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Representing states in iterated belief revision

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A B S T R A C T

Iterated belief revision requires information about the current beliefs. This information is represented by mathematical structures called doxastic states. Most literature concentrates on how to revise a doxastic state and neglects that it may exponentially grow. This problem is studied for the most common ways of storing a doxastic state. All four of them are able to store every doxastic state, but some do it in less space than others. In particular, the explicit representation (an enumeration of the current beliefs) is the more wasteful on space. The level representation (a sequence of propositional formulae) and the natural representation (a history of natural revisions) are more succinct than it. The lexicographic representation (a history of lexicographic revision) is even more succinct than them.

1. Introduction

A train runs between Rome and Viterbo. A bus runs between Rome and Viterbo. No, not exactly. The bus only runs when the train does not work. Something believed may turn wrong at later time. How to revise it is belief revision [22,21,40].

The first studies [2,22] deemed the factual beliefs like the train running and the bus running sufficient when facing new information like "either the train or the bus do not run". They are minimally changed to satisfy it. When more than one minimal change exists, they are all equally likely by default.

No actual agent revises like this. The train usually runs, except when it snows. Or not: the train line is in testing, and only runs occasionally. Beliefs are not sufficient. Their strength is necessary.

The strength of their combinations is necessary. That the train runs while the bus does not is more credible than the other way around, and both scenarios are more likely that the two services being both shut down. Beliefs are not independent to each other. Their combined strength may not be just the sum of their individual strength. The bus running is unlikely, but almost certain when the train does not work. The collection of this kind of information is called doxastic state. It tells how much each possible scenario is believed to be the case.

The simplest and most used form of doxastic state is a connected preorder between propositional models [28]. Each model stands for a possible situation; the preorder says which is more believed than which.

Even in the simplest setting, propositional logic, the models are exponentially many. Moreover, the doxastic state is not static: the revisions change it. Scenarios conflicting with new information decrease in credibility. Scenarios supported by new information increase. The problem is not only how to revise a doxastic state, but also how to store it in reasonable space. A list of comparisons "this scenario is more credible that this other one" is always exponentially longer than the number of individual beliefs. Exponential means intractable.

No actual agent holds an exponential amount of information. Artificial or otherwise: computers have limited memory; people do not memorize long lists easily.

How can a computer store a doxastic state? How do people remember a doxastic state? Not in the form of a list. Somehow else. In a short way. Maybe by some general rules, with exceptions.

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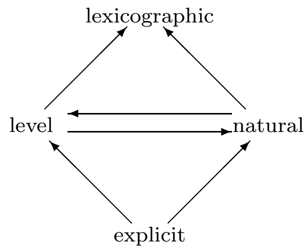

Fig. 1. Comparison of the four considered representations

How to store a doxastic state is not a new question. It showed up early in the history of iterated belief revision [44,55,18,19,57], resurfacing rarely from time to time [10,11,26,59,41] until attracting interest recently [23,50,46,48,4,45,49]. In spite of the many studies that concentrate on how to change the doxastic state neglecting its size [25,52,47,27,30], some solutions already exist.

Beside listing the strength of beliefs in each possible scenario one by one, the most common form of doxastic state is a list of logical formulae. The first is true in the most likely scenario. The second is true in the most likely among the remaining ones, and so on [55,18,57,19,26,36,41,48].

A common alternative is to not store the doxastic state itself but what creates it. The current beliefs come from what learned in the past. The past revisions make the doxastic state. Rather than strenuously compiling and storing the strength of belief in every single possible scenario, it is only computed when necessary from the sequence of previous revisions. The history is the doxastic state [44,10,11,29,43,58,24,49].

Of the many ways of changing a doxastic state [41], the two most studied ones are considered: lexicographic and natural revision [51,37,12,39,14]. They complete the list of the four representations compared:

Explicit representation: the mathematical representation of a connected preorder: a set of pairs of models; each model describes a possible scenario, a pair expresses a stronger belief in the first than in the second; every doxastic state can be represented by such a set or pairs, which may however be very large;

Level representation: a sequence of formulae; the first describes the most strongly believed scenarios, the second describes the most strongly believed remaining ones, and so on; these sequences represent every doxastic state, and may do that in less space than in the explicit representation;

Histories of natural revisions: they represent all doxastic states; they can be converted into the level representation and back without an exponential growth;

Histories of lexicographic revisions: they represent all doxastic states, and do that in the most space-saving way among the four considered methods: the others can be converted into lexicographic histories with a limited increase of size, while the inverse translation may not.

Fig. 1 shows the existence of polynomially-bounded translations between the four considered representations.

The following is a summary of the article section by section.

Section 2 formally defines the four considered representations of doxastic states and formalizes their equivalence.

Section 3 shows the equivalence classes of the connected preorders resulting from natural and lexicographic revisions. This proves that the inductive definitions of the previous section match the usual definitions in terms of equivalence classes, and also proves that both representations are translatable into the level representation.

Section 4 shows that the considered representations are universal, and completes their space efficiency comparison. Lexicographic histories are strictly more succinct than the level and natural histories representations, which compare the same and are strictly better than the explicit representation.

A comparison with the related literature is in Section 5, followed by a discussion of the results and the possible future directions of study in Section 6.

2. Definitions

Doxastic states may take several forms, such as connected preorders, rankings and systems of spheres [21]. Connected preorders [28] are studied in this article: a doxastic state is a comparison of the strength of belief in one scenario or another. It is quantitative, contrary to qualitative measures such as rankings [51] and ordinal conditional functions [56]. It says whether a scenario is more believed than another, but do say how much more; qualitative measures instead can say that a scenario is slightly more believed than another and much more than yet another.

A connected preorder is an order between propositional models that is reflexive ($I \leq I$), transitive ($I \leq J$ and $J \leq H$ entail $I \leq H$) and connected (either $I \leq J$ or $J \leq I$). They are also named total preorders.

Epistemically, $I \leq J$ means that I is a more believed scenario than J.

Mathematically, an order between propositional models is a set of pairs of models: it contains $\langle I, J \rangle$ if I is less than or equal than J, or $I \leq J$. This explicit representation of an order may take space linear in the number of models, which is exponential in the number of variables in propositional logics.

The alternative representations of an ordering considered in this article are by a formula for equivalence class and by a sequence of lexicographic or natural revisions. They all comprise a sequence of formulae. This calls for a wording simplification, where a sequence is identified with the order it represents. For example, "the lexicographic order $[S_1, \dots, S_m]$" is the order that results from the sequence of lexicographic revisions $S_m, \dots, S_1$ starting from a void doxastic state, where models are all equally believed.

The same sequence has different meanings in different representations: the natural order $[a \vee b, \neg a]$ differs from the lexicographic order $[a \vee b, \neg a]$. In the other way around, the same order is given by different sequences in different representation. For example, the lexicographic order $[a, b]$ is the same as the level order $[a \wedge b, a \wedge \neg b, \neg a \wedge b, \neg a \wedge \neg b]$. Technically, they are equivalent: $I \leq J$ holds in the first if and if it holds in the second. Equivalence is equality of $I \leq J$ for all pair of models.

2.1. The explicit order

The explicit order is the mathematical definition of an order between propositional models: a set of pairs of models. The set of all propositional models over the given alphabet is denoted by M.

Definition 1. The *explicit order* induced by $S \subseteq M \times M$ compares $I \leq_S J$ if $\langle I, J \rangle \in S$, where M is the set of all models.

A connected preorder is reflexive, transitive and connected:

reflexive: $\langle I, I \rangle \in S$ for every $I \in M$;
transitive: $\langle I, J \rangle \in S$ and $\langle J, H \rangle \in S$ imply $\langle I, H \rangle \in S$ for every $I, J, H \in M$;
connected: either $\langle I, J \rangle \in S$ or $\langle J, I \rangle \in S$ for every $I, J \in M$.

A connected preorder $S \subseteq M \times M$ is the same as a sequence of disjoint sets of models $S = [S_1, \dots, S_m]$, where every element S_i is a set of models: $S_i \subseteq M$ and $S_i \cap S_j$ if $i \neq j$. The correspondence is:

- $I \leq_S J$ if and only if $I \in M_i$, $J \in M_j$ and $i \leq j$;
- $S_i = \{ I \in M \backslash (S_1 \cup \dots \cup S_{i-1}) \mid \forall J \in M \backslash (S_1 \cup \dots \cup S_{i-1}) \, . \, I \leq J \}$

The first set S_1 comprises all minimal models according to $\leq$. The second set S_2 comprises the minimal models except for S_1. They are minimal among what remains: $M \backslash S_1$.

2.2. The level order

The explicit order takes quadratic space in the number of models, which is exponential in the number of variables. Space can often be significantly reduced by turning every set M_i into a propositional formula. This is the most used realistic representation of a doxastic state in iterated belief revision [55,18,57,19,26,36,41,48].

Definition 2 *(Level order)*. The *level order* induced by the sequence of formulae $S = [S_1, \dots, S_m]$ that are mutually inconsistent and whose disjunction is tautological compares $I \leq_S J$ if $i \leq j$ with $I \models S_i$ and $J \models S_j$.

Variants lift the condition of mutual inconsistency or tautological disjunction or add the requirement of no single inconsistent formula. In the first, i and j are the minimal indexes of formulae satisfied by I and J. In the second, the definition is added "or J does not satisfy any formula of S". The third does not require any modification. These changes are inessential:

1. the order $[S_1, \dots, S_m]$ is the same as $[S_1, S_2 \wedge \neg S_1, \dots, S_m \wedge \neg S_{m-1} \wedge \dots \wedge S_1]$, which comprises mutually inconsistent formulae;
2. the order $[S_1, \dots, S_m]$ is the same as $[S_1, \dots, S_m, \neg S_1 \vee \dots \vee \neg S_m]$, whose disjunction of formulae is tautological;
3. the order $[S_1, \dots, S_m]$ is the same as the same sequence with all inconsistent formulae removed.

2.3. The lexicographic order

The lexicographic order is what results from a sequence of lexicographic revisions [51,37,39] applied to a void doxastic state, where all models compare the same. A number of other iterated revision operators have been proved to be reducible to it [32], making it a good candidate for representing arbitrary doxastic states.

The first step in the definition of this order is the order induced by a single formula: believing a formula is the same as believing that every scenario where it is true is more likely than every scenario where it is false. Mathematically, its models are less than the others.

Definition 3. The *order induced by a formula* F compares $I \leq_F J$ if either $I \models F$ or $J \not\models F$.

The definition implies that $I \leq_F J$ holds in exactly three cases:

- $I \models F$ and $J \not\models F$ (strict order),
- $I \models F$ and $J \models F$ (equivalence, first case), or
- $I \not\models F$ and $J \not\models F$ (equivalence, second case).

The principle of the lexicographic order is that the last-coming formula makes the bulk of the ordering, separating its satisfying models from the others. The previous formulae matter only for ties. The following definition applies this principle to the condition $I \leq_S J$, where $S = [S_1, \ldots, S_m]$ is a sequence of lexicographic revisions in reverse order: S_m is the first, S_1 the last.

Definition 4. The *lexicographic order* induced by the sequence of formulae $S = [S_1, \ldots, S_m]$ compares $I \leq_S J$ if

- either $S = []$ or
  $I \leq_{S_1} J$ and either $J \not\leq_{S_1} I$ or $I \leq_R J$, where $R = [S_2, \ldots, S_m]$.

The sequence S is identified with the order, giving the simplified wording "the lexicographic order S".

The lexicographic order $I \leq_S J$ is equivalently defined in terms of the strict part and the equivalence relation of $\leq_F$:

- either $I <_{S_1} J$, or
- $I \equiv_{S_1} J$ and $I \leq_R J$, where $R = [S_2, \ldots, S_m]$.

2.4. The natural order

Like the lexicographic order is what results from a sequence of lexicographic revisions, every other revision gives a way to represent an ordering. One early and much studied such operator is the natural revision [51,12]. Along with lexicographic and restrained revision is one of the three elementary revision operators [14].

The founding principle of natural revision is to alter the doxastic state as little as possible to make the revising formula believed. A scenario becomes believed when it is one of the most believed scenario according to the formulae. The comparison is otherwise unchanged.

Definition 5 *(Natural order).* The natural order induced by the sequence of formulae $S = [S_1, \ldots, S_m]$ compares $I \leq_S J$ if either $S = []$ or:

- $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K$, or
- $I \leq_R J$ and either $J \notin Mod(S_1)$ or $\exists K \in Mod(S_1).J \not\leq_R K$,

where $R = [S_2, \ldots, S_m]$.

The simplified wording "the natural order S" stands for the order induced by the sequence S.

Being $\leq_R$ a connected preorder, the recursive subcondition $J \not\leq_R K$ is the same as $K <_R J$.

This definition implements its justification: that the minimal models of the revising formula are made minimal while not changing the relative order among the others. The next section will prove it by expressing the natural order on equivalence classes.

2.5. Different sequences, same order

The same order can be represented in different ways. The explicit order S may be the same as the level order R, the lexicographic orders Q and T and the natural order V. Same order, different representations or different sequences in the same representation. These sequences are equivalent on their induced order.

Definition 6 *(Equivalence).* Two orders S and R are equivalent, denoted $S \equiv R$, if $I \leq_S J$ and $I \leq_R J$ coincide for all pairs of models I and J.

This definition allows writing "the level order R is equivalent to the lexicographic order Q and to the lexicographic order T", meaning that the three sequences R, Q and T represent the same order.

Such statements are used when comparing two different representations, like when proving that every natural order S is equivalent to a lexicographic order R.

Sometimes, non-equivalence is easier to handle than equivalence: two sequences S and R are not equivalent if $I \leq_S J$ and $I \not\leq_R J$ or the same with S and R swapped for some models I and J. The same conditions with I and J swapped are not necessary because I and J are arbitrary.

3. Classes

Four representations of doxastic states are given in the previous section: explicit, level, lexicographic and natural. They are all defined in terms of whether $I \leq J$ holds or not.

Iterated belief revision is often defined in terms of how they change the doxastic state expressed in terms of its equivalence classes [41]. For example, natural revision moves the models of the first class having models consistent with the new information to a new, first equivalence class.

A sequence of equivalence classes is the same as the level order. These definitions say how lexicographic and natural revision change a level order. This section does that for the definitions in the previous section. It shows how to translate lexicographic and natural orders into level orders. This also proves that the definitions of the lexicographic and natural revisions match the definitions in terms of equivalence classes from the literature.

The proof scheme is:

- the natural order [] is equivalent to the level order [true] since $I \leq J$ holds for all models in both;
- the two orders are kept equivalent while adding formulae at the front of the natural order; this requires:
  - showing the order resulting from adding a single formula in front of the natural order;
  - expressing that order in the level representation.

The lexicographic representation is treated similarly. Details are in Appendix A.

3.1. From natural to level orders

The reduction from natural orders to level orders follows the scheme outlined above: the base case is a correspondence between the level order [true] and the natural order []; the induction case maintains the correspondence while adding a formula at time to the natural order. The first proof step shows the correspondence for the natural order [].

Lemma 1. *The natural order* [] *is equivalent to the level order* [*true*].

The induction step starts from the equivalence of a natural and a level order and maintains the equivalence while adding a formula at time at the front of the natural order, which is the same as a single natural revision.

This requires a property of natural orders: the models of the first formula that is consistent with a new formula are also the minimal models of the new formula according to the order.

Lemma 2. *If Q_c is the first formula of the level order Q that is consistent with the formula S_1, then $I \models S_1 \wedge Q_c$ is equivalent to $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_Q K$.*

The following lemma shows how a single natural revision by a formula S_1 changes a level order Q. Technically, "a single natural revision" is formalized as an addition to the front of a natural order. Since a natural order is a sequence of natural revisions, revising $[S_2, \dots, S_m]$ makes it $S = [S_1, S_2, \dots, S_m]$. The lemma expresses $I \leq_S J$. The expression is in terms of a natural order $[S_2, \dots, S_m]$ equivalent to a level order Q. In other words, it tells $I \leq_S J$ where S is equivalent to naturally revising Q by a formula S_1.

Lemma 3. *If $S = [S_1, S_2, \dots, S_m]$ is a natural order, Q is a level order equivalent to the natural order $[S_2, \dots, S_m]$ and Q_c is the first formula of Q that is consistent with S_1, then $I \leq_S J$ is:*

- *true if $I \models S_1 \wedge Q_c$;*
- *false if $I \not\models S_1 \wedge Q_c$ and $J \models S_1 \wedge Q_c$;*
- *same as $I \leq_Q J$ otherwise.*

The plan of the proof is to start from the equivalent natural order [] and level order [true] and to keep adding a single formula at time to the front of the first while keeping the second equivalent to it. This requires the level order that results from applying a natural revision. The previous lemma shows the order in terms of a condition equivalent to $I \leq J$. The following shows this order in the level representation.

Lemma 4. *If the natural order $[S_2, \dots, S_m]$ is equivalent to the level order $Q = [Q_1, \dots, Q_k]$, then the natural order $S = [S_1, S_2, \dots, S_m]$ is equivalent to the level order $R = [R_1, R_2, \dots, R_{k+1}]$, where $R_1 = S_1 \wedge Q_c$, $R_i = \neg R_1 \wedge Q_{i-1}$ for every $i > 1$ and Q_c is the first formula of Q such that $S_1 \wedge Q_c$ is consistent.*

The two requirements for induction are met: the base step by Lemma 1 and the induction step by Lemma 4. Starting from the natural order [] and the level order [true] and adding S_m, then S_{m-1}, and continuing until S_1 to the first results in a level order equivalent to $[S_1, \dots, S_{m-1}, S_m]$.

This is enough for translating a natural order into a level order of comparable size. Yet, it is not the way natural revision is normally expressed in terms of equivalence class. That would prove that the definition of natural revision of the previous section matches that commonly given. The following theorem provides that.

Theorem 1. *If the natural order $[S_2, \dots, S_m]$ is equivalent to the level order $Q = [Q_1, \dots, Q_k]$, then the natural order $S = [S_1, S_2, \dots, S_m]$ is equivalent to the following level order R, where Q_c is the first formula of Q that is consistent with S_1.*

$$R = [S_1 \wedge Q_c, Q_1, \dots, Q_{c-1}, \neg S_1 \wedge Q_c, Q_{c+1}, \dots, Q_k]$$

The level order $R = [S_1 \wedge Q_c, Q_1, \dots, Q_{c-1}, \neg S_1 \wedge Q_c, Q_{c+1}, \dots, Q_k]$ expresses the natural revision of a level order $[Q_1, \dots, Q_k]$. This is the same as naturally revising an order given as its sequence of equivalence classes [41].

The first resulting equivalence class comprises some of the models of the revising formula S_1. Namely, they are the ones in the first class that contains some models of S_1. All subsequent classes comprise the remaining models in the same relative order as before.

This correspondence of definitions was a short detour in the path from expressing a natural order by a level order. The proof was already in place, since both the base and the induction steps are proved.

Theorem 2. *Every natural order is equivalent to a level order of size bounded by a polynomial in the size of the natural order.*

3.2. From lexicographic to level orders

The reduction from lexicographic to level orders follows the same scheme as that of natural orders: base case and induction case. Only the reduction is shown, details are in Appendix A.

The base case proves the level order [true] equivalent to the empty lexicographic order [].

The induction step changes the level order to maintain the equivalence when adding a formula to the lexicographic order. Namely, prefixing a formula S_1 to the lexicographic order $[S_2, \dots, S_m]$ is the same as turning the corresponding level ordering $[Q_2, \dots, Q_k]$ into $[S_1 \wedge Q_2, \dots, S_1 \wedge Q_k, \neg S_1 \wedge Q_2, \dots, \neg S_1 \wedge Q_k]$. This proves that every lexicographic order is equivalent to some level order.

Theorem 3. *Every lexicographic order is equivalent to a level order.*

The theorem proves that every lexicographic order can be translated into a level order, but neglects size. It does not say that the level order is polynomial in the size of the lexicographic order. As a matter of facts, it is not. Some lexicographic orders explode into exponentially larger level orders. The next section proves this.

4. Comparison

Which representations are able to represent all doxastic states? Which do it shortly?

4.1. Expressivity

In propositional logic on a finite alphabet, all considered four representations are universal [23,49]: each represents all connected preorders.

The explicit representation is actually just the mathematical formalization of a connected preorder. Every connected preorder is representable by definition. The explicit representation is universal.

Theorem 4. *Every connected preorder is the level ordering of a sequence of mutually inconsistent formulae.*

The natural and lexicographic representations are proved universal indirectly: the level representation is translated into each of them. Since the level representation is universal, these are as well. These two translations are in the next section.

4.2. Succinctness

The translations from natural and lexicographic orders to level orders are in the previous section. A translation from natural to lexicographic orders is in a previous article [32], which however neglects size.

Since natural and lexicographic orders are defined inductively, an inductive definition of level orders facilitates the translations.

Theorem 5. *It holds $I \leq_S J$ holds if and only if the following condition holds, where S is a level order, S_1 is its first formula and R the sequence of the following ones.*

$$S = [] \text{ or } I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

4.2.1. From level to natural orders

Level orders translate into natural orders in polynomial time and space: every level order of a sequence of mutually inconsistent formulae is the natural order of the same sequence.

The proof comprises two steps. The first is a technical result: the models that falsify all formulae of a natural order are greater than or equal to every other model. This is the case because the formulae of a natural order state a belief in the truth of their models. The falsifying models are unsupported, and therefore unbelieved.

The second step of the proof is an inductive expression of the natural order $I \leq_S J$: it holds if S is either empty, or the first formula of S supports I, it denies J, or I is less than or equal to J according to the rest of S. This expression is the same as the level order of the same sequence.

Theorem 6. *Level orders translate into natural orders in polynomial time and space.*

4.2.2. From level to lexicographic orders

That level orders translate to lexicographic orders is a consequence of the translation from level to natural orders shown above and the translation from natural to lexicographic orders proved in a previous article [32]. Yet, the latter translation is not polynomial in time. What shown next is one that is.

The translation is the identity: every level order of a sequence of mutually inconsistent formulae is the lexicographic order of the same sequence. This is proved by showing that the lexicographic order $I \leq_S J$ holds if and only if either S is empty, its first formula makes I true, or it makes J false or the rest of the order compares it greater than or equal to I. This is the same as the expression of level orders proved by Theorem 5.

Theorem 7. *Level orders translate into lexicographic orders in polynomial time and space.*

4.2.3. From natural to lexicographic orders

This translation follows from two previous results: Theorem 2 shows a polynomial translation from natural to level orders; Corollary 6 shows the same from level to lexicographic orders.

Theorem 8. *Natural orders translate into lexicographic orders in polynomial space.*

4.3. From lexicographic to level and natural orders

All three representations are universal: they express all connected preorders. Therefore, they translate to each other. Whether they do in polynomial time or space is another story. What proved next is that not only polynomiality is unattainable in time, but also in space: some lexicographic orders are only equivalent to exponentially long natural orders.

The troublesome lexicographic orders are not even complicated: $[x_1, \ldots, x_n]$ is an example. The equivalence classes of this lexicographic order contain one model each. Therefore, they are exponentially many. The equivalence classes of level and natural orders are bounded by their number of formulae. Exponentially many classes equal exponentially many formulae.

The proof comprises two parts: the classes of lexicographic orders may be exponentially many; they never are for level and natural orders. Level orders are first.

Theorem 9. *The level order of a sequence of m formulae has at most $m + 1$ equivalence classes.*

Combining this result with the translation of Theorem 1 proves the same statement for natural orders.

Lemma 5. *The natural order of a sequence of m formulae has at most $m + 1$ equivalence classes.*

The second part of the proof is that the lexicographic order $[x_1, \ldots, x_n]$ comprises 2^n equivalence classes. A preliminary result is necessary.

Theorem 10. *The lexicographic comparisons $I \leq_S J$ and $J \leq_S I$ hold at the same time only if $I \leq_{S_k} J$ and $J \leq_{S_k} I$ both hold for every formula S_k of S.*

The number of equivalence classes of $[x_1, \ldots, x_n]$ can now be proved.

Lemma 6. *The lexicographic order $S = [x_1, \ldots, x_n]$ has 2^m equivalence classes.*

This result negates translations from lexicographic orders to level and natural orders of polynomial size.

Theorem 11. *The lexicographic order $[x_1, \ldots, x_n]$ is only equivalent to level and natural orders comprising at most $2^n - 1$ formulae.*

5. Related work

Most work in the iterated belief revision literature are purely semantical, but computational aspects are not neglected. An early example is the work by Ryan [44], who wrote: “Belief states are represented as deductively closed theories. This means that they are (in general) impossible to write down fully, or to store on a computer”; he employed a partial order between a finite number of formulae to represent the doxastic state. Williams [55] and Dixon [18] represented doxastic states by ordered partitions of formulae. Williams [57] later introduced partial entrenchment rankings, functions from a set of formulae to integers. Dixon and Wobcke [19] observed: “it is not possible to represent all entrenchments directly: some entrenchments allow infinitely many degrees of strength of beliefs. Moreover, it is impossible for the user of a system to enter all entrenchment relations: a more succinct representation must be found”; their solution is to allow for a partial specification, an ordered partition of formulae.

Computational issues are kept into account rarely [10,11,26,59,41], but recently attracted interest [23,50,46,48,4,45,49]. Most solutions employ structures equivalent or analogous to level orders [55,18,57,19,26,36,41,48], lexicographic orders [44,11,58], or histories of revisions [10,49]; the history of revisions may also be necessary for semantical, rather than computational, reasons [29,43,24]. Some other solutions change or extend these three solutions, and some others steer away from them. Souza, Moreira and Vieira [50] employ priority graphs [33], strict partial orders over a set of formulae. Aravanis [4] follow Areces and Becher [1] in their semantics based on a fixed ordering on the models.

Schwind, Konieczny and Pino Pérez [49] introduced a concept of doxastic state equivalence: “two epistemic states are strongly equivalent according to a revision operator if they cannot be distinguished from each other by any such successive revision steps, which means that these epistemic states have the same behavior for that revision operator”. This abstract definition generalizes Definition 6 to representations that are not connected preorders among propositional models.

6. Conclusions

How large a doxastic state is? It depends on how it is stored. The literature shows four ways: explicit, by a list of formulae expressing equivalence classes and by a history of revisions, either lexicographic or natural.

The comparison between different representation is simplified by a common style of definition: a sequence of revisions is inductively expressed from the same sequence but its last element. The definitions for the history of revisions are equivalent to the definitions based on equivalence classes. An immediate consequence is that they can be converted into the representation by equivalence classes.

The comparison is completed by the other reductions, both their existence and their succinctness, their ability to store doxastic states in little space. All four representations are universal: each represents all possible connected preorders on propositional models. They radically differ on succinctness. The explicit representation is the more wasteful: it is always exponential in the number of state variables, unlike the others. The representation by equivalence classes and by natural revisions is more succinct, and equally so. The most succinct of the four representations is that by lexicographic revisions. The other three representations can always be converted into it with a polynomial increase in size, while the converse reduction may produce exponentially large results.

Investigation can proceed in many directions. Succinctness does not only matter when comparing different representations. It also matters within the same. The same doxastic state has multiple lexicographic representations, for example. Some are short, some are long. The question is whether one can be made shorter. This problem is similar to Boolean formulae minimization [34,42,15,53,16,54]. A related question is how to shrink a connected preorder below a certain size while minimizing the loss of information. A subcase of interest is revision redundancy, whether a revision can be removed from a history without changing the resulting doxastic state.

The level, lexicographic and natural representations of the doxastic states are the most common in iterated belief revisions, but others are possible. An example is a single formula over an alphabet $Y \cup Z$ that is true on a model I if and only if $I[X/Y]$ is less than or equal than $I[X/Z]$. Other representations of the doxastic state have been proposed [3,5,23,50], such as prioritized bases [13,38,9], weighted knowledge bases [9,6,20] and conditionals [31,5,47]. Prioritized and weighted knowledge bases include lexicographic histories as the subcase where each priority class comprises a single formula and weights are exponentially increasing, like $1, 2, 4, \ldots$; whether they are strictly more succinct than lexicographic histories is however an open question. The preference reasoning field offers many alternatives [17]. An order among models may not suffice [7,8]. Similar representation issues also arise in belief merging [35].

CRediT authorship contribution statement

Paolo Liberatore: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs

Lemma 1. *The natural order* [] *is equivalent to the level order* [*true*].

Proof. The definition of the level order is satisfied by every pair of models I, J since both models satisfy the first formula of the level order [true]; as a result, both i and j are equal to 1, and they therefore satisfy $i \leq j$.

The definition of the natural order is satisfied because of its first part $S = []$. □

Lemma 2. *If Q_c is the first formula of the level order Q that is consistent with the formula S_1, then $I \models S_1 \wedge Q_c$ is equivalent to $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_Q K$.*

Proof. The two cases are considered in turn: either $I \models S_1 \wedge Q_c$ holds or it does not.

- $I \models S_1 \wedge Q_c$
  The claim is made of two parts: $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_Q K$.
  The first part $I \in Mod(S_1)$ holds because I satisfies $S_1 \wedge Q_c$.
  The second part $\forall K \in Mod(S_1).I \leq_Q K$ is now proved.
  Let Q_i be the only formula of Q such that $I \models Q_i$. The assumption $I \models Q_c$ implies $c = i$.
  Let K be an arbitrary model of S_1. The claim is $I \leq_Q K$.
  Only one formula S_k is satisfied by K. Since Q satisfies S_1 and Q_k, it also satisfies $S_1 \wedge Q_k$. As a result, $S_1 \wedge Q_k$ is consistent. Since Q_c is the first formula of S that is consistent with S_1, and Q_k is also consistent with S_1, it holds $c \leq k$. The equality of c and i proves $i \leq k$, which defines $I \leq_Q K$.
- $I \not\models S_1 \wedge Q_c$
  The claim is that either $I \in Mod(S_1)$ or $\forall K \in Mod(S_1).I \leq_Q K$ is false.
  If $I \not\models S_1$ the first part of this condition is falsified and the claim is therefore proved. It remains to be proved when $I \models S_1$.
  Let Q_i be the only formula satisfied by I. Since I satisfies S_1, it also satisfies $S_1 \wedge Q_i$. This formula is therefore satisfiable. Since Q_c is the first formula of the sequence that is consistent with S_1, the index c is less than or equal than the index i. If c were equal to i, then $S_1 \wedge Q_c$ would be $S_1 \wedge Q_i$. Yet, I is proved to satisfy the latter and assumed not to satisfy the former. The conclusion is $c < i$.
  Since Q_c is by assumption the first formula of Q that is consistent with S_1, the conjunction $S_1 \wedge Q_c$ is consistent. Let K be one of its models. Let Q_k be the only formula of Q that K satisfies. Since K satisfies $S_1 \wedge Q_c$, it also satisfies Q_c. As a result, k coincides with c.
  The conclusions of the last two paragraphs $c < i$ and $k = c$ imply $k < i$. This is the opposite of $i \leq k$, which defined $I \leq_Q K$. The conclusion is that a model K of S_1 that falsifies $I \leq_Q K$ exists. This proves the falsity of $\forall K \in Mod(S_1).I \leq_Q K$, as the claim requires. □

Lemma 3. *If $S = [S_1, S_2, \ldots, S_m]$ is a natural order, Q is a level order equivalent to the natural order $[S_2, \ldots, S_m]$ and Q_c is the first formula of Q that is consistent with S_1, then $I \leq_S J$ is:*

- *true if $I \models S_1 \wedge Q_c$;*
- *false if $I \not\models S_1 \wedge Q_c$ and $J \models S_1 \wedge Q_c$;*
- *same as $I \leq_Q J$ otherwise.*

Proof. The definition of natural order is that $I \leq_S J$ holds if and only if:

$$S = [] \text{ or}$$

$$(I \in Mod(S_1) \text{ and } \forall K \in Mod(S_1).I \leq_R K) \text{ or}$$

$$I \leq_R J \text{ and } (J \notin Mod(S_1) \text{ or } \exists K \in Mod(S_1).J \nleq_R K))$$

Since the statement of the lemma predicates about a first formula S_1 of S, whose existence falsifies the first part of this definition. The statement also assumes that a formula of Q is consistent with S_1, which proves its satisfiability. The statement also assumes that $\leq_R$ is the same as $\leq_Q$. The definition of natural order therefore becomes:

$$(I \in Mod(S_1) \text{ and } \forall K \in Mod(S_1).I \leq_Q K) \text{ or}$$

$$(I \leq_Q J \text{ and } (J \notin Mod(S_1) \text{ or } \exists K \in Mod(S_1).J \nleq_Q K))$$

The three cases are considered in turn.

$I \models S_1 \wedge Q_c$ Lemma 2 proves that $I \models S_1 \wedge Q_c$ implies $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_Q K$. This is the first part of the rewritten definition of $I \leq_S J$, which therefore holds.

$I \not\models S_1 \wedge Q_c$ and $J \models S_1 \wedge Q_c$ Lemma 2 proves that $I \not\models S_1 \wedge Q_c$ implies that $I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_Q K$ is false. This is the first part of the rewritten definition of $I \leq_S J$, which is therefore equivalent to its second part:

$$I \leq_Q J \text{ and } (J \notin Mod(S_1) \text{ or } \exists K \in Mod(S_1).J \not\leq_Q K)$$

Lemma 2 also applies to $J \models S_1 \wedge Q_c$ with J in place of I. It proves $J \in Mod(S_1)$ and $\forall K \in Mod(S_1).J \leq_Q K$. Its negation is therefore false: $J \notin Mod(S_1)$ or $\exists K \in Mod(S_1).J \not\leq_Q K$. This is the second part of the rewritten definition of $I \leq_S J$, which is therefore false, as the conclusion of the lemma requires in this case.

$I \not\models S_1 \wedge Q_c$ and $J \not\models S_1 \wedge Q_c$ As proved above, the assumption $I \not\models S_1$ transforms the definition of $I \leq_S J$ into $I \leq_Q J$ and $(J \notin Mod(S_1)$ or $\exists K \in Mod(S_1).J \not\leq_Q K)$.

Lemma 2 applies to $J \models S_1 \wedge Q_c$ with J in place of I. It proves that $J \not\models S_1 \wedge Q_c$ implies the falsity of $J \in Mod(S_1)$ and $\forall K \in Mod(S_1).J \leq_Q K$. This condition is the second part of the rewritten definition of $I \leq_S J$, which is therefore equivalent to the first part $I \leq_Q J$, as the conclusion of the lemma requires in this case. □

Lemma 4. *If the natural order $[S_2, \ldots, S_m]$ is equivalent to the level order $Q = [Q_1, \ldots, Q_k]$, then the natural order $S = [S_1, S_2, \ldots, S_m]$ is equivalent to the level order $R = [R_1, R_2, \ldots, R_{k+1}]$, where $R_1 = S_1 \wedge Q_c$, $R_i = \neg R_1 \wedge Q_{i-1}$ for every $i > 1$ and Q_c is the first formula of Q such that $S_1 \wedge Q_c$ is consistent.*

Proof. The sequence R is proved to be a level order and the disjunction of all its formulae to be tautologic. The sequence Q has the same properties by assumption. Each formula R_i with $i > 1$ is $\neg R_1 \wedge Q_i$. Its models are the models of Q_i minus some models. Since Q_i and Q_j do not share models, R_i and R_j do not either. Since the models subtracted from each Q_i when forming R_i are moved to R_1, which is also in R, the union of the models of R is exactly the union of the models of Q, the set of all models.

Lemma 3 proves that $I \leq_S J$ is:

- true if $I \models S_1 \wedge Q_c$;
- false if $I \not\models S_1 \wedge Q_c$ and $J \models S_1 \wedge Q_c$;
- same as $I \leq_Q J$ otherwise.

The proof shows that $\leq_R$ has the same values in the same cases.

$I \models S_1 \wedge Q_c$ The first formula of R is $R_1 = S_1 \wedge Q_c$ by definition. Since I satisfies $S_1 \wedge Q_c$ by assumption, it satisfies R_1. This proves that $I \models R_i$ with $i = 1$.

Let R_j be the formula of R such that $J \models R_j$. Since indexes start at 1, it holds $1 \leq j$. The equality $i = 1$ proves $i \leq j$. The claim $I \leq_R J$ follows.

$I \not\models S_1 \wedge Q_c$ and $J \models S_1 \wedge Q_c$

As in the previous case, $J \models S_1 \wedge Q_c$ implies $J \models R_j$ with $j = 1$.

Let R_i be the formula of R satisfies by I. If i were 1, then I would satisfy R_1, which is $S_1 \wedge Q_c$. Since I does not satisfy this formula by assumption, i is not 1. Since indexes start at one, i is strictly greater than one: $i > 1$.

The conclusions $j = 1$ and $i > 1$ prove $i > j$, which is the exact opposite of $i \leq j$. The claimed falsity of $I \leq_R J$ is therefore proved.

$I \not\models S_1 \wedge Q_c$ and $J \not\models S_1 \wedge Q_c$

Let R_i and R_j be the formulae respectively satisfied by I and J. Since neither model satisfies $R_1 = S_1 \wedge Q_c$ by assumption, both i and j are strictly greater than one: $i > 1$ and $j > 1$.

The formulae R_i and R_j for indexes greater than one are respectively defined as $\neg R_1 \wedge Q_{i-1}$ and $\neg R_1 \wedge Q_{j-1}$. Since I and J respectively satisfy them, they respectively satisfy Q_{i-1} and Q_{j-1}. The level order $I \leq_Q J$ is therefore equivalent to $i - 1 \leq j - 1$, which is the same as $i \leq j$. This is also the definition of $I \leq_S J$.

This proves the claimed equality of $I \leq_S J$ and $I \leq_Q J$. □

Theorem 1. *If the natural order $[S_2, \ldots, S_m]$ is equivalent to the level order $Q = [Q_1, \ldots, Q_k]$, then the natural order $S = [S_1, S_2, \ldots, S_m]$ is equivalent to the following level order R, where Q_c is the first formula of Q that is consistent with S_1.*

$$R = [S_1 \wedge Q_c, Q_1, \ldots, Q_{c-1}, \neg S_1 \wedge Q_c, Q_{c+1}, \ldots, Q_k]$$

Proof. Lemma 4 proves that the natural order S is equivalent to the level order $R = [R_1, R_2, \ldots, R_{k+1}]$, where $R_1 = S_1 \wedge Q_c$ and $R_i = \neg R_1 \wedge Q_{i-1}$ for every $i > 1$.

This is the same sequence as in the statement of the lemma.

The first formula is the same in both sequences: $S_1 \wedge Q_c$.

The formula R_i for $i = c + 1$ in the sequence of the lemma is also the same as the formula in the sequence of the theorem. Since R_1 is $S_1 \wedge Q_c$, the formula $R_i = \neg R_1 \wedge Q_{i-1}$ for $i = c + 1$ in the sequence of the lemma is the same as $\neg(S_1 \wedge Q_c) \wedge Q_c$, which is equivalent to $(\neg S_1 \vee \neg Q_c) \wedge Q_c$, in turn equivalent to $(\neg S_1 \wedge Q_c) \vee (\neg Q_c \wedge Q_c)$ and to $\neg S_1 \wedge Q_c$, which is the formula in the sequence of the theorem.

The formulae R_i with i strictly greater than one and different from $c+1$ are $R_i = \neg R_1 \wedge Q_{i-1}$ in the sequence of the lemma. Since R_1 is $S_1 \wedge Q_c$, this formula $R_i = \neg R_1 \wedge Q_{i-1}$ is the same as $R_i = \neg(S_1 \wedge Q_c) \wedge Q_{i-1}$, which is equivalent to $(\neg S_1 \wedge Q_{i-1}) \vee (\neg Q_c \vee Q_{i-1})$. The two formulae Q_c and Q_{i-1} are mutually inconsistent since i is not equal to $c+1$. As a result, Q_{i-1} implies $\neg Q_c$. This proves $\neg Q_c \vee Q_{i-1}$ equivalent to Q_{i-1}. Therefore, R_i in the sequence of the lemma is equivalent to $(\neg S_1 \wedge Q_{i-1}) \vee Q_{i-1}$, which is equivalent to Q_{i-1}, as in the sequence of the theorem. □

Theorem 2. *Every natural order is equivalent to a level order of size bounded by a polynomial in the size of the natural order.*

Proof. How to translate a natural order S into a level order R is shown by induction.

The base is case is $S = []$, which translates into $R = [\mathsf{true}]$ by Lemma 1.

The induction case requires a way to translate a natural order comprising at least a formula into a level order. Let $S = [S_1, S_2, \ldots, S_m]$ be the natural order. By the inductive assumption, $[S_2, \ldots, S_m]$ translates into a level order Q. Theorem 1 proves S equivalent to the level order R:

$$R = [S_1 \wedge Q_c, Q_1, \ldots, Q_{c-1}, \neg S_1 \wedge Q_c, Q_{c+1}, \ldots, Q_k]$$

This order is larger than R only by $2 \times |S_1|$.

This inductively proves that the translation is possible and produces a sequence that is at most twice the size of the original. □

The following lemmas and theorems show that every lexicographic order is equivalent to a level order. The proof scheme is the same as that of natural orders: base case and induction case. The base case is that the level order [true] is equivalent to the empty lexicographic order []. The induction case adds a single formula at the front of the lexicographic sequence and shows how it changes the corresponding level order.

Lemma 7. *The lexicographic order [] is equivalent to the level order [true].*

Proof. The two definitions are:

level order:

$i \leq j$ where $I \models S_i$ and $J \models S_j$

lexicographic order:

$S = []$ or $(I \leq_{S_1} J$ and $(J \not\leq_{S_1} I$ or $I \leq_R J))$

The first definition is satisfied by every pair of models I, J since both models satisfy the first formula of the level order [true]; as a result, both i and j are equal to 1, and they therefore satisfy $i \leq j$.

The second definition is satisfied because of its first part $S = []$. □

The induction step changes the level order to keep it equivalent to the lexicographic order while adding a formula at time to the latter.

Lemma 8. *If $S = [S_1, S_2, \ldots, S_m]$ is a lexicographic order and Q is a level order equivalent to the lexicographic order $[S_2, \ldots, S_m]$, then $I \leq_S J$ is:*

- *true if $I \models S_1$ and $J \not\models S_1$*
- *false if $I \not\models S_1$ and $J \models S_1$*
- *same as $I \leq_Q J$ otherwise*

Proof. The definition of the lexicographic order $I \leq_S J$ is:

$S = []$ or $(I \leq_{S_1} J$ and $(J \not\leq_{S_1} I$ or $I \leq_R J))$

The lemma implicitly assumes S not empty. What results from removing the case $S = []$ is:

$I \leq_{S_1} J$ and $(J \not\leq_{S_1} I$ or $I \leq_R J)$

The comparison $I \leq_S J$ is evaluated in the four cases where I or J satisfy S_1 or not.

$I \models S_1$ and $J \not\models S_1$ The definition of $I \leq_{S_1} J$ is $I \models S_1$ or $J \not\models S_1$, and is therefore satisfied. This is the first part of $I \leq_S J$.
Similarly, $J \leq_{S_1} I$ is $J \models S_1$ or $I \not\models S_1$. Both are false. Therefore, $J \leq_{S_1} I$ is false. Its negation $J \not\leq_{S_1} I$ is true. The second part of $I \leq_S J$ is therefore true, being $J \not\leq_{S_1} I$ or $I \leq_R J$

$I \not\models S_1$ and $J \models S_1$ The definition of $I \leq_{S_1} J$ is $I \models S_1$ or $J \not\models S_1$; both conditions are false. Since $I \leq_{S_1} J$ is false, its conjunction with $J \not\leq_{S_1} I$ or $I \leq_R J$ is also false. Since this conjunction is equivalent to $I \leq_S J$, this comparison is false as well.

$I \models S_1$ and $J \models S_1$ The first assumption $I \models S_1$ implies $I \leq_{S_1} J$.
The second assumption $J \models S_1$ implies $J \leq_{S_1} I$.
The condition $I \leq_{S_1} J$ and $(J \not\leq_{S_1} I$ or $I \leq_R J)$ simplifies to the equivalent condition true and (false or $I \leq_R J$), which is the same as $I \leq_R J$.

$I \not\models S_1$ and $J \not\models S_1$: The first assumption $I \not\models S_1$ implies $J \leq_{S_1} I$.
The second assumption $J \not\models S_1$ implies $I \leq_{S_1} J$.
The condition $I \leq_{S_1} J$ and $(J \not\leq_{S_1} I$ or $I \leq_R J)$ simplifies to the equivalent condition true and (false or $I \leq_R J$), which is the same as $I \leq_R J$. □

The second part of the induction step is representing the order $I \leq_S J$ in the previous lemma as a level order.

Lemma 9. *If the lexicographic order $[S_2, \dots, S_m]$ is equivalent to the level order $Q = [Q_1, \dots, Q_k]$, then the lexicographic order $[S_1, S_2, \dots, S_m]$ is equivalent to the following level order.*

$$R = [S_1 \wedge Q_1, \dots, S_1 \wedge Q_k, \neg S_1 \wedge Q_1, \dots, \neg S_1 \wedge Q_k]$$

Proof. The lemma assumes that Q is a level order (its formulae are mutually inconsistent) and the disjunction of all its formulae is tautologic. The models of every formula Q_i are split among $S_1 \wedge Q_i$ and $\neg S_1 \wedge Q_i$. Therefore, R has the same properties of Q.

The rest of the proof shows that S is equivalent to R.

Lemma 8 proves that $I \leq_S J$ is:

- true if $I \models S_1$ and $J \not\models S_1$
- false if $I \not\models S_1$ and $J \models S_1$
- same as $I \leq_Q J$ otherwise

The same is proved for $I \leq_R J$. The starting point is the definition of level order: $I \leq_R J$ is $i \leq j$ and $I \models R_i$ and $J \models R_j$. It is evaluated in the three cases above.

$I \models S_1$ **and** $J \not\models S_1$ Let R_i and R_j be the formulae of R respectively satisfied by I and J.
Since I also satisfies S_1 by assumption, it satisfies $S_1 \wedge R_i$. Since J falsifies S_1 by assumption, it satisfies $\neg S_1$ and therefore also $\neg S_1 \wedge R_j$.
These formulae $S_1 \wedge R_i$ and $\neg S_1 \wedge R_j$ are in the positions i and $k + j$ in the sequence R. Since i is an index a sequence of length k, it holds $i \leq k$. As a result, $i < k + j$. This inequality implies $i \leq k + j$, which defines $I \leq_S J$.

$I \not\models S_1$ **and** $J \models S_1$ Let R_i and R_j be the formulae of R respectively satisfied by I and J. The assumptions $I \not\models S_1$ imply $I \models \neg S_1$. As a result, I satisfies $\neg S_1 \wedge R_i$. Since J satisfies $J \models S_1$ by assumption, it also satisfies $S_1 \wedge R_j$.
The formulae $\neg S_1 \wedge R_i$ and $S_1 \wedge R_j$ are in the positions $k + i$ and j in the sequence R. Since j is an index a sequence of length k, it holds $j \leq k$. As a result, $j < k + i$. This inequality is the opposite of $k + i \leq j$, which defines $I \leq_S J$. This comparison is therefore false, as required.

otherwise The two remaining cases are $I \models S_1$ and $J \models S_1$ and $I \not\models S_1$ and $J \not\models S_1$.
Let R_i and R_j be the formulae of R respectively satisfied by I and J.
The conditions $I \models S_1$ and $J \models S_1$ imply $I \models S_1 \wedge R_i$ and $J \models S_1 \wedge R_j$. These formulae are in the sequence R at positions i and j. The definition of $I \leq_S J$ is $i \leq j$, which is also the definition of $I \leq_R J$ in this case.
The conditions $I \not\models S_1$ and $J \not\models S_1$ imply $I \models \neg S_1$ and $J \models \neg S_1$, which imply $I \models \neg S_1 \wedge R_i$ and $J \models \neg S_1 \wedge R_j$. These formulae are in the sequence R at positions $k + i$ and $k + j$. The definition of $I \leq_S J$ is $i \leq j$, which is equivalent to $k + i \leq k + j$, which defines $I \leq_R J$ in this case. □

This lemma shows how to keep a level order equivalent to a lexicographic order while adding a formula to the latter.

Theorem 3. *Every lexicographic order is equivalent to a level order.*

Proof. The claim is proved by induction on the length of the lexicographic order S.

The base case is $S = []$. Its equivalent level order is $R = [\text{true}]$. It is equivalent because both compare $I \leq J$ for all models. The first because $S = []$ is one of the conditions of its definition. The second because both I and J always satisfy true, the first formula of R.

In the induction case, the sequence S has length one or more. Let $S_1, S_2, \dots, S_m$ be its formulae. Lemma 9 requires a level order Q to be equivalent to $[S_2, \dots, S_m]$; it exists by the induction assumption. As a result, the claim of the lemma holds: S is equivalent to a level order R. □

Theorem 4. *Every connected preorder is the level ordering of a sequence of mutually inconsistent formulae.*

Proof. Every connected preorder corresponds to a sequence of disjoint sets $E = [E_1, \dots, E_m]$, where $I \leq J$ corresponds to $I \in E_i$, $J \in E_j$ and $i \leq j$. In the specific case of propositional models, every set of models E_i is the set of models of a formula S_i. Therefore, a connected preorder is also the level order $[S_1, \dots, S_m]$. □

Theorem 5. *It holds $I \leq_S J$ holds if and only if the following condition holds, where S is a level order, S_1 is its first formula and R the sequence of the following ones.*

$$S = [] \text{ or } I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

Proof. The definition of the level order $I \leq_S J$ is:

$$\forall j . J \not\models S_j \text{ or } i \leq j \text{ where } I \models S_i \text{ and } J \models S_j$$

This definition is proved to coincide with the condition in the statement of the theorem.

Since the condition is "$S = []$ or something else", it is true when $S = []$. This is also the case for the definition, since no formula of the sequence is satisfied by a model J.

If S is not empty, the condition of the statement of the theorem becomes:

$$I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

This is proved to coincide with the definition of the level orders in all cases: I satisfies S_1, it does not and J does, and none of them do. The proof is by induction: it is assumed true on sequences strictly shorter than S.

$I \models S_1$ The condition is true because it is "$I \in Mod(S_1)$ or something else", and $I \in Mod(S_1)$ is true because of the assumption $I \models S_1$.
The definition is also true. The assumption $I \models S_1$ implies that $I \models S_i$ with $i = 1$. Let S_j be the formula such that $J \models S_j$. Since the sequence starts at 1, this index j is larger or equal than 1. Since i is equal to one, $j \geq i$ follows. This is the same as $i \leq j$, which defines $I \leq_S J$.

$I \not\models S_1$ **and** $J \models S_1$ The condition simplifies from $I \in Mod(S_1)$ or $(J \notin Mod(S_1)$ and $I \leq_R J)$ to false or ($\cancel{t}$rue and $I \leq_R J$, which is false.
The definition is not met either. Since $J \models S_1$, the first part of the definition $\forall j . J \not\models S_j$ is false. Since $J \models S_1$, the index j such that $J \models S_j$ is 1. Since I does not satisfy S_1, it either satisfies S_i with $i > 1$ or it does not satisfy any formula of S. In the first case, $j > i$ implies that $i \leq j$ is false. In the second case, $I \models S_i$ is false for all formulae S_i. The second part of the definition $i \leq j$ where $I \models S_i$ and $J \models S_j$ is false either way.

$I \not\models S_1$ **and** $J \not\models S_1$ These two assumptions imply that $I \in Mod(S_1)$ and $J \in Mod(S_1)$ are false. The condition $I \in Mod(S_1)$ or $(J \notin Mod(S_1)$ and $I \leq_R J)$ simplifies to false or (true and $I \leq_R J$)), which is equivalent to $I \leq_R J$, where $R = [S_2, \dots, S_m]$.
The definition of the level order also simplifies.
Its first part $\forall j . J \not\models S_j$ is equivalent to $\forall j > 1 . J \not\models S_j$, since J is known not to satisfy S_1. This is the same as $\forall j . J \not\models R_j$.
Its second part $i \leq j$ where $I \models S_i$ and $J \models S_j$ may only hold with $i > 1$ and $j > 1$ since neither I nor J satisfy S_1. As a result, it simplifies to $i + 1 \leq j + 1$ where $I \models R_i$ and $J \models R_j$ where $R = [S_2, \dots, S_m]$. This is equivalent to $i \leq j$ where $I \models R_i$ and $J \models R_j$.
The conclusion is that the definition of $I \leq_S J$ is the same as $\forall j . J \not\models R_j$ or $i \leq j$ where $I \models R_i$ and $J \models R_j$, the definition of $I \leq_R J$. □

The next results prove that level orders translate into natural orders in polynomial time and space.

Theorem 5 proves that $I \leq_S J$ equates a certain inductive condition on the level order S. The following theorem proves the same for natural orders when S comprises mutually inconsistent formulae. For these sequences the identity translates level orders into equivalent natural orders. This suffices since level orders can be restricted to mutually inconsistent formulae.

A preliminary lemma on natural orders is necessary.

Lemma 10. *If a model J falsifies all formulae of the natural order S, then $I \leq_S J$ holds for every model I.*

Proof. The definition of $I \leq_S J$ is:

$$S = [] \text{ or}$$

$$(I \in Mod(S_1) \text{ and } \forall K \in Mod(S_1) . I \leq_R K) \text{ or}$$

$$I \leq_R J \text{ and } (J \notin Mod(S_1) \text{ or } \exists K \in Mod(S_1) . J \nleq_R K))$$

In the base case $S = []$ this definition is met. As a result, $I \leq_S J$ holds by definition.

The induction case is proved as follows. Since J does not satisfy any formula of $S = [S_1, S_2, \dots, S_m]$, it does not satisfy S_1 and does not satisfy any formula of $R = [S_2, \dots, S_m]$. The latter implies $I \leq_R J$ by the induction assumption. The definition of $I \leq_S J$ simplifies as follows when replacing $J \notin Mod(S_1)$ and $I \leq_R J$ by true.

$S = []$ or

$(I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K)$

or $(I \leq_R J$ and $(J \notin Mod(S_1)$ or $\exists K \in Mod(S_1).J \nleq_R K))$

$\equiv$ false or $(I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K)$ or

(true and (true or $\exists K \in Mod(S_1).J \nleq_R K$))

$\equiv (I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K)$ or (true and true)

$\equiv (I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K)$ or true

true □

This lemma allows expressing a natural order in the same way of a level order when its formulae are mutually inconsistent.

Lemma 11. *If S is a sequence of mutually inconsistent formulae, the natural order $I \leq_S J$ holds if and only if the following condition holds, where S_1 is the first formula of S and R the sequence of the following formulae of S.*

$S = []$ *or* $I \in Mod(S_1)$ *or* $(J \notin Mod(S_1)$ *and* $I \leq_R J)$

Proof. Since the formulae of S do not share models, if a model K satisfies S_1 it falsifies all other formulae $S_2, \ldots, S_m$. The latter implies $I \leq_R K$ by Lemma 10 since $R = [S_2, \ldots, S_m]$. In other words, $I \leq_R K$ holds for all formulae that satisfy S_1. In formulae, $\forall K \in Mod(S_1).I \leq_R K$.

For the same reason, $\forall K \in Mod(S_1).J \leq_R K$ holds as well. This is the contrary of $\exists K \in Mod(S_1).J \nleq_R K$, which is therefore false.

Replacing $\forall K \in Mod(S_1).I \leq_R K$ with true and $\exists K \in Mod(S_1).J \nleq_R K$ with false in the definition of the natural order $I \leq_S J$ yields:

$S = []$ or

$(I \in Mod(S_1)$ and $\forall K \in Mod(S_1).I \leq_R K)$

or $(I \leq_R J$ and $(J \notin Mod(S_1)$ or $\exists K \in Mod(S_1).J \nleq_R K))$

$\equiv S = []$ or $(I \in Mod(S_1)$ and true) or $(I \leq_R J$ and $(J \notin Mod(S_1)$ or false))

$\equiv S = []$ or $I \in Mod(S_1)$ or $(I \leq_R J$ and $J \notin Mod(S_1))$

The final condition is the claim of the lemma. □

Since both level orders and natural orders are equivalent to the same condition when their formulae are mutually inconsistent, they are equivalent.

Corollary 1. *Every level order of a sequence of mutually inconsistent formulae is the natural order of the same sequence.*

The translation is the identity. It takes polynomial time and space. This concludes the proof.

The following results prove that level orders translate into lexicographic order in polynomial time and space. This is proved by showing that every level order of a sequence of mutually inconsistent formulae is the lexicographic order of the same sequence.

The first step is a preliminary lemma.

Lemma 12. *If a model J falsifies all formulae of the sequence S, then the lexicographic ordering $I \leq_S J$ holds for every model I.*

Proof. The definition of the lexicographic order $I \leq_S J$ is:

$S = []$ or $(I \leq_{S_1} J$ and $(J \nleq_{S_1} I$ or $I \leq_R J))$

This definition is the disjunction of $S = []$ with another condition. The base case of induction $S = []$ therefore meets the definition. In the induction case, $S = []$ is false. The definition reduces to:

$I \leq_{S_1} J$ and $(J \nleq_{S_1} I$ or $I \leq_R J)$

Since J does not satisfy any formula of S, it does not satisfy its first formula S_1. In turn, $J \not\models S_1$ imply $I \models S_1$ or $J \not\models S_1$, the definition of $I \leq_{S_1} J$. Since J does not satisfy any formula of S, it does not satisfy any formula of its subsequence R. By the induction assumption, $I \leq_R J$ holds. The definition of the lexicographic order further simplifies to:

true and $(J \nleq_{S_1} I$ or true) $\equiv$ true and true $\equiv$ true □

This lemma allows proving the equivalent condition.

Lemma 13. *If S is a sequence of mutually inconsistent formulae, the lexicographic order $I \leq_S J$ holds if and only if the following condition holds, where S_1 is the first formula of S and R the sequence of the following formulae of S.*

$$S = [] \text{ or } I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

Proof. The condition in the statement of the lemma is proved to coincide with the definition of the lexicographic order $I \leq_S J$:

$$S = [] \text{ or } (I \leq_{S_1} J \text{ and } (J \not\leq_{S_1} I \text{ or } I \leq_R J))$$

Both the definition and the condition are disjunctions comprising $S = []$. Therefore, they are both true in the base case of induction $S = []$.

In the induction case, S is not empty. The definition and the condition respectively became:

$$I \leq_{S_1} J \text{ and } (J \not\leq_{S_1} I \text{ or } I \leq_R J)$$

$$I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

The induction assumption implies that the two occurrences of $I \leq_R J$ coincide since R is strictly shorter than S.

Two cases are considered: either I satisfies S_1 or not.

$I \models S_1$ The condition in the statement of the lemma is true since it is a disjunction comprising $I \in Mod(S_1)$.

The assumption $I \models S_1$ implies $I \leq_{S_1} J$ by definition. It also simplifies the definition of $J \leq_{S_1} I$ from $J \models S_1$ or $I \not\models S_1$ into just $J \models S_1$.

The definition of $I \leq_S J$ therefore simplifies:

$$I \leq_{S_1} J \text{ and } (J \not\leq_{S_1} I \text{ or } I \leq_R J)$$

$$\equiv \text{true and } (J \not\models S_1 \text{ or } I \leq_R J)$$

$$\equiv J \not\models S_1 \text{ or } I \leq_R J$$

This is true if $J \not\models S_1$, and is now proved true in the other case, $J \models S_1$.

Since the formulae S_i do not share models and J is a model of S_1, it is not a model of any other formula $S_2, \ldots, S_m$. These formulae make R: no formula of R is satisfied by J. Lemma 12 proves $I \leq_R J$.

Since the definition of $I \leq_S J$ is equivalent to $J \not\leq_{S_1} I$ or $I \leq_R J$, it is met.

$I \not\models S_1$ This assumption implies $J \leq_{S_1} I$, which makes $J \not\leq_{S_1} I$ false. Since $I \leq_{S_1} J$ is defined as $I \models S_1$ or $J \not\models S_1$, it becomes the same as $J \not\models S_1$.

The definition of $I \leq_S J$ simplifies:

$$I \leq_{S_1} J \text{ and } (J \not\leq_{S_1} I \text{ or } I \leq_R J)$$

$$\equiv J \not\models S_1 \text{ and (false or } I \leq_R J)$$

$$\equiv J \not\models S_1 \text{ and } I \leq_R J$$

The condition in the statement of the lemma also simplifies thanks to the current assumption $I \not\models S_1$:

$$I \in Mod(S_1) \text{ or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

$$\equiv \text{false or } (J \notin Mod(S_1) \text{ and } I \leq_R J)$$

$$\equiv J \not\models S_1 \text{ and } I \leq_R J$$

This is the same as the definition of $I \leq_S J$. $\square$

The translation is a corollary.

Corollary 2. *Every level order of a sequence of mutually inconsistent formulae is the natural order of the same sequence.*

Since the translation is the identity, it takes linear space and time. This concludes the proof that level orders translate into lexicographic orders in polynomial time and space.

Theorem 9. *The level order of a sequence of m formulae has at most $m + 1$ equivalence classes.*

Proof. As shown in Section 2, a level order is equivalent to another where every model is satisfied by exactly one formula by the addition of a single formula. This is an increase from m to $m+1$ formulae.

In this other order, the condition $\forall j. J \not\models S_j$ is always false. As a result, $I \leq_S J$ holds if and only if $i \leq j$, where $I \models S_i$ and $J \models S_j$. The reverse comparison $J \leq_S I$ holds if and only if $j \leq i$. As a result, I and J are compared the same if and only if $i = j$, where $I \models S_i$ and $J \models S_j$. Two models are equivalent if and only if they satisfy the same formula of S. This is an isomorphism between the equivalence classes and the formulae of the sequence. □

Lemma 5. *The natural order of a sequence of m formulae has at most $m+1$ equivalence classes.*

Proof. Theorem 1 translates a natural order into an equivalent level order by iterating over the formulae of the sequence, each time turning a sequence Q into a sequence R:

$$R = [S_1 \wedge Q_c, Q_1, \ldots, Q_{c-1}, \neg S_1 \wedge Q_c, Q_{c+1}, \ldots, Q_k]$$

The formulae of R are the same of Q except for the absence of Q_c and the presence of $S_1 \wedge Q_c$ and $\neg S_1 \wedge Q_c$. The number of formulae therefore increases by one at each step. Since the process starts from $Q = []$ and iterates over the m formulae of the natural order, it produces a level order of the same length.

By Theorem 9, this level order has at most $m+1$ equivalence classes. Since the natural order is equivalent, it has the same equivalent classes. □

Theorem 10. *The lexicographic comparisons $I \leq_S J$ and $J \leq_S I$ hold at the same time only if $I \leq_{S_k} J$ and $J \leq_{S_k} I$ both hold for every formula S_k of S.*

Proof. The claim is that $I \leq_S J$ and $J \leq_S I$ imply $I \leq_{S_k} J$ and $J \leq_{S_k} I$ for every formula S_k of S.

The definition of $I \leq_S J$ is:

$$S = [] \text{ or } (I \leq_{S_1} J \text{ and } (J \not\leq_{R_1} I \text{ or } I \leq_R J))$$

The claim is proved by induction. If S is the empty sequence, the conclusion is vacuously true since S contains no formula S_k.

The inductive assumption is that the claim holds for every sequence R shorter than S. The premise of the theorem is that both $I \leq_S J$ and $J \leq_S I$ hold. The claim can be split in three parts: $I \leq_{S_1} J$, $J \leq_{S_1} I$, and the same for every S_k with $k > 1$.

In the inductive case S is not empty. The definition of $I \leq_S J$ becomes:

$$I \leq_{S_1} J \text{ and } (J \not\leq_{R_1} I \text{ or } I \leq_R J)$$

Since $I \leq_S J$ holds by the premise of the theorem, its first conjunct $I \leq_{S_1} J$ holds as well. For the same reason, $J \leq_S I$ implies $J \leq_{S_1} I$. This proves the first two parts of the claim.

Since $I \leq_S J$ holds by the premise of the theorem, its second conjunct $J \not\leq_{R_1} I$ or $I \leq_R J$ holds as well. Its first disjunct is contradicted by $J \leq_{S_1} I$, proved above. What is left is $I \leq_R J$. The same argument proves that $J \leq I$ implies $J \leq_R I$.

By the induction assumption, $I \leq_R J$ and $J \leq_R I$ imply $I \leq_{S_k} J$ and $J \leq_{S_k} I$ for every formula S_k of R. These are the formulae S_k of S with $k > 1$. □

Lemma 6. *The lexicographic order $S = [x_1, \ldots, x_n]$ has 2^m equivalence classes.*

Proof. Two different models differ on a variable x_k at least: $I \models x_k$ and $J \not\models x_k$.

If I and J were equivalent according to S, then Theorem 10 would apply. It implies $I \leq_{x_k} J$ and $J \leq_{x_k} I$ for every k. The second conclusion $J \leq_{x_k} I$ is the same as $J \models x_k$ or $I \not\models x_k$, both of which are false. Therefore, I and J are not equivalent according to S.

The conclusion is that two different models are not equivalent. Every model is in its own class of equivalence. Since the models are 2^m, the classes of equivalence are the same number. □

Theorem 11. *The lexicographic order $[x_1, \ldots, x_n]$ is only equivalent to level and natural orders comprising at most $2^n - 1$ formulae.*

Proof. The lexicographic order $[x_1, \ldots, x_n]$ is assumed equivalent to a level or natural order S by contradiction. By Lemma 6, the lexicographic order $[x_1, \ldots, x_n]$ has 2^n equivalence classes. Since the level or natural order S is equivalent to it, it has the same equivalence classes. By Theorem 9 or Lemma 5, S comprises at least $2^m - 1$ formulae. □

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