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Improved maximin guarantees for subadditive and fractionally subadditive fair allocation problem [☆]

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ABSTRACT

In this work, we study the maximin share fairness notion (MMS) for allocation of indivisible goods in the subadditive and fractionally subadditive settings. While previous work refutes the possibility of obtaining an allocation which is better than $1/2$-MMS, the only positive result for the subadditive setting states that when the number of items is equal to m, there always exists an $\Omega(1/\log m)$-MMS allocation. Since the number of items may be larger than the number of agents (n), such a bound can only imply a weak bound of $\Omega(\frac{1}{n\log n})$-MMS allocation in general.

In this work, we improve this bound exponentially to $\Omega(\frac{1}{\log n\log\log n})$-MMS guarantee. In addition to this, we prove that when the valuation functions are fractionally subadditive, a 0.2192235-MMS allocation is guaranteed to exist. This also improves upon the previous bound of $1/5$-MMS guarantee for the fractionally subadditive setting.

1. Introduction

Fair division is a fundamental problem which has received significant attention in economics [1–5], political science [6–8], mathematics [9,10], and more recently in computer science and artificial intelligence [11–21]. In this problem the goal is to divide a resource among a set of agents in a fair manner. Both divisible and indivisible settings have been subject to several studies, though recent years have seen a plethora of developments in the indivisible setting; see [22] and [23] for surveys on recent developments in allocation of indivisible items.

Unfortunately, most of the guarantees that hold in the divisible setting do not carry over to the indivisible setting. For example, well-known fairness criteria such as *envy-freeness*[1] and proportionality[2] that are known to exist in the divisible setting may be impossible to guarantee in the indivisible setting. This led the community to develop more relaxed fairness notions that are better suited for the indivisible setting.

In this paper, we investigate the maximin-share (MMS) notion which is one of the central measures of fairness in the indivisible setting. This notion is introduced by Budish [2] as a relaxation of proportionality for the case of indivisible goods. Let $\mathcal{N} = \{a_1, a_2, \dots, a_n\}$ be a set of size n that contains the agents. For a set $\mathcal{M}$ of m indivisible goods and an agent a_i, $\mathsf{MMS}_i^n(\mathcal{M})$ is defined as

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1 An allocation is called envy-free, if no agent prefers to exchange her bundle with another agent.
2 An allocation is called proportional, if each agent receives a bundle which is worth at least $1/n$ of the entire resource to her.

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$$\mathsf{MMS}_i^n(\mathcal{M}) = \max_{\pi_1,\pi_2,\ldots,\pi_n\in\Pi} \min_j U_i(\pi_j),$$

where Π is the set of all partitionings of $\mathcal{M}$ into n bundles and $U_i(\pi_j)$ is the valuation of agent a_i for a bundle π_j. In other words, among all n partitionings of the items, the one that maximizes the minimum value of the partitions for agent a_i gives the MMS value of that agent. When the goal is to allocate the items to n agents, maximin-share of agent a_i is defined to be equal to $\mathsf{MMS}_i^n(\mathcal{M})$. For brevity, we denote this value by MMS_i. An allocation is then said to be MMS, if it guarantees each agent a_i a bundle with utility at least MMS_i to agent a_i.

MMS-allocations have received significant attention both in the additive and non-additive settings. While it may seem that in the additive setting, an MMS-allocation always exists, an elegant counter-example by Kurokawa et al. [13] reveals that some additive instances admit no MMS allocation. On the positive side, it has been shown that a 2/3-approximate MMS allocation (or 2/3-MMS in short, i.e., an allocation that guarantees each agent a_i a bundle with utility at least $2\mathsf{MMS}_i/3$) always exists [13]. This bound is improved by Ghodsi et al. [24] to a 3/4-MMS guarantee. Currently, the best known approximation guarantee for maximin-share is $3/4 + O(1/n)$ [25]. No counter-example refutes the possibility of obtaining a better bound and therefore whether or not a more efficient algorithm can guarantee a better bound remains an open question.

The importance of fair allocation problems goes well beyond the additive setting. For instance, it is quite natural to expect that an agent prefers to receive two items of value 400, rather than receiving 1000 items of value 1. Such a constraint cannot be imposed in the additive setting. However, subadditive and fractionally subadditive functions are strong tools for modeling such constraints. Previous work has already made some progress in generalizing the Maxmin-share fair allocation problem to non-additive settings. Barman and Krishna Murthy [26] prove that when the valuation functions are submodular, a 1/10-MMS allocation can be guaranteed for the fair allocation problem. The bound was later improved by Ghodsi et al. [24] to a 1/3-MMS guarantee for submodular functions. They also prove a 1/5-MMS guarantee for the fractionally subadditive setting and an $\Omega(1/\log m)$-MMS guarantee for the subadditive setting.

In this paper, we improve the previous results on subadditive and fractionally subadditive settings. Our proof gives an improved guarantee of $\Omega(\frac{1}{\log n \log\log n})$-MMS in the subadditive setting which exponentially improves the prior work of Ghodsi et al. [24]. In addition to this, we also improve the 1/5-MMS guarantee of Ghodsi et al. [24] for the fractionally subadditive setting to $0.2192235 \simeq 1/4.6$-MMS.

2. Related work

In addition to the studies we mentioned in the introduction [26,24], there are other works on maximin-share for settings more general than additive [27–31]. For a special case of fractionally subadditive valuations where the items form a hereditary set system, Li and Vetta [27] prove a 0.3667-MMS guarantee. Recently, in a preprint, Uziahu and Feige [28] improve the approximation ratio for submodular valuations to 10/27-MMS. Kulkarni et al. [30] also consider maximin-share for OXS valuations. OXS is a class of valuation functions that lies between submodular and additive set functions. Cousins et al. [31] consider maximin-share for agents with bivalued submodular valuations, that is, each good provides a marginal gain of either a or b to each set.

Subadditive and fractionally subadditive valuations are studied extensively for various allocation scenarios and objectives, including maximizing social welfare [32–34], maximizing Nash social welfare [18,35–39], combinatorial auctions [40–43], and envy based fairnesses [44,36,45]. Here we mention some of these studies that are related to Nash social welfare.

For Nash social welfare and submodular valuations, Li and Vondrák [37] obtain an allocation algorithm with a 380 approximation ratio. For maximizing Nash social welfare under Rado valuations, Grag et al. [38] gave a constant factor allocation algorithm. For subadditive and fractionally subadditive valuations, the best known approximation guarantee for Nash social welfare $\Omega(n)$ [35,36,38]. Barman et al. [39] develop a polynomial-time algorithm that finds a 288-approximation for the optimal Nash social welfare for binary fractionally subadditive and binary subadditive valuation functions.

3. Preliminaries

Throughout this paper, we assume the set of agents is denoted by $\mathcal{N}$ and the set of items is referred to by $\mathcal{M}$. Let $|\mathcal{N}| = n$ and $|\mathcal{M}| = m$. We refer to the i'th agent by a_i and to the i'th item by b_i, i.e., $\mathcal{N} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{M} = \{b_1, b_2, \ldots, b_m\}$. We denote the valuation of an agent a_i for a set S of items by $U_i(S)$. Our interest is in valuation functions that are monotone and non-negative. More precisely, we assume $U_i(S) \geq 0$ for every agent a_i and set $S \subseteq \mathcal{M}$, and for every two sets S_1 and S_2 and every agent a_i we have $U_i(S_1 \cup S_2) \geq \max\{U_i(S_1), U_i(S_2)\}$.

We restrict our attention to two classes of set functions:

- **Fractionally subadditive (XOS)**: A fractionally subadditive set function $V(\cdot)$ can be shown via a finite set of additive functions $\{V_1, V_2, \ldots, V_\alpha\}$ where $V(S) = \max_{i=1}^{\alpha} V_i(S)$ for any subset S of the ground set.
- **Subadditive**: A set function $V(\cdot)$ is subadditive if $V(S_1) + V(S_2) \geq V(S_1 \cup S_2)$ for every two subsets S_1, S_2 of the ground set.

Let Π_r be the set of all partitionings of $\mathcal{M}$ into r disjoint subsets. For every r-partitioning $P^* \in \Pi_r$, we denote the partitions by $P_1^*, P_2^*, \ldots, P_r^*$. For a set function $V(\cdot)$, we define $\mathsf{MMS}_V^r(\mathcal{M})$ as follows:

$$\mathsf{MMS}_V^r(\mathcal{M}) = \max_{P^*\in\Pi_r} \min_{1\leq j\leq r} V(P_j^*).$$

Table 1
A summary of the results of this paper.

setting	previous guarantee	our improvement
fractionally subadditive	$1/5$ Ghodsi et al. [24]	$1/4.6$ Theorem 4.3
subadditive	$\Omega(\frac{1}{n\log n})$ Ghodsi et al. [24]	$\Omega(\frac{1}{\log n \log\log n})$ Theorem 4.2

For brevity we refer to $\mathsf{MMS}^n_{U_i}(\mathcal{M})$ by MMS_i. Since scaling the valuation functions does not affect the optimality of an allocation, we assume without loss of generality that $\mathsf{MMS}_i = 1$ holds for all agents.

An allocation of items to the agents is a vector $A = \langle A_1, A_2, \ldots, A_n\rangle$ where $\bigcup A_i = \mathcal{M}$ and $A_i \cap A_j = \emptyset$ for every two agents $a_i \neq a_j \in \mathcal{N}$. An allocation A is α-approximate MMS or α-MMS in short, if every agent a_i receives a subset of the items whose value to that agent is at least α times MMS_i. More precisely, A is α-MMS if and only if $U_i(A_i) \geq \alpha\mathsf{MMS}_i$ for every $a_i \in \mathcal{N}$.

We may sometimes give an item to several agents in which case we call it a multiallocation. A multiallocation of items to the agents is $\langle \mathsf{MMS}, k\rangle$ if each agent receives a bundle which is worth at least her MMS value and each item is allocated to at most k agents. Similarly, a multiallocation is $\langle \alpha\text{-}\mathsf{MMS}, k\rangle$ if each agent receives an α fraction of her MMS value and no item is allocated to more than k agents.

A well-known technique in finding approximate MMS allocations is reducing [24,13,46]. Here we bring a consequence of this technique, stated in Lemma 3.1.

Lemma 3.1 *([46]). Given that an α-MMS allocation exists under the assumption that the value of each item for each agent is bounded by α, the same guarantee carries over to the general setting without any bounds on the valuations.*

For a threshold $0 < t \leq 1$ and a set function V, Ghodsi et al. [24] define V^t as follows:

$$\forall S \subseteq M \quad V^t(S) = \min\{t, V(S)\}. \tag{1}$$

Ghodsi et al. [24] prove that V^t is structurally similar to V. More precisely, they prove Proposition 3.2.

Proposition 3.2 *(Ghodsi et al. [24]). For a valuation function V and any $0 < t < 1$,*

- *If V is submodular, then so is V^t.*
- *If V is fractionally subadditive, then so is V^t.*
- *If V is subadditive, then so is V^t.*

4. Our contribution

Our main contribution is an improved MMS guarantee for the fair allocation problem under subadditive valuations. The previous work of Ghodsi et al. [24] provides a guarantee of $\Omega(1/\log m)$ which we improve in this work.

We would like to compare our result to previous work before proceeding to the techniques and results. First, m denotes the number of items and can be exponentially large in terms of the number of agents. Thus, the $\Omega(1/\log m)$ guarantee of Ghodsi et al. [24] does not explicitly give any bound in terms of the number of agents n. We show in Section 5 that any guarantee that holds for $m = n^n$ items also carries over to $m > n^n$ items. Unfortunately, this only gives us a weak bound of $\Omega(1/(n\log n))$-MMS when plugging the reduction into the bound of Ghodsi et al. [24].

We improve this bound exponentially and obtain an $\Omega(\frac{1}{\log n\log\log n})$-MMS guarantee in the subadditive setting. In addition, we improve the analysis of Ghodsi et al. [24] for fractionally subadditive valuations, yielding a $1/4.6$-MMS guarantee for the fractionally subadditive setting. See Table 1.

4.1. Subadditive setting

Let us first point out to the main difficulty of the subadditive setting. Unlike the previously studied settings such as additive, submodular, and fractionally subadditive settings, the subadditive setting seems to be particularly challenging to tackle when it comes to randomized and probabilistic methods. Let us show this with an example: Let V be a monotone subadditive set function and S be a subset of the ground elements. It follows from the subadditivity of V that if we put each element of S in a set S' uniformly at random with probability α then $\mathbb{E}[V(S')] \geq \alpha V(S)$ holds. This is a strong bound that has been used in previous studies [32] when the goal is to bound the expected value of the outcome. For our problem, the goal is to bound the MMS guarantee in the worst case and therefore instead of a bound on the expected utilities of the agents, we need a bound on the utilities of the agents in the worst case. Thus, a question that becomes relevant to our analysis is how well is the value of $V(S')$ concentrated around its expectation?

While the answer to the above question is positive for additive, submodular, and fractionally subadditive functions, there are several counterexamples that show the value of $V(S')$ may well deviate from its expectation. That is, with a considerable probability,

$V(S')$ may be smaller than $(1-\epsilon)\mathbb{E}[V(S')]$ which is a highly undesirable situation in our analysis. Moreover, lower tail bounds on subadditive functions of i.i.d. chosen sets are not well-understood. Indeed, the authors are not aware of any bound that guarantees for some **constant** values $c_1, c_2 > 1$, $\Pr[V(S') \geq \frac{\mathbb{E}[V(S')]}{c_1}] \geq 1/c_2$.

For reasons that will become clear later in the section, our analysis needs such a bound in the subadditive setting. As part of our analysis, we show a weaker lower tail bound for subadditive functions which is of independent interest. We prove Lemma 4.1 in Section 6.4.

Lemma 4.1. *Let V be a monotone subadditive function with non-negative values such that for a set S we have $V(S)=1$. In addition, assume that for some value $0 < t < 1/256$, for every element $e_i \in S$ we have $V(\{e_i\}) \leq \frac{t}{\log 1/t}$. Let S' be a set made randomly from S such that each element of S appears in S' independently with probability at least t. Then we have $\Pr[V(S') \leq t/3] \leq 0.77$.*

Notice that Lemma 4.1 gives us a tail-bound on the valuation of a randomly chosen subset of items but this bound only holds with constant probability. Therefore, another challenge that we have in our analysis is to improve the guarantee of the bound down to $1 - 1/n - \epsilon$ such that by taking the union bound on the undesirable possibilities we can prove that a desired scenario exists for all agents at once. In what follows, we show how we prove such a guarantee.

Our algorithm consists of two steps. In the first step, we find a multiallocation of the items to the agents such that each item is given to at most $O(\log n)$ agents and moreover, each agent can divide her items into $\Omega(\log n)$ bundles such that each bundle is worth at least $1/8$ to her (recall that we assume all the MMS values are equal to 1). In the second step, we make an allocation out of our multiallocation by giving each item to one of the agents that receives the item in the multiallocation uniformly at random. The bound of Lemma 4.1 then implies that the bundle given to each agent is worth at least $\Omega(1/\log n)$ to her with probability more than $1 - 1/n$. Intuitively, this follows since in a bad event, each of the independent $\Omega(\log n)$ high-value bundles of an agent in the multiallocation should provide a small utility to that agent and thus the probability that none of the bundles provides such a utility decreases exponentially.

Therefore, the main algorithmic difficulty is to show that there exists a multiallocation of items to the agents with the desired properties. To this end, we leverage two techniques: First we define a modified utility function for each agent in a way that for an integer $c \geq 1$, the value of a set S is at least c if and only if items of S can be divided into $\Omega(c)$ disjoint subsets each having a large value for the corresponding agent. We then write a configuration LP that fractionally allocates the items to the agents in a way that meets our conditions. We then leverage the proof of Feige [32] that shows the integrality gap of the LP is bounded by 2. This implies that there is an integer solution for the LP in which for a considerable portion of the agents, the allocated bundle maintains our property. Finally, we show that by repeating the same procedure $O(\log n)$ times we can obtain the desired multiallocation.

Theorem 4.2. *Let $\mathcal{I}$ be an instance of the fair allocation problem such that the valuation of all agents are subadditive. Then, $\mathcal{I}$ admits an $\Omega(\frac{1}{\log n \log\log n})$-MMS allocation.*

The additional $\log\log n$ term in the denominator of the guarantee in Theorem 4.2 comes from the reducibility argument. Since the bound of Lemma 4.1 holds only if each item is worth no more than $O(\frac{1}{\log n \log\log n})$ to each agent, then we lose an additional $\log\log n$ factor in the guarantee.

4.2. Fractionally subadditive setting

Fractionally subadditive functions are special cases of subadditive functions. Ghodsi et al. [24] show that when the valuation functions are fractionally subadditive, there always exists a $1/5$-MMS allocation. We improve this result to 0.2192235-MMS.

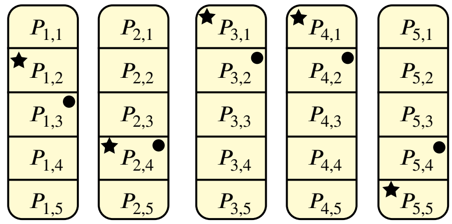
The structure of our proof is similar to that of [24]: we assume without loss of generality that the MMS values of the agents are equal to 1. For a certain threshold $0 < t$, we prove that an allocation A that maximizes $\sum_i U_i^t(A_i)$ is $t/2$-MMS. Ghodsi et al. [24] prove this claim for $t = 2/5$ and thus imply that a $1/5$-MMS allocation always exists. Via a more in-depth analysis, we prove that this holds for a slightly larger $t > 2/5$ but the analysis involves a more intricate process and a deeper analysis of the valuation functions.

Theorem 4.3. *Let $\mathcal{I}$ be an instance of the fair allocation problem such that the valuation of all agents are fractionally subadditive. Then, $\mathcal{I}$ admits a 0.219225-MMS allocation.*

5. An upper bound on the number of items

The goal of this section is to prove that we can assume without loss of generality that the number of items is upper-bounded by n^n. This in turn implies that the $\Omega(1/\log m)$-MMS guarantee of Ghodsi et al. [24] for subadditive valuations is also an $\Omega(\frac{1}{n \log n})$-MMS guarantee.

Observation 5.1. *For every instance I of the fair allocation problem with n agents and $m > n^n$ items, there exists an instance I' with n agents and m' items, such that*



$\vec{v}_\star = [2,4,1,1,5]$ $\qquad$ $\vec{v}_\bullet = [3,4,2,2,4]$

Fig. 1. An example for the representation vector of items.

- $m' \le n^n$.
- *Any α-MMS allocation for I' has an equivalent α-MMS allocation for I.*

Proof. For each item b_i in instance I, define an n-dimensional vector $\vec{v}_i$ where $\vec{v}_{i,j} = k$, if and only if b_i belongs to the k'th bundle of the optimal MMS-partition of agent a_j (see Fig. 1 for an example). Since each $\vec{v}_{i,j}$ is an integer in range $[1,n]$, the total number of different combinations of vectors is n^n. Now, define instance I' as follows: for each vector $\vec{v}'_k \in \{1,2,\dots,n\}^n$, there is an item b_k in I'. In addition, for every agent a'_i in I', the valuation function of a'_i is defined as

$$U'_i(S) = U_i\big(\{b_j \in I \mid \vec{v}_j = \vec{v}'_k \text{ for some } b'_k \in S\}\big).$$

Roughly speaking, in I', all the items whose corresponding vectors are the same are combined. Since the combined items belong to the same bundle for all the agents, this combination has no effect on the MMS values of the agents. In addition, it is easy to observe that any solution to I' has a corresponding solution to I with the same utility for the agents. Finally, we note that assuming U_i is subadditive, U'_i is also subadditive. Indeed, for every subset T_1 and T_2 of the items in instance I' we have

$$\begin{aligned}
U'_i(T_1 \cup T_2) &= U_i(\{b_j \in I \mid \vec{v}_j = \vec{v}'_k \text{ for some } b'_k \in T_1 \cup T_2\}) \\
&\le U_i(\{b_j \in I \mid \vec{v}_j = \vec{v}'_k \text{ for some } b'_k \in T_1\}) \\
&+ U_i(\{b_j \in I \mid \vec{v}_j = \vec{v}'_k \text{ for some } b'_k \in T_2\}) \\
&= U'_i(T_1) + U'_i(T_2),
\end{aligned}$$

where the second line is because U_i is subadditive. □

6. Subadditive valuations

In this section, we prove that an $\Omega(\frac{1}{\log n \log\log n})$-MMS allocation is guaranteed to exist when the valuations are subadditive. We have already explained the high-level ideas of our algorithms in Section 4.1. Here we discuss the proof in detail. Recall that we assume without loss of generality that the MMS value for each agent is equal to 1. From a technical point of view, our proof relies on two combinatorial and probabilistic techniques which we bring in the following.

The first observation implies that no matter what the MMS values are, we can always allocate the items to the agents in a way that a constant fraction of the agents receive a bundle whose value to them is at least a constant fraction of their MMS values.

Lemma 6.1. *For any instance of the fair allocation problem with subadditive valuations there always exists an allocation that guarantees $1/4$-MMS to at least $\lfloor n/3 \rfloor$ of the agents.*

We use Lemma 6.1 in an indirect way. As explained in Section 4.1, the first step of our algorithm is to find a multiallocation in a way that each item is given to at most $O(\log n)$ agents and that each agent can divide her items into $\Omega(\log n)$ bundles such that the value of each bundle to her is at least a constant fraction of her MMS value. In order to prove such a multiallocation exists, we first introduce a modified valuation function U'_i for each agent a_i such that (i) for each subset of size $O(n/\log n)$ of agents, the MMS values of the agents with respect to the modified valuation functions are $O(\log n)$ times larger than their original MMS values. (ii) if an agent receives a bundle of items whose value to her is a constant fraction of her new MMS value, then she can divide her bundle into $O(\log n)$ parts such that her original valuation for each part is at least some constant value. Using Lemma 6.1 in an iterative manner, we prove that such a multiallocation exists. We then leverage Lemma 4.1 to turn our multiallocation into a desired allocation.

Lemma 4.1. *Let V be a monotone subadditive function with non-negative values such that for a set S we have $V(S) = 1$. In addition, assume that for some value $0 < t < 1/256$, for every element $e_i \in S$ we have $V(\{e_i\}) \le \frac{t}{\log 1/t}$. Let S' be a set made randomly from S such that each element of S appears in S' independently with probability at least t. Then we have* $\Pr[V(S') \le t/3] \le 0.77$.

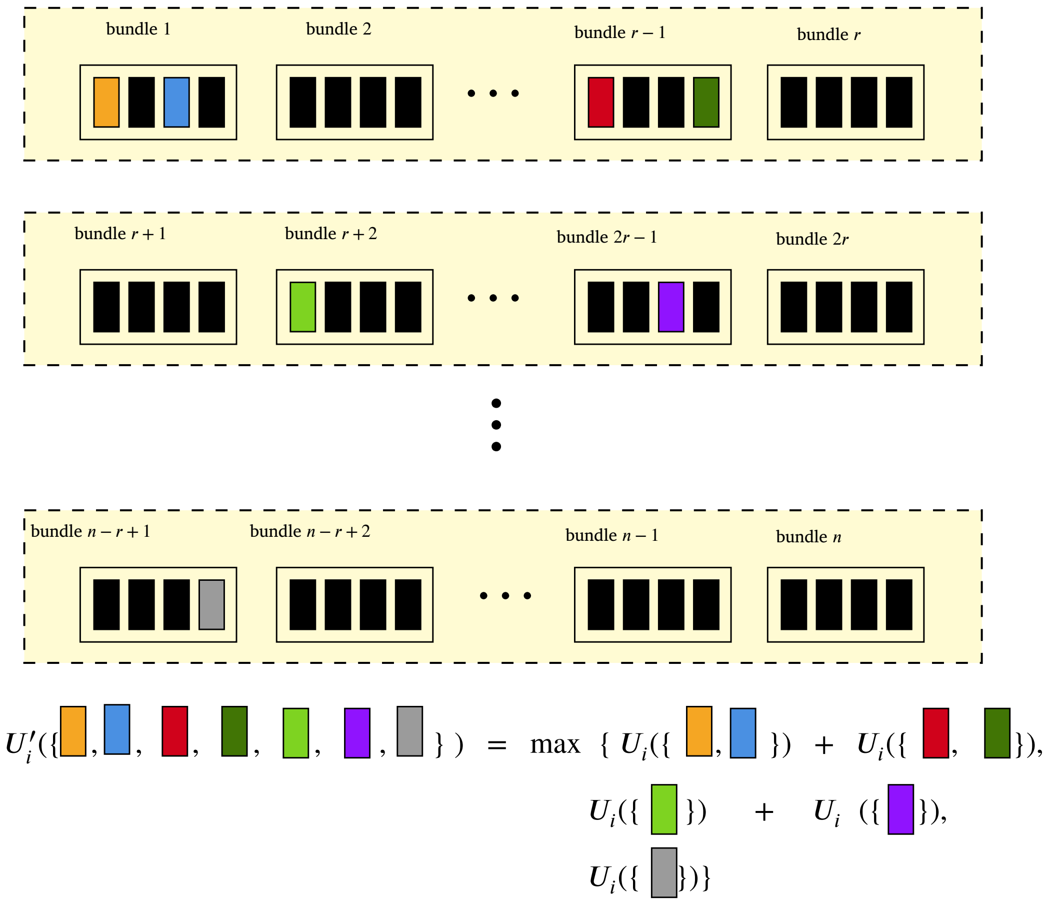


Fig. 2. The figure shows how the value of U_i' for each set is determined based on the values of U_i for each subset of elements. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

We defer the proofs of Lemmas 4.1 and 6.1 to Section 6.4 and Section 6.3. The first step and the second step of our algorithm are explained respectively in Sections 6.1 and 6.2.

6.1. *Constructing the multiallocation*

In the first part of our algorithm, we construct a multiallocation A with the following properties:

- Each agent a_i can partition her items into $\lfloor 6\log n\rfloor$ bundles each with value at least $1/8$ to her.
- No item is allocated to more than $\lceil 168\log n\rceil$ agents.

Let us begin our discussion in this section with a corollary of Lemma 6.1. Let $r < n$ be a parameter, and let $\mathcal{N}'$ be an arbitrary subset of $\mathcal{N}$ with size $\lfloor n/r\rfloor$. For each agent $a_i \in \mathcal{N}'$, let $P_{i,1}, P_{i,2}, \dots, P_{i,n}$ be the optimal MMS-partitioning of agent a_i, that is $U_i(P_{i,j}) \geq 1$ for all $1 \leq j \leq n$. We construct $\lfloor n/r\rfloor$ super-bundles $P^*_{i,1}, P^*_{i,2}, \dots, P^*_{i,\lfloor n/r\rfloor}$ such that each super-bundle $P^*_{i,j}$ contains at least r and at most $r+1$ bundles among $P_{i,1}, P_{i,2}, \dots, P_{i,n}$ and each bundle $P_{i,j}$ belongs to exactly one super-bundle. Next, we define a new valuation function U_i' for agent a_i as follows:

$$\forall S \subseteq M \qquad U_i'(S) = \max_{0\leq j<\lfloor n/r\rfloor}\left(\sum_{P_{i,l}\in P^*_{i,j}} U_i(S\cap P_{i,l})\right). \tag{2}$$

See Fig. 2 for a representation of $U_i'(\cdot)$. We show in Lemma 6.2 that the new valuation is subadditive.

Lemma 6.2. *For each agent $a_i \in \mathcal{N}'$, U_i' is subadditive.*

Proof. For two subsets S, T of items, we show that $U'_i(S \cup T) \le U'_i(S) + U'_i(T)$. Let j be such that

$$U'_i(S \cup T) = \sum_{P_{i,l} \in P^*_{i,j}} U_i\big((S \cup T) \cap P_{i,l}\big).$$

Since U_i is subadditive, for every $P_{i,l} \in P^*_{i,j}$, we have

$$U_i(S \cap P_{i,l}) + U_i(T \cap P_{i,l}) \ge U_i((S \cup T) \cap P_{i,l}).$$

Therefore,

$$\sum_{P_{i,l} \in P^*_{i,j}} U_i(S \cap P_{i,l}) + \sum_{P_{i,l} \in P^*_{i,j}} U_i(T \cap P_{i,l}) \ge \sum_{P_{i,l} \in P^*_{i,j}} U_i\big((S \cup T) \cap P_{i,l}\big). \tag{3}$$

The right-hand side of Inequality (3) equals to $U'_i(S \cup T)$ and the left-hand side is at most $U'_i(S) + U'_i(T)$. □

Now, consider an instance of the fair allocation problem with agents in $\mathcal{N}'$, valuation U'_i for each agent a_i, and all the items. Also, let MMS'_i be the maximin-share value of agent a_i in this instance. By the way we define the valuations for this instance, we know that for each agent a_i, we have $\mathsf{MMS}'_i \ge r$. To see why this holds, note that there are $\lfloor n/r \rfloor$ super partitions, and by Equation (2), for every super-partition $P^*_{i,j}$ we have

$$\begin{aligned} U'_i(P^*_{i,j}) &= \sum_{P_{i,l} \in P^*_{i,j}} U_i\left((P_{i,l} \cap P^*_{i,j}\right) \\ &= \sum_{P_{i,l} \in P^*_{i,j}} U_i(P_{i,l}) \\ &\ge \sum_{P_{i,l} \in P^*_{i,j}} 1 \\ &= r. \end{aligned}$$

By Lemma 6.1, we can allocate to $\lfloor |\mathcal{N}'|/3 \rfloor$ of the agents in $\mathcal{N}'$, a subset of items with value at least $r/4$ to them. Let a_i be one of these agents. For agent a_i, let A_i be the bundle of agent i in such allocation and define set Q_i as set of bundles in the original MMS partitioning of a_i, that contribute a subset with value at least $1/8$ to A_i, that is $Q_i = \{P_{i,j} \mid U_i(P_{i,j} \cap A_i) \ge 1/8\}$. We claim $|Q_i| \ge r/8$.

Lemma 6.3. *Let $a_i \in \mathcal{N}'$ be an agent that has received a bundle A_i with $U'_i(A_i) \ge r/4$. We have $|Q_i| \ge r/8$.*

Proof. Let $P^*_{i,j}$ be a super-bundle such that $U'_i(A_i) = \sum_{P_{i,l} \in P^*_{i,j}} U'_i(A_i \cap P_{i,l})$. We have

$$\begin{aligned} r/4 &\le U'_i(A_i) \\ &= \sum_{P_{i,l} \in P^*_{i,j}} U_i(A_i \cap P_{i,l}) \\ &< |\{P_{i,l} \mid U_i(P_{i,l} \cap A_i) \ge 1/8\}| + |\{P_{i,l} \mid U_i(P_{i,l} \cap A_i) < 1/8\}|/8 \\ &\le |Q_i| + (r + 1 - |Q_i|)/8. \end{aligned} \tag{4}$$

Inequality (4) implies $|Q_i| \ge r/7 - 1/56$. Since $r \ge 1$, we have $|Q_i| \ge r/8$. □

Corollary 6.4 *(Lemmas 6.1 and 6.3). Given a set of n agents with subadditive valuations. For any arbitrary subset $\mathcal{N}'$ of the agents with size at most $\lfloor n/r \rfloor$, it is possible to select a subset of at least $\lfloor \frac{|\mathcal{N}'|}{3} \rfloor$ of the agents in $\mathcal{N}'$, and allocate each agent a_i a bundle A_i of the items such that $|Q_i| \ge r/8$.*

Based on Corollary 6.4, we perform the first stage of our allocation algorithm by choosing $r = 48 \log n$ and iteratively running the following steps until no agent remains:

- Select a set $\mathcal{N}'$ of the remaining agents with size $\lfloor n/r \rfloor$. If the total number of the remaining agents is less than $\lfloor n/r \rfloor$, select all the remaining agents.
- Using Corollary 6.4, find a subset of size at least $\lfloor |\mathcal{N}'|/3 \rfloor$ of the agents in $\mathcal{N}'$ and allocate to each agent a_i in this subset a bundle A_i of items such that $|Q_i| \ge r/8$.

Algorithm 1: Finding a multiallocation.

Procedure `Allocate`(*$\mathcal{N}$: set of remaining agents, M: set of goods*):
  if $\mathcal{N} = \emptyset$ **then**
    return
  $\mathcal{N}'$ = a subset of size $\min(\lfloor n/r \rfloor, |\mathcal{N}|)$ of $\mathcal{N}$
  foreach $a_i \in \mathcal{N}'$ **do**
    Construct U_i'.
  A = Allocation defined in Corollary 6.4
  $\mathcal{N}''$ = agents that receive a bundle in A.
  foreach $a_i \in \mathcal{N}''$ **do**
    Allocate A_i to a_i
    Remove a_i from $\mathcal{N}$
  `Allocate`$(\mathcal{N}, M)$;

Algorithm 2: Random allocation algorithm.

Input: A multiallocation A obtained in the first part.
Output: An $\Omega(\log n)$-MMS allocation.
foreach $b \in M$ **do**
  Let $S = \{a_i \mid b \in A_i\}$
  Allocate b to one of the agents in S uniformly at random.

- Remove the agents that receive a bundle in the previous step and repeat these steps for the remaining agents and **all the items**. Note that, the goal is to find a multiallocation, so an item might be allocated in multiple rounds.

Algorithm 1 shows a pseudocode of our method for this step.

Lemma 6.5. *At the end of Algorithm 1, the following properties hold:*

- *Each agent a_i can partition her items into $6 \log n$ bundles each with value at least $1/8$ to her.*
- *Each item is allocated to at most $168 \log n$ agents.*

Proof. The first property follows from Lemma 6.3. For the second property, note that as long as the number of the remaining agents is more than $\lfloor n/r \rfloor$, our algorithm satisfies at least $\lfloor n/(3r) \rfloor$[3] agents in each step. Therefore, after a number of steps which is less than $3r$, the number of remaining agents shrinks to less than $\lfloor n/r \rfloor$. After that, in each step, at least $1/3$ of the remaining agents are removed from the process. Therefore, it takes no more than $\log_{3/2} \lfloor n/r \rfloor + 1$ steps to remove all the agents.[4] Hence, the total number of steps is

$$\begin{aligned} 3r + \log_{3/2}(\lfloor n/r \rfloor) + 1 &\le 3r + \log_{3/2} n && r \ge 2 \\ &\le 4r && r \ge \log_{3/2} n \\ &= 192 \log n && r = 48 \log n. \end{aligned}$$

Thus, no item is allocated to more than $192 \log n$ agents. □

6.2. From multiallocation to allocation

Recall that in a multiallocation, we might allocate a good to multiple agents. Let A be the multiallocation obtained in Section 6.1. We know by Lemma 6.5 that in A, each item belongs to at most $192 \log n$ agents. In this step, we convert A into an allocation via a simple procedure: for each item that is allocated to multiple agents, we select one of them independently and uniformly at random and allocate the item to her. Algorithm 2 shows a pseudocode for this procedure.

In Lemma 6.6 we prove that assuming that the items are small enough, with a non-zero probability, this process guarantees for each agent a bundle with a value at least $\Omega(1/\log n)$ to her.

Lemma 6.6. *Let A be the multiallocation of Algorithm 1, and let A' be the allocation obtained by running Algorithm 2 on A and let $t = \frac{1}{1536 \log n}$. Then, assuming that the value of each item for each agent is less than $\frac{t}{\log \frac{1}{t}}$, in A' with probability more than $1 - 1/n$ each agent receives a bundle with a value of $\frac{1}{4608 \log n}$ to her.*

[3] Note that, for a real number $x \in \mathbb{R}^+$ and integer $y \in \mathbb{Z}^+$, we always have $\lfloor \lfloor x \rfloor / y \rfloor = \lfloor x/y \rfloor$.

[4] If two agents are remained, we can simply allocate the items to them using a cut and choose method.

Proof. Consider an arbitrary agent a_i. By Corollary 6.4 we know that a_i can partition her share into $k \geq 6\log n$ bundles, each with value at least $1/8$ to her. Let $B_1, B_2, \dots, B_k$ be these bundles. By definition, for every B_j we have $U_i(B_j) \geq 1/8$. Let B'_j be the items in B_j that remain for agent a_i after running Algorithm 2. Since each good belongs to at most $192\log n$ agents, each item remains for a_i with probability at least $1/(192\log n)$ and therefore,

$$\mathbb{E}[U_i(B'_j)] \geq \frac{U_i(B_j)}{192\log n} \geq \frac{1}{1536\log n}.$$

By Lemma 4.1, assuming that the value of each item to each agent is smaller than $\frac{1}{1536\log n\log(1536\log n)}$, for every $1 \leq j \leq k$ we have $\Pr\left[U_i(B'_j) \leq \frac{1}{4608\log n}\right] \leq 0.77$. Therefore, with probability at least $1-(0.77)^k$, for at least one bundle $1 \leq j \leq k$ we have

$$U_i(A'_i) \geq U_i(B'_j) \geq \frac{1}{4608\log n}. \tag{5}$$

Since $k \geq 6\log n$ and,

$$\begin{aligned}1 - 0.77^k &\geq 1 - 0.77^{6\log n}\\ &\geq 1 - 0.77^{2\log_{0.77}(1/2)\log n} \geq 1 - \frac{1}{n^2},\end{aligned}$$

using union bound we conclude that with probability at least $1 - n(1/n^2) = 1 - 1/n$, Inequality (5) holds for all the agents. This in turn implies that with non-zero probability, our allocation is $\frac{1}{4608\log n}$-MMS. Therefore, such an allocation always exists. □

Finally, note that in order for Lemma 6.6 to hold, we need the value of each item for each agent to be upper bounded by $\frac{1}{1536\log n\log(1536\log n)}$. To resolve this, we choose the objective to find a $\frac{1}{1536\log n\log(1536\log n)}$-MMS allocation. By Lemma 3.1, for this objective we can assume that the value of each item for each agent is upper bounded by $\frac{1}{1536\log n\log(1536\log n)}$, and hence, the condition of Lemma 6.6 is satisfied. This reduces the final approximation factor to $\Omega(1/(\log n\log\log n))$.

Theorem 4.2. *Let $\mathcal{I}$ be an instance of the fair allocation problem such that the valuation of all agents are subadditive. Then, $\mathcal{I}$ admits an $\Omega(\frac{1}{\log n\log\log n})$-MMS allocation.*

6.3. Satisfying a fraction of agents

In this section, we prove Lemma 6.1. This Lemma states that for an instance of the fair allocation problem with subadditive valuations, there always exists an allocation that allocates to at least $\lfloor n/3\rfloor$ of the agents a bundle with value at least $1/4$. Here we bring the statement of Lemma 6.1.

Lemma 6.1. *For any instance of the fair allocation problem with subadditive valuations there always exists an allocation that guarantees $1/4$-MMS to at least $\lfloor n/3\rfloor$ of the agents.*

In our proof, we use the method of Feige [32] for maximizing welfare when the valuations are subadditive. Assume that the agents' valuations are subadditive and the objective is to maximize social welfare. This problem can be formulated as the following integer program:

$$\begin{aligned}\max \quad & \sum_{i,S} x_{i,S}\cdot U_i(S)\\ \text{s.t.} \quad & \sum_{i,S|b_j\in S} x_{i,S} \leq 1, \qquad \forall b_j\\ & \sum_{S\subseteq M} x_{i,S} \leq 1, \qquad \forall a_i\\ & x_{i,S} \in \{0,1\}, \ \forall a_i \text{ and } S\subseteq M\end{aligned} \tag{6}$$

Roughly speaking, Program (6) allocates the items to the agents in a way that each item is allocated to at most one agent (first set of constraints) and each agent receives at most one subset (second set of constraints). The linear relaxation of Program (6) is a famous linear program, especially in allocation problems. This LP is known as *configuration LP*.

$$\begin{aligned}\max \quad & \sum_{i,s} x_{i,S}\cdot U_i(S)\\ \text{s.t.} \quad & \sum_{i,S|b_j\in S} x_{i,S} \leq 1, \qquad \forall b_j\\ & \sum_{S\subseteq M} x_{i,S} \leq 1, \qquad \forall a_i\\ & x_{i,S} \geq 0, \qquad \forall a_i \text{ and } S\subseteq M\end{aligned} \tag{7}$$

Note that, despite the exponential number of constraints, assuming demand queries can be answered in polynomial time, it is possible to find a solution to LP (7) in polynomial time. Feige in [32] proposes a randomized rounding technique to produce a feasible integer allocation with expected welfare at least half of the value of LP (7). In other words, Feige [32] proves that the integrality gap of the configuration LP is at most 2 for subadditive valuations. Here, we use this fact to prove that there always exists an allocation that satisfies the conditions of Lemma 6.1.

Recall the definition of V^t. In Proposition 3.2, we state a very useful property of these valuations: for a monotone and subadditive set function V, V^t is also subadditive. According to this fact, consider the following LP:

$$\begin{aligned} \max \sum_{i,s} \quad & x_{i,S} \cdot U_i^1(S) \\ \text{s.t.} \sum_{i,S|j\in S} \quad & x_{i,S} \leq 1, \qquad \forall b_j \\ \sum_{S\in P(M)} \quad & x_{i,S} \leq 1, \qquad \forall a_i \\ & x_{i,S} \geq 0, \qquad \forall a_i \text{ and } S \subseteq M \end{aligned} \tag{8}$$

Note that LP (8) is similar to LP (7), except that U_i is replaced by U_i^1. Since for any subset S of items, we know that $U_i^1(S) \leq 1$, the objective of LP (8) is upper bounded by n. Also, consider the following fractional solution: for every set S and agent a_i, if S is one of the bundles in the optimal MMS partitioning of agent a_i, set $x_{i,S} = 1/n$ and set $x_{i,S} = 0$ otherwise. One can easily verify that this is a feasible solution to LP (8), with an expected welfare of n. Therefore, the answer of LP (8) is exactly n. Since the integrality gap of the configuration LP is bounded by 2, there exists an integral solution (an allocation) that obtains an objective of at least $n/2$. Denote such an allocation by $A = \langle A_1, A_2, \dots, A_n\rangle$. We know that

$$\sum_{1\leq i\leq n} U_i^1(A_i) \geq n/2.$$

Based on the above observation, we prove that allocation A satisfies the conditions of Lemma 6.1. Let S be the set of agents that receive a bundle with value at least $1/4$ to them, and assume for contradiction that $|S| < \lfloor n/3 \rfloor$. The contribution of these agents to the bounded social welfare is at most $|S|$. Also, the contribution of the rest of the agents to the social welfare is less than $(n-|S|)/4$. Therefore, the social welfare is upper bounded by

$$\begin{aligned} (n-|S|)/4+|S| &= n/4+3|S|/4 \\ &< n/4+n/4 \\ &= n/2. \end{aligned}$$

But we already know that the bounded social welfare of A is at least $n/2$, which is a contradiction.

6.4. A lower-tail bound for subadditive valuations

As mentioned earlier, one restriction of subadditive valuations is that no lower-tail concentration inequality is currently known in the literature for them. For the upper-tail case, the following inequality is shown by Schechtman [47] which we use here.

Theorem 6.7 *(Corollary 12, [47]). Let $Z = f(X_1, X_2, \dots, X_r)$ where f is a non-negative and subadditive function with marginal values in $[0,1]$, and $X_i \in \{0,1\}$ are i.i.d. random variables. Then, for all $a > 0$, any $1 \leq k \leq r$, and any integer $q \in \mathbb{N}$ we have*

$$\Pr[Z \geq (q+1)a + k] \leq \Pr[f \leq a]^{-q} q^{-k}. \tag{9}$$

In particular, if $a = \mathbb{E}[Z]/3$, and $q = 2$, Inequality (9) implies

$$\Pr[Z \geq \mathbb{E}[Z] + k] \leq \Pr[f \leq \mathbb{E}[Z]/3]^{-2} 2^{-k} \tag{10}$$

We use this bound to obtain a relatively weaker lower-tail bound for subadditive valuation functions, which we state in Lemma 4.1.

Lemma 4.1. *Let V be a monotone subadditive function with non-negative values such that for a set S we have $V(S) = 1$. In addition, assume that for some value $0 < t < 1/256$, for every element $e_i \in S$ we have $V(\{e_i\}) \leq \frac{t}{\log 1/t}$. Let S' be a set made randomly from S such that each element of S appears in S' independently with probability at least t. Then we have $\Pr[V(S') \leq t/3] \leq 0.77$.*

Proof. Let $|S| = r$ and denote its elements by $e_1, e_2, \dots, e_r$. We define a function f with r arguments such that for $X_1, X_2, \dots, X_r \in \{0,1\}$ we have

$$f(X_1, X_2, \dots, X_r) = V(\{e_i \mid X_i = 1\}) \frac{\log 1/t}{t}. \tag{11}$$

Let the value of X_i be chosen independently from $\{0,1\}$ in a way that $\Pr[X_i = 1] = t$. For $Z = f(X_1, X_2, \ldots, X_r)$ the following conditions hold:

- Since for each element e_j we have $V(\{e_j\}) \le \frac{t}{\log 1/t}$, the marginal contribution of each X_i is in range $[0,1]$, that is

$$\forall_{1\le k\le r} \max f(X_{-k}, X_k = 1) - f(X_{-k}, X_k = 0) \in [0,1]$$

  where the max is taken over all combinations for $X_1, X_2, \ldots, X_r$ and $X_{-k} = X_1, X_2, \ldots, X_{k-1}X_{k+1}, \ldots, X_r$.
- Since $V(S) = 1$, we have

$$f(1,1,\ldots,1) = \frac{\log(1/t)}{t}. \tag{12}$$

- Since V is subadditive and each X_i is equal to 1 with probability t, by [48] we have

$$\mathbb{E}[Z] = \mathbb{E}[V(S')]\frac{\log 1/t}{t} \ge t\frac{\log 1/t}{t} = \log 1/t. \tag{13}$$

By choosing $k = \log \frac{32\log 1/t}{t}$ in Inequality (10) we obtain

$$\Pr[Z \ge \mathbb{E}[Z] + \log(32\frac{\log 1/t}{t})] \le \frac{1}{\Pr[Z \le \mathbb{E}[Z]/3]^2 \frac{32\log 1/t}{t}}.$$

Note that by Inequality (13), we know that $\mathbb{E}[Z] \ge \log(1/t)$ and hence,

$$\begin{aligned}\mathbb{E}[Z] + \log(32\frac{\log 1/t}{t}) &= \mathbb{E}[Z] + \log(32\log 1/t) + \log 1/t\\ &\le 2\mathbb{E}[Z] + 5 + \log\log 1/t.\\ &\le 3\mathbb{E}[Z]\end{aligned}$$

where the second line is due to Inequality (13) and the third line is by our assumption that $\log(1/t) \ge 8$. Therefore,

$$\Pr[Z \ge 3\mathbb{E}[Z]] \le \frac{1}{\Pr[Z \le \mathbb{E}[Z]/3]^2 \frac{32\log 1/t}{t}}. \tag{14}$$

Now, let Π be the set of all 2^n combinations for variables $X_1, X_2, \ldots, X_n$. By definition, we know that

$$\mathbb{E}[Z] = \sum_{\pi\in\Pi} \Pr[\pi] f(\pi).$$

We divide Π into three subsets, Π_1, Π_2, and Π_3 as follows:

$$\begin{aligned}\Pi_1 &= \{\pi \in \Pi \mid f(\pi) < \mathbb{E}[Z]/3\},\\ \Pi_2 &= \{\pi \in \Pi \mid \mathbb{E}[Z]/3 \le f(\pi) \le 3\mathbb{E}[Z]\},\\ \Pi_3 &= \{\pi \in \Pi \mid f(\pi) > 3\mathbb{E}[Z]\}.\end{aligned}$$

Let $\Pr[\Pi] = \sum_{\pi\in\Pi} \Pr[\pi]$. We have

$$\mathbb{E}[Z] \le \Pr[\Pi_1]\mathbb{E}[Z]/3 + 3\Pr[\Pi_2]\mathbb{E}[Z] + \Pr[\Pi_3]\frac{\log 1/t}{t}. \tag{15}$$

The factor $\frac{\log 1/t}{t}$ in Inequality (15) is due to Inequality (12). Plugging Inequality (14) into Inequality (15) yields that

$$\mathbb{E}[Z] \le \Pr[\Pi_1]\mathbb{E}[Z]/3 + 3\Pr[\Pi_2]\mathbb{E}[Z] + \frac{\frac{\log 1/t}{t}}{\Pr[Z \le \mathbb{E}[Z]/3]^2 32\frac{\log 1/t}{t}}$$

and therefore

$$\begin{aligned}1 &\le \Pr[\Pi_1]/3 + 3\Pr[\Pi_2] + \frac{1}{32\Pr[\Pi_1]^2\mathbb{E}(Z)}\\ &\le \Pr[\Pi_1]/3 + 3\Pr[\Pi_2] + \frac{1}{32\Pr[\Pi_1]^2} \qquad \mathbb{E}(Z) \ge 1\end{aligned}$$

Since $\Pr[\Pi_1] \le 1 - \Pr[\Pi_2]$, we have

$$1 \le \Pr[\Pi_1]/3 + 3(1 - \Pr[\Pi_1]) + \frac{1}{32\Pr[\Pi_1]^2}. \tag{16}$$

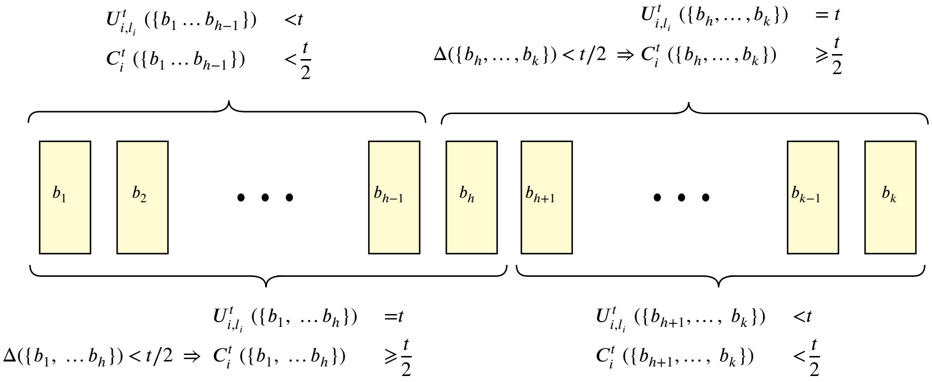


Fig. 3. Bundle S contains k items. In this bundle, h is the smallest index such that $U^t_{i,l_i}(b_1, b_2, \dots, b_h) = t$.

Inequality (16) for $\Pr[\Pi_1]$ implies

$$\Pr[\Pi_1] \le \frac{1}{8}\left(2 + \sqrt[3]{11 - \sqrt[2]{57}} + \sqrt[3]{11 + \sqrt[2]{57}}\right) < 0.77.$$

As a consequence, by Equation (11) we obtain

$$\Pr[V(S') \le t/3] < 0.77. \quad \square$$

7. Fractionally subadditive valuations

We improve the result of Ghodsi et al. [24] for fractionally subadditive valuations and show that a 0.2192235-MMS allocation always exists. Our method is based on the notion of bounded welfare, introduced by Ghodsi et al. [24].

Recall that for a subadditive function V, $V^t(S)$ is defined as $\min(V(S), t)$. Fix a constant t (we later determine the exact value of t) and let A be an allocation that maximizes the bounded social welfare, that is, $w = \sum_j U^t_j(A_j)$. Since for every agent a_i, the value of $U^t_i(S)$ for any set S of goods is upper bounded by t, a trivial upper bound on the value of w is nt. We show that for a properly chosen threshold $2/5 < t < 1/2$, we can guarantee that every agent receives a bundle in A whose value for the agent is at least $t/2$. We first define the contribution of the items to w.

Definition 7.1. For every agent a_j let $\{U^t_{j,1}, U^t_{j,2}, \dots, U^t_{j,\alpha_j}\}$ be the set of additive functions such that for every subset S of items, $U^t_j = \max_{1\le l\le \alpha_j} U^t_{j,l}(S)$. Then, for every $S \subseteq M$, we define the contribution of S to w, denoted by $C(S)$ as

$$C(S) = \sum_{1\le j\le n} U^t_{j,l_j}(S \cap A_j),$$

where $l_j = \arg\max_{1\le l\le \alpha_j} U^t_{j,l}(A_j)$.

One can easily observe that function $C(\cdot)$ is additive. Also, since for every agent a_j, U^t_j is fractionally subadditive, we have

$$\forall S \subseteq A_j \qquad U^t_j(A_j \setminus S) \ge U^t_j(A_j) - C(S). \tag{17}$$

Now, assume that there exists an agent a_i such that $U^t_i(A_i) < t/2$. Since $\mathsf{MMS}_i = 1$ (under valuation $U_i(\cdot)$), agent a_i can partition the goods into n sets with value at least 1 to her. Since $w < nt$[5] and $C(\cdot)$ is additive, the contribution of at least one of these bundles to the value of w is less than t. Let $S = \{b_1, b_2, \dots, b_k\}$ be the set of goods in this bundle. Also, let U_{i,l_i} be the additive function that defines the value of $U_i(S)$, i.e., $U_{i,l_i}(S) = U_i(S)$. We assume without loss of generality that the goods in S are sorted according to their U_{i,l_i} *value per contribution* that is,

$$\frac{U_{i,l_i}(\{b_1\})}{C(\{b_1\})} \ge \frac{U_{i,l_i}(\{b_2\})}{C(\{b_2\})} \ge \dots \ge \frac{U_{i,l_i}(\{b_k\})}{C(\{b_k\})} \tag{18}$$

(see Fig. 3). For a set $T \subseteq S$, we define $\Delta(T) := U^t_{i,l_i}(T) - C(T)$. Since allocation A maximizes the bounded social welfare, there is no way to increases w by modifying A. This yields Observation 7.1.

[5] Note that, at least one agent has received a bundle with value strictly less than t.

Observation 7.1. *For every subset $T \subseteq S$, $\Delta(T) < t/2$.*

Proof. If $\Delta(T) \geq t/2$, we can take the goods in T back from their owners and allocate them to a_i. We have

$$\begin{aligned}\sum_{1\leq j\leq n} U_j^t(A_j \setminus T) &= \sum_{1\leq j\leq n} U_j^t(A_j \setminus (T \cap A_j))\\ &\geq \sum_{1\leq j\leq n} \left(U_j^t(A_j) - C(T \cap A_j)\right)\\ &\geq w - \sum_{1\leq j\leq n} C(T\cap A_j)\\ &\geq w - C(T),\end{aligned}$$

where the second line is due to Inequality (17) and the fourth line is because $C(\cdot)$ is additive. On the other hand, the value of A_i to agent a_i is less than $t/2$. Thus, allocating T to agent a_i increases her utility by a value more than $U_i^t(T) - t/2$. Therefore, the total bounded welfare would be at least

$$\begin{aligned}w - C(T) + U_i^t(T) - U_i^t(A_i) &> w - C(T) + U_i^t(T) - t/2\\ &\geq w - C(T) + U_{i,l_i}^t(T) - t/2\\ &= w + \Delta(t) - t/2\\ &\geq w.\end{aligned}$$

This contradicts the maximality of w. $\square$

A simple corollary of Observation 7.1 is that agent a_i cannot divide the goods in S into two subsets T_1 and T_2 ($T_1 \cap T_2 = \emptyset$), such that $U_i^t(T_1), U_i^t(T_2) = t$. Otherwise, since $C(S) \leq t$ we have $U_i^t(T_1) + U_i^t(T_2) - C(S) \geq 2t - t = t$. By additivity of $C(\cdot)$ we can write $U_i^t(T_1) + U_i^t(T_2) - C(T_1) - C(T_2) \geq t$, which means at least one of $\Delta(T_1)$ or $\Delta(T_2)$ is at least $t/2$.

Corollary 7.2 *(Observation 7.1). There are no subsets $T_1, T_2 \subseteq S$ such that $T_1 \cap T_2 = \emptyset$, $U_{i,l_i}^t(T_1) = U_{i,l_i}^t(T_2) = t$.*

Let h be the smallest index such that $U_{i,l_i}^t(\{b_1, b_2, \dots, b_h\}) = t$. By Corollary 7.2, we know that $U_{i,l_i}^t(\{b_{h+1}, b_{l+2}, \dots, b_k\}) < t$. Let

$$\gamma = t - U_{i,l_i}^t(\{b_1, \dots, b_{h-1}\}), \gamma' = t - U_{i,l_i}^t(\{b_{h+1}, \dots, b_k\}). \tag{19}$$

Notice that both γ and γ' are larger than 0. Since the value of S to agent a_i is at least 1, $U_{i,l_i}^t(\{b_h\}) \geq 1 - 2t + \gamma + \gamma'$.

Observation 7.2. *We have $C(\{b_1, \dots, b_{h-1}\}) < t/2$ and $C(\{b_{h+1}, \dots, b_k\}) < t/2$.*

Proof. Since these two cases are symmetric, we only prove $C(\{b_1, \dots, b_{h-1}\}) < t/2$. Since $t < 1/2$ and $U_i(\{b_1, b_2, \dots, b_k\}) \geq 1$, we have $U_{i,l_i}^t(\{b_h, b_{h+1}, \dots, b_k\}) = t$ and by Observation 7.1, $\Delta(\{b_h, b_{h+1}, \dots, b_k\}) < t/2$. Therefore, we have $C(\{b_h, b_{h+1}, \dots, b_k\}) \geq t/2$. Since $C(S) < t$, by additivity of $C(\cdot)$, we have $C(\{b_1, b_2, \dots, b_{h-1}\}) < t/2$. $\square$

Based on Observation 7.2 define $\delta, \delta' > 0$ such that

$$\delta = t/2 - C(\{b_1, \dots, b_{h-1}\}), \delta' = t/2 - C(\{b_{h+1}, \dots, b_k\}). \tag{20}$$

Note that by Observation 7.1, $\delta < \gamma$ and $\delta' < \gamma'$. Also, since $C(S) < t$, $C(\{b_h\}) \leq \delta + \delta'$, and by Inequality (18), we have

$$\frac{t-\gamma}{t/2-\delta} \geq \frac{U_{i,l_i}^t(\{b_h\})}{C(\{b_h\})} \geq \frac{t-\gamma'}{t/2-\delta'}. \tag{21}$$

Finally, assuming that the goal is to find a $t/2$-MMS allocation, by Lemma 3.1, we can restrict our attention to the cases that the value of each good to each agent is less than $t/2$. Therefore,

$$1 - 2t + \gamma + \gamma' < t/2, \qquad \delta + \delta' < t/2. \tag{22}$$

To conclude, if for every subset T of goods $\Delta(T) < t/2$ holds, the following inequalities must be satisfied:

$$\begin{array}{ll}
\dfrac{t-\gamma}{t/2-\delta} \geq \dfrac{U^t_{i,l_i}(\{b_h\})}{C(\{b_h\})}, & \text{Inequality (21)} \\
\dfrac{U^t_{i,l_i}(\{b_h\})}{C(\{b_h\})} \geq \dfrac{t-\gamma'}{t/2-\delta'} & \text{Inequality (21)} \\
1-2t+\gamma+\gamma' \leq U^t_{i,l_i}(\{b_h\}) & \\
C(\{b_h\}) \leq \delta+\delta' & \\
C(\{b_h\}) \geq \delta, \delta' & \text{Observation 7.1} \\
U^t_{i,l_i}(\{b_h\}), C(\{b_h\}) < t/2 & \text{Inequality (22)} \\
t > \gamma, \quad t > \gamma' & \\
\gamma > \delta, \quad \gamma' > \delta' & \text{Observation 7.1} \\
\gamma, \gamma', t, \delta, \delta' > 0 &
\end{array}$$

We show in Section 7.1 that in order for all the above inequalities to hold, the value of t cannot be arbitrarily small. Indeed, we show that the answer of the following program is at least $t \simeq 0.438447$.

$$\begin{array}{l}
\min\ t \\
\qquad \dfrac{t-\gamma}{t/2-\delta} \geq \dfrac{U^t_{i,l_i}(\{b_l\})}{C(\{b_l\})}, \\
\qquad \dfrac{U^t_{i,l_i}(\{b_h\})}{C(\{b_h\})} \geq \dfrac{t-\gamma'}{t/2-\delta'} \\
\qquad 1-2t+\gamma+\gamma' \leq U^t_{i,l_i}(\{b_l\}) \\
\qquad C(\{b_h\}) \leq \delta+\delta' \\
\qquad C(\{b_h\}) \geq \delta, \delta' \\
\qquad U^t_{i,l_i}(\{b_h\}), C(\{b_h\}) < t/2 \\
\qquad t > \gamma, \quad t > \gamma' \\
\qquad \gamma > \delta, \quad \gamma' > \delta' \\
\qquad \gamma, \gamma', t, \delta, \delta' > 0
\end{array} \tag{23}$$

This means that for any threshold t less than 0.438447, the set of inequalities in Optimization Program (23) cannot be simultaneously met and therefore, there is a subset T with $\Delta(T) \geq t/2$. This contradicts Observation 7.1. Thus, Lemma 7.3 holds for $t = 0.438447$.

Lemma 7.3. *Let $t \leq 0.438447$, and let A be an allocation that maximizes $\sum_i U^t_i(A_i)$. Then, every agent i in A receives a bundle with value at least $t/2$ to her.*

Lemma 7.3 states that for any $t \leq 0.438447$, there exists a $t/2$-MMS allocation. Therefore, Theorem 4.3 holds.

Theorem 4.3. *Let $\mathcal{I}$ be an instance of the fair allocation problem such that the valuation of all agents are fractionally subadditive. Then, $\mathcal{I}$ admits a 0.219225-MMS allocation.*

7.1. The optimal value of Program (23)

Recall that in order to prove Theorem 4.3, we need to find an optimal answer to Program (23). which we also bring in the following:

$$\begin{array}{ll}
\text{minimize}\ t & \\
\qquad \dfrac{t-\gamma}{t/2-\delta} \geq \dfrac{U^t_{i,l_i}(\{b_h\})}{C(\{b_h\})}, & \text{Inequality (21)} \\
\qquad \dfrac{U^t_{i,l_i}(\{b_h\})}{C(\{b_h\})} \geq \dfrac{t-\gamma'}{t/2-\delta'} & \text{Inequality (21)} \\
\qquad 1-2t+\gamma+\gamma' \leq U^t_{i,l_i}(\{b_h\}) & \\
\qquad C(\{b_h\}) \leq \delta+\delta' & \\
\qquad C(\{b_h\}) \geq \delta, \delta' & \text{Observation 7.1} \\
\qquad U^t_{i,l_i}(\{b_h\}), C(\{b_h\}) \leq t/2 & \text{Inequality (22)} \\
\qquad t \geq \gamma, \quad t \geq \gamma' & \\
\qquad \gamma \geq \delta, \quad \gamma' \geq \delta' & \text{Observation 7.1} \\
\qquad \gamma, \gamma', t, \delta, \delta' \geq 0 &
\end{array} \tag{24}$$

For the sake of simplicity, all the strict inequalities in Program (23) are converted into loose ones in Program (24), i.e., all $>$ and $<$ operators are replaced respectively by $\geq$ and $\leq$. Since any solution to Program (23) is also a solution to Program (24), we are guaranteed that the optimal value of Program (24) is less than or equal to the optimal value of Program (23). In this section, we prove that the objective value of Program (24) is more than 0.4384477. We denote this value by t^* throughout the section. To prove our claim, we first simplify Program (24) using several reductions and then find the optimal answer using standard calculus.

In the first step, we omit $U^t_{i,l_i}(\{b_h\})$ and $C(\{b_h\})$ from program 24 using Observation 7.3.

Observation 7.3. *Let t^* be the optimal answer of Program (24) with the additional constraints that $U^t_{i,l_i}(\{b_h\}) = 1 - 2t + \gamma + \gamma'$ and $C(\{b_h\}) = \delta + \delta'$. Also, suppose that for some threshold t the following holds: and bundle $S = \{b_1, b_2, \dots, b_k\}$, we have*

$$\forall T \subseteq S \qquad \Delta(T) < t/2.$$

Then, we have $t \geq t^*$.

Proof. Recall the definition of $\gamma, \gamma', \delta, \delta'$ from Equations (19) and (20).

If for an answer of Program 23, both $U^t_{i,l_i}(\{b_h\}) = 1 - 2t + \gamma + \gamma'$ and $C(\{b_h\}) = \delta + \delta'$ hold, then $\gamma, \gamma', \delta, \delta', t, U^t_{i,l_i}(\{b_h\})$ and $C(\{b_h\})$ for this instance constitute a feasible solution to Program (24) and hence, $t \geq t^*$. If one of these equations does not hold, we create a new item b'_h such that $U^t_{i,l_i}(\{b'_h\}) = 1 - 2t + \gamma + \gamma', C(\{b'_h\}) = \delta + \delta'$. Let $S' = \{b_1, b_2 \dots, b_{h-1}, b'_h, b_{h+1}, \dots, b_k\}$.[6] Note that since for item b_h we know $U^t_{i,l_i}(\{b_h\}) \geq 1 - 2t + \gamma + \gamma'$ and $C(\{b_h\}) \leq \delta + \delta'$, we have $U^t_{i,l_i}(\{b_h\}) \geq U^t_{i,l_i}(\{b'_h\})$ and $C(\{b_h\}) \leq C(\{b'_h\})$. Therefore, if for any subset $T' \subseteq S'$, $\Delta(T') \geq t/2$ holds, then replacing b'_h by b_h in T' results in a subset $T \subseteq S$ with $\Delta(T) \geq t/2$. This implies that for every subset T' of S', we have $\Delta(T') < t/2$ and hence, if we define δ, δ', γ, and γ' according to Equations (19) and (20) for set S', parameters $\delta, \delta', \gamma, \gamma', t, U^t_{i,l_i}(\{b_h\})$, and $C(\{b_h\})$ constitutes a feasible answer to Program (24). Therefore, $t \geq t^*$. This completes the proof. □

Observation 7.3 reduces Program (24) to Program (25).

$$\begin{array}{lll}
\text{minimize} & t & \\
\text{subject to} & \dfrac{t-\gamma}{t/2-\delta} \geq \dfrac{1-2t+\gamma+\gamma'}{\delta+\delta'}, & \text{Equation (21)} \\
& \dfrac{1-2t+\gamma+\gamma'}{\delta+\delta'} \geq \dfrac{t-\gamma'}{t/2-\delta'} & \text{Equation (21)} \\
& 1-2t+\gamma+\gamma' \leq t/2 & \text{Equation (22)} \\
& \delta+\delta' \leq t/2 & \text{Equation (22)} \\
& t \geq \gamma, \gamma', \quad t \geq 2/5 & \\
& \gamma \geq \delta, \gamma' \geq \delta' & \\
& \gamma, \gamma', t, \delta, \delta' \geq 0 &
\end{array} \tag{25}$$

Now we show that for t^*, there exist γ, δ, δ', and γ' such that both the first and the second inequalities of Program (25) turn into equations.

Lemma 7.4. *Let t^* be the optimal answer to Program (25). Then, there exist γ, γ', δ, and δ' such that all the constraints of Program (25) are satisfied for $t = t^*$, and furthermore,*

$$\frac{t^*-\gamma}{t^*/2-\delta} = \frac{1-2t^*+\gamma+\gamma'}{\delta+\delta'} \tag{26}$$

and

$$\frac{1-2t^*+\gamma+\gamma'}{\delta+\delta'} = \frac{t^*-\gamma'}{t^*/2-\delta'}. \tag{27}$$

Proof. We prove the statement of Lemma 7.4 for Equation (26). The proof for Equation (27) is the same. Assume that

$$\frac{t^*-\gamma}{t^*/2-\delta} > \frac{1-2t^*+\gamma+\gamma'}{\delta+\delta'}. \tag{28}$$

Now, let $\gamma'' = \gamma - \epsilon$ and $\delta'' = \delta - \epsilon$, where ϵ is a small constant. Since $(t^* - \gamma)/(t^*/2 - \delta) > 1$, we have

$$\frac{t^*-\gamma}{t^*/2-\delta} > \frac{t^*-\gamma''}{t^*/2-\delta''}$$

and

$$\frac{1-2t^*+\gamma+\gamma'}{\delta+\delta'} < \frac{1-2t^*+\gamma''+\gamma'}{\delta''+\delta'}.$$

[6] Note that, the value per contribution order of the items in S' might be different.

In other words, with gradually decreasing the values of γ and δ equally and by a small extent ϵ, the left-hand side of Inequality (28) decreases and the right hand side increases. Therefore, for a properly chosen value of ϵ, the inequality can be turned into an equation. The same also holds for Equation (27). $\square$

As a result of Lemma 7.5, we can simplify Program (25) to the following program:

$$\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad \frac{t-\gamma}{t/2-\delta} = \frac{1-2t+\gamma+\gamma'}{\delta+\delta'}, \\ &\qquad\qquad \frac{1-2t+\gamma+\gamma'}{\delta+\delta'} = \frac{t-\gamma'}{t/2-\delta'} \\ &\qquad\qquad 1-2t+\gamma+\gamma' \le t/2 \\ &\qquad\qquad \delta+\delta' \le t/2 \\ &\qquad\qquad t \ge \gamma,\gamma', \quad t \ge 2/5 \\ &\qquad\qquad \gamma \ge \delta, \gamma' \ge \delta' \\ &\qquad\qquad \gamma,\gamma',t,\delta,\delta' \ge 0 \end{aligned} \tag{29}$$

Lemma 7.5. *Let t^* be the optimal answer to Program (29). There exist γ, γ', δ, and δ' such that all the constraints of Program (29) are satisfied for $t = t^*$, and furthermore, $\gamma = \gamma'$ and $\delta = \delta'$.*

Proof. Assume that $\gamma \ne \gamma'$. Note that this in turn implies that $\delta \ne \delta'$. We have

$$\begin{aligned} \frac{t^*-\gamma}{t^*/2-\delta} &= \frac{t^*-\gamma'}{t^*/2-\delta'} \\ &= \frac{2t^*-\gamma-\gamma'}{t^*-\delta-\delta'} \\ &= \frac{t^*-(\gamma-\gamma')/2}{t^*/2-(\delta-\delta')/2} \end{aligned} \tag{30}$$

Now, let $\gamma'' = (\gamma+\gamma')/2$ and $\delta'' = (\delta+\delta')/2$. Since $\gamma+\gamma' = 2\gamma''$ and $\delta+\delta' = 2\delta''$, we have

$$\frac{1-2t^*+\gamma+\gamma'}{\delta+\delta'} = \frac{1-2t^*+2\gamma''}{2\delta''}.$$

Therefore, parameters t^*, δ'' (instead of both δ, δ') and γ'' (instead of both γ, γ') also constitute a feasible answer to Program (29). $\square$

Lemmas 7.4 and 7.5 imply that the objective function of Program 29 is equal to that of Program 31.

$$\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad \frac{t-\gamma}{t/2-\delta} = \frac{1-2t+2\gamma}{2\delta}, \\ &\qquad\qquad 1-2t+2\gamma \le t/2 \qquad \text{Equation (22)} \\ &\qquad\qquad 2\delta \le t/2 \qquad\qquad\quad \text{Equation (22)} \\ &\qquad\qquad t \ge \gamma, \quad t \ge 2/5 \\ &\qquad\qquad \gamma \ge \delta \\ &\qquad\qquad \gamma, t, \delta >= 0 \end{aligned} \tag{31}$$

Lemma 7.6. *Let t^* be the optimal answer to Program (31). Then, there exists a solution with objective value t^* in which $\gamma = \delta$.*

Proof. Let t^*, γ, and δ be a solution to Program (31), and assume that $\gamma > \delta$ and let $\epsilon = \gamma - \delta$. By the first equation we have:

$$\frac{t-\gamma}{t^*/2-\delta} = \frac{1-2t^*+2\gamma}{2\delta} \tag{32}$$

Now, we replace t^* by $t^* - \epsilon$ and γ by $\gamma - \epsilon = \delta$. We have

$$\frac{t^*-\gamma}{t^*/2-\delta} = \frac{(t^*-\epsilon)-(\gamma-\epsilon)}{t^*/2-\delta} \le \frac{(t^*-\epsilon)-(\gamma-\epsilon)}{(t^*-\epsilon)/2-\delta}.$$

Also, we have

$$\frac{1-2t+2\gamma}{2\delta} = \frac{1-2(t-\epsilon)+2(\gamma-\epsilon)}{2\delta}.$$

Therefore,

$$\frac{(t^*-\epsilon)-(\gamma-\epsilon)}{(t^*-\epsilon)/2-\delta} \geq \frac{1-2(t-\epsilon)+2(\gamma-\epsilon)}{2\delta}.$$

By Lemma 7.4, we can find ϵ' such that

$$\frac{(t^*-\epsilon)-(\gamma-\epsilon+\epsilon')}{(t^*-\epsilon)/2-(\delta+\epsilon')} = \frac{1-2(t^*-\epsilon)+2(\gamma-\epsilon+\epsilon')}{2(\delta+\epsilon')}$$

Therefore, $t^*-\epsilon$, $\gamma-\epsilon+\epsilon'$, and $\delta+\epsilon'$ are also a feasible solution to Program 31 with objective value less than t^*. This contradicts the optimality of t^*. □

By Lemma 7.6, we can further simplify Program 31 to obtain the following program:

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && \frac{t-\delta}{t/2-\delta} = \frac{1-2t+2\delta}{2\delta}, \\ & && 1-2t+2\delta \leq t/2 \\ & && 2\delta \leq t/2 \\ & && t \geq 2/5 \\ & && \delta, t >= 0 \end{aligned} \tag{33}$$

The first Inequality of Program (33) implies that

$$t = 1/4(1+2\delta+\sqrt{1-12\delta+4\delta^2}).$$

By replacing this value instead of t in the second and the third inequalities of Program (33) and applying $1-12\delta+4\delta^2 \geq 0$, we imply that $0<\delta<1/8(21-5\sqrt{17})$. The minimum value for t (t^*) is achieved when $\delta = \frac{1}{8}(21-5\sqrt{17})$ which is

$$1/4+\delta/2+1/4\sqrt{1-12\delta+4\delta^2} \geq 0.4384477.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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