



Gödel–Dummett linear temporal logic

Juan Pablo Aguilera^{a,b}, Martín Diéguez^c, David Fernández-Duque^{a,d},
Brett McLean^{a,*}

^a Department of Mathematics WE16, Ghent University, Ghent, Belgium

^b Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria

^c LERIA, University of Angers, Angers, France

^d Department of Philosophy, University of Barcelona, Barcelona, Spain

ARTICLE INFO

Keywords:

Gödel–Dummett logic
Linear temporal logic
Automated temporal reasoning
PSPACE-complete
Hilbert-style deductive calculus

ABSTRACT

We investigate a version of linear temporal logic whose propositional fragment is Gödel–Dummett logic (which is well known both as a superintuitionistic logic and a t-norm fuzzy logic). We define the logic using two natural semantics: first a real-valued semantics, where statements have a degree of truth in the real unit interval, and second a ‘bi-relational’ semantics. We then show that these two semantics indeed define one and the same logic: the statements that are valid for the real-valued semantics are the same as those that are valid for the bi-relational semantics. This Gödel temporal logic does not have any form of the finite model property for these two semantics: there are non-valid statements that can only be falsified on an infinite model. However, by using the technical notion of a quasimodel, we show that every falsifiable statement is falsifiable on a finite quasimodel, yielding an algorithm for deciding if a statement is valid or not. Later, we strengthen this decidability result by giving an algorithm that uses only a polynomial amount of memory, proving that Gödel temporal logic is PSPACE-complete. We also provide a deductive calculus for Gödel temporal logic, and show this calculus to be sound and complete for the above-mentioned semantics, so that all (and only) the valid statements can be proved with this calculus.

1. Introduction

The importance of temporal logics and, independently, of fuzzy logics in computing is well established [1,2]. The potential usefulness of their combination is clear: for instance, it would provide a natural framework, in symbolic artificial intelligence, for the specification and verification of systems dealing with vague or incomplete data [3].

One of the most thoroughly studied fuzzy logics is *Gödel logic* (also called Gödel–Dummett logic) [4,5]. Gödel logic was introduced by Gödel [6] in his proof that intuitionistic logic is not finite-valued and later axiomatised by Dummett [7]. It is an extremely useful and natural framework because it is both a t-norm fuzzy logic and an extension of intuitionistic logic, thus capturing both the former’s approach to reasoning with vagueness as well as the latter’s approach to evidence-based reasoning. Gödel logic has been considered as a foundation for logic programming [8,9], answer set programming [10,11], and parallel λ -calculus [12,13].

* Corresponding author.

E-mail addresses: aguilera@logic.at (J.P. Aguilera), martin.dieguezlodeiro@univ-angers.fr (M. Diéguez), fernandez-duque@ub.edu (D. Fernández-Duque), brett.mclean@ugent.be (B. McLean).

<https://doi.org/10.1016/j.artint.2024.104236>

Received 27 June 2023; Received in revised form 8 October 2024; Accepted 13 October 2024

Available online 18 October 2024

0004-3702/© 2024 Elsevier B.V. All rights reserved, including those for text and data mining, AI training, and similar technologies.

Applications of fuzzy temporal reasoning in computer science, engineering, and artificial intelligence are numerous and include scheduling [14], robotics [15], quality control systems [16], and computer vision [17].

One very high-profile and promising current focus of artificial intelligence research is *autonomous driving*, where fuzzy logic is often the underlying framework for the controllers used to operate vehicles [18,19]. The use of fuzzy logic allows for the specification of control rules in natural language that can be written and understood by non-specialists. Rules applied to autonomous vehicles include if-then rules that are triggered under certain preconditions. For instance, these rules could include the following:

- (a) If the vehicle ahead is *close* and the car's speed is *high*, then the car must *decelerate*.

Usually the type of rules declared are reactive, i.e. the trigger depends only on the current state, and the action is implemented at the immediately following instant. Expressing rules whose evaluation implicates a possible infinite number of time instants is very complicated to do with existing approaches. Thus a more powerful framework is needed.

Perhaps the best-known and most successful formalism for temporal reasoning is *linear-time temporal logic* (or propositional linear-temporal logic; LTL) [20]: the extension of propositional logic with a variety of temporal modalities such as 'eventually', 'henceforth', and 'until'. Indeed the success of LTL in program and systems verification resulted in the 1996 Turing Award being conferred on Pnueli.

Several variants of LTL lacking the classical, binary conception of truth have been investigated, with a variety of application domains. In logic programming and non-monotonic reasoning, equilibrium logic [11] was given an LTL extension in [21,22], based on the Gödel logic with three truth values G_3 (also known as the logic of here-and-there [23]). Such an LTL extension of G_3 is axiomatised in [24]. Combinations of LTL with intuitionistic propositional logic have been considered as a framework for reasoning about dynamical systems [25]. Combinations of fuzzy logic with LTL have been used to enable vague time references, for example in [26,27]. Those approaches have many fields of application: disaster management, robot control, or smart home systems; that is, to dynamic systems whose evolution depends on uncertain conditions [28].

Motivated by the many applications of Gödel logic and its temporal extensions, particularly LTL extensions, in this paper we provide a detailed investigation of a linear-temporal extension of the real-valued Gödel logic (where degrees of truth can take any value in the unit interval $[0, 1]$). Specifically, we investigate a temporal extension of propositional Gödel logic whose syntax can express all the familiar LTL *future* and *past* modalities, 'until', 'henceforth', 'since', and so on. We call this logic GTL (short for *Gödel temporal logic*). This investigation extends that of [29,30], whose results apply to the language with only the *unary*, *future* modalities 'eventually' and 'henceforth'.

This language allows us to express statements whose satisfaction may require arbitrarily large amounts of time. For instance, we can express rules like:

- (b) If the petrol tank is *low*, it will remain low *until* either the tank is refilled or it becomes empty.
- (c) If *rockfall* is observed near Moutiers, then *henceforth* drive with caution in the vicinity of Moutiers.

Unlike example (a), expressing (b) and (c) requires a language with both a vague component and an elaborate temporal component capable of referring to actions potentially in the distant future.

The need for reasoning over potentially unbounded periods of time arises in many other applications. Consider a fraud-detection system for credit card usage. Suspicious activity may prompt a system to automatically require the user to re-enter their PIN, with such activity potentially occurring over long periods of time. Whether a given event is suspicious involves a fuzzy component, as for example the card being used in a different continent is more suspicious than it being used in a different city. The level of alert also varies and may thus be regarded as fuzzy, prompting the PIN request once a certain level of alert is reached. A typical rule in such a scenario may be:

- (d) If suspicious activity has been observed since the PIN was last entered, raise the level of alert.

In this setting we need a framework powerful enough to express the notion that a certain quantity is changing. This, as we will see, can be captured with the co-implication connective of bi-intuitionistic logic [56] (see Example 2.2).

GTL possesses two natural semantics, corresponding to whether it is viewed as a fuzzy logic or a superintuitionistic logic. As a fuzzy logic, propositions take values in $[0, 1]$, and truth values of compound propositions are defined using standard operations on the reals. As a superintuitionistic logic, models consist of *bi-relational structures*: a set equipped with a partial order to interpret implication as on an intuitionistic Kripke frame and a function to interpret the LTL tenses. Notably, the two semantics give rise to the same set of valid formulas (Theorem 3.3).

One may be concerned that the passage from two-valued to infinite-valued LTL could lead to an explosion in computational complexity, as has been known to happen when combining sub-classical logics with transitive modal logics: for these combinations it is often algorithmically undecidable if a formula is valid [31], or decidable with only a superexponential space upper bound being known [32]. As we will see, this need not be the case: our combination of Gödel–Dummett logic with linear temporal logic remains PSPACE-complete (Theorem 11.11), the minimal possible complexity given that classical LTL embeds into it. This is true even when the syntax is enriched with the dual implication connective [33], which has been argued in [34] to be useful for reasoning with incomplete or inconsistent information. In contrast, the LTL extension of equilibrium logic mentioned earlier is EXSPACE-complete [35].

Even the decidability of GTL is quite surprising, since its modal companion¹ $S4.3 \times \text{LTL}$ is not recursively axiomatisable [37], and, as we show, GTL does not enjoy the finite model property for either of its two semantics (i.e. there are non-valid formulas whose falsifiability can only be seen using an infinite model).

For Gödel logics, it is often possible to prove decidability despite the lack of finite model property by considering alternative semantics (see, e.g., [38]). For example, the logic GS4 does not have the real-valued finite model property, but it does have the bi-relational finite model property [32]. Since GTL does not enjoy *either* version of the finite model property, we instead introduce *quasimodels*, which do satisfy their own version of the finite model property. Quasimodels are not ‘true’ models, because the relation indicating the ‘next’ instant of time is no longer necessarily a function, but quasimodels give rise to standard bi-relational models by *unwinding*. Unwinding consists of transforming a structure into a new one whose worlds are *paths* selected from the original, a variant of the classic construction used for establishing the tree model property for standard modal logics [39]. Quasimodels are a well-established tool in proving e.g. decidability results for modal logics [39], but ours are specifically based on a variant tailored for proving upper complexity bounds for dynamic topological logic [40,41] and intuitionistic temporal logic [25]. These turn out to be particularly effective in the setting of Gödel temporal logic, yielding the optimal PSPACE upper bound.

A natural deductive calculus for GTL can readily be defined by combining axioms for Gödel logic (with dual implication) together with axioms for linear temporal logic. However, as in the classical setting, a standard attempt to prove the completeness of these axioms via a *canonical model* will not yield a structure where the tenses are interpreted correctly. For LTL, one may use the technique of *filtration* to obtain a finite model and then use the method of associating a *characteristic formula* to each world to characterise it up to bisimulation, using these to reason that infinitary tenses (such as ‘henceforth’) are indeed interpreted correctly (see, e.g., [42]). For GTL, we are able to prove completeness by using quasimodels to provide an analogue of the finite filtrated model (Theorem 10.4), with the additional complication that in this sub-classical setting it is necessary to assign *two* characteristic formulas to each world.

Related work

As already mentioned, many different extensions of temporal reasoning beyond the classical Boolean propositional framework have been studied, for many different reasons. Here we briefly describe the work most closely related to our own, and highlight the relationships with our approach and results.

In [43], Baaz and Preining consider a fragment of monadic *first-order* LTL. By a reduction from monadic first-order Gödel logic, they show that the valid sentences of this monadic first-order LTL are not computably enumerable. This makes our own result all the more surprising: in the classical setting, monadic first-order (indeed, second-order) logic is already decidable, from which decidability for LTL follows, although a PSPACE complexity bound requires additional proof. In the Gödel setting, we lose the decidability of monadic first-order logic, suggesting that Gödel LTL should at the very least also suffer in terms of complexity. Yet this is not the case, and the PSPACE upper bound is preserved.

Almagor et al. [44] introduce the parameterised family $\text{LTL}[F]$ of logics for reasoning about the *quality* of classical, Boolean computations. Quality is intended to be a measure of *how well* a computation meets a logically formalised specification. Accordingly, valuations yield values in the unit interval. The temporal operators are those of LTL, but the parameter F can be any set of (computable) functions over $[0, 1]$, which replace, in the syntax and semantics, the Boolean propositional operators of LTL. Thus F may contain the connectives of Gödel–Dummett logic, in which case GTL will *appear* to be a syntactic fragment of $\text{LTL}[F]$. However, since it is Boolean computational events that are modelled, there is the strong restriction that *atomic* propositions take values in $\{0, 1\}$. Since Gödel–Dummett connectives act classically on $\{0, 1\}$, the ‘GTL fragment’ of $\text{LTL}[F]$ has classical LTL semantics. Hence, contrary to initial appearances, $\text{LTL}[F]$ does not subsume GTL.

In [44], the authors investigate the validity problem (among others) for $\text{LTL}[F]$, and they determine (by means of automata-based techniques) that this is in PSPACE (assuming polynomial-space computability of the functions in F). Bouyer et al. [45] further extend $\text{LTL}[F]$ to $\text{SL}[F]$ by replacing LTL with *strategy logic*.

Temporal equilibrium logic [22] is a non-monotonic LTL that is one of the most popular semantics for temporal answer set programming. Its definition consists of two main components: a monotonic basis in terms of an LTL extension of the Gödel logic G_3 and a selection criterion to select minimal models. Such a selection criterion can be seen as a second-order quantification over the set of temporal G_3 models. As a result, the complexity of the satisfiability problem of temporal equilibrium logic turns out to be EXPSPACE [35]. There exists, nevertheless, an interesting fragment, called *splittable temporal logic programs* [46], whose complexity remains open. The approach presented in this paper could help to determine the complexity of the satisfiability problem for the case of such programs.

Finally, there have been several investigations into fuzzy *interval temporal logics*, for example [47,48]. This work focuses on the study of fuzzy time periods and reasoning about them (e.g. “ X ends before Y begins”). They do not work with a non-classical implication, but rather focus on disjunctions of atomic expressions. Accordingly, the complexity of the tasks they consider is NP. In principle, the relations they consider can be represented in a language such as ours, although they work in continuous time. Indeed, a continuous-time analogue of our framework would be a natural avenue for future exploration.

Structure of the article

Section 2: Syntax and semantics. We introduce the temporal language that we work with, and then introduce both the real-valued semantics and bi-relational semantics for Gödel temporal logic.

¹ Each superintuitionistic logic L has a family of *modal companions*: a normal modal logic M such that $\varphi \in L \iff T(\varphi) \in M$, where T is the well-known Gödel–McKinsey–Tarski translation [36]. The smallest modal companion of (propositional) Gödel logic is $S4.3$. Similarly, $\varphi \in \text{GTL} \iff T(\varphi) \in S4.3 \times \text{LTL}$ (where the modality introduced by the translation should be distinct from the GTL modalities). This is immediate from the bi-relational semantics of GTL (Definition 2.5).

Section 3: Real-valued versus bi-relational validity. We prove the equivalence of these two semantics, that is, that they yield the same validities (Theorem 3.3).

Section 4: Labelled systems and quasimodels. We first note that we do not have a finite model property for either of these semantics. But then we define quasimodels (Definition 4.6), and in later sections show that our Gödel temporal logic is sound and complete for the class of finite quasimodels.

Section 5: From quasimodels to bi-relational models. We show that Gödel temporal logic is sound for (all) quasimodels, constructing a bi-relational model from an arbitrary quasimodel by unwinding selected paths within the quasimodel.

Section 6: From bi-relational models to finite quasimodels. Given a bi-relational model falsifying a formula, we use a quotient construction to produce a finite (exponential in the length of the formula) quasimodel also falsifying the formula. This completes the proof that the semantics of Gödel temporal logic can be reduced to finite quasimodels, yielding the decidability of Gödel temporal logic (Theorem 6.7).

Section 7: A Hilbert-style deductive calculus. Introduces our Hilbert calculus for the valid formulas of Gödel temporal logic.

Section 8: The canonical system and canonical quasimodel. Introduces the canonical system, essentially a standard canonical model familiar from modal logic. However, this system is not a ‘true’ model, so we proceed to construct the canonical quasimodel by re-using the quotient construction from Section 6.

Section 9: Characteristic formulas. For the purpose of proving that the canonical quasimodel is indeed a quasimodel, we define characteristic formulas, which define models locally up to bisimulation.

Section 10: Completeness. Characteristic formulas are used to prove that the canonical quasimodel satisfies all required properties of a quasimodel, from which completeness of our axiomatisation follows (Theorem 10.4).

Section 11: PSPACE-complete complexity. We refine our decidability result, showing that Gödel temporal logic is in fact PSPACE-complete (Theorem 11.11).

Section 12: Concluding remarks. We summarise our results, contrast them with the properties of similar logics, and suggest some directions for future work.

There are several routes through the paper for readers interested only in a portion of the results. Section 2 is common background needed for the rest of the paper. Section 3 may be skipped by readers not interested in real-valued semantics, or alternatively read after the rest of the paper. Sections 5–6 provide a full proof of decidability, and readers interested only in complexity may then skip to Section 11. Readers interested in completeness of the Hilbert calculus should read Sections 5–10; note that the proof of completeness builds on the proof of decidability.

2. Syntax and semantics

In this section we first introduce the temporal language we work with and then two possible semantics for this language: real-valued semantics and bi-relational semantics.

Fix a countably infinite set \mathbb{P} of propositional variables. Then the **Gödel temporal language** \mathcal{L} is defined by the grammar (in Backus–Naur form):

$$\varphi, \psi := p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \varphi \Leftarrow \psi \mid X\varphi \mid Y\varphi \mid G\varphi \mid H\varphi \mid \varphi \cup \psi \mid \varphi S \psi,$$

where $p \in \mathbb{P}$. The intended meaning of the temporal modalities is the following: $X\varphi$ means that φ is true at the ‘next’ time instant; $Y\varphi$, read as ‘yesterday φ ’, means that φ holds at the previous time instant; $G\varphi$, read as ‘going to always be φ ’, means that φ holds at all future time instants; $H\varphi$, read as ‘has always been φ ’, means that φ holds at all past time instants; $\varphi \cup \psi$ means that ‘ φ holds until ψ holds’, and $\varphi S \psi$ means that ‘ φ holds since ψ holds’. The connective \Leftarrow is called ‘co-implication’ and it is the order dual to standard (intuitionistic) implication [49], in much the same way as disjunction is dual to conjunction. Roughly, $\varphi \Leftarrow \psi$ states that the implication $\psi \Rightarrow \varphi$ would fail if all truth values were ‘turned upside down’ (see Definition 2.3). We also use the following abbreviations:

- \top abbreviates $p \Rightarrow p$, for some fixed, but unspecified, $p \in \mathbb{P}$;
- \perp abbreviates $p \Leftarrow p$;
- $\neg\varphi$ abbreviates $\varphi \Rightarrow \perp$;
- $\varphi \Leftrightarrow \psi$ abbreviates $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (not the formula $(\varphi \Rightarrow \psi) \wedge (\varphi \Leftarrow \psi)$);
- $F\varphi$, read as ‘future’, abbreviates $\top \cup \varphi$;
- $P\varphi$, read as ‘past’, abbreviates $\top S \varphi$.

Although we will not need them, note that we can also define ‘release’ as $\varphi R \psi := (\psi \cup \varphi) \vee G\psi$ and the past analog of release similarly. Often the alternative names ‘eventually’ and ‘henceforth’ are used for ‘future’ and ‘going’ respectively.

Example 2.1. The rules for an autonomous vehicle presented in the introduction correspond to the following temporal formulas:

(a) $G(\text{aheadClose} \wedge \text{speedHigh} \Rightarrow \text{decelerate})$;

- (b) $G(\text{tankLow} \Rightarrow (\text{tankLow} \cup (\text{refill} \vee \text{tankEmpty})))$;
 (c) $G(\text{rockfall} \wedge \text{moutiers} \Rightarrow G(\text{moutiers} \Rightarrow \text{caution}))$.

Example 2.2. In the setting of a credit card fraud-detection system, we considered a rule stating that the level of alert must be raised if suspicious activity has been recorded since the PIN was last requested. This may be formalised as

- (d) $(\text{suspicious} \Rightarrow (X\text{alert} \Leftarrow \text{alert}))\text{SPIN}$

Here, the expression $X\text{alert} \Leftarrow \text{alert}$ is crucial, as it indicates that the level of alert should be *strictly* higher at the next moment. Properties of this sort (i.e., values changing or being different from each other) are expressed using co-implication.

We now introduce the first of our semantics for the Gödel temporal language: *real-valued semantics*, which views \mathcal{L} as a *fuzzy logic* (enriched with temporal modalities). In the definition, $[0, 1]$ denotes the real unit interval.

Definition 2.3 (*real-valued semantics*). A **flow** is a pair $\mathcal{T} = (T, S)$, where T is a set and $S : T \rightarrow T$ is a bijection. A **real valuation** on \mathcal{T} is a function $V : \mathcal{L} \times T \rightarrow [0, 1]$ such that, for all $t \in T$, the following equalities hold, where \mathbb{N} denotes the set of non-negative integers.

$$\begin{aligned}
 V(\varphi \wedge \psi, t) &= \min\{V(\varphi, t), V(\psi, t)\} & V(\varphi \vee \psi, t) &= \max\{V(\varphi, t), V(\psi, t)\} \\
 V(\varphi \Rightarrow \psi, t) &= \begin{cases} 1 & \text{if } V(\varphi, t) \leq V(\psi, t) \\ V(\psi, t) & \text{otherwise} \end{cases} & V(\varphi \Leftarrow \psi, t) &= \begin{cases} 0 & \text{if } V(\varphi, t) \leq V(\psi, t) \\ V(\varphi, t) & \text{otherwise} \end{cases} \\
 V(X\varphi, t) &= V(\varphi, S(t)) & V(Y\varphi, t) &= V(\varphi, S^{-1}(t)) \\
 V(G\varphi, t) &= \inf_{n \in \mathbb{N}} V(\varphi, S^n(t)) & V(H\varphi, t) &= \inf_{n \in \mathbb{N}} V(\varphi, S^{-n}(t)) \\
 V(\varphi \cup \psi, t) &= \sup_{n \in \mathbb{N}} \min\{V(\varphi, S^0(t)), \dots, V(\varphi, S^{n-1}(t)), V(\psi, S^n(t))\} \\
 V(\varphi \text{ S } \psi, t) &= \sup_{n \in \mathbb{N}} \min\{V(\varphi, S^0(t)), \dots, V(\varphi, S^{-(n-1)}(t)), V(\psi, S^{-n}(t))\}
 \end{aligned}$$

A flow \mathcal{T} equipped with a valuation V is a **real-valued (Gödel temporal) model**.

Informally, the real number $V(\varphi, t)$ records the degree of truth of φ at time t .

Example 2.4. Following on from Example 2.1, let us examine the semantic conditions imposed by the first two formulas.

Let $\mathcal{M} = (T, S, V)$ be a real-valued model and $t \in T$. Suppose that

$$V(G(\text{aheadClose} \wedge \text{speedHigh} \Rightarrow \text{decelerate}), t) = 1.$$

By the semantic clause for G , this may only occur if

$$V(\text{aheadClose} \wedge \text{speedHigh} \Rightarrow \text{decelerate}, S^n(t)) = 1$$

for all $n \in \mathbb{N}$, which is equivalent to the inequality

$$\min\{V(\text{aheadClose}, S^n(t)), V(\text{speedHigh}, S^n(t))\} \leq V(\text{decelerate}, S^n(t)).$$

In other words, the vehicle must decelerate with a higher intensity when it has a higher speed and a vehicle is nearby.

We may perform a similar analysis to check when

$$V(G(\text{tankLow} \Rightarrow (\text{tankLow} \cup (\text{refill} \vee \text{tankEmpty}))), t) = 1.$$

This formula holds when $V(\text{tankLow}, S^n(t))$ is non-decreasing with respect to n , until a value m is reached with

$$V(\text{tankLow}, S^m(t)) \leq \max\{V(\text{refill}, S^m(t)), V(\text{tankEmpty}, S^m(t))\}.$$

As autonomous vehicles presumably are programmed to avoid an empty tank, this prompts the vehicle to eventually refill.

Note that in this example, tankEmpty should arguably be Boolean, but this is readily incorporated into our framework by replacing it by $\neg \text{tankEmpty}$, which always takes values in $\{0, 1\}$ (other combinations such as $\sim \text{tankEmpty}$ or $\sim \neg \text{tankEmpty}$ will also do the trick). This allows us to combine classical and fuzzy reasoning without having to extend our framework.

The second semantics, *bi-relational semantics*, views \mathcal{L} as a (consistent) *superintuitionistic logic*, temporally enriched. Below, define $\tilde{S}(w, t) = (w, S(t))$, and if X is a subset of the domain of \tilde{S} , then $\tilde{S}X = \{\tilde{S}(x) \mid x \in X\}$.

Definition 2.5 (*bi-relational semantics*). A **(Gödel temporal) bi-relational frame** is a quadruple $\mathcal{F} = (W, T, \leq, S)$ where (W, \leq) is a linearly ordered set and (T, S) is a flow. A **bi-relational valuation** on \mathcal{F} is a function $\llbracket \cdot \rrbracket : \mathcal{L} \rightarrow 2^{W \times T}$ such that, for each $p \in \mathbb{P}$, the set $\llbracket p \rrbracket$ is *downward closed* in its first coordinate, and the following equalities hold.

$$\begin{aligned} \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket & \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket \\ \llbracket \varphi \Rightarrow \psi \rrbracket &= \{(w, t) \in W \times T \mid \forall v \leq w ((v, t) \in \llbracket \varphi \rrbracket \text{ implies } (v, t) \in \llbracket \psi \rrbracket)\} \\ \llbracket \varphi \Leftarrow \psi \rrbracket &= \{(w, t) \in W \times T \mid \exists v \geq w ((v, t) \in \llbracket \varphi \rrbracket \text{ and } (v, t) \notin \llbracket \psi \rrbracket)\} \\ \llbracket X\varphi \rrbracket &= \bar{S}^{-1} \llbracket \varphi \rrbracket & \llbracket Y\varphi \rrbracket &= \bar{S} \llbracket \varphi \rrbracket \\ \llbracket G\varphi \rrbracket &= \bigcap_{n \in \mathbb{N}} \bar{S}^{-n} \llbracket \varphi \rrbracket & \llbracket H\varphi \rrbracket &= \bigcap_{n \in \mathbb{N}} \bar{S}^n \llbracket \varphi \rrbracket \\ \llbracket \varphi \cup \psi \rrbracket &= \bigcup_{n \in \mathbb{N}} \left(\bar{S}^{-n} \llbracket \psi \rrbracket \cap \bigcap_{0 \leq i < n} \bar{S}^{-i} \llbracket \varphi \rrbracket \right) & \llbracket \varphi \text{ S } \psi \rrbracket &= \bigcup_{n \in \mathbb{N}} \left(\bar{S}^n \llbracket \psi \rrbracket \cap \bigcap_{0 \leq i < n} \bar{S}^i \llbracket \varphi \rrbracket \right) \end{aligned}$$

A bi-relational frame \mathcal{F} equipped with a valuation $\llbracket \cdot \rrbracket$ is a **(Gödel temporal) bi-relational model**.

Informally, the set $\{w \in W \mid (w, t) \in \llbracket \varphi \rrbracket\}$ records the degree of truth of φ at time t . This semantics combines standard intuitionistic Kripke frame semantics for the implications based on the order \leq (read downward) and standard semantics for the tenses based on S : for example, $(w, t) \in \llbracket \varphi \cup \psi \rrbracket$ if and only if there exists $n \geq 0$ such that $(w, S^i(t)) \in \llbracket \varphi \rrbracket$ for all $0 \leq i < n$, and $(w, S^n(t)) \in \llbracket \psi \rrbracket$. Note that by structural induction, the valuation of *any* formula φ in \mathcal{L} is downward closed in its first coordinate, in the sense that if $(w, t) \in \llbracket \varphi \rrbracket$ and $v \leq w$, then $(v, t) \in \llbracket \varphi \rrbracket$.

For both real-valued semantics and bi-relational semantics, validity of \mathcal{L} -formulas is defined in the usual way.

Definition 2.6 (*validity*). Given a real-valued model $\mathcal{X} = (T, S, V)$ and a formula $\varphi \in \mathcal{L}$, we say that φ is **globally true** on \mathcal{X} , written $\mathcal{X} \models \varphi$, if for all $t \in T$ we have $V(\varphi, t) = 1$. Given a bi-relational model $\mathcal{X} = (\mathcal{F}, \llbracket \cdot \rrbracket)$ and a formula $\varphi \in \mathcal{L}$, we say that φ is **globally true** on \mathcal{X} , written $\mathcal{X} \models \varphi$, if $\llbracket \varphi \rrbracket = W \times T$.

If \mathcal{X} is a flow or a bi-relational frame, we write $\mathcal{X} \models \varphi$ and say φ is **valid** on \mathcal{X} , if φ is globally true for every valuation on \mathcal{X} . If Ω is a class of flows, frames, or models, we say that $\varphi \in \mathcal{L}$ is **valid** on Ω if, for every $\mathcal{X} \in \Omega$, we have $\mathcal{X} \models \varphi$. If φ is not valid on Ω , it is **falsifiable** on Ω .

We define:

- the logic $\text{GTL}_{\mathbb{R}}$ to be the set of \mathcal{L} -formulas that are valid over the class of all *flows*;
- the logic GTL_{rel} to be the set of \mathcal{L} -formulas that are valid over the class of all *bi-relational frames*.

3. Real-valued versus bi-relational validity

In this section, we show that an arbitrary \mathcal{L} -formula is real valid if and only if it is bi-relationally valid. That is, $\text{GTL}_{\mathbb{R}} = \text{GTL}_{\text{rel}}$. The two directions of this equivalence are Lemma 3.1 and Lemma 3.2 below.

Lemma 3.1. *Suppose that φ is an \mathcal{L} -formula that is not real valid. Then φ is not bi-relationally valid.*

Proof. Let (T, S) be a flow, V a real valuation on (T, S) , and $t_0 \in T$ be such that $V(\varphi, t_0) < 1$. Since we are only concerned with the valuation at t_0 , we may assume without loss of generality that $T = \mathbb{Z}$, $t_0 = 0$, and $S(x) = x + 1$; in particular, that T is countable. Let X_0 be the set of all real numbers x such that $V(\psi, t) = x$ for some \mathcal{L} -formula ψ and some $t \in T$. Thus, X_0 is a countable subset of $[0, 1]$. Let $X = (0, 1) \setminus X_0$.

We consider the bi-relational frame $\mathcal{F} = (X, T, \leq, S)$, where \leq is the usual order on real numbers, and the bi-relational valuation $\llbracket \cdot \rrbracket$ given by

$$(x, t) \in \llbracket p \rrbracket \text{ if and only if } V(p, t) > x. \quad (1)$$

We then prove by induction that (1) holds for arbitrary \mathcal{L} -formulas ψ and arbitrary $x \in X$ and $t \in T$. Then letting $x \in X$ be such that $V(\varphi, t_0) < x$ (this exists because X_0 is countable), we have $(x, t_0) \notin \llbracket \varphi \rrbracket$, so that $(\mathcal{F}, \llbracket \cdot \rrbracket)$ is a bi-relational countermodel for φ , as needed. \square

Lemma 3.2. *Suppose that φ is an \mathcal{L} -formula that is not bi-relationally valid. Then φ is not real valid.*

Proof. Suppose that there is a bi-relational frame $\mathcal{F} = (W, T, \leq, S)$ and a valuation $\llbracket \cdot \rrbracket$ such that $(w, t_0) \notin \llbracket \varphi \rrbracket$. As above, we may assume that $T = \mathbb{Z}$, that S is the standard successor function, and that $t_0 = 0$. By a routine downward Löwenheim–Skolem-type

argument, we may assume that W is countable. To do so, let us build a suborder (W^*, \leq^*) of (W, \leq) by induction on the structure of φ as follows: start with $\{w\}$ and inductively decompose φ according to its outermost connective. When considering subformulas ψ of the form $\varphi_0 \Rightarrow \varphi_1$ or $\varphi_0 \Leftarrow \varphi_1$, we add to W^* (if necessary), for each world w^* in the suborder being built and for each $s \in T$, a new element of W witnessing the quantifier in the definition of $\llbracket \psi \rrbracket$ in a way that ensures that ' $(w^*, s) \in \llbracket \psi \rrbracket$ ' holds in (W, T, \leq, S) if and only if it holds in (W^*, T, \leq^*, S) .

We define a binary relation on $\mathcal{L} \times T$ by

$$(\psi, t_1) \leq (\chi, t_2) \text{ if and only if } \forall w \in W ((w, t_1) \in \llbracket \psi \rrbracket \implies (w, t_2) \in \llbracket \chi \rrbracket),$$

that is, if the valuation of ψ at t_1 is contained in that of χ at t_2 . Since valuations of formulas are downward closed in their first coordinates, this reflexive transitive relation is total (any two elements are comparable). Let (L, \leq) denote the linear order of all equivalence classes $[\psi, t]$ under \leq , and let 0 and 1 denote the equivalence classes of $[\perp, t]$ and $[\top, t]$, respectively (these are independent of the choice of t).

We claim that this linear order respects valuations, in the sense that it satisfies the following properties for each $t \in T$:

- 0 is the least element and 1 is the greatest element of L ;
- $[\psi \wedge \chi, t] = \min\{[\psi, t], [\chi, t]\}$ and $[\psi \vee \chi, t] = \max\{[\psi, t], [\chi, t]\}$;
- $[\chi \Rightarrow \psi, t] = 1$ if $[\chi, t] \leq [\psi, t]$, and $[\chi \Rightarrow \psi, t] = [\psi, t]$ otherwise;
- $[\chi \Leftarrow \psi, t] = 0$ if $[\chi, t] \leq [\psi, t]$, and $[\chi \Leftarrow \psi, t] = [\chi, t]$ otherwise;
- $[X\psi, t] = [\psi, t + 1]$ and $[Y\psi, t] = [\psi, t - 1]$;
- $[G\psi, t] = \inf_{n \in \mathbb{N}} [\psi, t + n]$ and $[H\psi, t] = \inf_{n \in \mathbb{N}} [\psi, t - n]$;
- $[\varphi \cup \psi, t] = \sup_{n \in \mathbb{N}} \min\{[\varphi, t], \dots, [\varphi, t + (n - 1)], [\psi, t + n]\}$;
- $[\varphi \cap \psi, t] = \sup_{n \in \mathbb{N}} \min\{[\varphi, t], \dots, [\varphi, t - (n - 1)], [\psi, t - n]\}$.

These properties easily follow from the definitions.

Now, (L, \leq) is a countable linear order with endpoints, so it can be continuously (with respect to the order topology) embedded into the interval $[0, 1]$ in such a way that the images of 0 and 1 are, respectively, 0 and 1. Let ρ be such an embedding. (We refer the reader to for example the proof of [5, Theorem 5.1] for an explicit construction of such a ρ .) We define a real valuation V by setting $V(\psi, t) = \rho([\psi, t])$. By the properties above, V is indeed a real valuation and $V(\varphi, t_0) = \rho([\varphi, t_0]) < 1$. \square

Theorem 3.3. $\text{GTL}_{\mathbb{R}} = \text{GTL}_{\text{rel}}$.

4. Labelled systems and quasimodels

Our decidability proof for the set of Gödel temporal validities $\text{GTL}_{\mathbb{R}}$ is based on (nondeterministic) quasimodels. In this section we will introduce labelled spaces, labelled systems, and finally, quasimodels. Quasimodels can be viewed as a sort of temporally nondeterministic generalisation of bi-relational models.

The method of quasimodels was originally introduced in several papers on related to combinations of description logics with modal and temporal logics [50,51] and later extended to products of modal logics in [52]. In [53] quasimodels are a key tool to prove decidability in a wide range of products of modal logics. Regarding constructive extensions of temporal logics, the method of quasimodels was also extended in [40] for *dynamic topological logic*, a classical predecessor of *intuitionistic temporal logic*, for which quasimodels were also used in [25]. As the bi-relational semantics makes evident, Gödel temporal logic is closely related to intuitionistic temporal logic.

Definition 4.1. The **strong finite model property** for GTL_{rel} is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a bi-relational model, then it is falsifiable on a bi-relational model $\mathcal{F} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ where both W and T are finite.

The **order finite model property** for GTL_{rel} is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a bi-relational model, then it is falsifiable on a bi-relational model $\mathcal{F} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ where W is finite.

The **temporal finite model property** for GTL_{rel} is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a bi-relational model, then it is falsifiable on a bi-relational model $\mathcal{F} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ where T is finite.

Of course, many decidability proofs for classical modal logics are obtained via the finite model property, so it is worthwhile to first note that this strategy cannot work for $\text{GTL}_{\mathbb{R}}$ because finite model properties do not hold, in either semantics. The finite model properties we define are of the form: *falsifiable* implies *falsifiable in a finite model*. It is worth remarking that in sub-classical logics, it is indeed the notion of falsifiability that is relevant, as it is falsifiability that is (by definition) the complement to validity. However, in view of our inclusion of co-implication, we may define $\sim\varphi \equiv T \Leftarrow \varphi$, and then it is not hard to check that $\sim\varphi$ is *satisfiable* (in the sense of having a non-zero truth value) if and only if φ is falsifiable. Moreover, as we will see, our logic is PSPACE-complete, and this class is closed under complements. Thus validity, satisfiability, and falsifiability for $\text{GTL}_{\mathbb{R}}$ are all inter-reducible.

Proposition 4.2. None of the finite model properties for GTL_{rel} listed in Definition 4.1 hold. In particular, $F(p \Rightarrow Xp)$ is falsifiable, yet it is valid over the class of finite bi-relational models.

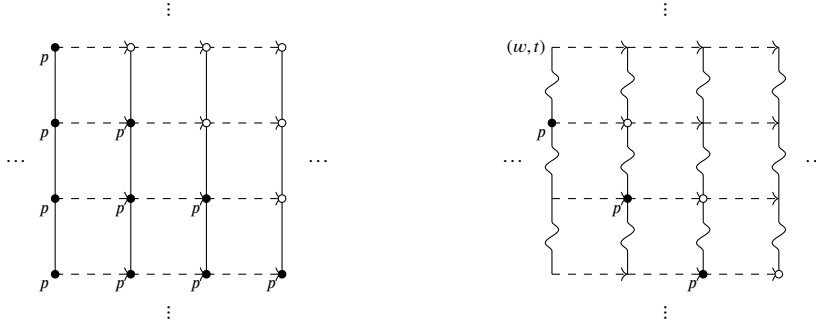


Fig. 1. Left: A bi-relational model falsifying $F(p \Rightarrow Xp)$; right: W and T are necessarily infinite. Time is represented by the *horizontal* axis and truth by the *vertical*.

Proof. To see that $F(p \Rightarrow Xp)$ is falsifiable, take the flow (\mathbb{Z}, S) , where S is the standard successor function. Consider the model $(\mathbb{Z}, \mathbb{Z}, \leq, S, \llbracket \cdot \rrbracket)$, where $\llbracket p \rrbracket = \{(n, t) \in \mathbb{Z} \times \mathbb{Z} \mid n \leq -t\}$. (See Fig. 1 (left).) Then at each $(-i, i)$, the formula p holds but Xp does not. Hence at each $(0, i)$ with $i \in \mathbb{Z}_{\geq 0}$, the formula $p \Rightarrow Xp$ is falsified. Thus $F(p \Rightarrow Xp)$ is falsified at $(0, 0)$.

To see that $F(p \Rightarrow Xp)$ can only be falsified on a model $(W, T, \leq, S, \llbracket \cdot \rrbracket)$ for which both W and T are infinite, suppose $F(p \Rightarrow Xp)$ is falsified at (w, t) . Then there must be a sequence $w \geq w_0 > w_1 > \dots$ such that for each $i \in \mathbb{Z}_{\geq 0}$, the formula p holds on each $(w_i, S^i(t))$ but not on $(w_i, S^{i+1}(t))$. This clearly forces W to be infinite, and by downward closure of $\llbracket \cdot \rrbracket$, it forces T to be infinite too. (See Fig. 1 (right).) \square

The same example as in Proposition 4.2 shows that under real-valued semantics it is also the case that some formulas can only be falsified on an infinite flow with infinitely many realised truth values, as falsification of $F(p \Rightarrow Xp)$ forces $V(p, t) > V(p, S(t))$ for all t .

Note that we have refuted all these finite model properties without using many of the connectives (in particular without \Leftarrow and without any past connectives), thus in fact proving the stronger result that the finite model properties fail for the fragment standardly used for (future time) linear temporal logic.

We now begin to introduce the structures we will use to mitigate the failure of these finite model properties.

Definition 4.3. Given a set $\Sigma \subseteq \mathcal{L}$ that is closed under subformulas, we say that $\Phi \subseteq \Sigma$ is a Σ -**type** if the following occur.

1. If $\varphi \wedge \psi \in \Sigma$, then $\varphi \wedge \psi \in \Phi$ if and only if $\varphi \in \Phi$ and $\psi \in \Phi$.
2. If $\varphi \vee \psi \in \Sigma$, then $\varphi \vee \psi \in \Phi$ if and only if either $\varphi \in \Phi$ or $\psi \in \Phi$.
3. If $\varphi \Rightarrow \psi \in \Sigma$, then
 - (a) $\varphi \Rightarrow \psi \in \Phi$ implies either $\varphi \notin \Phi$ or $\psi \in \Phi$,
 - (b) $\psi \in \Phi$ implies $\varphi \Rightarrow \psi \in \Phi$.
4. If $\varphi \Leftarrow \psi \in \Sigma$, then
 - (a) $\varphi \Leftarrow \psi \in \Phi$ implies $\varphi \in \Phi$,
 - (b) $\varphi \in \Phi$ and $\psi \notin \Phi$ implies $\varphi \Leftarrow \psi \in \Phi$.

The set of Σ -types will be denoted by \mathbb{T}_Σ .

A partially ordered set (A, \leq) is **locally linear** if it is a disjoint union of linear posets. If $a, b \in A$, we write $a \lesseqgtr b$ if $a \leq b$ or $b \leq a$. We call the set $\{b \in A : b \lesseqgtr a\}$ the **linear component** of a ; by assumption, the linear components partition A .

Definition 4.4. Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas. A Σ -**labelled space** is a triple $\mathcal{W} = (W, \leq_{\mathcal{W}}, \ell_{\mathcal{W}})$, where $(W, \leq_{\mathcal{W}})$ is a locally linear poset and $\ell : W \rightarrow \mathbb{T}_\Sigma$ an anti-monotone function, in the sense that

$$w \leq_{\mathcal{W}} v \text{ implies } \ell_{\mathcal{W}}(w) \supseteq \ell_{\mathcal{W}}(v),$$

and such that for all $w \in W$:

- (a) whenever $\varphi \Rightarrow \psi \in \Sigma \setminus \ell_{\mathcal{W}}(w)$, there is $v \leq w$ such that $\varphi \in \ell_{\mathcal{W}}(v)$ and $\psi \notin \ell_{\mathcal{W}}(v)$;
- (b) whenever $\varphi \Leftarrow \psi \in \ell_{\mathcal{W}}(w)$, there is $v \geq w$ such that $\varphi \in \ell_{\mathcal{W}}(v)$ and $\psi \notin \ell_{\mathcal{W}}(v)$.

The Σ -labelled space \mathcal{W} **falsifies** $\varphi \in \mathcal{L}$ if $\varphi \in \Sigma \setminus \ell_{\mathcal{W}}(w)$ for some $w \in W$. The **height** of \mathcal{W} is the supremum of all n such that there is a chain $w_1 <_{\mathcal{W}} w_2 <_{\mathcal{W}} \dots <_{\mathcal{W}} w_n$.

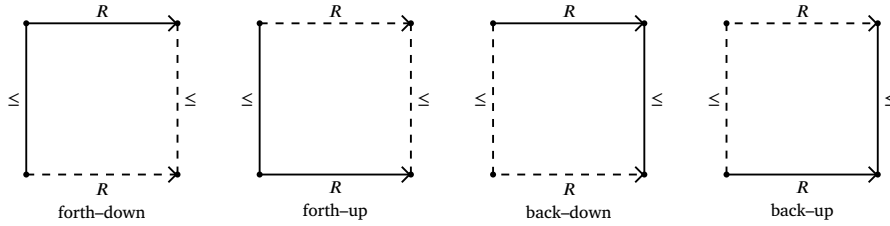


Fig. 2. Confluence conditions.

If \mathcal{W} is a labelled space, elements of its domain W will sometimes be called **worlds**.² When clear from the context we will omit subscripts and write, for example, \leq instead of $\leq_{\mathcal{W}}$.

We now enrich labelled spaces with a relation capturing *temporal* information.

Definition 4.5. Recall that a subset S of a poset (P, \leq) is **convex** if $s \in S$ whenever $a, b \in S$ and $a \leq s \leq b$. In this article, a **convex relation** between posets (A, \leq_A) and (B, \leq_B) is a binary relation $R \subseteq A \times B$ such that for each $x \in A$ the image set $\{y \in B \mid x R y\}$ is convex with respect to \leq_B , and for each $y \in B$ the preimage set $\{x \in A \mid x R y\}$ is convex with respect to \leq_A .

The relation R is **fully confluent** if it validates the four following conditions:

- forth-down** if $x \leq_A x' R y'$, then there is y such that $x R y \leq_B y'$;
- forth-up** if $x' \geq_A x R y$, then there is y' such that $x' R y' \geq_B y$;
- back-down** if $x' R y' \geq_B y$, then there is x such that $x' \geq_A x R y$;
- back-up** if $x R y \leq_B y'$, then there is x' such that $x \leq_A x' R y'$.

Fig. 2 depicts the four confluence conditions that a fully confluent relation must validate.

Definition 4.6. Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas. Suppose that $\Phi, \Psi \in \mathbb{T}_{\Sigma}$. The ordered pair (Φ, Ψ) is **sensible** if

- (1) for all $X\varphi \in \Sigma$, we have $X\varphi \in \Phi$ if and only if $\varphi \in \Psi$;
- (2) for all $\forall\varphi \in \Sigma$, we have $\forall\varphi \in \Psi$ if and only if $\varphi \in \Phi$;
- (3) for all $G\varphi \in \Sigma$, we have $G\varphi \in \Phi$ if and only if $\varphi \in \Phi$ and $G\varphi \in \Psi$;
- (4) for all $H\varphi \in \Sigma$, we have $H\varphi \in \Psi$ if and only if $\varphi \in \Psi$ and $H\varphi \in \Phi$;
- (5) for all $\varphi \cup \psi \in \Sigma$, we have $\varphi \cup \psi \in \Phi$ if and only if either $\psi \in \Phi$ or $(\varphi \in \Phi$ and $\varphi \cup \psi \in \Psi)$;
- (6) for all $\varphi S \psi \in \Sigma$, we have $\varphi S \psi \in \Psi$ if and only if either $\psi \in \Psi$ or $(\varphi \in \Psi$ and $\varphi S \psi \in \Phi)$.

A pair (w, v) of worlds in a Σ -labelled space \mathcal{W} is **sensible** if $(\ell(w), \ell(v))$ is sensible. A relation $R \subseteq W \times W$ is **sensible** if every pair in R is sensible. It is **bi-serial**, if for each $w \in W$ there exist both $v_1 \in W$ with $w R v_1$ and $v_{-1} \in W$ with $v_{-1} R w$. Further, R is **ω -sensible** if:

- whenever $G\varphi \in \Sigma \setminus \ell(w)$, there are $n \geq 0$ and v such that $w R^n v$ and $\varphi \in \Sigma \setminus \ell(v)$;
- whenever $H\varphi \in \Sigma \setminus \ell(w)$, there are $n \geq 0$ and v such that $v R^n w$ and $\varphi \in \Sigma \setminus \ell(v)$;
- whenever $\varphi \cup \psi \in \ell(w)$, there are $n \geq 0$ and v such that $w R^n v$ and $\psi \in \ell(v)$ ³;
- whenever $\varphi S \psi \in \ell(w)$, there are $n \geq 0$ and v such that $v R^n w$ and $\psi \in \ell(v)$.

A Σ -labelled system is a labelled space \mathcal{W} equipped with a bi-serial, fully confluent, convex, sensible relation $R_{\mathcal{W}} \subseteq W \times W$. If moreover $R_{\mathcal{W}}$ is ω -sensible, we say that \mathcal{W} is a Σ -quasimodel. The Σ -quasimodel $Q = (Q, \leq_Q, \ell_Q, R_Q)$ falsifies $\varphi \in \mathcal{L}$ if $\varphi \in \Sigma \setminus \ell_Q(w)$ for some $w \in Q$.

Any bi-relational model can be regarded as a Σ -quasimodel. If $\mathcal{X} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ is a bi-relational model and $x \in W \times T$, we can assign a Σ -type $\ell_{\mathcal{X}}(x)$ to x given by $\ell_{\mathcal{X}}(x) = \{\psi \in \Sigma \mid x \in \llbracket \psi \rrbracket\}$. We also set $R_{\mathcal{X}} = \{((w, t), (w, S(t))) \mid w \in W, t \in T\}$; it is obvious that $R_{\mathcal{X}}$ is ω -sensible. Henceforth we will tacitly identify \mathcal{X} with its associated Σ -quasimodel.

Example 4.7. Recall that in Example 2.4, we provided conditions under which

$$\varphi := G(\text{tankLow} \Rightarrow (\text{tankLow} \cup (\text{refill} \vee \text{tankEmpty})))$$

² Note that the conditions on a labelled space ensure that every type $\ell_{\mathcal{W}}(w)$ used to label a world is necessarily ‘consistent’ in the sense that it cannot contain \perp (which implies further that it can never contain both φ and $\neg\varphi$).

³ Note that because R is assumed to be sensible, this condition is equivalent to the existence of $m \geq 0$ and $v_0 R v_1 R \dots R v_m$ such that $v_0 = w$ and $\varphi \in \ell(v_0), \dots, \varphi \in \ell(v_{m-1}), \psi \in \ell(v_m)$. Similarly for the following S condition.

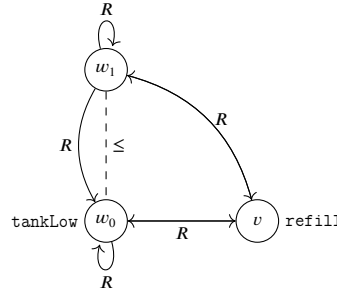


Fig. 3. Quasimodel associated with Example 4.7. The dashed line represents the relation \leq (for which reflexive lines are not displayed), while solid arrows represent the relation R .

is true in a real-valued model. Alternately, quasimodels allow us for a succinct witness that this formula is satisfiable.

Let Σ be the set of subformulas of φ and consider the quasimodel of Fig. 3, which corresponds to the structure $Q = (Q, \leq, \ell, R)$, where $Q = \{w_0, w_1, v\}$. Suppose that $w_0 < w_1$ (and $x \leq x$ for all x) and define R so that $R(w_0) = \{w_0, v\}$, $R(w_1) = \{w_0, w_1, v\}$ and $R(v) = \{w_0, w_1\}$.

The labelling ℓ is such that `tankLow` is the only atomic proposition in $\ell(w_0)$, `refill` is the only atomic proposition in $\ell(v)$, and no atomic proposition belongs to $\ell(w_1)$. It is not too hard to check that such an ℓ can be defined so that φ belongs to every label. In particular, letting $\psi := \text{tankLow} \cup (\text{refill} \vee \text{tankEmpty})$, we would have

$$\ell(w_0) = \{\text{tankLow}, \psi, \text{tankLow} \Rightarrow \psi, \varphi\}.$$

The ω -sensibility condition for \cup is witnessed by the transition $w_0 R v$.

This quasimodel not only makes φ globally true, but it does so in a way that does not allow the tank to become empty, thus making the required condition valid while also representing desirable behaviour for an autonomous vehicle.

5. From quasimodels to bi-relational models

If $Q = (Q, \leq_Q, \ell_Q, R_Q)$ is a quasimodel, then we cannot necessarily view Q directly as a bi-relational model for the primary reason that R_Q is not necessarily a function. However, we can extract bi-relational models from quasimodels via an unwinding and selection construction. More precisely, given a $\Sigma \subseteq \mathcal{L}$ that is finite and closed under subformulas, suppose φ is falsified on the Σ -quasimodel Q . In this section we show how to obtain from Q a bi-relational model \vec{Q}_φ falsifying φ . We call the resulting bi-relational model a *limit model* of Q . This proves GTL_{rel} is sound for the class of quasimodels.

The general idea for determinising Q is to consider infinite paths on Q as points in the limit model. However, we will only select paths \vec{w} with the property that, if $\varphi \cup \psi$ occurs in \vec{w} , then ψ must also occur at a later time, with similar conditions for S , G , and H . These are the *realising* paths of Q .

Definition 5.1. A path in a Σ -quasimodel $Q = (Q, \leq_Q, \ell_Q, R_Q)$ is any sequence $(w_i)_{\alpha < i < \beta}$ with $-\infty \leq \alpha < \beta \leq \infty$ such that $w_i R_Q w_{i+1}$ for all $\alpha < i < \beta - 1$. An infinite path $\vec{w} = (w_i)_{i \in \mathbb{Z}}$ is **realising** if for all $i \in \mathbb{Z}$:

- for all $\text{G}\psi \in \Sigma \setminus \ell(w_i)$, there exists $j \geq i$ such that $\psi \in \Sigma \setminus \ell(w_j)$;
- for all $\text{H}\psi \in \Sigma \setminus \ell(w_i)$, there exists $j \leq i$ such that $\psi \in \Sigma \setminus \ell(w_j)$;
- for all $\varphi \cup \psi \in \ell(w_i)$, there exists $j \geq i$ such that $\psi \in \ell(w_j)$;
- for all $\varphi \text{S} \psi \in \ell(w_i)$, there exists $j \leq i$ such that $\psi \in \ell(w_j)$.

Denote the set of realising paths by $|\vec{Q}|$, and let $(v_i)_{i \in \mathbb{Z}} \leq (w_i)_{i \in \mathbb{Z}}$ if and only if $v_i \leq w_i$ for all $i \in \mathbb{Z}$. The worlds of the limit model will be a subset of $|\vec{Q}|$ that is linearly ordered (with respect to \leq).

Given our $\Sigma \subseteq \mathcal{L}$ that is finite and closed under subformulas and a formula φ falsified in Q , the limit model \vec{Q}_φ will be of the form $(W, \mathbb{Z}, \leq, S, \llbracket \cdot \rrbracket)$, where $W \subseteq |\vec{Q}|$, the flow function $S : \mathbb{Z} \rightarrow \mathbb{Z}$ is the standard successor function, and $\llbracket p \rrbracket = \{(\vec{w}, i) \mid p \in \ell(w_i)\}$ (extended to compound formulas in accordance with Definition 2.5). We now describe how to select the linearly ordered subset of realising paths W .

Definition 5.2. A **finite grid** is a finite set of paths in Q with uniform index bounds $l < k \in \mathbb{Z}$ that is linearly ordered (by the pointwise ordering).

A finite grid P' of paths with bounds $l' < k'$ **extends** a finite grid P of paths with bounds k , if $l' \leq l < k \leq k'$ and $P \subseteq P'|_{l,k}$, where $P'|_{l,k}$ is the restriction of the paths in P' to their segments within the bounds l and k .

Definition 5.3 (defects). Let P be a finite grid.

- A **G-defect** of P is a pair $((w_i)_{l < i < k}, G\varphi) \in P \times \Sigma$ such that $G\varphi \notin \ell(w_{k-1})$, but $\varphi \in \ell(w_{k-1})$.
- A **H-defect** of P is a pair $((w_i)_{l < i < k}, H\varphi) \in P \times \Sigma$ such that $H\varphi \notin \ell(w_{l+1})$, but $\varphi \in \ell(w_{l+1})$.
- An **U-defect** of P is a pair $((w_i)_{l < i < k}, \varphi \cup \psi) \in P \times \Sigma$ such that $\varphi \cup \psi \in \ell(w_{k-1})$, but $\psi \notin \ell(w_{k-1})$.
- A **S-defect** of P is a pair $((w_i)_{l < i < k}, \varphi \text{ S } \psi) \in P \times \Sigma$ such that $\varphi \text{ S } \psi \in \ell(w_{l+1})$, but $\psi \notin \ell(w_{l+1})$.
- An **\Rightarrow -defect** of P is a triple $((w_i)_{l < i < k}, \varphi \Rightarrow \psi, j) \in P \times \Sigma \times \{l+1, \dots, k-1\}$ such that $\varphi \Rightarrow \psi \notin \ell(w_j)$, but there is no $(v_i)_{l < i < k} \leq (w_i)_{l < i < k}$ also in P such that $\varphi \in \ell(v_j)$, but $\psi \notin \ell(v_j)$.
- A **\Leftarrow -defect** of P is a triple $((w_i)_{l < i < k}, \varphi \Leftarrow \psi, j) \in P \times \Sigma \times \{l+1, \dots, k-1\}$ such that $\varphi \Leftarrow \psi \in \ell(w_j)$, but there is no $(v_i)_{l < i < k} \geq (w_i)_{l < i < k}$ also in P such that $\varphi \in \ell(v_j)$, but $\psi \notin \ell(v_j)$.
- A **seriality defect** is a pair $((w_i)_{l < i < k}, b) \in P \times \{-, +\}$. The symbols $-$ and $+$ indicate a defect affecting the past or future, respectively.

Note that because Σ is finite, any finite grid has a finite number of defects. We select the set $W \subseteq |\vec{Q}|$ as follows. We maintain a (finite) first-in-first-out queue D of defects and a finite grid P that is extended each time we process a defect. (We will update D at each iteration to ensure that it only contains all the defects of the grid after each update.

Choose some $w \in Q$ falsifying φ . We initialise the grid to the single sequence (w) (of length 1, using the index 0) and add all defects of this grid to D .

Using the properties of quasimodels, we may extend P to a new grid P' , so that the defect at the head of the queue D is not a defect of P' . (Because of seriality defects, D is never empty.) The construction of P' is realised via a case-by-case analysis.

- If the defect is an U-defect $((w_i)_{l < i < k}, \varphi \cup \psi)$ of P , then because $R = R_Q$ is ω -sensible we know that there exist $j > 0$ and $v \in R^j(w_{k-1})$ such that $\varphi \in \ell(v)$. We then define $k' = k + j$ and choose

$$w_{k-1} R w_k R \dots R w_{k'-1} = v.$$

By the forth-up confluence property, we can extend every $(v_i)_{l < i < k} > (w_i)_{l < i < k}$ in P to a sequence with upper index bound k' in a way that preserves the (linear) ordering on sequences. Similarly for every $(v_i)_{l < i < k} < (w_i)_{l < i < k}$ in P using the forth-down confluence property.

- If the defect is a G-defect $((w_i)_{l < i < k}, G\varphi)$, we can find $j > 0$ and $v \in R^j(w_{k-1})$ such that $\varphi \notin \ell(v)$, and proceed as for the U case.
- The cases of S- and H-defects are the temporal duals of U- and G-defects, respectively.
- If the defect is an \Rightarrow -defect $((w_i)_{l < i < k}, \varphi \Rightarrow \psi, j)$, in view of Definition 4.4(a) we may choose $v_j \in Q$ such that $v_j < w_j$ and $\varphi \in \ell(v_j)$, but $\psi \notin \ell(v_j)$. Let $(u_i)_{l < i < k}$ be the minimum sequence in P with $u_j > v_j$ and $(t_i)_{l < i < k}$ be the maximum sequence in P with $t_j < v_j$, if it exists. We will assume that $(t_i)_{l < i < k}$ is defined, since the case where v_j is not bounded below is similar but simpler. We will complete the sequence $(v_i)_{l < i < k}$ so that $(t_i)_{l < i < k} < (v_i)_{l < i < k} < (u_i)_{l < i < k}$. Suppose we have defined $v_{j'}$ for $j \leq j' < k-1$. To define $v_{j'+1}$, choose y with $v_{j'} R y$ and $y \leq u_{j'+1}$, which exists by forth-down confluence. If $y \geq t_{j'+1}$ we can set $v_{j'+1} = y$ and we are done. Otherwise, by local linearity of \leq , we have $y < t_{j'+1}$. In this case, choose z with $v_{j'} R z$ and $z \geq t_{j'+1}$, which exists by forth-up confluence. Then $y < t_{j'+1} \leq z$, so as the image set of $v_{j'}$ under R is convex (by convexity of R), we have $v_{j'} R t_{j'+1}$ and we can set $v_{j'+1} = t_{j'+1}$. In this way we can define $(v_i)_{l < i < k}$ inductively for all indices greater than j . The process for indices less than j is similar, using back-up and back-down confluence and the convexity of preimage sets under R . By construction, $(v_i)_{l < i < k}$ sits strictly between $(t_i)_{l < i < k}$ and $(u_i)_{l < i < k}$.
- The case of a \Leftarrow -defect is similar to that of an \Rightarrow -defect, except that we choose $v_j > w_j$.
- If the defect is a seriality defect $((w_i)_{l < i < k}, +)$, we extend $(w_i)_{l < i < k}$ to $(w_i)_{l < i < k+1}$, relying on the fact that R is serial, and then we extend the other sequences to length $k+1$ using forth-up and forth-down confluence. A seriality defect $((w_i)_{l < i < k}, -)$ is analogous: we extend $(w_i)_{l < i < k}$ to $(w_i)_{l-1 < i < k}$, relying on converse seriality of R .

Next we must update D so that it contains all defects of P' , and all the elements of D that are not defects of P' (which by design includes the head of D) are removed. Finally, we update $P := P'$.

We set W to be the limit of this sequence of finite grids. More precisely, let the sequence of grids be P_0, P_1, \dots , containing paths with bounds $(l_0, k_0), (l_1, k_1), \dots$, respectively. Then W is the set of paths $(w_i)_{i \in \mathbb{Z}}$ for which there exists N such that for all $n > N$ the finite segment $(w_i)_{l_n < i < k_n}$ is in P_n . (This gives us the limit we would expect and forbids ‘diagonal’ sequences whose finite segments appear in P_0, P_1, \dots merely infinitely often.)

By construction, (W, \leq) is a linearly ordered set, and for each $p \in \mathbb{P}$, the set $\llbracket p \rrbracket$ is downward closed in its first coordinate. Thus the limit model $\vec{Q}_\varphi = (W, \mathbb{Z}, \leq, S, \llbracket \cdot \rrbracket)$ indeed defines a bi-relational model. Of course \vec{Q}_φ is only useful if $\llbracket \cdot \rrbracket$ ‘matches’ ℓ on all formulas in Σ , not just propositional variables. Fortunately, this turns out to be the case.

Lemma 5.4. Let $\Sigma \subseteq \mathcal{L}$ be finite and closed under subformulas, Q be a Σ -quasimodel, $\varphi, \psi \in \Sigma$, and $\vec{Q}_\varphi = (W, \mathbb{Z}, \leq, S, \llbracket \cdot \rrbracket)$ be as described above. Then $\llbracket \psi \rrbracket = \{(\vec{w}, i) \mid \psi \in \ell(w_i)\}$.

Proof. The proof goes by standard induction of formulas. The induction steps for \wedge and \vee are immediate. The cases of the remaining connectives follow straightforwardly from the construction of W , because every defect is eventually eliminated. Indeed every w_i in

the limit is also in some P_n , at which point all the defects associated to w_i are present in the finite first-in-first-out queue D ; thus they will eventually reach the front of D and be eliminated. \square

We obtain the main result of this section, which in particular implies that GTL_{rel} is sound for the class of quasimodels.

Proposition 5.5. *Let $\Sigma \subseteq \mathcal{L}$ be finite and closed under subformulas and \mathcal{Q} be any Σ -quasimodel, and suppose φ is falsified on \mathcal{Q} . Then there exists a bi-relational model $\tilde{\mathcal{Q}}_\varphi$ that falsifies φ .*

It is interesting to note that although we assumed in this section that Σ was finite, this restriction can be removed. Since \mathcal{L} is countable, for an arbitrary subformula-closed $\Sigma \subseteq \mathcal{L}$, there can only be a countable number of defects in any finite grid. Thus with appropriate scheduling all defects can be eliminated in the limit.

6. From bi-relational models to finite quasimodels

As we noted earlier, every bi-relational model can be naturally viewed as a quasimodel. However, we wish to show that, given a finite and subformula-closed Σ , we can from each bi-relational model \mathcal{X} produce a *finite* quasimodel falsifying exactly the same formulas from Σ as \mathcal{X} . In this section we do just this by describing a quotient construction that given an arbitrary Σ -labelled system produces a new Σ -labelled system \mathcal{Q} . This construction does not require Σ to be finite, but when it is, \mathcal{Q} is finite. Since bi-relational models can be viewed as Σ -labelled systems, our quotient construction applies to them. We then observe that the construction preserves ω -sensitivity, so that when applied to a bi-relational model, we do indeed obtain a quasimodel. An illustrative example will follow (Example 6.4).

The process of constructing of \mathcal{Q} consists of two steps. The first is to take a quotient to obtain a Σ -labelled space equipped with a bi-serial, fully confluent, sensible relation. The second step is to enlarge the equipped relation to make it also convex, so all the conditions for a Σ -labelled space hold.

Let Σ be a subformula-closed subset of \mathcal{L} , and let $\mathcal{W} = (W, \leq, \ell, R)$ be a Σ -labelled system. For $w \in W$, define $L_{\mathcal{W}}(w) = \{\ell(v) \mid v \preceq w\}$. Define the equivalence relation \sim on W by

$$w \sim v \iff (\ell(w), L(w)) = (\ell(v), L(v)).$$

If Σ is finite, then clearly W/\sim is finite.⁴

Now define a partial order \leq_Q on the equivalence classes W/\sim of \sim by

$$[w] \leq_Q [v] \iff L(w) = L(v) \text{ and } \ell(w) \supseteq \ell(v),$$

noting that this is well-defined and is indeed a partial order.

Since each set $L(w)$ can be linearly ordered by inclusion and $\ell(w) \in L(w)$, the poset $(W/\sim, \leq_Q)$ is a disjoint union of linear posets. By defining ℓ_Q by $\ell_Q([w]) = \ell(w)$ we obtain a Σ -labelled space $(W/\sim, \leq_Q, \ell_Q)$; it is not hard to check that this labelling is anti-monotone and that the clauses for \Rightarrow and \Leftarrow in the definition of a labelled space hold with this labelling.

Now define the binary relation R_Q on W/\sim to be the smallest relation such that $w R v$ implies $[w] R_Q [v]$.

Lemma 6.1. *The relation R_Q is bi-serial, sensible, and fully confluent.*

Proof. It is clear that R_Q is bi-serial and sensible.

For confluence, suppose $[w] R_Q [v]$, where we may assume that $w R v$. To see that the forth-up condition holds, suppose further that $[w] \leq_Q [u]$. Then, as $l(u) \in L(u) = L(w)$, there is some $u' \geq w$ with $[u] = [u']$. By forth-up confluence of R , there exists z with $u' R z$ and $v \leq z$. Then, we have $[u'] R_Q [z]$ and $[v] \leq_Q [z]$, as required for the forth-up condition. The proofs of the remaining three confluence conditions are entirely analogous. \square

As promised, we now have a Σ -labelled space equipped with a bi-serial, fully confluent, sensible relation. We now transform this structure into a Σ -labelled system by making the equipped relation convex by fiat.

Define R_Q^+ by $X R_Q^+ Y$ if and only if there exist $X_1 \leq_Q X \leq_Q X_2$ and $Y_1 \leq_Q Y \leq_Q Y_2$ such that $X_2 R_Q Y_1$ and $X_1 R_Q Y_2$. Now define $\mathcal{Q} = (W/\sim, \leq_Q, \ell_Q, R_Q^+)$.

Proposition 6.2. *The structure \mathcal{Q} is a Σ -labelled system.*

Proof. We already know that $(W/\sim, \leq_Q, \ell_Q)$ is a Σ -labelled space. First we must check that the enlarged relation R_Q^+ is still bi-serial, sensible, and fully confluent. Bi-seriality is clear, since this is a monotone condition.

⁴ Note that if \mathcal{W} is a bi-relational model, then \sim is the largest relation that is simultaneously a bisimulation with respect to the relations \leq and \geq , with Σ treated as the set of atomic propositions that bisimilar worlds must agree on.

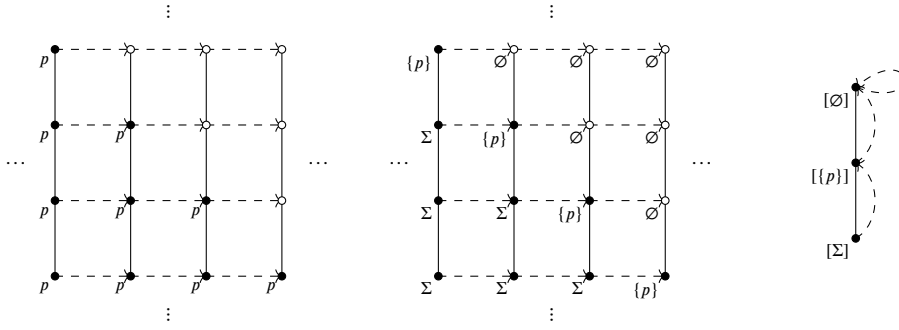


Fig. 4. Left: The bi-relational model \mathcal{X} ; centre: values of $\ell_{\mathcal{X}}$ when $\Sigma = \{F(p \Rightarrow Xp), p \Rightarrow Xp, Xp, p\}$; right: the quotient quasimodel \mathcal{Q} obtained from \mathcal{X} and Σ .

To see that R_Q^+ is sensible, suppose $X R_Q^+ Y$ and that $X\varphi \in \Sigma$. Take $X_1 \leq_Q X \leq_Q X_2$ and $Y_1 \leq_Q Y \leq_Q Y_2$ such that $X R_Q Y_1$. Then

$$\begin{aligned} X\varphi \in \ell_Q(X) &\implies X\varphi \in \ell_Q(X_1) &\implies \varphi \in \ell_Q(Y_2) &\implies \varphi \in \ell_Q(Y) \\ &\implies \varphi \in \ell_Q(Y_1) &\implies X\varphi \in \ell_Q(X_2) &\implies X\varphi \in \ell_Q(X), \end{aligned}$$

so $X\varphi \in \ell_Q(X) \iff \varphi \in \ell_Q(Y)$. The cases of the remaining modalities are similar.

For the forth-down condition, suppose $X \leq_Q X' R_Q^+ Y'$. Then by the definition of R_Q^+ , there exist $X_2 \geq_Q X'$ and $Y_1 \leq_Q Y'$ such that $X_2 R_Q Y_1$. Since $X \leq_Q X' \leq_Q X_2$, by the forth-down condition for R_Q there is some $Y \leq_Q Y_1$ with $X R_Q Y$ and therefore $X R_Q^+ Y$. Since $Y \leq_Q Y_1 \leq_Q Y'$, we are done. The proof that the forth-up condition holds is just the order dual of that for forth-down. The proofs of the back-down and back-up conditions are similar. Thus R_Q^+ is fully confluent.

Finally, we show that R_Q^+ is convex. Firstly, for the image condition, if $X R_Q^+ Y_1$ and $X R_Q^+ Y_2$ with $Y_1 \leq_Q Y \leq_Q Y_2$, then by the definition of R_Q^+ we can find $X_2 \geq_Q X$ and $Y'_1 \leq_Q Y_1$ with $X_2 R_Q Y'_1$, and similarly $X_1 \leq_Q X$ and $Y'_2 \geq_Q Y_2$ with $X_1 R_Q Y'_2$. Since then $X_1 \leq_Q X \leq_Q X_2$ and $Y'_1 \leq_Q Y \leq_Q Y'_2$, by the definition of R_Q^+ we conclude that $X R_Q^+ Y$. The preimage condition is completely analogous. This completes the proof that \mathcal{Q} is a Σ -labelled system. \square

Proposition 6.2 will be of independent interest later for proving the completeness of our deductive calculus. For now, however, we are only interested in the case that we had an ω -sensible R to begin with, in which case we are certain to obtain a quasimodel, as the next lemma shows.

Lemma 6.3. *If R is ω -sensible, then R_Q^+ is ω -sensible.*

Proof. We first show that R_Q is ω -sensible. Suppose that $\varphi \cup \psi \in \ell_Q([w])$. Then by definition, $\varphi \cup \psi \in \ell(w)$. Thus as R is ω -sensible, there exists $v \in R^n(w)$ such that $\psi \in \ell(v)$ for some $n > 0$. It follows that $[v] \in R_Q^n([w])$ and $\psi \in \ell_Q([v])$. Similar reasoning applies when we suppose that $\varphi \cap \psi \in \ell_Q([w])$, $G\varphi \notin \ell_Q([w])$, or $H\varphi \notin \ell_Q([w])$; thus R_Q is ω -sensible. It now follows immediately that R_Q^+ is ω -sensible, since the conditions for a relation to be ω -sensible are monotone. \square

Example 6.4. We return to the example seen in the proof of Proposition 4.2. Let Σ be the set of all subformulas of $F(p \Rightarrow Xp)$. So $\Sigma = \{F(p \Rightarrow Xp), p \Rightarrow Xp, Xp, p\}$. We take as bi-relational model \mathcal{X} the model $(\mathbb{Z}, \mathbb{Z}, \leq, S, \llbracket \cdot \rrbracket)$ seen in the proof of Proposition 4.2 and depicted again on the left of Fig. 4. Examining \mathcal{X} , we see that there are three possible values that $\ell_{\mathcal{X}}(x)$ can take: Σ , $\{p\}$, or \emptyset . These values are depicted in the centre of Fig. 4. There is only one possible value for $\ell_{\mathcal{X}}(x)$: $\{\Sigma, \{p\}, \emptyset\}$, so the quotient quasimodel \mathcal{Q} is linearly ordered by \leq_Q . The quasimodel \mathcal{Q} is depicted on the right of Fig. 4 (with the constant L -data omitted).

Proposition 6.5. *Let $\mathcal{X} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ be a bi-relational model and $\mathcal{Q} = ((W \times T)/\sim, \leq_Q, \ell_Q, S_Q^+)$ the quotient Σ -quasimodel obtained from \mathcal{X} . Let $\varphi \in \Sigma$. Then \mathcal{X} falsifies φ if and only if \mathcal{Q} falsifies φ .*

Proof. We have: \mathcal{X} falsifies φ if and only if $\exists(w, t) \in (W \times T) \setminus \llbracket \varphi \rrbracket$ if and only if $\exists[(w, t)] \in (W \times T)/\sim$ with $\varphi \in \Sigma \setminus \ell_Q([(w, t)])$ if and only if \mathcal{Q} falsifies φ . \square

In order to use Proposition 6.5 to prove decidability, we need to compute a bound on the size of the quasimodel \mathcal{Q} in terms of the size of Σ , when Σ is finite.

Lemma 6.6. *Suppose Σ is finite, and write $|\Sigma|$ for its cardinality. Then the height of \mathcal{Q} is bounded by $|\Sigma| + 1$, and the cardinality of the domain $(W \times T)/\sim$ of \mathcal{Q} is bounded by $(|\Sigma| + 1) \cdot 2^{|\Sigma|(|\Sigma|+1)+1}$.*

Proof. Each element of the domain of Q is a pair (ℓ, L) where L is a (nonempty) subset of $\mathcal{P}(\Sigma)$ and $\ell \in L$. Since L is linearly ordered by inclusion, it has height at most $|\Sigma| + 1$. There are $(2^{|\Sigma|})^i$ subsets of $\mathcal{P}(\Sigma)$ of size i , so there are at most $\sum_{i=1}^{|\Sigma|+1} (2^{|\Sigma|})^i$ distinct L . The sum is bounded by $2^{|\Sigma|(|\Sigma|+1)+1}$. The factor of $|\Sigma| + 1$ corresponds to choice of an $\ell \in L$, for each L . \square

Thus we have an exponential bound on the size of Q . Hence any falsifiable formula is falsifiable on an effectively bounded quasimodel, and it follows that GTL_{rel} , which we know coincides with $\text{GTL}_{\mathbb{R}}$, is decidable.

Theorem 6.7. *The logic $\text{GTL}_{\mathbb{R}}$ of \mathcal{L} -formulas that are valid on all flows and the logic GTL_{rel} of \mathcal{L} -formulas that are valid on all bi-relational frames are equal and decidable.*

Proof. By Theorem 3.3, $\text{GTL}_{\mathbb{R}} = \text{GTL}_{\text{rel}}$. Since falsifiability is the complement of validity, it suffices to show that it is decidable whether a formula φ is falsifiable over the class of all bi-relational frames. Let Σ be the set of subformulas of φ . If φ is falsifiable in a Σ -quasimodel of size at most $(|\Sigma| + 1) \cdot 2^{|\Sigma|(|\Sigma|+1)+1}$, then by Proposition 5.5, φ is falsified in a bi-relational frame. Conversely, if φ is falsified in a bi-relational frame, then by Proposition 6.5 and Lemma 6.6, φ is falsified in a Σ -quasimodel of size at most $(|\Sigma| + 1) \cdot 2^{|\Sigma|(|\Sigma|+1)+1}$. Hence it suffices to check falsifiability of φ on the set of all Σ -quasimodels of size at most $(|\Sigma| + 1) \cdot 2^{|\Sigma|(|\Sigma|+1)+1}$. It is clear that this check can be carried out within a computable time bound; hence the problem is decidable. \square

7. A Hilbert-style deductive calculus

Now begins the second half of this article, in which we prove that a certain Hilbert-style deductive calculus is sound and complete for the validities of Gödel temporal logic (under both real-valued and bi-relational semantics). We start by presenting the calculus. It is obtained by adapting the standard axioms and inference rules of LTL [54], as well as their order dual and temporal dual versions.

Definition 7.1. The logic GTL is the least set of \mathcal{L} -formulas closed under the following axioms and rules.

I All substitution instances of intuitionistic tautologies (see e.g. [55])

II Axioms and rules of Heyting–Brouwer logic:

$$\begin{array}{lll} \text{A} & \varphi \Rightarrow (\psi \vee (\varphi \Leftarrow \psi)) & \text{B} \quad \frac{\varphi \Rightarrow \psi}{(\varphi \Leftarrow \theta) \Rightarrow (\psi \Leftarrow \theta)} \quad \text{C} \quad \frac{\varphi \Rightarrow \psi \vee \gamma}{(\varphi \Leftarrow \psi) \Rightarrow \gamma} \end{array}$$

III Linearity axioms:

$$\text{A} \quad (\varphi \Rightarrow \psi) \vee (\psi \Rightarrow \varphi) \quad \text{B} \quad \neg((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$$

IV Temporal axioms:

$$\begin{array}{ll} \text{A} \quad \neg X \perp & \text{A}' \quad \neg Y \perp \\ \text{B} \quad X(\varphi \vee \psi) \Rightarrow (X\varphi \vee X\psi) & \text{B}' \quad Y(\varphi \vee \psi) \Rightarrow (Y\varphi \vee Y\psi) \\ \text{C} \quad (X\varphi \wedge X\psi) \Rightarrow X(\varphi \wedge \psi) & \text{C}' \quad (Y\varphi \wedge Y\psi) \Rightarrow Y(\varphi \wedge \psi) \\ \text{D} \quad X(\varphi \Rightarrow \psi) \Leftrightarrow (X\varphi \Rightarrow X\psi) & \text{D}' \quad Y(\varphi \Rightarrow \psi) \Leftrightarrow (Y\varphi \Rightarrow Y\psi) \\ \text{E} \quad G(\varphi \Rightarrow \psi) \Rightarrow (G\varphi \Rightarrow G\psi) & \text{E}' \quad H(\varphi \Rightarrow \psi) \Rightarrow (H\varphi \Rightarrow H\psi) \\ \text{F} \quad G(\varphi \Rightarrow \psi) \Rightarrow (\theta \cup \varphi \Rightarrow \theta \cup \psi) & \text{F}' \quad H(\varphi \Rightarrow \psi) \Rightarrow (\theta \text{ S } \varphi \Rightarrow \theta \text{ S } \psi) \\ \text{G} \quad G(\varphi \Rightarrow \psi) \Rightarrow (\varphi \cup \theta \Rightarrow \psi \cup \theta) & \text{G}' \quad H(\varphi \Rightarrow \psi) \Rightarrow (\varphi \text{ S } \theta \Rightarrow \psi \text{ S } \theta) \\ \text{H} \quad G\varphi \Rightarrow \varphi \wedge XG\varphi & \text{H}' \quad H\varphi \Rightarrow \varphi \wedge YH\varphi \\ \text{I} \quad \psi \vee (\varphi \wedge X(\varphi \cup \psi)) \Rightarrow \varphi \cup \psi & \text{I}' \quad \psi \vee (\varphi \wedge Y(\varphi \text{ S } \psi)) \Rightarrow \varphi \text{ S } \psi \\ \text{J} \quad G(\varphi \Rightarrow X\varphi) \Rightarrow (\varphi \Rightarrow G\varphi) & \text{J}' \quad H(\varphi \Rightarrow Y\varphi) \Rightarrow (\varphi \Rightarrow H\varphi) \\ \text{K} \quad G(\psi \wedge X\varphi \Rightarrow \varphi) \Rightarrow (\psi \cup \varphi \Rightarrow \varphi) & \text{K}' \quad H(\psi \wedge Y\varphi \Rightarrow \varphi) \Rightarrow (\psi \text{ S } \varphi \Rightarrow \varphi) \end{array}$$

V Connection axioms

$$\text{A} \quad \varphi \Leftrightarrow XY\varphi \quad \text{A}' \quad \varphi \Leftrightarrow YX\varphi$$

VI Standard modal rules:

$$\text{MP} \quad \frac{\varphi, \varphi \Rightarrow \psi}{\psi} \quad \text{NEC}_X \quad \frac{\varphi}{X\varphi} \quad \text{NEC}_Y \quad \frac{\varphi}{Y\varphi} \quad \text{NEC}_G \quad \frac{\varphi}{G\varphi} \quad \text{NEC}_H \quad \frac{\varphi}{H\varphi}$$

Axiom group II concerns the relationship between \Rightarrow and \Leftarrow . In particular, Axiom II.A is used in C. Rauszer's axiomatisation of intuitionistic logic with co-implication (called Heyting–Brouwer logic) [56]. The Gödel–Dummett axiom III.A and its order dual III.B reflect the fact that the connectives \Rightarrow and \Leftarrow are implemented on locally linear posets (which conversely is *necessary* for these axioms to be valid).

Axioms IV.A to IV.K concern the future. Axioms IV.C, IV.D, and IV.E are standard modal axioms (viewing X as a box-type modality). In particular they hold in any normal modal logic, although of course GTL is not itself normal by virtue of being strictly sub-classical. Axiom IV.F is an order dual version of IV.E; such dual axioms are often needed in intuitionistic modal logic, since F and G are not typically interdefinable. The axioms IV.A and IV.B have to do with the passage of time being deterministic in linear temporal logic: IV.A characterises seriality and IV.B characterises (partial) functionality, thus together they constrain temporal accessibility to be a total function.

The co-inductive axiom IV.H states that if something will henceforth be the case, then it is true now and, moreover, in the next moment, it will still henceforth be the case. Axiom IV.J is successor induction, as future moments are indexed by the natural numbers with their usual ordering. Axioms IV.I and IV.K are analogues of IV.H and IV.J respectively. Axioms IV.A' to IV.K' are the past analogues of axioms IV.A to IV.K. The connection axioms V.A and V.A' relate past and future tenses by combining the connectives X and Y .

All rules of group VI are standard modal logic deduction rules, and in particular any normal modal logic is closed under these rules.

Most of the axioms are either included in the axiomatisation of intuitionistic LTL [57] or a variant of one of them (e.g. a contrapositive). From this, we easily derive the following.

Proposition 7.2. *The above calculus is sound for the class of real-valued models, as well as for the class of bi-relational models.*

Proof. The rules II.B and II.C are readily seen to preserve validity. We check Axiom III.B; all other rules or axioms have been shown to be sound for intuitionistic or bi-relational models in the literature (see e.g. [56,58]).

III.B: Let us assume towards a contradiction that $\neg((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$ is not valid with respect to bi-relational models, so we can find $\mathcal{M} = (W, T, \leq, S, \llbracket \cdot \rrbracket)$ and $(w, t) \in \mathcal{M}$ such that $(w, t) \notin \llbracket \neg((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi)) \rrbracket$. Therefore there exists $(v, t) \leq (w, t)$ such that $(v, t) \in \llbracket (\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi) \rrbracket$. Then $(v, t) \in \llbracket \varphi \Leftarrow \psi \rrbracket$ and $(v, t) \in \llbracket \psi \Leftarrow \varphi \rrbracket$. Hence there exist $(v', t) \geq (v, t)$ and $(v'', t) \geq (v, t)$ such that $(v', t) \in \llbracket \varphi \rrbracket \setminus \llbracket \psi \rrbracket$ and $(v'', t) \in \llbracket \psi \rrbracket \setminus \llbracket \varphi \rrbracket$. Since (W, \leq) is a linear order, either $(v', t) \leq (v'', t)$ or $(v'', t) \leq (v', t)$. In the former case we get that $(v', t) \in \llbracket \psi \rrbracket$ and in the latter case we get that $(v'', t) \in \llbracket \varphi \rrbracket$; in any case we reach a contradiction. \square

Our main objective is to show that our calculus is indeed complete; proving this will take up the remainder of this article.

As we show next, we can also derive the converses of some of these axioms. Below, for a set of formulas Γ we define $X\Gamma = \{X\varphi \mid \varphi \in \Gamma\}$, and empty conjunctions and disjunctions are defined by $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \perp$.

When reasoning about GTL, it is useful to note that by I and MP (modus ponens), the following weak form of the deduction theorem holds: for $\varphi, \psi \in \mathcal{L}$, we have $\varphi \Rightarrow \psi \in \text{GTL}$ if (and only if) we can deduce ψ from φ in the system with GTL as its axioms and modus ponens as its only deduction rule. We will often implicitly use this to simplify reasoning.

Lemma 7.3. *Let $\varphi \in \mathcal{L}$ and $\Gamma \subseteq \mathcal{L}$ be finite. Then the following formulas belong to GTL.*

- | | |
|---|---|
| (a) $X(\bigvee \Gamma) \Leftrightarrow \bigvee X\Gamma$ | (g) $Y(\bigvee \Gamma) \Leftrightarrow \bigvee Y\Gamma$ |
| (b) $X(\bigwedge \Gamma) \Leftrightarrow \bigwedge X\Gamma$ | (h) $Y(\bigwedge \Gamma) \Leftrightarrow \bigwedge Y\Gamma$ |
| (c) $\varphi \cup \psi \Rightarrow F\psi$ | (i) $\varphi S \psi \Rightarrow P\psi$ |
| (d) $\varphi \cup \psi \Rightarrow \psi \vee (\varphi \wedge X(\varphi \cup \psi))$ | (j) $\varphi S \psi \Rightarrow \psi \vee (\varphi \wedge Y(\varphi S \psi))$ |
| (e) $\varphi \wedge XG\varphi \Rightarrow G\varphi$ | (k) $\varphi \wedge YH\varphi \Rightarrow H\varphi$ |
| (f) $(\varphi \Leftarrow \varphi) \Rightarrow \varphi$ | (l) $(\varphi \Leftarrow \psi) \Rightarrow \varphi$ |

Proof. For claims (a) and (b), one direction is obtained from repeated use of Axioms IV.B or IV.C and the other is proven using NEC_X and IV.D; note that the first claim requires IV.A to treat the case when $\Gamma = \emptyset$. Details are left to the reader. For (c), we recall that $F\psi$ is shorthand for $\top \cup \psi$. Then one uses IV.G and the tautology $\varphi \rightarrow \top$ to obtain $\varphi \cup \psi \rightarrow \top \cup \psi$.

For (d), let $\theta = \psi \vee (\varphi \wedge X(\varphi \cup \psi))$. We claim that $\varphi \wedge X\theta \Rightarrow \theta$ is derivable, for which it suffices to show $X\theta \Rightarrow X(\varphi \cup \psi)$. Using (a) and (b), this amounts to showing

$$X\psi \vee (X\varphi \wedge XX(\varphi \cup \psi)) \Rightarrow X(\varphi \cup \psi),$$

which would follow from both $X\psi \Rightarrow X(\varphi \cup \psi)$ and $X\varphi \wedge XX(\varphi \cup \psi) \Rightarrow X(\varphi \cup \psi)$ via propositional reasoning. But these follow from IV.I, which yields $\psi \Rightarrow (\varphi \cup \psi)$ and $\varphi \wedge X(\varphi \cup \psi) \Rightarrow (\varphi \cup \psi)$, along with routine reasoning involving X .

From $\varphi \wedge X\theta \Rightarrow \theta$, we use necessitation and IV.K to obtain $\varphi \cup \theta \Rightarrow \theta$. Now we note that $\psi \Rightarrow \theta$ is a tautology; hence using IV.F, we obtain $\varphi \cup \psi \Rightarrow \theta$, i.e. $\varphi \cup \psi \Rightarrow \psi \vee (\varphi \wedge X(\varphi \cup \psi))$, as needed.

Claim (e) is similar. Note that $G\varphi \Rightarrow \varphi$ holds by IV.H, so we have $XG\varphi \Rightarrow X\varphi$ by NEC_X , IV.D, and modus ponens as before. Similarly, $G\varphi \Rightarrow XG\varphi$ holds by IV.H, so $XG\varphi \Rightarrow XXG\varphi$ holds by NEC_X , IV.D, and modus ponens. Hence we have

$$XG\varphi \Rightarrow X\varphi \wedge XXG\varphi.$$

Using IV.C and some propositional reasoning we obtain

$$\varphi \wedge XG\varphi \Rightarrow X(\varphi \wedge XG\varphi).$$

By G-necessitation and IV.J, we have

$$\varphi \wedge \text{XG}\varphi \Rightarrow \text{G}(\varphi \wedge \text{XG}\varphi).$$

Since $\text{G}(\varphi \wedge \text{XG}\varphi) \Rightarrow \text{G}\varphi$ can be proven using IV.E, we obtain $\varphi \wedge \text{XG}\varphi \Rightarrow \text{G}\varphi$, as needed.

Claim (f) can be derived from the intuitionistic tautology $\varphi \Rightarrow \psi \vee \varphi$ and Rule II.C.

The proofs of claims (g)–(l) follow similar lines of reasoning as for their future versions. \square

8. The canonical system and canonical quasimodel

In this section we first construct a standard canonical structure for GTL. In the presence of U, S, G, and H, this canonical structure is only a labelled system, rather than a proper bi-relational model. Nevertheless, it will be a useful ingredient in our completeness proof, for we can apply the quotient construction of Section 6 to it and show that the resulting labelled system is in fact a quasimodel.

Since we are working over an intermediate logic, the role of maximal consistent sets will be played by complete types, as defined below (see Definition 4.3 for the definition of *types*). Below, recall that by convention, $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \perp$.

Definition 8.1. Given two sets of formulas $\Gamma, \Delta \subseteq \mathcal{L}$, we say that Δ is a **consequence** of Γ , denoted by $\Gamma \vdash \Delta$, if there exist finite (possibly empty) $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\bigwedge \Gamma' \Rightarrow \bigvee \Delta' \in \text{GTL}$. We say that the pair (Γ, Δ) is **consistent** if $\Gamma \not\vdash \Delta$. We will call a Σ -type Φ consistent if $(\Phi, \Sigma \setminus \Phi)$ is consistent.

Note that we are using the standard interpretation of $\Gamma \vdash \Delta$ in Gentzen-style calculi. When working within a turnstile, we will follow the usual proof-theoretic conventions of writing Γ, Δ instead of $\Gamma \cup \Delta$, and writing φ instead of $\{\varphi\}$.

We will often want to think of a consistent Σ -type Φ as representing the pair $(\Phi, \Sigma \setminus \Phi)$. Note that for such a Φ , the set $\Sigma \setminus \Phi$ cannot contain any formula in GTL since $\varphi \in \text{GTL}$ implies $\emptyset \vdash \varphi$.

Lemma 8.2. If $\Sigma \subseteq \mathcal{L}$ is closed under subformulas, $\Phi \subseteq \Sigma$, and $(\Phi, \Sigma \setminus \Phi)$ is consistent, then Φ is a (consistent) Σ -type.

Proof. We check the conditions of Definition 4.3.

- Condition 1: If $\varphi \wedge \psi \in \Phi$ then since $\varphi \wedge \psi \Rightarrow \varphi$ is an intuitionistic tautology, we cannot have $\varphi \in \Sigma \setminus \Phi$; thus $\varphi \in \Phi$. Similarly, $\psi \in \Phi$. The converse is even more trivial, using the tautology $\varphi \wedge \psi \Rightarrow \varphi \wedge \psi$.
- Condition 2: If $\varphi \vee \psi \in \Phi$ then since $\varphi \vee \psi \Rightarrow \varphi \vee \psi$ is an intuitionistic tautology, we cannot have both $\varphi \in \Sigma \setminus \Phi$ and $\psi \in \Sigma \setminus \Phi$; thus either $\varphi \in \Phi$ or $\psi \in \Phi$. The converse uses the tautologies $\varphi \Rightarrow \varphi \vee \psi$ and $\psi \Rightarrow \varphi \vee \psi$.
- Condition 3: (a) If $\varphi \Rightarrow \psi \in \Phi$ and $\varphi \in \Phi$, then since $(\varphi \Rightarrow \psi) \wedge \varphi \Rightarrow \psi$ is an intuitionistic tautology, we cannot have $\psi \in \Sigma \setminus \Phi$; thus $\psi \in \Phi$.
(b) If $\psi \in \Phi$ then since $\psi \Rightarrow (\varphi \Rightarrow \psi)$ is an intuitionistic tautology, we cannot have $\varphi \Rightarrow \psi \in \Sigma \setminus \Phi$; thus $\varphi \Rightarrow \psi \in \Phi$.
- Condition 4: (a) By Lemma 7.3(l), we know that $\varphi \Leftarrow \psi \vdash \varphi$. Hence if $\varphi \Leftarrow \psi \in \Phi$ we cannot have $\varphi \in \Sigma \setminus \Phi$; thus $\varphi \in \Phi$.
(b) By Axiom II.A we know that $\varphi \vdash \psi \vee (\varphi \Leftarrow \psi)$. Hence if $\varphi \in \Phi$ and $\psi \in \Sigma \setminus \Phi$, we cannot have $\varphi \Leftarrow \psi \in \Sigma \setminus \Phi$; thus $\varphi \Leftarrow \psi \in \Phi$. \square

As with maximal consistent sets, consistent types satisfy a Lindenbaum property. Below, if (Γ, Δ) and (Γ', Δ') are pairs of sets of formulas, we say that (Γ', Δ') **extends** (Γ, Δ) if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. A consistent Σ -type Φ extends (Γ, Δ) if $(\Phi, \Sigma \setminus \Phi)$ extends (Γ, Δ) .

Lemma 8.3 (Lindenbaum lemma). Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas, and $\Gamma, \Delta \subseteq \Sigma$. If $\Gamma \not\vdash \Delta$, then there exists a consistent Σ -type extending (Γ, Δ) .

Proof. The proof is standard, but we provide a sketch. Let $\varphi \in \Sigma$. Note that either $\Gamma, \varphi \not\vdash \Delta$ or $\Gamma \not\vdash \Delta, \varphi$, for otherwise by an application of the cut rule (which is intuitionistically derivable) we would have $\Gamma \vdash \Delta$. Thus we can add φ to Γ or to Δ and remain consistent. By repeating this process for each of the countable number of formulas of Σ (or using Zorn's lemma) we can find a consistent (Γ', Δ') extending (Γ, Δ) such that $\Gamma' \cup \Delta' = \Sigma$. Since (Γ', Δ') is consistent, Δ' is also disjoint from Γ' ; hence $\Delta' = \Sigma \setminus \Gamma'$. Thus by Lemma 8.2, Γ' is a consistent Σ -type. \square

Before defining the canonical system, recall that for a set of formulas Γ , we have $\text{X}\Gamma := \{\text{X}\varphi \mid \varphi \in \Gamma\}$. We also define

$$\text{X}^{-1}\Gamma := \{\varphi \mid \text{X}\varphi \in \Gamma\}.$$

Given a set A , let \mathbb{I}_A denote the identity function on A . The canonical system \mathfrak{C} is defined as the labelled structure

$$\mathfrak{C} = (\mathbb{T}_\infty, \leq_{\mathfrak{C}}, \ell_{\mathfrak{C}}, S_{\mathfrak{C}}),$$

where

- \mathbb{T}_∞ is the set of consistent \mathcal{L} -types,
- $\Phi \leq_{\mathcal{C}} \Psi$ if $\Phi \supseteq \Psi$,
- $\ell_{\mathcal{C}}(\Phi) = \Phi$,
- $S_{\mathcal{C}}(\Phi) = X^{-1}\Phi$.

We will usually omit writing $\ell_{\mathcal{C}}$, as it has no effect on its argument. We will verify shortly that $X^{-1}\Phi$ is indeed a consistent \mathcal{L} -type.

Next we show that \mathcal{C} is an \mathcal{L} -labelled system. We begin by showing that it is based on a labelled space.

Lemma 8.4. $(\mathbb{T}_\infty, \leq_{\mathcal{C}}, \ell_{\mathcal{C}})$ is an \mathcal{L} -labelled space.

Proof. It is trivial that $\leq_{\mathcal{C}}$ is a partial order. Moreover, $\ell_{\mathcal{C}}$ is the identity, so $\Phi \leq_{\mathcal{C}} \Psi$ implies that $\ell_{\mathcal{C}}(\Phi) \supseteq \ell_{\mathcal{C}}(\Psi)$; that is, $\ell_{\mathcal{C}}$ is anti-monotone.

To prove that $(\mathbb{T}_\infty, \leq_{\mathcal{C}})$ is locally linear, assume towards a contradiction that it is not. We consider two cases:

1. There exist Φ, Ψ and Θ such that $\Phi \subseteq \Psi$ and $\Phi \subseteq \Theta$, but $\Psi \not\subseteq \Theta$ and $\Theta \not\subseteq \Psi$. Thus there exist two formulas $\varphi \in \Theta \setminus \Psi$ and $\psi \in \Psi \setminus \Theta$. Then it is easy to see that $\varphi \Rightarrow \psi \notin \Theta$ and $\psi \Rightarrow \varphi \notin \Psi$. Thus neither of these formulas is in Φ , so by condition 2 of Definition 4.3 their disjunction, Axiom III.A, is in $\mathcal{L} \setminus \Phi$ —a contradiction.
2. There exist Φ, Ψ and Θ such that $\Phi \supseteq \Psi$ and $\Phi \supseteq \Theta$, but $\Psi \not\supseteq \Theta$ and $\Theta \not\supseteq \Psi$. Then it is easy to see that there exist two formulas $\varphi \in \Psi \setminus \Theta$ and $\psi \in \Theta \setminus \Psi$ such that $\varphi \Leftarrow \psi \in \Psi$ and $\psi \Leftarrow \varphi \in \Theta$. From $\Phi \supseteq \Psi$ and $\Phi \supseteq \Theta$, we conclude that $\varphi \Leftarrow \psi, \psi \Leftarrow \varphi \in \Phi$. But this contradicts the consistency of Φ , since by Axiom III.B we have $\varphi \Leftarrow \psi, \psi \Leftarrow \varphi \vdash \perp$.

We finish by considering the conditions on \Rightarrow and \Leftarrow in the definition of a labelled space. Let us consider $\Phi \in \mathbb{T}_\infty$.

- Suppose $\varphi \Rightarrow \psi \notin \Phi$. Then by condition 3b of Definition 4.3, $\psi \notin \Phi$. Let us define $u = (\Phi \cup \{\varphi\}, \{\psi\})$, and let us assume for a contradiction that u is inconsistent. This means that there exists $\gamma \in \Phi$ such that $\gamma \wedge \varphi \Rightarrow \psi \in \text{GTL}$. By propositional reasoning, $\gamma \Rightarrow (\varphi \Rightarrow \psi) \in \text{GTL}$. Since $\gamma \in \Phi$ and Φ is consistent, $\varphi \Rightarrow \psi \notin \mathcal{L} \setminus \Phi$, that is, $\varphi \Rightarrow \psi \in \Phi$ —a contradiction. Hence u is consistent and by Lemma 8.3 can be extended to a consistent \mathcal{L} -type Ψ . From the definition of u we can conclude that $\Psi \leq_{\mathcal{C}} \Phi$, $\varphi \in \Psi$, and $\psi \notin \Psi$ as required.
- Suppose $\varphi \Leftarrow \psi \in \Phi$. Then by condition 4a of Definition 4.3, $\varphi \in \Phi$. Let us define $u = (\{\varphi\}, (\mathcal{L} \setminus \Phi) \cup \{\psi\})$, and let us assume by contradiction that u is inconsistent. This means that there exists $\gamma \in \mathcal{L} \setminus \Phi$ such that $\varphi \Rightarrow \psi \vee \gamma \in \text{GTL}$. By Rule II.C, we get $(\varphi \Leftarrow \psi) \Rightarrow \gamma \in \text{GTL}$. Since $\gamma \in \mathcal{L} \setminus \Phi$, we deduce that $\varphi \Leftarrow \psi \in \mathcal{L} \setminus \Phi$ —a contradiction. By Lemma 8.3, u can be extended to a complete type Ψ . It is easy to check that $\Phi \leq_{\mathcal{C}} \Psi$, $\varphi \in \Psi$, and $\psi \notin \Psi$ as required. \square

Lemma 8.5. The function $S_{\mathcal{C}} : \mathbb{T}_\infty \rightarrow \mathbb{T}_\infty$ is well defined. That is, $S_{\mathcal{C}}(\Phi) \in \mathbb{T}_\infty$ for every $\Phi \in \mathbb{T}_\infty$.

Proof. Let $\Phi \in \mathbb{T}_\infty$ and $\Psi = S_{\mathcal{C}}(\Phi) = X^{-1}\Phi$. By Lemma 8.2, we only need to check that $(\Psi, \mathcal{L} \setminus \Psi)$ is consistent. Suppose not; then let $\Gamma \subseteq \Psi$ and $\Delta \subseteq \mathcal{L} \setminus \Psi$ be finite and such that $\bigwedge \Gamma \Rightarrow \bigvee \Delta \in \text{GTL}$. Using NEC_X and IV.D we see that $X \bigwedge \Gamma \Rightarrow X \bigvee \Delta \in \text{GTL}$, which in view of Lemma 7.3 implies that $\bigwedge X\Gamma \Rightarrow \bigvee X\Delta \in \text{GTL}$ as well. But $X\Gamma \subseteq \Phi$ and $X\Delta \subseteq \mathcal{L} \setminus \Phi$ (since $\delta \in \Delta$ implies $\delta \notin X^{-1}\Phi$, which implies $X\delta \notin \Phi$), contradicting the fact that Φ is consistent. We conclude that $\Psi \in \mathbb{T}_\infty$. \square

In order to see that the function $S_{\mathcal{C}}$ is a fully confluent, convex relation, it will be useful to note that $S_{\mathcal{C}}$ is a bijection.

Lemma 8.6. The function $S_{\mathcal{C}}$ is a bijection.

Proof. Define $Y^{-1}\Phi := \{\varphi \mid Y\varphi \in \Phi\}$. We claim that $\Phi \mapsto Y^{-1}\Phi$ is the inverse of $S_{\mathcal{C}}$. By Axiom V.A

$$\varphi \in Y^{-1}X^{-1}\Phi \iff XY\varphi \in \Phi \iff \varphi \in \Phi,$$

and by Axiom V.A'

$$\varphi \in X^{-1}Y^{-1}\Phi \iff YX\varphi \in \Phi \iff \varphi \in \Phi. \quad \square$$

Lemma 8.7. The function $S_{\mathcal{C}}$ is a fully confluent relation.

Proof. We check the four confluence conditions.

Forth-down, forth-up: Let Φ and Ψ be such that $\Phi \supseteq \Psi$. Since $S_{\mathcal{C}}$ is a function, these two confluence properties amount to showing that $S_{\mathcal{C}}(\Phi) \supseteq S_{\mathcal{C}}(\Psi)$. If $\varphi \in S_{\mathcal{C}}(\Psi)$ then $X\varphi \in \Psi$, which since $\Phi \supseteq \Psi$, implies that $X\varphi \in \Phi$, and hence $\varphi \in S_{\mathcal{C}}(\Phi)$.

Back-down, back-up: Let Φ and Ψ be such that $\Phi \supseteq \Psi$. Since, by Lemma 8.6, $S_{\mathcal{C}}^{-1}$ is a function, these two confluence properties amount to showing that $S_{\mathcal{C}}^{-1}(\Phi) \supseteq S_{\mathcal{C}}^{-1}(\Psi)$, that is, $Y^{-1}(\Phi) \supseteq Y^{-1}(\Psi)$. If $\varphi \in Y^{-1}(\Psi)$ then $Y\varphi \in \Psi$, which since $\Phi \supseteq \Psi$, implies that $Y\varphi \in \Phi$, and hence $\varphi \in Y^{-1}(\Phi)$. \square

Lemma 8.8. *The function $S_{\mathcal{C}}$ is a convex relation.*

Proof. Since $S_{\mathcal{C}}$ is a function, images of points are singletons and hence automatically convex. The same is true for preimages of points, since by Lemma 8.6, $S_{\mathcal{C}}$ is an injection. \square

Lemma 8.9. *The function $S_{\mathcal{C}}$ is a sensible relation.*

Proof. Let us consider Φ and Ψ such that $\Psi = S_{\mathcal{C}}(\Phi)$. We verify the conditions required for (Φ, Ψ) to be sensible (Definition 4.6).

We have $\varphi \in \Psi$ if and only if $X\varphi \in \Phi$ by the definition of $S_{\mathcal{C}}$. Since $\Phi = Y^{-1}(\Psi)$, we similarly have $\varphi \in \Phi$ if and only if $Y\varphi \in \Psi$.

If $G\varphi \in \Phi$ then, by Axiom IV.H we get $\varphi, XG\varphi \in \Phi$. Since $\Psi = S_{\mathcal{C}}(\Phi)$, we get $G\varphi \in \Psi$. Conversely, assume that $G\varphi \in \Psi^-$. By Lemma 7.3(e), $\varphi \wedge XG\varphi \in \Phi^-$, so either $\varphi \in \Phi^-$ or $XG\varphi \in \Phi^-$ (giving in the second case $G\varphi \in \Psi^-$). In either case we reach the desired conclusion.

For the case of U, by Lemma 7.3(d), we know $\varphi \cup \psi \vdash \psi$, $\varphi \wedge X(\varphi \cup \psi)$. Hence if $\varphi \cup \psi \in \Phi$ then either $\psi \in \Phi$ or $\varphi \wedge X(\varphi \cup \psi) \in \Phi$. In the first case we are done immediately. In the second, since Φ is an \mathcal{L} -type we deduce $\varphi \in \Phi$ and $X(\varphi \cup \psi) \in \Phi$, the latter of which gives $\varphi \cup \psi \in \Psi$, and we are done. Conversely, by Axiom IV.I and some intuitionistic reasoning, both $\psi \vdash \varphi \cup \psi$ and $\varphi, X(\varphi \cup \psi) \vdash \varphi \cup \psi$. From the first, we see that $\psi \in \Phi$ implies $\varphi \cup \psi \in \Phi$. If, alternatively, both $\varphi \in \Phi$ and $\varphi \cup \psi \in \Psi$, then $\varphi \in \Phi$ and $X(\varphi \cup \psi) \in \Phi$, which together with $\varphi, X(\varphi \cup \psi) \vdash \varphi \cup \psi$, also yields $\varphi \cup \psi \in \Phi$.

The cases for H and S are the temporal converses of those for G and U respectively (using again the fact that $S_{\mathcal{C}}^{-1}$ is to Y as $S_{\mathcal{C}}$ is to X). \square

We note the general fact that given a Σ_1 -labelled system and a subformula-closed $\Sigma_2 \subseteq \Sigma_1$, one can restrict the labelling to Σ_2 in the natural way (by replacing its value at any point by its intersection with Σ_2). Doing so yields a Σ_2 -labelled system. This is easily verifiable from the definitions.

Proposition 8.10. *The canonical system \mathcal{C} is an \mathcal{L} -labelled system. Restricting the labelling to any subformula-closed $\Sigma \subseteq \mathcal{L}$ yields a Σ -labelled system.*

Proof. For the first claim, we need for the following three properties to hold:

1. $(\mathbb{T}_{\infty}, \leq_{\mathcal{C}}, \ell_{\mathcal{C}})$ is a labelled space;
2. $S_{\mathcal{C}}$ is a bi-serial, fully confluent, convex, sensible relation; and
3. $\ell_{\mathcal{C}}$ has $\mathbb{T}_{\mathcal{L}}$ as its codomain.

The first item is Lemma 8.4. The relation $S_{\mathcal{C}}$ is bi-serial since it is a well defined bijection, and it is a fully confluent, convex, sensible relation by Lemmas 8.7, 8.8, and 8.9. Finally, if $\Phi \in \mathbb{T}_{\infty}$ then $\ell_{\mathcal{C}}(\Phi) = \Phi$, which is an element of $\mathbb{T}_{\mathcal{L}}$ by Lemma 8.2.

The second claim follows from the observation preceding the proposition. \square

By Proposition 6.2, the structure $\mathcal{C}/\Sigma := (\mathbb{T}_{\infty}/\sim, \leq_Q, R_Q^+, \ell_Q)$ formed with the quotient construction of Section 6 is a Σ -labelled system. We call \mathcal{C}/Σ the **canonical quasimodel**, although we do not yet know it is a quasimodel, since we do not yet know that R_Q^+ is ω -sensible. In the sequel, we show that this is the case. From this we will obtain completeness of our deductive calculus, since any formula that is not derivable is falsified in the canonical quasimodel and hence on some model by unwinding.

9. Characteristic formulas

In this section, we show that there exist formulas defining points in the canonical quasimodel $\mathcal{C}/\Sigma = (\mathbb{T}_{\infty}/\sim, \leq_Q, R_Q^+, \ell_Q)$, i.e. to each $w \in \mathbb{T}_{\infty}/\sim$ we assign two formulas that together ‘distinguish’ w . In the next section, these formulas will play an essential role in the proof that R_Q^+ is ω -sensible, and hence that \mathcal{C}/Σ is a quasimodel.

First, we define a formula $\chi_{\Sigma}^+(w)$ (or $\chi^+(w)$ when Σ is clear from context) such that for all $\Gamma \in \mathbb{T}_{\infty}$, $\chi^+(w) \in \Gamma$ if and only if $[\Gamma] \leq w$. Dually, we define $\chi^-(w)$ so that for all $\Gamma \in \mathbb{T}_{\infty}$, $\chi^-(w) \notin \Gamma$ if and only if $w \leq [\Gamma]$. Compared to [59], these formulas require co-implication, as they must look ‘up’ as well as ‘down’ the model. In this section, we write $\mathcal{C}/\Sigma = (\mathbb{T}_{\infty}/\sim, \leq, R, \ell)$. We will omit subscript indices on the ℓ and L functions.

Definition 9.1. Fix a $\Sigma \subseteq \mathcal{L}$ that is finite and closed under subformulas. Given $\Delta \in \mathbb{T}_{\Sigma}$, define $\bar{\Delta} := \bigwedge \Delta \Rightarrow \bigvee (\Sigma \setminus \Delta)$ and $\bar{\Delta} := \bigwedge \Delta \Leftarrow \bigvee (\Sigma \setminus \Delta)$. Given $w \in \mathbb{T}_{\infty}/\sim$, we define a formula $\chi^0(w)$ by

$$\chi^0(w) := \bigwedge_{\Delta \in L(w)} \sim \bar{\Delta} \wedge \bigwedge_{\Delta \in \mathbb{T}_{\Sigma} \setminus L(w)} \neg \bar{\Delta}.$$

Then define $\chi^+(w)$ by

$$\chi^+(w) := \overline{\ell(w)} \wedge \chi^0(w)$$

and $\chi^-(w)$ by

$$\chi^-(w) := \chi^0(w) \Rightarrow \overline{\ell(w)}.$$

Proposition 9.2. *Given $w \in \mathbb{T}_\infty / \sim$ and $\Gamma \in \mathbb{T}_\infty$,*

- 1) $\chi^0(w) \in \Gamma$ if and only if $L(\Gamma) = L(w)$,
- 2) $\chi^+(w) \in \Gamma$ if and only if $[\Gamma] \leq w$, and
- 3) $\chi^-(w) \notin \Gamma$ if and only if $[\Gamma] \geq w$.

Proof. Let $w \in \mathbb{T}_\infty / \sim$ and $\Gamma \in \mathbb{T}_\infty$.

1) For the left-to-right implication, assume that $\chi^0(w) \in \Gamma$, so that both $\bigwedge_{\Delta \in L(w)} \sim \bar{\Delta} \in \Gamma$ and $\bigwedge_{\Delta \in \mathbb{T}_\Sigma \setminus L(w)} \neg \bar{\Delta} \in \Gamma$, by the defining conditions of \mathcal{L} -types. For the inclusion $L(w) \subseteq L(\Gamma)$, let $\Delta \in L(w)$. From $\sim \bar{\Delta} \in \Gamma$, we obtain $\Phi \geq \Gamma$ such that $\bar{\Delta} \notin \Phi$. Hence there is $\Phi_\Delta \leq \Phi$ with $\bigwedge \Delta \in \Phi_\Delta$ and $\bigvee (\Sigma \setminus \Delta) \notin \Phi_\Delta$; i.e. $\ell(\Phi_\Delta) = \Delta$. From local linearity we see that $\Phi_\Delta \leq \Gamma$; hence $\Delta = \ell(\Phi_\Delta) \in L(\Gamma)$.

For the inclusion $L(w) \supseteq L(\Gamma)$, let $\Delta \in \mathbb{T}_\Sigma \setminus L(w)$. Then for any $\Psi \leq \Gamma$ we have that $\bar{\Delta} \notin \Psi$, so that there is no $\Psi_\Delta \geq \Psi$ with $\bigwedge \Delta \in \Psi_\Delta$ and $\bigvee (\Sigma \setminus \Delta) \notin \Psi_\Delta$. Thus there is no $\Psi_\Delta \leq \Gamma$ with $\bigwedge \Delta \in \Psi_\Delta$ and $\bigvee (\Sigma \setminus \Delta) \notin \Psi_\Delta$ (for the $\Psi_\Delta \geq \Gamma$ case, set $\Psi = \Gamma$; for $\Psi_\Delta \leq \Gamma$ set $\Psi = \Psi_\Delta$). We deduce that $\Delta \notin L(\Gamma)$.

For the right-to-left implication, assume that $L(\Gamma) = L(w)$. Then for $\Delta \in L(\Gamma)$ we readily obtain $\sim \bar{\Delta} \in \Gamma$, and similarly for $\Delta \in \mathbb{T}_\Sigma \setminus L(\Gamma)$ we obtain $\neg \bar{\Delta} \in \Gamma$, from which we obtain by propositional reasoning that $\chi^0(w) \in \Gamma$.

2) If $\chi^+(w) \in \Gamma$ then $\chi^0(w) \in \Gamma$, and so by item (1), $L(\Gamma) = L(w)$, while $\overline{\ell(w)} \in \Gamma$ implies there is some $\Gamma' \geq \Gamma$ with $\ell(\Gamma') = \ell(w)$. This shows that $w = [\Gamma'] \geq [\Gamma]$, as claimed.

Conversely, if $[\Gamma] \leq w$, then $L(\Gamma) = L(w)$, and so by item (1), $\chi^0(w) \in \Gamma$. Further, there exists Γ' such that $[\Gamma'] = w$ and $\Gamma' \geq \Gamma$, which implies $\overline{\ell(w)} = \ell([\Gamma']) = \ell(\Gamma') \in \Gamma$ also.

3) If $\chi^-(w) \notin \Gamma$ then there exists $\Phi \leq \Gamma$ with $\chi^0(w) \in \Phi$ and $\overline{\ell(w)} \notin \Phi$. From $\chi^0(w) \in \Phi$ we deduce $L(\Gamma) = L(\Phi) = L(w)$. From $\overline{\ell(w)} \notin \Phi$ we deduce there exists $\Gamma' \leq \Phi$ with $\ell(\Gamma') = \ell(w)$. Since also $L(\Gamma') = L(\Phi) = L(w)$, we have $[\Gamma'] = w$. Then since $\Gamma \geq \Phi \geq \Gamma'$ we have $[\Gamma] \geq [\Gamma'] = w$.

Conversely, if $[\Gamma] \geq w$, then there exists $\Gamma' \leq \Gamma$ with $[\Gamma'] = w$. Now we have $\chi^0(w) \in \Gamma'$ but $\overline{\ell(w)} \notin \Gamma'$. It follows that $\chi^-(w) \notin \Gamma$. \square

Remark 9.3. Note that the formula $\chi_\Sigma^+(w)$ makes essential use of co-implication, as properties of $w \geq [\Gamma]$ do not affect truth values in Γ in the language with \Rightarrow alone. In contrast, the formulas χ_Σ^- are similar to the formulas $\text{Sim}(w)$ of [59], although we remark that co-implication is still needed to describe the full linear component of w .

Next we establish some provable properties of each of χ_Σ^+ and χ_Σ^- . We begin with the former.

Proposition 9.4. *Given $w \in \mathbb{T}_\infty / \sim$ and $\psi \in \Sigma$:*

- 1) If $\psi \in \Sigma \setminus \ell(w)$, then $\chi^+(w) \Rightarrow (\chi^+(w) \Leftarrow \psi) \in \text{GTL}$.
- 2) If $\psi \in \ell(w)$, then $\chi^+(w) \Rightarrow \psi \in \text{GTL}$.
- 3) For any $w \in \mathbb{T}_\infty / \sim$, $\chi^+(w) \Rightarrow \bigvee_{wRv} \chi^+(v) \in \text{GTL}$.

Proof. 1) Let $\psi \in \Sigma \setminus \ell(w)$. It suffices to show that for an arbitrary $\Gamma \in \mathbb{T}_\infty$, we have that $\chi^+(w) \in \Gamma$ implies $\chi^+(w) \Leftarrow \psi \in \Gamma$, for this demonstrates, by Lemma 8.3, that $(\chi^+(w), \chi^+(w) \Leftarrow \psi)$ is inconsistent, giving $\chi^+(w) \Rightarrow (\chi^+(w) \Leftarrow \psi) \in \text{GTL}$. From $\chi^+(w) \in \Gamma$ and Proposition 9.2 we obtain $\Gamma' \leq \Gamma$ such that $[\Gamma'] = w$, and hence $\chi^+(w) \in \Gamma'$ and $\psi \notin \Gamma'$, yielding $\chi^+(w) \Leftarrow \psi \in \Gamma$.

2) With similar reasoning as for 1), let $\psi \in \ell(w)$ and $\Gamma \in \mathbb{T}_\infty$, and suppose $\chi^+(w) \in \Gamma$. Let $\Gamma' \leq \Gamma$ with $[\Gamma'] = w$. Then $\psi \in \Gamma'$, yielding $\psi \in \Gamma$.

3) Let Γ be such that $\chi^+(w) \in \Gamma$, so that there is $\Gamma' \geq \Gamma$ with $[\Gamma'] = w$. Then $wR[S_\ell(\Gamma')]$ by definition, and moreover $\chi^+([S_\ell(\Gamma')]) \in S_\ell(\Gamma')$, which implies that $\bigvee_{wRv} \chi^+(v) \in \Gamma'$. Thus $\bigvee_{wRv} \chi^+(v) \in \Gamma$ since $\Gamma' \subseteq \Gamma$, so that $\bigvee_{wRv} \chi^+(v) \in \Gamma$. \square

The formula χ_Σ^- behaves ‘order dually’, as established below.

Proposition 9.5. *Given $w \in \mathbb{T}_\infty / \sim$ and $\psi \in \Sigma$:*

- 1) If $\psi \in \Sigma \setminus \ell(w)$, then $\psi \Rightarrow \chi^-(w) \in \text{GTL}$.
- 2) If $\psi \in \ell(w)$, then $(\psi \Rightarrow \chi^-(w)) \Rightarrow \chi^-(w) \in \text{GTL}$.

3) For any $w \in \mathbb{T}_\infty / \sim$, $X(\bigwedge_{wRv} \chi^-(v)) \Rightarrow \chi^-(w) \in \text{GTL}$.

Proof. 1) Assume that $\psi \in \Sigma \setminus \ell(w)$ and $\psi \in \Gamma$ and write $w = [\Gamma']$. Then $\psi \notin \Gamma'$, which means $\Gamma \not\leq \Gamma'$, so $[\Gamma] \not\leq w$. Hence Proposition 9.2(3) implies that $\chi^-(w) \in \Gamma$.

2) Suppose that $\psi \in \ell(w)$ and proceed to prove the claim by contraposition. If $\chi^-(w) \notin \Gamma$ for some $\Gamma \in \mathbb{T}_\infty$, then there exists $\Gamma' \leq \Gamma$ such that $w = [\Gamma']$. But then $\chi^-(w) \notin \Gamma'$ and $\psi \in \Gamma'$, which implies that $\psi \Rightarrow \chi^-(w) \notin \Gamma'$, and hence also $\psi \Rightarrow \chi^-(w) \notin \Gamma$, as required.

3) Proceed by contraposition. If $\chi^-(w) \notin \Gamma$ for some $\Gamma \in \mathbb{T}_\infty$, then there exists $\Gamma' \leq \Gamma$ such that $w = [\Gamma']$. We have that $w R [S_{\mathcal{C}}(\Gamma')]$ by definition. Letting $v = [S_{\mathcal{C}}(\Gamma')]$, we have that $\chi^-(v) \in S_{\mathcal{C}}(\Gamma')$, and hence $X\chi^-(v) \notin \Gamma'$. Then since $\Gamma' \geq \Gamma$, we have $X\chi^-(v) \notin \Gamma$. Hence $X\bigwedge_{wRv} \chi^-(v) \notin \Gamma$. \square

10. Completeness

In this section, we use the formulas χ_Σ^\pm to show that the relation R_Q^+ on \mathcal{C}/Σ is ω -sensible and hence \mathcal{C}/Σ is a quasimodel. Since validity over the class of quasimodels is equivalent to bi-relational validity, by Propositions 5.5 and 6.5, and bi-relational validity is equivalent real-valued validity, by Theorem 3.3, completeness will follow.

The following lemma is the first step towards establishing ω -sensitivity. Once again, we write $\mathcal{C}/\Sigma = (\mathbb{T}_\infty / \sim, \leq, R, \ell)$, and as usual R^* is the transitive, reflexive closure of R .

Lemma 10.1. *If $\Sigma \subseteq \mathcal{L}$ is finite and closed under subformulas and $w \in \mathbb{T}_\infty / \sim$, then*

1. $\bigvee_{wR^*v} \chi^+(v) \Rightarrow X\bigvee_{wR^*v} \chi^+(v) \in \text{GTL}$;
2. $X\bigwedge_{wR^*v} \chi^-(v) \Rightarrow \bigwedge_{wR^*v} \chi^-(v) \in \text{GTL}$;
3. $\bigvee_{vR^*w} \chi^+(v) \Rightarrow Y\bigvee_{vR^*w} \chi^+(v) \in \text{GTL}$;
4. $Y\bigwedge_{vR^*w} \chi^-(v) \Rightarrow \bigwedge_{vR^*w} \chi^-(v) \in \text{GTL}$.

Proof. Item 1 follows from Proposition 9.4(3), as for any $v \in R^*(w)$ we have that

$$\chi^+(v) \Rightarrow X\bigvee_{vRu} \chi^+(u) \in \text{GTL}.$$

Since $v R u$ implies that $w R^* u$ by transitivity,

$$\chi^+(v) \Rightarrow X\bigvee_{wR^*u} \chi^+(u) \in \text{GTL}.$$

Since v was arbitrary, we obtain

$$\bigvee_{wR^*v} \chi^+(v) \Rightarrow X\bigvee_{wR^*u} \chi^+(u) \in \text{GTL},$$

which by a change of variables yields the original claim.

Item 2 is similar, but uses Proposition 9.5(3). Items 3 and 4 are the temporal duals of items 1 and 2. \square

In order to complete our proof that R is ω -sensible, it suffices to apply induction to the formulas of Lemma 10.1.

Proposition 10.2.

1. If $w \in \mathbb{T}_\infty / \sim$ and $\varphi \cup \psi \in \ell(w)$, then there exists $v \in R^*(w)$ such that $\psi \in \ell(v)$.
2. If $w \in \mathbb{T}_\infty / \sim$ and $\varphi \cup \psi \in \ell(w)$, then there exists $v \in (R^{-1})^*(w)$ such that $\psi \in \ell(v)$.
3. If $w \in \mathbb{T}_\infty / \sim$ and $G\psi \in \Sigma \setminus \ell(w)$, then there exists $v \in R^*(w)$ such that $\psi \notin \ell(v)$.
4. If $w \in \mathbb{T}_\infty / \sim$ and $H\psi \in \Sigma \setminus \ell(w)$, then there exists $v \in (R^{-1})^*(w)$ such that $\psi \notin \ell(v)$.

Proof. 1. Towards a contradiction, assume that $w \in \mathbb{T}_\infty / \sim$ and $\varphi \cup \psi \in \ell(w)$ but, for all $v \in R^*(w)$, $\psi \notin \ell(v)$. By Lemma 10.1(2), $X(\bigwedge_{wR^*v} \chi^-(v)) \Rightarrow \bigwedge_{wR^*v} \chi^-(v) \in \text{GTL}$. By the F-induction axiom IV.K and standard modal reasoning, $F(\bigwedge_{wR^*v} \chi^-(v)) \Rightarrow \bigwedge_{wR^*v} \chi^-(v) \in \text{GTL}$; in particular,

$$F(\bigwedge_{wR^*v} \chi^-(v)) \Rightarrow \chi^-(w) \in \text{GTL}. \quad (2)$$

Now let $v \in R^*(w)$. By Proposition 9.5(1) and the assumption that $\psi \notin \ell(v)$, we have that $\psi \Rightarrow \chi^-(v) \in \text{GTL}$, and since v was an arbitrary element of $R^*(w)$, we have $\psi \Rightarrow \bigwedge_{wR^*v} \chi^-(v) \in \text{GTL}$. Using G-necessitation and the distributivity axiom IV.F we then

have that $F\psi \Rightarrow F \bigwedge_{wR^*v} \chi^-(v) \in \text{GTL}$. This, along with (2), shows that $F\psi \Rightarrow \chi^-(w) \in \text{GTL}$. Since by Lemma 7.3(c), $\varphi \cup \psi \Rightarrow F\psi$, we can obtain $\varphi \cup \psi \Rightarrow \chi^-(w) \in \text{GTL}$. However, by Proposition 9.5(2) and our assumption that $\varphi \cup \psi \in \ell(w)$ we have that $(\varphi \cup \psi \Rightarrow \chi^-(w)) \Rightarrow \chi^-(w) \in \text{GTL}$. Hence by modus ponens we obtain $\chi^-(w) \in \text{GTL}$. Writing $w = [\Gamma]$, Proposition 9.2 yields $\chi^-(w) \notin \Gamma^+$, but this contradicts $\chi^-(w) \in \text{GTL}$. We conclude that there exists $v \in R^*(w)$ with $\psi \in \ell(v)$, as needed.

2. This is the temporal dual of item 1 (so in particular we use Lemma 10.1(4) in place of Lemma 10.1(2)).

3. This is similar to the U case, but order dualised. Towards a contradiction, assume that $w \in \mathbb{T}_\infty / \sim$ and $G\psi \in \Sigma \setminus \ell(w)$, but for all $v \in R^*(w)$, $\psi \in \ell(v)$. By Lemma 10.1(1), $\bigvee_{wR^*v} \chi^+(v) \Rightarrow X \bigvee_{wR^*v} \chi^+(v) \in \text{GTL}$. By the G-induction axiom IV.J, $\bigvee_{wR^*v} \chi^+(v) \Rightarrow G \bigvee_{wR^*v} \chi^+(v) \in \text{GTL}$; in particular,

$$\chi^+(w) \Rightarrow G \bigvee_{wR^*v} \chi^+(v) \in \text{GTL}. \quad (3)$$

Now let $v \in R^*(w)$. By Proposition 9.4(2) and the assumption that $\psi \in \ell(v)$, we have that $\chi^+(v) \Rightarrow \psi \in \text{GTL}$, and since v was arbitrary, $\bigvee_{wR^*v} \chi^+(v) \Rightarrow \psi \in \text{GTL}$. Using the distributivity axiom IV.E we then have that $G \bigvee_{wR^*v} \chi^+(v) \Rightarrow G\psi \in \text{GTL}$. This, along with (3), shows that $\chi^+(w) \Rightarrow G\psi \in \text{GTL}$. Hence by Rule II.B we obtain

$$(\chi^+(w) \Leftarrow G\psi) \Rightarrow (G\psi \Leftarrow G\psi) \in \text{GTL}. \quad (4)$$

By Proposition 9.4(1) and our assumption that $G\psi \in \Sigma \setminus \ell(w)$ we have that $\chi^+(w) \Rightarrow (\chi^+(w) \Leftarrow G\psi) \in \text{GTL}$. From this together with (4) we obtain $\chi^+(w) \Rightarrow (G\psi \Leftarrow G\psi) \in \text{GTL}$. In view of Lemma 7.3(f), this gives $\chi^+(w) \Rightarrow \perp \in \text{GTL}$, so that the pair $(\chi^+(w), \emptyset)$ is inconsistent. Writing $w = [\Gamma]$, Proposition 9.2 yields $\chi^+(w) \in \Gamma$, contradicting the deduced inconsistency. We conclude that there exists $v \in R^*(w)$ with $\psi \in \Sigma \setminus \ell(v)$.

4. This is the temporal dual of item 3 (so in particular we use Lemma 10.1(3) in place of Lemma 10.1(1)). \square

Corollary 10.3. *If $\Sigma \subseteq \mathcal{L}$ is finite and closed under subformulas, then \mathcal{C}/Σ is a quasimodel.*

Proof. By Proposition 8.10 and Proposition 6.2, \mathcal{C}/Σ is a labelled system, while by Proposition 10.2, R is ω -sensible. By definition, these facts make \mathcal{C}/Σ a quasimodel. \square

We are now ready to prove that our calculus is complete.

Theorem 10.4. *If $\varphi \in \mathcal{L}$ is valid, then $\varphi \in \text{GTL}$.*

Proof. We prove the contrapositive. Suppose φ is an unprovable formula and let Σ be the set of subformulas of φ . Since φ is unprovable, there is $\Gamma \in \mathbb{T}_\infty$ with $\varphi \notin \Gamma$. Then $[\Gamma] \in \mathbb{T}_\infty / \sim$ is a point in a quasimodel falsifying φ , so that by Proposition 5.5, φ is not valid. \square

11. PSPACE-complete complexity

The proof of Theorem 6.7 yields only a NEXPTIME upper bound for the validity problem. In this section, we prove that this can be improved to PSPACE.

First, we recall that the validity problem for LTL is PSPACE-complete [60, Theorem 4.1]. Thus to prove PSPACE-hardness of the GTL validity problem, it suffices to give a reduction from LTL validity to GTL validity. Consider the (negative) translation $(\cdot)^*$ [61] defined as follows:

- 1) $p^* = \neg p$, for each propositional variable p , and
- 2) homomorphic for the operators.

In what follows we may assume that $T = \mathbb{Z}$, equipped with the standard successor function.

To any given LTL model (\mathbb{Z}, S, V) , we associate a ‘crisp’ Gödel model (\mathbb{Z}, S, V') where $V'(p, t) = 1$ if $t \in V(p)$ and 0 otherwise.

Proposition 11.1. *For any $\varphi \in \mathcal{L}$ and for all $t \in \mathbb{Z}$,*

- 1) if $(\mathbb{Z}, S, V), t \models \varphi$ then $V'(\varphi, t) = 1$, and
- 2) if $(\mathbb{Z}, S, V), t \not\models \varphi$ then $V'(\varphi, t) = 0$.

Proof. By structural induction on φ . \square

Conversely, to any given real-valued Gödel temporal model (\mathbb{Z}, S, V') , we associate the crisp Gödel temporal model (\mathbb{Z}, S, V) by fixing $V(p, t) = V'(\neg p, t) \in \{0, 1\}$.

Proposition 11.2. *For any $\varphi \in \mathcal{L}$ and any $t \in \mathbb{Z}$, we have $V'(\varphi^*, t) = V(\varphi, t)$.*

Proof. By structural induction. The case $\varphi \in \mathbb{P}$ follows from the definitions: $V'(p^*, t) = V'(\neg p, t) = V(p, t)$. The other cases follow from $(\cdot)^*$ being homomorphic. \square

Corollary 11.3. *For any $\varphi \in \mathcal{L}$, we have $\text{LTL} \models \varphi$ if and only if $\text{GTL} \models \varphi^*$.*

Proof. For the left-to-right direction, assume by contraposition that $\text{GTL} \not\models \varphi^*$. Therefore there exists a Gödel temporal model (\mathbb{Z}, S, V') and $t \in \mathbb{Z}$ such that $V'(\varphi^*, t) \neq 1$. By Proposition 11.2 there exists a crisp Gödel temporal model (\mathbb{Z}, S, V) such that $V(\varphi, t) \neq 1$. This latter model can be viewed as an LTL model with $(\mathbb{Z}, S, V), t \not\models \varphi$, since the real-valued semantics coincide with classical truth definitions when values are in $\{0, 1\}$. Therefore $\text{LTL} \not\models \varphi$.

Conversely, assume by contraposition that $\text{LTL} \not\models \varphi$. This means that there exists an LTL model (\mathbb{Z}, S, V) and $t \in \mathbb{Z}$ such that $(\mathbb{Z}, S, V), t \not\models \varphi$. Then for the crisp Gödel temporal model (\mathbb{Z}, S, V') as defined in Proposition 11.1 we have $V'(\varphi^*, t) = 0$. As a consequence $\text{GTL} \not\models \varphi^*$. \square

For the membership of PSPACE, we adapt the proof of LTL satisfiability from [1, Chapter 6] to the case of GTL. Define an **ultimately periodic quasimodel** to be a quasimodel $Q = (W, \leq, \ell, S)$ such that there is a ‘double lasso’ graph (T, R) with $T = \{-(i' + l' - 1), \dots, i + l - 1\}$, $k R (k + 1)$ for $-(i' + l' - 1) \leq k \leq i + l - 2$, and also $(i + l) R i$ and $-i' R -(i' + l')$, and a ‘projection’ function $\pi: W \rightarrow T$ such that each $\pi^{-1}(t)$ is an entire linear component of W , and $w S v$ implies $\pi(w) R \pi(v)$.

In other words, Q has an underlying nondeterministic ‘flow’ T consisting of a loop, followed by a middle segment (containing 0), and finally a second loop, and each $t \in T$ is assigned a linear order $\pi^{-1}(t)$, which we may also write as W_t .

Theorem 11.4 (ultimately periodic quasimodel property). *Every falsifiable \mathcal{L} -formula is falsifiable in an ultimately periodic quasimodel of height bounded by $|\Sigma| + 1$.*

Proof. We sketch the construction. By Lemma 6.6, if φ is falsifiable, it is falsifiable on some quasimodel $Q' = (W', \leq', \ell', S')$ of height at most $|\Sigma| + 1$. Choose $w_0 \in W'$ such that $\varphi \notin \ell_0(w_0)$, and let W_0 be the linear component of w_0 (i.e. $W_0 = \{v \in W' \mid v \leq w_0 \text{ or } w_0 \leq v\}$) and \leq_0 be the restriction of \leq' to W_0 . By a priority method similar to that of Section 5, we define a bi-infinite sequence $\dots, (W_{-1}, \leq_{-1}), (W_0, \leq_0), (W_1, \leq_1), \dots$ and sensible relations $S_k \subseteq W_k \times W_{k+1}$, such that $Q^\infty = (W^\infty, \leq^\infty, \ell^\infty, S^\infty)$ is a quasimodel, where $W^\infty = \bigsqcup_{k \in \mathbb{Z}} W_k$ (\bigsqcup denotes a disjoint union), $\leq^\infty = \bigsqcup_{k \in \mathbb{Z}} \leq_k$, and so on.

Note that there are at most $2^{|\Sigma|(|\Sigma|+1)}$ possible choices of W_k , since each W_k consists of at most $|\Sigma| + 1$ types, and there are at most $2^{|\Sigma|}$ types. This in particular implies that some W_i repeats infinitely often for arbitrarily large values of i . Let $l > 0$ be such that $W_{i+l} = W_i$ and every U or G defect of W_i has been realised before W_{i+l} ; such an l exists because W_i has finitely many defects. Similarly, some $W_{-i'}$ repeats for infinitely many values of i' , and we choose $l' > 0$ such that $W_{-(i'+l')} = W_{-i'}$ and all S or H defects of $W_{-i'}$ have been realised after $W_{-(i'+l')}$. We define $Q = (W, \leq, \ell, S)$ to be the restriction of Q^∞ to $\bigcup_{k=-(i'+l'-1)}^{i+l-1} W_k$, but with S redefined on W_{i+l-1} so that it maps to W_i and redefined on $W_{-(i'+l'-1)}$ so that it maps from $W_{i'}$.

It remains to check that Q is a quasimodel. We only check that it is ω -sensible, as the other properties are easy to check. Consider the case of $\varphi \cup \psi \in \ell(w)$ (the cases for other temporal modalities are similar). Then $w \in W_k$ for some k , which means that for some j (namely, $j = i + l - k$), there is $v \in W_i$ such that $w S^j v$. If $\varphi \cup \psi \notin \ell(v)$, then this defect must already have been realised. Otherwise, by construction, there are some j' and some u such that $v S^{j'} u$ and $\psi \in \ell(u)$. Thus $w S^{j+j'} u$ and $\psi \in \ell(u)$, as required. \square

Ultimately periodic models can be represented using sets of *moments*.

Definition 11.5. A Σ -**moment** is a sequence of the form $m = (m_0, \dots, m_m)$, where

1. each m_i is a Σ -type,
2. $m_i \supsetneq m_{i+1}$ for $i < m - 1$,
3. for every $\varphi \Rightarrow \psi \in \Sigma \setminus m_i$ there is some $j \leq i$ with $\varphi \in m_j$ but $\psi \notin m_j$,
4. for every $\varphi \Leftarrow \psi \in m_i$ there is some $j \geq i$ with $\varphi \in m_j$ but $\psi \notin m_j$.

We write $|\mathfrak{M}|$ for the set $\{m_0, \dots, m_m\}$. The set of Σ -moments is denoted M_Σ .

We define the labelled space $(m_0, \dots, m_m) + (n_0, \dots, n_n)$ to be the parallel sum of the two linear posets (m_0, \dots, m_m) and (n_0, \dots, n_n) with labelling given by the identity.

Definition 11.6. The moment n is a **temporal successor** of m , denoted $m S_\Sigma n$, if there exists a fully confluent convex sensible relation $R \subseteq |m| \times |n|$ on the labelled space $m + n$.

Definition 11.7. We define $\mathbb{M}_\Sigma = (M_\Sigma, S_\Sigma)$.

Because of condition (2) in Definition 11.5, if Σ is finite then so is \mathbb{M}_Σ .

Definition 11.8. A **small falsifiability witness** for an \mathcal{L} -formula φ is a finite sequence of Σ -moments $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^i, \dots, m^{i+l}$ with distinguished positions $-i', i$ and binary relations $\emptyset \neq S_k \subseteq |m^k| \times |m^{k+1}|$ for each k with $-i' - l' \leq k < i + l$ such that

- (A) $\varphi \notin \Phi$ for some $\Phi \in m^0$
- (B) $m^{-i'} = m^{-i'-l'}$ and $m^i = m^{i+l}$,
- (C) each S_k is fully confluent, convex, and sensible,
- (D) if $\varphi \cup \psi \in m_j^i$ then there are $r < l$ and a sequence $(j_k)_{k \leq r}$ with $j_0 = j$ such that $m_{j_k}^{i+k} S_{i+k} m_{j_{k+1}}^{i+k+1}$ and $\varphi \in m_{j_k}^{i+k}$ for all $k < r$, and $\psi \in m_{j_r}^{i+r}$,
- (E) if $G\psi \in \Sigma \setminus m_j^i$ then there are $r < l$ and a sequence $(j_k)_{k \leq r}$ with $j_0 = j$ such that $m_{j_k}^{i+k} S_{i+k} m_{j_{k+1}}^{i+k+1}$ for all $k < r$ and $\psi \notin m_{j_r}^{i+r}$,
- (F) if $\varphi S \psi \in m_j^{i'}$ then there are $r < l'$ and a sequence $(j_k)_{k \leq r}$ with $j_0 = j$ such that $m_{j_{k+1}}^{-i'-k-1} S_{-i'-k} m_{j_k}^{-i'-k}$ and $\varphi \in m_{j_k}^{-i'-k}$ for all $k < r$, and $\psi \in m_{j_r}^{-(i'+r)}$,
- (G) if $H\psi \in \Sigma \setminus m_j^{-i'}$ then there are $r < l'$ and a sequence $(j_k)_{k \leq r}$ with $j_0 = j$ such that $m_{j_{k+1}}^{-i'-k-1} S_{-i'-k} m_{j_k}^{-i'-k}$ for all $k < r$, and $\psi \notin m_{j_r}^{-i'-r}$.

Theorem 11.4 implies that if an \mathcal{L} -formula is falsifiable then it has a small falsifiability witness. Moreover, the converse is also true, because the small falsifiability witness can be viewed as a quasimodel. As a consequence, we obtain an equivalence between the existence of a possibly infinite structure (a model of φ) and the existence of a finite structure (a small falsifiability witness) for a given \mathcal{L} -formula φ .

Theorem 11.9. An \mathcal{L} -formula is falsifiable if and only if it has a small falsifiability witness.

Proof. For the left-to-right direction, assume that the formula φ is falsifiable. By Theorem 11.4, there exists an ultimately periodic quasimodel $Q = (W, \leq, \ell, S)$ such that $W = \bigcup_{k=-i'-l'}^{i+l} W_k$ and $\varphi \notin \ell(w)$ for some $w \in W_0$. For each W_k , let $W_k = \{v_0^k, \dots, v_{m_k}^k\}$ in increasing order, and let $m^k = (\ell(v_0^k), \dots, \ell(v_{m_k}^k))$ (deleting repeating types if needed). It is easy to check that the sequence $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^0, \dots, m^i, \dots, m^{i+l}$, with sensible relations S_k defined in the obvious way, yields a small falsifiability witness.

Conversely, we will show that if a formula has a small falsifiability witness $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^i, \dots, m^{i+l}$ then it is falsifiable. Write $m^k = (m_0^k, \dots, m_{m_k}^k)$ and consider the labelled space $Q = (W, \leq, \ell, S)$, where $W = \{(m_s^k, k) \mid -(i' + l') < k < i + l \text{ and } 0 \leq s \leq m_k\}$ and \leq, ℓ , and S are defined in the obvious way. It is not hard to check that Q is a quasimodel falsifying φ . Hence by Lemma 5.4, φ is falsifiable. \square

From Theorem 11.9 we may define Algorithm 1, which nondeterministically checks for falsifiability in PSPACE. According to Savitch's theorem [62], nondeterministic polynomial space is equal to deterministic polynomial space; thus we obtain the following.

Theorem 11.10. There exists a deterministic algorithm for falsifiability checking of an \mathcal{L} -formula that is correct and works in space that is polynomial in the size of the input formula.

Proof. We argue that Algorithm 1 is a correct and complete nondeterministic polynomial-space algorithm for falsifiability. If φ is falsifiable, then by Theorem 11.9, φ has a small falsifiability witness $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^0, \dots, m^i, \dots, m^{i+l}$. We initialise both m and m' to m^0 . With the first while loop, we initialise m^s to $m^{i'}$, initialise m^f to m^l . During each iteration of the second while loop, we choose n to be either the successor of m or the predecessor of m' . This yields an accepting computation of Algorithm 1. In particular, since $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^0, \dots, m^i, \dots, m^{i+l}$ has no defects, for each element of Δ , at some stage a 'cure' is found witnessing it is not in fact a defect. That is, we eventually obtain $\Delta \subseteq \Gamma$.

Conversely, if Algorithm 1 has an accepting computation, let $m^{-i'-l'}, \dots, m^{-i'}, \dots, m^0, \dots, m^i, \dots, m^{i+l}$ enumerate the values taken by m and m' where i' and i are the least natural numbers such that $m^{i'} = m^s$ and $m^i = m^f$, respectively. It is not hard to check that this sequence yields a small falsifiability witness.

To check that the nondeterministic algorithm uses polynomial space, it is sufficient to observe that each subset of Σ can be encoded by a polynomial number of bits. Since m, m', m^s and n have height at most $|\Sigma| + 1$ we need $7|\Sigma| + 7$ of those sets (at most $|\Sigma| + 1$ for each of $m, m', m^s, m^f, n, \Delta$, and Γ). Checking that n is a temporal successor (respectively predecessor) of m (respectively m') can be done nondeterministically by guessing a relation R and checking that it is a sensible and bi-serial relation; but the size of R is bounded by the product of the sizes of m (respectively m') and n . Similarly, both S^* and $(S^{-1})^*$ have at most $|\Sigma|^2$ elements, so also only require polynomial space. \square

With this we conclude that the validity problem is PSPACE-complete.

Algorithm 1: GTL falsifiability algorithm.

```

1 input  $\varphi$ 
2 set  $\Sigma$  to be the set of subformulas of  $\varphi$ 
3 guess three moments,  $m^p$ ,  $m$ , and  $m^f$ , of heights  $s, m, n \leq |\Sigma| + 1$  such that  $\varphi \notin m_m$ 
4  $m' \leftarrow m$ 
5 while  $m \neq m^f$  or  $m' \neq m^p$  do
6   guess a moment  $n$  of height at most  $|\Sigma| + 1$ 
7   if  $m \neq m^f$  and  $n$  is a temporal successor of  $m$  then
8      $m \leftarrow n$ 
9   else if  $m' \neq m^p$  and  $n$  is a temporal predecessor of  $m'$  then
10     $m' \leftarrow n$ 
11   else
12     reject
13   end
14 end
// Initialise defects
15  $\Delta \leftarrow \{(k, F\psi) \mid \varphi \cup \psi \in m_k, 0 \leq k \leq n\} \cup \{(k, P\psi) \mid \varphi \cup \psi \in m'_k, 0 \leq k \leq s\} \cup \{(k, G\psi) \mid G\psi \in \Sigma \setminus m_k, 0 \leq k \leq n\} \cup \{(k, H\psi) \mid H\psi \in \Sigma \setminus m'_k, 0 \leq k \leq s\}$ 
// Initialise cured defects
16  $\Gamma \leftarrow \{(k, F\psi) \mid \psi \in m_k, 0 \leq k \leq n\} \cup \{(k, P\psi) \mid \psi \in m'_k, 0 \leq k \leq s\} \cup \{(k, G\psi) \mid \psi \in \Sigma \setminus m_k, 0 \leq k \leq n\} \cup \{(k, H\psi) \mid \psi \in \Sigma \setminus m'_k, 0 \leq k \leq s\}$ 
// Initialise reachability relations
17  $S^* \leftarrow \{(k, k) \mid 0 \leq k \leq n\}$ 
18  $(S^{-1})^* \leftarrow \{(k, k) \mid 0 \leq k \leq s\}$ 
19 while  $m^f \neq m$  or  $m^p \neq m'$  or  $\Delta \not\subseteq \Gamma$  do
20   guess a moment  $n$  of height at most  $|\Sigma| + 1$ 
21   if  $m \neq m^f$  and  $n$  is a temporal successor of  $m$ , witnessed by  $R$  then
22      $S^* \leftarrow \{(k, z) \mid \exists y : (k, y) \in S^* \text{ and } (m_y, n_z) \in R\}$ 
23      $\Gamma \leftarrow \Gamma \cup \{(k, F\psi) \mid \exists z : (k, z) \in S^* \text{ and } \psi \in n_z\}$ 
24      $\Gamma \leftarrow \Gamma \cup \{(k, G\psi) \mid \exists z : (k, z) \in S^* \text{ and } \psi \in \Sigma \setminus n_z\}$ 
25      $m \leftarrow n$ 
26   else if  $m' \neq m^p$  and  $n$  is a temporal predecessor of  $m'$ , witnessed by  $R$  then
27      $(S^{-1})^* \leftarrow \{(z, k) \mid \exists y : (y, k) \in (S^{-1})^* \text{ and } (m'_y, n_y) \in R^{-1}\}$ 
28      $\Gamma \leftarrow \Gamma \cup \{(k, P\psi) \mid \exists z : (k, z) \in (S^{-1})^* \text{ and } \psi \in n_z\}$ 
29      $\Gamma \leftarrow \Gamma \cup \{(k, H\psi) \mid \exists z : (k, z) \in (S^{-1})^* \text{ and } \psi \in \Sigma \setminus n_z\}$ 
30      $m' \leftarrow n$ 
31   else
32     reject
33   end
34 end
35 accept

```

Theorem 11.11. *The validity problem for GTL is PSPACE-complete.*

12. Concluding remarks

We have defined a natural version of *linear temporal logic* with a Gödel–Dummett base, suitable for reasoning with vague or incomplete information, and shown that it may equivalently be characterised as a fuzzy logic or as a superintuitionistic logic using standard semantics in each case. Despite the lack of a finite model property for either of the two semantics, we have introduced a class of *quasimodels* for which GTL *does* satisfy a version of the finite model property, and moreover shown how these quasimodels can be used to adapt the classical proof that the validity problem is PSPACE-complete.

This PSPACE-complete complexity puts Gödel temporal logic in sharp contrast to other fuzzy logics, whose transitive modal versions are undecidable [31], or intuitionistic temporal logics, where systems are known to be decidable only with non-elementary upper bounds, if at all [25,63]. This places Gödel–Dummett logic as *the* premier base for computational applications of sub-classical modal and temporal logics, as far as complexity is concerned.

We have also provided a sound and complete calculus for the Gödel temporal logic GTL. This result further cements GTL as a privileged logic for temporal reasoning with non-binary degrees of truth and paves the way for a proof-theoretic treatment of this logic. Among the challenges in this direction is the design of cut-free or cyclic calculi.

In proving our main results, we have developed tools for the treatment of superintuitionistic temporal logics, specifically identifying the usefulness of combining the presence of ‘henceforth’ with *co-implication*. We believe that this insight will lead to completeness proofs for related logics, including intuitionistic LTL, where complete calculi for ‘eventually’ are available, but not for ‘henceforth’. Regarding this, it should be remarked that the techniques of [59] should lead to a sound and complete calculus for the logic with implication, ‘next’, and ‘eventually’ (but no co-implication or henceforth), although such a result does not follow immediately from the present work. The techniques we have used to prove completeness and decidability are quite robust and we expect that they may be applicable to more expressive logics such as the linear-time μ -calculus (see [64]) or the branching-time logics PDL, CTL*, or even the full modal μ -calculus. This represents a milestone in the program pioneered by Caicedo et al. [38,65] of extending results from classical modal and temporal logics to their Gödel counterparts.

Another subject that would be worth studying in the near future is *bisimulation* in Gödel temporal logic. This tool has been used to determine that temporal operators are not interdefinable in the intuitionistic temporal setting [58,63]. For the class of temporal here-and-there models, ‘henceforth’ is a basic operator that cannot be defined, while ‘eventually’ becomes definable in terms of ‘henceforth’, ‘next’, and implication [58,63]. When including co-implication, results on definability exist in the literature: for a combination of the logic of here-and-there, co-implication and the basic modal logic K, it was proven in [66] that the ‘possibly’ and ‘necessarily’ modalities become interdefinable. We do not know if co-implication has a similar effect on our Gödel temporal logic; a negative answer would require a suitable notion of bisimulation preserving both implications as well as the temporal operators.

CRedit authorship contribution statement

Juan Pablo Aguilera: Conceptualization, Investigation, Writing – original draft, Writing – review & editing. **Martín Diéguez:** Conceptualization, Investigation, Writing – original draft, Writing – review & editing. **David Fernández-Duque:** Conceptualization, Investigation, Writing – original draft, Writing – review & editing. **Brett McLean:** Conceptualization, Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Brett McLean reports financial support was provided by Research Foundation Flanders. Juan Pablo Aguilera reports financial support was provided by Research Foundation Flanders. Juan Pablo Aguilera reports financial support was provided by Austrian Science Fund. David Fernandez-Duque reports financial support was provided by Research Foundation Flanders. David Fernandez-Duque reports financial support was provided by European Cooperation in Science and Technology. Martin Dieguez reports financial support was provided by European Cooperation in Science and Technology. Martin Dieguez reports financial support was provided by Regional Council of Pays de la Loire.

Acknowledgements

This work has been partially supported by FWO–FWF grant G030620N/I4513N (Aguilera and Fernández-Duque), FWO grant 3E017319 (Aguilera), the projects EL4HC and étoiles montantes CTASP at Région Pays de la Loire, France (Diéguez), the COST action CA17124 (Diéguez and Fernández-Duque), SNSF–FWO Lead Agency Grant 200021L_196176/G0E2121N (McLean and Fernández-Duque), and the FWO Senior Postdoctoral Fellowship 1280024N (McLean).

Data availability

No data was used for the research described in the article.

References

- [1] S. Demri, V. Goranko, M. Lange, *Temporal Logics in Computer Science: Finite-State Systems*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2016.
- [2] J. Mendel, Fuzzy logic systems for engineering: a tutorial, *Proc. IEEE* 83 (3) (1995) 345–377, <https://doi.org/10.1109/5.364485>.
- [3] R. Kruse, E. Schwecke, J. Heinssohn, *Uncertainty and Vagueness in Knowledge Based Systems: Numerical Methods*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1991, pp. 29–44, Ch. Vague Data.
- [4] N. Preining, Gödel logics – a survey, in: C.G. Fermüller, A. Voronkov (Eds.), *Logic for Programming, Artificial Intelligence, and Reasoning*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010, pp. 30–51.
- [5] M. Baaz, N. Preining, R. Zach, First-order Gödel logics, *Ann. Pure Appl. Log.* 147 (2007) 23–47.
- [6] K. Gödel, Zum intuitionistischen Aussagenkalkül, *Anz. Akad. Wiss. Wien Math.-Nat.wiss. Kl.* 69 (1932) 65–66.
- [7] M. Dummett, A propositional calculus with denumerable matrix, *J. Symb. Log.* 24 (2) (1959) 97–106.
- [8] T. Alsinet, C.I. Chesñevar, L. Godo, S.A. Sandri, G.R. Simari, Formalizing argumentative reasoning in a possibilistic logic programming setting with fuzzy unification, *Int. J. Approx. Reason.* 48 (3) (2008) 711–729.
- [9] D. Dubois, J. Lang, H. Prade, Towards possibilistic logic programming, in: K. Furukawa (Ed.), *Logic Programming, Proceedings of the Eighth International Conference, Paris, France, June 24–28, 1991*, MIT Press, 1991, pp. 581–595.
- [10] D. Pearce, A new logical characterisation of stable models and answer sets, in: J. Dix, L.M. Pereira, T.C. Przymusiński (Eds.), *Non-Monotonic Extensions of Logic Programming, Selected Papers, NMELP '96*, Bad Honnef, Germany, September 5–6, 1996, in: *Lecture Notes in Computer Science*, vol. 1216, Springer, 1996, pp. 57–70.
- [11] D. Pearce, Equilibrium logic, *Ann. Math. Artif. Intell.* 47 (1–2) (2006) 3–41.
- [12] A. Avron, Hypersequents, logical consequence and intermediate logics for concurrency, *Ann. Math. Artif. Intell.* 4 (1991) 225–248.
- [13] F. Aschieri, A. Ciabatonni, F.A. Genco, Gödel logic: from natural deduction to parallel computation, in: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20–23, 2017*, IEEE Computer Society, 2017, pp. 1–12.
- [14] D. Dubois, H. Prade, Processing fuzzy temporal knowledge, *IEEE Trans. Syst. Man Cybern.* 19 (4) (1989) 729–744.
- [15] K.B. Lamine, F. Kabanza, History checking of temporal fuzzy logic formulas for monitoring behavior-based mobile robots, in: *12th IEEE International Conference on Tools with Artificial Intelligence (ICTAI 2000)*, 13–15 November 2000, IEEE Computer Society, Vancouver, BC, Canada, 2000, pp. 312–319.
- [16] S. Mukherjee, P. Dasgupta, A fuzzy real-time temporal logic, *Int. J. Approx. Reason.* 54 (9) (2013) 1452–1470.
- [17] R. Gerber, H.-H. Nagel, Representation of occurrences for road vehicle traffic, *Artif. Intell.* 172 (4) (2008) 351–391, <https://doi.org/10.1016/j.artint.2007.07.001>.
- [18] J.E. Naranjo, C. Gonzalez, R. Garcia, T. de Pedro, Lane-change fuzzy control in autonomous vehicles for the overtaking maneuver, *IEEE Trans. Intell. Transp. Syst.* 9 (3) (2008) 438–450, <https://doi.org/10.1109/TITS.2008.922880>.

- [19] S. Shukla, M. Tiwari, Fuzzy logic of speed and steering control system for three dimensional line following of an autonomous vehicle, *Int. J. Comput. Sci. Inf. Secur.* 7 (2010).
- [20] A. Pnueli, The temporal logic of programs, in: *Proceedings 18th IEEE Symposium on the Foundations of CS*, 1977, pp. 46–57.
- [21] F. Aguado, P. Cabalar, M. Diéguez, G. Pérez, T. Schaub, A. Schuhmann, C. Vidal, Linear-time temporal answer set programming, *Theory Pract. Log. Program.* 23 (1) (2021) 2–56.
- [22] F. Aguado, P. Cabalar, M. Diéguez, G. Pérez, C. Vidal, Temporal equilibrium logic: a survey, *J. Appl. Non-Class. Log.* 23 (1–2) (2013) 2–24.
- [23] A. Heyting, Die formalen Regeln der intuitionistischen Logik, *Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse, Deutsche Akademie der Wissenschaften zu Berlin, Mathematisch-Naturwissenschaftliche Klasse*, 1930.
- [24] P. Balbiani, M. Diéguez, Temporal here and there, in: M. Loizos, A. Kakas (Eds.), *Logics in Artificial Intelligence*, Springer, 2016, pp. 81–96.
- [25] D. Fernández-Duque, The intuitionistic temporal logic of dynamical systems, *Log. Methods Comput. Sci.* 14 (3) (2018).
- [26] H. Thiele, S. Kalenka, On fuzzy temporal logic, in: *Proceedings of Second IEEE International Conference on Fuzzy Systems*, vol. 2, 1993, pp. 1027–1032.
- [27] A. Frigeri, L. Pasquale, P. Spoletini, Fuzzy time in linear temporal logic, *ACM Trans. Comput. Log.* 15 (4) (2014).
- [28] Z. Lu, J. Liu, J.C. Augusto, H. Wang, *Computational Intelligence in Complex Decision Systems*, Atlantis Press, Paris, 2010, pp. 125–146, Ch. A Many-Valued Temporal Logic and Reasoning Framework for Decision Making.
- [29] J.P. Aguilera, M. Diéguez, D. Fernández-Duque, B. McLean, Time and Gödel: fuzzy temporal reasoning in pspace, in: A. Ciabattoni, E. Pimentel, R.J.G.B. de Queiroz (Eds.), *Logic, Language, Information, and Computation*, Springer International Publishing, Cham, 2022, pp. 18–35.
- [30] J.P. Aguilera, M. Diéguez, D. Fernández-Duque, B. McLean, A Gödel calculus for linear temporal logic, in: *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning (KR)*, IJCAI, 2022, pp. 2–11.
- [31] A. Vidal, On transitive modal many-valued logics, *Fuzzy Sets Syst.* 407 (2021) 97–114.
- [32] P. Balbiani, M. Diéguez, D. Fernández-Duque, Some constructive variants of S4 with the finite model property, in: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, IEEE, 2021, pp. 1–13.
- [33] C. Rauszer, An Algebraic and Kripke-Style Approach to a Certain Extension of Intuitionistic Logic, *Instytut Matematyczny Polskiej Akademii Nauk, Warsaw*, 1980.
- [34] M. Bílková, S. Frittella, D. Kozhemiachenko, Constraint tableaux for two-dimensional fuzzy logics, in: A. Das, S. Negri (Eds.), *Automated Reasoning with Analytic Tableaux and Related Methods - 30th International Conference, Proceedings, TABLEUX 2021, Birmingham, UK, September 6–9, 2021*, in: *Lecture Notes in Computer Science*, vol. 12842, Springer, 2021, pp. 20–37.
- [35] L. Bozzelli, D. Pearce, On the complexity of temporal equilibrium logic, in: *Proceedings of the 2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, LICS '15, IEEE Computer Society, USA, 2015, pp. 645–656.
- [36] K. Gödel, Eine interpretation des intuitionistischen aussagenkalküls, *Ergebnisse eines mathematischen Kolloquiums*, vol. 4, 1933, pp. 39–40.
- [37] M. Reynolds, M. Zakharyashev, On the products of linear modal logics, *J. Log. Comput.* 11 (6) (2001) 909–931, <https://doi.org/10.1093/logcom/11.6.909>.
- [38] X. Caicedo, G. Metcalfe, R.O. Rodríguez, J. Rogger, Decidability of order-based modal logics, *J. Comput. Syst. Sci.* 88 (2017) 53–74.
- [39] P. Blackburn, M. de Rijke, Y. Venema, *Modal Logic*, Cambridge University Press, 2001.
- [40] D. Fernández-Duque, Non-deterministic semantics for dynamic topological logic, *Ann. Pure Appl. Log.* 157 (2–3) (2009) 110–121.
- [41] D. Fernández-Duque, A sound and complete axiomatization for dynamic topological logic, *J. Symb. Log.* 77 (3) (2012) 947–969.
- [42] S. Demri, V. Goranko, M. Lange, *Temporal Logics in Computer Science: Finite-State Systems*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2016.
- [43] M. Baaz, N. Preining, Gödel logics and the fully boxed fragment of LTL, in: T. Eiter, D. Sands (Eds.), *LPAR-21. 21st International Conference on Logic for Programming, Artificial Intelligence and Reasoning*, in: *EPiC Series in Computing*, vol. 46, 2017, pp. 404–416.
- [44] S. Almagor, U. Boker, O. Kupferman, Formally reasoning about quality, *J. ACM* 63 (3) (2016), <https://doi.org/10.1145/2875421>.
- [45] P. Bouyer, O. Kupferman, N. Markey, B. Maubert, A. Murano, G. Perelli, Reasoning about quality and fuzziness of strategic behaviors, *ACM Trans. Comput. Log.* 24 (3) (2023), <https://doi.org/10.1145/3582498>.
- [46] F. Aguado, P. Cabalar, G. Pérez, C. Vidal, Loop formulas for splittable temporal logic programs, in: J.P. Delgrande, W. Faber (Eds.), *Logic Programming and Nonmonotonic Reasoning*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011, pp. 80–92.
- [47] S. Badaloni, M. Giacomini, The algebra IAFuz: a framework for qualitative fuzzy temporal reasoning, *Artif. Intell.* 170 (10) (2006) 872–908, <https://doi.org/10.1016/j.artint.2006.04.001>.
- [48] S. Schockaert, M. De Cock, Temporal reasoning about fuzzy intervals, *Artif. Intell.* 172 (8) (2008) 1158–1193, <https://doi.org/10.1016/j.artint.2008.01.001>.
- [49] F. Wolter, On logics with coimplication, *J. Philos. Log.* 27 (4) (1998) 353–387.
- [50] F. Wolter, M. Zakharyashev, On the decidability of description logics with modal operators, in: A.G. Cohn, L.K. Schubert, S.C. Shapiro (Eds.), *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, Trento, Italy, June 2–5, 1998, Morgan Kaufmann, 1998, pp. 512–523.
- [51] F. Wolter, M. Zakharyashev, Temporalizing description logics, in: D.M. Gabbay, M. de Rijke (Eds.), *Frontiers of Combining Systems, Second International Workshop, Proceedings, FroCoS 1998, Amsterdam, the Netherlands, October 2–4, 1998*, Research Studies Press/Wiley, 1998, pp. 379–402.
- [52] F. Wolter, The product of converse PDL and polymodal K, *J. Log. Comput.* 10 (2) (2000) 223–251, <https://doi.org/10.1093/LOGCOM/10.2.223>.
- [53] A. Kurucz, F. Wolter, M. Zakharyashev, D. Gabbay, *Many-Dimensional Modal Logics: Theory and Applications*, 1st edition, *Studies in Logic and the Foundations of Mathematics*, vol. 148, North Holland, 2003.
- [54] O. Lichtenstein, A. Pnueli, Propositional temporal logics: decidability and completeness, *Log. J. IGPL* 8 (1) (2000) 55–85.
- [55] G. Mints, *A Short Introduction to Intuitionistic Logic*, University Series in Mathematics, Springer, 2000.
- [56] C. Rauszer, A formalization of the propositional calculus of H-B logic, *Stud. Log.* 33 (1) (1974) 23–34.
- [57] J. Boudou, M. Diéguez, D. Fernández-Duque, A decidable intuitionistic temporal logic, in: *26th EACSL Annual Conference on Computer Science Logic, CSL 2017, August 20–24, 2017, Stockholm, Sweden*, 2017, 14.
- [58] P. Balbiani, J. Boudou, M. Diéguez, D. Fernández-Duque, Bisimulations for intuitionistic temporal logics, *J. Appl. Log., IfCoLog J. Log. Appl.* 8 (8) (2021) 2265–2285.
- [59] M. Diéguez, D. Fernández-Duque, An intuitionistic axiomatization of ‘Eventually’, in: G. Bezhanishvili, G. D’Agostino, G. Metcalfe, T. Studer (Eds.), *Advances in Modal Logic 12, Proceedings of the 12th Conference on Advances in Modal Logic, Held in Bern, Switzerland, August 27–31, 2018*, College Publications, 2018, pp. 199–218.
- [60] A.P. Sistla, E.M. Clarke, The complexity of propositional linear temporal logics, *J. ACM* 32 (3) (1985) 733–749.
- [61] A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics: An Introduction*, vol. 1, *Studies in Logic and the Foundations of Mathematics*, vol. 121, North-Holland, 1988.
- [62] W.J. Savitch, Relationships between nondeterministic and deterministic tape complexities, *J. Comput. Syst. Sci.* 4 (2) (1970) 177–192.
- [63] P. Balbiani, J. Boudou, M. Diéguez, D. Fernández-Duque, Intuitionistic linear temporal logics, *ACM Trans. Comput. Log.* 21 (2) (2020) 14.
- [64] M.Y. Vardi, A temporal fixpoint calculus, in: *Proceedings of the 15th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '88, Association for Computing Machinery, New York, NY, USA, 1988*, pp. 250–259.
- [65] X. Caicedo, R.O. Rodríguez, Standard Gödel modal logics, *Stud. Log.* 94 (2) (2010) 189–214.
- [66] P. Balbiani, M. Diéguez, Here and there modal logic with dual implication, in: G. Bezhanishvili, G. D’Agostino, G. Metcalfe, T. Studer (Eds.), *Advances in Modal Logic 12, Proceedings of the 12th Conference on Advances in Modal Logic, Held in Bern, Switzerland, August 27–31, 2018*, College Publications, 2018, pp. 63–82.