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Discrete preference games with logic-based agents: Formal framework, complexity, and islands of tractability

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ABSTRACT

Analyzing and predicting the dynamics of opinion formation in the context of social environments are problems that attracted much attention in literature. While grounded in social psychology, these problems are nowadays popular within the artificial intelligence community, where opinion dynamics are often studied via *game-theoretic* models in which individuals/agents hold opinions taken from a fixed set of *discrete* alternatives, and where the goal is to find those configurations where the opinions expressed by the agents emerge as a kind of compromise between their innate opinions and the social pressure they receive from the environments. As a matter of facts, however, these studies are based on very high-level and sometimes simplistic formalizations of the social environments, where the mental state of each individual is typically encoded as a variable taking values from a Boolean domain. To overcome these limitations, the paper proposes a framework generalizing such *discrete preference games* by modeling the reasoning capabilities of agents in terms of weighted propositional logics. It is shown that the framework easily encodes different kinds of earlier approaches and fits more expressive scenarios populated by conformist and dissenter agents. Problems related to the existence and computation of stable configurations are studied, under different theoretical assumptions on the structural shape of the social interactions and on the class of logic formulas that are allowed. Remarkably, during its trip to identify some relevant tractability islands, the paper devises a novel technical machinery whose significance goes beyond the specific application to analyzing opinion formation and diffusion, since it significantly enlarges the class of Integer Linear Programs that were known to be tractable so far.

1. Introduction

1.1. Opinion formation in social environments

Understanding how some kind of global behavior emerges from local interactions among individuals is a well-established topic of research in a number of different areas, including economics, finance, epidemiology, social psychology, political science, and public governance. Due to the rapid proliferation of social networking services, such as Facebook and Twitter, which created novel and highly-dynamic forms of techno-social ecosystems [1], within the last two decades the problem has attracted much attention in the Artificial Intelligence community too (see, e.g., [2–6] and the references therein).

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By looking at social environments from the AI perspective, we usually model them as networks whose nodes correspond to the individuals and whose edges encode their social interconnections which give rise to influence phenomena. In fact, a plethora of approaches have already been proposed to analyze scenarios where neighbors communicate by propagating and diffusing "conceptual items", such as the adoption of a technology, because of reasons ranging from similarity and social ties [7], to conformity [8], and to compliance [9], just to name a few. For instance, well-known diffusion models are the cascade [10], the tipping/threshold [11], and the homophilic model [12]. Moreover, in some cases, these models are mixed together (e.g., [13]) or they are extended to accommodate the diffusion of different and competing information [14–17].

More recently, richer models of social environments have been proposed in the literature, which are tailored to study how "immaterial" things, in particular *opinions*, form and diffuse over the network. Many of these works [18–21] build on a basic model of DeGroot [22], where each individual is equipped with a real number (for example, representing a position on a political spectrum or a probability assigned to a certain belief), which is updated, at each time step, to be a weighted average of that opinion with the current opinions of the neighbors. By doing so, the diffusion processes will converge to a state of consensus where all individuals hold the same opinion. A natural extension, first proposed by [23], is to equip each individual with an innate opinion in addition to the expressed opinion. At each time step, the expressed opinion is then updated to minimize the disagreement with the innate opinion and the opinions expressed by the neighbors—see, e.g., [24–27]. Chierichetti et al. [28,29] further elaborated this approach, by arguing that, in many settings, there is no natural way to average among the available options. Indeed, they conducted a systematic study of social environments where a set of strategic agents take the available opinions from a fixed set of discrete alternatives. Such environments have a clear game-theoretic flavor, and they have been formalized as the class of *discrete preference games*, where the main question is whether it is possible to end up with a *Nash equilibrium*, that is, a configuration of the network where all agents are happy/stable with their current opinion, and they do not want to adopt a different opinion because of the social pressure of their neighbors/friends.

In fact, discrete preference games share the spirit of other game-theoretic model proposed in the literature to study networks populated by strategic agents, such as, *graphical games* (see, e.g., [30–32]) and *coordination games on graphs* (see, e.g., [33–35]), and they have been influential for subsequent literature. For instance, studies have been conducted to characterize the dynamics converging to Nash equilibria [36], to assess the robustness with respect to manipulability [37,38], to analyze seed-selection related problems [6], and to understand the role that can be played by pressure to reach an agreement [39]. By abstracting from their technical peculiarities, however, these studies typically found on very high-level formalizations of the social environments, where the mental state of each individual is just encoded as a variable taking values from some fixed discrete domain, with this domain being binary in many cases. Indeed, very few efforts have been spent in the literature to enhance the expressiveness of these settings, and noticeable exceptions include the definition of approaches where agents have more complex constraints they want to satisfy in addition to conforming to their neighbors/friends [27], or where seed-selection problems are studied when three opinions are available to them [40], or where different neighbors/friends of an agent can have different levels of influence on her [41].

As a matter of facts, a comprehensive framework where strategic aspects related to opinion formation are studied for agents equipped with their own inference mechanisms has not been proposed and studied in the literature so far. Our goal is to fill this gap, by proposing and studying a framework that abstracts and generalizes earlier approaches in the literature, for it being equipped with the following two knowledge representation features:

- First, the framework goes far beyond the "binary/ternary" setting, in that it imposes no bound on the number of available opinions and equips each agent with a logical theory that can be used to succinctly express rich preferences over them. This feature overcomes a relevant limitation of earlier approaches to model opinion formation via discrete preference games, and it paves the way for analyzing a wide spectrum of application scenarios. As an example, our framework can be naturally used to reason about strategic issues arising in election campaigns with an arbitrary number of candidates (see, e.g., [42]) or, more generally, about settings where agents express preferences over combinatorial domains [43].
- Second, the framework allows agents to reason—in a very flexible way—about the opinion expressed by their neighbors. In fact, classical approaches to model discrete preference games and opinion diffusion scenarios deal with *conformist* agents only, that is, with agents that tend to adapt their opinions to the opinion expressed by their neighbors. Instead, our framework uses a logic theory not only to define the innate preferences of the agents (as discussed in the point above), but also to define "how" agents are influenced by their neighbors; in particular, it allows us to model agents willing to distinguish their opinions from the opinions of their neighbors. For instance, many of us have some friend (especially in virtual environments, such as Twitter or Facebook) who likes to support "controversial" arguments and who is inclined to manifest "unpopular" viewpoints. This friend can be called a *dissenter* (a.k.a. contrarian, anticonformist, or nonconformist), in that she/he often finds her/himself supporting the opinion taken by the *minority* of her/his neighbors. In fact, earlier approaches in the literature do not allow to model environments populated by dissenters, thereby reflecting at a macroscopic level the intrinsic *monotonicity/submodularity* of the conformist behavior (cf. [44]). Instead, our framework is more flexible, for it allows to deal with environments populated by conformist agents coexisting with dissenters, which is a more realistic model of social interactions (e.g., [45,46]).[1]

[1] In the game-theory jargon, we say that our framework supports *coordination* [33–35] and *anti-coordination* [47,48].

1.2. Contribution

In the rest of the paper we shall elaborate and study the formal ingredients allowing us to define environments populated by agents that can express rich form of preferences over an arbitrary number of alternatives and over the opinions expressed by their neighbors. In more details, our contribution can be summarized as follows:

- We propose the setting of *discrete* LB*-preference games*, which is reminiscent of the works on preference games, but where the reasoning capabilities of the agents are modeled by using *weighted propositional logics* [49]. Indeed, this (Logic Based) modeling language has been shown to express all common classes of utility functions, and it also provides a convenient means to elicit user's preferences while balancing expressivity and complexity [50]. Furthermore, it practically enables to exploit SAT-Solvers to deal with instances involving tens of thousands of variables and formulas consisting of millions of symbols, today [51].
- We study discrete LB-preference games from the knowledge representation viewpoint, by showing that they can easily encode different kinds of frameworks proposed in the literature to model strategic interactions among agents, by precisely exploiting two key features discussed above. In particular, we show that discrete LB-preference games can be used to reason not only about (standard) discrete preference games [29,41], but also about *opinion diffusion on social networks* (see, e.g., [6,11,22,37]), *about anonymous games* (e.g., [52–55]) and *graphical games* (e.g., [30–32]).
- We analyze discrete LB-preference games from the computational viewpoint, by providing a clear and comprehensive picture of the complexity of questions related to the existence of Nash equilibria. Our analysis has been conducted by considering various parameters, ranging from the number of variables available in the knowledge bases of the agents to the kinds of preferences they are able to express. Moreover, we deal with networks where the neighborhood relation is symmetric as well as with arbitrary networks. By looking at our results, it emerges that deciding the existence of Nash equilibria becomes quickly an intractable problem, formally **NP**-complete (and even $\mathbf{\Sigma_2^P}$-complete, in some circumstances). Moreover, our results shed lights on the impact of the different parameters (and, in particular, of the knowledge representation features being used) on the complexity of the setting. It emerges that a crucial source of intractability is precisely given by the interplay of conformist and dissenter agents.
- We conduct a fine-grained analysis to identify large classes of tractable environments, by considering agents that do not have preferences on the specific opinions expressed by the neighbors, so that their utilities only depend on their innate preferences and on the *number* of neighbors that agree/disagree with them. For this setting, which we call *linear*, we show that, over networks that are symmetric, discrete LB-preference games are tractable whenever they model environment where conformists and dissenters do not interact; instead, over arbitrary networks, discrete LB-preference games are tractable for environments populated by conformists only. For these tractable cases we have identified, constructive algorithms have been exhibited that are capable of producing profiles that are stable.
- With the aim of identifying even large classes of discrete LB-preference games that are tractable, we then study discrete LB-preference games over networks that enjoy some specific structural properties. In particular, we analyze networks that are *nearly-acyclic*, formally that have bounded *treewidth* [56]. Indeed, many **NP**-hard problems arising in AI are known to be tractable when restricted to structures that can be modeled in terms of acyclic or nearly-acyclic graphs; and we know that such results have not only just a clear theoretical interest, but they can be also useful in practice (see Section 6.1). Our analysis evidenced that bounded treewidth is a key to identify tractable discrete LB-preference games, too. Notably, our tractability results are clearly inherited by all the settings discussed above and that can be modeled in terms of discrete LB-preference games.
- Finally, an important contribution of the paper is provided on a technical level. Indeed, while the reader might naturally envisage that bounded treewidth is also a key to identify tractable discrete LB-preference games, establishing the result posed a number of challenges that have been not addressed in earlier literature. In particular, by using classical machineries for treewidth (e.g., [57,58]), we could have established in our setting some tractability results only by considering bounded-treewidth networks that satisfy the additional property of having bounded *degree* (that is, each individual can have a bounded number of influencers only). To overcome this limitation, we considered a different approach by devising a novel technical tool based on the concept of *decomposition* for an *Integer Linear Program*—into which we easily recast the problems of interest—and on a method to solve in polynomial time "decomposable" ILPs. Importantly, the approach we propose generalizes known machineries for treewidth of ILPs that have recently attracted much attention in the AI literature (e.g., [59–61]). Hence, its significance clearly goes beyond the exploitation for identifying tractable classes of discrete LB-preference games, and, given the flexibility of ILPs to encode different kinds of real-world problems, we naturally foresee that our results will be used to identify large islands of tractability in several concrete application scenarios not necessarily related to reasoning about strategic issues over a network.

Note that the analysis we carry out in this paper extends some preliminary results we discussed in [62]. In particular, the analysis of the knowledge representation capabilities of the framework, a number of complexity results related to qualitative restrictions over discrete LB-preference games, and the ILP-based structural tractability results are entirely novel contributions.

1.3. Organization

The rest of the paper is organized as follows. The framework of discrete LB-preference games is formalized and discussed in Section 2, whereas its knowledge representation capabilities are illustrated in Section 3. An overview of the complexity issues arising with discrete LB-preference games is presented in Section 4, whereas a closer look at the complexity of Nash equilibria is taken in Section 5 where a number of islands of tractability have been identified based on certain natural qualitative restriction posed on the games. Instead, structural tractability results are elaborated in Section 6. Eventually, conclusions are drawn in Section 7, where a number of avenues for further research are also illustrated.

2. Formal framework for reasoning about social environments

In this section, we introduce a logic-based framework for modeling strategic interactions on social environments, by discussing the basic concepts and notations that will be used throughout the rest of the paper. We start the exposition by illustrating some background notions on *weighted propositional logic*, which we use to encode the reasoning capabilities of the agents [50].

2.1. Preliminaries on weighted propositional logic and goalbases

We assume that a universe $\mathcal{V}$ of propositional variables is given and, for any structure ζ (such as a formula or a set of formulas) defined on $\mathcal{V}$, we denote by $\mathtt{dom}(\zeta)$ the set of all the variables occurring in ζ. We consider the propositional language $\mathcal{L}$ consisting of all formulas built over $\mathcal{V}$ by using the Boolean connectives $\wedge$, $\vee$, and $\neg$, plus the Boolean constants $\top$ (*true*) and $\bot$ (*false*). Moreover, given two formulas φ_1 and φ_2 in $\mathcal{L}$, we use $\varphi_1 \rightarrow \varphi_2$ as a shorthand for $\neg\varphi_1 \vee \varphi_2$, and $\varphi_1 \leftrightarrow \varphi_2$ as a shorthand for $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$.

An *interpretation* I is a function $\{x \mapsto I(x)\}_{x\in\mathtt{dom}(I)}$ assigning a Boolean value to each variable in its domain, that is, $I(x) \in \{\top, \bot\}$ holds for each $x \in \mathtt{dom}(I)$. We often describe I extensively, that is, as the set of literals $\{x \mid x \in \mathtt{dom}(I) \wedge I(x) = \top\} \cup \{\neg x \mid x \in \mathtt{dom}(I) \wedge I(x) = \bot\}$. We deal with I under the usual semantics for propositional logic, and we denote by $I \models \varphi$ the fact that I satisfies a formula $\varphi \in \mathcal{L}$ with $\mathtt{dom}(I) \supseteq \mathtt{dom}(\varphi)$.

In the paper, the reasoning capabilities of agents will be modeled via the notion of goalbase that is next formalized based on the concept of weighted formulas [50,63].

Definition 1. A *goalbase* is a finite set G of weighted formulas, that is pairs (φ, w) where $\varphi \in \mathcal{L}$ and where $w \in \mathbb{Q}$ is its weight. For any interpretation I with $\mathtt{dom}(I) \supseteq \mathtt{dom}(G)$, we define $G(I)$ as the sum of the weights over all the pairs $(\varphi, w) \in G$ such that $I \models \varphi$ holds:

$$G(I) = \sum_{(\varphi,w)\in \text{ s.t. } I\models\varphi} w. \quad \square$$

As an example, if we have the goalbase $G = \{(x \wedge y, 3), (x \vee z, 2), (\neg y \wedge \neg z, 7)\}$ and the interpretation $I = \{x, y, \neg z\}$, then we have $G(I) = 3 + 2 = 5$. As will be used to model the reasoning capabilities of the agents, the value $G(I)$ will be interpreted as the *utility* an agent gets for selecting the interpretation I (see [50]). Formal properties of this logic-based approach to model agents' preferences are discussed, for instance, in the work by Coste-Marquis et al. [49].

Example 1. Consider a group of agents that have to decide whether to meet each other in the evening and have a pizza in some pizzeria. Within this context, consider the goalbase $G = \{(pizza \leftrightarrow go\_to\_Ciro's, 1), (\neg pizza, 1)\}$. Intuitively, G is meant to encode an agent whose favorite pizzeria is $Ciro's$, but that today would prefer not to have a pizza. Indeed, her utility over all possible interpretations ranges between 0 and 2, with the maximum value being achieved over the interpretation $\{\neg pizza, \neg go\_to\_Ciro's\}$. Moreover, note that if she eventually decides to have a pizza at $Ciro's$ with her friends, then her utility will be $G(\{pizza, go\_to\_Ciro's\}) = 1$. $\lhd$

2.2. Discrete LB-preference games

We assume that a set N of agents is given. A *discrete* LB*-preference game* is a triple $\mathcal{G} = \langle N, E, \kappa\rangle$, such that (N, E) is a graph whose nodes are the agents in N and where, for each $i \in N$, every agent $j \in N$ such that the edge (i, j) belongs to E is called a *neighbor* of i. The set of all the neighbors of i is hereinafter denoted as $\mathtt{neigh}_{\mathcal{G}}(i)$. In particular, the presence/absence of certain edges in E can encode different kinds of neighborhood relationships, such as physical limitations, constraints, legal banishments and friendships, just to name a few. The game $\mathcal{G}$ is also characterized by a function κ associating each agent $i \in N$ with a pair $\kappa(i) = \langle G_i^0, G_i^1\rangle$ of goalbases, which encodes her preferences as follows.

Let us first define a *profile* Π as a function $\{i \mapsto \Pi_i\}_{i\in N}$ mapping each agent $i \in N$ to an interpretation Π_i over $\mathtt{dom}(\kappa(i))$, which plays the role of the opinion selected by agent i. Note that, similarly to *discrete preference games* [24,29], agents have a finite and discrete set of possible opinions available at hand. But, in our Logic-Based setting, such opinions are implicitly defined in terms of the interpretations over the underlying knowledge bases.

Let us now define the set of agents *compatible* with i in Π as $\mathtt{compatible}_{\mathcal{G},i}(\Pi) = \{j \in \mathtt{neigh}_{\mathcal{G}}(i) \mid \forall x \in \mathtt{dom}(\kappa(i)) \cap \mathtt{dom}(\kappa(j)), \Pi_i(x) = \Pi_j(x)\}$, that is, as the set including all neighbors j for which the interpretations Π_i and Π_j agree over each variable x occurring in both $\kappa(i)$ and $\kappa(j)$. Then, the *utility* of agent i with respect to Π is the following rational number:

$$u_{\mathcal{G},i}(\Pi) = G_i^0(\Pi_i) + G_i^1(\Pi_i) \cdot |\mathtt{compatible}_{\mathcal{G},i}(\Pi)|. \tag{1}$$

In words, for each agent $i \in N$, $u_{\mathcal{G},i}(\Pi)$ defines the utility function of i by summing up two terms. The first term $G_i^0(\Pi_i)$ quantifies the innate preferences of the agent on the given profile Π. Instead, the second term is built on top of G_i^1 and takes into account the

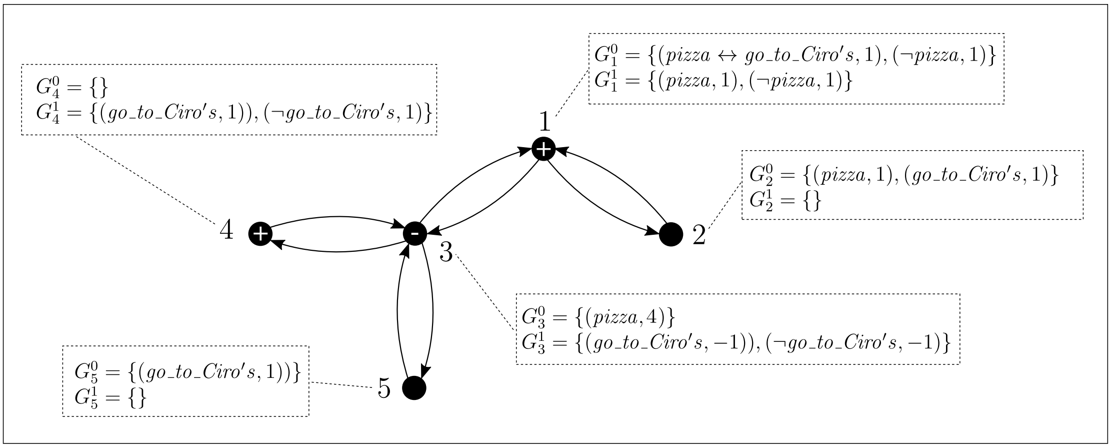

Fig. 1. Knowledge bases and interactions for the discrete LB-preference game of Example 2, with $N = \{1, ..., 5\}$. Agents 1 and 4 are marked with '+', since they would like to be compatible with as many neighbors as possible. Instead, agent 3 is marked with '-', since she prefers to select interpretations in which she minimizes the number of compatible neighbors. Eventually, agents 2 and 5 are indifferent towards their neighbors.

social relationships of the agent, by actually filtering out the neighbors that are not compatible with i in Π. Indeed, the value $G_i^1(\Pi_i)$ is amplified by the factor $|\texttt{compatible}_{\mathcal{G},i}(\Pi)|$.[2]

Note that the notion of compatibility, on top of which the utility function is defined, is reminiscent of approaches in game theory based on coordination among agents (see, e.g., [64]), which have been recently considered for games played on social environments too [33–35]. In fact, since the weights associated with the goalbases can be negative, utility functions can also model a behavior based on anti-coordination, that is, agents that would like to act differently than their neighbors [47,48]. This is next exemplified.

Example 2. Let us consider the discrete LB-preference game $\mathcal{G} = \langle N, E, \kappa\rangle$ illustrated in Fig. 1, according to an intuitive graphical notation. We have five agents ($N = \{1, 2, 3, 4, 5\}$) related by some friendship relations. In particular, such relationships are symmetric, that is, for each $i \in N$ and $j \in \texttt{neigh}(i)$, we have that $i \in \texttt{neigh}(j)$ holds too. Moreover, note that $\texttt{dom}(\kappa(1)) = \texttt{dom}(\kappa(2)) = \texttt{dom}(\kappa(3)) = \{pizza, go\_to\_Ciro's\}$ and $\texttt{dom}(\kappa(4)) = \texttt{dom}(\kappa(5)) = \{go\_to\_Ciro's\}$. That is, agents 1, 2 and 3 reason on having a pizza as well as on the specific pizzeria they would like to go. Instead, agents 4 and 5 just reason about going to *Ciro's*, without caring of having a pizza there—for instance, they might well order a salad.

Consider, now, agent 1 and note that $\kappa(1) = \langle G_1^0, G_1^1\rangle$, where $G_1^0 = \{(pizza \leftrightarrow go\_to\_Ciro's, 1), (\neg pizza, 1)\}$ and $G_1^1 = \{(pizza, 1), (\neg pizza, 1)\}$. That is, G_1^0 is precisely the goalbase discussed in Example 1, which encodes the innate preferences of agent 1. However, in the light of G_1^1, the agent now gets an additional utility if she agrees with some neighbor on the decision about having pizza. Hence, depending on the choices of her neighbors 2 and 3, she can reconsider the idea of going to *Ciro's* for dinner. Moreover, note that agent 3 would like to have a pizza, but has no innate preference about *Ciro's*. Indeed, since the weights in G_3^1 are negative, then agent 3 prefers to minimize the number of neighbors that have dinner with her. Conversely, agent 4 is happy when she is compatible with agent 3, whereas agent 4 and agent 2 are not influenced at all by their neighbors (agent 1 and 3, respectively)—see, again, Fig. 1 and the notation used for the agents.

Consider the profile Π such that $\Pi_1 = \{go\_to\_Ciro's, pizza\}$, $\Pi_2 = \{go\_to\_Ciro's,\ pizza\}$, $\Pi_3 = \{pizza, \neg go\_to\_Ciro's\}$, $\Pi_4 = \{\neg go\_to\_Ciro's\}$, and $\Pi_5 = \{go\_to\_Ciro's\}$. According to Π, agents 1, 2 and 5 will go to *Ciro's*, whereas agents 3 and 4 will not go there. In particular, for the utility of agent 1, we have $\texttt{compatible}_1(\Pi) = \{2\}$ and $u_1(\Pi) = G_1^0(\Pi_1) + G_1^1(\Pi_1) \cdot |\{2\}| = 1 + 1 \cdot 1 = 2$. For the others, we have that $u_2(\Pi) = 2$, $u_3(\Pi) = 3$, $u_4(\Pi) = 1$, and $u_5(\Pi) = 1$. ◁

2.3. Nash stability

Social environments will be studied in the paper from the Nash-stability viewpoint, that is, with the aim of assessing whether there exists some profile in which all agents are satisfied and do not want to change their current interpretation. Formally, for each agent $i \in N$, the *restriction* Π_{-i} of a profile Π to the agents in $N \setminus \{i\}$ is the function $\{j \mapsto \Pi_j\}_{j \in N\setminus\{i\}}$. Given a profile Π, an agent $i \in N$, and an interpretation I, the mapping $\{i \mapsto I\}$ is a *best response move* for i (w.r.t. Π) if $u_i(\Pi_{-i} \cup \{i \mapsto I\}) \geqslant u_i(\Pi_{-i} \cup \{i \mapsto J\})$, for each interpretation J.

Then, a profile Π is said *(Nash) stable* if each agent is playing one of her best response moves. That is, for each $i \in N$ and for each interpretation I, it holds that $u_i(\Pi) \geqslant u_i(\Pi_{-i} \cup \{i \mapsto I\})$.

[2] In the following, to keep notation simple, the subscript $\mathcal{G}$ will be omitted whenever the discrete LB-preference game $\mathcal{G}$ is clearly understood from the context.

Example 3. Consider again the discrete LB-preference game described in Example 2 and note that the profile Π introduced there is a stable one. Indeed, agents 2 and 5 are not influenced by the other agents in the environment, and they are already getting their maximum possible utility, because $u_2(\Pi) = 2$ and $u_5(\Pi) = 1$. Moreover, given the goalbase $G_3^0 = \{(pizza, 4)\}$ and since agent 3 has only three neighbors, we can conclude that the agent necessarily adopts an interpretation where she will have a pizza for dinner. In addition, note that agent 3 is satisfied by the fact that she does not go to *Ciro′s* (while still willing to have a pizza), because the majority of her neighbors want to go there and because the goalbase G_3^1 just states that she wants to go to a place/pizzeria different from the choice of 1, 4 and 5. Eventually, agent 1 (resp., 4) is also happy with Π, precisely because she agrees with her neighbors on having a pizza (resp., on going to a different place than *Ciro′s*).

Note that the profile obtained from Π by setting the interpretation associated with agent 1 to $\{\neg go\_to\_Ciro's, \neg pizza\}$ (in place of $\{go\_to\_Ciro's, pizza\}$) is stable, too. ◁

Clearly enough, stable profiles are in general not guaranteed to exist. A discrete LB-preference game without stable profiles is next discussed.

Example 4. Consider the restriction on agents 3 and 4 of the environment depicted in Fig. 1. That is, agents 3 and 4 are the only agents in the resulting environment, they are still connected with each other ($3 \in \texttt{neigh}(4)$ and $4 \in \texttt{neigh}(3)$), and their goalbases are still those in the figure. Note that agent 3 acts as a "challenger" in that she always would like to adopt an interpretation (including *pizza* and either *go_to_Ciro′s* or ¬*go_to_Ciro′s*) where she is not compatible with 4. Conversely, agent 4 is happy only when she is compatible with agent 3. Therefore, they cannot be satisfied simultaneously.

More formally, assume that Π is a profile such that $\Pi_3 = \{pizza, go\_to\_Ciro's\}$. We distinguish two cases. In the case where $\Pi_4 = \{go\_to\_Ciro's\}$, then $u_3(\Pi) = 4 - 1 \cdot |\{4\}|$ and agent 3 finds convenient to adopt the interpretation $\{\neg go\_to\_Ciro's\}$, by getting utility $u_3(\Pi_{-3} \cup \{3 \mapsto \{\neg go\_to\_Ciro's\}\}) = 4$. Instead, in the case where $\Pi_4 = \{\neg go\_to\_Ciro's\}$, we have $u_4(\Pi) = 0$ and agent 4 has an incentive to adopt interpretation $\{go\_to\_Ciro's\}$, by getting utility $u_4(\Pi_{-4} \cup \{4 \mapsto \{go\_to\_Ciro's\}\}) = 1$. Hence, Π is not stable. Eventually, the same line of reasoning applies with the other possible profiles. ◁

3. A closer look at discrete LB-preference games

The framework of discrete LB-preference games we have introduced in Section 2 can be used to conveniently reason about strategic agents in a wide spectrum of application domains. In fact, our framework is expressive enough to generalize several settings and models proposed and studied in earlier literature. In this section, we focus on four of these settings—which we selected based of their features that make them representative of a wide range of scenarios[3]—and we show that they can be encoded as discrete LB-preference games.

In particular, we stress that our encodings can be implemented in polynomial time and that they lead to establish precise correspondences between the (Nash) stability concepts adopted in the four settings and the stability concept we have formalized in Section 2.3.

3.1. Opinion diffusion on social networks

The study of diffusion processes on social networks has attracted attention since the late seventies, when researchers proposed the first formal models to explain how opinions can form and diffuse because of the existence of social ties among the individuals/agents (e.g., [11,22]). By abstracting from their specific peculiarities, all diffusion models proposed in the literature so far can be grouped in two classes [65], *progressive* and *non-progressive* models. In a model of the former type, once an agent has been "influenced" in the diffusion process, she remains influenced forever. In a non-progressive model, instead, it is possible that an "influenced" agent will change again her mind during the diffusion process. In fact, non-progressive models exhibit a game-theoretic flavor and they are appropriate to deal with discrete LB-preference games hosting opinions that compete with each other (see, e.g., [66]). Practical applications of these models have been pointed out in the context of the diffusion of competing (product) innovations, in the usage of mobile apps, and for analyzing cycles of opinions that are in fashion [67]. The first setting we would like to consider in order to shed lights on the features of our framework is precisely a basic (majority-based) non-progressive model of opinion diffusion (see, e.g., [6,37] and the references therein).

Formalization. Let $\mathsf{S} = (N, E)$ be a graph encoding the interactions of a set N of agents. Assume that two opinions, say white and black, are available to them. A *configuration* for S is a function c mapping each agent i to her opinion $\mathsf{c}(i) \in \{\mathsf{white}, \mathsf{black}\}$. For any set $V \subseteq N$ of agents, we define $V_{\mathsf{white}/\mathsf{c}}$ as the set of all agents in V with opinion white in c. The set $V_{\mathsf{black}/\mathsf{c}}$ is defined analogously. Agent $i \in N$ is *stable with respect to* c if her opinion agrees with the opinion held by a (non-strict) majority of her neighbors; that is, either $|\texttt{neigh}(i)_{\mathsf{white}/\mathsf{c}}| \leq |\texttt{neigh}(i)_{\mathsf{black}/\mathsf{c}}|$ and $\mathsf{c}(i) = \mathsf{black}$; or $|\texttt{neigh}(i)_{\mathsf{black}/\mathsf{c}}| \leq |\texttt{neigh}(i)_{\mathsf{white}/\mathsf{c}}|$ and $\mathsf{c}(i) = \mathsf{white}$. A configuration c is *stable* if all agents in N are stable. Note, for instance, that the two configurations where all agents hold the same opinion are trivially stable.

[3] In fact, the reader can check that, with minor modifications, our encoding schemes can easily accommodate more complicated variants of the considered settings.

Encoding. The distinguishing feature of the setting of opinion diffusion is that agents reason about their own opinions and the opinions hold by their neighbors. This can be naturally encoded in our framework. In particular, we shall next establish a precise correspondence between stable configurations for S and stable profiles for a properly built discrete LB-preference game $\mathcal{G}^{\mathsf{S}}$.

Let us first view opinion white as a propositional variable, in fact the only one occurring in the knowledge bases of $\mathcal{G}^{\mathsf{S}}$. Hence, the literal $\neg$white naturally stands for the opinion black. Based on S, let us build the discrete LB-preference game $\mathcal{G}^{\mathsf{S}} = \langle N, E, \kappa \rangle$ such that, for each $i \in N$, $\kappa(i) = \langle G_i^0, G_i^1 \rangle$ with $G_i^0 = \emptyset$ and $G_i^1 = \{(\mathsf{white}, 1), (\neg\mathsf{white}, 1)\}$. Note that the utility of agent i with respect to the profile Π is given by $u_i(\Pi) = G_i^0(\Pi_i) + G_i^1(\Pi_i) \cdot |\mathtt{compatible}_i(\Pi)| = |\mathtt{compatible}_i(\Pi)|$. That is, $u_i(\Pi)$ is the number of neighbors j of i such that $\Pi_j = \Pi_i$ (that is, they have the same opinion).

With each configuration c, let us associate the profile Π^{c} such that: $\Pi_i^{\mathsf{c}} = \{\mathsf{white}\}$ if $\mathsf{c}(i) = \mathsf{white}$, and $\Pi_i^{\mathsf{c}} = \{\neg\mathsf{white}\}$ if $\mathsf{c}(i) = \mathsf{black}$. Then, the properties of the construction are as follows.

Theorem 1. *Let $\mathcal{G}^{\mathsf{S}}$ be the discrete* LB*-preference game associated with a diffusion scenario* S.

(1) *Assume that* c *is a stable configuration for* S*. Then, the profile Π^{c} is stable for $\mathcal{G}^{\mathsf{S}}$.*
(2) *Assume that Π is a stable profile for $\mathcal{G}^{\mathsf{S}}$. Then, there exists a stable configuration* c *for* S *such that $\Pi = \Pi^{\mathsf{c}}$.*

Proof. *(1)* Let c be a stable configuration. Assume, for the sake of contradiction, that an agent $i \in N$ is not playing one of her best response moves in Π^{c}. Consider the case where $\mathsf{c}(i) = \mathsf{white}$ and $\mathsf{white} \in \Pi_i^{\mathsf{c}}$—the case where $\mathsf{c}(i) = \mathsf{black}$ and $\neg\mathsf{white} \in \Pi_i^{\mathsf{c}}$ can be addressed with the same line of reasoning. Then, we have $u_i(\Pi^{\mathsf{c}}) < u_i(\Pi_{-i}^{\mathsf{c}} \cup \{i \mapsto \neg\mathsf{white}\})$. Recall that, for each profile Π, $u_i(\Pi)$ is the number of neighbors j of i such that $\Pi_j = \Pi_i$. Therefore, $u_i(\Pi^{\mathsf{c}}) = |\mathtt{neigh}(i)_{\mathsf{white/c}}|$ whereas $u_i(\Pi_{-i}^{\mathsf{c}} \cup \{i \mapsto \neg\mathsf{white}\}) = |\mathtt{neigh}(i)_{\mathsf{black/c}}|$. So, $|\mathtt{neigh}(i)_{\mathsf{white/c}}| < |\mathtt{neigh}(i)_{\mathsf{black/c}}|$, which is impossible because c is stable and $\mathsf{c}(i) = \mathsf{white}$.

(2) Let Π be a stable profile and consider the configuration c for S such that $\Pi = \Pi^{\mathsf{c}}$. Note that c is well defined. Indeed, for each $i \in N$, we have that $\mathsf{c}(i) = \mathsf{white}$ (resp., $\mathsf{c}(i) = \mathsf{black}$) if $\Pi_i = \{\mathsf{white}\}$ (resp., $\Pi_i = \{\neg\mathsf{white}\}$). Assume now, for the sake of contradiction, that an agent $i \in N$ is not stable with respect to c. In particular, assume that $\mathsf{c}(i) = \mathsf{white}$, as the case where $\mathsf{c}(i) = \mathsf{black}$ can be addressed with the same line of reasoning. Then, since i is not stable, we have $|\mathtt{neigh}(i)_{\mathsf{white/c}}| < |\mathtt{neigh}(i)_{\mathsf{black/c}}|$. By construction of Π^{c}, this entails that $u_i(\Pi^{\mathsf{c}}) < u_i(\Pi_{-i}^{\mathsf{c}} \cup \{i \mapsto \neg\mathsf{white}\})$, which is impossible since Π is stable. □

3.2. Anonymous hedonic games with enemies

The second setting we would like to consider is related to strategic agents that are capable of reasoning about forming *coalitions*. Whenever coalition formation has to be approached from the strategic viewpoint, *hedonic games* play a prominent role and, in fact, they received much attention from both an economic and an algorithmic perspective (see, e.g., [68,69]). In the following, we focus on the well-known class of *anonymous hedonic games*, where agents' preferences depend only on the size of the coalitions where they belong to [70,71]. In particular, we next consider a setting where neighbors are interpreted as *enemies* and agents would like to avoid forming coalitions with them. Note that this is a special setting of hedonic games with friends, enemies, and neutrals [52–55]—we focus on enemies only, since we would like to discuss a setting where agents minimize their compatibility with their neighbors, contrasted to the setting discussed in Section 3.1 where we have already modeled agents that tend to conform with the opinion of their friends.

Formalization. We are given a set N of agents, and we define *coalition* as any subset of N. Then, an anonymous hedonic game with enemies H is simply a graph (N, E) where each agent $i \in N$ has a preference on the coalitions she may belong to: agent i prefers $C_1 \subseteq N$ to $C_2 \subseteq N$ if, and only if, $|C_1 \cap \mathtt{neigh}(i)| < |C_2 \cap \mathtt{neigh}(i)|$. An *outcome* $\mathcal{C}$ for H is a set of coalitions that forms a partition of N. Then, we say that the outcome $\mathcal{C}$ is (Nash-)*stable* if, for each coalition $C \in \mathcal{C}$, for each agent $i \in C$ and for each coalition $C' \in \mathcal{C} \cup \{\emptyset\}$, i does not prefer $\{i\} \cup C'$ to C. Note that the outcome where each agent forms a singleton coalition is trivially stable.

Encoding. Reasoning about coalition formation poses a number of questions that cannot be addressed with the machinery we have used in Section 3.1. First, we need a mechanism to encode the coalition where agents belong to. To this end, by letting $n = |N|$, we can assume that all knowledge bases are defined over the set $\{x_1, \ldots, x_{\lceil \log n \rceil}\}$ of propositional variables. So, for each profile Π and agent i, the coalition where agent i belongs to is $coal(i, \Pi) = \{j \in N \mid \Pi_j = \Pi_i\}$.

Moreover, we observe that the preference relations of every agent $i \in N$ can be encoded as a function μ_i such that $\mu_i(C) = -|C \cap \mathtt{neigh}(i)|$. Indeed, agent i prefers $C_1 \subseteq N$ to $C_2 \subseteq N$ if, and only if, $\mu_i(C_1) > \mu_i(C_2)$. Accordingly, in order to encode H, we build the discrete LB-preference game $\mathcal{G}^{\mathsf{H}} = \langle N, E, \kappa \rangle$ where, for each $i \in N$, $\kappa(i)$ is such that:

- $G_i^0 = \{(\bigwedge_{q \leqslant \lceil \log n \rceil} (x_q \vee \neg x_q), 0)\}$, and
- $G_1^1 = (\top, -1)$.

Note that if Π is a profile, then $u_i(\Pi) = \mu_i(coal(i, \Pi)) = -|\mathtt{compatible}_i(\Pi)|$. On the other way round, if $\mathcal{C}$ is an outcome for H, then we can define $\Pi^{\mathcal{C}}$ as the profile such that $\mathcal{C} = \{coal(i, \Pi^{\mathcal{C}}) \mid i \in N\}$—in fact, note that $\Pi^{\mathcal{C}}$ is well defined. Armed with the these notions, we now prove the following result.

Theorem 2. *Let $\mathcal{G}^{\mathsf{H}}$ be the discrete* LB*-preference game associated with the hedonic game* H.

(1) *Assume that $\mathcal{C}$ is a stable outcome for* H. *Then, the profile $\Pi^{\mathcal{C}}$ is stable for $\mathcal{G}^{\mathsf{H}}$.*
(2) *Assume that Π is a stable profile for $\mathcal{G}^{\mathsf{H}}$. Then, there exists stable outcome $\mathcal{C}$ for* H *such that $\Pi = \Pi^{\mathcal{C}}$.*

Proof. *(1)* Let $\mathcal{C}$ be a stable outcome for H and assume, by contradiction, that $\Pi^{\mathcal{C}}$ is not stable. Then, there is an agent $i \in N$ such that $u_i(\Pi^{\mathcal{C}}) < u_i(\Pi')$, where Π' coincides with Π over all the agents in $N \setminus \{i\}$. Therefore, if C is the coalition in $\mathcal{C}$ to which agent i belongs, then we have $\mu_i(C) < \mu_i(coal(i,\Pi'))$. In particular, note that $coal(i,\Pi') \setminus \{i\}$ is a coalition that belongs to $\mathcal{C}$. Hence, $\mathcal{C}$ is not stable, which is impossible.

(2) Assume that Π is a stable profile for $\mathcal{G}^{\mathsf{H}}$. Consider the outcome $\mathcal{C}$ such that $\Pi^{\mathcal{C}} = \Pi$ and assume, for the sake of contradiction, that $\mathcal{C}$ is not stable. Let $i \in N$ be an agent belonging to a coalition $C \in \mathcal{C}$ and such that $\mu_i(C) < \mu_i(\{i\} \cup C')$, for some given coalition $C' \in \mathcal{C} \cup \{\emptyset\}$. Consider, then, the profile Π' that coincides with Π on all agents but i, and such that $\{i\} \cup C' = coal(i,\Pi')$. In fact, note that $\Pi_i \neq \Pi'_i$. Eventually, $u_i(\Pi) = \mu_i(coal(i,\Pi)) = \mu_i(C) < \mu_i(\{i\} \cup C') = \mu_i(coal(i,\Pi')) = u_i(\Pi')$ holds, which is impossible. □

3.3. Graphical games

The settings we have discussed so far are representative of two extreme scenarios, where agents are willing to conform with their neighbors (cf. Section 3.1) and where they are enemies with them (cf. Section 3.2), respectively. Indeed, we have already noticed that some trivial stable configurations exist in these cases, where agents stay altogether and where they stay in isolation, respectively. We next consider a more flexible setting, based on *strategic games*, which can be used to model intermediate circumstances (and where, therefore, stable configurations do not necessarily exist). In fact, while being grounded in the game-theory literature, strategic games [72] have received considerable attention in the Artificial Intelligence community, because they have been recognized as appropriate models to formalize and reason about environments where selfish agents interact with each other in order to achieve their own goals (see, e.g., [73]). In particular, the third setting we consider is the setting of *graphical games* [30–32], that is, of those strategic games where we do not require that every agent directly influences all the others.

Formalization. Let us consider a set $N = \{1, \ldots, n\}$ of agents and a set S of strategies (for instance, actions that the agents might want to perform). A *joint strategy* σ for a set $V \subseteq N$ of agents is a function $\{j \mapsto \sigma(j)\}_{j \in V}$ mapping each agent to a strategy. A graphical game GG on N and S can be viewed as tuple $\langle N, S, \mathit{influence}, P\rangle$, where *influence* is a function and P is a set of n payoff functions. In particular, for each agent $i \in N$, *influence* provides the set of players $\mathit{influence}(i) \subseteq N \setminus \{i\}$ that influence i, whereas P contains her utility function pay_i associating each possible joint strategy σ for $\{i\} \cup \mathit{influence}(i)$ to a rational number $pay_i(\sigma)$. A joint strategy σ for N is a *Nash equilibrium* for GG if, for each agent $i \in N$ and each strategy $s \in S$,

$$pay_i\left(\{i \mapsto \sigma(i)\} \cup \{j \mapsto \sigma(j)\}_{j \in \mathit{influence}(i)}\right) \geq pay_i\left(\{i \mapsto s\} \cup \{j \mapsto \sigma(j)\}_{j \in \mathit{influence}(i)}\right).$$

Encoding. A distinguished feature of graphical games is that, for each agent $i \in N$, all agents in $\mathit{influence}(i)$ have an effect on i, no matter of the strategies they actually adopt. This feature cannot be immediately rephrased in terms of the notion of "compatibility" we used for defining the utility in a discrete LB-preference game (see Equation (1)). Hence, a careful encoding is in order.

Based on GG, let us build the discrete LB-preference game $\mathcal{G}^{\mathrm{GG}} = \langle N, E, \kappa\rangle$ as follows. First, for each agent $i \in N$ and $j \in \mathit{influence}(i)$, the edge (i, j) is in E; and no further edge is in E. Moreover, for each agent $i \in N$, by taking a large enough natural number $M > 0$,[4] we define:

$$\begin{aligned} G_i^0 \;=\; & \{(st_{j,s} \wedge st_{j,s'}, -3M^2) \mid j \in \{i\} \cup \mathit{influence}(i), s \in S, s' \in S, s \neq s'\} \bigcup \\ & \{(\textstyle\bigwedge_{s \in S} \neg st_{j,s}, -3M^2) \mid j \in \{i\} \cup \mathit{influence}(i)\}. \end{aligned}$$

Note that, in the above goalbase, propositional variables have the form $st_{j,s}$, with the intended meaning that agent $j \in N$ is playing strategy $s \in S$. In fact, the role of G_i^0 is to ensure that it is always convenient for agent i to select interpretations that immediately correspond to joint strategies for $\{i\} \cup \mathit{influence}(i)$, that is, where each agent $j \in \{i\} \cup \mathit{influence}(i)$ is univocally associated with a strategy. Indeed, thanks to the negative weight $-3M^2$, just note that according to the first group of weighted formulas, i will not associate j with two distinct strategies, whereas the last weighted formula guarantees that i associates j with at least one strategy. Instead, if $J_i = \mathit{influence}(i)$, then G_i^1 is defined as the set:

$$G_i^1 = \{(st_{i,s_i} \wedge \bigwedge_{j \in J_i} st_{j,s_j}, pay_i(\{i \mapsto s_i\} \cup \{j \mapsto s_j\}_{j \in J_i}) + M^2) \mid s_i \in S,\ s_j \in S\ \forall j \in J_i\}$$

In order to understand the role played by G_i^1, recall that an interpretation I that is desirable to i corresponds to a joint strategy for $\{i\} \cup \mathit{influence}(i)$. Hence, if I satisfies $st_{i,s_i} \wedge \bigwedge_{j \in \mathit{influence}(i)} st_{j,s_j}$, then I actually corresponds to the joint strategy $\{i \mapsto s_i\} \cup \bigcup_{j \in \mathit{influence}(i)} \{j \mapsto s_j\}$. Then, the corresponding weight is the payoff that i receives over this joint strategy, plus a fixed term M^2.

That fixed term plays a crucial role in the construction. Indeed, consider two agents i and $j \in \mathit{influence}(i)$. We know that preferred interpretations I for i define a strategy, say s_j, for j; however, there is no a-priory guarantee that s_j corresponds to the strategy that j encodes in her preferred interpretations. In fact, i and j might well select interpretations in which they are not compatible with

[4] For instance, it suffices that M is larger than $1 + |N|$ and than the maximum utility value in the definition of GG.

each other. The role of the term M^2 is precisely to avoid such scenarios, by providing an incentive to agent i when she selects an interpretation in which she is compatible with all the agents that have an influence on her. We next formalize our intuition.

Lemma 3. *Let Π be a profile for $\mathcal{G}^{\mathrm{GG}}$. If $u_i(\Pi) \geq M^2 \cdot |\mathtt{neigh}(i)|$, then*

(1) *there is precisely one strategy $s_i \in S$ such that $st_{i,s_i} \in \Pi_i$;*
(2) *for each agent $j \in influence(i)$, there is precisely one strategy $s_j \in S$ such that $st_{j,s_j} \in \Pi_i$;*
(3) *for each agent $j \in influence(i)$, $st_{j,s_j} \in \Pi_i$ implies that $st_{j,s_j} \in \Pi_j$.*

Proof. Recall first that the utility of agent i in a profile Π is given by:

$$u_i(\Pi) = G_i^0(\Pi_i) + G_i^1(\Pi_i) \cdot |\mathtt{compatible}_i(\Pi)|$$

Since M is greater than the maximum utility value in GG, by definition of G_i^1—where exactly one weighted formula is true—, we derive that $G_i^1(\Pi_i) < M + M^2$. Hence, $u_i(\Pi) \geq M^2 \cdot |\mathtt{neigh}(i)|$ and $|\mathtt{compatible}_i(\Pi)| \leq |\mathtt{neigh}(i)|$ imply that either $|\mathtt{neigh}(i)| = 0$ and $G_i^0(\Pi_i) \geq 0$, or $G_i^0(\Pi_i) > -M \cdot |\mathtt{neigh}(i)|$. Because of the definition of G_i^0 and by the fact that we have chosen $|\mathtt{neigh}(i)| < M$, in both cases, properties *(1)* and *(2)* are then seen to hold, and $G_i^0(\Pi_i) = 0$ actually hold.

So, we know that $u_i(\Pi) = G_i^1(\Pi_i) \cdot |\mathtt{compatible}_i(\Pi)|$. Assume, for the sake of contradiction, that an agent $j \in influence(i)$ exists such that $st_{j,s_j} \in \Pi_i$ and $st_{j,s_j} \notin \Pi_j$. This means that $|\mathtt{compatible}_i(\Pi)| \leq |\mathtt{neigh}(i)| - 1$. Hence, $u_i(\Pi) < (M + M^2) \cdot (|\mathtt{neigh}(i)| - 1)$. However, this is impossible by our choice of M and since $u_i(\Pi) \geq M^2 \cdot |\mathtt{neigh}(i)|$ by hypothesis. □

With each joint strategy $\sigma = \{i \mapsto s_i\}_{i \in N}$, we now associate a profile Π^σ such that, for each $i \in N$, $\Pi_i = \{st_{i,\sigma(i)}\} \cup \{st_{j,\sigma(j)} \mid j \in influence(i)\}$. Armed with the above definition and with Lemma 3, we can prove the following relationship.

Theorem 4. *Let $\mathcal{G}^{\mathrm{GG}}$ be the discrete* LB*-preference game associated with the graphical game* GG.

(1) *Assume that σ is a Nash equilibrium for* GG. *Then, the profile Π^σ is stable for $\mathcal{G}^{\mathrm{GG}}$.*
(2) *Assume that Π is a stable profile for $\mathcal{G}^{\mathrm{GG}}$. Then, there exists Nash equilibrium σ for* GG *such that $\Pi = \Pi^\sigma$.*

Proof. *(1)* If σ is a joint strategy for N, then $u_i(\Pi^\sigma) \geq M^2 \cdot |\mathtt{neigh}(i)|$ holds for each agent $i \in N$. Assume, now, that σ is actually a Nash equilibrium and, for the sake of contradiction, that an agent $i \in N$ is not playing one of her best response moves in $\mathcal{G}^{\mathrm{GG}}$. This means that there is an interpretation $I \neq \Pi_i^\sigma$ such that $u_i(\Pi^\sigma) < u_i(\Pi_{-i}^\sigma \cup \{i \mapsto I\})$. Hence, we can apply Lemma 3 on $\Pi_{-i}^\sigma \cup \{i \mapsto I\}$, in order to derive that I must be of the following form: $I = \{st_{i,s_i}\} \cup \bigcup_{j \in influence(i)} \{st_{j,s_j}\}$. Moreover, for each agent $j \in influence(i)$, we know that $st_{j,s_j} \in \Pi_j^\sigma$ and, hence, $\sigma(j) = s_j$. Therefore, I defines a joint strategy for N, say σ', that coincides with σ on the restriction over the agents in $N \setminus \{i\}$. Let $\bar{\sigma}$ and $\bar{\sigma}'$ be the restrictions of σ and σ' over $\{i\} \cup influence(i)$. Then, observe that $u_i(\Pi^\sigma) = (pay_i(\bar{\sigma}) + M^2) \cdot |\mathtt{neigh}(i)|$ and $u_i(\Pi_{-i}^\sigma \cup \{i \mapsto I\}) = (pay_i(\bar{\sigma}') + M^2) \cdot |\mathtt{neigh}(i)|$. That is, $pay_i(\bar{\sigma}) < pay_i(\bar{\sigma}')$. This is impossible, since σ is a Nash equilibrium.

(2) Assume that Π is a stable profile for $\mathcal{G}^{\mathrm{GG}}$. We first claim that $u_i(\Pi^\sigma) \geq M^2 \cdot |\mathtt{neigh}(i)|$ holds for each $i \in N$. To prove the claim, let us start first notice that we can assume that, for each agent $i \in N$, Π_i encodes a joint strategy over $\{i\} \cup \mathtt{neigh}(i)$ that we denote by τ_i: for each $j \in \{i\} \cup \mathtt{neigh}(i)$, $\tau_i(j) = s$ if, and only if, $st_{j,s} \in \Pi_i$. Indeed, if this were not the case, then $u_i(\Pi) < 0$ would hold (because of the form of G_i^0 and G_i^1) and i would get an incentive to select a different interpretation corresponding to a proper joint strategy (see Lemma 3). Moreover, note that every agent $i \in N$ can get a utility value greater than $M^2 \cdot |\mathtt{neigh}(i)|$, by just selecting τ_i so that, for each $j \in influence(i)$, $\{j \mapsto s_j\} \in \tau_i \cap \tau_j$ holds. Hence, we can apply Lemma 3 on Π, by noticing that a joint strategy σ for N exists such that $\Pi^\sigma = \Pi$. In particular, $\sigma = \{i \mapsto \tau_i(i)\}_{i \in N}$.

Assume now, for the sake of contradiction, that σ is not a Nash equilibrium. Then, there exists an agent $i \in N$ and strategy $s_i' \in S$ such that $pay_i(\bigcup_{j \in influence(i)} \{j \mapsto \sigma(j)\} \cup \{i \mapsto \sigma(i)\}) < pay_i(\bigcup_{j \in influence(i)} \{j \mapsto \sigma(j)\} \cup \{i \mapsto s_i'\})$. If σ' is the joint strategy built from σ by changing the strategy of i to s_i', then from the above inequality and by construction of the goalbases we derive $u_i(\Pi^\sigma) = (pay_i(\bigcup_{j \in influence(i)} \{j \mapsto \sigma(j)\} \cup \{i \mapsto \sigma(i)\}) + M^2) \cdot |\mathtt{neigh}(i)| < u_i(\Pi^{\sigma'}) = (pay_i(\bigcup_{j \in influence(i)} \{j \mapsto \sigma(j)\} \cup \{i \mapsto s_i'\}) + M^2) \cdot |\mathtt{neigh}(i)|$, which is impossible. □

3.4. Discrete preference games

The final setting we consider is that of discrete preferences games, as they were originally defined in the work by Chierichetti et al. [29]—in fact, the reader can check that our encoding straightforwardly accommodates the variants proposed in recent works [41]. The encoding uses some ideas discussed in Section 3.3 and, therefore, its illustration should be rather easy to follow.

Formalization. Let $N = \{1, \ldots, n\}$ be a set of agents and S a set of strategies. Each agent $i \in N$ has a preferred strategy $pr_i \in S$ and there is a metric $d(\cdot, \cdot)$ on S measuring how different are two given strategies in S. A discrete preference game DPG is then a tuple $\langle N, S, influence, d \rangle$, where *influence* is the influence function. A joint strategy σ is a function $\{j \mapsto \sigma(j)\}_{j \in N}$, and the *cost* to agent $i \in N$ in σ is given by $c_i(\sigma) = \alpha \cdot d(\sigma(i), pr_i) + (1-\alpha) \cdot \sum_{j \in \mathit{influence}(i)} d(\sigma(i), \sigma(j))$, where α is a rational number with $1 > \alpha \geq 0$. Then, σ is a *Nash equilibrium* for DPG if, for each agent $i \in N$ and each strategy $s \in S$, $c_i(\sigma) \leq c_i(\{i \mapsto s\} \cup \{j \mapsto \sigma(j)\}_{j \in N \setminus \{i\}})$.

Encoding. Based on DPG, we build a discrete LB-preference game $\mathcal{G}^{\mathsf{DPG}} = \langle N, E, \kappa \rangle$ as follows.[5] First, for each agent $i \in N$ and $j \in \mathit{influence}(i)$, the edge (i, j) is in E; and no further edge is in E. Moreover, for each agent $i \in N$, we define (for a large enough natural number $M > 0$):

$$\begin{aligned} G_i^0 \;=\; & \{(st_{j,s} \wedge st_{j,s'}, -M^2) \mid j \in \{i\} \cup \mathit{influence}(i), s \in S, s' \in S, s \neq s'\} \bigcup \\ & \{(\textstyle\bigwedge_{s \in S} \neg st_{j,s}, -M^2) \mid j \in \{i\} \cup \mathit{influence}(i)\} \bigcup \\ & \{(st_{i,s}, -\alpha \cdot d(pr_i, s) - |\mathit{influence}(i)| \cdot M) \mid s \in S\} \end{aligned}$$

$$G_i^1 \;=\; \{(st_{i,s_i} \wedge st_{j,s_j}, -(1-\alpha) \cdot d(s_i, s_j) + M) \mid s_i \in S, s_j \in S\}$$

So, the encoding is rather straightforward: First, we just force that desirable profiles are such that each agent chooses precisely one strategy. In particular, the following is easily seen to hold.

Lemma 5. *Let Π be a profile for $\mathcal{G}^{\mathsf{DPG}}$. If $u_i(\Pi) > -M$, then*

(1) *there is precisely one strategy $s_i \in S$ such that $st_{i,s_i} \in \Pi_i$;*
(2) *for each agent $j \in \mathit{influence}(i)$, there is precisely one strategy $s_j \in S$ such that $st_{j,s_j} \in \Pi_i$;*
(3) *for each agent $j \in \mathit{influence}(i)$, $st_{j,s_j} \in \Pi_i$ implies that $st_{j,s_j} \in \Pi_j$.*

Proof. By definition of G_i^0 and M, properties *(1)* and *(2)* are immediate. Assume, now, that *(3)* does not hold. Then, $u_i(\Pi) = G_i^0(\Pi_i) + G_i^1(\Pi_i) \cdot |\mathtt{compatible}_i(\Pi)| \leq -|\mathit{influence}(i)| \cdot M + (|\mathtt{neigh}(i)| - 1) \cdot M \leq -M$, which is impossible. □

Now, note that the values in the other goalbases are defined as to mimic the costs in DPG. Indeed, as in Section 3.3, with each joint strategy $\sigma = \{i \mapsto s_i\}_{i \in N}$, let us associate a profile Π^σ such that, for each $i \in N$, $\Pi_i = \{st_{i,\sigma(i)}\} \cup \{st_{j,\sigma(j)} \mid j \in \mathit{influence}(i)\}$. In particular, profile Π^σ is such that $u_i(\Pi^\sigma) > -M$, for each $i \in N$. Thus, by Lemma 5, it holds that $u_i(\Pi^\sigma) = -\alpha \cdot d(pr_i, s) - |\mathit{influence}(i)| \cdot M + \sum_{j \in \mathtt{neigh}(i)} (-(1-\alpha) \cdot d(\sigma(i), \sigma(j)) + M) = -c_i(\sigma)$ and the following can be shown with the same line of reasoning as in the proof of Theorem 4.

Theorem 6. *Let $\mathcal{G}^{\mathsf{DPG}}$ be the discrete LB-preference game associated with the game* DPG.

(1) *Assume that σ is a Nash equilibrium for* DPG. *Then, the profile Π^σ is stable for $\mathcal{G}^{\mathsf{DPG}}$.*
(2) *Assume that Π is a stable profile for $\mathcal{G}^{\mathsf{DPG}}$. Then, there exists Nash equilibrium σ for* DPG *such that $\Pi = \Pi^\sigma$.*

We leave the section by noticing that a setting related DPG is the one of *generalized discrete preferences games* (GDPG) proposed by Auletta et al. [27]. In GDPG, each agent has an internal constraint/goalbase plus a set of social constraints/goalbases defined over variables that are shared with other agents, and the utility of each agent is defined as a non-decreasing function over the sum of the weights of her constraints that are satisfied. In fact, this setting is incomparable with ours, since, on the one hand, we cannot model arbitrary non-decreasing functions and since, on the other hand, utilities in GDPG do not depend on the number of compatible agents.

4. On the complexity of reasoning about discrete LB-preference games

After that our logic-based framework to reason about discrete LB-preference games has been formalized, we can initiate the study of its computational properties. In particular, we already know from Example 4 that discrete LB-preference games exist that do not admit stable profiles and, in fact, our goal is to study questions related to the stability notion we have formalized in Section 2.3. More formally, we would like to analyze the following two decision problems, both of them receiving as input a discrete LB-preference game $\mathcal{G}$:

- IS-NASH: Given a profile Π together with $\mathcal{G}$, is Π stable in $\mathcal{G}$?
- $\exists$-NASH: Does $\mathcal{G}$ admit any stable profile?

In the analysis, we assume a standard encoding[6] (see, e.g., [74]) for the various components of $\mathcal{G} = \langle N, E, \kappa \rangle$, whose size is denoted by $||\mathcal{G}||$. We next overview the results we have derived, and we eventually elaborate the corresponding proofs.

[5] Now, one can take M as an integer larger than $1 + |N|$ and the maximum absolute value of the costs.
[6] For instance, κ is encoded by explicitly listing all agents in N with their associated knowledge bases.

Table 1
Summary of the results discussed in Section 4. Hardness results hold even on discrete LB-preference games defined over *symmetric directed* graphs.

maxDomSize($\mathcal{G}$)	IS-NASH	∃-NASH
$O(1)$	in **PTIME** [Theorem 9]	**NP**-complete [Theorem 11]
$O(\log \|\|\mathcal{G}\|\|)$	in **PTIME** [Theorem 9]	**NP**-complete [Theorem 12]
no restriction	**coNP**-complete [Theorem 7]	$\mathbf{\Sigma_2^P}$-complete [Theorem 8]

Table 2
Knowledge base κ in the proof of Theorem 8.

i	G_i^0	G_i^1
1	$\{(z \Leftrightarrow \phi(\mathbf{x}, \mathbf{y}), 3/2)\}$	$\{(\top, 1)\}$
2	$\emptyset$	$\{((x_1 \vee \neg x_1) \wedge \cdots \wedge (x_p \vee \neg x_p), 1)\}$
3	$\emptyset$	$\{(z, -1), (\neg z, -1)\}$
4	$\emptyset$	$\{(z, 1), (\neg z, 1)\}$
5	$\{(z, 1)\}$	$\emptyset$

4.1. Overview of the results

In the light of the precise correspondences we have established in Section 3, the careful reader might have already derived by herself some complexity results for ∃-NASH. For instance, the problem of deciding whether Nash equilibria exist in graphical games is known to be **NP**-complete (in the specific formulation we have considered), as shown by Gottlob et al. [30]. Therefore, Theorem 4 (and the fact that the encoding can be implemented in polynomial time) immediately implies that ∃-NASH is **NP**-hard, too. Here, we would like to embark on a finer grained analysis, by assessing how that complexity is affected by the maximum cardinality of the domain over all the knowledge bases in $G = \langle N, E, \kappa \rangle$, denoted as $\mathtt{maxDomSize}(\mathcal{G}) = \max_{i \in N} |\mathtt{dom}(\kappa(i))|$. Indeed, this analysis constitutes an important ingredient to guide our search for islands of tractability, since it allows us to understand whether the intrinsic complexity of reasoning about stable profiles is related to the individual capabilities of the agents (measured in terms of the number of variables on which they reason), or rather it comes from their interplay over social networks.

A summary of our complexity results, derived parametrically w.r.t. the value maxDomSize($\mathcal{G}$), is presented in Table 1. Let us focus on ∃-NASH. Note that in the general case, that is, when no restriction is provided to maxDomSize($\mathcal{G}$), reasoning about the existence of stable profiles is even harder than just reasoning about Nash equilibria in graphical games. Indeed, ∃-NASH emerges to be complete for the second level of the polynomial hierarchy (cf. $\mathbf{\Sigma_2^P}$-complete). The complexity drops down one level in the polynomial hierarchy when a logarithmic bound on this parameter is defined. Moreover, we show that intractability still holds on games $\mathcal{G}$ such that $\mathtt{maxDomSize}(\mathcal{G}) \in O(1)$, that is, when agents reason about a constant number of variables (in fact, even when they reason about one variable only, as we shall see in Lemma 10).

Eventually, the complexity of IS-NASH emerges to be precisely related to that of ∃-NASH, with the problem being feasible in polynomial time whenever ∃-NASH is **NP**-complete.

4.2. Proofs of the intractable cases

We start illustrating the proofs of the results summarized in Table 1 by considering *arbitrary* games $\mathcal{G}$, that is, where no restriction is imposed on maxDomSize($\mathcal{G}$). In this case, IS-NASH and ∃-NASH emerge to be hard for the complexity classes **coNP** and $\mathbf{\Sigma_2^P}$, respectively.

Theorem 7. IS-NASH *is **coNP**-complete. Hardness holds even on discrete* LB*-preference games defined over one agent only.*

Proof. *(Membership)* Let $\mathcal{G}$ be a discrete LB-preference game and let Π be a profile given as input. Let us consider the complementary problem of deciding whether Π is not a stable profile for $\mathcal{G}$. We shall show that this problem belongs to the complexity class **NP**. Indeed, the problem can be solved by a non-deterministic Turing machine that first (a) guesses an agent $i \in N$ and an interpretation I and, then (b) checks that $u_i(\Pi) < u_i(\Pi_{-i} \cup \{i \mapsto I\})$. In particular, note that the utility function can be computed in polynomial time.

(Hardness) Consider the prototypical **coNP**-complete problem of deciding whether a propositional formula ϕ with $\mathtt{dom}(\phi) = \{x_1, \ldots, x_m\}$ is not satisfiable [74,75]. Based on ϕ, we construct in polynomial time the discrete LB-preference game $\mathcal{G} = \langle \{1\}, \emptyset, \kappa \rangle$ with $\kappa(1) = \langle G_1^0, \emptyset \rangle$ and $G_1^0 = \{(x_0 \wedge \phi, 1)\}$. Consider the profile $\Pi = \{1 \mapsto \Pi_1\}$ with $\Pi_1 = \{\neg x_0, \neg x_1, \ldots, \neg x_m\}$, and note that $u_1(\Pi) = 0$ since $\Pi_1 \not\models x_0 \wedge \phi$. Eventually, Π is stable, if and only if, ϕ is unsatisfiable. □

Theorem 8. ∃-NASH *is* $\mathbf{\Sigma_2^P}$*-complete. Hardness holds even on discrete* LB*-preference games defined over symmetric directed graphs.*

Proof. *(Membership)* The problem can be solved in polynomial time by a non-deterministic Turing machine that first guesses a profile Π, and then checks whether Π is stable. This latter task belongs to **coNP** because of Theorem 7; hence, it can be solved (in polynomial time) by invoking an **NP** oracle. Therefore, $\exists$-NASH belongs to $\mathbf{\Sigma_2^P}$.

(Hardness) Let $\phi(\mathbf{x},\mathbf{y})$ be a propositional formula over the variables in $\mathbf{x} = \{x_1,...,x_p\}$ and in $\mathbf{y} = \{y_1,...,y_q\}$. Consider the quantified formula $F = \exists x_1,...,x_p\ \forall y_1,...,y_q\ \phi(\mathbf{x},\mathbf{y})$, and note that deciding whether F is satisfiable is a well-known $\mathbf{\Sigma_2^P}$-complete problem [74]. Based on F, we build in polynomial time the discrete LB-preference game $\mathcal{G} = \langle\{1,2,3,4,5\}, E, \kappa\rangle$ by keeping the underlying graph of Fig. 1 and where κ is now specified in Table 2. Note that the knowledge bases are defined over the variables in $\mathbf{x}$ and in $\mathbf{y}$, plus the distinguished variable z.

We claim that: *F is satisfiable $\Leftrightarrow$ $\mathcal{G}$ admits a stable profile.*

($\Rightarrow$) Assume that I is an interpretation with $\mathtt{dom}(I) = \mathbf{x}$ witnessing that F is satisfiable. That is, for each possible interpretation J with $\mathtt{dom}(J) = \mathbf{y}$, it holds that $I \cup J \vDash \phi(\mathbf{x},\mathbf{y})$. Let $\bar{J}$ be any arbitrary interpretation with $\mathtt{dom}(\bar{J}) = \mathbf{y}$, and let $\Pi = \{i \mapsto \Pi_i\}_{i\in[5]}$ such that:
- $\Pi_1 = I \cup \bar{J} \cup \{z\}$;
- $\Pi_2 = I$;
- $\Pi_3 = \Pi_4 = \{\neg z\}$; and
- $\Pi_5 = \{z\}$.

According to the knowledge bases depicted in Table 2, it can be checked that: $u_1(\Pi) = 5/2$, $u_2(\Pi) = 1$, $u_3(\Pi) = -1$, $u_4(\Pi) = 1$, and $u_5(\Pi) = 1$. Now, we shall show that each agent is playing one of her best response moves in Π. Indeed, agents 2, 4 and 5 are already getting their maximum possible utility values. Consider now agent 3. She is getting utility -1, because in Π she is compatible with agent 4 (just notice that $\neg z \in \Pi_3 \cap \Pi_4$), and she is stable because she would get -2 in any interpretation where z evaluates true (because $z \in \Pi_5 \cap \Pi_1$ holds). Finally, consider agent 1. She is compatible with agent 2 but not with agent 3, and is such that $G_1^0(\Pi_1) = 3/2$ (which is the maximum possible value for this goalbase). Assume, for the sake of contradiction, that 1 is not stable (i.e., she is not playing her best response move). Then, she has an incentive to deviate to an interpretation having the form $I \cup \bar{J}' \cup \{\neg z\}$, because in the novel interpretation she must be compatible with both 2 and 3. However, $G_1^0(I \cup \bar{J}' \cup \{\neg z\}) = 0$ holds—recall that I witnesses that F is satisfiable. Hence, 1 would get utility $2 < 5/2$, which is impossible.

($\Leftarrow$) Assume that Π is a stable profile. Note first that $\Pi_5 = \{z\}$ holds. Let us then consider agent 3, for which we claim that $\Pi_3 = \{\neg z\}$. Indeed, assume by contradiction that $\Pi_3 = \{z\}$. Then, since Π is stable, we immediately derive that $\Pi_4 = \{z\}$. Hence, $u_3(\Pi) \leq -2$ and agent 3 will clearly deviate to the interpretation where z evaluates false, since $u_3(\Pi_{-3} \cup \{3 \mapsto \{\neg z\}) \geq -1$ holds (just note that agent 1 is the only that can be compatible). Therefore, we conclude that agent 3 is not stable, which is impossible. So, we know that $\Pi_3 = \{\neg z\}$, which entails that $\Pi_4 = \{\neg z\}$. Moreover, it is immediate to check that, for the stability of agent 2, it must be the case that agent 2 is compatible with agent 1. In the following, let $I = \Pi_2$ (which is an interpretation with $\mathtt{dom}(I) = \{x_1,...,x_p\}$). Consider agent 1. We first claim that $z \in \Pi_1$ holds. Indeed, if $\neg z \in \Pi_1$, then $u_3(\Pi) = -2$ and agent 3 finds convenient to deviate to an interpretation where z is true, which is impossible.

Hence, $z \in \Pi_1$ and, in fact, $\Pi_1 \setminus \{z\}$ must satisfy the formula $\phi(\mathbf{x},\mathbf{y})$, otherwise agent 1 would be not stable in that she might find convenient to deviate to an interpretation where z is false. At this point, we know that $u_1(\Pi) \geq 3/2 + 1$, since $G_1^0(\Pi_1) = 3/2$ and since agent 1 is compatible only with agent 2. Now, for each possible interpretation J such that $\mathtt{dom}(J) = \{y_1,....,y_q\}$, consider the profile $\Pi' = \Pi_{-1} \cup \{1 \mapsto I \cup J \cup \{\neg z\}\}$. Since 1 is stable in Π, it must be the case that $u_1(\Pi') = G_1^0(I \cup J \cup \{\neg z\}) + 2 \leq u_1(\Pi) = 5/2$. Hence, for each interpretation J, $G_1^0(I \cup J \cup \{\neg z\}) = 0$ must actually hold. That is, for each interpretation J with $\mathtt{dom}(J) = \{y_1,....,y_q\}$, $I \cup J \vDash \phi(\mathbf{x},\mathbf{y})$. Thus, I witnesses the satisfiability of F. $\square$

Let us now move to the setting where $\mathtt{maxDomSize}(\mathcal{G}) \in O(\log||\mathcal{G}||)$. In this case, there are polynomially-many interpretations associated with the various knowledge bases, so that the following is immediately established.

Theorem 9. *On classes of games $\mathcal{G}$ such that $\mathtt{maxDomSize}(\mathcal{G}) \in O(\log||\mathcal{G}||)$, IS-NASH is in **PTIME**, whereas $\exists$-NASH is in **NP**.*

Proof. Consider a game $\mathcal{G}$ and a profile Π. For each agent $i \in N$, we have just to exhaustively enumerate all possible interpretations I for $\kappa(i)$, and check whether if $u_i(\Pi) \geq u_i(\Pi_{-i} \cup \{i \mapsto I\})$ actually holds. Therefore, IS-NASH is clearly feasible in polynomial time.

Concerning $\exists$-NASH, it can be solved in polynomial time by a non-deterministic Turing machine that first guesses the profile Π and then check, as discussed above, whether it is stable. $\square$

The above result is clearly inherited by those scenarios where the number of variables in knowledge bases is bounded by some given constant, as it emerges by looking at Table 2. To complete the picture reported there, it now suffices to prove the hardness results with a constant or logarithmic number of variables. We start with a lemma focusing on $\mathtt{maxDomSize}(\mathcal{G}) = 1$.

In the following, we use some reductions for coloring problems and it is convenient to introduce some further notation. Let $C = \{c_1,...,c_k\}$ be a set of k colors. Let $\mathcal{H} = (V, H)$ be a *hypergraph* on a set of nodes V and where, for each $h \in H$, $h \subseteq V$ holds; in the special case where $|h| = 2$, for each $h \in H$, $\mathcal{H}$ is an undirected graph. A *k-coloring* for $\mathcal{H}$ is a function γ^k mapping each node $i \in V$ to a color $\gamma^k(i) \in C$. We say that $\mathcal{H}$ is *k-colorable*, if it admits a k-coloring γ^k that is *valid*, i.e., such that for each $h \in H$, there are (at least) two nodes i and j in h with $\gamma^k(i) \neq \gamma^k(j)$.

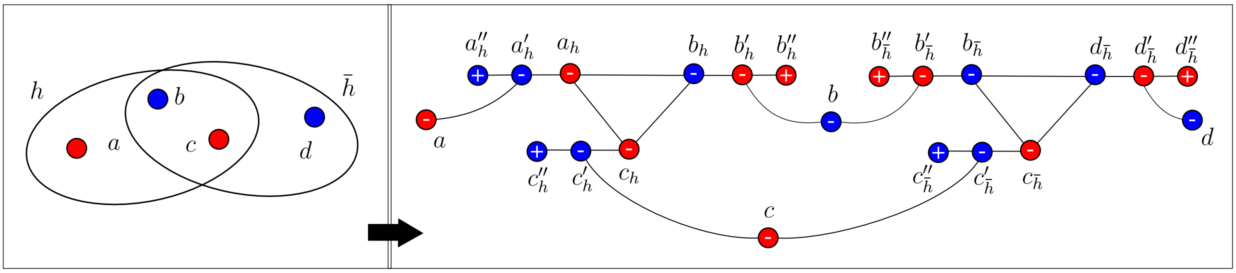

Fig. 2. Example reduction in the proof of Lemma 10—the directed graph is symmetric and edge orientation is omitted, for the sake of simplicity. According to the reduction, the coloring of the nodes induces a stable profile.

Lemma 10. *On classes of games $\mathcal{G}$ such that* $\texttt{maxDomSize}(\mathcal{G}) = 1$, $\exists$-NASH *is* ***NP****-complete. Hardness holds even on games defined over symmetric directed graphs.*

Proof. Membership is inherited from Theorem 9. For the hardness part, consider a *3-uniform hypergraph* $\mathcal{H} = (V, H)$, where V is the set of its nodes whereas H is the set of its 3-uniform hyperedges—that is, for each $h \in H$, we have $|h| = 3$. Deciding whether $\mathcal{H}$ is 2-colorable is a well-known **NP**-complete problem [76].

Based on $\mathcal{H}$, we build the environment $\mathcal{G} = \langle N, E, \kappa \rangle$ such that:

(1) N contains all the nodes of V, here viewed as agents; also, for each hyperedge $h \in H$ and for each node $a \in h$, N contains the agents in $\{a_h, a'_h, a''_h\}$; and no further agent is in N;
(2) for each hyperedge $h \in H$ with $h = \{a, b, c\}$, E contains the following edges and their opposite, i.e., E is symmetric: (a_h, b_h), (a_h, c_h), (b_h, c_h), (a_h, a'_h), (a'_h, a''_h), (a'_h, a), (b_h, b'_h), (b'_h, b''_h), (b'_h, b), (c_h, c'_h), (c'_h, c''_h), and (c'_h, c); no further edge is in E;
(4) for each hyperedge $h \in H$ and for each node $a \in h$, agent a''_h would like to be compatible with her neighbors, i.e., $\kappa(a''_h) = \langle G^0_{a''_h}, G^1_{a''_h} \rangle$ with $G^0_{a''_h} = \{(c_1, 0)\}$ and $G^1_{a''_h} = \{(\top, 1)\}$;
(5) all remaining agents $i \in N$ prefer not to be compatible with their neighbors, i.e., $\kappa(i) = \langle G^0_i, G^1_i \rangle$ with $G^0_i = \{(c_1, 0)\}$ and $G^1_i = \{(\top, -1)\}$.

The above reduction is illustrated in Fig. 2, over a hypergraph consisting of two hyperedges only. Note that agents reason about the propositional variable c_1 corresponding to one of the two colors available in C, so that $\neg c_1$ is meant to encode the other color c_2.

Note that the reduction is feasible in polynomial time; hence, we conclude by claiming that: *$\mathcal{H}$ admits a valid 2-coloring $\Leftrightarrow$ $\mathcal{G}$ admits a stable profile.*

($\Rightarrow$) Assume that γ^2 is a valid 2-coloring, and consider the profile Π such that:
- for each node $a \in V$ with $\gamma^2(a) = c_1$ and for each hyperedge $h \in H$ with $a \in h$, $\Pi_a = \Pi_{a_h} = \{c_1\}$ and $\Pi_{a'_h} = \Pi_{a''_h} = \{\neg c_1\}$;
- for each node $a \in V$ with $\gamma^2(a) = c_2$ and for each hyperedge $h \in H$ with $a \in h$, $\Pi_a = \Pi_{a_h} = \{\neg c_1\}$ and $\Pi_{a'_h} = \Pi_{a''_h} = \{c_1\}$.

Consider an agent $a''_h \in N$ and note that she is stable, as she is compatible in Π with her unique neighbor a'_h. In her turn, agent a'_h is stable, because she has three neighbors: a, a_h and a''_h and, by construction of Π, she is compatible with one of them only. Consider now an agent $a \in N \cap V$. She prefers not to be compatible with her neighbors; these neighbors have the form a'_h, where h is a hyperedge and are such that $\Pi_{a'_h} \neq \Pi_a$. That is, a is stable too. In order to conclude, let us now show that all agents having the form a_h, with $a \in V$ and $h \in H$ such that $h = \{a, b, c\}$, are stable. Consider agent a_h and note that she is adjacent to a'_h, which is such that $\Pi_{a_h} \neq \Pi_{a'_h}$. Moreover, since γ^2 is a valid coloring, nodes a, b and c are not colored with the same color, so that we can assume, w.l.o.g., that $\Pi_{a_h} \neq \Pi_{b_h}$. That is, agent a_h has three neighbors and her interpretation is compatible in Π with at most one of them. This implies that a_h is stable.

($\Leftarrow$) Assume that Π is a stable interpretation. Consider any pair of agents having the form a and a_h, with $a \in V$ and $h \in H$. Note that these two agents plus a''_h are the neighbors of the agent a'_h, which would like not to be compatible with them. Since a'_h is stable, then at most one of her neighbors is compatible with her. In particular, a''_h is necessarily compatible with her, for otherwise she would not be stable. This means that $\Pi_{a'_h} \neq \Pi_{a_h}$ and $\Pi_{a'_h} \neq \Pi_a$: therefore, $\Pi_a = \Pi_{a_h}$. In words, a'_h just acts in a way to propagate the color of a to the agent a_h. Based on Π, consider now the 2-coloring γ^2 such that $\gamma^2(a) = c_1$ if, and only if, $\Pi_a = \{c_1\}$. Assume for the sake of contradiction that γ^2 is not valid and, w.l.o.g., let $h = \{a, b, c\}$ be such that $\gamma^2(a) = \gamma^2(b) = \gamma^2(c) = c_1$. Then, we know that $\Pi_{a_h} = \Pi_{b_h} = \Pi_{c_h} = \{c_1\}$. Consider now agent a_h. She is compatible with b_h and c_h, and she has three neighbors (the other one is a'_h). Hence, she is not stable because she prefers not to be compatible with her neighbors, which is impossible.

The proof is now complete and the statement is proven. $\square$

Theorem 11. *Let $h \geq 1$ be a fixed natural number. On the class of discrete* LB*-preference games $\mathcal{G}$ such that* $\texttt{maxDomSize}(\mathcal{G}) = h$, $\exists$-NASH *is* ***NP****-complete. Hardness holds even on games defined over symmetric directed graphs.*

Proof. Membership is inherited from Theorem 9. Moreover, after Lemma 10, we can just focus on considering the case where $h > 1$. Let $k = 2^h$. Consider a graph $G = (N, E)$ and recall that deciding whether G is k-colorable is **NP**-hard [77]. Based on G, we build the discrete LB-preference game $\mathcal{G} = \langle N', E', \kappa\rangle$ such that:

(1) for each node $v \in N$ having $\delta(v) \in \mathbb{N}$ incident edges in E, N' contains the agents $v, v_1, ..., v_{\delta(v)}$; no further agent is in N';
(2) E' contains all edges in E; for each node $v \in N$, E' also contains the edges (v, v_j) and (v_j, v) with $j \in \{1, ..., \delta(v)\}$; no further edge is in E';
(3) for each agent $a \in N'$, $\kappa(a) = \langle G_a^0, G_a^1\rangle$ is such that:
   - $G_a^0 = (\top, 0)$;
   - $G_a^1 = ((y_1 \vee \neg y_1) \wedge ... \wedge (y_h \vee \neg y_h), \alpha)$, where $\alpha = -1$ if $a \in N$, and $\alpha = 1$ otherwise.

.

Note that the reduction is feasible in polynomial time. Moreover, note that agents reason about the propositional variables in $\{y_1, ..., y_h\}$. In particular, every interpretation I can be associated with the color in $C = \{c_1, ..., c_k\}$ having index $c(I) = \sum_{y_i \in I} 2^{i-1}$. Then, we conclude by claiming that: *G admits a valid k-coloring $\Leftrightarrow$ $\mathcal{G}$ admits a stable profile.*

($\Rightarrow$) Assume that γ^k is a valid k-coloring. Consider the profile Π such that, for each node $v \in N$ and each $j \in \{1, ..., \delta(v)\}$, $\Pi_v = \Pi_{v_j} = I$ with $\gamma^k(v) = c_{c(I)}$. Note that each agent having the form v_j is stable, since in Π she is compatible with her neighbor v. Moreover, since γ^k is valid, for each $(v, v') \in E$, it holds that $\gamma^k(v) \neq \gamma^k(v')$. Hence, for each $v \in N$, $|\mathtt{compatible}_v(\Pi)| = \delta(v)$, whereas $|\mathtt{neigh}(v)| = 2 \cdot \delta(v)$, which implies that v is stable.

($\Leftarrow$) Assume that Π is a stable profile. Consider an agent $v \in N$. Each agent v_j, for each $j \in \{1, ..., \delta(v)\}$, is clearly such that $\Pi_v = \Pi_{v_j}$. Hence, $|\mathtt{compatible}_v(\Pi)| = \delta(v)$. Then, given that v prefers not to be compatible with her neighbors and given that she is stable, we conclude that $\Pi_v \neq \Pi_{v'}$, whenever $(v, v') \in E$. Hence, the coloring γ^k such that, for each $v \in N$, $\gamma^k(v)$ returns the color with index $c(\Pi_v)$ is valid. □

We conclude the analysis by observing that the problem of deciding whether a graph $G = (N, E)$ is k-colorable remains intractable, even if $k = O(\log |N|)$—indeed, just inspect the proof of Maffray and Preissman [77]. Hence, by looking at the proof of our result (and at Theorem 9), the following is immediately established.

Theorem 12. *On the class of games $\mathcal{G}$ such that* $\mathtt{maxDomSize}(\mathcal{G}) = O(\log ||\mathcal{G}||)$, ∃-NASH *is **NP**-complete. Hardness holds even on games defined over symmetric directed graphs.*

5. Islands of tractability based on qualitative restrictions

Given the bad news emerged from the analysis we conducted so far, and moving with the aim of identifying islands of tractability, we next embark on the study of a restricted and special kind of discrete LB-preference games, called *linear*, where for each agent $i \in N$ with $\kappa(i) = \langle G_i^0, G_i^1\rangle$, $G_i^1(\Pi_i) = G_i^1(\Pi_i')$ holds on any pair of interpretations Π_i and Π_i'. Accordingly, in the setting we shall next elaborate, by denoting as $\mathtt{grad}(i)$ the value returned by G_i^1 on any given interpretation, the utility of each agent $i \in N$ (as given by Equation (1)) can be more conveniently rewritten as follows:

$$u_i(\Pi) = G_i^0(\Pi_i) + \mathtt{grad}(i) \cdot |\mathtt{compatible}_i(\Pi)|. \tag{2}$$

Therefore, in this setting, it makes sense to classify every agent i based on the associated value $\mathtt{grad}(i)$, by saying that i is: *autonomous*, if $\mathtt{grad}(i) = 0$; *conformist*, if $\mathtt{grad}(i) > 0$; and *dissenter*, if $\mathtt{grad}(i) < 0$. Indeed, an autonomous agent i does not take care of the neighbors; a conformist agent i would like to be compatible with as many neighbors as possible; and, a dissenter agent i prefers to avoid being compatible with her neighbors. Intuitively, linear games define a natural and simple setting to define the concepts of autonomous, conformist and dissenter agents, and for reasoning about environments in which they interact, while still offering a rich way to model their innate opinions. For instance, note that the environment defined in Example 2 is linear; in particular, agents 2 and 5 are autonomous, agents 4 and 1 are conformists, whereas agent 3 is a dissenter.

Armed with the above definitions, let us now take a closer look at results we provided in Section 4. By inspecting the various proofs, the following is immediately shown.

Theorem 13. *All hardness results in Table 1 hold on linear discrete* LB*-preference games where conformists coexist with dissenters.*

That is, our construction in Section 4 already evidence that the interplay of conformists and dissenters quickly leads to intractability. And, of course, this poses the question about what happens if we are able to prevent such interactions. In this section, we precisely address this question by reconsidering ∃-NASH and by imposing qualitative restrictions on the kinds of agents that are allowed. A summary of our results is reported in Fig. 3, where we distinguish the two cases of games built on top of symmetric and arbitrary graphs. In particular, since our goal is to look for islands of tractability where we can even compute in polynomial time a stable profile if one exists, it makes sense to focus on games $\mathcal{G}$ such that $\mathtt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$. Indeed, just notice that computing

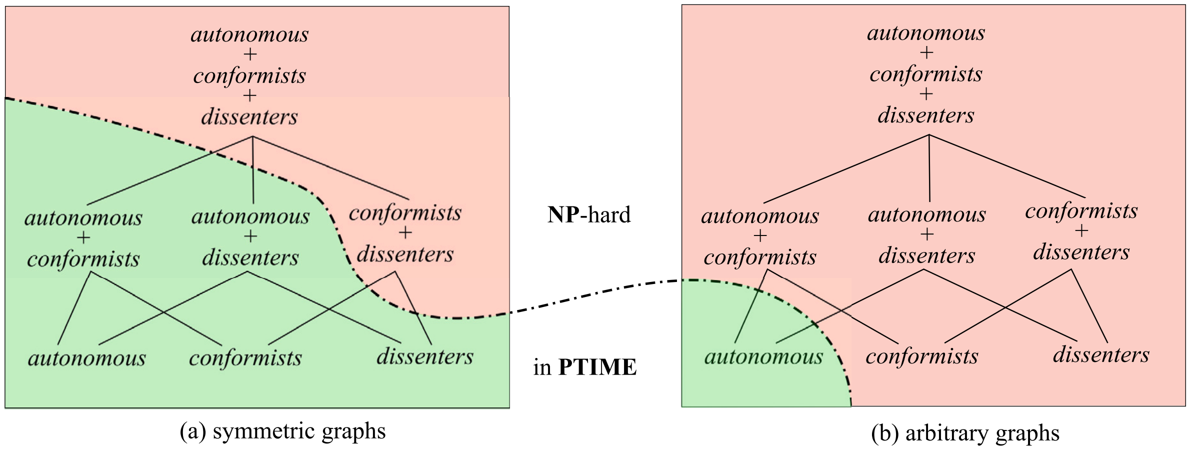

Fig. 3. Summary of complexity results on linear games $\mathcal{G}$, with $\texttt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$.

a stable interpretation is intractable even on games with one agent only, if that agent can reason about an arbitrary number of variables—in fact, the problem is equivalent to computing an interpretation with maximum weight [50].

Note in Fig. 3 that, over arbitrary graphs, reasoning about the types of the agents provides little benefits. Instead, on symmetric directed graphs, we are able to prove a number of rather interesting tractability results. In particular, such results are established in a constructive way, by showing that any sequence of best response moves will eventually converge, after polynomially many steps, to a stable profile. This is a noticeable behavior. For instance, a behavior of this kind has been already known [29,37] to characterize the opinion diffusion scenarios discussed in Section 3.1.[7]

Proofs of the results in Fig. 3 are next elaborated, by first focusing on symmetric directed graphs and then completing the picture by discussing the case of games built on arbitrary graphs.

5.1. Symmetric directed graphs and normal forms

Let us start our analysis by recalling from Equation (2) that the utility function of an agent $i \in N$ in a linear environment $\mathcal{G} = \langle N, E, \kappa \rangle$ can be written as $u_i(\Pi) = G_i^0(\Pi_i) + \texttt{grad}(i) \cdot |\texttt{compatible}_i(\Pi)|$. We now further elaborate this equation in order to derive an expression that can be more conveniently used to prove the results. This is captured by the following definition.

Definition 2. Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear game, and let $h = \texttt{maxDomSize}(\mathcal{G})$. We say that $\mathcal{G}$ is in *normal form* if, for each agent $i \in N$, it holds that: $\texttt{grad}(i) \in \{-2^h, 0, 2^h\}$ and $\{G_i^0(I) \mid I$ is an interpretation over $\texttt{dom}(\kappa(i))\} \subset \{0, ..., M\}$, with $M = (|N| + 2) \cdot 2^{2h}$. □

We next show that we can focus, w.l.o.g., on linear games in normal form.

Theorem 14. *Given a linear discrete* LB*-preference game* $\bar{\mathcal{G}} = \langle N, E, \bar{\kappa} \rangle$*, we can build a linear discrete* LB*-preference game in normal form* $\mathcal{G} = \langle N, E, \kappa \rangle$ *with* $\texttt{dom}(\kappa(i)) = \texttt{dom}(\bar{\kappa}(i))$*, for each* $i \in N$*, and such that: a profile* Π *is stable for* $\bar{\mathcal{G}}$ *if, and only if,* Π *is stable for* $\mathcal{G}$*. Moreover, if* $\texttt{maxDomSize}(\bar{\mathcal{G}}) \in O(\log ||\bar{\mathcal{G}}||)$*, then* $\mathcal{G}$ *can be built in polynomial time (w.r.t.* $||\bar{\mathcal{G}}||$*).*

Proof. Let $\bar{\mathcal{G}} = \langle N, E, \bar{\kappa} \rangle$ be a linear game and let us focus on an agent $i \in N$ with $\bar{\kappa}(i) = \langle \bar{G}_i^0, \bar{G}_i^1 \rangle$—our construction can be applied to the agents independently on each other.

Let $\{\bar{I}_1, ..., \bar{I}_m\}$ be the set of all possible interpretations over $\texttt{dom}(\bar{\kappa}(i))$. For each profile Π, recall that the utility of agent i is given by $\bar{G}_i^0(\Pi_i) + \bar{G}_i^1(\Pi_i) \cdot |\texttt{compatible}_i(\Pi)|$. Hereinafter, let $\bar{g} = \bar{G}_i^1(\bar{I}_1) = ... = \bar{G}_i^1(\bar{I}_m)$ and $\bar{b}_j = \bar{G}_i^0(\bar{I}_j)$, for each $j \in \{1, ..., m\}$. Moreover, let us define $\kappa(i) = \langle G_i^0, G_i^1 \rangle$ such that:

- $G_i^0 = \{(\bigwedge_{x \in \bar{I}_j} x \wedge \bigwedge_{\neg x \in \bar{I}_j} \neg x, b_j) \mid j \in \{1, ..., m\}\}$, and
- $G_i^1 = \{(\bigwedge_{x \in \bar{I}_j} x \wedge \bigwedge_{\neg x \in \bar{I}_j} \neg x, g) \mid j \in \{1, ..., m\}\}$,

where b_j and g, for each $j \in \{1, ..., m\}$, are rational numbers that we shall shortly define.

In particular, note that, for each $j \in \{1, ...m\}$, we have $b_j = G_i^0(\bar{I}_j)$ and $g = G_i^1(\bar{I}_j)$. In order to prove the result, we have then to show that it is possible to define $b_j \in \{0, ..., M\}$, with $M = 2^h - 1 + (|N| + 1) \cdot 2^{2h}$ and $g \in \{-2^h, 0, 2^h\}$, where $h = \texttt{maxDomSize}(\bar{\mathcal{G}}) = \texttt{maxDomSize}(\mathcal{G})$ and $n = |N|$, in a way that preserves the preferences that agent i has in $\bar{\mathcal{G}}$. More formally, we shall show that

[7] It can be checked that the encoding we have proposed for these scenarios produces games that are linear and that are populated by conformist agents, only.

such values can be defined in a way that, for each pair $j_1, j_2 \in \{1, ...m\}$ and each pair of natural numbers $D, D' \in \{0, ..., n-1\}$, the following property holds:

$$b_{j_1} + g \cdot D > b_{j_2} + g \cdot D' \text{ if, and only if, } \bar{b}_{j_1} + \bar{g} \cdot D > \bar{b}_{j_2} + \bar{g} \cdot D'. \tag{3}$$

For the analysis that we shall present, note that if $W = w_1, ..., w_m$ is a sequence of rational numbers with $m \leq 2^h$, then it is easy to define a function $discr_W$ mapping each element w of W to $discr_W(w) \in \{0, ..., 2^h - 1\}$ and such that $discr_W(w_{j_1}) > discr_W(w_{j_2})$ if, and only if, $w_{j_1} > w_{j_2}$. Let $w'_1, ..., w'_m$ be the new sequence obtained from W by arranging its element in non-decreasing order. Then, we just define $discr_W(w'_1) = 0$. And, for each $j \in \{2, ..., m\}$, if $w'_j = w'_{j-1}$ (resp., $w'_j > w'_{j-1}$), then we set $discr_W(w'_j) = discr_W(w'_{j-1})$ (resp., $discr_W(w'_j) = discr_W(w'_{j-1}) + 1$).

Let us distinguish three cases, and note that the constructions below are all feasible in polynomial time provided that $\mathtt{maxDomSize}(\bar{\mathcal{G}}) \in O(\log ||\bar{\mathcal{G}}||)$.

Assume that $\bar{g} = 0$. Then, we set $g = 0$ and $b_j = discr_{\{\bar{b}_1, ..., \bar{b}_m\}}(\bar{b}_j)$, for each $j \in \{1, ..., m\}$. We trivially have that $\{b_1, ..., b_m\} \subseteq \{0, ..., 2^h - 1\} \subseteq \{0, ..., M\}$ and Property (3) is immediately seen to hold.

Assume that $\bar{g} > 0$. Assume, w.l.o.g., that $\bar{b}_1, ..., \bar{b}_m$ are arranged in non-decreasing order. For each $j \in \{1, ..., m\}$, let us define $t_j = {}^{(\bar{b}_j - \bar{b}_1)}/_{\bar{g}}$, and let $t_0 = 0$. Note that, for each pair $j_1, j_2 \in \{1, ...m\}$ and each pair $D, D' \in \{0, ..., n-1\}$, the following trivially holds:

$$\bar{b}_{j_1} + \bar{g} \cdot D > \bar{b}_{j_2} + \bar{g} \cdot D' \text{ if, and only if, } t_{j_1} + D > t_{j_2} + D'. \tag{4}$$

Let us define the sequence of pairs of rational numbers $(F_0, I_0) = (0, 0)$, (F_1, I_1), ..., (F_m, I_m) such that, for each $j \in \{1, ..., m\}$, if $\lfloor t_j \rfloor > \lfloor t_{j-1} \rfloor + n$ (resp., $\lfloor t_j \rfloor \leq \lfloor t_{j-1} \rfloor + n$), then we set $(F_j, I_j) = (t_j - \lfloor t_j \rfloor, I_{j-1} + n + 1)$ (resp., $(F_j, I_j) = (t_j - \lfloor t_j \rfloor, I_{j-1} + \lfloor t_j \rfloor - \lfloor t_{j-1} \rfloor)$). Note that $I_m \leq (n+1) \cdot m$ holds, whereas $1 > F_j \geq 0$ and $I_j \geq 0$ hold, for each $j \in \{1, ..., m\}$. Moreover, for each pair $j_1, j_2 \in \{1, ...m\}$ and each pair $D, D' \in \{0, ..., n-1\}$, we claim that:

$$t_{j_1} + D > t_{j_2} + D' \text{ if, and only if, } F_{j_1} + I_{j_1} + D > F_{j_2} + I_{j_2} + D'. \tag{5}$$

Indeed, take two indices j_1 and j_2 with $t_{j_2} \geq t_{j_1}$. Consider, first, the case where $\lfloor t_{j_2} \rfloor > \lfloor t_{j_1} \rfloor + n$. Since $|D| \leq n-1$ and $|D'| \geq 0$, we have $t_{j_2} + D' > t_{j_1} + D$. So, $t_{j_2} - t_{j_1} > D - D'$ and we have to show that $F_{j_2} + I_{j_2} + D' > F_{j_1} + I_{j_1} + D$. In fact, $F_{j_2} + I_{j_2} - F_{j_1} - I_{j_1} = (I_{j_2} - I_{j_1}) + (F_{j_2} - F_{j_1}) > (I_{j_2} - I_{j_1}) - 1 \geq n + 1 - 1 > D' - D$. Consider, instead, the case where $\lfloor t_{j_2} \rfloor \leq \lfloor t_{j_1} \rfloor + n$. Then, note that $t_{j_2} - t_{j_1} = t_{j_2} - \lfloor t_{j_2} \rfloor + \lfloor t_{j_2} \rfloor - t_{j_1} + \lfloor t_{j_1} \rfloor - \lfloor t_{j_1} \rfloor = F_{j_2} - F_{j_1} + (I_{j_2} - I_{j_1})$, and Property (5) then holds.

Let $F = F_1, ..., F_m$ and define $\delta_j = {}^{discr_F(F_j)}/_{2^h}$, for each $j \in \{1, ..., m\}$. Recall that $F_j < 1$ holds, for each $j \in \{1, ..., m\}$, and that I_j, D and D' are natural numbers. Given that $|discr_F(F_j)| < 2^h$, the following then holds:

$$F_{j_1} + I_{j_1} + D > F_{j_2} + I_{j_2} + D' \text{ if, and only if, } \delta_{j_1} + I_{j_1} + D > \delta_{j_2} + I_{j_2} + D'. \tag{6}$$

Eventually, let us define $b_j = (\delta_j + I_j) \cdot 2^h$, for each $j \in \{1, ..., m\}$, and $g = 2^h$. Then, Property (3) follows by combining Property (6), Property (5), and Property (4). In order to conclude the proof of this case, just check that since $\delta_j \cdot 2^h \in \{0, ..., 2^h - 1\}$ and $I_j \in \{0, ..., (n+1) \cdot m\} \subseteq \{0, ..., (n+1) \cdot 2^h\}$, we have that $b_j \in \{0, ..., 2^h - 1 + (n+1) \cdot 2^{2h}\} \subseteq \{0, ..., M\}$.

Assume that $\bar{g} < 0$. Assume, w.l.o.g., that $\bar{b}_1, ..., \bar{b}_m$ are arranged in non-decreasing order. For each $j \in \{1, ..., m\}$, let us define $t_j = {}^{(\bar{b}_j - \bar{b}_1)}/_{(-\bar{g})}$, and let $t_0 = 0$. With the same approach and notation used in the case $\bar{g} > 0$, for each pair $j_1, j_2 \in \{1, ...m\}$ and each pair $D, D' \in \{0, ..., n-1\}$, the following properties are seen to hold:

$$\bar{b}_{j_1} + \bar{g} \cdot D > \bar{b}_{j_2} + \bar{g} \cdot D' \text{ if, and only if, } t_{j_1} - D > t_{j_2} - D'. \tag{7}$$

$$t_{j_1} - D > t_{j_2} - D' \text{ if, and only if, } F_{j_1} + I_{j_1} - D > F_{j_2} + I_{j_2} - D' \tag{8}$$

$$F_{j_1} + I_{j_1} - D > F_{j_2} + I_{j_2} - D' \text{ if, and only if, } \delta_{j_1} + I_{j_1} - D > \delta_{j_2} + I_{j_2} - D'. \tag{9}$$

Eventually, let us define $b_j = (\delta_j + I_j) \cdot 2^h$, for each $j \in \{1, ..., m\}$, and $g = -2^h$. Then, Property (3) follows by combining Property (9), Property (8), and Property (7). In order to conclude the proof of this case, just check that since $\delta_j \cdot 2^h \in \{0, ..., 2^h - 1\}$ and $I_j \in \{0, ..., (n+1) \cdot m\} \subseteq \{0, ..., (n+1) \cdot 2^h\}$, we have that $b_j \in \{0, ..., 2^h - 1 + (n+1) \cdot 2^{2h}\} \subseteq \{0, ..., M\}$. □

At this point, we can show that linear discrete LB-preference games in normal form where dissenters do not coexist with conformists always admit a Nash equilibrium, which can be moreover computed in polynomial time by just leaving the agents play their best response moves.

More formally, let us define a *(Nash) dynamics* as a sequence of profiles $\Pi^1, ..., \Pi^s$ such that, for each $j \in \{2, ..., s\}$, $\Pi^j = \Pi^{j-1}_{-i} \cup \{i \mapsto I\}$, with $\{i \mapsto I\}$ being a best response move for some agent i w.r.t. the profile Π^j. Then, the following result is established by showing that, in linear discrete LB-preference games in normal form, any dynamics will eventually converge after polynomially-many best response moves to a Nash equilibrium.

Theorem 15. *On linear discrete* LB*-preference games in normal form* $\mathcal{G} = \langle N, E, \kappa \rangle$ *where* E *is symmetric and there is no pair of agents with one of them being a conformist and the other a dissenter, a stable profile is always guaranteed to exist. Moreover, if* $\mathtt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$*, then a stable profile can be computed in polynomial time (w.r.t.* $||\mathcal{G}||$*).*

Proof. Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear environment in normal form such that E is symmetric and there is no pair of agents with one of them being a conformist and the other a dissenter.

If $\mathcal{G}$ does not contain dissenters (resp., conformists), then let $g = 2^h$ (resp., $g = -2^h$). Note that, for each agent $i \in N$, $\texttt{grad}(i) \in \{0, g\}$ holds. Moreover, for each profile Π, let $com(\Pi)$ denote set of all unordered pairs of agents that are neighbors of each other and are compatible in Π, that is, $com(\Pi) = \{\{i, j\} \in E \mid j \in \texttt{compatible}_i(\Pi)\}$—note that since E is symmetric, $j \in \texttt{compatible}_i(\Pi)$ if, and only if, $i \in \texttt{compatible}_j(\Pi)$. Consider, then, the function Φ such that:

$$\Phi(\Pi) = g \cdot |com(\Pi)| + \sum_{x \in N} G_x^0(\Pi).$$

For each agent $i \in N$, let $E_i = \{\{i, j\} \mid j \in \texttt{neigh}(i)\}$ and note that $|com(\Pi)| = |com(\Pi) \setminus E_i| + |\texttt{compatible}_i(\Pi)|$. Hence, $\Phi(\Pi)$ can be rewritten as

$$\Phi(\Pi) = \left(G_i^0(\Pi) + g \cdot |\texttt{compatible}_i(\Pi)|\right) + \left(g \cdot |com(\Pi) \setminus E_i| + \sum_{x \in N, x \neq i} G_x^0(\Pi)\right)$$

Whenever $\texttt{grad}(i) \neq 0$, the first term in the above expression $G_i^0(\Pi) + g \cdot |\texttt{compatible}_i(\Pi)|$ coincides with the utility of agent i in Π; the other term, instead, does not depend at all on agent i and on her current interpretation. This means that if $\Pi' = \Pi_{-i} \cup \{i \mapsto I\}$ is the result of a best response move of agent i, then we get $\Phi(\Pi') - \Phi(\Pi) = u_i(\Pi') - u_i(\Pi)$, implying that Φ behaves as an *exact potential function* [78]. Note also that, since $\mathcal{G}$ is in normal form, $u_i(\Pi') - u_i(\Pi) > 0$ actually implies that $u_i(\Pi') - u_i(\Pi) \geq 1$ and, furthermore, that $\Phi(\Pi') - \Phi(\Pi) \leq (n+1) \cdot 2^h \cdot (2^h - 1) + 2^h \cdot (n-1)$, where $n = |N|$ and $h = \texttt{maxDomSize}(\mathcal{G})$.

Clearly enough, the above bound on $\Phi(\Pi') - \Phi(\Pi)$ defines the maximum number of best response moves played in any dynamics by agents that are not autonomous. The result then follows by just observing that autonomous agents can move at most once in any Nash dynamics, so that even when they coexist with agents i with $\texttt{grad}(i) \neq 0$, we are guaranteed that Nash dynamics converge in a number of steps that is polynomial w.r.t. n and 2^h. $\square$

By putting it all together, we get our desired tractability result.

Corollary 16. *Consider a linear discrete* LB*-preference game $\bar{\mathcal{G}}$ where the underlying graph is symmetric and having no pair of agents with one being a conformist and the other a dissenter. Then, $\bar{\mathcal{G}}$ always admits a stable profile. Moreover, if $\texttt{maxDomSize}(\bar{\mathcal{G}}) \in O(\log ||\bar{\mathcal{G}}||)$, then a stable profile can be computed in polynomial time (w.r.t. $||\bar{\mathcal{G}}||$).*

Proof. The fact that $\bar{\mathcal{G}}$ admits a stable profile follows by Theorem 14 and Theorem 15. In particular, Property (3) in the proof of Theorem 14 guarantees that Nash dynamics are preserved when transforming any linear environment $\bar{\mathcal{G}}$ into the equivalent one $\mathcal{G}$ in normal form. So, if $\texttt{maxDomSize}(\bar{\mathcal{G}}) \in O(\log ||\bar{\mathcal{G}}||)$, then $\mathcal{G}$ can be computed in polynomial time (cf. Theorem 14) and, by starting with an arbitrary profile for $\mathcal{G}$, after polynomially-many best response moves we get a stable profile for $\mathcal{G}$ (which is also a stable profile for $\bar{\mathcal{G}}$). $\square$

5.2. Intractability over arbitrary graphs

We next complete the picture by considering the case where linear discrete LB-preference games are defined over graphs that are not necessarily symmetric. In this scenario, further bad news emerge since the problem of deciding the existence of Nash equilibria remains intractable even if dissenters do not coexist with conformists. The only—rather tiny—island of tractability is given by games populated by autonomous agents, only.

Theorem 17. *Consider a linear discrete* LB*-preference game $\mathcal{G}$ populated by autonomous agents only. Then, $\mathcal{G}$ always admits a stable profile. Moreover, if $\texttt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$, then a stable profile can be computed in polynomial time (w.r.t. $||\mathcal{G}||$).*

Proof. In an environment $\mathcal{G} = \langle N, E, \kappa \rangle$ populated only by autonomous agents, Nash dynamics converge in at most $|N|$ steps after that each agent has played (independently on the others) her best response move. In particular, for each agent $i \in N$, note that the number of her possible interpretation is polynomial w.r.t. $||\mathcal{G}||$, whenever $\texttt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$. $\square$

When either conformists or dissenters populate a linear environment defined over a graph that is not necessarily symmetric, stable profiles are no longer guaranteed to exist and the $\exists$-NASH problem becomes intractable. Rather than providing ad-hoc **NP**-hardness proofs, we would like to prove the result with an approach that sheds more lights into the difference between the symmetric and the arbitrary case. Indeed, we next show that the behavior of conformists/dissenters can be "simulated" by exploiting the orientation of the neighborhood relation, so that intractability will be then implied by the results in Section 4.

In the constructions that follow and also hereinafter in the paper, for two interpretations I and I', we write $I \simeq I'$ if $I(x) = I'(x)$ holds for each $x \in \texttt{dom}(I) \cap \texttt{dom}(I')$.

Dissenters $\mapsto$ Conformists. Let us start by showing how dissenters can be "simulated" via conformists. Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear environment and let $i \in N$ be a dissenter agent. Based on $\mathcal{G}$, we define a discrete LB-preference game $\dot{\mathcal{G}}^i = \langle \dot{N}, \dot{E}, \dot{\kappa} \rangle$ as follows. Let $\{I_1, \ldots, I_m\}$ be the set of all possible interpretations of agent i, hence with $m \leq 2^h$ and $h = \texttt{maxDomSize}(\mathcal{G})$. Then,

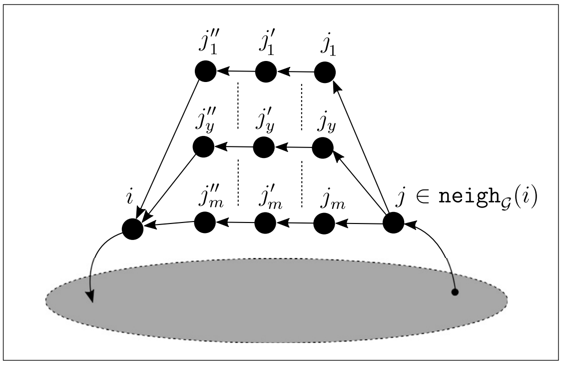

Fig. 4. Illustrations for the construction "Dissenters $\mapsto$ Conformists" in Section 5.2.

- $\dot{N} = N \cup \{j_y, j'_y, j''_y \mid j \in \texttt{neigh}_{\mathcal{G}}(i) \wedge y \in \{1,...,m\}\}$. That is, for each neighbor j of agent i and for each interpretation I_y, we include the fresh agents j_y, j'_y, and j''_y.
- $\dot{E} = E \setminus \{(i,j) \mid j \in \texttt{neigh}_{\mathcal{G}}(i)\} \cup \{(i,j''_y),(j''_y,j'_y),(j'_y,j_y),(j_y,j) \mid j \in \texttt{neigh}_{\mathcal{G}}(i) \wedge y \in \{1,...m\}\}$. That is, i is disconnected from each of her neighbors, and the influence of j on i is mediated by the chains over the fresh agents as illustrated in Fig. 4.
- For each $x \in N \setminus \{i\}$, agent x keeps her knowledge base without modifications, namely $\dot{\kappa}(x) = \kappa(x)$.
- For the agent i with $\kappa(i) = \langle G_i^0, G_i^1 \rangle$, we define $\dot{\kappa}(i) = \langle \dot{G}_i^0, \dot{G}_i^1 \rangle$ such that:
  - $\dot{G}_i^0 = \{(\neg\texttt{ok} \wedge \bigwedge_{\ell \in I_y} \ell, G_i^0(I_y) + |\texttt{neigh}_{\mathcal{G}}(i)| \cdot G_i^1(I_y)) \mid y \in \{1,...,m\}\} \cup \{(\texttt{ok}, -M)\}$, where $M > 0$ is a large enough natural number;
  - $\dot{G}_i^1 = \{(\neg\texttt{ok} \wedge \bigwedge_{\ell \in I_y} \ell, -G_i^1(I_y)) \mid y \in \{1,...,m\}\}$.

  That is, we introduce the novel variable ok, which agent i finds always convenient to evaluate false. Moreover, the utility she gets for the interpretation $I_y \cup \{\neg\texttt{ok}\}$ in a profile $\dot{\Pi}$ for $\dot{\mathcal{G}}^i$ is $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}) = G_i^0(I_y) + |\texttt{neigh}_{\mathcal{G}}(i)| \cdot G_i^1(I_y) - G_i^1(I_y) \cdot |\texttt{compatible}_{\dot{\mathcal{G}}^i,i}(\dot{\Pi})| = G_i^0(I_y) + G_i^1(I_y) \cdot (|\texttt{neigh}_{\mathcal{G}}(i)| - |\texttt{compatible}_{\dot{\mathcal{G}}^i,i}(\dot{\Pi})|)$; in particular, note that agent i is a conformist in $\dot{\mathcal{G}}^i$, since $G_i^1(I_y) < 0$ holds—because she is a dissenter in $\mathcal{G}$.
- For each agent $j \in \texttt{neigh}_{\mathcal{G}}(i)$ with $\kappa(j) = \langle G_j^0, G_j^1 \rangle$ and where $\{K_1,...,K_r\}$ is the set of her possible interpretations over $\texttt{dom}(\kappa(j))$, and for each $y \in \{1,...,m\}$, we define $\dot{\kappa}(j_y) = \langle \emptyset, \dot{G}_{j_y}^1 \rangle$, $\dot{\kappa}(j'_y) = \langle \emptyset, \dot{G}_{j'_y}^1 \rangle$, and $\dot{\kappa}(j''_y) = \langle \emptyset, \dot{G}_{j''_y}^1 \rangle$ where,
  - $\dot{G}_{j_y}^1 = \{(\bigwedge_{\ell \in K_x} \ell \wedge \texttt{ok}, 1) \mid x \in \{1,...,r\} \text{ and } K_x \backsimeq I_y\} \cup$ $\{(\bigwedge_{\ell \in K_x} \ell \wedge \neg\texttt{ok}, 1) \mid x \in \{1,...,r\} \text{ and not } K_x \backsimeq I_y\}$;
  - $\dot{G}_{j'_y}^1 = \{(\texttt{ok} \vee \neg\texttt{ok}, 1)\}$;
  - $\dot{G}_{j''_y}^1 = \{(\texttt{ok} \vee (\neg\texttt{ok} \wedge \bigwedge_{\ell \in I_y} \ell), 1)\}$.

  In words, agent j_y is a conformist that tends to replicate the interpretation K_x selected by her unique neighbor j and to select ok (resp., ¬ok) if $K_x \backsimeq I_y$ holds (resp., $K_x \backsimeq I_y$ does not hold). Agent j'_y just tends to propagate the selection of agent j_y (concerning the truth value of ok). Eventually, since j''_y is a conformist, she tends to propagate again that selection, by actually guaranteeing that the interpretation I_y is adopted whenever ok evaluates false.

We next show that $\dot{\mathcal{G}}^i$ preserves the stable profiles of $\mathcal{G}$ in a rather strong sense.

Theorem 18. *Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear environment and let $i \in N$ be a dissenter agent.*

(1) *Assume that Π is a stable profile for $\mathcal{G}$. Then, there is a stable profile $\dot{\Pi}$ for $\dot{\mathcal{G}}^i$ such that $\Pi_x = \dot{\Pi}_x$, for each agent $x \in N$.*
(2) *Assume that $\dot{\Pi}$ is a stable profile for $\dot{\mathcal{G}}^i$. Then, the profile Π such that $\Pi_x = \dot{\Pi}_x$, for each agent $x \in N$, is stable for $\mathcal{G}$.*

Proof. Before embarking on the proof of *(1)* and *(2)*, we need to point out ⊢ important property of the construction. Let $\{I_1,...,I_m\}$ be the set of all possible interpretations of agent i.

Claim A. *Let $\bar{\Pi}$ be a profile in $\dot{\mathcal{G}}^i$ such that all agents in $\dot{N} \setminus N$ are stable and where $\neg\texttt{ok} \in \bar{\Pi}_i$. Let $\bar{\Pi}[N]$ denote the restriction of $\bar{\Pi}$ over the agents in N and where $\bar{\Pi}[N]_i = \bar{\Pi}_i \setminus \{\texttt{ok}, \neg\texttt{ok}\}$. Then, for each agent $v \in N$, $u_{\dot{\mathcal{G}}^i,v}(\bar{\Pi}) = u_{\mathcal{G},v}(\bar{\Pi}[N])$.*

Proof. Given the construction of the knowledge bases, for each $y \in \{1,...,m\}$, if $\bar{\Pi}_j \backsimeq I_y$, then $\texttt{ok} \in \bar{\Pi}_{j''_y}$. Otherwise, i.e., if $\bar{\Pi}_j \backsimeq I_y$ does not hold, then $\bar{\Pi}_{j''_y} = I_y \cup \{\neg\texttt{ok}\}$. Now, observe that, for each agent $j \in \texttt{neigh}_{\mathcal{G}}(i)$, agent i can be compatible with at most one agent taken from the set $\{j''_1,...,j''_m\}$. In particular, if i is compatible with an agent j''_y in $\bar{\Pi}$, then $\bar{\Pi}_{j''_y} = \bar{\Pi}_i = I_y \cup \{\neg\texttt{ok}\}$ and $\bar{\Pi}_i \backsimeq \bar{\Pi}_j$ does not hold. Instead, if i is compatible with none of the agents in $\{j''_1,...,j''_m\}$, then $\bar{\Pi}_i \backsimeq \bar{\Pi}_j$. That is, $|\texttt{compatible}_{\dot{\mathcal{G}},i}(\bar{\Pi})| =$

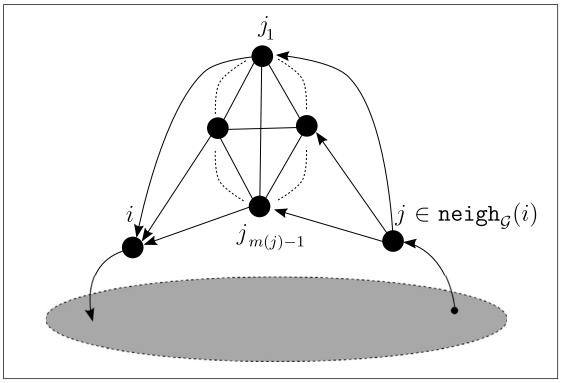

Fig. 5. Illustrations for the construction "Conformists $\mapsto$ Dissenters" in Section 5.2—edges in the clique are depicted without orientation.

$|\mathtt{neigh}_{\mathcal{G}}(i)| - |\mathtt{compatible}_{\mathcal{G},i}(\bar{\Pi}[N])|$. Therefore, given the form of the utility function we have discussed in the construction of $\dot{\mathcal{G}}^i$, we get $u_{\dot{\mathcal{G}}^i,i}(\bar{\Pi}) = u_{\mathcal{G},i}(\bar{\Pi}[N])$. More generally, since agents in $N \setminus \{i\}$ are not affected by the transformation (from $\mathcal{G}$ to $\dot{\mathcal{G}}^i$), we derive that for each agent $v \in N$, $u_{\dot{\mathcal{G}}^i,v}(\bar{\Pi}) = u_{\mathcal{G},v}(\bar{\Pi}[N])$. $\diamond$

(1) Given the stable profile Π, let us build $\dot{\Pi}$ such that $\dot{\Pi}_x = \Pi_x$, for each agent $x \in N \setminus \{i\}$, and $\dot{\Pi}_i = \Pi_i \cup \{\neg\mathtt{ok}\}$. Moreover, if $\{I_1, ..., I_m\}$ is the set of all possible interpretations of agent i, then for each agent $j \in \mathtt{neigh}_{\mathcal{G}}(i)$ and for each $y \in \{1, ..., m\}$, we define:

- $\dot{\Pi}_{j_y} = \Pi_j \cup \{\mathtt{ok}\}$ (resp., $\dot{\Pi}_{j_y} = \Pi_j \cup \{\neg\mathtt{ok}\}$) if $\Pi_j \simeq \Pi_i$ (resp., not $\Pi_j \simeq \Pi_i$);
- $\dot{\Pi}_{j'_y} = \dot{\Pi}_{j_y} \cap \{\mathtt{ok}, \neg\mathtt{ok}\}$;
- $\dot{\Pi}_{j''_y} = I_j \cup \{\neg\mathtt{ok}\}$ (resp., $\dot{\Pi}_{j''_y} = I_j \cup \{\mathtt{ok}\}$) if $\neg\mathtt{ok} \in \dot{\Pi}_{j'_y}$ (resp., $\mathtt{ok} \in \dot{\Pi}_{j'_y}$).

Note that agents j_y, j'_y, and j''_y get their maximum possible utility and they are stable in $\dot{\Pi}$. Moreover, all agents in $N \setminus \{i\}$ are not affected by the transformation (from $\mathcal{G}$ to $\dot{\mathcal{G}}^i$), hence they are stable in $\dot{\Pi}$ too. Assume now, by contradiction, that $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}_{-i} \cup \{i \mapsto I\}) > u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi})$. Observe that $\neg\mathtt{ok} \in I$ necessarily holds, and that all agents in $\dot{N} \setminus N$ are still stable in $\dot{\Pi}_{-i} \cup \{i \mapsto I\}$, because they are not directly influenced by agent i. Therefore, in the light of Claim A, we know that $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}_{-i} \cup \{i \mapsto I\}) = u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\})$ and $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}) = u_{\mathcal{G},i}(\Pi)$. That is, $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) > u_{\mathcal{G},i}(\Pi)$, which is impossible.

(2) Assume that $\dot{\Pi}$ is a stable profile for $\dot{\mathcal{G}}^i$, and consider the profile Π such that $\Pi_x = \dot{\Pi}_x$, for each agent $x \in N$. Note that all agents in $N \setminus \{i\}$ are stable in $\mathcal{G}$. Assume, for the sake of contradiction, that i is not stable, that is, $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) > u_{\mathcal{G},i}(\Pi)$. Consider the profile $\dot{\Pi}' = \dot{\Pi}_{-i} \cup \{i \mapsto I \cup \{\neg\mathtt{ok}\}\}$. Note that all agents in $\dot{N} \setminus N$ are stable in $\dot{\Pi}'$, as they are not directly influenced by i. Hence, after Claim A, we know that $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) = u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}_{-i} \cup \{i \mapsto I \cup \{\neg\mathtt{ok}\})$ and $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}) = u_{\mathcal{G},i}(\Pi)$. That is, $u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi}_{-i} \cup \{i \mapsto I \cup \{\neg\mathtt{ok}\}\}) > u_{\dot{\mathcal{G}}^i,i}(\dot{\Pi})$, which is impossible because $\dot{\Pi}$ is stable. $\square$

Corollary 19. *Hardness results in Theorem 11 and Theorem 12 hold even on linear discrete* LB*-preference games with conformist agents only (but defined over graphs that are not symmetric).*

Proof. Given a linear environment $\mathcal{G}$, we can just repeatedly apply Theorem 18, by removing in succession all dissenter agents. If $\mathtt{maxDomSize}(\mathcal{G}) = O(\log ||\mathcal{G}||)$, then each transformation step is feasible in polynomial time. $\square$

Conformists $\mapsto$ Dissenters. We next complete the picture by showing how conformists can be "simulated" via dissenters. Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear environment and let $i \in N$ be a conformist. Based on $\mathcal{G}$, we define a discrete LB-preference game $\ddot{\mathcal{G}}^i = \langle \ddot{N}, \ddot{E}, \ddot{\kappa} \rangle$ as follows. Let $\{I_1, ..., I_{m(j)}\}$ be the set of all possible interpretations defined over the set of variables in $\mathtt{dom}(\kappa(i)) \cap \mathtt{dom}(\kappa(j))$, hence with $m(j) \leq 2^h$ and $h = \mathtt{maxDomSize}(\mathcal{G})$. Then,

- $\ddot{N} = N \cup \{j_1, ..., j_{m(j)-1} \mid j \in \mathtt{neigh}_{\mathcal{G}}(i)\}$. That is, for each neighbor j of i in $\mathcal{G}$, we add $m(j) - 1$ fresh agents.
- $\ddot{E} = E \setminus \{(i, j) \mid j \in \mathtt{neigh}_{\mathcal{G}}(i)\} \cup \{(i, j_y), (j_y, j) \mid j \in \mathtt{neigh}_{\mathcal{G}}(i) \wedge y \in \{1, ..., m(j) - 1\}\} \cup \{(j_y, j_{y'}) \mid j \in \mathtt{neigh}_{\mathcal{G}}(i) \wedge y \in \{1, ..., m(j) - 1\} \wedge y' \in \{1, ..., m(j) - 1\} \wedge y \neq y'\}$. That is, i is disconnected from each of her original neighbors, and the influence of j on i is mediated by a clique over agents $j_1, ..., j_{m(j)-1}$ as illustrated in Fig. 5.
- For each agent $x \in N \setminus \{i\}$, $\ddot{\kappa}(x) = \kappa(x)$.
- For the agent i with $\kappa(i) = \langle G_i^0, G_i^1 \rangle$, if $\{K_1, ..., K_r\}$ is the set of her possible interpretations, then we define $\ddot{\kappa}(i) = \langle \ddot{G}_i^0, \ddot{G}_i^1 \rangle$ such that:
  - $\ddot{G}_i^0 = \{(\bigwedge_{\ell \in K_s} \ell, G_i^0(K_s) + |\mathtt{neigh}_{\mathcal{G}}(i)| \cdot G_i^1(K_s)) \mid s \in \{1, ..., r\}\}$;
  - $\ddot{G}_i^1 = \{(\bigwedge_{\ell \in K_s} \ell, -G_i^1(K_s)) \mid s \in \{1, ..., r\}\}$.

That is, agent i is a dissenter, with utility $u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}) = G_i^0(\ddot{\Pi}_i) + |\mathtt{neigh}_{\mathcal{G}}(i)| \cdot G_i^1(\ddot{\Pi}_i) - G_i^1(\ddot{\Pi}_i) \cdot |\mathtt{compatible}_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi})| = G_i^0(\ddot{\Pi}_i) + G_i^1(\ddot{\Pi}_i) \cdot (|\mathtt{neigh}_{\mathcal{G}}(i)| - |\mathtt{compatible}_{\ddot{\mathcal{G}}^i,i}(\dot{\Pi})|)$.

- For each agent $j \in \mathtt{neigh}_{\mathcal{G}}(i)$ and for each $y \in \{1, \ldots, m(j) - 1\}$, $\ddot{\kappa}(j_y) = \langle \emptyset, \ddot{G}^1_{j_y} \rangle$ is such that $\ddot{G}^1_{j_y} = \{(\bigvee_{x \in \mathtt{dom}(\kappa(i)) \cap \mathtt{dom}(\kappa(j))}(x \vee \neg x), -1)\}$. That is, j_y is a dissenter agent that prefers not to be compatible with her neighbors in $\mathtt{neigh}_{\ddot{\mathcal{G}}}(j_y) = \{j_{y'} \in \ddot{N} \mid y' \neq y\} \cup \{j\}$.

Theorem 20. *Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be a linear environment and let $i \in N$ be a dissenter agent.*

(1) *Assume that Π is a stable profile for $\mathcal{G}$. Then, there is a stable profile $\ddot{\Pi}$ for $\ddot{G}^i$ such that $\Pi_x = \ddot{\Pi}_x$, for each agent $x \in N$.*
(2) *Assume that $\ddot{\Pi}$ is a stable profile for $\ddot{\mathcal{G}}^i$. Then, the profile Π such that $\Pi_x = \ddot{\Pi}_x$, for each agent $x \in N$, is stable for $\mathcal{G}$.*

Proof. We start by pointing out an important property of the construction.

Claim B. *Let $\bar{\Pi}$ be a profile in $\ddot{\mathcal{G}}^i$ such that all agents in $\ddot{N} \setminus N$ are stable. Let $\bar{\Pi}[N]$ denote the restriction of $\bar{\Pi}$ over the agents in N. Then, for each agent $v \in N$, $u_{\ddot{\mathcal{G}}^i,v}(\bar{\Pi}) = u_{\mathcal{G},v}(\bar{\Pi}[N])$.*

Proof. We have to focus on agent i, since all other agents in N are not affected by the transformation. Let j be in $\mathtt{neigh}_{\mathcal{G}}(i)$ and note that agents $j_1, \ldots, j_{m(j)-1}$ are dissenters. Since they are stable in $\bar{\Pi}$, they select an interpretation that is not selected by some of their neighbors, that is, $\{\bar{\Pi}_j\} \cup \bigcup_{y \in \{1,\ldots,m(j)-1\}} \{\bar{\Pi}_{j_y}\}$ is precisely the set of all possible interpretations defined over the set of variables in $\mathtt{dom}(\kappa(i)) \cap \mathtt{dom}(\kappa(j))$. This means that agent i can be compatible with at most one of her neighbors taken from the set $\{j_1, \ldots, j_{m(j)-1}\}$. In particular, if i is compatible with an agent j_y in $\bar{\Pi}$, then $\bar{\Pi}_i \simeq \bar{\Pi}_j$ does not hold. Instead, if i is compatible with none of her neighbors, then $\bar{\Pi}_i \simeq \bar{\Pi}_j$. That is, $|\mathtt{compatible}_{\ddot{\mathcal{G}},i}(\bar{\Pi})| = |\mathtt{neigh}_{\mathcal{G}}(i)| - |\mathtt{compatible}_{\mathcal{G},i}(\bar{\Pi}[N])|$. Thus, $u_{\ddot{\mathcal{G}}^i,i}(\bar{\Pi}) = u_{\mathcal{G},i}(\bar{\Pi}[N])$. ⋄

Armed with the property above, the proof of *(1)* and *(2)* is similar to that of Theorem 20, and is next reported for the sake for completeness only.

(1) Given the stable profile Π, let us build $\ddot{\Pi}$ such that $\ddot{\Pi}_x = \Pi_x$, for each agent $x \in N$. Moreover, for each $j \in \mathtt{neigh}_{\mathcal{G}}(i)$, $\ddot{\Pi}$ assigns to agents in $\{j_1, \ldots, j_{m(j)-1}\}$ interpretations such that $\{\ddot{\Pi}_j\} \cup \bigcup_{y \in \{1,\ldots,m(j)-1\}} \{\ddot{\Pi}_{j_y}\}$ is precisely the set of all possible interpretations defined over the set of variables in $\mathtt{dom}(\kappa(i)) \cap \mathtt{dom}(\kappa(j))$. Note that all agents in $\ddot{N} \setminus N$ are stable, and that agents in $N \setminus \{i\}$ are not affected by the transformation.

Assume now, by contraction, that $u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}_{-i} \cup \{i \mapsto I\}) > u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi})$. By Claim B, we know that $u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}_{-i} \cup \{i \mapsto I\}) = u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\})$ and $u_{\ddot{\mathcal{G}}^i,i}(\dot{\Pi}) = u_{\mathcal{G},i}(\Pi)$. That is, $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) > u_{\mathcal{G},i}(\Pi)$, which is impossible.

(2) Assume that $\ddot{\Pi}$ is a stable profile for $\ddot{\mathcal{G}}^i$, and consider the profile Π such that $\Pi_x = \ddot{\Pi}_x$, for each $x \in N$. Note that all agents in $N \setminus \{i\}$ are stable in $\mathcal{G}$. Assume, for the sake of contradiction, that i is not stable, that is, $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) > u_{\mathcal{G},i}(\Pi)$. Consider the profile $\ddot{\Pi}' = \ddot{\Pi}_{-i} \cup \{i \mapsto I\}\}$. Note that all agents in $\ddot{N} \setminus N$ are stable in $\ddot{\Pi}'$, as they are not directly influenced by i. Hence, after Claim B, we know that $u_{\mathcal{G},i}(\Pi_{-i} \cup \{i \mapsto I\}) = u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}_{-i} \cup \{i \mapsto I\})$ and $u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}) = u_{\mathcal{G},i}(\Pi)$. That is, $u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi}_{-i} \cup \{i \mapsto I\}) > u_{\ddot{\mathcal{G}}^i,i}(\ddot{\Pi})$, which is impossible because $\ddot{\Pi}$ is stable. □

Corollary 21. *Hardness results in Theorem 11 and Theorem 12 hold even on linear discrete* LB*-preference games with dissenter agents only (but defined over graphs that are not symmetric).*

Proof. Given a linear environment $\mathcal{G}$, we can just repeatedly apply Theorem 20, by removing in succession all dissenter agents. If $\mathtt{maxDomSize}(\mathcal{G}) = O(\log ||\mathcal{G}||)$, then each transformation step is feasible in polynomial time. □

6. Conformists meet dissenters: structural tractability beyond linearity

On linear discrete LB-preference games, we have noticed that preventing the interactions of conformists and dissenters is a necessary condition to guarantee that stable profiles (exists and) can be computed in polynomial time. Identifying tractable scenarios where these two types of agents coexist thus requires looking for other kinds of restrictions. In this section, we embark on the identification of such islands of tractability by taking a *structural* perspective, that is, by considering classes of games where interactions among agents can be kept "under control" by exploiting the topological properties of the underlying graphs.

In order to analyze such structurally restricted games, rather than defining ad-hoc algorithms, we introduce a novel and general technical tool based on the decomposition of *Integer Linear Programs* (ILPs) [79]—into which we easily recast the problems of interest—and on a method to solve in polynomial time "decomposable" ILPs. In particular, our approach founds and generalizes some useful results recently proven in the literature about the tractability of certain types of "decomposable" ILPs (see [59–61], and the references therein). Hence, the significance of our machinery goes beyond the specific setting discussed in the paper, and applies to any setting that can be recast in terms of ILPs.

We start the exposition of our structural tractability results, by first recalling the notion of *tree decomposition* [56], which is a natural generalization of graph *acyclicity* and which will play a crucial role in our structural analysis, and by reporting a few preliminaries on ILPs.

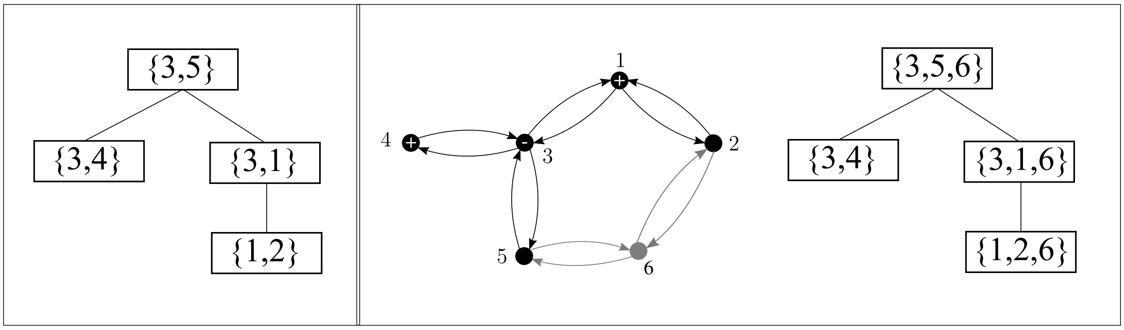

Fig. 6. Illustration of Example 5.

6.1. Tree decompositions

Let (N, E) be a graph.[8] A *tree decomposition* of (N, E) is a pair $\langle T, \chi \rangle$, where $T = (V, F)$ is a *tree* and χ is a function assigning to each vertex p of V a set of nodes $\chi(p) \subseteq N$, such that the following conditions are satisfied:

(1) for each $i \in N$, there is a vertex $p \in V$ such that $i \in \chi(p)$;
(2) for each $(i, j) \in E$, there is a vertex $p \in V$ such that $\{i, j\} \subseteq \chi(p)$; and,
(3) for each $i \in N$, the set $\{p \in V \mid i \in \chi(p)\}$ induces a connected subtree of T.

The *width* of $\langle T, \chi \rangle$ is $\max_{p \in V} |\chi(p)| - 1$, and the *treewidth* of (N, E) (short: $tw(N, E)$) is the minimum width over all its tree decompositions (see, e.g., [56]). For any fixed natural number $k > 0$, deciding whether a graph has treewidth bounded by k and computing a width-k tree decomposition, if any exists, are feasible in linear time [81].

Example 5. Let us consider again the discrete LB-preference game $\mathcal{G} = \langle N, E, \kappa \rangle$ discussed in Example 2. Fig. 6 depicts on the left a tree decomposition of (N, E) whose width is 1. Note that the decomposition is *rooted* at a vertex r with $\chi(r) = \{3, 5\}$, and that this vertex has two *children*. Consider now the environment $\mathcal{G}' = \langle N', E', \kappa' \rangle$ derived from $\mathcal{G}$ by adding agent 6 as illustrated in the right part of Fig. 6. A width-2 tree decomposition of (N', E') is depicted in the figure too. In fact, the treewidth of (N', E') is 2, since the graph admits no tree decomposition with smaller width. $\triangleleft$

An analysis of the treewidth of a number of real and synthetic social graphs (and other kinds of real-world graphs) has been recently conducted by Maniu, Sannelart, and Jog [82]. The authors evidenced that such graphs tend to be *scale-free* and are, hence, characterized by large values of treewidth. However, they showed that tree decompositions can be still profitably used within an algorithmic scheme where graphs are decomposed into a fringe of low treewidth components and a smaller core of high treewidth. A different approach to use tree decompositions in practice is to exploit them after that graphs have been appropriately preprocessed. The idea is to identify some formal property (or, at least, heuristic) that can guide the removal of some nodes and edges, in order to reduce the treewidth. This approach is very powerful over medium-sized networks (say, up to some thousands of nodes), since low values of treewidth can be achieved with rather few simplifications in these cases. An example has been presented in [83,84], to decompose the co-authorship graph of the "La Sapienza" University of Rome, the largest Italian University.

Therefore, the tractability results we shall next provide for discrete LB-preference games with low treewidth not only have a clear theoretical interest, but they can also represent the basis on top of which practical approaches can be implemented to scale in real-world scenarios.

6.2. Preliminaries on ILPs

As we have anticipated, our approach to identify islands of tractability is to first encode the problem of deciding the existence of stable profiles in terms of an Integer Linear Program. To our ends (similarly to [59,61]), it is useful to define an ILP instance P as a tuple $\langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$, where:

- $\mathcal{X}$ is a set of variables and $\mathbb{D}$ is a function associating each variable $x \in \mathcal{X}$ with a *domain* $\mathbb{D}(x) = \{low_x, ..., up_x\}$ of integers; that is, we require that variables are domain bounded;
- Γ is a set of *constraints*; each constraint $\gamma \in \Gamma$ has the form $\sum_{x \in \mathcal{X}} a_{\gamma,x} \cdot x \leq b_\gamma$, where $a_{\gamma,x}$, for each $x \in \mathcal{X}$, and b_γ are rational numbers;

[8] The definition of tree decomposition typically considers graphs that are symmetric, and we apply it to the directed graphs associated to discrete LB-preference games by implicitly considering their symmetric closures. Note that this is different from the application of the notion of *direct treewidth* [80], which has been proven to play a crucial role in flow-based or routing problems—while appearing hardly useful in our context.

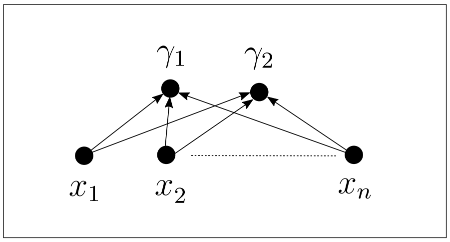

Fig. 7. Incidence Graph in Example 6.

- η is an expression of the form $\sum_{x\in\mathcal{X}} c_x \cdot x$ where, for each $x \in \mathcal{X}$, c_x is a rational number.

An *assignment* α maps every variable $x \in \mathcal{X}$ to an integer $\alpha(x) \in \mathbb{D}(x)$. We say that α is a (feasible) *solution* to P if $\sum_{x\in\mathcal{X}} a_{\gamma,x} \cdot \alpha(x) \leq b_\gamma$ holds, for each $\gamma \in \Gamma$. The solution is *optimal* if there is no other feasible solution α' such that $\sum_{x\in\mathcal{X}} c_x \cdot \alpha(x) < \sum_{x\in\mathcal{X}} c_x \cdot \alpha'(x)$. Then, by solving an instance P, we just mean deciding whether it admits a feasible solution and compute an optimal one maximizing the value of η, if any feasible exists. Whenever $c_x = 0$, for each $x \in \mathcal{X}$, i.e., if we are just interested in the feasibility problem for ILPs, we omit to specify η in P.

In the following, we denote by $||x|| = |up_x - low_x|$ the maximum *domain span* of $x \in \mathcal{X}$, and by $||\mathbb{D}||_\infty = \max_{x\in\mathcal{X}} ||x||$ the maximum domain span over all the variables. Moreover, $||\Gamma||_\infty$ denotes the maximum absolute value of the coefficients $a_{\gamma,x}$, over all $\gamma \in \Gamma$ and $x \in \mathcal{X}$. And, finally, we denote by $||\mathrm{P}||$ the size of the encoding of P.

It is well-known that ILPs can encode **NP**-hard problems. This is exemplified below.

Example 6 *(cf. [59]).* Consider the *subset sum* **NP**-hard problem [75]. We are given a set $S = \{s_1, ..., s_n\}$ of integers and an integer s, and we have to decide whether there is a set $S' \subseteq S$ with $\sum_{s'\in S'} s' = s$. The problem can be encoded as an instance $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma\rangle$ with $\mathcal{X} = \{x_1, ..., x_n\}$ such that $\mathbb{D}(x_i) = \{0, 1\}$, for each $x_i \in \mathcal{X}$, and Γ is the set of the following two constraints:

$$\begin{aligned} \gamma_1 :&\quad s_1 \cdot x_1 + ... + s_n \cdot x_n \leq s \\ \gamma_2 :&\quad (-s_1) \cdot x_1 + ... + (-s_n) \cdot x_n \leq -s \end{aligned}$$

The correctness of the encoding is immediate—for instance, if α is a solution, then $\alpha(x_i) = 1$ (resp., $\alpha(x_i) = 0$) denotes that s_i is included (resp., is not included) in the set S'. Note also that $||x_i|| = 1$, for each $x_i \in \mathcal{X}$. Instead, $||\Gamma||_\infty$ coincides with the value of the largest absolute value of the integers in S, which might be well exponential w.r.t. n. And, in fact, we know that on classes of instances where this value is polynomially bounded the subset problem can be solved in polynomial time [75]. $\lhd$

With each ILP instance $\mathrm{P} = \langle \mathcal{X}, \Gamma, \mathbb{D}, \eta\rangle$, we can naturally associate an *incidence* graph $\mathtt{IG}(\mathrm{P}) = (\mathcal{X} \cup \Gamma, E)$ where variables and constraints are transparently viewed as nodes and where an edge (x_i, γ_j) is in E if, and only if, $a_{j,i} \neq 0$; and no further edge is in E. For instance, the incidence graph associated to the instance we have presented in Example 6 is depicted in Fig. 7. Note that its treewidth is 2 (consider, to this aim, the tree decomposition where the root is labeled by $\{\gamma_1, \gamma_2\}$ and each child is of the form $\{\gamma_1, \gamma_2, x_i\}$). In fact, having an incidence graph with bounded treewidth does not imply that the corresponding instance can be solved in polynomial time—just recall that the subset problem is in general **NP**-hard. Ganian et al. [59] showed the following result.

Theorem 22 *(cf. Theorem 11 in [59]). Integer Linear Programs can be solved in polynomial time on classes of instances $\mathrm{P} = \langle \mathcal{X}, \Gamma, \mathbb{D}, \eta\rangle$ such that $tw(\mathtt{IG}(\mathrm{P}))$ is bounded by some fixed natural number, and both $||\mathbb{D}||_\infty$ and $||\Gamma||_\infty$ are bounded by some polynomial in $||\mathrm{P}||$.*

6.3. *Encoding discrete* LB*-preference games as ILPs*

Having all notions and definitions in place, we can show how any discrete LB-preference game $\mathcal{G} = \langle N, E, \kappa\rangle$ can be encoded in terms of an Integer Linear Program $\mathrm{P}^{\mathcal{G}} = \langle \mathcal{X}, \Gamma, \mathbb{D}\rangle$.

To simplify the notation, let us assume, w.l.o.g., that for each agent $i \in N$, $|\mathtt{dom}(\kappa(i))| = h$ holds (hence $h = \mathtt{maxDomSize}(\mathcal{G})$). In particular, let $m = 2^h$ and let $\{I_{i,1}, ..., I_{i,m}\}$ denote the set of all possible interpretations for i. Then, for each agent $i \in N$, for each $c \in \{1, ..., m\}$, and for each agent $j \in \mathtt{neigh}(i)$, $\mathcal{X}$ contains the variables $x_{i,c}$ and $x_{i,c,j}$ such that $\mathbb{D}(x_{i,c}) = \mathbb{D}(x_{i,c,j}) = \{0, 1\}$. No further variable is in $\mathcal{X}$. Intuitively, $x_{i,c} = 1$ (resp., $x_{i,c} = 0$) means that agent i selects (resp., does not select) the interpretation $I_{i,c}$. Moreover, $x_{i,c,j}$ is an auxiliary variable such that $x_{i,c,j} = 1$ (resp., $x_{i,c,j} = 0$) means that agent j has (resp., has not) selected an interpretation, say $I_{j,\bar{c}}$, being compatible with $I_{i,c}$, namely $I_{j,\bar{c}} \simeq I_{i,c}$.

Finally, Γ consists of the three kinds of constraints:

(C1) For each $i \in N$, Γ contains the following two constraints, whose role is to guarantee that precisely one interpretation is selected for agent i:

$$\begin{aligned} \gamma_i :&\quad x_{i,1} + ... + x_{i,m} \leq 1 \\ \gamma'_i :&\quad -x_{i,1} - .... - x_{i,m} \leq -1 \end{aligned}$$

(C2) For each agent $i \in N$, for each $c \in \{1, ..., m\}$, and for each agent $j \in \texttt{neigh}(i)$, Γ contains the following two constraints, whose role is to encode the intended semantics of $x_{i,c,j}$:

$$\begin{array}{ll} \gamma_{i,c,j}: & x_{i,c,j} - \sum_{\bar{c}|I_{j,\bar{c}} \simeq I_{i,c}} x_{j,\bar{c}} \leq 0 \\ \gamma'_{i,c,j}: & -x_{i,c,j} + \sum_{\bar{c}|I_{j,\bar{c}} \simeq I_{i,c}} x_{j,\bar{c}} \leq 0 \end{array}$$

(C3) Finally, for each agent $i \in N$, and for each pair $c, c' \in \{1, ..., m\}$, Γ contains the following constraint, whose role is to encode the stability condition, namely that if agent i selected interpretation $I_{i,c}$, then it is not convenient for her to select interpretation $I_{i,c'}$:

$$\bar{\gamma}_{i,c,c'}: \quad x_{i,c} \cdot M + G_i^1(I_{i,c'}) \cdot \sum_j x_{i,c',j} - G_i^1(I_{i,c}) \cdot \sum_j x_{i,c,j} \leq M + G_i^0(I_{i,c}) - G_i^0(I_{i,c'}),$$

where $M > 0$ is a sufficiently large natural number. Intuitively, M serves to trivially satisfy the constraint $\bar{\gamma}_{i,c,c'}$ whenever $x_{i,c} = 0$; conversely, in case $x_{i,c} = 1$ it plays a neutral role.

Note that, given the constraints **(C1)**, with each solution α to $\mathrm{P}^{\mathcal{G}}$ we can associate a profile Π^α such that, for each $i \in N$, Π_i coincides with the interpretation $I_{i,c}$ such that $\alpha(x_{i,c}) = 1$. This is the key to establish a correspondence between solutions to $\mathrm{P}^{\mathcal{G}}$ and stable profiles for $\mathcal{G}$.

Theorem 23. *Let $\mathrm{P}^{\mathcal{G}}$ be the Integer Linear Program associated with the environment $\mathcal{G} = \langle N, E, \kappa \rangle$.*

(1) *Assume that α is a solution to $\mathrm{P}^{\mathcal{G}}$. Then, the profile Π^α is stable for $\mathcal{G}$.*
(2) *Assume that Π is a stable profile for $\mathcal{G}$. Then, there exists a solution α for $\mathrm{P}^{\mathcal{G}}$ such that $\Pi = \Pi^\alpha$.*

Proof. *(1)* Let α be a solution for $\mathrm{P}^{\mathcal{G}}$ and assume, by contradiction, that there is an agent $i \in N$ such that $u_i(\Pi^\alpha_{-i} \cup \{i \mapsto I_{i,c'}\}) > u_i(\Pi^\alpha_i)$. Let $I_{i,c} = \Pi^\alpha_i$. Note that $\alpha(x_{i,c}) = 1$ holds, so that, by the constraint $\bar{\gamma}_{i,c,c'}$, we know that

$$G_i^1(I_{i,c'}) \cdot \sum_j \alpha(x_{i,c',j}) - G_i^1(I_{i,c}) \cdot \sum_j \alpha(x_{i,c,j}) \leq G_i^0(I_{i,c}) - G_i^0(I_{i,c'}).$$

Moreover, in the light of the constraints **(C2)**, $\sum_j \alpha(x_{i,c',j})$ (resp., $\sum_j \alpha(x_{i,c,j})$) coincides with the number of agents compatible with i in the profile where i selects $I_{i,c'}$ (resp., $I_{i,c}$), so we have:

$$G_i^0(I_{i,c'}) + G_i^1(I_{i,c'}) \cdot \texttt{compatible}_i(\Pi^\alpha_{-i} \cup \{i \mapsto I_{i,c'}\}) \leq G_i^0(I_{i,c}) + G_i^1(I_{i,c}) \cdot \texttt{compatible}_i(\Pi^\alpha).$$

Thus, $u_i(\Pi^\alpha_{-i} \cup \{i \mapsto I_{i,c'}\}) \leq u_i(\Pi^\alpha_i)$, which gives a contradiction.

(2) Assume that Π is a stable profile for $\mathcal{G}$, and let us build an assignment α for the variables in $\mathrm{P}^{\mathcal{G}}$ as follows. For each agent $i \in N$, $x_{i,c} = 1$ if, and only if, $\Pi_i = I_{i,c}$; and, for each agent $j \in \texttt{neigh}(i)$, $x_{i,c,j} = 1$ only if $j \in \texttt{compatible}_i(\Pi)$. By construction, α trivially satisfies constraints of kind **(C1)** and **(C2)**. Moreover, every constraint of the form $\bar{\gamma}_{i,\bar{c},c'}$, for $\bar{c} \neq c$, is trivially satisfied because $\alpha(x_{i,\bar{c}}) = 0$—just note that the term M occurs on the right-hand side of the inequality. So, assume, by contradiction, that α does not satisfy a constraint $\bar{\gamma}_{i,c,c'}$. This means that:

$$G_i^0(I_{i,c'}) + G_i^1(I_{i,c'}) \cdot \texttt{compatible}_i(\Pi_{-i} \cup \{i \mapsto I_{i,c'}\}) > G_i^0(I_{i,c}) + G_i^1(I_{i,c}) \cdot \texttt{compatible}_i(\Pi),$$

which implies that Π is not stable. $\square$

The following result sheds further light on the encoding, by relating the treewidth of $\mathtt{IG}(\mathrm{P}^{\mathcal{G}})$ with that of the underlying graph (N, E) on top of which $\mathcal{G}$ is built.

Theorem 24. *Consider a discrete* LB*-preference game $\mathcal{G} = \langle N, E, \kappa \rangle$ such that $tw(N, E) = \bar{k}$. Let $m = 2^{\mathtt{maxDomSize}(\mathcal{G})}$. Then, $tw(\mathtt{IG}(\mathrm{P}^{\mathcal{G}})) \leq (m^2 + m + 2) \cdot (\bar{k} + 1) + 3m \cdot (\bar{k} + 1)^2 - 1$.*

Proof. Let $\langle T, \chi \rangle$ be a tree decomposition of (N, E) having width $\bar{k}$. Let us build the pair $\langle T, \chi' \rangle$ such that, for each vertex p in T, $\chi'(p) = \chi'_N(p) \cup \chi'_E(p)$, where:

$$\begin{array}{ll} \chi'_N(p) = & \{x_{i,1}, ..., x_{i,m}, \gamma_i, \gamma'_i, \gamma_{i,1,1}, ..., \gamma_{i,m,m} \mid i \in \chi(p)\} \\ \chi'_E(p) = & \{x_{i,1,j}, ..., x_{i,m,j}, \gamma_{i,1,j}, ..., \gamma_{i,m,j}, \gamma'_{i,1,j}, ..., \gamma'_{i,m,j} \mid i \in \chi(p), j \in \texttt{neigh}(i) \cap \chi(p)\} \end{array}$$

Note that $|\chi'_N(p)| \leq (m^2 + m + 2) \cdot |\chi(p)|$ whereas $|\chi'_E(p)| \leq 3m \cdot |\chi(p)|^2$. Hence, $|\chi'(p)| \leq (m^2 + m + 2) \cdot (\bar{k} + 1) + 3m \cdot (\bar{k} + 1)^2$. Hence, it remains to show that $\langle T, \chi' \rangle$ is, indeed, a tree decomposition of $\mathtt{IG}(\mathrm{P}^{\mathcal{G}})$.

Concerning the first two conditions in the definition of a tree decomposition, consider the constraints γ_i and γ'_i, and the variables $x_{i,1}, ..., x_{i,m}$. Let p be a vertex such that $i \in \chi(p)$, which exists since $\langle T, \chi \rangle$ is a tree decomposition of (N, E). Then, by construction of χ', we have $\{x_{i,1}, ..., x_{i,m}, x_{i,1,1}, ..., x_{i,m,m}, \gamma_i, \gamma'_i\} \subseteq \chi'(p)$. Consider the constraints $\gamma_{i,c,j}$ and $\gamma'_{i,c,j}$, and the variables $x_{i,1,j}, ..., x_{i,m,j}, x_{j,1}, ...x_{j,m}$. Since $j \in \texttt{neigh}(i)$, there exists a vertex p such that $\{i, j\} \in \chi(p)$ holds. Then, by construction of χ', we have in particular:

$$\{x_{i,1,j},\dots,x_{i,m,j},\gamma_{i,1,j},\dots,\gamma_{i,m,j},\gamma'_{i,1,j},\dots,\gamma'_{i,m,j}\}\cup\{x_{j,1},\dots,x_{j,m}\}\subseteq\chi'(p).$$

Finally, consider the constraint $\bar{\gamma}_{i,c,c'}$, and the variables $x_{i,c}$ and $x_{i,1,j},\dots,x_{i,m,j}$. Again, since $j\in\mathtt{neigh}(i)$, there exists a vertex p such that $\{i,j\}\in\chi(p)$ holds. Then, by construction of χ', we have $\{x_{i,1},\dots,x_{i,m},x_{i,1,j},\dots,x_{i,m,j},\bar{\gamma}_{i,1,1},\dots,\bar{\gamma}_{i,m,m}\}\subseteq\chi'(p)$. So, $\langle T,\chi'\rangle$ satisfies the first two conditions of a tree decomposition.

Concerning the third condition, assume that p and q are two vertices of T such that $\chi'(p)\cap\chi'(q)\neq\emptyset$. In particular, let $Y\in\chi'(p)\cap\chi'(q)$ be any element in this intersection and let v any vertex occurring in the unique path connecting p and q in T. We have to show that $Y\in\chi'(v)$ holds. Indeed, we distinguish two cases. In the case where Y has the form $x_{i,c}$, or γ_i, or γ'_i, or $\bar{\gamma}_{i,c,c'}$ for some agent $i\in N$, then $i\in\chi(p)\cap\chi(q)$ holds by the construction of χ'. Then, because of the connectedness condition of $\langle T,\chi\rangle$, we know that $i\in\chi(v)$ and, hence, $Y\in\chi'(v)$. Similarly, if Y has the form $x_{i,c,j}$, or $\gamma_{i,c,j}$, or $\gamma'_{i,c,j}$ for some $i\in N$ and $j\in\mathtt{neigh}(j)$, then $\{i,j\}\subseteq\chi(p)\cap\chi(q)$ holds by construction of χ'. So, we have $\{i,j\}\subseteq\chi(v)$ and, eventually, $Y\in\chi'(v)$. □

At this point, we know that the ILP encoding we have proposed preserves the stable profiles (cf. Theorem 23) as well as the treewidth (cf. Theorem 24) in a precise formal sense: whenever the treewidth of (N,E) is bounded by some constant, say $\bar{k}$, and if m is a constant, then the treewidth of $\mathtt{IG}(\mathrm{P}^{\mathcal{G}})$ is bounded by a constant too, namely $(m^2+m+2)\cdot(\bar{k}+1)+3m\cdot(\bar{k}+1)^2-1$.

Hence, we can apply Theorem 22 on encoding $\mathrm{P}^{\mathcal{G}}=\langle\mathcal{X},\Gamma,\mathbb{D}\rangle$. In particular, note that $||\mathbb{D}||_\infty\leq 1$ and that $||\Gamma||_\infty$ coincides with the maximum absolute value returned by the goalbases in κ. Therefore, the following is established after Theorem 23, Theorem 24, and Theorem 22.

Corollary 25. *Let $\bar{k}>0$ and $h>0$ be fixed natural numbers. Let $\mathcal{G}=\langle N,E,\kappa\rangle$ be a discrete* LB*-preference game with $tw(N,E)\leq\bar{k}$ and $h=\mathtt{maxDomSize}(\mathcal{G})$, and such that, for each agent i and interpretation I, $|G_i^0(I)|$ and $|G_i^1(I)|$ are polynomially bounded w.r.t. $|N|$ and $|E|$. Then, $\exists$-*NASH *can be solved in polynomial time w.r.t. $||\mathcal{G}||$ (and a stable profile, if any exists, can be computed in polynomial time too).*

Unfortunately, Corollary 25 has been established under a rather technical condition on the values returned by the goalbases, which limits the island of tractability it can identify. Our main achievement in the section will be to get rid of this requirement, by actually proving a result on Integer Linear Programs generalizing significantly the known result on ILPs (that is, Theorem 22) we used to establish Corollary 25.

In order to state our main result, we need to introduce some further notation. Consider a class C of ILP instances and a function $f:\mathbb{N}^+\rightarrow\mathbb{N}^+$. In what follows, C is said to be *bounded by* f if, for each $\mathrm{P}=\langle\mathcal{X},\mathbb{D},\Gamma,\eta\rangle$ in C, it holds that $||\mathbb{D}||_\infty\leq f(||\mathrm{P}||)$. Moreover, a rational number r is said to be *bounded w.r.t.* $||\mathrm{P}||$ *via* f if $|r|\leq f(||\mathrm{P}||)$. Consider a class C of ILP instances bounded by some polynomial function $poly$. Let $\mathrm{P}=\langle\mathcal{X},\mathbb{D},\Gamma,\eta\rangle$ be an element of C. Assume that each constraint $\gamma\in\Gamma$ has the form $\sum_{x\in\mathcal{X}}a_{\gamma,x}\cdot x\leq b_\gamma$. For each constraint $\gamma\in\Gamma$, let $unbound(\gamma,\mathrm{P})=\{s_{\gamma,1},\dots,s_{\gamma,\ell_\gamma}\}\subseteq\{a_{\gamma,x}\mid x\in\mathcal{X}\}$ be the set of all distinct coefficients occurring in γ that are not bounded w.r.t. $||\mathrm{P}||$ via $poly$, namely such that $|s_{\gamma,i}|>poly(||\mathrm{P}||)$ holds for each $i\in\{1,\dots,\ell_\gamma\}$. Let $||unbound(\mathrm{P})||_\infty$ denote the maximum cardinality of the set $unbound(\gamma,\mathrm{P})$ over all constraints $\gamma\in\Gamma$, implying that $\ell_\gamma\leq||unbound(\mathrm{P})||_\infty$, for each $\gamma\in\Gamma$.

We have now all ingredients in place to state our main result.

Theorem 26. *Let $k_1>0$ and $k_2>0$ be fixed natural numbers. Consider a class C of ILP instances bounded by some polynomial function such that, for each $\mathrm{P}=\langle\mathcal{X},\mathbb{D},\Gamma,\eta\rangle$ of C, both $tw(\mathtt{IG}(\mathrm{P}))\leq k_1$ and $||unbound(P)||_\infty\leq k_2$ hold. Then, Integer Linear Programs can be solved in polynomial time on C.*

The remaining part of the section is devoted to present and prove the result, which has an interest in its own and has a wider spectrum of possible applications, beyond the study of discrete LB-preference games. In fact, for the sake of presentation, we anticipate here that an immediate consequence of Theorem 26 will be the following (much larger) island of tractability.

Corollary 27. *Let $\bar{k}>0$ and $h>0$ be fixed natural numbers. Let $\mathcal{G}=\langle N,E,\kappa\rangle$ be an environment with $tw(N,E)\leq\bar{k}$ and $h=\mathtt{maxDomSize}(\mathcal{G})$. Then, $\exists$-*NASH *can be solved in polynomial time w.r.t. $||\mathcal{G}||$ (and a stable profile, if any exists, can be computed in polynomial time too).*

Note, in particular, that the island of tractability identified by the above corollary is rather large and fits several interesting application scenarios. Indeed, we have already observed that a number of real-world social environments have bounded treewidth (or this notion can be used as a heuristic method to decompose more complex structures); moreover, concerning the bound on the maximum domain size, we observe that the setting has already found relevant applications in some of the domains we considered in Section 3 (for instance, it is a standard assumption in the works analyzing dynamics of opinion diffusion).

6.4. Larger islands of tractability for ILPs: proof of Theorem 26

In order to establish Theorem 26, we need to elaborate two specific technical ingredients.

Ingredient (1). The first ingredient is to show that we can always transform P in a way that each constraint either does not contain unbounded coefficients, or it contains at most $||unbound(\mathrm{P})||_\infty+1$ non-zero coefficients. Indeed, based on P, let us define $\mathrm{P}'=\langle\mathcal{X}',\mathbb{D}',\Gamma',\eta\rangle$ as the ILP such that:

- All variables in $\mathcal{X}$ are also in $\mathcal{X}'$. For each constraint $\gamma \in \Gamma$, $\mathcal{X}'$ includes the fresh variables in $\{x_{\gamma,1}, ..., x_{\gamma,\ell_\gamma+1}\}$. And no further variable is in $\mathcal{X}'$.
- $\mathbb{D}'$ coincides with $\mathbb{D}$ over the variables in $\mathcal{X}$. Moreover, for each $\gamma \in \Gamma$ and $y \in \{1, ..., \ell_\gamma\}$, we set $\mathbb{D}'(x_{\gamma,y}) = \{-\Omega_{\gamma,y}, ..., \Omega_{\gamma,y}\}$, with $\Omega_{\gamma,y} = \sum_{x\in\mathcal{X}|a_{\gamma,x}=s_{\gamma,y}} ||\mathbb{D}||_\infty$. And, finally, we set $\mathbb{D}'(x_{\gamma,\ell_\gamma+1}) = \{-\Omega_{\gamma,\ell_\gamma+1}, ..., \Omega_{\gamma,\ell_\gamma+1}\}$, with $\Omega_{\gamma,\ell_\gamma+1} = \sum_{x\in\mathcal{X}|a_{\gamma,x}\notin\{s_{\gamma,1},...s_{\gamma,\ell_\gamma}\}} |a_{\gamma,x}| \cdot ||\mathbb{D}||_\infty$.
- For each $\gamma \in \Gamma$, Γ' contains the following constraint:

$$\hat{\gamma}: \quad s_{\gamma,1} \cdot x_{\gamma,1} + ... + s_{\gamma,\ell_\gamma} \cdot x_{\gamma,\ell_\gamma} + x_{\gamma,\ell_\gamma+1} \leq b_\gamma.$$

  For each $y \in \{1, ..., \ell_\gamma\}$, Γ' contains the following two constraints defined over polynomially bounded coefficients (actually, each such coefficient is either -1 or 1):

$$\begin{aligned} \gamma_y: &\quad x_{\gamma,y} + \textstyle\sum_{x\in\mathcal{X}|a_{\gamma,x}=s_{\gamma,y}} x \leq 0 \\ \gamma'_y: &\quad -x_{\gamma,y} - \textstyle\sum_{x\in\mathcal{X}|a_{\gamma,x}=s_{\gamma,y}} x \leq 0 \end{aligned}$$

  And, finally, it contains the following two constraints defined over coefficients that are bounded w.r.t. $||\mathrm{P}||$ via *poly*:

$$\begin{aligned} \gamma_{\ell_\gamma+1}: &\quad x_{\gamma,\ell_\gamma+1} + \textstyle\sum_{x\in\mathcal{X}|a_{\gamma,x}\notin\{s_{\gamma,1},...s_{\gamma,\ell_\gamma}\}} a_{\gamma,x} \cdot x \leq 0 \\ \gamma'_{\ell_\gamma+1}: &\quad -x_{\gamma,\ell_\gamma+1} - \textstyle\sum_{x\in\mathcal{X}|a_{\gamma,x}\notin\{s_{\gamma,1},...s_{\gamma,\ell_\gamma}\}} a_{\gamma,x} \cdot x \leq 0 \end{aligned}$$

Lemma 28. *Consider a class C of ILP instances bounded by some polynomial function poly. Let $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$ be an instance of C, let $k_1 = tw(\mathtt{IG}(\mathrm{P}))$, and let $k_2 = ||unbound(\mathrm{P})||_\infty$. The following properties hold on $\mathrm{P}' = \langle \mathcal{X}', \mathbb{D}', \Gamma', \eta \rangle$.*

(1) *If P admits a feasible solution, then P' admits a feasible solution too. Moreover, if α' is an optimal solution to P', then its restriction on the variables in $\mathcal{X}$ is an optimal solution to P.*
(2) $tw(\mathtt{IG}(\mathrm{P}')) \leq (3 \cdot k_2 + 5) \cdot (k_1 + 1) - 1$.
(3) $||\mathbb{D}'||_\infty \leq 2 \cdot |\mathcal{X}| \cdot poly^2(||\mathrm{P}||)$*, implying that also the class $C' = \{\mathrm{P}' \mid \mathrm{P} \in C\}$ of ILP instances is bounded by some polynomial function $poly'$.*
(4) $||unbound(\mathrm{P}')||_\infty = k_2$.
(5) *For each $\gamma' \in \Gamma'$, either $unbound(\gamma', \mathrm{P}') = \emptyset$ or γ' contains at most $k_2 + 1$ non-zero coefficients.*

Proof. *(1)* The property trivially follows by substituting in $\hat{\gamma}$ the expressions for the variables $x_{\gamma,1}, ..., x_{\gamma,\ell_\gamma+1}$ that are implied by the constraints γ_1, γ'_1, ..., $\gamma_{\ell_\gamma+1}$, $\gamma'_{\ell_\gamma+1}$. In particular, note that the domains for such fresh variables (in $\mathbb{D}'$) accommodate all possible values that the corresponding expressions can take.

(2) Let $k_1 = tw(\mathtt{IG}(\mathrm{P}))$ and $\langle T, \chi \rangle$ be a tree decomposition of $\mathtt{IG}(\mathrm{P})$ whose width is k_1. Consider the labeling χ' for the vertices of T such that, for each vertex p,

$$\chi'(p) = \chi(p) \cap \mathcal{X} \cup \{\hat{\gamma}, \gamma_1, ..., \gamma_{\ell_\gamma+1}, \gamma'_1, ..., \gamma'_{\ell_\gamma+1}, x_{\gamma,1}, ..., x_{\gamma,\ell_\gamma+1} \mid \gamma \in \chi(p)\}.$$

Note first that every variable $x \in \mathcal{X}$ occurs in some χ'-labeling because χ' preserves the variables in χ and because $\langle T, \chi \rangle$ is a tree decomposition of $\mathtt{IG}(\mathrm{P})$. Similarly, each constraint $\gamma' \in \{\hat{\gamma}, \gamma_1, ..., \gamma_{\ell_\gamma+1}, \gamma'_1, ..., \gamma'_{\ell_\gamma+1}\}$ is covered in a vertex p such that $\gamma \in \chi(p)$, which exists again because $\langle T, \chi \rangle$ is a tree decomposition of $\mathtt{IG}(\mathrm{P})$. Consider now an edge connecting a constraint $\gamma' \in \{\hat{\gamma}, \gamma_1, ..., \gamma_{\ell_\gamma+1}, \gamma'_1, ..., \gamma'_{\ell_\gamma+1}\}$ and a variable x. If $x \in \mathcal{X}' \setminus \mathcal{X}$, then $\{\gamma, x\} \subseteq \chi'(p)$ where p is any vertex such that $\gamma \in \chi(p)$. Instead, if $x \in \mathcal{X}$, then a vertex q exists in T such that $\chi(q) \supseteq \{x, \gamma\}$. By construction, $\chi'(q) \supseteq \{x, \gamma'\}$ holds too.

So, it remains to show that $\langle T, \chi' \rangle$ satisfies the connectedness condition of a tree decomposition. In fact, the property holds on the variables in $\mathcal{X}$, as it holds on $\langle T, \chi \rangle$ and χ' entirely preserves such variables. Consider then a constraint $\gamma' \in \{\hat{\gamma}, \gamma_1, ..., \gamma_{\ell_\gamma+1}, \gamma'_1, ..., \gamma'_{\ell_\gamma+1}\}$. Let p and q be two vertices such that $\gamma' \in \chi'(p) \cap \chi'(q)$. By construction, γ' belongs to $\chi(p) \cap \chi(q)$. Consider now any vertex v in the unique path connecting p and q in T. We clearly have $\gamma \in \chi(v)$, as the connectedness condition holds on χ, and hence $\gamma' \in \chi'(v)$.

To complete the proof of this second statement, observe that $|\chi'(p)| = |\chi(p) \cap \mathcal{X}| + (3 \cdot \ell_\gamma + 4) \cdot |\chi(p)| \leq |\chi(p)| + (3 \cdot k_2 + 4) \cdot |\chi(p)| \leq (k_1 + 1) + (3 \cdot k_2 + 4) \cdot (k_1 + 1) = (3 \cdot k_2 + 4 + 1) \cdot (k_1 + 1) = (3 \cdot k_2 + 5) \cdot (k_1 + 1)$. Hence, $tw(\mathtt{IG}(\mathrm{P}')) \leq (3 \cdot k_2 + 5) \cdot (k_1 + 1) - 1$.

(3) Note that $\mathbb{D}'$ coincides with $\mathbb{D}$ on the variables in $\mathcal{X}$. Consider then a variable $x_{\gamma,y}$, for $\gamma \in \Gamma$ and $y \in \{1, ..., \ell_\gamma\}$. By construction, $\mathbb{D}'(x_{\gamma,y}) = \{-\Omega_{\gamma,y}, ..., \Omega_{\gamma,y}\}$, with $\Omega_{\gamma,y} = \sum_{x\in\mathcal{X}|a_{\gamma,x}=s_{\gamma,y}} ||\mathbb{D}||_\infty$. So, the domain span of $x_{\gamma,y}$ is bounded by $2 \cdot |\mathcal{X}| \cdot poly(||\mathrm{P}||)$, because $||\mathbb{D}||_\infty$ is bounded by $poly(||\mathrm{P}||)$. Similarly, for a variable $x_{\gamma,\ell_\gamma+1}$, it is sufficient to recall that $\mathbb{D}'(x_{\gamma,\ell_\gamma+1}) = \{-\Omega_{\gamma,\ell_\gamma+1}, ..., \Omega_{\gamma,\ell_\gamma+1}\}$, with $\Omega_{\gamma,\ell_\gamma+1} = \sum_{x\in\mathcal{X}|a_{\gamma,x}\notin\{s_{\gamma,1},...s_{\gamma,\ell_\gamma}\}} |a_{\gamma,x}| \cdot ||\mathbb{D}||_\infty$. Hence, the domain span of $x_{\gamma,\ell_\gamma+1}$ is bounded by $2 \cdot |\mathcal{X}| \cdot poly^2(||\mathrm{P}||)$, since we are assuming that $|a_{\gamma,x}| \leq poly(||\mathrm{P}||)$, whenever $a_{\gamma,x} \notin \{s_{\gamma,1}, ...s_{\gamma,\ell_\gamma}\}$.

(4 & 5) Consider a constraint $\gamma' \in \{\gamma_1, ..., \gamma_{\ell_\gamma+1}, \gamma'_1, ..., \gamma'_{\ell_\gamma+1}\}$. By construction, we clearly have $unbound(\gamma', \mathrm{P}') = \emptyset$. For a constraint $\hat{\gamma}$, instead, the number of non-zero coefficients is $\ell_\gamma + 1 \leq ||unbound(\mathrm{P})||_\infty + 1$. And, in particular, $\ell_\gamma \leq ||unbound(\mathrm{P})||_\infty$ of them are not polynomially bounded, so that $k_2 = ||unbound(\mathrm{P})||_\infty = ||unbound(\mathrm{P}')||_\infty$ also holds. □

Ingredient (2). Consider a class C of ILP instances bounded by some polynomial function *poly*. Let $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$ be an element of C. Let us consider a constraint $\gamma \in \Gamma$ having the form $\sum_{x\in\mathcal{X}} a_{\gamma,x} \cdot x \leq b_\gamma$ and such that $unbound(\gamma, \mathrm{P}) = \emptyset$. Moreover, let $k_1 = tw(\mathtt{IG}(\mathrm{P}))$ and $\langle T, \chi \rangle$ be a tree decomposition of $\mathtt{IG}(\mathrm{P})$ whose width is k_1, and assume, w.l.o.g., that T is a nearly complete binary tree [85]. Let T_γ be the subtree of T whose vertices are such that their χ-labels contain γ. To navigate T_γ, we use $root(T_\gamma)$ to denote its root and, for each vertex v in T_γ, we use $p(v)$ and $c(v)$ to select the parent of v and the set of its (at most two) children, respectively. If v is a leaf (resp., the root) of T_γ, then $c(v) = \emptyset$ (resp., $p(v) = \mathsf{nil}$ with $\chi(\mathsf{nil}) = \emptyset$). Finally, for each v in T_γ, let $core(v) = \{x \in \mathcal{X} \mid x \in \chi(v) \setminus \chi(p(v)) \wedge a_{\gamma,x} \neq 0\}$ be the set of all variables for which v is the vertex closest to the root where they occur.

The second ingredient is to show that γ can be always reformulated in terms of a set of equivalent constraints in which the number of non-zero coefficients is bounded by some constant depending on k_1 only. Indeed, consider the program $\mathrm{P}' = \langle \mathcal{X}', \mathbb{D}', \Gamma', \eta \rangle$ built as follows:

- $\mathcal{X}'$ contains all variables in $\mathcal{X}$ plus a fresh variable x_v, for each vertex[9] v in T_γ.
- $\mathbb{D}'$ coincides with $\mathbb{D}$ over the variables in $\mathcal{X}$. Moreover, for each variable $x_v \in \mathcal{X}' \setminus \mathcal{X}$, we set $\mathbb{D}'(x_v) = \{-\Omega_v, ..., \Omega_v\}$ with $\Omega_v = \Sigma_{x\in\mathcal{X}} |a_{\gamma,x}| \cdot ||\mathbb{D}||_\infty$.
- Γ' contains all constraints in Γ but γ, which is replaced by the constraints $\bar{\gamma}$ such that $x_{root(T_\gamma)} \leq b_\gamma$. Moreover, for each vertex v in T_γ, the following two constraints are added:

$$\begin{array}{ll} \gamma_v: & \Sigma_{x\in core(v)} a_{\gamma,x} \cdot x + \Sigma_{\alpha\in c(v)} x_\alpha - x_v \leq 0 \\ \gamma'_v: & -\Sigma_{x\in core(v)} a_{\gamma,x} \cdot x - \Sigma_{\alpha\in c(v)} x_\alpha + x_v \leq 0 \end{array}$$

Lemma 29. *Consider a class C of ILP instances bounded by some polynomial function poly. Let $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$ be an element of C and let $k_1 = tw(\mathtt{IG}(\mathrm{P}))$. The following properties hold on P'.*

(1) *If P admits a feasible solution, then P' admits a feasible solution too. Moreover, if α' is an optimal solution to P', then its restriction on the variables in $\mathcal{X}$ is an optimal solution to P.*
(2) $||\mathbb{D}'||_\infty \leq 2 \cdot |\mathcal{X}| \cdot poly^2(||\mathrm{P}||)$, *implying that also the class $C' = \{\mathrm{P}' \mid \mathrm{P} \in C\}$ of ILP instances is bounded by some polynomial function $poly'$.*
(3) *For each $\gamma' \in \{\bar{\gamma}\} \cup \bigcup_{v\in T_\gamma} \{\gamma_v, \gamma'_v\}$, γ' contains at most $k_1 + 4$ non-zero coefficients.*

Proof. *(1)* By repeatedly substituting in $\bar{\gamma}$ the expressions for the variables x_v, for each vertex v in T_γ, that are implied by the constraints γ_v and γ'_v, we derive the following constraint:

$$\sum_{v\in T_\gamma} \sum_{x\in core(v)} a_{\gamma,x} \cdot x \quad \leq \quad b_\gamma$$

Indeed, note that for each $x \in \mathcal{X}$ occurring in the χ-labeling of some vertex in T_γ, there is precisely one vertex v such that $x \in core(v)$. Moreover, recall that T_γ is the subtree of T whose vertices are such that their χ-labels contain γ. This entails that T_γ also covers all variables x such that $a_{\gamma,x} \neq 0$, because there is an edge connecting x and γ in $\mathtt{IG}(\mathrm{P})$, which is covered in T_γ by definition of tree decomposition. Therefore, the above constraint is actually equivalent to the original constraint γ.

(2) Note that $\mathbb{D}'$ coincides with $\mathbb{D}$ on the variables in $\mathcal{X}$. Consider then a variable x_v, for some vertex v in T_γ. By construction, $\mathbb{D}'(x_v) = \{-\Omega_v, ..., \Omega_v\}$ with $\Omega_v = \Sigma_{x\in\mathcal{X}} |a_{\gamma,x}| \cdot ||\mathbb{D}||_\infty$. The result then follows since $||\mathbb{D}||_\infty$ is bounded by $poly(||\mathrm{P}||)$ and $unbound(\gamma, \mathrm{P}) = \emptyset$.

(3) The property clearly holds for $\bar{\gamma}$. Consider, then, a constraint $\gamma' \in \{\gamma_v, \gamma'_v\}$, for some vertex v in T_γ. The number of non-zero coefficients is $|core(v)| + |c(v)| + 1 \leq |\chi(v)| + 2 + 1 \leq k_1 + 4$. □

Now, based on $\langle T, \chi \rangle$, let us build a tuple $\langle T, \chi' \rangle$ as follows: for each vertex v not in T_γ, $\chi'(v) = \chi(v)$; for the root v of T_γ, $\chi'(v) = (\chi(v) \setminus \{\gamma\}) \cup \{x_v, \bar{\gamma}, \gamma_v, \gamma'_v\} \cup \{x_\alpha \mid \alpha \in c(v)\}$; for each other vertex v, $\chi'(v) = (\chi(v) \setminus \{\gamma\}) \cup \{x_v, \gamma_v, \gamma'_v\} \cup \{x_\alpha \mid \alpha \in c(v)\}$.

Lemma 30. *The following properties hold on P' and $\langle T, \chi' \rangle$.*

1. *$\langle T, \chi' \rangle$ is a tree decomposition of $\mathtt{IG}(\mathrm{P}')$.*
2. *For each vertex v not in T_γ, $|\chi'(v)| = |\chi(v)|$; whereas, for each vertex v in T_γ, $|\chi'(v)| \leq |\chi(v)| + 5$.*

Proof. *(1)* We have to show that the three conditions on tree decomposition hold on $\langle T, \chi' \rangle$. First, every variable $\mathcal{X}$ and constraint $\Gamma \setminus \{\gamma\}$ is clearly covered by χ', as it was originally covered by χ. Moreover, for each vertex $v \in T_\gamma$, x_v, γ_v and γ'_v are covered in $\chi'(v)$. And, $\bar{\gamma}$ is covered in the χ'-label of the root of T_γ. For the second condition, consider a constraint γ' and a variable x connected with an edge in $\mathtt{IG}(\mathrm{P}')$. If γ' belongs to Γ, then there is a vertex p such that $\chi(p) \supseteq \{\gamma', x\}$ and, hence, $\chi'(p) \supseteq \{\gamma', x\}$. Assume, then, that $\gamma' \in \{\bar{\gamma}\} \cup \bigcup_{v\in T_\gamma} \{\gamma_v, \gamma'\}$. In the case where $\gamma' = \bar{\gamma}$, then x actually coincides with $x_{root(T_\gamma)}$ and we have $\chi(root(T_\gamma)) \supseteq \{\bar{\gamma}, x_{root(T_\gamma)}\}$.

[9] Vertices of T_γ are transparently viewed as indices for variables.

In the case where $\gamma' = \gamma_v$, instead, either $x = x_v$ or $x = x_\alpha$ with $\alpha \in c(v)$. In both cases, $\chi'(v) \supseteq \{\gamma_v, x_v\}$—the same line of reasoning applies in the case where $\gamma' = \gamma'_v$. Finally, we have to show that the connectedness condition holds on the fresh variables x_v and $\gamma' \in \{\bar{\gamma}, \gamma_v, \gamma'_v\}$. In fact, γ' occurs precisely in one vertex, by construction. Instead, for each vertex v, x_v occurs in $\chi'(v)$ and, whenever v is not the vertex of T_γ, it also occurs in $\chi'(p(v))$.

(2) Vertices outside T_γ are not affected by the transformation. For a vertex v in T_γ, instead, $|\chi'(v)| \leq |\chi(v)| - 1 + 6$ clearly holds, by construction. □

Putting It All Together. We are now ready to combine the two ingredients and derive the proof of Theorem 26. For the sake of presentation, we repeat here its statement.

Theorem 26. *Let $k_1 > 0$ and $k_2 > 0$ be fixed natural numbers. Consider a class C of ILP instances bounded by some polynomial function such that, for each $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$ of C, both $tw(\mathtt{IG}(\mathrm{P})) \leq k_1$ and $||unbound(P)||_\infty \leq k_2$ hold. Then, Integer Linear Programs can be solved in polynomial time on C.*

Proof. Let $\mathrm{P} = \langle \mathcal{X}, \mathbb{D}, \Gamma, \eta \rangle$ be an ILP of C. Assume that C is bounded by the polynomial function *poly*. We apply on P the transformation in the Ingredient (1), and build a novel ILP $\mathrm{P}' = \langle \mathcal{X}', \mathbb{D}', \Gamma', \eta \rangle$ such that $||\mathbb{D}'||_\infty \leq 2 \cdot |\mathcal{X}| \cdot poly^2(||\mathrm{P}||)$ and $tw(\mathtt{IG}(\mathrm{P}')) \leq k'_1$, where $k'_1 = (3 \cdot k_2 + 5) \cdot (k_1 + 1) - 1$ is the constant provided by Lemma 28. In particular, by the same Lemma, we know that, for each $\gamma' \in \Gamma'$, either $unbound(\gamma', \mathrm{P}') = \emptyset$, or γ' contains at most $||unbound(\mathrm{P}')||_\infty + 1 = ||unbound(\mathrm{P})||_\infty + 1 = k_2 + 1$ non-zero coefficients. Let $C' = \{\mathrm{P}' \mid \mathrm{P} \in C\}$ be the new class of ILP instances obtained by applying the transformation in the Ingredient (1); by Lemma 28, C' is bounded by some polynomial function $poly_1$.

Consider now the ILP instance $\mathrm{P}' = \langle \mathcal{X}', \mathbb{D}', \Gamma', \eta \rangle$ of C'. We repeatedly apply the transformation in the Ingredient (2) to every constraint that does not contain unbounded coefficients. By Lemma 29, at the end of this process, we get an ILP $\mathrm{P}^* = \langle \mathcal{X}^*, \mathbb{D}^*, \Gamma^*, \eta \rangle$ such that $||\mathbb{D}^*||_\infty \leq 2 \cdot |\mathcal{X}'| \cdot poly_1^2(||\mathrm{P}'||)$, implying that also the class $C^* = \{\mathrm{P}^* \mid \mathrm{P}' \in C'\}$ of ILP instances is bounded by some polynomial function $poly_2$. Moreover, all constraints $\gamma^* \in \Gamma^*$ are such that γ^* contains k non-zero elements, where k is the maximum between $k_2 + 1$ and $k'_1 + 4$. In addition, we claim that $tw(\mathtt{IG}(\mathrm{P}^*)) \leq k''_1$ holds for another constant k''_1. Indeed, note that each vertex v of T can be processed by at most $k'_1 + 1$ transformations in Ingredient (2), and each time its cardinality grows by a constant factor only, because of Lemma 30. And, finally, note that by combining Lemma 28 and Lemma 29, we get that: *If* P *admits a feasible solution, then* P^* *admits a feasible solution too. Moreover, if α' is an optimal solution to* P^**, then its restriction on the variables in $\mathcal{X}$ is an optimal solution to* P.

In order to conclude the proof, it remains to show that P^* can be solved in polynomial time. To this end, let us associate with ILP P^* the *primal* graph $\mathtt{PG}(\mathrm{P}^*) = (\mathcal{X}^*, E^*)$, whose nodes are the variables and where there is an edge between any pair of variables occurring in the same constraints with associated non-zero coefficients. Let $\langle T, \chi^* \rangle$ be a tree decomposition of $\mathtt{IG}(\mathrm{P}^*)$, and consider the labeling $\bar{\chi}^*$ such that $\bar{\chi}^*(p) = \chi^*(p) \setminus \Gamma^* \cup \{x \mid \gamma^* \in \chi^*(p) \wedge a_{\gamma^*,x} \neq 0\}$, for each vertex p of T. Note that $|\bar{\chi}^*(p)| \leq |\chi^*(p)| \cdot k$ holds, for each vertex p. Furthermore, we claim that $\langle T, \bar{\chi}^* \rangle$ is a tree decomposition of $\mathtt{PG}(\mathrm{P}^*)$, so that we would immediately get that the treewidth of $\mathtt{PG}(\mathrm{P}^*)$ is bounded by a constant, too. To prove the claim, let us consider the three requirements in the definition of tree decomposition.

First, for each variable $x \in \mathcal{X}^*$, there is a vertex p such that $x \in \bar{\chi}^*(p)$, since this condition already holds for $\langle T, \chi^* \rangle$ and since $\bar{\chi}^*(p) \cap \mathcal{X}^* \supseteq \chi(p) \cap \mathcal{X}^*$. Second, consider two variables, say x and x', connected by an edge in E^*, and let γ be the constraint where they occur (with non-zero coefficients). Let p be a vertex of T such that $\gamma \in \chi^*(p)$, which exists since $\langle T, \chi^* \rangle$ is a tree decomposition. By construction, $\bar{\chi}'(p) \supseteq \{x, x'\}$ and, hence, $\langle T, \chi^* \rangle$ satisfies the second requirement in the definition of tree decomposition, too. Finally, to prove that the connectedness condition holds, consider a variable x and two distinct vertices, say p and p', such that $x \in \bar{\chi}^*(p) \cap \bar{\chi}^*(p')$. If $x \in \chi^*(p) \cap \chi^*(p')$, then any vertex in the path connecting p and p' (in T) contains x in its χ^*-labeling (and, hence, x in its $\bar{\chi}^*$-labeling), because $\langle T, \chi^* \rangle$ is a tree decomposition. Consider, then, the case where $x \notin \chi^*(p)$. By construction of $\bar{\chi}^*$, this means that there is a constraint $\gamma \in \chi^*(p)$ where the variable x occurs in it with a non-zero coefficient. Then, there exists some vertex p_γ such that $\chi^*(p_\gamma) \supseteq \{x, \gamma\}$ (as $\langle T, \chi^* \rangle$ is a tree decomposition of the incidence graph) and every vertex in the path connecting p and p_γ contains γ in its χ^*-labeling (and, hence, x in its $\bar{\chi}^*$-labeling, by construction). Now, if $x \in \chi^*(p')$, then we have concluded because every vertex connecting p_γ and p' must contains x in its χ^*-labeling. So, let us assume that $x \notin \chi^*(p')$, that is, there is a constraint $\gamma' \in \chi^*(p')$ such that the variable x occurs in it with a non-zero coefficient. With the same line of reasoning as above, we derive that there is a vertex $p_{\gamma'}$ such that $\chi^*(p_{\gamma'}) \supseteq \{x, \gamma'\}$ and every vertex in the path connecting p' and $p_{\gamma'}$ contains γ' in its χ^*-labeling (and, hence, x in its $\bar{\chi}^*$-labeling). The result then follows since $x \in \chi^*(p_{\gamma'}) \cap \chi^*(p_\gamma)$ and, hence, any vertex connecting $p_{\gamma'}$ and p_γ contains x in its χ^*-labeling (and in its $\bar{\chi}^*$-labeling, too).

Therefore, we have shown that P reformulated in terms of the ILP P^* whose associated primal graph $\mathtt{PG}(\mathrm{P}^*)$ has treewidth bounded by some constant. In order to conclude the proof, let us just notice that the properties of ILPs whose treewidth of the primal graph is bounded by a constant have been studied in the work by Jansen and Kratsch [86]. In particular, it has been shown that they can be solved in polynomial time. Hence, P^* can be solved in polynomial time. □

In fact, the above result is the key to establish Corollary 27.

Proof of Corollary 27. Let $\bar{k} > 0$ and $h > 0$ be fixed natural numbers. Let $\mathcal{G} = \langle N, E, \kappa \rangle$ be an environment with $tw(N, E) \leq \bar{k}$ and $h = \mathtt{maxDomSize}(\mathcal{G})$. After Theorem 23, Theorem 24, and Theorem 26, we have just to show that $||unbound(P^{\mathcal{G}})||_\infty$ is bounded by some given constant. Indeed, just notice that the encoding $P^{\mathcal{G}}$ does not contain unbounded coefficients, but in constraints **(C3)** each of them with the distinct coefficients M, $G_i^1(I_{i,c'})$, and $G_i^1(I_{i,c})$. $\square$

Note that the above result has an interest in its own and improves on earlier results in the literature about the structural tractability of ILPs, which require that both $||\mathbb{D}||_\infty$ and $||\Gamma||_\infty$ are bounded by some polynomial [59]. Applications can be envisaged in all cases where we need to relax the assumption that all coefficients in the program have to be polynomially bounded. This was particularly elegant in our context, where the ILPs naturally exhibit a constant number of distinct unbounded coefficients. In other cases, the extension might be more "syntactic", but still technically interesting. For instance, by Theorem 26 and Example 6, we easily derive that the subset sum problem is tractable not only if all given integers $\{s_1, ..., s_n\}$ are polynomially bounded (as we would derive by known results in the literature), but also assuming that some of them (up to a constant number) take exponential values w.r.t. n.

7. Conclusion and discussion

We have proposed and studied the setting of discrete LB-preference games for analyzing influence phenomena over social networks, where the reasoning capabilities of the agents are modeled via weighted propositional logic. We have studied the expressiveness of the framework and we have embarked in a thorough complexity analysis precisely isolating those scenarios for which stable profiles can be easily computed. Our results evidence that one of the main sources of the intractability of discrete LB-preference games is the interplay between conformist and dissenter agents, and that this source of complexity can be kept under control on classes of games that enjoy some desirable structural property, namely near-acyclicity. To establish the latter result we have proposed an encoding of discrete LB-preference games in terms of Integer Linear Programs, and we have devised an island of structural tractability for them which generalizes earlier approaches in the literature and have a wider spectrum of applicability. Our work opens several avenues of further research, which will be analyzed and discussed in the rest the section.

7.1. Asymmetric compatibility and Boolean games

One natural avenue of further research is to enrich our basic framework with a number of features that can enhance its expressiveness. In particular, observe that discrete LB-preference games rely on the following notion of compatibility given in Section 2.2

$$\texttt{compatible}_{\mathcal{G},i}(\Pi) = \{j \in \texttt{neigh}_{\mathcal{G}}(i) | \ \forall x \in \texttt{dom}(\kappa(i)) \cap \texttt{dom}(\kappa(j)),\ \Pi_i(x) = \Pi_j(x)\},$$

which is defined w.r.t. all variables in the domain of each agent. Indeed, according to our perspective, variables in $\texttt{dom}(\kappa(i))$ collectively reflect what matters to agent i, regardless of whether they belong to the G_i^0 or G_i^1. Nonetheless, one might adopt the following variant

$$\texttt{compatible}_{\mathcal{G},i}(\Pi) = \{j \in \texttt{neigh}_{\mathcal{G}}(i) | \ \forall x \in \texttt{dom}(G_i^1) \cap \texttt{dom}(\kappa(j)),\ \Pi_i(x) = \Pi_j(x)\}, \tag{10}$$

where the compatibility check for agent i only focuses on the variables in the domain of her social goalbase G_i^1, hence disregarding the variables in G_i^0. According to this perspective, G_i^1 would model the "social" knowledge that is relevant to i when interacting with the other agents; instead, the evaluation of i's neighbors of the variables that only belong to G_i^0 do not affect the utility of that agent i (hence, G_i^0 would be a kind of "private" knowledge). In fact, this variant is a proper generalization of the setting considered in our paper. Indeed, to mimic our notion, for each variable x occurring in $\texttt{dom}(G_i^0) \setminus \texttt{dom}(G_i^1)$, one can add the dummy formula $(x \vee \neg x, 0)$ to G_i^1 so that $\texttt{dom}(G_i^1) = \texttt{dom}(\kappa(i))$. Hence, all hardness results reported in Table 1 would still hold under the considered variant (and memberships hold as well, with the same arguments used for our setting). On the other hand, the proposed variant breaks the symmetry that is naturally induced when our setting is applied over symmetric directed graphs; in fact, just notice that with the proposed variant it is no longer the case that $j \in \texttt{compatible}_i(\Pi)$ if, and only if, $i \in \texttt{compatible}_j(\Pi)$. Hence, for example, Theorem 15 would no longer hold, and a specific complexity study over symmetric graphs (or the definition of a different notion of "symmetry" that takes into account the knowledge bases of the agents and that is more adherent with the variant) would be in order.

The "asymmetric" behavior of the extension presented above paves the way for reasoning on strategic settings that do not naturally fit our "symmetric" framework. A noticeable example is given by the well-known setting of *Boolean games* (see, e.g., [87–90]). Informally, in such games, agents' goals are represented by a propositional logic formula: each agent is associated with a set of Boolean variables, and there is a formula or goal associated with each agent that is composed of variables that may not be in her control. The main objective of each agent is to achieve its goal while minimizing its total cost. We next formalize the setting, by subsequently evidencing how it can be easily encoded in the extension we have discussed.

Formalization. As in [88], a Boolean game is as a tuple $\mathsf{B} = \langle N, V, c, \gamma_1, ..., \gamma_n, V_1, ..., V_n\rangle$, where we have that (1) N is a set of n agents, (2) $V \subseteq \mathcal{V}$ is a set of propositional variables, (3) $c : V \cup \{\neg x \mid x \in V\} \rightarrow \mathbb{Q}^+ \cup \{0\}$ is a cost function associating a nonnegative rational number to each literal over V, (4) $\gamma_1, ..., \gamma_n$ are propositional formulas over V, with γ_i being the goal of agent i, (5) $V_1, ..., V_n$ is a partition of V, and (6) each agent i can only control the truth values of the variables in V_i. Given an interpretation I over V, extensively given in terms of a set of positive and negative literals, the *total marginal cost* payed by agent i with respect to I is

$$c_i(I) = \sum_{x \in V_i \wedge x \in I} c(x) + \sum_{x \in V_i \wedge \neg x \in I} c(\neg x).$$

Let μ_i be the maximum total marginal cost that an agent i may pay by considering all possible interpretations for the variables in V_i. The utility of agent i with respect to an interpretation I is

$$u_i^{\mathsf{B}}(I) = \begin{cases} 1 + \mu_i - c_i(I) & \text{if } I \vDash \gamma_i \\ -c_i(I) & \text{otherwise} \end{cases}$$

Consider an interpretation I. The restriction of I to a given set $X \subseteq \mathtt{dom}(I)$ of variables is denoted by $I[X]$. Then, I is a *Nash equilibrium* for B if, for each agent $i \in N$ and each interpretation J_i for the variables in V_i, it holds that $u_i^{\mathsf{B}}(I) \geq u_i^{\mathsf{B}}(I')$ with $I' = (I \setminus I[V_i]) \cup J_i$.

Encoding. Based on B, we build a discrete LB-preference game $\mathcal{G}^{\mathsf{B}} = \langle N, E, \kappa \rangle$ as follows. Given an agent $i \in N$ and its associated goal γ_i, let $\tilde{\gamma}_i$ denote the formula obtained from γ_i by replacing each variable x with x_i. For example, if $\gamma_i = x \wedge \neg y \wedge z$, then $\tilde{\gamma}_i = x_i \wedge \neg y_i \wedge z_i$. Essentially, we *clone* each variable in V by taking into account who are the agents that use that variable in their goals. Moreover, $(j, i) \in E$ if, and only if, agent i controls some variable that occurs in γ_j. Finally, for each agent $i \in N$, we define:

$$\begin{aligned} G_i^0 \; = \; & \{(\tilde{\gamma}_i, 1 + \mu_i)\} \bigcup \\ & \{(x_i, -c(x)), (\neg x_i, -c(\neg x)) \mid x \in V_i\} \bigcup \\ & \{(x_j \leftrightarrow x_i, 1) \mid (j, i) \in E \wedge x \in V_i\} \end{aligned}$$

$$G_i^1 \; = \; \{(x_i \vee \neg x_i, \tfrac{2+\mu_i}{|\mathtt{dom}(\gamma_i) \setminus V_i|}) \mid x \in \mathtt{dom}(\gamma_i) \setminus V_i)\}$$

By construction, the asymmetry in the notion of compatibility given by Equation (10) can be exploited to guarantee that each agent gets an incentive to fix the truth values of the variables that are not under her control according to the valuation provided by the owners of that variables. This paves the way to encode Boolean games in our framework.

More formally, given an interpretation I for B, let us first define the profile $\Pi^I = \{i \mapsto \Pi_i\}_{i \in N}$ where, for each $i \in N$, the interpretation Π_i is

$$\rho_i(I[\mathtt{dom}(\gamma_i)]) \cup \{x_j \mid (j, i) \in E \wedge x \in V_i \wedge x \in I\} \cup \{\neg x_j \mid (j, i) \in E \wedge x \in V_i \wedge \neg x \in I\},$$

where $\rho_i(I[\mathtt{dom}(\gamma_i)])$ is the interpretation obtained from $I[\mathtt{dom}(\gamma_i)]$ by renaming each variable x with x_i. For example, if $I[\mathtt{dom}(\gamma_i)] = \{x, \neg y\}$, then $\rho_i(I[\mathtt{dom}(\gamma_i)]) = \{x_i, \neg y_i\}$. Then, the following result can be established.

Theorem 31. *Let $\mathcal{G}^{\mathsf{B}} = \langle N, E, \kappa \rangle$ be the discrete LB-preference game associated with B, and under the "asymmetric" variant of the notion of compatibility in Equation (10).*

(1) *For each $i \in N$, $u_{\mathcal{G}^{\mathsf{B}}, i}(\Pi^I) = u_i^{\mathsf{B}}(I) + |V_i| \cdot |\{j \mid (j, i) \in E\}| + (2 + \mu_i) \cdot |\mathtt{neigh}(i)|$.*
(2) *If I is a Nash equilibrium for B, then Π^I is stable for $\mathcal{G}^{\mathsf{B}}$.*
(3) *If Π is a stable profile for $\mathcal{G}^{\mathsf{B}}$, then there is a Nash equilibrium I for B such that $\Pi = \Pi^I$.*

Proof. (1) Consider an interpretation I for B and its associated profile $\Pi^I = \{i \mapsto \Pi_i\}_{i \in N}$. Since $\Pi_i \supseteq \rho_i(I[\mathtt{dom}(\gamma_i)])$, by construction, $I \vDash \gamma_i$ holds if, and only if, $\Pi_i \vDash \rho_i(\gamma_i)$ holds. This means that $G_i^0(\Pi_i) = u_i^{\mathsf{B}}(I) + k_i$, where k_i depends on the evaluation of the formulas of the form $(x_j \leftrightarrow x_i, 1)$. But since, by construction of Π_i, each x_j has the same value as x_i, then $k_i = |V_i| \cdot |\{j \mid (j, i) \in E\}|$. Hence, $G_i^0(\Pi_i) = u_i^{\mathsf{B}}(I) + |V_i| \cdot |\{j \mid (j, i) \in E\}|$. Concerning $G_i^1(\Pi_i)$, since by definition, it contains exactly $\mathtt{dom}(\gamma_i) \setminus V_i$ formulas that are always true, its value is clearly $(2 + \mu_i)$; moreover, since, by construction of Π_i, each variable occurring in $\mathcal{G}^{\mathsf{B}}$ has the same value in all the interpretations of Π^I, then i is compatible with all her neighbors. Hence, $G_i^1(\Pi_i) \cdot |\mathtt{compatible}_i(\Pi)| = (2 + \mu_i) \cdot |\mathtt{neigh}(i)|$ and the statement holds.

(2) Consider a Nash equilibrium I for B. For the sake of contradiction, assume that Π^I is not stable. This means that there exists some agent i who is not playing one of her best response moves. But this is not possible because, as we discussed above concerning (1), i is always compatible with all her neighbors, all the formulas in G_i^1 are always trivially satisfied, all the formulas of the form $(x_j \leftrightarrow x_i, 1)$ in G_i^0 are always satisfied by construction of Π_i, and the evaluation of the remaining formulas of G_i^0 leads exactly to $u_i^{\mathsf{B}}(I)$. Since $u_i^{\mathsf{B}}(I)$ is strictly smaller than $2 + \mu_i$, there is no incentive for i to change the value of some variable in $\{x_i \mid x \in \mathtt{dom}(\gamma_i) \setminus V_i\}$ to reduce the number of her compatible agents and increase the evaluation of G_i^0. Hence, the only chance would be to change the value of some variable in $\{x_i \mid x \in V_i\}$, but this would mean that $u_i^{\mathsf{B}}(I)$ can be improved, implying that I is not a Nash equilibrium for B, which is a contradiction.

(3) Consider a stable profile $\Pi = \{i \mapsto \Pi_i\}_{i \in N}$ for $\mathcal{G}^{\mathsf{B}}$. First, we claim that $J = \bigcup_{i \in N} \Pi_i$ is a consistent interpretation, namely no variable occurs both as positive and negated. Indeed, if $x \in V_i$ (i.e., x is controlled by i), then its clone x_i occurs in G_i^0 only; moreover, if $x \in \mathtt{dom}(\gamma_i) \setminus V_i$, then x is controlled by some agent $j \neq i$ and its clone x_i occurring in G_i^1 is always compatible with the clone x_i present in G_j^0. This is true because $2 + \mu_i$ is always bigger than the maximum utility of i in B (i.e., $1 + \mu_i$). Now, we claim that the set I of literals obtained from J by removing the subscripts from all the variables is a valid interpretation for B. Indeed, given a variable $x \in V_i$, its clones x_j and x_i occurring in the formula $(x_j \leftrightarrow x_i, 1)$ do always agree. For the sake of contradiction, assume that I is not a Nash equilibrium for B. This means that there exists J such that $u_i^{\mathsf{B}}(J) > u_i^{\mathsf{B}}(I)$ for some agent i. However, this is not possible since $u_{\mathcal{G}^{\mathsf{B}}, i}(\Pi^J) - u_{\mathcal{G}^{\mathsf{B}}, i}(\Pi^I) = u_i^{\mathsf{B}}(J) - u_i^{\mathsf{B}}(I)$, as it would imply that Π^I is not stable, which is a contradiction. Finally, since I is obtained from Π by simply removing all subscripts, by definition of Π^I, we have that $\Pi = \Pi^I$. $\square$

7.2. From decision to computation problems

Another avenue for further research naturally arises by observing that ∃-NASH is a decision problem, but the islands of tractability we have identified in Section 5 are such that stable profiles are always guaranteed to exist and can be even computed efficiently. However, this property does not hold for the structural tractability results in Section 6, thereby calling to study the optimization version of ∃-NASH, say MAX-STABLE, where we would like to compute the profile Π with the maximum number of stable agents over all the possible profiles. It fact, it is easy to see that a slight modification to the encoding in Section 6.3 can be designed in order to optimize the number of agents that are stable, rather than just checking that none of them is unstable.

Theorem 32. *Let $\bar{k} > 0$ and $h > 0$ be fixed natural numbers. Let $\mathcal{G} = \langle N, E, \kappa\rangle$ be a discrete* LB*-preference game with $tw(N, E) \leq \bar{k}$ and $h = \mathtt{maxDomSize}(\mathcal{G})$. Then,* MAX-STABLE *can be solved in polynomial time w.r.t. $||\mathcal{G}||$.*

Sketch. In the encoding $\mathrm{P}^{\mathcal{G}} = \langle \mathcal{X}, \Gamma, \mathbb{D}\rangle$, it is sufficient to consider, for each agent $i \in N$, a fresh variable x_i^{stable} with domain $\{0, 1\}$, whose intended meaning is to check whether i is stable in the current profile. Then, we can modify the constraint $\bar{\gamma}_{i,c,c'}$ as follows:

$$x_i^{stable} \cdot M^2 + x_{i,c} \cdot M + G_i^1(I_{i,c'}) \cdot \sum_j x_{i,c',j} - G_i^1(I_{i,c}) \cdot \sum_j x_{i,c,j} \leq M^2 + M + G_i^0(I_{i,c}) - G_i^0(I_{i,c'}),$$

where the terms with factor M^2 make the constraint trivially satisfied when x_i^{stable} is mapped to 0. Eventually, the encoding can be completed by adding the objective $\sum_{i\in N} x_i^{stable}$. □

More generally, however, it would be interesting to conduct a systematic study of MAX-STABLE, in particular on its approximability. For instance, a preliminary, but very appealing result in this direction is the following.

Theorem 33. *On linear $\mathcal{G} = \langle N, E, \kappa\rangle$ defined over graphs that are symmetric and such that $\mathtt{maxDomSize}(\mathcal{G}) \in O(\log ||\mathcal{G}||)$,* MAX-STABLE *is $1/2$-approximable. Moreover, for any fixed $\varepsilon > 0$,* MAX-STABLE *is not $(1/2 - \varepsilon)$-approximable, unless* **PTIME = NP**.

Sketch. Let n_c, n_d, and n_a be the number of conformist, dissenter, and autonomous agents, respectively. Consider the case where $n_c \geq n_d$. Then, let us build the environment $\mathcal{G}' = \langle N, E, \kappa'\rangle$ such that, for each agent $i \in N$, $\kappa'(i) = \kappa(i)$ whenever i is a conformist or an autonomous agent; and $\kappa'(i) = \langle \emptyset, \emptyset\rangle$ if i was a dissenter. Note that we are in the position of applying Theorem 15 on $\mathcal{G}'$, hence computing in polynomial time a profile Π' that is stable in $\mathcal{G}'$. The approximability then follows since the number of agents that are not stable with Π' in $\mathcal{G}'$ is precisely bounded by $n_d \leq n_c + n_a \leq |N|/2$. The case where $n_d \leq n_c$ can be addressed with the same argument, this time overriding the knowledge bases of the conformists. Finally, the result is tight. Indeed, given $\varepsilon > 0$, we can take a discrete LB-preference game $\mathcal{G} = \langle N, E, \kappa\rangle$ from a **NP**-hard class, such as the one in the proof of Lemma 10. Then, we can build $\mathcal{G}'$ by just adding a set $2 \cdot M$ of agents, of which only M can be stable in any profile (e.g., a dissenter is paired with a conformist). The result eventually follows, by just defining M large enough so that a profile achieves a $(1/2 - \varepsilon)$-approximation if, and only if, all agents in N are stable with it. □

7.3. Discussion on implementation and experimental evaluations

Yet another line for further research can be traced by considering an application viewpoint. Indeed, while our work is foundational in nature and does not include an experimental evaluation, we clearly foresee that the effectiveness of our approach heavily relies on having “small” values of $tw(N, E)$ (see, e.g., [82–84]). However, the different reductions we have used to establish our tractability results cause an explosion of the exponent in the polynomial defining the running time, which can be unpractical even with small treewidth values. In fact, assessing the practical feasibility of our approach and, eventually, defining specialized algorithms and ad-hoc solution approaches are interesting research issues, particularly when focusing on large social networks. Moreover, it would be intriguing to evaluate whether our tractability results can be integrated with machine learning techniques designed to discern agents' attitudes based on their logged interactions. This integration could pave the way for hybrid (logic-based and inductive) approaches to simulate/predict the spread of opinions in segments of real-world socio-technical systems, such as those observed on platforms like Facebook or Twitter.

CRediT authorship contribution statement

Gianluigi Greco: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Marco Manna:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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References

[1] D.A. Easley, J.M. Kleinberg, Networks, Crowds, and Markets - Reasoning About a Highly Connected World, Cambridge University Press, 2010.
[2] A. Salehi-Abari, C. Boutilier, K. Larson, Empathetic decision making in social networks, Artif. Intell. 275 (2019) 174–203, https://doi.org/10.1016/J.ARTINT.2019.05.004.
[3] L. Sless, N. Hazon, S. Kraus, M.J. Wooldridge, Forming k coalitions and facilitating relationships in social networks, Artif. Intell. 259 (2018) 217–245, https://doi.org/10.1016/J.ARTINT.2018.03.004.
[4] C. Kang, S. Kraus, C. Molinaro, F. Spezzano, V.S. Subrahmanian, Diffusion centrality: a paradigm to maximize spread in social networks, Artif. Intell. 239 (2016) 70–96, https://doi.org/10.1016/J.ARTINT.2016.06.008.
[5] M.T. Irfan, L.E. Ortiz, On influence, stable behavior, and the most influential individuals in networks: a game-theoretic approach, Artif. Intell. 215 (2014) 79–119, https://doi.org/10.1016/J.ARTINT.2014.06.004.
[6] V. Auletta, D. Ferraioli, G. Greco, On the complexity of reasoning about opinion diffusion under majority dynamics, Artif. Intell. 284 (2020) 103288, https://doi.org/10.1016/J.ARTINT.2020.103288.
[7] A. Anagnostopoulos, R. Kumar, M. Mahdian, Influence and correlation in social networks, in: Y. Li, B. Liu, S. Sarawagi (Eds.), Proceedings of the 14th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, Las Vegas, Nevada, USA, August 24-27, 2008, ACM, 2008, pp. 7–15.
[8] J. Tang, S. Wu, J. Sun, Confluence: conformity influence in large social networks, in: I.S. Dhillon, Y. Koren, R. Ghani, T.E. Senator, P. Bradley, R. Parekh, J. He, R.L. Grossman, R. Uthurusamy (Eds.), Proceedings of the 19th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD 2013, Chicago, IL, USA, August 11-14, 2013, ACM, 2013, pp. 347–355.
[9] R.B. Cialdini, N.J. Goldstein, Social influence: compliance and conformity, Annu. Rev. Psychol. 55 (2004) 591–621, https://doi.org/10.1146/annurev.psych.55.090902.142015.
[10] D. Kempe, J.M. Kleinberg, É. Tardos, Maximizing the spread of influence through a social network, in: L. Getoor, T.E. Senator, P.M. Domingos, C. Faloutsos (Eds.), Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, Washington, DC, USA, August 24-27, 2003, ACM, 2003, pp. 137–146.
[11] M. Granovetter, Threshold models of collective behavior, Am. J. Sociol. 83 (6) (1978) 1420–1443.
[12] S. Aral, L. Muchnik, A. Sundararajan, Distinguishing influence-based contagion from homophily-driven diffusion in dynamic networks, Proc. Natl. Acad. Sci. USA 106 (51) (2009) 21544–21549, https://doi.org/10.1073/pnas.0908800106.
[13] M. Cha, A. Mislove, P.K. Gummadi, A measurement-driven analysis of information propagation in the Flickr social network, in: J. Quemada, G. León, Y.S. Maarek, W. Nejdl (Eds.), Proceedings of the 18th International Conference on World Wide Web, WWW 2009, Madrid, Spain, April 20-24, 2009, ACM, 2009, pp. 721–730.
[14] A. Borodin, Y. Filmus, J. Oren, Threshold models for competitive influence in social networks, in: A. Saberi (Ed.), Internet and Network Economics - Proceedings of the 6th International Workshop, WINE 2010, Stanford, CA, USA, December 13-17, 2010, in: Lecture Notes in Computer Science, vol. 6484, Springer, 2010, pp. 539–550.
[15] X. He, G. Song, W. Chen, Q. Jiang, Influence blocking maximization in social networks under the competitive linear threshold model, in: Proceedings of the Twelfth SIAM International Conference on Data Mining, Anaheim, California, USA, April 26-28, 2012, SIAM / Omnipress, 2012, pp. 463–474.
[16] S. Simon, K.R. Apt, Choosing products in social networks, in: P.W. Goldberg (Ed.), Internet and Network Economics - Proceedings of the 8th International Workshop, WINE 2012, Liverpool, UK, December 10-12, 2012, in: Lecture Notes in Computer Science, vol. 7695, Springer, 2012, pp. 100–113.
[17] T. Carnes, C. Nagarajan, S.M. Wild, A. van Zuylen, Maximizing influence in a competitive social network: a follower's perspective, in: M.L. Gini, R.J. Kauffman, D. Sarppo, C. Dellarocas, F. Dignum (Eds.), Proceedings of the 9th International Conference on Electronic Commerce: The Wireless World of Electronic Commerce, 2007, University of Minnesota, Minneapolis, MN, USA, August 19-22, 2007, in: ACM International Conference Proceeding Series, vol. 258, ACM, 2007, pp. 351–360.
[18] D. Acemoglu, A.E. Ozdaglar, Opinion dynamics and learning in social networks, Dyn. Games Appl. 1 (1) (2011) 3–49, https://doi.org/10.1007/S13235-010-0004-1.
[19] P.M. DeMarzo, D. Vayanos, J. Zwiebel, Persuasion bias, social influence, and unidimensional opinions, Q. J. Econ. 118 (3) (2003) 909–968.
[20] B. Golub, M.O. Jackson, Naïve learning in social networks and the wisdom of crowds, Am. Econ. J. Microecon. 2 (1) (2010) 112–149.
[21] M.O. Jackson, Social and Economic Networks, Princeton University Press, Princeton, NJ, USA, 2008.
[22] M.H. DeGroot, Reaching a consensus, J. Am. Stat. Assoc. 69 (345) (1974) 118–121, https://doi.org/10.1080/01621459.1974.10480137.
[23] N.E. Friedkin, E.C. Johnsen, Social influence and opinions, J. Math. Sociol. 15 (3–4) (1990) 193–206, https://doi.org/10.1080/0022250X.1990.9990069.
[24] D. Bindel, J.M. Kleinberg, S. Oren, How bad is forming your own opinion?, Games Econ. Behav. 92 (2015) 248–265, https://doi.org/10.1016/J.GEB.2014.06.004.
[25] K. Bhawalkar, S. Gollapudi, K. Munagala, Coevolutionary opinion formation games, in: D. Boneh, T. Roughgarden, J. Feigenbaum (Eds.), Proceedings of the Symposium on Theory of Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4, 2013, ACM, 2013, pp. 41–50.
[26] U. Grandi, E. Lorini, L. Perrussel, Propositional opinion diffusion, in: G. Weiss, P. Yolum, R.H. Bordini, E. Elkind (Eds.), Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2015, Istanbul, Turkey, May 4-8, 2015, ACM, 2015, pp. 989–997.
[27] V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, G. Persiano, Generalized discrete preference games, in: S. Kambhampati (Ed.), Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016, IJCAI/AAAI Press, 2016, pp. 53–59.
[28] F. Chierichetti, J.M. Kleinberg, S. Oren, On discrete preferences and coordination, in: M.J. Kearns, R.P. McAfee, É. Tardos (Eds.), Proceedings of the Fourteenth ACM Conference on Electronic Commerce, EC 2013, Philadelphia, PA, USA, June 16-20, 2013, ACM, 2013, pp. 233–250.
[29] F. Chierichetti, J.M. Kleinberg, S. Oren, On discrete preferences and coordination, J. Comput. Syst. Sci. 93 (2018) 11–29, https://doi.org/10.1016/J.JCSS.2017.11.002.
[30] G. Gottlob, G. Greco, F. Scarcello, Pure Nash equilibria: hard and easy games, J. Artif. Intell. Res. 24 (2005) 357–406, https://doi.org/10.1613/JAIR.1683.
[31] M.J. Kearns, M.L. Littman, S. Singh, Graphical models for game theory, in: J.S. Breese, D. Koller (Eds.), Proceedings of the 17th Conference in Uncertainty in Artificial Intelligence, UAI'01, University of Washington, Seattle, Washington, USA, August 2-5, 2001, Morgan Kaufmann, 2001, pp. 253–260.

[32] C. Daskalakis, C.H. Papadimitriou, Computing pure Nash equilibria in graphical games via Markov random fields, in: J. Feigenbaum, J.C. Chuang, D.M. Pennock (Eds.), Proceedings 7th ACM Conference on Electronic Commerce (EC-2006), Ann Arbor, Michigan, USA, June 11-15, 2006, ACM, 2006, pp. 91–99.

[33] S. Simon, D. Wojtczak, Efficient local search in coordination games on graphs, in: S. Kambhampati (Ed.), Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016, IJCAI/AAAI Press, 2016, pp. 482–488.

[34] K.R. Apt, B. de Keijzer, M. Rahn, G. Schäfer, S. Simon, Coordination games on graphs, Int. J. Game Theory 46 (3) (2017) 851–877, https://doi.org/10.1007/S00182-016-0560-8.

[35] S. Simon, K.R. Apt, Social network games, J. Log. Comput. 25 (1) (2015) 207–242, https://doi.org/10.1093/LOGCOM/EXT012.

[36] D. Ferraioli, P.W. Goldberg, C. Ventre, Decentralized dynamics for finite opinion games, Theor. Comput. Sci. 648 (2016) 96–115, https://doi.org/10.1016/J.TCS.2016.08.011.

[37] R. Bredereck, E. Elkind, Manipulating opinion diffusion in social networks, in: C. Sierra (Ed.), Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, 2017, pp. 894–900, ijcai.org.

[38] V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, G. Persiano, Robustness in discrete preference games, in: K. Larson, M. Winikoff, S. Das, E.H. Durfee (Eds.), Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2017, São Paulo, Brazil, May 8-12, 2017, ACM, 2017, pp. 1314–1322.

[39] D. Ferraioli, C. Ventre, Social pressure in opinion games, in: C. Sierra (Ed.), Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, 2017, pp. 3661–3667, ijcai.org.

[40] V. Auletta, D. Ferraioli, V. Fionda, G. Greco, Maximizing the spread of an opinion when tertium datur est, in: E. Elkind, M. Veloso, N. Agmon, M.E. Taylor (Eds.), Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS '19, Montreal, QC, Canada, May 13-17, 2019, International Foundation for Autonomous Agents and Multiagent Systems, 2019, pp. 1207–1215.

[41] P.R. Lolakapuri, U. Bhaskar, R. Narayanam, G.R. Parija, P.S. Dayama, Computational aspects of equilibria in discrete preference games, in: S. Kraus (Ed.), Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019, 2019, pp. 471–477, ijcai.org.

[42] M. Castiglioni, D. Ferraioli, N. Gatti, G. Landriani, Election manipulation on social networks: seeding, edge removal, edge addition, J. Artif. Intell. Res. 71 (2021) 1049–1090, https://doi.org/10.1613/JAIR.1.12826.

[43] J. Lang, From preference representation to combinatorial vote, in: D. Fensel, F. Giunchiglia, D.L. McGuinness, M. Williams (Eds.), Proceedings of the Eights International Conference on Principles and Knowledge Representation and Reasoning (KR-02), Toulouse, France, April 22-25, 2002, Morgan Kaufmann, 2002, pp. 277–290.

[44] D. Kempe, J.M. Kleinberg, É. Tardos, Influential nodes in a diffusion model for social networks, in: L. Caires, G.F. Italiano, L. Monteiro, C. Palamidessi, M. Yung (Eds.), Automata, Languages and Programming - Proceedings of the 32nd International Colloquium, ICALP 2005, Lisbon, Portugal, July 11-15, 2005, in: Lecture Notes in Computer Science, vol. 3580, Springer, 2005, pp. 1127–1138.

[45] M.A. Javarone, Social influences in opinion dynamics: the role of conformity, Phys. A, Stat. Mech. Appl. 414 (2014) 19–30, https://doi.org/10.1016/j.physa.2014.07.018.

[46] H. Fang, X. Li, J. Zhang, Integrating social influence modeling and user modeling for trust prediction in signed networks, Artif. Intell. 302 (2022) 103628, https://doi.org/10.1016/J.ARTINT.2021.103628.

[47] Y. Bramoullé, D. López-Pintado, S. Goyal, F. Vega-Redondo, Network formation and anti-coordination games, Int. J. Game Theory 33 (1) (2004) 1–19, https://doi.org/10.1007/S001820400178.

[48] J. Kun, B. Powers, L. Reyzin, Anti-coordination games and stable graph colorings, in: B. Vöcking (Ed.), Algorithmic Game Theory - Proceedings of the 6th International Symposium, SAGT 2013, Aachen, Germany, October 21-23, 2013, in: Lecture Notes in Computer Science, vol. 8146, Springer, 2013, pp. 122–133.

[49] S. Coste-Marquis, J. Lang, P. Liberatore, P. Marquis, Expressive power and succinctness of propositional languages for preference representation, in: D. Dubois, C.A. Welty, M. Williams (Eds.), Principles of Knowledge Representation and Reasoning: Proceedings of the Ninth International Conference (KR2004), Whistler, Canada, June 2–5, 2004, AAAI Press, 2004, pp. 203–212.

[50] J. Uckelman, Y. Chevaleyre, U. Endriss, J. Lang, Representing utility functions via weighted goals, Math. Log. Q. 55 (4) (2009) 341–361, https://doi.org/10.1002/MALQ.200810024.

[51] C.P. Gomes, H. Kautz, A. Sabharwal, B. Selman, Satisfiability solvers, in: F. van Harmelen, V. Lifschitz, B. Porter (Eds.), Handbook of Knowledge Representation, in: Foundations of Artificial Intelligence, vol. 3, Elsevier, 2008, pp. 89–134.

[52] K. Ohta, N. Barrot, A. Ismaili, Y. Sakurai, M. Yokoo, Core stability in hedonic games among friends and enemies: impact of neutrals, in: C. Sierra (Ed.), Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, 2017, pp. 359–365, ijcai.org.

[53] A.M. Kerkmann, J. Rothe, Stability in FEN-hedonic games for single-player deviations, in: E. Elkind, M. Veloso, N. Agmon, M.E. Taylor (Eds.), Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS '19, Montreal, QC, Canada, May 13–17, 2019, International Foundation for Autonomous Agents and Multiagent Systems, 2019, pp. 891–899.

[54] M. Flammini, B. Kodric, G. Varricchio, Strategyproof mechanisms for friends and enemies games, in: Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2020, New York, NY, USA, February 7-12, 2020, AAAI Press, 2020, pp. 1950–1957.

[55] D. Dimitrov, P. Borm, R. Hendrickx, S.C. Sung, Simple priorities and core stability in hedonic games, Soc. Choice Welf. 26 (2) (2006) 421–433, https://doi.org/10.1007/S00355-006-0104-4.

[56] N. Robertson, P.D. Seymour, Graph minors. III. Planar tree-width, J. Comb. Theory, Ser. B 36 (1) (1984) 49–64, https://doi.org/10.1016/0095-8956(84)90013-3.

[57] B. Courcelle, Graph rewriting: an algebraic and logic approach, in: J. van Leeuwen (Ed.), Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics, Elsevier and MIT Press, 1990, pp. 193–242.

[58] H.L. Bodlaender, Dynamic programming on graphs with bounded treewidth, in: T. Lepistö, A. Salomaa (Eds.), Automata, Languages and Programming - Proceedings of the 15th International Colloquium, ICALP88, Tampere, Finland, July 11–15, 1988, in: Lecture Notes in Computer Science, vol. 317, Springer, 1988, pp. 105–118.

[59] R. Ganian, S. Ordyniak, M.S. Ramanujan, Going beyond primal treewidth for (M)ILP, in: S. Singh, S. Markovitch (Eds.), Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence, February 4-9, 2017, AAAI Press, San Francisco, California, USA, 2017, pp. 815–821.

[60] M. Koutecký, A. Levin, S. Onn, A parameterized strongly polynomial algorithm for block structured integer programs, in: I. Chatzigiannakis, C. Kaklamanis, D. Marx, D. Sannella (Eds.), Proceedings of the 45th International Colloquium on Automata, Languages, and Programming, ICALP, 2018, July 9-13, 2018, Prague, Czech Republic, in: LIPIcs, vol. 107, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, pp. 85:1–85:14.

[61] R. Ganian, S. Ordyniak, Solving integer linear programs by exploiting variable-constraint interactions: a survey, Algorithms 12 (12) (2019) 248, https://doi.org/10.3390/A12120248.

[62] E. Acar, G. Greco, M. Manna, Group reasoning in social environments, in: K. Larson, M. Winikoff, S. Das, E.H. Durfee (Eds.), Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2017, São Paulo, Brazil, May 8-12, 2017, ACM, 2017, pp. 1296–1304.

[63] C. Lafage, J. Lang, Logical representation of preferences for group decision making, in: A.G. Cohn, F. Giunchiglia, B. Selman (Eds.), KR 2000, Principles of Knowledge Representation and Reasoning - Proceedings of the Seventh International Conference, Breckenridge, Colorado, USA, April 11-15, 2000, Morgan Kaufmann, 2000, pp. 457–468.

[64] R. Cooper, Coordination Games, Cambridge University Press, 1999.

[65] D. Kempe, J.M. Kleinberg, É. Tardos, Maximizing the spread of influence through a social network, Theory Comput. 11 (4) (2015) 105–147, https://doi.org/10.4086/TOC.2015.V011A004.

[66] M. Fazli, M. Ghodsi, J. Habibi, P.J. Khalilabadi, V.S. Mirrokni, S.S. Sadeghabad, On non-progressive spread of influence through social networks, Theor. Comput. Sci. 550 (2014) 36–50, https://doi.org/10.1016/J.TCS.2014.07.009.
[67] V.Y. Lou, S. Bhagat, L.V.S. Lakshmanan, S. Vaswani, Modeling non-progressive phenomena for influence propagation, in: A. Sala, A. Goel, K.P. Gummadi (Eds.), Proceedings of the Second ACM Conference on Online Social Networks, COSN 2014, Dublin, Ireland, October 1-2, 2014, ACM, 2014, pp. 131–138.
[68] H. Aziz, R. Savani, Hedonic games, in: F. Brandt, V. Conitzer, U. Endriss, J. Lang, A.D. Procaccia (Eds.), Handbook of Computational Social Choice, Cambridge University Press, 2016, pp. 356–376.
[69] E. Elkind, A. Fanelli, M. Flammini, Price of Pareto optimality in hedonic games, Artif. Intell. 288 (2020) 103357, https://doi.org/10.1016/J.ARTINT.2020.103357.
[70] C. Ballester, NP-completeness in hedonic games, Games Econ. Behav. 49 (1) (2004) 1–30, https://doi.org/10.1016/J.GEB.2003.10.003.
[71] D. Peters, Complexity of hedonic games with dichotomous preferences, in: D. Schuurmans, M.P. Wellman (Eds.), Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence, February 12-17, 2016, AAAI Press, Phoenix, Arizona, USA, 2016, pp. 579–585.
[72] J. Nash, Non-cooperative games, Ann. Math. 54 (2) (1951) 286–295.
[73] N. Nisan, T. Roughgarden, É. Tardos, V.V. Vazirani (Eds.), Algorithmic Game Theory, Cambridge University Press, 2007.
[74] C.H. Papadimitriou, Computational Complexity, Addison-Wesley, 1994.
[75] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, 1979.
[76] L. Lovász, Coverings and colorings of hypergraphs, in: Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Mathematica, 1973, pp. 3–12.
[77] F. Maffray, M. Preissmann, On the NP-completeness of the k-colorability problem for triangle-free graphs, Discrete Math. 162 (1–3) (1996) 313–317, https://doi.org/10.1016/S0012-365X(97)89267-9.
[78] D. Monderer, L.S. Shapley, Potential games, Games Econ. Behav. 14 (1) (1996) 124–143, https://doi.org/10.1006/game.1996.0044.
[79] A. Schrijver, Theory of Linear and Integer Programming, John Wiley & Sons, Inc., New York, NY, USA, 1986.
[80] T. Johnson, N. Robertson, P.D. Seymour, R. Thomas, Directed tree-width, J. Comb. Theory, Ser. B 82 (1) (2001) 138–154, https://doi.org/10.1006/JCTB.2000.2031.
[81] H.L. Bodlaender, A linear-time algorithm for finding tree-decompositions of small treewidth, SIAM J. Comput. 25 (6) (1996) 1305–1317, https://doi.org/10.1137/S0097539793251219.
[82] S. Maniu, P. Senellart, S. Jog, An experimental study of the treewidth of real-world graph data, in: P. Barceló, M. Calautti (Eds.), Proceedings of the 22nd International Conference on Database Theory, ICDT 2019, Lisbon, Portugal, March 26–28, 2019, in: LIPIcs, vol. 127, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, pp. 12:1–12:18.
[83] G. Greco, F. Lupia, F. Scarcello, The tractability of the Shapley value over bounded treewidth matching games, in: C. Sierra (Ed.), Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, 2017, pp. 1046–1052, ijcai.org.
[84] C. Demetrescu, F. Lupia, A. Mendicelli, A. Ribichini, F. Scarcello, M. Schaerf, On the Shapley value and its application to the Italian VQR research assessment exercise, J. Informetr. 13 (1) (2019) 87–104, https://doi.org/10.1016/J.JOI.2018.11.008.
[85] V. Heun, E.W. Mayr, Embedding graphs with bounded treewidth into their optimal hypercubes, J. Algorithms 43 (1) (2002) 17–50, https://doi.org/10.1006/JAGM.2002.1217.
[86] B.M.P. Jansen, S. Kratsch, A structural approach to kernels for ILPs: treewidth and total unimodularity, in: N. Bansal, I. Finocchi (Eds.), Algorithms - Proceedings of the 23rd Annual European Symposium, ESA '15, Patras, Greece, September 14–16, 2015, in: Lecture Notes in Computer Science, vol. 9294, Springer, 2015, pp. 779–791.
[87] P. Harrenstein, W. Hoek, J.-j. Meyer, C. Witteveen, Boolean games, in: Proceedings of the 8th Conference on Theoretical Aspects of Rationality and Knowledge (TARK '01), Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2001, pp. 287–298.
[88] M.J. Wooldridge, U. Endriss, S. Kraus, J. Lang, Incentive engineering for Boolean games, Artif. Intell. 195 (2013) 418–439, https://doi.org/10.1016/J.ARTINT.2012.11.003.
[89] T. Ågotnes, P. Harrenstein, W. van der Hoek, M.J. Wooldridge, Verifiable equilibria in Boolean games, in: F. Rossi (Ed.), IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013, IJCAI/AAAI, 2013, pp. 689–695.
[90] A. Adiga, S. Kraus, S.S. Ravi, Boolean games: inferring agents' goals using taxation queries, in: A.E.F. Seghrouchni, G. Sukthankar, B. An, N. Yorke-Smith (Eds.), Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems, AAMAS '20, Auckland, New Zealand, May 9-13, 2020, International Foundation for Autonomous Agents and Multiagent Systems, 2020, pp. 1735–1737.