frac{n(m-n+1)}{m}$$

To show that this bound is tight for every m and n, we consider the graph that is the hub and n-armed star connected by a path of length $m-n$ (see Fig. 5 for an illustration). Formally,

$$G = (\{h, v_1, \dots, v_m\}, \{(h, v_1), (v_1, v_2), \dots, (v_{m-n-1}, v_{m-n}), (v_{m-n}, v_{m-n+1}), \dots, (v_{m-n}, v_m)\}).$$

Observe that in an MMS allocation, each vertex $v_{m-n+1}, \dots, v_m$ has to be served by a different agent (otherwise the cost of an agent serving two or more of such vertices would be greater than $m-n+1$, which is the value of the MMS). Thus, the cost of every MMS allocation matches the established upper bound.

Price of EF1. Recall that the price of EF1 is the ratio of the social cost of the minimum cost EF1 allocation and m. In trying to give an upper bound on this, we again aim to maximize the number of vertices that are visited by multiple agents, conditioned on the allocation having minimum cost over all EF1 allocations.

Let $I = \langle [n], G, h\rangle$ be an arbitrary instance and A an arbitrary minimum cost EF1 allocation. We shall prove by induction that for every $k \in [n]$ there exists a set of k vertices, S_k, such that $V \setminus S_k$ induces a connected graph and the vertices in S_k are visited by at most k agents, i.e., $|\{i \in [n] : V(G|A_i) \cap S_k \neq \varnothing\}| \leq k$. For $k = 1$, observe that there has to exist at least one leaf in graph G and the leaf is visited only by the agent that serves it. Also, removing this leaf retains a connected graph. Hence, the induction basis holds.

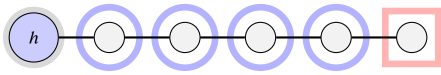


Fig. 6. An example with $m = 5$ and $n = 2$. A minimum cost EF1 allocation is depicted. Here PoEF1 $= \frac{9}{5} = \frac{2\cdot(10-2+1)}{10}$.

Now, assume that the induction hypothesis holds for some $k \in [n-1]$ and consider $k+1$. We know that there exists a set S_k such that $|S_k| = k$ and graph induced by $V \setminus S$, i.e., $G' = G[V \setminus S]$ is connected. Let v be an arbitrary leaf in G'. Observe that apart from the agent serving v it can be visited only by agents serving vertices in S. Hence, v is visited by at most $k+1$ agents. Then, $S \cup \{v\}$ contains $k+1$ vertices visited by at most $k+1$ agents. Observe also that removing this set of vertices retains a connected graph. Hence, by induction, the thesis holds.

Consequently, we get that there is an agent not visiting $n-1$ vertices in S_{n-1}, an agent not visiting $n-2$ vertices in S_{n-2} and so on. This yields the following upper bound on the total cost in minimum cost EF1:

$$\text{PoEF1} \le \frac{n \cdot m - \sum_{k \in [n]}(k-1)}{m} = \frac{n \cdot m - n(n-1)/2}{m} = \frac{n \cdot (2m - n + 1)}{2m}.$$

For an example of an instance where this bound is tight for every m and n, we consider a path graph with hub at one of the ends of the path (see Fig. 6 for an illustration). Formally, let $G = (\{h, v_1, \ldots, v_m\}, \{(h, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)\})$. Observe that in order for the allocation to be EF1, each vertex $v_m, v_{m-1}, \ldots, v_{m-n+1}$ has to be served by a different agent. This yields the cost equal to the established upper bound. □

We note that the bound for the price of MMS is lower, which matches our intuition that MMS is more compatible with efficiency in our setting. Recall from Theorem 6.1 that finding a min cost MMS or EF1 allocation is NP-hard. Further, from Proposition 6.1, a min cost MMS allocation has to be PO (which is not the case for EF1). This allows us to use Algorithm 1 to calculate the price of MMS and study it in numerical experiments in Section 7.

7. Experimental results

We now present our experimental results on the running time of our algorithm, the existence of fair and efficient allocations and investigate the efficiency loss of fair solutions through their *price of fairness*. In each experiment, we generated trees, uniformly at random, based on Prüfer sequences [60] using NetworkX Python library [35].[4] For each experiment and a graph size, we sampled 1,000 trees.

We note that the experiments were conducted on Dell Latitude E5570 computer with Intel(R) Core(TM) i7-6600U CPU @ 2.60GHz 2.80 GHz processor, 16.0 GB of RAM, and Windows 10 Education operating system. We used Python version 3.10.7 with NetworkX library version 2.8.8 (the plots were created using MatPlotLib v. 3.6.2).

7.1. *Running time of Algorithm 1*

In our first experiment, we observe the running time of our algorithm to find the Pareto frontier of a given delivery instance. We run Algorithm 1 for graphs of sizes $10, 20, \ldots, 100$ and every number of agents from 2 to 6. The running times are reported in Fig. 7a (the running time for 2 agents is not reported as it would be indistinguishable from the running times for 3 agents in the picture). The power in the running time of our algorithm for $n = 6$ agents is significant, hence the sharp increase in the running time with growing graph size in this case.

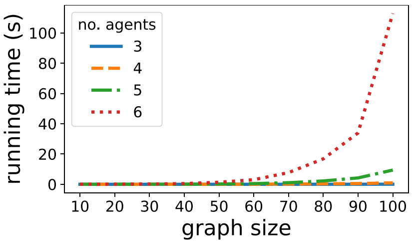
7.2. *Existence of fair and efficient allocations*

In this experiment, we check how often fair and efficient allocations exist for randomly generated trees. As we have seen, MMS and PO allocations always exist. Consequently, we focus on the three other combinations of fairness and efficiency.

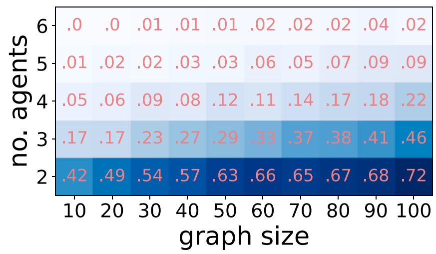
First, we checked how often there exists an EF1 and PO allocation. To this end, we generated trees of sizes $10, 20, \ldots, 100$ and for each tree we run Algorithm 1 for each number of agents from 2 to 6. Based on the output, we checked the number of trees that admit an EF1 and PO allocation. As shown in Fig. 7b, the probability of finding an EF1 and PO allocation increases steadily when we increase the size of the graph, but drops sharply when we increase the number of agents. Intuitively, on larger graphs we have more flexibility in how we fairly and efficiently split the vertices. However, when there are more agents, it may still be difficult to satisfy fairness for each of them. We repeat a similar experiment for EF1 and SO as well as MMS and SO allocations.

The results for EF1 and SO allocations are presented in Fig. 7c. There, we see a sharp decrease in probability with the increase in either number of agents or the size of a graph. For the former it is clear, as with larger number of agents it is difficult to be fair to all of them. For the size of a graph, recall Proposition 4.5 in which we have shown that EF1 and SO allocation exists only if the hub is in the center of a tree. Moreover, the center of a tree always contains one or two vertices. Hence, with the increase in the size, the probability that the hub will be in the center decreases.

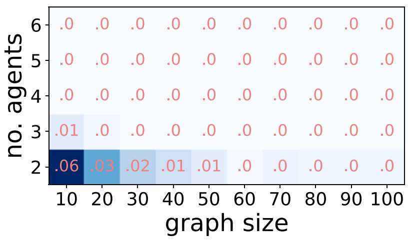
[4] The code for our experiments is available at: https://doi.org/10.5281/zenodo.11149658.



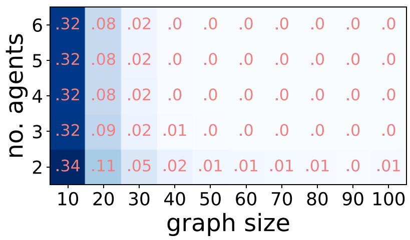
(a) Running time.



(b) EF1 and PO existence.

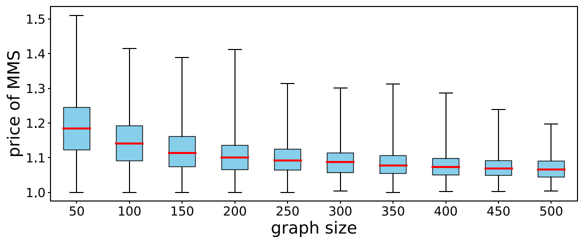


(c) EF1 and SO allocation existence.

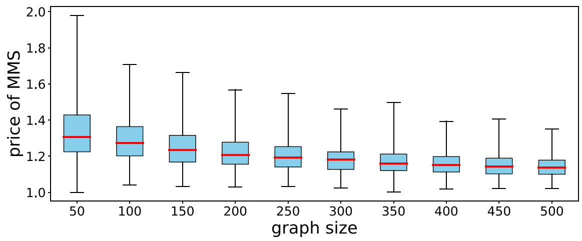


(d) MMS and SO allocation existence.

Fig. 7. Fig. 7a illustrates the average running time of Algorithm 1 on graphs of different sizes with different number of agents. Figs. 7b to 7d show the fraction of instances admitting EF1 and PO, EF1 and SO and MMS and SO allocation, respectively, for different number of agents and sizes of graphs.



(a) Price of MMS for 2 agents.



(b) Price of MMS for 3 agents.

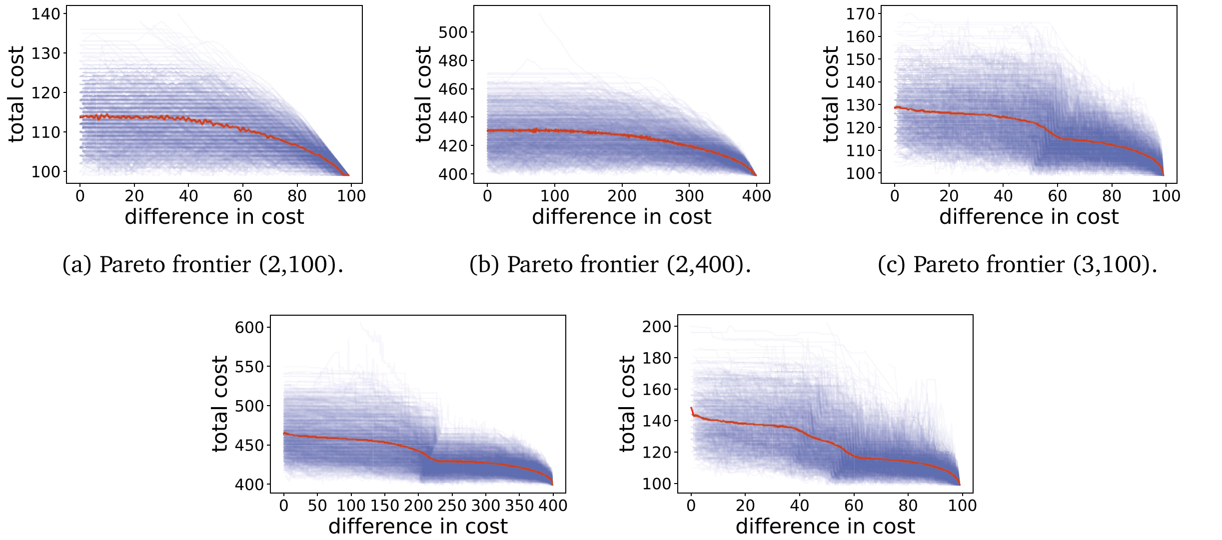
Fig. 8. Price of MMS for graphs of different sizes for 2 and 3 agents.

The results for MMS and SO allocations are presented in Fig. 7c. As can be seen in the picture, the probability does not really vary much between different numbers of agents (especially for small graphs). A plausible explanation for that phenomenon is that for MMS we only have to care to not be unfair to the worst off agent. If we have a small graph and a lot of agents, then probably some of them will not be assigned to any vertex either way. However, the number of such agents does not impact whether an allocation is MMS or not. Hence the visible effect.

7.3. *Price of* MMS

Next, we analyze the price of MMS observed in randomly generated trees. Recall that the price of fairness is the ratio of the sum of the agents' costs under a min cost fair allocation to the minimum possible social costs. From Proposition 6.1, a minimum cost MMS allocation must be PO. However, an EF1 and PO allocation need not exist. Furthermore, we do not have an exact algorithm for finding a minimum cost EF1. Recall that this problem is NP-hard, as shown in Theorem 6.1. Consequently, we only pursue the price of MMS experimentally.

In this experiment, we observe the price of MMS randomly generated trees of different sizes. In particular, we observe the change in the median price of fairness observed for each tree size. To this end, we generated trees of sizes $50, 100, 150, \dots, 500$ and looked for a min cost MMS allocation for 2 and 3 agents using Algorithm 1. The results are reported in boxplots in Figs. 8a and 8b. For 2 agents, Fig. 8a illustrates that the median price is around 1.19 for the graphs of size 50 and it steadily decreases for the larger graphs, to 1.05 for graphs of size 500. For 3 agents, Fig. 8b illustrates that the median price is around 1.3 for the graphs of size 50 and here too, it



(a) Pareto frontier (2,100).

(b) Pareto frontier (2,400).

(c) Pareto frontier (3,100).

(d) Pareto frontier (3,400).

(e) Pareto frontier (4,100).

Fig. 9. "Pareto frontier (n, m)", shows the distribution of Pareto frontiers for n agents and graphs of size m.

steadily decreases for the larger graphs, to 1.17 for graphs of size 500. These results suggest that as the size of the instance grows, the efficiency loss due to MMS becomes negligible in most cases (at least for a small number of agents). For a fixed graph size, the price of MMS seems to be higher for more agents than fewer, which is intuitive.

7.4. Pareto frontiers

In our final experiment, we analyze the trajectories of Pareto frontiers in randomly generated instances of fixed sizes in order to establish an empirical trade-off between fairness (the maximum difference in the costs) and efficiency (the total cost of agents).

We focus on graphs of size 100 or 400, with 2-4 agents (for 4 agents, we only consider graphs of size 100). For each size and the number of agents and for each of the 1000 trees generated, we look at each allocation in the Pareto frontier and report the total cost of all agents on y-axis and the maximum difference in the costs of the agents on x-axis (in case of multiple allocations in the frontier with the same total cost, we report only the one in which the maximum difference in the costs is the smallest). Then, we connect all such points for one Pareto frontier to form a partially transparent blue line. By superimposition of all 1000 of such blue lines, we obtain a general view on the distribution of Pareto frontiers. With the thick red line we denote the average total cost, for each difference in costs. We see that particular Pareto frontiers can behave very differently, but the general pattern is quite strong:

Two agents. For 2 agents and graphs of size 400, shown in Fig. 9b the total cost does not vary much when the difference in costs is between 0 and 250, however it is much steeper for the larger differences. These findings imply that it is usually not effective to focus on partial fairness as the additional total cost that we incur by guarantying complete fairness instead of partial is not that big. The plot for the size 100 and 2 agents, shown in Fig. 9a looks very similar. Again, we can say, that the total cost for agents increases sharply when we decrease the difference in costs of agents from 100, but the further we go, this increase is slower.

Three agents. We now look at the Pareto frontiers with 3 agents, shown in Fig. 9c and Fig. 9d. For both graph sizes, the plots are not as smooth as the case for 2 agents. Specifically, there is a sharp change in the total cost near the middle of the plot (where the x-axis values are between 55 and 60 for Fig. 9c and between 210 and 240 for Fig. 9d). We offer the following interpretation of this fact: in the rightmost part of the plot we begin with an allocation where the first agent is serving all of the vertices, which gives us the minimum total cost, but also the maximum difference between the costs of two agents. Then, as we move towards the left, in the direction of decreasing differences in the agents' costs, some of the vertices served by the first agent are now given to some other agent.

Initially, we do not observe a dramatic increase in the total costs as the difference in costs decreases. In the allocations in this area, the graph seems to be serviced only between the first two agents, which results in a smaller total cost for a higher difference in costs. However, when we reach the difference in cost that is around the half of the graph size, this is no longer possible. To decrease the difference further, vertices must be assigned to the third agent, which sharply increases the total cost. This leads to the sharp increase seen in so many of the individual Pareto frontiers.

Four agents. Turning to the case of four agents, shown in Fig. 9e, we observe two sharp change in the Pareto frontiers as opposed to a single change for 3 agents (one for the x-axis values between 35 and 45 and another between 55 and 60). We believe that the explanation for 4 agents is similar to that of 3 agents. Again moving from the right most point with minimum total cost and maximum difference in cost, towards the left, we see a small increase in the total cost as at first only two agents are servicing the vertices. Then around the middle (half the number of vertices in the graph), we see a sharp increase in total costs, as now three agents are servicing the whole graph. Another sharp increase is seen when the fourth agent also starts delivering items.

8. Conclusions and future work

We introduced a novel problem of fair distribution of delivery orders on graphs. We showed that even when for unweighted tree graphs, the problem of finding fair and efficient solutions proves to be pretty hard. We provided a comprehensive characterization of the space of instances that admit fair (EF1 or MMS) and efficient (SO or PO) allocations and—despite proving their hardness—developed an XP algorithm parameterized by the number of agents for each combination of fairness-efficiency notions. We found that between MMS and EF1, MMS is more compatible with efficiency: an MMS and PO allocation always exists and the worst case price of MMS is lower than that of EF1.

Since a preliminary version of this work was published [40], there has been significant interest in this model and its extensions. Some subsequent work is already available online. [36] extends our model to weighted graphs and pursues MMS and a relaxation of efficiency for different types of trees. One notable result there shows that in fact no FPT algorithm can find MMS and efficient allocations, making our XP algorithm the optimal. A different follow up considers fairness in multi stage deliveries [45]. Specifically, this is an offline model where delivery tasks are known some time in advance and fairness needs to be maintained throughout.

Even beyond these recent follow ups, our work paves the way for future research on developing approximation schemes or perhaps algorithms parameterized by graph characteristics (e.g., maximum degree or diameter) in this domain. Another natural direction is extending our results to cyclic graphs. While the current impossibility results will continue to hold, multiplicative approximations of MMS would be an important question to pursue in this space.

An alternate way to extend the current model would be by allowing agents to have heterogeneous cost functions, instead of being identical. Similarly, it would make sense to consider a setting where agents have capacity constraints on how many orders they may be allocated. Another extension of the model could involve multiple hubs.

It would also be useful to study delivery tasks settings where tasks arrive in an online fashion. Alternatively, the underlying graph itself may be dynamic and some branches may be added or removed across time. Note that any model that is more general will still experience the same intractability hurdles as our model, and thus, the problem will be quite non-trivial.

Another avenue for future work would be to do the experiments we have done for price of fairness for alternate objectives like minimum cost EF1. While the problem is NP-hard, an exponential time exact algorithm, which is yet unknown, would allow such experiments. In a similar vein, it would be useful to run the price of fairness experiments on real world transportation networks.

CRediT authorship contribution statement

Hadi Hosseini: Writing – original draft, Writing – review & editing, Validation, Investigation, Supervision. **Shivika Narang:** Writing – original draft, Writing – review & editing, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Tomasz Wąs:** Writing – original draft, Writing – review & editing, Validation, Methodology, Software, Investigation, Formal analysis.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Hadi Hosseini reports financial support was provided by National Science Foundation under grants #2144413, #2052488, and #2107173. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Additional related work

Fair division of indivisible items has garnered much attention in recent years. Several notions of fairness have been explored here with EF1 [8,9,21,24,50] and MMS [5,9,32,37,59] being among the most prominent ones. A significant amount of work has gone into studying the coexistence of these fairness notions with efficiency, through the lens of Pareto optimality [8,24,39] or that of social welfare [4,9,22].

An important result from this space is from Caragiannis et al. [24] showing that an EF1 and Pareto Optimal allocation is guaranteed to exist. In our setting we find that this is no longer the case. We use the leximin optimal allocation to provide a characterization for settings where EF1 and PO can simultaneously be satisfied. In prior work, Plaut and Roughgarden [58] use a specific type of leximin optimal allocation to find EFX allocations under subadditive valuations. We also provide a characterization of when EF1 and socially optimal (SO) allocations exist. Such an allocation would maximize the social welfare (sum of all agents' values) and also be fair. Prior work has looked at maximizing SW over EF1 allocations [9,14,22].

Submodular valuations and their subclasses have been well studied in prior work. Typically, submodular valuations require oracle access as the functions tend to be too large to be sent as an input to the algorithm. Depending on the type of oracle access, the abilities and efficiency of the algorithm changes. Subclasses of submodular valuations have also been explored in fair division literature, especially in the context of MMS [5–7,14,32,48]. Additionally, while the majority of the work in this space has looked at settings with goods, some recent work also looks at chores, either alone or in conjunction with goods [3,16,22,31,38,44].

Submodular costs have been studied in various algorithmic settings, be it combinatorial auctions, facility location or other graph problems like shortest cycles. Of these the only study to look at a cost model similar to ours is Svitkina and Tardos [66] where the authors consider submodular facility costs using a rooted tree whose leaves are the facilities. The facility costs of opening a certain set of facilities was the sum of the weights of the vertices needed to be crossed from the root to reach these facilities from the root. While this is very similar to the cost functions in our model, it is important to note that this work has no fairness considerations, only aiming to minimize the total costs. Needless to say, this can cause a large discrepancy in the workloads of different facilities.

Price of fairness has also been studied in both additive and submodular valuation settings [10,13,17,23,49,65]. Recently, Li et al. [49] provide a complete characterization of the price of envy-based fairness notions under divisible and indivisible goods. Sun et al. [65] looks at the price of EF1 and approximate MMS among other notions in chore division settings. They also show how the notions relate to the each other. In particular, they demonstrate that the price of fairness can be arbitrarily large. This is not the case in our setting where the price of fairness is bounded for a given m and n (Proposition 6.2).

A.1. Fair task allocation

Task allocation is a bit of an overloaded term and has been used outside of fair allocations of chores. For the most part, it refers to efficiently allocating tasks to robots (see [55] for an overview) and is not concerned with fairness. When pursuing fair task allocations, the tasks considered are independent of each other, unlike our model where tasks lie on a graph. Most settings studied require exactly one agent to complete a task [12,64,71,74]. Some others may require groups to be allocated [2] or allow tasks to be divided among multiple agents [18].

Fairness here is pursued in different ways. For the most part, it focuses on either i) reducing variance in the assigned loads of agents [12,18] or maximizing the minimum number of tasks assigned [71]. For the most part, no costs are considered here, with the exception of the costs of the agents commissioning the tasks [71]. Most works assume prior knowledge of upcoming tasks [12,18,71,74] and give algorithms that allocate these tasks in advance. Amador et al. [2] also looks at a setting where tasks arrive dynamically, but the tasks here are largely independent. Each task has an associated start and end time. As a result, an agent can only be allocated tasks whose start and end times do not overlap. In contrast, our model does not how such restrictions.

A.2. Fair food delivery

Work on fairness in delivery settings has been almost entirely empirical [34,54,62,69,70] and no positive theoretical guarantees have been provided. While the models pursued are not identical, one common assumption is that an agent services one request at a time. Further, the cost of servicing a set of requests is simply the sum of the costs of the individual requests. This clearly differs from our setting where agents service a large set of orders together and the cost of the set is a submodular function of the costs of the individual orders. To the best of our knowledge, no other paper has studied interdependent costs of delivery tasks.

Additionally, under prior work on food delivery, the aim is typically to achieve fairness by income distribution [34,54,62]. Specifically, the goal is reducing the variance in the income earned by the various delivery workers. In contrast, we look for fairness in the workload of the delivery agents, as there can be many settings where agents receive no or fixed monetary compensation for their work. In our setting, fairness is achieved by a fair division of the underlying graph.

A.3. Fairness with graphs

Some prior work has also looked at fair division on graphs [19,20,51,53,68]. With the exception of Misra and Nayak [51], all other papers looked at settings where items are on a graph and each agent must receive a *connected* bundle. Our model does not have such a restriction. The purpose behind the prior work done in fair division on graph is to partition the graph into n vertex-disjoint connected subgraphs. The value of the agents comes only from the vertices in their bundle, and does not depend on vertices outside of their bundle. In our case, the subgraphs will always intersect in the hub. Additionally, for us, the bundle that an agent receives need not form a connected subgraph. The cost depends on the distances of these vertices to the hub, which is not allocated. Thus, agents can incur costs from having to traverse vertices that are not in their bundles. Misra and Nayak [51] look at a setting where agents are connected by a social network and only envy those agents they have an edge to.

Recent work, initiated by Christodoulou et al. [26] introduces *graphical valuations* where agents are vertices on a graph and the edges are the items [1,15,28,41,42,52,72,73]. An incident edge can only be assigned to one of its two endpoints. It is easy to see that there is little overlap between this setting and the one introduced in this paper.

While fairness has been studied in routing problems, the aim has been to balance the amount of traffic on each edge [47,57]. This does not capture the type of delivery instances that we look at. Some work has also looked at ride-hailing platforms and aiming to achieve group and individual fairness in these settings [30,61]. However, these studies are largely experimental and do not provide any theoretical guarantees. Further, these models also do not look at models with submodular costs. In fact, they use linear programs to achieve their experimental results.

Appendix B. Graph preliminaries

A *graph* is defined by a pair $G = (V, E)$, where V (or $V(G)$) is a set of *vertices* and E (or $E(G)$) is a set of (undirected) *edges*.

We are often interested in some specific sub elements of a given graph. We shall now define those relevant to this paper. A *walk* is a sequence of vertices $(v_0, \dots, v_k)$ such that every two consecutive vertices are connected by an edge. The *length* of a walk is the number of edges in it, so the number of vertices minus 1. A *path* is a walk in which all vertices are pairwise distinct.

A walk or a path between two vertices allows us to define several useful properties. We say that a graph is *connected* if there exists a walk between every pair of vertices in V. In a connected graph, the distance between vertex u and v, i.e., $\mathrm{dist}(u, v)$, is the minimum possible length of a walk connecting u and v.

We now discuss some graph notation. A *subgraph* of graph G is any graph $H = (U, M)$ such that $U \subseteq V$ and $M \subseteq E$. A subgraph is *induced* if for every edge $(u, v) \in E$ such that $u, v \in U$ we have $(u, v) \in M$.

A *tree* is a graph in which between every pair of vertices there exists exactly one path. Some properties of a tree $G = (V, E)$ with n vertices are as follows:

- G is connected
- G is acyclic
- G contains exactly $n - 1$ edges.
- Given a subset of vertices $S \subset V$, a minimum length walk containing all the vertices in S can be found in polynomial time.

We now finally define some terms related to trees that enable convenient reference in our proofs. A *rooted tree*, (G, h), is a tree with one vertex, $h \in V$, designated as the *root*. In our paper we shall assume that our trees are rooted at the hub.

For every vertex $u \in V$, every vertex in the path from u to h except for u is called its *ancestor*. For such vertices u is a *descendant*. An ancestor (or descendant) that is connected to u by an edge is called a *parent* (or *child*).

A vertex without children is called a *leaf* (vertex); otherwise, if it does have children, it is called an *internal* vertex. A *subtree rooted in* u is an induced subgraph $T_u = (U, M)$ rooted in u, where U contains u and all its descendants. By a *branch* outgoing from h we understand a set of vertices in a subtree rooted in a child of h.

Appendix C. Relation between MMS and EF

In this section, we consider the relation between MMS and envy-freeness.

It is well-known that EF implies MMS for additive items, via Proportional share. However, EF1 does not imply MMS, even for identical additive valuations and vice versa. However, for identical additive settings an allocation that is simultaneously EFX and MMS must exist.

Proposition C.1. *Given a n identical agents and m indivisible goods M and additive valuation function $v : 2^M \to \mathbb{R}^+$, there exists an allocation A s.t. A is* EFX *and* MMS*. However, every* EFX *allocation is not* MMS *and every* MMS *allocation is not* EF1.

Proof. Consider the leximin optimal allocation A^* for the n agents over M under v. Clearly, A^* is leximin optimal. We now show that A^* must be EFX. Without loss of generality let $v(A_1^*) \geq v(A_2^*) \geq \cdots \geq v(A_n^*)$. Further, we assume that for all $g \in M$, $v(g) > 0$

For contradiction, let A^* not be EFX. That is, there exists i s.t. for some $g \in A_i$, $v(A_i^* \setminus g) > v(A_n^*)$. Fix $g \in A_i^*$ s.t. $v(A_i^* \setminus g) > v(A_n^*)$. Consider the allocation A where for $j \notin \{i, n\}$ $A_j = A_j^*$ and $A_i = A_i^* \setminus \{g\}$ and $A_n = A_n^* \cup \{g\}$. Now, by assumption $v(A_i) = v(A_i^* \setminus \{g\}) > v(A_n^*)$ and clearly $v(A_n) > v(A_n^*)$. Thus, A leximin dominates A^*. This contradicts the assumption that A^* is leximin optimal.

As a result, a leximin optimal allocation must be MMS and EFX.

EFX is not MMS. We show that EFX does not imply MMS for identical valuations (EFX is a stricter notion than EF1 that requires that on the removal of *any* item, envy-freeness must be satisfied). Take a simple example. Let $n = 2$ and $m = 4$, where $v(g_1) = v(g_2) = 2$ and $v(g_3) = v(g_4) = 1$. Here an MMS allocation would give both agents a value of 3. Now, the allocation A where $A_1 = \{g_1, g_2\}$ and $A_2 = \{g_3, g_4\}$ is EFX (and hence, EF1) but not MMS.

MMS is not EF1. Consider an example with 3 agents and 5 indivisible goods where $v(g) = 1$ for each good g. Here, for an allocation to be MMS, we need that $v(A_i) \geq 1$ for each i. Consider the allocation A where $|A_1| = 3$ and $|A_2| = |A_3| = 1$. Clearly this is MMS but not EF1. □

An analogous proof also shows that for identical additive cost functions over indivisible chores, an allocation that satisfies MMS and EF1 always exists.

Unfortunately, even EF need not imply MMS under our setting. Consider the instance and allocation depicted in Fig. 10. Both agents incur a combined cost of 3, but the MMS cost is 2.

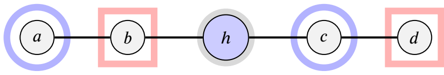

Fig. 10. An example graph showing that EF does not imply MMS. The visible allocation, i.e., $(\{a, c\}, \{b, d\})$, is EF but not MMS.

Algorithm 1 FindParetoFrontier(n, G, h).

$\mathcal{F} \leftarrow [(\emptyset, \dots, \emptyset)]$
for $u \in$ children of h **do**
  $T_u \leftarrow$ a subtree rooted in u
  $\mathcal{F}' \leftarrow$ FindParetoFrontier(n, T_u, u)
  for $A \in \mathcal{F}'$ **do** add u to A_1
  $\mathcal{F}'' \leftarrow [\,]$, empty list of allocations
  for $A \in \mathcal{F}, B \in \mathcal{F}', \pi \in \Pi_n$ **do**
    $C = \text{sort}(A \oplus \pi(B))$
    if there is no $D \in \mathcal{F}''$ s.t. $C \leq_{PO} D$ **then**
      add C to $\mathcal{F}''$
      while there is $D \in \mathcal{F}''$ s.t. $D <_{PO} C$ **do**
        remove D from $\mathcal{F}''$
      end while
    end if
  end for
  $\mathcal{F} \leftarrow \mathcal{F}''$
end for
return $\mathcal{F}$

Appendix D. Additional material for Section 6

In this section, we describe Algorithm 1 in more detail and formally prove its correctness. We further show how it can be used to find allocations satisfying a combination of fairness and efficiency notions whenever they exist.

We begin by introducing some additional notation. By Π_n let us denote the set of all permutations of set $[n]$. For a permutation $\pi \in \Pi_n$ and allocation $A = (A_1, \dots, A_n)$, by $\pi(A)$ we understand allocation $(A_{\pi(1)}, \dots, A_{\pi(n)})$, i.e., allocation where agent i receives the original bundle of agent $\pi(i)$. For two partial allocations, $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$, with disjoint set of distributed vertices, i.e., $\bigcup_{i \in [n]} A_i \cap \bigcup_{i \in [n]} B_i = \emptyset$, by $C = A \oplus B$, we understand allocation $C = (A_1 \cup B_1, \dots, A_n \cup B_n)$. For two allocations A and B we write $A <_{PO} B$, when A is Pareto dominated by B. We write $A \leq_{PO} B$, when it is *weakly Pareto dominated*, i.e., $c(A_i) \geq c(B_i)$ for every $i \in [n]$. Now, we are ready to describe Algorithm 1 with details on how we combine the two lists of allocations, $\mathcal{F}$ and $\mathcal{F}'$.

Algorithm. Throughout the algorithm, we keep allocations in the list $\mathcal{F}$, each allocation sorted in non-increasing cost order. First, we initialize it with just one empty allocation. Then, we look at vertices directly connected to the hub. For each such u, we run our algorithm on a smaller instance where the graph is just the branch outgoing from h that u is on, and u is the hub. In each allocation in the output, $\mathcal{F}'$, we add u to the bundle of the first agent. Finally, we combine the allocations returned by each child of h.

To this end, we first initialize $\mathcal{F}''$ with an empty list of allocations. We then iterate over all possible triples (A, B, π), where A and B are allocations from input lists $\mathcal{F}$ and $\mathcal{F}'$, respectively, and π is a permutation of agents $[n]$. For each such triple, we consider allocation $C = \text{sort}(A \oplus \pi(B))$, i.e., the allocations in which agent i, receives bundle $A_i \cup B_{\pi(i)}$, sorted in non-increasing cost order.

In the next step, we check if there exists an allocation D in $\mathcal{F}''$ that weakly Pareto dominates C. If this is the case, we disregard C and move to the next triple. If this is not the case, then we add allocation C to the list $\mathcal{F}''$ and remove all allocations D from $\mathcal{F}''$ that are Pareto dominated by C. We note that these operations can be performed more efficiently if we keep allocations in Res in a specific ordering, but since it is not necessary for our results, we do not go into details for the sake of simplicity. After considering all pairs and permutations, we put $\mathcal{F}''$ for $\mathcal{F}$.

Theorem 5.1. *Given a delivery instance $I = \langle [n], G, h \rangle$, Algorithm 1 computes its Pareto frontier and runs in time $O((n+2)! m^{3n+2})$, where $m = |E(G)|$.*

Proof. Let us start by showing that the output is a Pareto frontier, which we will prove by induction on the number of edges in a graph. If there is only one edge, the graph consists of the hub, h, and one vertex connected to it, say u. Then, there is only one possible allocation (up to a permutation of agents), namely, $(\{u\}, \emptyset, \dots, \emptyset)$, and it is PO. Observe that this is also the only allocation returned by our algorithm for such a graph. Hence, the inductive basis holds.

Now, assume that our algorithm outputs a Pareto frontier for every instance, in which the number of edges is smaller or equal to M for some $M \in \mathbb{N}$ and consider an instance $\langle [n], G, h \rangle$ with $M + 1$ edges. Take arbitrary PO allocation A. We will show that in the output of Algorithm 1 for this instance, $\mathcal{F}$, there exists allocation B such that $c(A_i) = c(B_{\pi(i)})$ for some permutation $\pi \in \Pi_n$.

If h has only one child, u, then observe that every agent that services some vertex in A has to visit u. Also, the agent that services u services also other vertices (otherwise giving u to an agent that services some other vertex would be a Pareto improvement). Hence, partial allocation A' obtained from A by removing u is still PO and the cost of each agent is the same in both allocations. Observe that A' is also a PO allocation in instance $\langle [n], T_u, u \rangle$. Let $\mathcal{F}'$ be the output of Algorithm 1 for instance $\langle [n], T_u, u \rangle$. Then, since T_u has M edges, from inductive assumption we know that there exists $B' \in \mathcal{F}'$ and $\pi \in \Pi_n$ such that $c(A'_i) = c(B'_{\pi(i)})$, for every $i \in [n]$. Then, let B be an allocation obtained from B' by adding u to the bundle of agent 1. Since we sort allocation so that consecutive agents have nonincreasing costs, we know that $B'_1 \neq \emptyset$. Hence, cost of each agent in B is the same as in B'. Hence, $c(A_i) = c(A'_i) = c(B'_{\pi(i)}) = c(B_{\pi(i)})$. Since B is in the output of the algorithm, the induction thesis holds.

Now, assume that h has more than one child. Let us denote them as $u^1, \dots, u^k$ and by $U^1, \dots, U^k$ let us denote the respective branches outgoing from h. Let $A^1, \dots, A^k$ be partial allocations obtained from A by restricting A to one branch from $U^1, \dots, U^k$, respectively (i.e., removing all vertices not in the branch from all of the bundles). Observe that since A is PO, each allocation $A^1, \dots, A^k$ is also PO (otherwise a Pareto improvement in A^j for some $j \in [k]$ would be a Pareto improvement also in A). With the same reasoning as in the previous paragraph, by line 6 of Algorithm 1 in the iteration of the loop for each child u^j, list $\mathcal{F}'$ contains an allocation B^j such that $c(A^j_i) = c(B^j_{\pi^j(i)})$ for some $\pi^j \in \Pi_n$ and every $i \in [n]$. Now, when we combine $\mathcal{F}$ with $\mathcal{F}'$ we consider all possible combinations of allocations in $\mathcal{F}$ and $\mathcal{F}'$ along with all possible permutations of agents. Hence, in the output of the algorithm, there will be allocation B and permutation $\pi \in \Pi_n$ such that $B_{\pi(i)} = B^1_{\pi^1(i)} \cup \dots \cup B^k_{\pi^k(i)}$ for every $i \in [n]$, unless there is some allocation D that weakly Pareto dominates B. Since $A^1, \dots, A^k$ are partial allocations of separate branches and $B^1, \dots, B^k$ as well, we have

$$c(B_{\pi(i)}) = \sum_{j \in [k]} c(B^j_{\pi^j(i)}) = \sum_{j \in [k]} c(A^j_i) = c(A_i),$$

for every $i \in [n]$. Hence, B is PO. Thus, if there is D that weakly Pareto dominates B, then $c(D_i) = c(B_i)$ for every $i \in [n]$. Either way, there exists an allocation in $\mathcal{F}$ that for corresponding agents gives the same costs as allocation A. Therefore, $\mathcal{F}$ is a Pareto frontier, which concludes the induction proof.

In the rest of the proof, let us focus on showing that the running time of Algorithm 1 is $O((n+2)!m^{3n+1})$. To this end, recall that in the nested loop of the algorithm (lines 7-15) we consider all triples (A, B, π), where A is an allocation in $\mathcal{F}$, B an allocation in $\mathcal{F}'$, and π a permutation in Π_n. Since the cost of an agent is an integer between 0 and m and at any moment we cannot have two allocation with the same cost for every agent in $\mathcal{F}$ or $\mathcal{F}'$ (because one is weakly Pareto dominated by the other), the sizes of $\mathcal{F}$ and $\mathcal{F}'$ are bounded by $(m+1)^n$. Hence, the nested loop in lines 7–15 will have at most $n!(m+1)^{2n}$ iterations. For each triple, we have to sort the resulting allocation, which can take time $n\log(n)$. Moreover, we may need to check whether resulting allocation C Pareto dominates or is Pareto dominated by all of the allocations already kept in $\mathcal{F}''$. The size of $\mathcal{F}''$ is also bounded by $(m+1)^n$ and checking Pareto domination can be done in time n. All in all, the running time of the nested loop is in $O((n+2)!m^{3n})$. Finally, observe that in the algorithm we go through the nested loop less than m times, thus final running time is in $O((n+2)!m^{3n+1})$. □

Now, let us show how we can use Algorithm 1 to find allocation satisfying certain fairness and efficiency requirements (or decide if they exist).

Theorem 5.2. *There exists an XP algorithm parameterized by n, that given a delivery instance $I = \langle [n], G, h \rangle$, computes an* MMS *and* PO *allocation, and decides whether there exist* MMS *and* SO, EF1 *and* PO, *and* EF1 *and* SO *allocations.*

Proof. Let us split the proof into four lemmas devoted to each combination of fairness and efficiency notions. We start with MMS and PO allocations.

Lemma D.1. *There exists an XP algorithm parameterized by n that for every delivery instance $\langle [n], G, h \rangle$ finds an* MMS *and* PO *allocation.*

Proof. Observe that the allocation in a Pareto frontier that leximin dominates all other allocations in the frontier is a leximin optimal allocation. Since Algorithm 1 returns a Pareto frontier, and Pareto frontier contains at most $O(m^n)$ allocations, we can find such an allocation in XP time by Theorem 5.1. Therefore, the lemma follows from the fact that leximin optimal allocation is MMS an PO. □

Next, let us move to EF1 and PO allocations.

Lemma D.2. *There exists an XP algorithm parameterized by n that for every delivery instance $\langle [n], G, h \rangle$ decides if there exists an* EF1 *and* PO *allocation and finds it if it exists.*

Proof. From the proof of Theorem 4.1 and Lemma 4.1 we know that an EF1 and PO allocation is leximin optimal and the pairwise differences in the costs of agents are at most 1. Observe that it is an equivalence, i.e., a leximin optimal allocation in which the pairwise differences in the costs of agents are at most 1 is EF1 and PO (PO because of leximin optimality, and EF1 because for every agent there exists a vertex that removed from a bundle of this agent reduces the cost by at least 1). By Lemma D.1, we can find a leximin optimal allocation in XP time with respect to n. Therefore, it remains to check if the pairwise differences in the costs of agents are at most 1. If it is true, then we know that this allocation is EF1 and PO. Otherwise, we know there is no EF1 and PO allocation. □

Now, let us consider MMS and SO allocations.

Lemma D.3. *There exists an XP algorithm parameterized by n that for every delivery instance $\langle [n], G, h \rangle$ decides if there exists an* MMS *and* SO *allocation and finds it if it exists.*

Proof. For MMS and SO, observe that an SO allocation is also necessarily a PO allocation. Having Pareto frontier from Theorem 5.1 and MMS value from Lemma D.1, we can simply check all allocation in the frontier if they satisfy MMS and SO. □

Finally, we focus on EF1 and SO allocations.

Lemma D.4. *There exists an XP algorithm parameterized by n that for every delivery instance $\langle [n], G, h\rangle$ decides if there exists an* EF1 *and* SO *allocation and finds it if it exists.*

Proof. For EF1 and SO, observe that Theorem 4.1 also implies that EF1 and SO allocation is MMS. From Lemma D.3 we know that we can find all MMS and SO allocations in Pareto frontier in XP time with respect to n. Hence, we can check if any one of them satisfies also EF1 and this will give us the solution. □

From Lemmas D.1 to D.4 we obtain the thesis. □

Appendix E. Extensions to weighted/cyclic graphs and future work

This work is the first to theoretically consider standard fairness objectives for delivery tasks. Trees, as we discuss in the introduction are an important setting to consider, not the least because they allow for tractable routing solutions. For this reason, we explore and exploit the structure of trees in order to get our results.

It is important to note that several of our results extend to weighted cyclic graphs as well. In particular, all our hardness results and the existence of EF1 allocations extend. Further, our hardness results show that when there are edge weights, looking for MMS allocations or allocations that satisfy any combination of fairness and efficiency is *strongly* NP-hard. Thus our results have strong implications for delivery settings with weights and/or cycles in the graph. We now remark on each of these settings separately and remark on the avenues for future work.

E.1. Weighted trees

The setting of weighted trees subsumes the standard fair chore division setting with identical additive costs. Any such chore division setting can be captured by a weighted star graph with a separate hub and a vertex for every chore, and the edge between the hub and the vertex has weight equal to the cost of the chore. Consequently, the setting is significantly more difficult than typical chore division settings.

As previously mentioned, all the results in Section 3 extend to weighted trees. Here, oracle access is not explicitly required to find EF1 allocations, unlike the case of cyclic graphs. Further, Algorithm 1 can easily extend to weighted trees, however the running time of such algorithm will have the sum of the edge weights across the tree in place of m.

Proposition E.1. *Given a delivery instance with a weighted tree, an* EF1 *allocation can be found in polynomial time.*

An important direction of further study would be to find a faster algorithm to find a fair and efficient algorithm, if possible. Also, studying the connection between EF1 and MMS in this space would be very interesting.

Additionally, for the setting of unweighted trees itself, our work paves the way for future research on developing approximation schemes or perhaps algorithms parameterized by graph characteristics (e.g., maximum degree or diameter) in this domain.

E.2. Cyclic graphs

While cyclic graphs are practically relevant, they bring intrinsic computational hardness. In fact, even the cost of servicing a given set of nodes in a weighted cyclic graph cannot be computed tractably.

Cyclic weighted graphs With edge weights, the fair delivery problem generalizes the traveling salesperson problem, which is hard to approximate within *any* polynomial factor. Therefore, while the envy-cycle elimination argument does guarantee the existence of EF1 allocations, to compute one would require access to a value oracle which will provide the cost of servicing a given set of nodes.

The remainder of our results rely on the graph being acyclic, hence do not extend to cyclic graphs. An important avenue of future work includes understanding if there is any connection between EF1 and MMS allocation. Further, it is relevant to note that an efficient way of verifying if a given allocation is PO may not exist in that setting (it is co-NP complete for standard settings). Hence, even a brute force approach to finding a fair and efficient allocation for cyclic graphs may take time $\Omega(m^n)$.

Data availability

We have shared a link to the code within the article.

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