Artificial Intelligence 331 (2024) 104117

Contents lists available at ScienceDirect

Artificial Intelligence

journal homepage: www.elsevier.com/locate/artint

Knowledge-driven profile dynamics

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ARTICLE INFO

Keywords:
Profile dynamics
Personalized systems
Belief revision
KM-update

ABSTRACT

In the last decades, user profiles have been used in several areas of information technology. In the literature, most research works, and systems focus on the creation of profiles (using Data Mining techniques based on user's navigation or interaction history). In general, the dynamics of profiles are made by means of a systematic recreation of the profiles, without using the previous profiles. In this paper we propose to formalize the creation, representation, and dynamics of profiles from a Knowledge-Driven perspective. We introduce and axiomatically characterize four operators for changing profiles using a belief change inspired approach.

1. Introduction

The rapid evolution of technology has led to a vast amount of information available to users, creating a need for personalized interactions and filtering of irrelevant data. User profiles, which encompass relevant characteristics and interests, are crucial for achieving personalization.

User profiles are collections of characteristics that describe how users interact with a system, their interests, and their needs [3]. User Modelling is the process of constructing and maintaining these profiles, and it involves determining the information to be stored and how the profiles are built, updated, and maintained [40, p.9].

In the last two decades, user profiles have been used in several areas of information technology. We can mention, for example, their use in recommendation systems [9,70], in adaptable user interfaces [10], in personalized systems [40], in cognitive or physical rehabilitation systems [15]; etc.

In general, the systems store personal preference profiles as a set of items, which, in the vast majority, are represented by numeric attributes or linguistic labels (e.g. "Very low / Low / Medium / High / Very high", "single / married / separated / widowed", etc.).

In the literature, the vast majority of research works and systems focus on the creation of profiles (using Machine Learning techniques based on user navigation history) and their dynamics are made by means of systematic recreation of the profiles, without using the previously created profile. These creations and dynamics are made *ad-hoc* for each particular implementation, without (as far as we know) a formal study on the creation of profiles and their dynamics. On the other hand, when using Data-Driven techniques, systems are unable to produce explanations neither on the creation of the profile nor on its dynamics. The focus of explainability in

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https://doi.org/10.1016/j.artint.2024.104117
Received 12 November 2022; Received in revised form 11 March 2024; Accepted 15 March 2024

Available online 21 March 2024

personalized systems was focused on the explanation of the recommendations, rather than explanations regarding the changes in the profile (e.g. [63,67,68]).

In this paper we propose to formalize the creation, representation and dynamics of profiles from a Knowledge-Driven perspective. In particular, we aim to create a formal profile representation structure, based on the definition of a formal language that allows to clearly represent a user profile and its attributes, as well as to describe properties of different types of operations over profiles. We will use this language to define dynamic operators for profiles. Roughly speaking we will define a profile as a tuple of attributes in which each entry is an element of an associated domain and we will define different operators:

(a) From one single profile to one single profile: in this case, the model represents the evolution of a single profile. This evolution is made by an external stimulus, for example an interaction with a system, a training program, etc.
(b) From one profile to a set of profiles: this case is similar to case (a) but there are several possible profiles which are natural outcomes of the considered change. This model can be reduced to case (a) if a tie-break method is applied.
(c) From a set of profiles to a set of profiles: in this case we analyze the changes produced by input in a collection of profiles and determine which new profiles are obtained as a result of this modification. This type of operators is particularly useful in two contexts: (i) to capture the change produced by a single event in a population (set of profiles) (ii) to analyze changes when a single individual is represented by a set of possible profiles due to lack of information. For example, if we know that the age of a person is between 18 and 20 years old, we can represent her means of a set of three profiles: one with the age 18, one with the age 19 and one with the age 20. We consider here two different kinds of change, *revision*, when a new piece of information is received, assuming a static world and *update*, when the new information reflects changes in the world.[1]

The proposed operators are based on some well-known operators of the belief revision literature.[2] Belief revision systems are logical frameworks for modelling the way how agents modify their beliefs when they receive new information (sometimes inconsistent with previous beliefs). To integrate the new information, the agent may need to reject some previous information, but he should preserve as much as possible of the original information. The AGM model [1] is the standard model for formalizing this kind of dynamics. By adapting the belief revision model we will formalize the dynamics of computational profiles for it to be possible to predict the evolution of a profile based on partial, incomplete, or inconsistent information.

1.1. Motivational examples

In this subsection we specify two systems in two different domains, the first one is a software for personalized training in neurorehabilitation therapy and the second one is a home banking system. We will use both examples throughout the rest of the paper.

Neurorehabilitation System:

In neurorehabilitation, the recovery of a patient that suffered a stroke is a complex and gradual process. Stroke commonly includes focal disorders such as aphasia, and some other more diffuse abnormalities such as slowed information processing and executive dysfunction [12]. Cognitive rehabilitation is designed to restore, substitute, or compensate for the loss of cognitive abilities, and is the treatment of choice for these deficits [6]. Information and Communications Technology (ICT) based solutions such as gaming, virtual reality or computer simulations have been shown to have an enormous potential for enhancing cognitive rehabilitation by supporting the ability to carry out controlled and highly adaptive valid tasks [5].

Assume a personalized training system, like NeuroAIreh@b [20,48]. In that system, the patient performs a battery of neuropsychological assessments (NPAs) that allow the psychologist to create a cognitive profile. Some of the characteristics considered in the profiles are Memory (Mem), Attention (Att), Language (Lan), Executive Function (EFu), Age, Sex, and School Years (ScY). The domains for each characteristic (i.e. the set of different values which may be assigned to each of the characteristics under consideration) of the profile are displayed in Table 1. In the case of numerical values, all of them are integers. The initial profile is used to personalize a set of interactive neurorehabilitation tasks (INTs) [49] that the patient will perform remotely. The system uses the patient's cognitive profile to set the difficulty and, according to the results of the patient's performance, updates the profile for the next iteration. It is relevant to mention that the performance of the INTs cannot be easily translated into a direct correlation with the cognitive domains of the profile. First, although one INT can target the training of a specific domain, for instance EFu, other domains, such as attention and memory, may be involved in the training because they are intrinsically interconnected. Second, a performance result may not always numerically correspond to a change in the profile. And third, in the current practices in cognitive rehabilitation, it is not common to use cognitive profiles characterized by numeric values in the domains. It is more common that the Health Professionals (HPs) use labels such as mild or severe to identify the cognitive deficits of a patient. Therefore, the use of the belief revision dynamics presented in this paper may be useful not only to explain and predict how a profile is susceptible to evolve over time but also to capture the change in "beliefs" of the HPs when characterizing the cognitive status of a patient.

The following is an example of a profile:

$$Patient_1 = \langle 75, 80, 50, 60, 67, \mathrm{M}, 12 \rangle$$

[1] For the difference between revision and update see [36].

[2] For an overview of some of the main works in the area of belief revision see [23].

Table 1
Domains for the cognitive profiles.

	Mem	Att	Lan	EFu	Age	Sex	ScY
Range	[0, 100]	[0, 100]	[0, 100]	[0, 100]	[0, 150]	{M, F}	[0,30]

Table 2
Values for customer profile. All values are Natural numbers.

	Trf	Marks	Inv	Cred
Range	[1, 5]	[1, 5]	[1, 5]	[1, 5]

The evaluation of the INTs results is translated into a change of patient's cognitive condition. For example "there is a small improvement in patient's general cognitive condition". This corresponds to a change in patient's profile. However there are several possible profiles that are subject to reflect that change.

Homebanking system:

Homebanking systems and their entire banking core are a tool that tracks the entire financial journey of a customer from early adulthood to seniority. In this way, they witness an entire social, economic, personal and even technological evolution of each user, which allows them to trace over time a profile evolution that, due to change, requires that its nearby tools adapt and follow this evolution. To categorize customers, it is useful for these platforms to profile their customers according to common needs and behaviours. In this way, processes, commercial approaches, digital interfaces are easily created, and the business model is adapted. All this is built given these information clusters (profiles) made from a set of common attributes.

In practice, we can portray homebanking as a platform composed of what will refer to as Major Bank Operations. These are primary operations (or areas) used by bank customers on their everyday activities such as Transfers, Credits, Consultation of Account Data, Market / Stock, Investments, among others.

As the customer environment and context evolve, they may use different types of operations and the frequency of access may vary, according to the profile needs. This way, a profile can be characterized and parameterized by the frequency of activity in each significant operation. Therefore, we can consider for this example the following characteristics: Transfers (Trf), Markets / Exchange (Marks), Investments (Inv) and Credits (Cred). The domain values for the profile are displayed in Table 2.

These domain values are associated with the following parameterization scale: 1. Rarely or never use the operation; 2. Uses once a month; 3. Uses between 2 to 5 times a month; 4. Uses once a week; 5. Uses almost every day. An example of a profile is:

$$Prof_1 = \langle 3, 4, 2, 1 \rangle$$

This parameterization will constantly be updated when an action is executed within an interaction in a Major operation. In other words, if the user performs a transfer, the frequency of this user's activity will be evaluated in the primary operation transfers and, consequently, it will be updated in the corresponding profile parameterization. One goal of maintaining a parameterization is to monitor the evolution of the profile, which creates the possibility of shaping and adapting the graphical interface according to the frequency of access to a major operation.

The main contributions of this paper are:

- The introduction of a formal definition for the concept of *profile* and the introduction of a formal language suitable for expressing assertions about profiles.
- The proposal of a novel dynamics for profiles grounded on a Knowledge-Driven perspective. Specifically, we propose four distinct profile change operators that, when given an initial profile or set of profiles, take a sentence from the language as input and produce a new profile or set of profiles that are consistent with the input sentence.
- The presentation of axiomatic characterizations for each of the newly proposed operators. These characterization theorems enable us to compare the behaviours of these various operators and can also be used for characterizing the change formulas that lead to obtain a desired new state from a given initial state through the corresponding operation.

The rest of the paper is organized as follows: In Section 2, we provide a literature review on relevant works concerning user profiles. In Section 3 we introduce the notations and recall the main background concepts that will be needed throughout this article. In Section 4 we introduce a formal definition of a profile and a formal language for expressing properties of profiles. In Section 5 we introduce and characterize four operator for dynamics of profiles. In Section 6 we summarize the main contributions of the paper and briefly discuss their relevance. In the Appendix we provide proofs for all the original results presented.

2. Literature review

In this section, we will describe different approaches to user profiling as outlined in the literature.[3]

User modelling plays a central role in personalized systems that adapt to user behaviour, needs, and preferences. It is important to note that user interests and needs are not static; they change over time. Static user profiles risk containing outdated information, leading to ineffective system personalization [31]. Therefore, Profile Dynamics techniques are developed to detect changes in user characteristics, integrating them into user profiles and personalized systems.

User profile dynamics techniques are essential in Recommender Systems (RS) [32] and personalized or adaptive systems [7]. RS collect user preferences for items and use them to generate relevant recommendations. This can be done explicitly (user ratings) or implicitly (user behaviour) [32]. Adaptive systems employ profile dynamics techniques to adapt interfaces based on user preferences and context [10].

The first step in User profile dynamics is the construction of the user profile. This process, named User Modelling, comprises information retrieval, construction, and maintenance of user profiles [55]. Inside the User Modelling, there are three sub-phases: Information Retrieval phase, User Profile Construction phase, and Preference Learning phase.

- **Information Retrieval:** The collection of user information is crucial for effective personalization in information retrieval. This process can be achieved through explicit (feedback and manual editing), implicit (analyzing user interactions and behaviour), or hybrid (combining explicit and implicit) methods [32]. Demographic filtering, utilizing information such as age, gender, location, and education, helps overcome data sparsity issues and enables baseline recommendations for similar users [4]. User behaviour and preferences are also considered to infer interests and needs. Features extracted from the collected information are used to create the user profile [19].
  The two main approaches for information retrieval are *Explicit Information Retrieval* and *Implicit Information Retrieval*. Explicit Information Retrieval involves collecting data directly provided by users, including demographic information and personal interests [27]. Users inform the system and assign weights to their preferences [16]. While simple and effective, this approach has limitations such as being time-consuming, requiring user comprehension of concepts and rating scales, and facing user reluctance to provide feedback [44]. Implicit Information Retrieval, on the other hand, automatically collects data without user intervention [7]. It analyzes browsing behaviour, query logs, selected options, and other indicators to infer user preferences and interests. However, this approach requires significant computational power due to the volume of data collected, and confidence in the results may vary [44]. *Hybrid Information Retrieval* combines both explicit and implicit techniques, leveraging the strengths of each approach. This combined approach allows for a more comprehensive understanding of users' preferences and interests [32].
- **User Profile Construction:** In user profile construction, there are three main approaches mentioned in the literature [19]. The first approach is the *Keyword Profile*, where user profiles are represented as vectors that consist of pairs of concepts and their respective weights [2]. The second approach is the *Semantic Network Profile*, where user profiles are represented as weighted semantic networks. Each node in the network represents a concept, and the weights indicate the strength of the association [27]. The third approach is the *Concept Profile*, which is similar to semantic profiles. Here, the profiles are represented by weighted nodes and relationships between concepts. However, the nodes in Concept Profiles represent abstract topics that are of interest to the user, rather than specific words or word sets [27].
- **Preference Learning:** It involves assigning weights to each user's preferences based on their collected information and chosen profile construction method. These weights can be static or dynamic. Common techniques for preference learning include supervised machine learning methods like k-Nearest Neighbours, Naive Bayes, and Support Vector Machine, as well as unsupervised machine learning techniques such as Agent-based and k-Means Clustering. Filtering techniques, including Content-based, Collaborative-based, Hybrid-based, and Rule-based approaches, are also utilized. Additionally, ontology-based techniques like Neighbourhood-based and Statistical modelling play a role in preference learning. A comprehensive overview and taxonomy of these techniques can be found in the work by Tang et al. [61].

Profile dynamics techniques are employed to create dynamic user profiles that evolve alongside users' changing interests and needs. Various techniques have been proposed and developed to achieve this process:

- **Long-Term vs. Short-Term Interests:** This involves creating separate profiles based on the user's stable, long-term interests and their ever-changing short-term interests. Short-term interests are typically tied to specific time periods, such as the previous month, week, or session [31]. Several studies have explored this approach and found that long-term interests are more useful at the beginning of a search process, while short-term interests are more valuable during extended searches [42,11,45,51,17]. Combining both short- and long-term profiles has shown to yield better results than using either alone. Enhancing the weighting technique by incorporating contextual information enables the system to adapt its behaviour based on the user's needs in a given context [66]. Additionally, visualizing long- and short-term profiles using temporal graphs has been proposed as an alternative to focusing solely on preference weights at specific time points [59].

[3] The information presented in this section is based on the study reported in [62].

- **Evolution vs. Recalculation:** Profile evolution can be achieved through two approaches: recalculation and evolution [44,54,46]. In the recalculation approach, the user profile is recalculated at fixed intervals, such as every month, based on the changes that occurred between two profiles. On the other hand, the evolution approach involves calculating a new user profile by taking into account the previous profiles and the information about what happened between them. This information is transformed into training rules, which are iteratively refined until a stable training set is obtained.
- **Evolutionary/Genetic Algorithms:** They are also employed for profile evolution, inspired by the principles of natural selection. These algorithms update users' interests using genetic operators such as selection, mutation, and crossover until an optimal solution is reached. Various studies have proposed genetic algorithms to update user interests based on queries, interests, or clustering algorithms that evolve to represent preferences accurately over time [18,53].
- **Adaptation Algorithms/Rules:** They dynamically adjust the system's behaviour based on the information provided before running. These algorithms analyze the options selected by the user to rank similar items higher and decrease the rank of non-selected items in the future [45,69,44]. Adaptation rules have been applied in interface development [39] and e-learning systems [50].
- **Context Awareness:** Another approach for achieving dynamic user profiles is context awareness. It involves incorporating information about the user's context, such as location, to adapt the system's behaviour accordingly [58,43,57,65,52,28,66,13]. Context-awareness systems have shown successful applications in various domains, including medical professionals [60].
- **Belief Revision:** In the present paper, we propose a novel formal framework for representing profiles and their dynamics based on belief revision techniques. Our framework focuses specifically on the logical and rational change of user profiles in response to new information that suggests a potential need for change. Our approach builds upon the core principle that user profiles and beliefs are dynamic entities, susceptible to modification as new experiences and information regarding the profiles are received. By employing belief revision techniques, we aim to capture this dynamic nature and ensure that user profiles remain accurate and reflective of their preferences or characteristics. We believe that integrating our approach with the aforementioned techniques for gathering information about profiles can result in a more comprehensive and robust modelling of the dynamics of user profiles. This, in turn, has the potential to enhance the user experience and satisfaction in relevant contexts, such as online platforms with recommendation systems.

  To the best of our knowledge there are not many works that relate belief revision with user's profiles. One of those works is [41]. In that paper, the authors developed a service recommendation agent based on belief revision logic to handle the non-monotonicity problem of web service recommendation. They applied belief revision-based reasoning to determine the most suitable context for the initial service request based on the beliefs stored in the user's profile. After service request reasoning, the set of potential web services is identified and ranked. The highest-ranked services are considered to be the most desirable ones that match the user's specific interests.

  Another topic that is related to the dynamics of profiles is the representation of the dynamics of computational systems with multi-context reputation and trust in Multi-Agent Systems. Currently, many online applications utilize user feedback to calculate trust values for products, services, and also other users. In those application domains, trust models are needed and widely employed. This issue has been addressed in [33] and [34]. In [33], the authors proposed a multi-context trust approach where contexts are related through a taxonomy. The primary objective of their approach is to extend an incomplete trust order with information acquired from other related contexts. In [34], the authors focus on credibility dynamics. The trust or credibility associated with a set of agents is represented through a pairwise comparison partial order of agents called credibility order. The authors formalize a prioritized multiple revision operator, which can be employed to revise one credibility order by another credibility order.

3. Background

3.1. Formal preliminaries

In this section, we present some mathematical concepts and notations that will be used throughout this article.

Given a set S, we will denote by $\mathcal{P}(S)$ the power set S, *i.e.* the set of all subsets of S.

Given a set A, a binary relation $\preceq$ on A is:

- reflexive if and only if $\alpha \preceq \alpha$ for all $\alpha \in A$;
- transitive if and only if it holds that if $\alpha \preceq \beta$ and $\beta \preceq \delta$, then $\alpha \preceq \delta$, for all $\alpha, \beta, \delta \in A$;
- antisymmetric if and only if it holds that if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$, for all $\alpha, \beta \in A$.
- total if and only if $\alpha \preceq \beta$ or $\beta \preceq \alpha$, for all $\alpha, \beta \in A$.
- irreflexive if and only if $\alpha \npreceq \alpha$, for all $\alpha \in A$.

A relation is a:

- pre-order if and only if it is reflexive and transitive.
- order if and only if it is a pre-order which is also antisymmetric.
- strict order if and only if it is irreflexive and transitive.[4]
- total strict order $\prec$ on A if and only if it is a strict order and it holds that if $\alpha \neq \beta$, then $\alpha \prec \beta$ or $\beta \prec \alpha$, for all $\alpha, \beta \in A$.

[4] Every irreflexive and transitive relation on a set A is also antisymmetric.

Given a binary relation $\preceq$ on a set A:
- $\alpha \simeq \beta$ will be used to denote that $\alpha \preceq \beta$ and $\beta \preceq \alpha$;
- $\alpha \prec \beta$ will be used to denote that $\alpha \preceq \beta$ and $\beta \npreceq \alpha$.

Given a pre-order $\preceq$ on a set A, the *strict part* of $\preceq$ is the strict order $\prec$ on A, that is defined by $\alpha \prec \beta$ if and only if $\alpha \preceq \beta$ and $\beta \npreceq \alpha$, for all $\alpha, \beta \in A$.

Let A be a set and Γ be a finite subset of A. Given a total strict order $\prec$ on A, the minimum of Γ with respect to $\prec$ is denoted by $min(\Gamma, \prec)$ and is defined as follows:

$$P = min(\Gamma, \prec) \text{ iff } P \in \Gamma \text{ and } P \prec Q \text{ for all } Q \in \Gamma \setminus \{P\}.$$

Given a pre-order $\preceq$ on A, the set of minimal elements of Γ with respect to $\preceq$ is denoted by $Min(\Gamma, \preceq)$ and is defined as follows:

$$Min(\Gamma, \preceq) = \{P \in \Gamma : Q \nprec P, \text{ for all } Q \in \Gamma\}.$$

Given a total pre-order $\preceq$ on A, the set of minimal elements of Γ with respect to $\preceq$ is denoted by $Min(\Gamma, \preceq)$ and is defined as follows:

$$Min(\Gamma, \preceq) = \{P \in \Gamma : P \preceq Q, \text{ for all } Q \in \Gamma\}.$$

3.2. AGM

The AGM model [1] is considered the standard model of belief change, and was created to model the dynamics of beliefs. One of the main goals underlying the area of belief change is to model how a rational agent changes her set of beliefs when confronted with new information. In the AGM model, each *belief* of an agent is represented by a sentence and the *belief state* of an agent is represented by a logically closed set of (belief-representing) sentences. These sets are called *belief sets*. A change consists in adding or removing a specific sentence from a belief set to obtain a new belief set. The AGM model considers three kinds of belief change operators, namely *expansion, contraction* and *revision*. An expansion occurs when new information is added to the set of the beliefs of an agent. The expansion of a belief set $\mathbf{K}$ by a sentence α (denoted by $\mathbf{K} + \alpha$) is the logical closure of $\mathbf{K} \cup \{\alpha\}$. A contraction occurs when information is removed from the set of beliefs of an agent. A revision occurs when new information is added to the set of the beliefs of an agent while retaining consistency if the new information is itself consistent. From the three operations, expansion is the only one that can be univocally defined. The other two operations are characterized by a set of postulates that determine the behaviour of each one of these operators, establishing conditions or constraints that they must satisfy. The Levi and Harper identities[5] define the AGM revision and contraction operators in terms of each other. Thus one can consider revision as a primitive operation and treat contraction as defined by the Harper identity in terms of revision.
The following postulates, which were originally presented in [24–26], are commonly known as *AGM postulates for revision*[6]:

($\star$1) $\mathbf{K}\star\alpha = Cn(\mathbf{K}\star\alpha)$ (*i.e.* $\mathbf{K}\star\alpha$ is a belief set). (Closure)
($\star$2) $\alpha \in \mathbf{K}\star\alpha$. (Success)
($\star$3) $\mathbf{K}\star\alpha \subseteq \mathbf{K}+\alpha$. (Inclusion)
($\star$4) If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K}+\alpha \subseteq \mathbf{K}\star\alpha$. (Vacuity)
($\star$5) If α is consistent, then $\mathbf{K}\star\alpha$ is consistent. (Consistency)
($\star$6) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}\star\alpha = \mathbf{K}\star\beta$. (Extensionality)
($\star$7) $\mathbf{K}\star\alpha \cap \mathbf{K}\star\beta \subseteq \mathbf{K}\star(\alpha \vee \beta)$. (Disjunctive overlap)
($\star$8) If $\neg\alpha \notin \mathbf{K}\star(\alpha \vee \beta)$, then $\mathbf{K}\star(\alpha \vee \beta) \subseteq \mathbf{K}\star\alpha$. (Disjunctive inclusion)

3.2.1. The Katsuno and Mendelzon revision approach

Considering a finitary propositional language, Katsuno and Mendelzon [35] proposed a framework where any belief set $\mathbf{K}$ is represented by a propositional formula ψ, designated by knowledge base, such that $\mathbf{K} = \{\varphi : \psi \vdash \varphi\}$. When considering a finite language, this is always possible. Suppose that $\{\psi_1, \dots, \psi_n\}$ is a collection of pairwise non-equivalent formulas in $\mathbf{K}$. Then $\mathbf{K}$ can be represented by $\psi_1 \wedge \dots \wedge \psi_n$. In this framework Katsuno and Mendelzon proposed the following set of postulates [35]:

(R1) $\kappa \odot \varphi \vdash \varphi$.
(R2) If $\kappa \wedge \varphi$ is satisfiable, then $\kappa \odot \varphi \equiv \kappa \wedge \varphi$.
(R3) If φ is satisfiable, then $\kappa \odot \varphi$ is also satisfiable.
(R4) If $\kappa_1 \equiv \kappa_2$ and $\varphi_1 \equiv \varphi_2$, then $\kappa_1 \odot \varphi_1 \equiv \kappa_2 \wedge \varphi_2$.
(R5) $(\kappa \odot \varphi) \wedge \psi \vdash \kappa \odot (\varphi \wedge \psi)$.
(R6) If $(\kappa \odot \varphi) \wedge \psi$ is satisfiable, then $\kappa \odot (\varphi \wedge \psi) \vdash (\kappa \odot \varphi) \wedge \psi$.

When considering a finite propositional language the above postulates are equivalent to the AGM ones [35]. Before presenting the revision function based on Katsuno and Mendelzon's approach, we will recall some concepts.

[5] **Harper identity:** [30] $\mathbf{K} \div \alpha = (\mathbf{K}\star\neg\alpha) \cap \mathbf{K}$.
Levi identity: [38] $\mathbf{K}\star\alpha = (\mathbf{K} \div \neg\alpha) + \alpha$.
[6] These postulates were previously presented in [1] but with slightly different formulations.

Definition 3.1. Let $\mathcal{L}$ be a finitary propositional language whose atomic formulas are $p_1, \dots, p_n$. An interpretation of $\mathcal{L}$ is a function $I : \{p_1, \dots, p_n\} \mapsto \{0, 1\}$.

An interpretation of a language $\mathcal{L}$ assigns to each atomic formula of $\mathcal{L}$ a truth value.

Definition 3.2. An interpretation I is said to satisfy a formula φ or that it is a model of φ, which is denoted by $I \models \varphi$, if it can be shown inductively to do so under the following conditions:

1. $I \models p_i$ iff $I(p_i) = 1$;
2. $I \models (\neg\varphi)$ iff $I \not\models \varphi$;
3. $I \models (\varphi \wedge \psi)$ iff $I \models \varphi$ and $I \models \psi$;
4. $I \models (\varphi \vee \psi)$ iff $I \models \varphi$ or $I \models \psi$;
5. $I \models (\varphi \to \psi)$ iff $I \not\models \varphi$ or $I \models \psi$;
6. $I \models (\varphi \leftrightarrow \psi)$ iff ($I \models \varphi$ iff $I \models \psi$).

An interpretation is a model of a set of formulas Γ, denoted by $I \models \Gamma$, if $I \models \varphi$ for all $\varphi \in \Gamma$. The set of all interpretations is denoted by $\mathbb{I}$. $Mod(\varphi) = \{I \in \mathbb{I} : I \models \varphi\}$ and $Mod(\Gamma) = \{I \in \mathbb{I} : I \models \Gamma\}$. A formula or a set of formulas is called satisfiable if there exist at least one model for it. We say that φ is a tautology, which is denoted by $\models \varphi$ is for all interpretation $I \in \mathbb{I}$ it holds that $I \models \varphi$. Two formulas φ and ψ are called equivalent, which is denoted by $\varphi \equiv \psi$ if and only if $Mod(\varphi) = Mod(\psi)$.

Definition 3.3. A faithful assignment is a function mapping each base φ to a pre-order $\leq_\varphi$ over interpretations such that:

1. If $I, I' \in Mod(\varphi)$, then $I <_\varphi I'$ does not hold.
2. If $I \in Mod(\varphi)$ and $I' \notin Mod(\varphi)$, then $I <_\varphi I'$ holds.
3. If $\psi \equiv \varphi$, then $\leq_\psi = \leq_\varphi$

If $I \leq_\varphi J$ is interpreted as meaning that the interpretation I is better than the interpretation J, then for every formula φ the binary relation $\leq_\varphi$ on $\mathbb{I}$ that is associated to it by a faithful assignment is such that no model of φ is better than any other model of φ, but every model of φ is better than any non-model of φ.

Observation 3.4 *([35]). An operator $\odot$ satisfies postulates (R1) to (R6) if and only if there exists a faithful assignment that maps each base φ to a total pre-order $\leq_\varphi$ such that $Mod(\varphi \odot \mu) = min(Mod(\mu), \leq_\varphi)$.*[7]

3.3. Update

In [36], Katsuno and Mendelzon introduced the concept of update as a distinct operation of change. Whereas revision operations are suitable to capture changes that reflect evolving belief about a static situation, update operations are intended to represent changes in beliefs that result from changes in the objects of belief. The difference was pointed out for the first time by Keller and Winslett [37] (in the context of relational databases) and is captured in the following example:

Example 3.5 *([64,36,29]).* Initially the agent knows that there is either a book on the table (p) or a magazine on the table (q), but not both.

Case 1: The agent is told that there is a book on the table. The agent concludes that there is no magazine on the table. This is revision.

Case 2: The agent is told that subsequently a book has been put on the table. In this case she should not conclude that there is no magazine on the table. This is update.

As stated in [36] one useful approach to updating consists in associating time to the sentences. In this case we obtain pair like $\ll p, t \gg$, meaning that p holds at time t. Let t_1 denote the instant that the first sentence refers to and t_2 the moment in which the second information is received. Initially the agent believes that $\ll \neg(p \leftrightarrow q), t_1 \gg$. Revision by p can be represented by the incorporation of $\ll p, t_1 \gg$, and updating by p by the incorporation of $\ll p, t_2 \gg$ into the agent's set of beliefs. Hence $\ll \neg q, t_1 \gg$ is implied by the outcome of the revision but not by outcome of the update.

Definition 3.6 *([36]).* A *local faithful assignment* is a function mapping each interpretation I to a pre-order $\leq_I$ such that for any $J \in \mathbb{I}$, if $I \neq J$, then $I <_I J$.

Definition 3.7 *([36]).* Let K be a finite-based belief set. Let $\varphi \in \mathcal{L}$ be such that $Cn(\varphi) = K$. An operation $\diamond$ on φ is an *update* if and only if there is a local faithful assignment such that: $Mod(\varphi \diamond \mu) = \bigcup_{I \in Mod(\varphi)} Min(Mod(\mu), \leq_I)$

[7] When considering a finite language, we can replace "$\odot$ satisfies postulates (R1) to (R6)" by "$\odot$ is an AGM revision operator".

The following set of postulates were proposed in [36] to characterize update on finite-based belief sets.

($U1$) $\varphi \diamond \mu \vdash \mu$
($U2$) If $\varphi \vdash \mu$, then $\varphi \diamond \mu \equiv \varphi$.
($U3$) If $\varphi \nvdash \bot$ and $\mu \nvdash \bot$, then $\varphi \diamond \mu \nvdash \bot$.
($U4$) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$, then $\varphi_1 \diamond \mu_1 \equiv \varphi_2 \diamond \mu_2$.
($U5$) $(\varphi \diamond \mu_1) \wedge \mu_2$ implies $\varphi \diamond (\mu_1 \wedge \mu_2)$.
($U6$) If $\varphi_1 \diamond \mu_1 \vdash \mu_2$ and $\varphi_2 \diamond \mu_2 \vdash \mu_1$, then $\varphi_1 \diamond \mu_1 \equiv \varphi_2 \diamond \mu_2$.
($U7$) If φ is complete,[8] then $(\varphi \diamond \mu_1) \wedge (\varphi \diamond \mu_2)$ implies $\varphi \diamond (\mu_1 \vee \mu_2)$.
($U8$) $(\varphi_1 \vee \varphi_2) \diamond \mu \equiv (\varphi_1 \diamond \mu) \vee (\varphi_2 \diamond \mu)$.
($U9$) If φ is complete and $(\varphi \diamond \mu) \wedge \phi \nvdash \bot$, then $\varphi \diamond (\mu_1 \wedge \phi) \vdash (\varphi \diamond \mu) \wedge \phi$.

Observation 3.8 *([36]). Let $\diamond$ be an update operator. The following conditions are equivalent:*

1. *$\diamond$ satisfies postulates (U1) to (U8).*
2. *There exists a local faithful assignment that maps each interpretation I to a pre-order $\leq_I$ such that:*

$$Mod(\varphi \diamond \mu) = \bigcup_{I \in Mod(\varphi)} Min(Mod(\mu), \leq_I).$$

3. *There exists a local faithful assignment that maps each interpretation I to an order $\leq_I$ such that:*

$$Mod(\varphi \diamond \mu) = \bigcup_{I \in Mod(\varphi)} Min(Mod(\mu), \leq_I).$$

Observation 3.9 *([36]). Let $\diamond$ be an update operator. The following conditions are equivalent:*

1. *$\diamond$ satisfies postulates (U1) to (U5), (U8) and (U9).*
2. *There exists a local faithful assignment that maps each interpretation I to a total pre-order $\leq_I$ such that:*

$$Mod(\varphi \diamond \mu) = \bigcup_{I \in Mod(\varphi)} Min(Mod(\mu), \leq_I).$$

3. *There exists a local faithful assignment that maps each interpretation I to a total order $\leq_I$ such that:*

$$Mod(\varphi \diamond \mu) = \bigcup_{I \in Mod(\varphi)} Min(Mod(\mu), \leq_I).$$

4. On the logic of profiles

In this section, we present the formal definition of a profile and introduce appropriate language and semantics for formalizing the dynamics of profiles. To formalize a profile dynamics, we need first to define a profile structure and second to define a language to express profile dynamics.

4.1. Profile definition

We define a profile as a tuple of elements of the associated domains of the labels. Formally:

Definition 4.1. Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels. For each $i \in \{1, \ldots, n\}$ let D_i be a finite set associated with label L_i, that we will designate by the domain of L_i.
A *profile*, associated with $\mathbb{L}$, denoted by $P_{\mathbb{L}}$ (or simply by P if the tuple of labels is clear from the context), is an element of $D_1 \times D_2 \times ... \times D_n$. The set of all profiles associated with $\mathbb{L}$ will be denoted by $\mathbb{P}_{\mathbb{L}}$ (or simply by $\mathbb{P}$ if the tuple of labels is clear from the context).

Example 4.2. Given the tuple of labels $\mathbb{L} = \ll$age, gender, civil status, nationality$\gg$; $D_{age} = \{0, 1, 2, ..., 150\}$, $D_{gender} = \{$male, female, other$\}$, $D_{civil\ status} = \{$single, married, divorced, widowed, other$\}$, $D_{nationality} = \{$English, Portuguese, Argentinian$\}$, the following are examples of profiles:

$$\langle 25, male, single, English \rangle$$

$$\langle 45, female, married, Portuguese \rangle$$

[8] φ is complete if and only if for all $p \in \mathcal{L}$, $\varphi \vdash p$ or $\varphi \vdash \neg p$.

4.2. Profile's language

To express characteristics of a profile (and therefore the possibility of changing it), we need to define a formal language:

Definition 4.3. (Alphabet) Given a tuple of labels $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$, for each $i \in \{1, ..., n\}$, let D_i be the domain associated with the label L_i. The alphabet of symbols of the language $\mathcal{L}_{\mathbb{L}}$ (or simply $\mathcal{L}$) associated with $\mathbb{L}$ that we will consider is:

1. $L_1, L_2, ..., L_n$ (labels);
2. = (symbol of equality);
3. (,) (punctuation symbols);
4. $a, b, ...$ (elements of $\bigcup_{i=1}^{n} D_i$);
5. $\perp$ (symbol of contradiction);
6. $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (symbols of connectives).

Definition 4.4. (Formulae of the language) Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels. For each $i \in \{1, ..., n\}$, let D_i be the domain associated with the label L_i.
$L_i = a$, where $a \in D_i$, for $i \in \{1, ..., n\}$ is an atomic formula of $\mathcal{L}_{\mathbb{L}}$.
A well-formed formula (wff) of $\mathcal{L}_{\mathbb{L}}$ is defined by:

1. Every atomic formula of $\mathcal{L}_{\mathbb{L}}$ is a wff of $\mathcal{L}_{\mathbb{L}}$.
2. If A and B are wffs of $\mathcal{L}_{\mathbb{L}}$, so are $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.[9]

The following example clarifies the above definition.

Example 4.5. Given the tuple of labels $\mathbb{L} = \ll$age, gender, civil status, nationality$\gg$; $D_{age} = \{0, 1, 2, ..., 150\}$, $D_{gender} = \{$male, female, other$\}$, $D_{civil\ status} = \{$single, married, divorced, widowed, other$\}$, $D_{nationality} = \{$English, Portuguese, Argentinian$\}$, the following are examples of formulae of the language $\mathcal{L}_{\mathbb{L}}$:

- age= 25;
- gender= female $\wedge$ civil status = single;
- age= 25$\wedge$ (nationality = Portuguese $\vee$ nationality = Argentinian).

For the purpose of the present paper, the above defined language is enough. However, it is possible to enrich the language if needed. For example, if D_i and D_j are both numerical, we can define $X_i + X_j = a$ in order to establish relations between different domains of the profile. Given a numeric value n we can also use, for example, $X > n$ as an abbreviation of the wff $\bigvee_{n_i \in \mathbb{N} \cap D_X \text{ s.t. } n_i > n} (X = n_i)$.

In the following definition, we introduce the semantics for profile dynamics.

Definition 4.6. Let $\mathbb{L} = \ll L_1, L_2, ..., L_k \gg$ be a tuple of labels. For each $i \in \{1, ..., n\}$, let D_i be the domain associated with the label L_i. A profile $P = \langle p_1, \ldots, p_k \rangle$ is said to satisfy a formula α or to be a model of α, which is denoted by $P \vDash \alpha$, if it can be shown inductively to do so under the following conditions:

1. $P \vDash L_i = a$ iff $p_i = a$;
2. $P \vDash (\neg\beta)$ iff $P \nvDash \beta$;
3. $P \vDash (\beta \wedge \delta)$ iff $P \vDash \beta$ and $P \vDash \delta$;
4. $P \vDash (\beta \vee \delta)$ iff $P \vDash \beta$ or $P \vDash \delta$;
5. $P \vDash (\beta \rightarrow \delta)$ iff $P \nvDash \beta$ or $P \vDash \delta$;
6. $P \vDash (\beta \leftrightarrow \delta)$ iff ($P \vDash \beta$ iff $P \vDash \delta$).

The set of models of α is denoted by $\|\alpha\|$. It holds that $\| \perp \| = \emptyset$. A set of profiles Γ is said to satisfy α if and only if every profile in Γ is a model of α. We say that α is a tautology if and only if $\|\alpha\| = \mathbb{P}_{\mathbb{L}}$. We will use $\vDash \alpha$ to denote that α is a tautology.
Let M be a subset of $\mathbb{P}_{\mathbb{L}}$. We will denote by $\|\alpha_M\|$ a formula such that $\|\alpha_M\| = M$. We will often omit the braces, by writing, for example, α_{P_i,P_j} instead of $\alpha_{\{P_i,P_j\}}$.

The following definition introduces the notion of Γ-faithful binary relation on $\mathbb{P}_{\mathbb{L}}$.

[9] As it is the case in the context of classical logic, the empty disjunction shall be considered as denoting a contradiction.

Definition 4.7. Let $\mathbb{L}$ be a tuple of labels and Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$. A binary relation $\preceq_{\Gamma}$ (or $\prec_{\Gamma}$) on $\mathbb{P}_{\mathbb{L}}$ is Γ-faithful if it satisfies:

1. If $P_i \in \Gamma$ and $P_j \in \Gamma$, then $P_i \prec_{\Gamma} P_j$ does not hold.
2. If $P_i \in \Gamma$ and $P_j \in \mathbb{P}_{\mathbb{L}} \setminus \Gamma$, then $P_i \prec_{\Gamma} P_j$.

If $P_i \prec_{\Gamma} P_j$ is interpreted as meaning that P_i is better than P_j, to say that a binary relation $\prec_{\Gamma}$ on $\mathbb{P}_{\mathbb{L}}$ is Γ-faithful is to say that (for that relation) no profile in Γ is better than any other profile in Γ, but any profile in Γ is better than any profile in $\mathbb{P}_{\mathbb{L}} \setminus \Gamma$.
If $\Gamma = \{P\}$ is singleton, then we will omit the braces in the subscript of the binary relation mentioned in the above definition by writing $\preceq_P$ instead of $\preceq_{\{P\}}$. We will also write P-faithful instead of $\{P\}$-faithful. Note that if $\Gamma = \{P\}$, and $\prec_P$ is a strict order on $\mathbb{P}_{\mathbb{L}}$, then the first condition of Definition 4.7 follows trivially, since $\prec_P$ is irreflexive, and the second condition can be rewritten as $P \prec_P P_i$ for all $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$.

Example 4.8. Let $\mathbb{L}$ be a tuple of labels such that $\mathbb{P}_{\mathbb{L}} = \{P_1, P_2, P_3, P_4\}$. Let $\Gamma = \{P_1, P_2\}$. The transitive closure of the binary relation $\preceq$ on $\mathbb{P}_{\mathbb{L}}$ defined by $P_1 \simeq P_2 \prec P_3 \prec P_4$ is Γ-faithful.

In the following definition we introduce the notion of UpdP-faithful assignment.

Definition 4.9. Let $\mathbb{L}$ be a tuple of labels. A UpdP-faithful assignment is a function from $\mathbb{P}_{\mathbb{L}}$ to $\mathbb{P}_{\mathbb{L}} \times \mathbb{P}_{\mathbb{L}}$ that assigns to each profile P in $\mathbb{P}_{\mathbb{L}}$ a P-faithful binary relation $\preceq_P$ over $\mathbb{P}_{\mathbb{L}}$.

Example 4.10. Let $\mathbb{L}$ be a tuple of labels such that $\mathbb{P}_{\mathbb{L}} = \{P_1, P_2, P_3, P_4\}$. Let $\preceq_{P_1}, \preceq_{P_2}, \preceq_{P_3}$ and $\preceq_{P_4}$ be the binary relations on $\mathbb{P}_{\mathbb{L}}$ defined by

- $P_1 \prec_{P_1} P_2 \simeq_{P_1} P_3 \prec_{P_1} P_4$;
- $P_2 \prec_{P_2} P_1 \simeq_{P_2} P_3 \simeq_{P_2} P_4$;
- $P_3 \prec_{P_3} P_1 \simeq_{P_3} P_2 \prec_{P_3} P_4$;
- $P_4 \prec_{P_4} P_1 \simeq_{P_4} P_2 \simeq_{P_4} P_3$.

For each $i \in \{1, 2, 3, 4\}$, the transitive closure of the binary relation $\preceq_{P_i}$ is P_i-faithful. Therefore, a function from $\mathbb{P}_{\mathbb{L}}$ to $\mathbb{P}_{\mathbb{L}} \times \mathbb{P}_{\mathbb{L}}$ that assigns to each profile $P_i \in \mathbb{P}_{\mathbb{L}}$ the transitive closure of the binary relation $\preceq_{P_i}$ is an UpdP-faithful assignment.

4.3. Hamming distance between two profiles

To implement the profile change operators that we will present in the following section, we need to establish an order among profiles based on some criteria. The following definition presents a method for ordering profiles based on the Hamming distance [8,14].

Definition 4.11. Let $\mathbb{L} = \ll L_1, \ldots, L_n \gg$ be a tuple of labels among which there is at least one whose domain is a set of numbers. Let $I \subseteq \{1, \ldots, n\}$ be the set formed by the indexes of the labels of $\mathbb{L}$ whose domain is a set of numbers. Let $P, P' \in \mathbb{P}_{\mathbb{L}}$ be such that $P = \langle p_1, \ldots, p_n \rangle$ and $P' = \langle p'_1, \ldots, p'_n \rangle$. The Hamming distance between P and P', denoted by $d_H(P, P')$, is given by

$$d_H(P, P') = \sum_{i \in I} |p_i - p'_i|$$

The Hamming distance between a profile P and a set of profiles $\Gamma \subseteq \mathbb{P}_{\mathbb{L}}$ is:

$$d_H(P, \Gamma) = \min(\{d_H(P, P') : P' \in \Gamma\})$$

Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels among which there is at least one whose domain is a set of numbers. Using the Hamming distance, we can define a total pre-order and a total strict order on $\mathbb{P}_{\mathbb{L}}$.

Let $P \in \mathbb{P}_{\mathbb{L}}$. The binary relation $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$ defined by $P' \preceq_P P''$ iff $d_H(P, P') \leq d_H(P, P'')$ (for all $P', P'' \in \mathbb{P}_{\mathbb{L}}$) is a total pre-order.[10]

To obtain a P-faithful total pre-order $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$, the following slightly more complex definition must be used:

1. For all $Q \in \mathbb{P}_{\mathbb{L}}$, $P \preceq_P Q$;
2. For all $P', P'' \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$, $P' \preceq_P P''$ iff $d_H(P, P') \leq d_H(P, P'')$.

[10] For each profile $P' \in \mathbb{P}_{\mathbb{L}}$, the Hamming distance $d_H(P, P')$ is a number. Since the binary relation $\leq$ on any set of numbers is a total pre-order, it follows immediately that the binary relation $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$ is also a total pre-order.

In order to define a total strict order on $\mathbb{P}_{\mathbb{L}}$, given a profile Let $P = \langle p_1, \dots, p_n \rangle \in \mathbb{P}_{\mathbb{L}}$, we start by introducing the two following conditions relating two profiles $P' = \langle p'_1, \dots, p'_n \rangle$ and $P'' = \langle p''_1, \dots, p''_n \rangle$ of $\mathbb{P}_{\mathbb{L}}$:

(i) There is $i \in \{1, \dots, n\}$ such that $|p_i - p'_i| < |p_i - p''_i|$ and for all $j < i$ it holds that $|p_j - p'_j| = |p_j - p''_j|$.
(ii) For all $k \in \{1, \dots, n\}$ it holds that $|p_k - p'_k| = |p_k - p''_k|$ and there is $i \in \{1, \dots, n\}$ such that $p_i - p'_i > p_i - p''_i$ and for all $j < i$ it holds that $p_j - p'_j = p_j - p''_j$.

The binary relation $\prec_P$ on $\mathbb{P}_{\mathbb{L}}$ defined by $P' \prec_P P''$ iff one of the two conditions (i)-(ii) above holds is a total strict order. Furthermore, this relation $\preceq_P$ is P-faithful.

Example 4.12. Let $\mathbb{P}$ be the set of profiles that can be considered in the context of the Neurorehabilitation System presented in Subsection 1.1. Let $Patient_1 = \langle 75, 79, 50, 60, 67, M, 9 \rangle$, $Patient_2 = \langle 80, 75, 50, 60, 67, M, 9 \rangle$, $\Gamma = \{Patient_1, Patient_2\}$ and $\alpha =$ "$Mem + Att = 155 \wedge L_6 = M$". Considering the Hamming distance between profiles presented in Definition 4.11 we have that:

(a) the models of α that are closest to $Patient_1$ (both at an Hamming distance of 1) are $\langle 75, 80, 50, 60, 67, M, 9 \rangle$ and $\langle 76, 79, 50, 60, 67, M, 9 \rangle$.
(b) the model of α that is closest to $Patient_2$ is $Patient_2$ itself (since $Patient_2$ is a model of α).
(c) the model of α that is closest to Γ is $Patient_2$.

As mentioned in item (a), the models of α that are closest to $Patient_1$ are $P_1 = \langle 75, 80, 50, 60, 67, M, 9 \rangle$ and $P_2 = \langle 76, 79, 50, 60, 67, M, 9 \rangle$. If $\preceq_{Patient_1}$ and $\prec_{Patient_1}$ are, respectively, the total pre-order and the total strict order on $\mathbb{P}$ that are defined, as proposed above, by means of the Hamming distance using $Patient_1$ as reference, it holds that:

- $P_1 \preceq_{Patient_1} P_2$ and $P_2 \preceq_{Patient_1} P_1$;
- $P_1 \prec_{Patient_1} P_2$ (and $P_2 \not\prec_{Patient_1} P_1$).

Finally we note that proceeding as exposed above but using a set of profiles Γ (instead of a single profile) as reference and the Hamming distance between a profile and a set of profiles (instead of the Hamming distance between two profiles), we can define a Γ-faithful total pre-order and a Γ-faithful total strict order on $\mathbb{P}_{\mathbb{L}}$.

5. Profile dynamics

In this section we will present the four different models for profiles dynamics that were mentioned in Section 1. Models 1 and 2 reflect the change of a single profile when new information is received. The outcome of the operators associated to Model 1 is a new profile that incorporates the new information whereas the operators associated with Model 2 return a set of possible profiles that incorporate the new information. Models 3 and 4 characterize the dynamics of a set of profiles when new information is received. Model 3 is based on the operators of belief revision [1,35] while Model 4 is based on the operators of KM-update [36]. All the models will be axiomatically characterized.

5.1. Model 1. From one profile to one profile

In this subsection we present the first model for profile dynamics. In this model, we revise a profile by a formula of the language obtaining as output a profile.

We can consider a P-faithful total strict assignment $\prec_P$ as a relation measuring the closeness of a given profile to P, *i.e.*, $P' \prec_P P''$ means that P' is closer to P than P''. In this sense, it is natural to define a profile revision operator such that the outcome of revising P by a consistent formula α returns the closest of the α models to P. If α is inconsistent the operator returns P. Formally:

Definition 5.1. Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, and let P be a profile associated with $\mathbb{L}$. Let $\prec_P$ be a P-faithful total strict order on $\mathbb{P}_{\mathbb{L}}$. The $PtoP$ profile revision induced by $\prec_P$ is the operation $\odot_{\prec_P}$ such that for all sentences α:

$$P \odot_{\prec_P} \alpha = \begin{cases} min(\|\alpha\|, \prec_P) & \text{if } \|\alpha\| \neq \emptyset \\ P & \text{otherwise} \end{cases}$$

An operator $\odot$ is a $PtoP$ profile revision on P if and only if there is a P-faithful total strict order on $\mathbb{P}_{\mathbb{L}}$, $\prec_P$, such that for all sentences α

$$P \odot \alpha = P \odot_{\prec_P} \alpha.$$

Example 5.2. Let $\mathbb{L} = \ll L_1, L_2 \gg$ and $D_1 = \{a, b\}$ and $D_2 = \{1, 2, 3\}$ be the domains of, respectively, L_1 and L_2. There are six profiles associated with $\mathbb{L}$. Let P be the profile $\langle a, 1 \rangle$. Consider the following P-faithful total strict order on $\mathbb{P}_{\mathbb{L}}$:

$\langle a,1\rangle \prec_P \langle b,1\rangle \prec_P \langle a,2\rangle \prec_P \langle b,2\rangle \prec_P \langle a,3\rangle \prec_P \langle b,3\rangle$.

When considering the $PtoP$ profile revision induced by $\prec_P$, the outcome of the revision of P by $L_1 = b$ is $\langle b,1\rangle$ and the outcome of the revision of P by $L_2 = 2$ is $\langle a,2\rangle$.

In the following example we revisit the Neurorehabilitation System mentioned in Subsection 1.1.

Example 5.3. Consider the context of Example 4.12, where $Patient_1 = \langle 75,79,50,60,67,M,9\rangle$, and assume that the INTs provide the new information $\alpha =$ "$Mem+Att = 155 \wedge L_6 = M$". Let $\prec_{Patient_1}$ be the $Patient_1$-faithful total strict order on $\mathbb{P}$ defined by means of the Hamming distance using $Patient_1$ as reference. As mentioned in Example 4.12 there are only two models of α whose Hamming distance to $Patient_1$ is 1, namely $P_1 = \langle 75,80,50,60,67,M,9\rangle$ and $P_2 = \langle 76,79,50,60,67,M,9\rangle$, and it holds that $P_1 \prec_{Patient_1} P_2$. Then, considering the $PtoP$ profile revision induced by $\prec_{Patient_1}$, the outcome of the revision of $Patient_1$ by α is $\langle 75,80,50,60,67,M,9\rangle$.

5.1.1. Postulates

The following postulates, which are based on the modified version of the AGM revision and update postulates proposed by Katsuno and Mendelzon [35,36] (which are recalled in Subsections 3.2 and 3.3), will be useful towards the axiomatic characterization of the $PtoP$ profile revision operators.

(P1) If $\|\alpha\| \neq \emptyset$, then $P \odot \alpha \in \|\alpha\|$
(P2) If $\|\alpha\| = \emptyset$, then $P \odot \alpha = P$
(P3) If $P \vDash \alpha$, then $P \odot \alpha = P$.
(P4) If $\|\alpha\| = \|\beta\|$, then $P \odot \alpha = P \odot \beta$.
(P5) $P \odot (\alpha \vee \beta) = P \odot \alpha$ or $P \odot (\alpha \vee \beta) = P \odot \beta$.
(P6) If $P \odot \alpha \vDash \beta$ and $P \odot \beta \vDash \alpha$, then $P \odot \alpha = P \odot \beta$.

Postulate **P1** states that the outcome of a profile revision by any consistent information is a profile that satisfies the incoming information. **P2** asserts that if the new information is contradictory, then the output and the original profiles coincide. **P3** states that if the starting profile P is a model of the input information α, then the outcome of the profile revision of P by α is P itself. This is natural, since, in this case, no changes to the original profile need to be made in order to obtain a profile that satisfies the received information. **P4** expresses the principle of irrelevance of the syntax, *i.e.*, revising a profile by two equivalent sentences produces the same output profile. **P5** states that in the case of a profile revision by a disjunction, one of the disjuncts will be preferred in the outcome. Is natural to expect that the PtoP profile revision operators satisfy this property since the $\alpha \vee \beta$-model closest to P must be either the α-model or the β-model closest to P. Finally, Postulate **P6** states that if the outcome of revising a profile P by α implies β and the outcome of revising that profile by β implies α, then the outcomes of those two revisions are identical. This condition appears as (U6) in KM-update [36], as condition (C7) in [24] in a belief revision context and as a conditional logic axiom (CSO) in [47].

The following observation illustrates that ($P6$) follows from the other above mentioned postulates.

Observation 5.4. *Let $\mathbb{L} = \ll L_1, L_2, \ldots, L_n \gg$ be a tuple of labels and let $P \in \mathbb{P}_{\mathbb{L}}$. Let $\odot : \mathcal{L}_{\mathbb{L}} \to \mathbb{P}_{\mathbb{L}}$ be a profile revision operator on P that satisfies postulates (P1) to (P5). Then $\odot$ also satisfies (P6).*

We now present an axiomatic characterization for the $PtoP$ profile revision operators.

Theorem 5.5. *Let $\mathbb{L} = \ll L_1, L_2, \ldots, L_n \gg$ be a tuple of labels and let P be a profile associated with $\mathbb{L}$. $\odot$ is a $PtoP$ profile revision operator on P if and only if $\odot$ satisfies the postulates (P1) to (P5).*

The above representation theorem characterizes in terms of postulates the PtoP profile revision operators. It is useful, for example, for having some knowledge about the outcome of a PtoP profile revision by a given sentence (even without knowing explicitly the binary relation underlaying that operator). This axiomatic characterization can also be used for identifying some properties that a formula must satisfy when only the outcome of a PtoP revision by that formula is known. The following example illustrates these two possible ways of using the above result.

Example 5.6. Let $\mathbb{P}_{\mathbb{L}} = \{P_1, P_2, P_3, P_4\}$, $\|\alpha\| = \{P_1, P_2\}$ and $\|\beta\| = \{P_3, P_4\}$. The outcome of a PtoP revision of P_1 by:
- α is P_1, according to postulate **(P3)**;
- β is either P_3 or P_4, according to postulate **(P1)**.
On the other hand, if the outcome of a PtoP revision of P_1 by a formula δ is P_3, then δ must be such that $P_3 \in \|\delta\| \subseteq \mathbb{P}_{\mathbb{L}} \setminus \{P_1\}$, according to postulates **(P1)**, **(P2)** and **(P3)**.

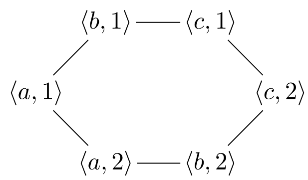

Fig. 1. Hasse diagram of pre-order $\preceq_P$ between profiles of $\mathbb{P}_{\mathbb{L}}$. A link between two profiles represents that the profile on the left precedes the one in the right. The reflexivity and transitivity of the pre-order is graphically omitted.

5.2. Model 2. From one profile to a set of profiles

Given a (fixed) profile P, the operator introduced in the following definition, designated by PtoSP profile revision receives a sentence and returns as output a set of profiles which satisfy that sentence. If a tie-break method is applied, model 2 can be seen as a variation of model 1.

The intuition regarding the PtoSP profile revision operators is similar to the one presented for PtoP profile revision operators, but in this case the output by a consistent formula α is the set of the α models that are closer to P and in the case of α being inconsistent, it is $\{P\}$.

Definition 5.7. Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let $P \in \mathbb{P}_{\mathbb{L}}$. Let $\preceq_P$ be a P-faithful pre-order on $\mathbb{P}_{\mathbb{L}}$. The $\preceq_P$-based $PtoSP$ profile revision on P is the operation $\odot_{\preceq_P} : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ such that for all sentences α:

$$P \odot_{\preceq_P} \alpha = \begin{cases} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

An operator $\odot$ is a PtoSP profile revision on P if and only if there exists a P-faithful pre-order $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$, such that, for all sentences α, $P \odot \alpha = P \odot_{\preceq_P} \alpha$.

At this point we clarify the main differences between operators of model 2 and of model 1. First of all the PtoP profile revisions are defined by means of a total strict order while PtoSP profile revisions are based on a pre-order. As a consequence of this fact, on the one hand the output of a PtoP profile revision is a single profile (specifically the minimum of the set of models of the input sentence) and, on the other hand, the output of a PtoSP profile revision is a set of profiles (specifically the set formed by the minimal elements of the set of models of the input sentence). Hence, while model 1 returns as output a totally determined profile, model 2 allows situations in which the agent's output profile is not completely determined. Furthermore, while model 1 can only be used in contexts where it is possible and makes sense to consider a total strict order among all profiles, model 2 can be used in situations where there are incomparable profiles and/or some profiles are considered to have the same level of preference.

Example 5.8. Let $\mathbb{L} = \ll L_1, L_2 \gg$ and $D_1 = \{a, b, c\}$ and $D_2 = \{1, 2\}$ be the domains of, respectively, L_1 and L_2. There are, in this case, six profiles associated with $\mathbb{L}$. Let P be the profile $\langle a, 1\rangle$. Consider the P-faithful pre-order $\preceq_P$ represented by the Hasse diagram in Fig. 1. When considering the $PtoSP$ profile revision on P induced by $\preceq_P$, the outcome of the revision of P by:

1. $L_2 = 2$ is $\{\langle a, 2\rangle\}$;
2. $L_1 = b$ is $\{\langle b, 1\rangle, \langle b, 2\rangle\}$;
3. $L_1 = c \vee L_2 = 2$ is $\{\langle a, 2\rangle, \langle c, 1\rangle\}$.

In the first case, we obtain complete information about the resulting new state of the agent, in the sense that we know that $L_1 = a$ and $L_2 = 2$. In the second case, we have only partial information about the new state of the agent. We know that $L_1 = b$ but L_2 can be either 1 or 2. In the third case, we also have partial information about the new state of the agent. We know that L_1 can be either a or c and L_2 can take any value from of the corresponding domain. But we also know that $L_1 = a$ if and only if $L_2 = 2$ and that $L_1 = c$ if and only if $L_2 = 1$.

In the following example we revisited once more the Neurorehabilitation System proposed in Subsection 1.1.

Example 5.9. Let $Patient_1 = \langle 75, 79, 50, 60, 67, M, 9\rangle$ and $\alpha =$ "$Mem + Att = 155 \wedge L_6 = M$". Let $\preceq_{Patient_1}$ be the $Patient_1$-faithful pre-order defined as proposed in Subsection 4.3 by means of the Hamming distance using $Patient_1$ as reference. As mentioned in Example 4.12 there are only two models of α whose Hamming distance to $Patient_1$ is 1, namely $P_1 = \langle 75, 80, 50, 60, 67, M, 9\rangle$ and $P_2 = \langle 76, 79, 50, 60, 67, M, 9\rangle$, and it holds that $P_1 \preceq_{Patient_1} P_2$ and $P_2 \preceq_{Patient_1} P_1$. Then, considering the $PtoSP$ profile revision induced by $\preceq_{Patient_1}$, the outcome of the revision of $Patient_1$ by α is $\{P_1, P_2\}$.

5.2.1. Postulates

The list of postulates presented below illustrates some properties satisfied by $PtoSP$ profile revision operators.

(S1) If $\|\alpha\| \neq \emptyset$, then $P \odot \alpha \subseteq \|\alpha\|$.

(S2) If $\|\alpha\| = \emptyset$, then $P \odot \alpha = \{P\}$.

(S3) $P \odot \alpha \neq \emptyset$.

(S4) If $\|\alpha\| = \|\beta\|$, then $P \odot \alpha = P \odot \beta$.

(S5) If $P \vDash \alpha$, then $P \odot \alpha = \{P\}$.

(S6) $P \odot (\alpha \vee \beta) = P \odot \alpha$ or $P \odot (\alpha \vee \beta) = P \odot \beta$ or $P \odot (\alpha \vee \beta) = P \odot \alpha \cup P \odot \beta$.

(S7) $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha \cup P \odot \beta$.

(S8) If $P \odot \alpha \vDash \beta$ and $P \odot \beta \vDash \alpha$, then $P \odot \alpha = P \odot \beta$.

(S9) $P \odot \alpha \cap P \odot \beta \subseteq P \odot (\alpha \vee \beta)$.

(S10) $P \odot \alpha \cap \|\beta\| \subseteq P \odot (\alpha \wedge \beta)$.

S1, **S2**, **S4**, **S5** and **S8** are the adapted versions of **P1**, **P2**, **P4**, **P3** and **P6** respectively. **S3** states that the outcome is always consistent, *i.e.* a PtoSP always returns a non-empty set of profiles. **S6** is a well-known postulate in AGM theory, there called "ventilation". The intuition behind this postulate is that if we wish to revise by a disjunction and there is some preference between the disjuncts, then this revision is equivalent to revising by the preferred disjunct. In the case of indifference, revising by the disjunction returns the set of profiles consisting of the union of the outcomes of revising by each member of the disjunction.**S7** is a weaker version of **S6**. It states that the outcome of a revision by a disjunction is a subset of the set of profiles consisting of the union of the outcomes of revising by each member of the disjunction. **S9** states that the profiles that are common to the outcomes of revising by each member of a given disjunction are also contained in the outcome of the revision by that disjunction. It has appeared as (R8) in [35] in the context of belief revision and in [56, p.113] in the context of non-monotonic reasoning. **S10** states that the profiles in the outcome of a revision by α that are models of β are also contained in the outcome of the revision by $\alpha \wedge \beta$.

The following observation illustrates some relations between the postulates presented above.

Observation 5.10. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let $P \in \mathbb{P}_{\mathbb{L}}$. Let $\odot : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ be a profile revision operator on P.*

1. *If $\odot$ satisfies (S1), (S2), (S4), (S5) and (S7), then it satisfies (S10).*
2. *If $\odot$ satisfies (S1), (S2), (S4) and (S10), then it satisfies (S7).*

In the following theorems, we present axiomatic characterizations for $PtoSP$ profile revision operators (based on a P-faithful pre-order, a P-faithful order and a P-faithful total pre-order).

Theorem 5.11. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, $P \in \mathbb{P}_{\mathbb{L}}$ and $\odot : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ be a profile revision operator on P. The following conditions are equivalent:*

1. *$\odot$ satisfies the postulates (S1) to (S5) and (S7) to (S9).*
2. *There exists a P-faithful pre-order $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_P$-based PtoSP profile revision on P.*
3. *There exists a P-faithful order $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_P$-based PtoSP profile revision on P.*

Theorem 5.12. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, $P \in \mathbb{P}_{\mathbb{L}}$ and $\odot : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ be a profile revision operator on P. The following conditions are equivalent:*

1. *$\odot$ satisfies the postulates (S1) to (S6).*
2. *There exists a P-faithful total pre-order $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_P$-based PtoSP profile revision on P.*

5.3. From a set of profiles to a set of profiles

In this subsection we will characterize the changes produced by an input in a collection of profiles Γ and which give rise to a (new) set of profiles. As we mentioned in the introduction, we recognized here two different kinds of change: *revision*, when new information is received, assuming a static world and *update*, when the new information reflects a change in the world. At the level of pre-orders, the main difference between the two models is that revision uses a unique pre-order where the profiles of Γ are at the bottom of the pre-order (as a whole), while update needs several pre-orders, more precisely it makes use of a pre-order for each profile that integrates Γ. We only make the distinction at this point since for Models 1 and 2 the notion of revision and update coincide.

$$\begin{array}{c}\langle a,1\rangle \\ \langle b,1\rangle\end{array} < \begin{array}{l}\langle a,3\rangle - \langle c,3\rangle - \langle c,1\rangle \\ \langle a,2\rangle - \langle c,2\rangle < \begin{array}{c}\langle b,3\rangle \\ \langle b,2\rangle\end{array}\end{array}$$

Fig. 2. Hasse diagram of pre-order $\preceq_\Gamma$ between profiles of $\mathbb{P}_{\mathbb{L}}$. The pairs of profiles $\langle a,1\rangle$ and $\langle b,1\rangle$ and $\langle b,3\rangle$ and $\langle b,2\rangle$ are at the same level (i.e., $\langle a,1\rangle \simeq \langle b,1\rangle$ and $\langle b,3\rangle \simeq \langle b,2\rangle$).

5.3.1. Model 3. Revising Sets of Profiles

We now present a model which addresses the problem of revising a set of profiles returning as output a set of profiles. We can see a Γ-faithful pre-order $\preceq_\Gamma$ as a relation that measures the distance of each profile to Γ, *i.e.*, $P' \preceq_\Gamma P''$ can be interpreted as meaning that P' is at least as close to Γ as P''. In this sense, it is natural to define a profile revision operator on a set of profiles Γ which is such that the result of revising by a consistent formula α is the set formed by the models of α closest to Γ. It is also natural to consider that when the input sentence α is inconsistent, then the outcome of the corresponding revision is simple the original set of profiles Γ.

Definition 5.13. Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$. Let $\preceq_\Gamma$ be a Γ-faithful pre-order on $\mathbb{P}_{\mathbb{L}}$. The $\preceq_\Gamma$-based SPtoSP profile revision on Γ is the operation $\odot_{\preceq_\Gamma} : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ such that for all sentences α:

$$\Gamma \odot_{\preceq_\Gamma} \alpha = \begin{cases} Min(\|\alpha\|, \preceq_\Gamma) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

An operator $\odot$ is a SPtoSP profile revision on Γ if and only if there exists a Γ-faithful pre-order $\preceq_\Gamma$ on $\mathbb{P}_{\mathbb{L}}$, such that, for all sentences α, $\Gamma \odot \alpha = \Gamma \odot_{\preceq_\Gamma} \alpha$.

Example 5.14. Let $\mathbb{L} = \ll L_1, L_2 \gg$ and $D_1 = \{a,b,c\}$ and $D_2 = \{1,2,3\}$ be the domains of, respectively, L_1 and L_2. There are, in this case, nine profiles associated with $\mathbb{L}$. Let $\Gamma = \{\langle a,1\rangle, \langle b,1\rangle\}$. Consider the Γ-faithful pre-order $\preceq_\Gamma$ represented by the Hasse diagram in Fig. 2. When considering the $SPtoSP$ profile revision on Γ induced by $\preceq_\Gamma$, the outcome of the revision of Γ by:

1. $L_2 = 2$ is $\{\langle a,2\rangle\}$;
2. $L_1 = c$ is $\{\langle c,2\rangle, \langle c,3\rangle\}$;
3. $L_1 = c \vee L_2 = 2$ is $\{\langle a,2\rangle, \langle c,3\rangle\}$.

In the following example we present an example of a SPtoSP revision in the context of the Neurorehabilitation System proposed in Subsection 1.1.

Example 5.15. Consider a patient characterized by the set of profiles $\Gamma = \{\langle 75, 79, 50, 60, 67, M, 9\rangle, \langle 80, 75, 50, 60, 67, M, 9\rangle\}$ and let $\alpha =$ "$Mem + Att = 155 \wedge L_6 = M$". Let $\preceq_\Gamma$ be the Γ-faithful pre-order defined as proposed in Subsection 4.3 by means of the Hamming distance between a profile and a set of profiles using Γ as reference. Since Γ contains some models of α, namely the profile $\langle 80, 75, 50, 60, 67, M, 9\rangle$, it holds that $Min(\|\alpha\|, \preceq_\Gamma) = \{\langle 80, 75, 50, 60, 67, M, 9\rangle\}$. Therefore, considering the $SPtoSP$ profile revision induced by $\preceq_\Gamma$, the outcome of the revision of Γ by α is $\{\langle 80, 75, 50, 60, 67, M, 9\rangle\}$.

Postulates.

The following list of postulates illustrates some of the properties of $SPtoSP$ profile revision operators.

(SP1) If $\|\alpha\| \neq \emptyset$, then $\Gamma \odot \alpha \subseteq \|\alpha\|$.
(SP2) If $\|\alpha\| = \emptyset$, then $\Gamma \odot \alpha = \Gamma$.
(SP3) $\Gamma \odot \alpha \neq \emptyset$.
(SP4) If $\|\alpha\| = \|\beta\|$, then $\Gamma \odot \alpha = \Gamma \odot \beta$.
(SP5) If $\Gamma \cap \|\alpha\| \neq \emptyset$ then $\Gamma \odot \alpha = \Gamma \cap \|\alpha\|$.
(SP6) $\Gamma \odot (\alpha \vee \beta) = \Gamma \odot \alpha$ or $\Gamma \odot (\alpha \vee \beta) = \Gamma \odot \beta$ or $\Gamma \odot (\alpha \vee \beta) = \Gamma \odot \alpha \cup \Gamma \odot \beta$.
(SP7) $\Gamma \odot (\alpha \vee \beta) \subseteq \Gamma \odot \alpha \cup \Gamma \odot \beta$.
(SP8) If $\Gamma \odot \alpha \vDash \beta$ and $\Gamma \odot \beta \vDash \alpha$, then $\Gamma \odot \alpha = \Gamma \odot \beta$.
(SP9) $\Gamma \odot \alpha \cap \Gamma \odot \beta \subseteq \Gamma \odot (\alpha \vee \beta)$.
(SP10) $\Gamma \odot \alpha \cap \|\beta\| \subseteq \Gamma \odot (\alpha \wedge \beta)$.

These postulates are similar to those proposed for model 2 except for **SP5**. This postulate corresponds to the AGM revision postulate vacuity and states that if there are profiles in Γ that satisfy the input sentence α, then the output of the revision of Γ by α is the set formed by those profiles. In Example 5.15 above, we presented a situation in which it holds that $\Gamma \cap \|\alpha\| \neq \emptyset$.

The following observation illustrates some relations between the postulates presented above.

Observation 5.16. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$ and $\odot$ be a profile revision operator on Γ.*

1. *If $\odot$ satisfies (SP1), (SP2), (SP4), (SP5) and (SP7), then it satisfies (SP10).*
2. *If $\odot$ satisfies (SP1), (SP2), (SP4) and (SP10), then it satisfies (SP7).*

Now we present two representation theorems.

Theorem 5.17. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$ and $\odot$ be a profile revision operator on Γ. The following conditions are equivalent:*

1. *$\odot$ satisfies the postulates (SP1) to (SP5) and (SP7) to (SP9).*
2. *There exists a Γ-faithful pre-order $\preceq_\Gamma$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_\Gamma$-based SPtoSP profile revision on Γ.*
3. *There exists a Γ-faithful order $\preceq_\Gamma$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_\Gamma$-based SPtoSP profile revision on Γ.*

Theorem 5.18. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, $P \in \mathbb{P}_{\mathbb{L}}$ and $\odot : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ be a profile revision operator on P. The following conditions are equivalent:*

1. *$\odot$ satisfies the postulates (SP1) to (SP6).*
2. *There exists a Γ-faithful total pre-order $\preceq_\Gamma$ on $\mathbb{P}_{\mathbb{L}}$ such that $\odot$ is the $\preceq_\Gamma$-based SPtoSP profile revision on P.*

The representation theorems above characterize the SPtoSP profile revision operators in terms of postulates. They are useful, for example, for having some knowledge about the outcome of a SPtoSP profile revision by a given sentence, even when the binary relation in which the operator is based is unknown.

Example 5.19. Let $\mathbb{P}_{\mathbb{L}} = \{P_1, P_2, P_3, P_4, P_5\}$, $\Gamma = \{P_2, P_3\}$, $\|\alpha\| = \{P_1, P_4\}$. The outcome of an SPtoSP revision of Γ by α must be either $\{P_1, P_4\}$, $\{P_1\}$ or $\{P_4\}$, according to postulates **SP1** and **SP3**. On the other hand, $\{P_1, P_2\}$ cannot be the outcome of a SPtoSP revision of Γ by any sentence. Indeed, let $\odot$ be an SPtoSP revision on Γ, it holds that:
- If β is such that $\Gamma \cap \|\beta\| \neq \emptyset$ then it follows from **SP5** that $\Gamma \odot \beta = \Gamma \cap \|\beta\|$ and, therefore, $P_1 \notin \Gamma \odot \beta$.
- If β is such that $\Gamma \cap \|\beta\| = \emptyset$, then it follows from **SP1** and **SP2** that either $\Gamma \odot \beta \subseteq \|\beta\|$ (if $\|\beta\| \neq \emptyset$) or $\Gamma \odot \beta = \Gamma$ (if $\|\beta\| = \emptyset$). Hence, in any case, $\Gamma \odot \beta \neq \{P_1, P_2\}$ (note that if $\Gamma \odot \beta \subseteq \|\beta\|$ then $P_2 \notin \Gamma \odot \beta$, because it follows from $P_2 \in \Gamma$ and $\Gamma \cap \|\beta\| = \emptyset$ that $P_2 \notin \|\beta\|$).

5.3.2. Model 4. Updating Profiles

In this subsection, we present another kind of profile change functions, namely the profile update operators.

The main difference between a SPtoSP profile revision operator on Γ and a profile update operator on the same set of profiles is that, while the former is based on only one binary relation on $\mathbb{P}_{\mathbb{L}}$ (namely a Γ-faithful binary relation), the latter uses several binary relation on $\mathbb{P}_{\mathbb{L}}$, more precisely one P_i-faithful binary relation for each profile P_i in Γ. The output of a profile update of Γ by a consistent formula α is the set formed by all the α-models closest to at least one of the profiles P_i in Γ (according to the relation $\preceq_{P_i}$). As it was the case in the operations proposed above, the outcome of updating Γ by an inconsistent formula is Γ.

Definition 5.20. Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$. Let $\preceq$ be a UpdP-faithful assignment (that maps each profile P to a P-faithful binary relation $\preceq_P$ on $\mathbb{P}_{\mathbb{L}}$). The $\preceq$-based profile update on Γ is the operation $\diamond_\preceq : \mathcal{L}_{\mathbb{L}} \to \mathcal{P}(\mathbb{P}_{\mathbb{L}})$ such that for all sentences α:

$$\Gamma \diamond_\preceq \alpha = \begin{cases} \bigcup\limits_{P \in \Gamma} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

An operator $\diamond$ is a profile update on Γ if and only if there exists an UpdP-faithful assignment $\preceq$, such that, for all sentences α, $\Gamma \diamond \alpha = \Gamma \diamond_\preceq \alpha$.

Example 5.21. Let $\mathbb{L} = \ll L_1, L_2 \gg$ and $D_1 = \{a, b\}$ and $D_2 = \{1, 2\}$ be the domains of, respectively, L_1 and L_2. There are, in this case, four profiles associated with $\mathbb{L}$, namely $P_1 = \langle a, 1\rangle$, $P_2 = \langle b, 1\rangle$, $P_3 = \langle a, 2\rangle$ and $P_4 = \langle b, 2\rangle$. Let $\Gamma = \{P_1, P_2\}$. Let $\preceq$ be an UpdP-faithful assignment that associates to profiles P_1 and P_2, the binary relations $\preceq_{P_1}$ and $\preceq_{P_2}$ (which are, respectively, P_1-faithful and P_2-faithful), respectively, defined by:

- $P_1 \prec_{P_1} P_2 \prec_{P_1} P_3 \prec_{P_1} P_4$;
- $P_2 \prec_{P_2} P_3 \prec_{P_2} P_1 \prec_{P_2} P4$.

When considering the $\preceq$-based profile update on Γ, the outcome of the update of Γ by:

1. $L_2 = 2$ is $\{\langle a, 2\rangle\}$;
2. $L_1 = a$ is $\{\langle a, 1\rangle, \langle a, 2\rangle\}$.

In the following example we revisit Example 5.15, this time to update Γ by α (instead of revising Γ by α).

Example 5.22. Let $P_1 = \langle 75, 79, 50, 60, 67, M, 9\rangle$, $P_2 = \langle 80, 75, 50, 60, 67, M, 9\rangle$, $\Gamma = \{P_1, P_2\}$, and $\alpha =$ "$Mem + Att = 155 \wedge L_6 = M$". Let $\preceq_{P_1}$ be the P_1-faithful pre-order defined as proposed in Subsection 4.3 by means of the Hamming distance using P_1 as reference. As mentioned in Example 4.12 there are only two models of α whose Hamming distance to P_1 is 1, namely $Q_1 = \langle 75, 80, 50, 60, 67, M, 9\rangle$ and $Q_2 = \langle 76, 79, 50, 60, 67, M, 9\rangle$, and it holds that $Q_1 \preceq_{P_1} Q_2$ and $Q_2 \preceq_{P_1} Q_1$. Hence, $Min(\|\alpha\|, \preceq_{P_1}) = \{Q_1, Q_2\}$. On the other hand, if $\preceq_{P_2}$ is the P_2-faithful pre-order defined in the same way but using P_2 as reference, then since $P_2 \in \|\alpha\|$, $Min(\|\alpha\|, \preceq_{P_2}) = \{P_2\}$. Let $\preceq$ be an UpdP-faithful assignment that associates to profiles P_1 and P_2, the binary relations $\preceq_{P_1}$ and $\preceq_{P_2}$. Then the outcome of the $\preceq$-based profile update of Γ by α is $\{\langle 76, 79, 50, 60, 67, M, 9\rangle, \langle 75, 80, 50, 60, 67, M, 9\rangle, \langle 80, 75, 50, 60, 67, M, 9\rangle\}$ (since $Min(\|\alpha\|, \preceq_{P_1}) \cup Min(\|\alpha\|, \preceq_{P_2}) = \{Q_1, Q_2, P_2\}$).

Postulates.

We now present the postulates that will be used in the representation theorems of profile update operators.

(U0) If $\|\alpha\| = \emptyset$, then $\Gamma \diamond \alpha = \Gamma$.
(U1) If $\|\alpha\| \neq \emptyset$, then $\Gamma \diamond \alpha \subseteq \|\alpha\|$.
(U2) If $\Gamma \subseteq \|\alpha\|$, then $\Gamma \diamond \alpha = \Gamma$.
(U3) $\Gamma \diamond \alpha \neq \emptyset$.
(U4) If $\|\alpha\| = \|\beta\|$, then $\Gamma \diamond \alpha = \Gamma \diamond \beta$.
(U5) $\Gamma \diamond \alpha \cap \|\beta\| \subseteq \Gamma \diamond (\alpha \wedge \beta)$.
(U6) If $\Gamma \diamond \alpha \vDash \beta$ and $\Gamma \diamond \beta \vDash \alpha$, then $\Gamma \diamond \alpha = \Gamma \diamond \beta$.
(U7) If P is a profile, then $\{P\} \diamond \alpha \cap \{P\} \diamond \beta \subseteq \{P\} \diamond (\alpha \vee \beta)$.
(U8) $(\Gamma_1 \cup \Gamma_2) \diamond \alpha = \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$.
(U9) If $\|\alpha \wedge \beta\| \neq \emptyset$, P is a profile and $(\{P\} \diamond \alpha) \cap \|\beta\| \neq \emptyset$, then $\{P\} \diamond (\alpha \wedge \beta) \subseteq (\{P\} \diamond \alpha) \cap \|\beta\|$.
(U10) If $\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_1 \diamond \alpha \subseteq \Gamma_2 \diamond \alpha$.
(U11) $\Gamma \cap \|\alpha\| \subseteq \Gamma \diamond \alpha$.

The postulates presented above are similar to those proposed for update operators in [36]. The formalization of the postulates **U0**, **U1**, **U2**, **U3**, **U4** and **U6** is similar to one of the postulates **SP2**, **SP1**, **P3**, **SP3**, **SP4** and **SP8**, respectively. **U2** is a weaker version of **SP5** (assuming that Γ is a non-empty set). **U5** states that if a profile is in the outcome of updating Γ by α and is also a model of β, then it must be also in the outcome of updating Γ by $\alpha \wedge \beta$. **U7** and **U9** are only applied when we have complete information about the agent's profile (i.e., when the agent's set of profiles is a singleton). **U7** states that any profile that is present in both the outcome of updating $\{P\}$ by α and the outcome of updating $\{P\}$ by β is also in the outcome of updating $\{P\}$ by $\alpha \vee \beta$. **U9** states that if α is consistent with β and the outcome of updating $\{P\}$ by α contains models of β, then the outcome of updating $\{P\}$ by $\alpha \wedge \beta$ is a subset of the set formed by the models of β that are in the outcome of updating $\{P\}$ by α. **U7** and **U9** can be seen as translations to the context of profiles of the namesake postulates presented for update operators in [36]. **U8** states that the outcome of updating $\Gamma_1 \cup \Gamma_2$ by α is identical to the union of the outcomes of updating Γ_1 by α and Γ_2 by α. **U8** can be regarded as the adapted profile version of the "disjunction rule" mentioned in the introduction of [36]. **U10** is an expression of monotonicity and follows directly from **U8**. **U11** states that models of α that are in Γ must be in the outcome of updating Γ by α.

The following observation illustrates some relations between the (update) postulates presented above.

Observation 5.23. *Let Γ, Γ_1 and Γ_2 be sets of profiles and $\diamond$ be a profile update operator.*

1. *If $\diamond$ satisfies (U8), then $\diamond$ satisfies (U10).*
2. *If $\diamond$ satisfies (U2) and (U8), then $\diamond$ satisfies (U11).*

We now present axiomatic characterizations for some profile update operators.

Theorem 5.24. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels, Γ be a non-empty subset of $\mathbb{P}_{\mathbb{L}}$ and $\diamond$ be a profile update operator. The following conditions are equivalent:*

1. *The profile update operator $\diamond$ satisfies postulates (U0) to (U8).*
2. *There exists a UpdP-faithful assignment $\preceq$, that maps each profile P to a P-faithful pre-order $\preceq_P$, such that $\diamond$ is the $\preceq$-based profile update on Γ.*
3. *There exists a UpdP-faithful assignment $\preceq$, that maps each profile P to a P-faithful order $\preceq_P$, such that $\diamond$ is the $\preceq$-based profile update on Γ.*

Theorem 5.25. *Let $\mathbb{L} = \ll L_1, L_2, ..., L_n \gg$ be a tuple of labels and let $\diamond$ be a profile update operator. The following conditions are equivalent:*

1. *The profile update operator $\diamond$ satisfies postulates (U0) to (U5), (U8) and (U9).*
2. *There exists a UpdP-faithful assignment $\preceq$, that maps each profile P to a P-faithful total pre-order $\preceq_P$, such that $\diamond$ is the $\preceq$-based profile update on Γ.*

6. Conclusions and future works

The main contributions of this paper are:

- Introduction of a formal definition of profile and of an associated formal language appropriate for expressing assertions about profiles;
- Presentation of several profile change operators. More precisely, given an initial profile or set of profiles, we have proposed four different operators which receive as input a sentence of the above mentioned formal language and return as output a new profile or set profiles consistent with the received sentence;
- Presentation of axiomatic characterizations for each of the newly proposed operators. These representations theorems allow us to compare the behaviours of these different observations and can also be used for characterizing the change formulas that lead to obtain a desired new state from a given initial state by means of the corresponding operation.

The operators proposed in this paper are essentially the result of adapting to the context of profiles, the well-known belief revision and belief update operators originally proposed in [1,35,36]. In fact, the notion of profile that we propose can be seen as a generalization of the notion of interpretation and, accordingly, the operators we have proposed can be considered generalizations of the revision and update operators defined in [35,36] by means of binary relations on the set of all interpretations. In this sense, the operators here proposed can be seen as an application of the belief change tools in a concrete scenario, namely for modelling profile dynamics. We also note that, as it is the case with all operators proposed in the context of knowledge representation, the outputs of the operators here proposed are more transparent, in the sense that it is possible to pinpoint the cause of an observed profile change or, at very least, characterize such a change. In the models here proposed each step of the evolution/change of a profile is justified by a certain input (sentence), which can be explicitly determined/characterized. On the contrary, in the case of the data driven profile dynamics models, which are based on machine learning techniques, it is not possible to extract from the model the explanation (or cause) for an observed change in a profile. In these latter models each profile is created/computed from a data set and the justification for the fact that a new profile is obtained is simply the fact that a different data set has been used for obtaining that profile.

There are many natural extensions of the study reported in the present paper. We have already started working on some of those potential extensions, namely:

(a) Targeted training: Given a profile, sometimes it is necessary to determine a set of interactions, tasks, or pieces of training to transform the profile of a user into a target profile. This is the case, for example, in a rehabilitation system, or in systems where it may be convenient to change a user's program interface in order to allow him to perform more complex interactions. In these cases, the changes must be small, minimal or imperceptible. Abrupt changes in systems generally lead to rejections by the end-users. This may be very relevant in cognitive rehabilitation where it is common practice to predetermine goals at the beginning of a training program. These goals are not easily translated into numerical improvements, therefore, the analysis of a cognitive profile and its evolution over time through belief revision dynamics will be a contribution to the explainable Artificial Intelligence field.

(b) Identification: Given a set of previous and current profiles for a community, the identification functions are used for identifying which inputs caused the changes. This is particularly important for example for determining the impact of spreading fake news or the impact of a specific set of tasks in a training system.

Some preliminary results in the context of the above mentioned topics have already been presented in [22,21].

CRediT authorship contribution statement

Eduardo Fermé: Writing – original draft, Writing – review & editing. **Marco Garapa:** Writing – original draft, Writing – review & editing. **Maurício D.L. Reis:** Writing – original draft, Writing – review & editing. **Yuri Almeida:** Writing – original draft, Writing – review & editing. **Teresa Paulino:** Writing – original draft, Writing – review & editing. **Mariana Rodrigues:** Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Eduardo Ferme reports financial support was provided by Foundation for Science and Technology - Portugal.

Data availability

No data was used for the research described in the article.

Acknowledgements

We want to thank the anonymous referees for their stimulating suggestions on an earlier version of the article. The text has also benefited from discussions during presentations at the Workshop of Machine Teaching for Humans (MT4H) in Funchal, at the Computer Science Department of the FernUniversität in Hagen, at the Computer Science Department of the Universidad Nacional del Sur, Bahía Blanca and the Centre de Recherche en Informatique de Lens (CRIL) at the Université d'Artois, Lens, France.

This paper was partially supported by FCT-Fundação para a Ciência e a Tecnologia, Portugal through project PTDC/CCI-COM/4464/2020. E.F. was partially supported by FCT through project UIDB/04516/2020 (NOVA LINCS). M.G. and M.R. were partially supported by the Centro de Investigação em Matemática e Aplicações (CIMA), through the project UIDB/04674/2020 of FCT (DOI 10.54499/UIDB/04674/2020 - https://doi.org/10.54499/UIDB/04674/2020). Y.A. was supported by FCT through SFRH/BD/138911/2018 and T.P. by FCT through SFRH/BD/147390/2019.

Appendix A. Proofs

Lemma 1. *Let $\preceq$ be a pre-order on a set A. Let $\preceq'$ be a relation such that: $\alpha \preceq' \beta$ if and only if $\alpha = \beta$ or $\alpha \prec \beta$, for all $\alpha, \beta \in A$. Then it holds that $\alpha \prec' \beta$ if and only if $\alpha \prec \beta$.*

Proof. Assume that $\alpha \prec' \beta$. Hence $\alpha \preceq' \beta$ and $\beta \npreceq' \alpha$, from which it follows that ($\alpha \prec \beta$ or $\alpha = \beta$) and $\beta \nprec \alpha$ and $\alpha \neq \beta$. Thus $\alpha \prec \beta$.
Consider now that $\alpha \prec \beta$. It holds that $\alpha \preceq' \beta$. Assume towards a contradiction that $\beta \preceq' \alpha$. Hence $\beta \prec \alpha$ or $\alpha = \beta$.
case 1) $\alpha = \beta$. From $\alpha \prec \beta$ it follows that $\beta \npreceq \alpha$. Hence $\alpha \npreceq \alpha$. Contradiction, since $\preceq$ is reflexive.
case 2) $\alpha \neq \beta$. Hence $\beta \prec \alpha$. Thus $\beta \preceq \alpha$ and $\alpha \npreceq \beta$. On the other hand, it follows from $\alpha \prec \beta$ that $\alpha \preceq \beta$ and $\beta \npreceq \alpha$. Contradiction. □

Proof of observation 5.4. Let $\odot : \mathcal{L}_{\mathbb{L}} \to \mathbb{P}_{\mathbb{L}}$ be a profile revision operator on P that satisfies **(P1)** to **(P5)**.
Assume that $P \odot \alpha \vDash \beta$ and $P \odot \beta \vDash \alpha$. Hence $P \odot \alpha \in \|\beta\|$ and $P \odot \beta \in \|\alpha\|$. By **(P4)** it holds that $P \odot \alpha = P \odot ((\alpha \wedge \neg\beta) \vee (\alpha \wedge \beta))$ and $P \odot \beta = P \odot ((\neg\alpha \wedge \beta) \vee (\alpha \wedge \beta))$. By **(P5)** it follows that either $P \odot \alpha = P \odot (\alpha \wedge \beta)$ or $P \odot \alpha = P \odot (\alpha \wedge \neg\beta)$ and either $P \odot \beta = P \odot (\alpha \wedge \beta)$ or $P \odot \beta = P \odot (\neg\alpha \wedge \beta)$. We will prove by cases:
Case 1) $P \odot \alpha = P \odot (\alpha \wedge \neg\beta)$. Assuming that $\|\alpha \wedge \neg\beta\| \neq \emptyset$ leads to a contradiction. Since by **(P1)** it holds that $P \odot (\alpha \wedge \neg\beta) \in \|\alpha \wedge \neg\beta\| = \|\alpha\| \cap \|\neg\beta\|$ and by hypothesis $P \odot \alpha \in \|\beta\|$. Hence $\|\alpha \wedge \neg\beta\| = \emptyset$. By **(P2)** it follows that $P \odot \alpha = P \odot (\alpha \wedge \neg\beta) = P$. Hence $P \vDash \beta$, from which it follows by **(P3)** that $P \odot \beta = P$. Hence $P \odot \alpha = P = P \odot \beta$.
Case 2) $P \odot \alpha = P \odot (\alpha \wedge \beta)$.
Sub-case 2.1) $P \odot \beta = P \odot (\neg\alpha \wedge \beta)$. Reasoning as in case 1) it follows that $P \odot \beta = P = P \odot \alpha$.
Sub-case 2.2) $P \odot \beta = P \odot (\alpha \wedge \beta)$. Hence $P \odot \alpha = P \odot (\alpha \wedge \beta) = P \odot \beta$. □

Proof of Theorem 5.5. (If part) The proofs for **(P1)**, **(P2)** and **(P4)** are straightforward.
(P3) Let $P \vDash \alpha$. Hence $\|\alpha\| \neq \emptyset$. Let $P' \in \|\alpha\| \setminus \{P\}$. By Definition 4.7 it follows that $P \prec_P P'$. Hence $P \odot \alpha = min(\|\alpha\|, \prec_P) = P$.
(P5) It holds that $\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$. We will prove by cases:
case 1) $\|\alpha\| = \emptyset$. Hence $\|\alpha \vee \beta\| = \|\beta\|$. From which it follows that $P \odot (\alpha \vee \beta) = P \odot \beta$.
case 2) $\|\beta\| = \emptyset$. Hence $\|\alpha \vee \beta\| = \|\alpha\|$. From which it follows that $P \odot (\alpha \vee \beta) = P \odot \alpha$.
case 3) $\|\alpha\| \neq \emptyset$ and $\|\beta\| \neq \emptyset$. Hence $\|\alpha \vee \beta\| \neq \emptyset$. Hence $P \odot (\alpha \vee \beta) = min(\|\alpha \vee \beta\|, \prec_P)$, $P \odot \alpha = min(\|\alpha\|, \prec_P)$ and $P \odot \beta = (min\|\beta\|, \prec_P)$.
Suppose towards a contradiction that $P \odot (\alpha \vee \beta) \neq P \odot \alpha$ and $P \odot (\alpha \vee \beta) \neq P \odot \beta$. It holds that $\prec_P$ is total. From which it follows from $P \odot (\alpha \vee \beta) \neq P \odot \alpha$ that either $P \odot (\alpha \vee \beta) \prec_P P \odot \alpha$ or $P \odot \alpha \prec_P P \odot (\alpha \vee \beta)$. In the second case, it holds that $P \odot (\alpha \vee \beta) \neq min(\|\alpha \vee \beta\|, \prec_P)$ (since $P \odot \alpha \in \|\alpha\| \subseteq \|\alpha \vee \beta\|$). Contradiction. Consider now that $P \odot (\alpha \vee \beta) \prec_P P \odot \alpha$. By symmetry of the case, it follows from $P \odot (\alpha \vee \beta) \neq P \odot \beta$ that $P \odot \beta \prec_P P \odot (\alpha \vee \beta)$ leads to a contradiction. Assume now that $P \odot (\alpha \vee \beta) \prec_P P \odot \beta$. It holds that either $P \odot (\alpha \vee \beta) \in \|\alpha\|$ or $P \odot (\alpha \vee \beta) \in \|\beta\|$. In the first case it follows from $P \odot (\alpha \vee \beta) \prec_P P \odot \alpha$ that $P \odot \alpha \neq min(\|\alpha\|, \prec_P)$ and in the second case, it follows from $P \odot (\alpha \vee \beta) \prec_P P \odot \beta$ that $P \odot \beta \neq min(\|\beta\|, \prec_P)$. Contradiction.
(Only if part) Assume that $\odot$ satisfies **(P1)** to **(P5)**. By Observation 5.4 it follows that $\odot$ also satisfies **(P6)**. We will define a relation $\prec_P$ for each P by using the operator $\odot$. For a profile P we define a relation $\prec_P$ as:

For any profiles P_i and P_j, $P_i \prec_P P_j$ if and only if $P_i \neq P_j$ and $P_i = P \odot \alpha_{P_i,P_j}$.

We need to show that $\prec_P$ is a total strict order.
(Totality:) Let P_i and P_j be profiles. Assume that $P_i \neq P_j$. It holds that $\alpha_{P_i,P_j} \nvDash \bot$. Hence by **(P1)** it follows that $P \odot \alpha_{P_i,P_j} \in \{P_i, P_j\}$. Hence $\prec_P$ is total.
(Irreflexivity:) By definition, $\prec_P$ is irreflexive.
(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \prec_P P_j$ and $P_j \prec_P P_k$. We intend to prove that $P_i \prec_P P_k$. It holds, by definition of $\prec$ that $P_i \neq P_j$, $P_j \neq P_k$, $P_i = P \odot \alpha_{P_i,P_j}$ and $P_j = P \odot \alpha_{P_j,P_k}$. If $P_i = P_k$, then by **(P4)** it follows that $P \odot \alpha_{P_i,P_j} = P \odot \alpha_{P_j,P_k}$. Hence $P_i = P_j$. Contradiction. Assume now that $P_i \neq P_k$. Assume towards a contradiction that $P_i \nprec_P P_k$. By definition of $\prec_P$ it follows from the latter that $P_i \neq P \odot \alpha_{P_i,P_k}$. It holds that $\alpha_{P_i,P_k} \nvDash \bot$ thus by **(P1)** it follows that $P \odot \alpha_{P_i,P_k} \in \{P_i, P_k\}$. Thus $P \odot \alpha_{P_i,P_k} = P_k$.
Now consider $P \odot \alpha_{P_i,P_j,P_k}$. By **(P4)** it follows that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_i,P_j} \vee \alpha_{P_k}$. From which it follows by **(P5)** that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_i,P_j}$ or $P \odot \alpha_{P_k}$. By **(P1)** it follows that $P \odot \alpha_{P_k} = P_k$ (and also that $P \odot \alpha_{P_i} = P_i$ and $P \odot \alpha_{P_j} = P_j$). Hence $P \odot \alpha_{P_i,P_j,P_k} \in$

$\{P_i, P_k\}$.
On the other hand it follows by **(P4)** and **(P5)** that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_j,P_k}$ or $P \odot \alpha_{P_i}$. Thus $P \odot \alpha_{P_i,P_j,P_k} \in \{P_i, P_j\}$. From which it follows that $P \odot \alpha_{P_i,P_j,P_k} = P_i$. Using again **(P4)** and **(P5)** it follows that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_i,P_k}$ or $P \odot \alpha_{P_j}$. From which it follows by **(P1)** that $P \odot \alpha_{P_i,P_j,P_k} \in \{P_k, P_j\}$. Contradiction.

The first condition of Definition 4.7 follows trivially. Now we will show that the second condition of that definition also holds. Let $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. And consider α_{P,P_i}. It holds that $P \vDash \alpha_{P,P_i}$. Thus by **(P3)** it follows that $P \odot \alpha_{P,P_i} = P$. Hence, by definition of $\prec_P$, $P \prec_P P_i$.

It remains to prove that:

$$P \odot \alpha = \begin{cases} min(\|\alpha\|, \prec_P) & \text{if } \|\alpha\| \neq \emptyset \\ P & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(P2)** it follows that $P \odot \alpha = P$. Assume now that $\|\alpha\| \neq \emptyset$. If $P \in \|\alpha\|$, then it follows by **(P3)** that $P \odot \alpha = P$. Assume towards a contradiction that $P \neq min(\|\alpha\|, \prec_P)$. Let $P_i = min(\|\alpha\|, \prec_P)$. Hence $P_i \prec_P P$. It follows, by definition of $\prec_P$, that $P \neq P \odot \alpha_{P,P_i}$. This contradicts **(P3)**.
Consider now that $P \notin \|\alpha\|$. By **(P1)** it follows that $P \odot \alpha \in \|\alpha\| \subseteq \mathbb{P}$. Let $P \odot \alpha = P_i$ and $P_j = min(\|\alpha\|, \prec_P)$. Assume by *reductio ad absurdum* that $P_i \neq P_j$. From which it follows that $P_j \prec_P P_i$. By definition of $\prec_P$ it follows that $P \odot \alpha_{P_i,P_j} = P_j$. Hence $P \odot \alpha = P_i \vDash \alpha_{P_i,P_j}$ and $P \odot \alpha_{P_i,P_j} = P_j \vDash \alpha$. By **(P6)** it follows that $P \odot \alpha = P \odot \alpha_{P_i,P_j}$. Contradiction. □

Proof of Observation 5.10. Let $\odot : \mathcal{L}_{\mathbb{L}} \to (\mathbb{P}_{\mathbb{L}})$ be a profile revision operator on P.

1. Assume that $\odot$ satisfies **(S1)**, **(S2)**, **(S4)**, **(S5)** and **(S7)**. If $\alpha \vDash \perp$, then $\alpha \wedge \beta \vDash \perp$. From which it follows by **(S2)** that $P \odot \alpha = P \odot (\alpha \wedge \beta) = \{P\}$. Thus $P \odot \alpha \cap \|\beta\| \subseteq P \odot (\alpha \wedge \beta)$. Assume now that $\alpha \nvDash \perp$. Let $P_i \in P \odot \alpha \cap \|\beta\|$. Hence $P_i \in P \odot \alpha$ and $P_i \in \|\beta\|$. By **(S1)** it follows that $P_i \in \|\alpha\|$. Hence $P_i \in \|\alpha \wedge \beta\|$. On the other hand, it holds, by **(S4)**, that $P \odot \alpha = P \odot ((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta))$. By **(S7)** it follows that $P \odot ((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)) \subseteq P \odot (\alpha \wedge \beta) \cup P \odot (\alpha \wedge \neg\beta)$. From which it follows that $P_i \in P \odot (\alpha \wedge \beta) \cup P \odot (\alpha \wedge \neg\beta)$. Hence either $P_i \in P \odot (\alpha \wedge \beta)$ or $P_i \in P \odot (\alpha \wedge \neg\beta)$. In the former case we are done. Assume now that $P_i \in P \odot (\alpha \wedge \neg\beta)$. Assuming that $\alpha \wedge \neg\beta \nvDash \perp$ leads to a contradiction since it would follow by **(S1)** that $P_i \in \|\alpha \wedge \neg\beta\| = \|\alpha\| \cap \|\neg\beta\|$ and $P_i \in |\beta\|$. Assume now that $\alpha \wedge \neg\beta \vDash \perp$. By **(S2)** it follows that $P \odot (\alpha \wedge \neg\beta) = \{P\}$. From which it follows that $P_i = P$. Thus $P \in \|\alpha \wedge \beta\|$. Hence, by **(S5)**, it follows that $P \odot \alpha = P \odot (\alpha \wedge \beta) = \{P\}$. Thus $P_i = P \in P \odot (\alpha \wedge \beta)$.
2. Assume that $\odot$ satisfies **(S1)**, **(S2)**, **(S4)** and **(S10)**. If $\alpha \vee \beta \vDash \perp$, then $\alpha \vDash \perp$ and $\beta \vDash \perp$. From which it follows by **(S2)**, that $P \odot (\alpha \vee \beta) = \{P\} = P \odot \alpha \cup P \odot \beta$. Assume now that $\alpha \vee \beta \nvDash \perp$. Let $P_i \in P \odot (\alpha \vee \beta)$. By **(S4)** it holds that $P \odot ((\alpha \vee \beta) \wedge \alpha) = P \odot \alpha$ and $P \odot ((\alpha \vee \beta) \wedge \beta) = P \odot \beta$. On the other hand, it holds by **(S10)** that $P \odot (\alpha \vee \beta) \cap \|\alpha||\subseteq P \odot ((\alpha \vee \beta) \wedge \alpha)$. Hence $P \odot (\alpha \vee \beta) \cap \|\alpha\| \subseteq P \odot \alpha$. By symmetry of the case it follows that $P \odot (\alpha \vee \beta) \cap \|\beta\| \subseteq P \odot \beta$. Hence $(P \odot (\alpha \vee \beta) \cap \|\alpha\|) \cup (P \odot (\alpha \vee \beta) \cap \|\beta\|) \subseteq P \odot \alpha \cup P \odot \beta$.
By set theory, it holds that $(P \odot (\alpha \vee \beta) \cap \|\alpha\|) \cup (P \odot (\alpha \vee \beta) \cap \|\beta\|) = (P \odot (\alpha \vee \beta) \cup (P \odot (\alpha \vee \beta) \cap \|\beta\|)) \cap (\|\alpha\| \cup (P \odot (\alpha \vee \beta) \cap \|\beta\|)) = P \odot (\alpha \vee \beta) \cap (\|\alpha\| \cup P \odot (\alpha \vee \beta)) \cap (\|\alpha\| \cup \|\beta\|) = P \odot (\alpha \vee \beta) \cap (\|\alpha\| \cup P \odot (\alpha \vee \beta)) \cap \|\alpha \vee \beta\|$. By **(S1)** it holds that $P \odot (\alpha \vee \beta) \subseteq \|\alpha \vee \beta\|$. Hence $P \odot (\alpha \vee \beta) \subseteq P \odot (\alpha \vee \beta) \cap (\|\alpha\| \cup P \odot (\alpha \vee \beta)) \cap \|\alpha \vee \beta\| = (P \odot (\alpha \vee \beta) \cap \|\alpha\|) \cup (P \odot (\alpha \vee \beta) \cap \|\beta\|) \subseteq P \odot \alpha \cup P \odot \beta$. Therefore $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha \cup P \odot \beta$. □

Proof of Theorem 5.11. **(1 to 2)** Assume that $\odot$ satisfies postulates **(S1)** to **(S5)** and **(S7)** to **(S9)**. By observation 5.10 it follows that $\odot$ satisfies **(S10)**. For any profiles P_i and P_j we define a relation $\preceq_P$ as follows:

$$P_i \preceq_P P_j \text{ if and only if } P \odot \alpha_{P_i,P_j} = \{P_i\} \text{ or } P_i = P.$$

We need to show that $\preceq_P$ is a pre-order.
(Reflexivity:) Let P_i be a profile. It holds that $\alpha_{P_i,P_i} \nvDash \perp$. It follows by **(S1)** and **(S3)** that $P \odot \alpha_{P_i,P_i} = \{P_i\}$. Therefore, by definition of $\preceq_P$, it follows that $P_i \preceq_P P_i$.

(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$. We intend to prove that $P_i \preceq_P P_k$. It follows trivially if $P_i = P$. Assume now that $P_i \neq P$. Hence $P \odot \alpha_{P_i,P_j} = \{P_i\}$. If $P_j = P$, then $P \in \|\alpha_{P_i,P_j}\|$. By **(S5)** it follows that $P \odot \alpha_{P_i,P_j} = \{P\}$. Thus $P_i = P$. Contradiction. Hence $P_j \neq P$. Assuming that $P_k = P$ leads to a contradiction (reasoning as above). Hence $P_k \neq P$.
It follows trivially that $P_i \preceq_P P_k$ if $P_i = P_j$ or $P_j = P_k$. It holds that $\preceq_P$ is reflexive (as shown above) hence it also follows that $P_i \preceq_P P_k$ if $P_i = P_k$. Consider now that $P_i \neq P_j$, $P_i \neq P_k$ and $P_j \neq P_k$. By definition of $\preceq_P$ it follows, from $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$, that $P \odot \alpha_{P_i,P_j} = \{P_i\}$ and $P \odot \alpha_{P_j,P_k} = \{P_j\}$. By **(S1)** and **(S3)** it follows that $\emptyset \neq P \odot \alpha_{P_i,P_k} \subseteq \{P_i, P_k\}$.
By **(S10)** it follows that $P \odot \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_i,P_j}\| \subseteq P \odot (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j})$. By **(S4)** it follows from $\vDash (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j}) \leftrightarrow \alpha_{P_i,P_j}$ that $P \odot (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j}) = P \odot \alpha_{P_i,P_j}$. Hence $P \odot \alpha_{P_i,P_j,P_k} \cap \{P_i, P_j\} \subseteq P \odot \alpha_{P_i,P_j} = \{P_i\}$. From which it follows that $P_j \notin P \odot \alpha_{P_i,P_j,P_k}$.
By **(S10)** it follows that $P \odot \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_j,P_k}\| \subseteq P \odot (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_j,P_k})$. By **(S4)** it follows that $P \odot (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_j,P_k}) = P \odot \alpha_{P_j,P_k}$.
Hence $P \odot \alpha_{P_i,P_j,P_k} \cap \{P_j, P_k\} \subseteq P \odot \alpha_{P_j,P_k} = \{P_j\}$. From which it follows that $P_k \notin P \odot \alpha_{P_i,P_j,P_k}$. Hence, by **(S1)** and **(S3)** it follows that $P \odot \alpha_{P_i,P_j,P_k} = \{P_i\}$.
Therefore, $P \odot \alpha_{P_i,P_j,P_k} = \{P_i\} \vDash \alpha_{P_i,P_k}$ and $P \odot \alpha_{P_i,P_k} \vDash \alpha_{P_i,P_j,P_k}$. By **(S8)** it follows that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_i,P_k}$. Hence $P \odot \alpha_{P_i,P_k} = \{P_i\}$. From which it follows by definition of $\preceq_P$ that $P_i \preceq_P P_k$.

Now we will prove that $\leq_P$ is P-faithful.
Let $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. By definition of $\leq_P$ it follows that $P \leq_P P_i$. Assume towards a contradiction that $P_i \leq_P P$. Hence it holds that $P \odot \alpha_{P_i,P} = \{P_i\}$. On the other hand it holds that $P \vDash \alpha_{P_i,P}$. Thus, by **(S5)**, it follows that $P \odot \alpha_{P_i,P} = \{P\}$. Thus $P = P_i$. Contradiction. Therefore $P_i \nleq_P P$. Hence $P \leq_P P_i$.

It remains to prove that:

$$P \odot \alpha = \begin{cases} Min(\|\alpha\|, \leq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(S2)** it follows that $P \odot \alpha = \{P\}$. Assume now that $\|\alpha\| \neq \emptyset$. Suppose that $P_i \in P \odot \alpha$. By **(S1)** it follows that $P_i \in \|\alpha\|$. Suppose towards a contradiction that $P_i \notin Min(\|\alpha\|, \leq_P)$. Hence there exists $P_j \in \|\alpha\|$ such that $P_j \prec_P P_i$. Thus $P_i \nleq_P P_j$. Therefore $P_i \neq P_j$ (since $\leq_P$ is reflexive).
From $P_j \prec_P P_i$ it follows, by definition of $\leq_P$, that either $P_j = P$ or $P \odot \alpha_{P_i,P_j} = \{P_j\}$. We will now show that in the former case it also follows that $P \odot \alpha_{P_i,P_j} = \{P_j\}$. Assume that $P_j = P$. By **(S5)** it follows from $P_j = P$ and $P_j \vDash \alpha_{P_i,P_j}$ that $P \odot \alpha_{P_i,P_j} = \{P_j\}$.
By **(S10)** it follows that $P \odot \alpha \cap \|\alpha_{P_i,P_j}\| \subseteq P \odot (\alpha \wedge \alpha_{P_i,P_j})$. It holds that $\vDash (\alpha \wedge \alpha_{P_i,P_j}) \leftrightarrow \alpha_{P_i,P_j}$. Thus by **(S4)** it follows that $P \odot \alpha \cap \{P_i, P_j\} \subseteq P \odot \alpha_{P_i,P_j}$. From which it follows that $P_i \in P \odot \alpha_{P_i,P_j}$. Contradiction.
Hence $P \odot \alpha \subseteq Min(\|\alpha\|, \leq_P)$. For the other inclusion, let $P_i \in Min(\|\alpha\|, \leq_P)$. Let $\|\alpha\| = \{P_1, ..., P_k\}$. For all $P_j \in \|\alpha\|$ it holds that $P_j \nprec_P P_i$. Hence $P_j \nleq_P P_i$ or $P_i \leq_P P_j$. From which it follows, by definition of $\leq_P$, that ($P \odot \alpha_{P_i,P_j} \neq \{P_j\}$ and $P_j \neq P$) or $P \odot \alpha_{P_i,P_j} = \{P_i\}$ or $P_i = P$. If $P_i = P$, then by **(S5)** it follows that $P \odot \alpha_{P_i,P_j} = \{P_i\}$, for all $j \in \{1, ..., k\}$. If $P \odot \alpha_{P_i,P_j} \neq \{P_j\}$, then by **(S1)** and **(S3)** it follows that $P_i \in P \odot \alpha_{P_i,P_j}$, for all $j \in \{1, ..., k\}$. Hence in all the three above mentioned cases it holds that $P_i \in P \odot \alpha_{P_i,P_j}$, for all $j \in \{1, ..., k\}$. On the other hand, it holds that $\vDash \alpha \leftrightarrow (\alpha_{P_i,P_1} \vee ... \vee \alpha_{P_i,P_k})$. Thus by **(S4)** it follows that $P \odot \alpha = P \odot (\alpha_{P_i,P_1} \vee ... \vee \alpha_{P_i,P_k})$. By repeated applications of **(S9)** it follows that $P \odot \alpha_{P_i,P_1} \cap ... \cap P \odot \alpha_{P_i,P_k} \subseteq P \odot (\alpha_{P_i,P_1} \vee ... \vee \alpha_{P_i,P_k})$. Hence $P_i \in P \odot \alpha$. Therefore $Min(\|\alpha\|, \leq_P) \subseteq P \odot \alpha$.

(2 to 3) For a P-faithful pre-order $\leq_P$, we define a relation $\leq'_P$ as:

$P_i \leq'_P P_j$ if and only if $P_i = P_j$ or $P_i \prec_P P_j$

We first show that $\leq'_P$ is an order.

(Reflexivity:) Follows immediately from the definition of $\leq'_P$.

(Transitivity:) Let $P_i \leq'_P P_j$ and $P_j \leq'_P P_k$. We intend to show that $P_i \leq'_P P_k$. If $P_i = P_j$ or $P_j = P_k$ or $P_i = P_k$, then it follows trivially that $P_i \leq'_P P_k$. Assume now that $P_i \neq P_j$, $P_j \neq P_k$ and $P_i \neq P_k$. It follows, from $P_i \leq'_P P_j$ and $P_j \leq'_P P_k$, by definition of $\leq'_P$ that $P_i \prec_P P_j$ and $P_j \prec_P P_k$. It holds that $\leq_P$ is transitive, hence its strict part is also transitive. Thus $P_i \prec_P P_k$. From which it follows by definition of $\leq'_P$ that $P_i \leq'_P P_k$.

We will now show that $\leq'_P$ is antisymmetric.
Let $P_i \leq'_P P_j$ and $P_j \leq'_P P_i$. Assume towards a contradiction that $P_i \neq P_j$. Hence, by definition of $\leq'_P$ it follows that $P_i \prec_P P_j$ and $P_j \prec_P P_i$. Hence $P_i \leq_P P_j$, $P_j \nleq_P P_i$, $P_j \leq_P P_i$ and $P_i \nleq_P P_j$. Contradiction.

Now we will show that $P \prec'_P P_i$, for all $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$.
Let $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. It holds, that $P \prec_P P_i$. From which follows by Lemma 1 that $P \prec'_P P_i$.

It remains to prove that:

$$P \odot \alpha = \begin{cases} Min(\|\alpha\|, \leq'_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

We know that:

$$P \odot \alpha = \begin{cases} Min(\|\alpha\|, \leq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

The proof follows trivially if we prove that:

$$Min(\|\alpha\|, \leq'_P) = Min(\|\alpha\|, \leq_P), \text{ whenever } \|\alpha\| \neq \emptyset.$$

We will prove by double inclusion. Assume that $\|\alpha\| \neq \emptyset$.
Let $P_i \in Min(\|\alpha\|, \leq'_P)$ and assume towards a contradiction that $P_i \notin Min(\|\alpha\|, \leq_P)$. From the latter, it follows that there exists $P_j \in \|\alpha\|$ such that $P_j \prec_P P_i$. Hence by Lemma 1 it follows that $P_j \prec'_P P_i$. From which it follows that $P_i \notin Min(\|\alpha\|, \leq'_P)$. Contradiction. Hence $Min(\|\alpha\|, \leq'_P) \subseteq Min(\|\alpha\|, \leq_P)$. The proof of the other inclusion is analogous.

(3 to 1) Assume that there exists a P-faithful order $\leq_P$ on $\mathbb{P}_{\mathbb{L}}$ such that:

$$P \odot \alpha = \begin{cases} Min(\|\alpha\|, \leq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

We need to show that $\odot$ satisfies postulates (S1) to (S5) and (S7) to (S9).
The proofs for **(S1)** to **(S4)** are straightforward.
(S5) Let $P \vDash \alpha$. Hence $\|\alpha\| \neq \emptyset$. Let $P' \in \|\alpha\|$. It holds that $\leq_P$ is reflexive. Hence, if $P' = P$ then $P \leq P'$. If $P' \neq P$, then $P \prec_P P'$. Hence $P \odot \alpha = Min(\|\alpha\|, \leq_P) = \{P\}$.

(S7) If $\|\alpha\| = \emptyset$, then $\|\alpha \vee \beta\| = \|\beta\|$. By definition of $\odot$ it follows that $P \odot (\alpha \vee \beta) = P \odot \beta$. By symmetry of the case it follows that, if $\|\beta\| = \emptyset$, then $P \odot (\alpha \vee \beta) = P \odot \alpha$. Thus it holds that $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha \cup P \odot \beta$ if either $\|\alpha\| = \emptyset$ or $\|\beta\| = \emptyset$. Assume now that $\|\alpha\| \neq \emptyset$ and $\|\beta\| \neq \emptyset$. Hence $\|\alpha \vee \beta\| \neq \emptyset$. Let $P_i \in P \odot (\alpha \vee \beta)$. Thus $P_i \in Min(\|\alpha \vee \beta\|, \preceq_P)$. Therefore $P_i \in \|\alpha \vee \beta\|$. Thus either $P_i \in \|\alpha\|$ or $P_i \in \beta\|$. We will consider those two cases separately.
Case 1) $P_i \in \|\alpha\|$. Let P_j be an arbitrary element of $\|\alpha\|$. Hence $P_j \in \|\alpha \vee \beta\|$. Thus $P_j \not\prec_P P_i$. Hence $P_i \in Min(\|\alpha\|, \preceq_P) = P \odot \alpha$.
Case 2) $P_i \in \|\beta\|$. By symmetry of the case it follows that $P_i \in P \odot \beta$.
In both cases it holds that $P_i \in P \odot \alpha \cup P \odot \beta$. From which it follows that $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha \cup P \odot \beta$.
(S8) Assume that $P \odot \alpha \vDash \beta$ and $P \odot \beta \vDash \alpha$. Hence $P \odot \alpha \subseteq \|\alpha\|$ and $P \odot \beta \subseteq \|\beta\|$. Assume towards a contradiction that $P \odot \alpha \neq P \odot \beta$. Assume, without loss of generality that there exists $P_i \in P \odot \alpha \subseteq \|\beta\|$ such that $P_i \notin P \odot \beta$. From $P_i \in \|\beta\|$ and $P_i \notin P \odot \beta$ it follows that there exists $P_j \in Min(\|\beta\|, \preceq_P) = P \odot \beta$ such that $P_j \prec_P P_i$. Hence it holds that $P_j \in \|\alpha\|$, from which it follows that $P_i \notin Min(\|\alpha\|, \preceq_P) = P \odot \alpha$. Contradiction.
(S9) If $\|\alpha\| = \emptyset$, then $P \odot \beta = P \odot (\alpha \vee \beta)$. Thus $P \odot \alpha \cap P \odot \beta \subseteq P \odot (\alpha \vee \beta)$. If $\|\beta\| = \emptyset$ then $P \odot \alpha = P \odot (\alpha \vee \beta)$. Thus $P \odot \alpha \cap P \odot \beta \subseteq P \odot (\alpha \vee \beta)$. Assume now that $\|\alpha\| \neq \emptyset$ and $\|\beta\| \neq \emptyset$. Let $P_i \in P \odot \alpha \cap P \odot \beta$. Hence $P_i \in \|\alpha \vee \beta\|$, $P_i \in Min(\|\alpha\|, \preceq_P)$ and $P_i \in Min(\|\beta\|, \preceq_P)$. Assume towards a contradiction that $P_i \notin P \odot (\alpha \vee \beta)$. Hence there exists $P_j \in \|\alpha \vee \beta\|$ such that $P_j \prec_P P_i$. It holds that either $P_j \in \|\alpha\|$ or $P_j \in \|\beta\|$. In the first case it follows that $P_i \notin Min(\|\alpha\|, \preceq_P)$ and in the second case that $P_i \notin Min(\|\beta\|, \preceq_P)$. Hence both cases lead to a contradiction. $\square$

Proof of Theorem 5.12. (2 to 1) The proofs for **(S1)** to **(S4)** are straightforward. The proof of **(S5)** is the same as the one presented in the (3 to 1) part of Theorem 5.11.
(S6) It holds that $\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$. We will prove by cases:
case 1) $\|\alpha\| = \emptyset$. Hence $\|\alpha \vee \beta\| = \|\beta\|$. From which it follows that $P \odot (\alpha \vee \beta) = P \odot \beta$.
case 2) $\|\beta\| = \emptyset$. Hence $\|\alpha \vee \beta\| = \|\alpha\|$. From which it follows that $P \odot (\alpha \vee \beta) = P \odot \alpha$.
case 3) $\|\alpha\| \neq \emptyset$ and $\|\beta\| \neq \emptyset$. Hence $\|\alpha \vee \beta\| \neq \emptyset$. Let $P_i \in P \odot (\alpha \vee \beta)$. Hence $P_i \in Min(\|\alpha \vee \beta\|, \preceq_P)$. Hence $P_i \in \|\alpha \vee \beta\|$. It holds that either $P_i \in \|\alpha\|$ or $P_i \in \|\beta\|$.
case 3.1) $P_i \in \|\alpha\|$. Let $P_j \in \|\alpha\|$. Hence $P_j \in \|\alpha \vee \beta\|$, from which it follows that $P_i \preceq_P P_j$. Hence $P_i \in Min(\|\alpha\|, \preceq_P) = P \odot \alpha$. Hence $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha$.
case 3.2) $P_i \in \|\beta\|$. This sub-case is symmetrical with the previous one. Hence $P \odot (\alpha \vee \beta) \subseteq P \odot \beta$.
From the previous sub-cases we can conclude that $P \odot (\alpha \vee \beta) \subseteq P \odot \alpha \cup P \odot \beta$.
If $P \odot \alpha \cup P \odot \beta \subseteq P \odot (\alpha \vee \beta)$ then $P \odot (\alpha \vee \beta) = P \odot \alpha \cup P \odot \beta$ and we are done.
Assume now that $P \odot \alpha \cup P \odot \beta \nsubseteq P \odot (\alpha \vee \beta)$. Hence there exists $P_i \in P \odot \alpha \cup P \odot \beta$ such that $P_i \notin P \odot (\alpha \vee \beta)$. From $P_i \in P \odot \alpha \cup P \odot \beta$ it follows that either $P_i \in P \odot \alpha$ or $P_i \in P \odot \beta$. In both cases it holds that $P_i \in \|\alpha \vee \beta\|$. From $P_i \notin P \odot (\alpha \vee \beta)$ it follows, by definition of $\odot$, that there exists $P_j \in Min(\|\alpha \vee \beta\|, \preceq_P)$ such that $P_i \npreceq_P P_j$. It holds that, either $P_j \in \|\alpha\|$ or $P_j \in \|\beta\|$. We will consider two cases:
Case 1) $P_i \in P \odot \alpha$. It follows that $P_j \in \|\beta\|$ (otherwise, it would follow that $P_i \notin Min(\|\alpha\|, \preceq_P) = P \odot \alpha$).
Let $P_k \in P \odot \beta$. Hence $P_k \in Min(\|\beta\|, \preceq_P)$, from which it follows that $P_k \preceq_P P_j$. On the other hand, it holds that $P_k \in \|\alpha \vee \beta\|$. Let $P_l \in \|\alpha \vee \beta\|$. Hence $P_j \preceq_P P_l$, from which it follows by the transitivity of $\preceq_P$ that $P_k \preceq_P P_l$. Hence $P_k \in Min(\|\alpha \vee \beta\|, \preceq_P) = P \odot (\alpha \vee \beta)$. Thus $P \odot \beta \subseteq P \odot (\alpha \vee \beta)$.
Let $P_m \in P \odot (\alpha \vee \beta)$. Hence $P_m \preceq_P P_j$ and $P_m \in \|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$. From which it follows by the transitivity of $\preceq_P$ that $P_i \npreceq_P P_m$. Hence $P_m \in \|\beta\|$ (otherwise, it would follow that $P_m \in \|\alpha\|$ and, consequently, that $P_i \notin P \odot \alpha$). Let $P_n \in \|\beta\|$. It holds that $P_n \in \|\alpha \vee \beta\|$. Hence $P_m \preceq_P P_n$. From which it follows that $P_m \in Min(\|\beta\|, \preceq_P) = P \odot \beta$. Thus $P \odot (\alpha \vee \beta) \subseteq P \odot \beta$.
Therefore $P \odot (\alpha \vee \beta) = P \odot \beta$.
Case 2) $P_i \in P \odot \beta$. This case is symmetrical with the previous one. Hence in this case it follows that $P \odot (\alpha \vee \beta) = P \odot \alpha$.
(1 to 2) Assume that $\odot$ satisfies **(S1)** to **(S6)**. We will define a relation $\preceq_P$ for each P by using the operator $\odot$. For any profile P we define a relation $\preceq_P$ as:

For any profiles P_i and P_j, $P_i \preceq_P P_j$ if and only if $P_i \in P \odot \alpha_{P_i,P_j}$.

We need to show that $\preceq_P$ is a total pre-order.
(Totality:) Let P_i and P_j be profiles. It holds that $\alpha_{P_i,P_j} \nvdash \perp$. Hence by **(S1)** it follows that $P \odot \alpha_{P_i,P_j} \subseteq \{P_i, P_j\}$. On the other hand by **(S3)** it follows that $P \odot \alpha_{P_i,P_j} \neq \emptyset$. Thus either $P_i \in P \odot \alpha_{P_i,P_j}$ or $P_j \in P \odot \alpha_{P_i,P_j}$. Therefore, by definition of $\preceq_P$, it follows that either $P_i \preceq_P P_j$ or $P_j \preceq_P P_i$.

(Reflexivity:) Let P_i be a profile. It holds that $\alpha_{P_i,P_i} \nvdash \perp$. It follows by **(S1)** and **(S3)** that $P \odot \alpha_{P_i,P_i} = \{P_i\}$. Therefore, by definition of $\preceq_P$, it follows that $P_i \preceq_P P_i$.

(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$. We intend to prove that $P_i \preceq_P P_k$. It follows trivially if $P_i = P_j$ or $P_j = P_k$. It holds that $\preceq$ is reflexive (as shown above) hence it also follows that $P_i \preceq_P P_k$ if $P_i = P_k$. Consider now that $P_i \neq P_j$, $P_i \neq P_k$ and $P_j \neq P_k$. By definition of $\preceq_P$ it follows, from $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$, that $P_i \in P \odot \alpha_{P_i,P_j}$ and $P_j \in P \odot \alpha_{P_j,P_k}$. Assume towards a contradiction that $P_i \npreceq_P P_k$. By definition of $\preceq_P$ it follows from the latter that $P_i \notin P \odot \alpha_{P_i,P_k}$. It holds that $\alpha_{P_i,P_k} \nvdash \perp$ thus by **(S1)** it follows that $P \odot \alpha_{P_i,P_k} \subseteq \{P_i, P_k\}$. By **(S3)** it follows that $P \odot \alpha_{P_i,P_k} \neq \emptyset$. Hence $P \odot \alpha_{P_i,P_k} = \{P_k\}$.
Now consider $P \odot \alpha_{P_i,P_j,P_k}$. By **(S4)** it follows that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_i,P_k} \vee \alpha_{P_j}$. From which it follows by **(S6)** that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_i,P_k}$, $P \odot \alpha_{P_j}$ or $P \odot \alpha_{P_i,P_k} \cup P \odot \alpha_{P_j}$. By **(S1)** and **(S3)** it follows that $P \odot \alpha_{P_j} = \{P_j\}$. Hence $P \odot \alpha_{P_i,P_j,P_k}$ is either $\{P_k\}, \{P_j\}$ or $\{P_j, P_k\}$.

Suppose that $P \odot \alpha_{P_i,P_j,P_k} = \{P_k\}$. Now consider $P \odot \alpha_{P_j,P_k} \vee \alpha_{P_i}$. It follows by **(S4)** that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_j,P_k} \vee \alpha_{P_i}$. On the other hand, it follows by **(S6)** that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_j,P_k}$, $P \odot \alpha_{P_i}$ or $P \odot \alpha_{P_j,P_k} \cup P \odot \alpha_{P_i}$. By **(S1)** and **(S3)** it follows that $P \odot \alpha_{P_i} = \{P_i\}$. Thus the last two cases lead to a contradiction. Assume now that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_j,P_k}$. In this case it follows that $P_j \notin P \odot \alpha_{P_j,P_k} = \{P_k\}$. Contradiction.

Suppose now that $P \odot \alpha_{P_i,P_j,P_k} = \{P_j\}$. Consider $P \odot \alpha_{P_i,P_j} \vee \alpha_{P_k}$. It follows by **(S4)** that $P \odot \alpha_{P_i,P_j,P_k} = P \odot \alpha_{P_i,P_j} \vee \alpha_{P_k}$. On the other hand, it follows by **(S6)** that $P \odot \alpha_{P_i,P_j,P_k}$ is either $P \odot \alpha_{P_i,P_j}$, $P \odot \alpha_{P_k}$ or $P \odot \alpha_{P_i,P_j} \cup P \odot \alpha_{P_k}$. Thus the first and third cases lead to a contradiction (since $P_i \in P \odot \alpha_{P_i,P_j}$ and $P \odot \alpha_{P_i,P_j,P_k} = \{P_j\}$). By **(S1)** and **(S3)** it follows that $P \odot \alpha_{P_k} = \{P_k\}$. Thus the second case also leads to a contradiction.

Finally, suppose that $P \odot \alpha_{P_i,P_j,P_k} = \{P_j, P_k\}$. It holds that $P \odot \alpha_{P_i,P_j} \vee \alpha_{P_k} = P \odot \alpha_{P_i,P_j,P_k}$. By **(S6)** it follows that $P \odot \alpha_{P_i,P_j} \vee \alpha_{P_k}$ is either $P \odot \alpha_{P_i,P_j}$, $P \odot \alpha_{P_k}$ or $P \odot \alpha_{P_i,P_j} \cup P \odot \alpha_{P_k}$. By **(S1)** the first two cases lead to a contradiction. In the third case it follows from $P_i \in P \odot \alpha_{P_i,P_j}$ that $P_i \in P \odot \alpha_{P_i,P_j,P_k} = \{P_j, P_k\}$. Contradiction.

Now we will prove that $\preceq_P$ is P-faithful.

Let $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. And consider α_{P,P_i} It holds that $P \vDash \alpha_{P,P_i}$. Thus by **(S5)** it follows that $P \odot \alpha_{P,P_i} = \{P\}$. Hence, by definition of $\preceq_P$, $P \prec P_i$.

It remains to prove that:

$$P \odot \alpha = \begin{cases} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \{P\} & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(S2)** it follows that $P \odot \alpha = \{P\}$. Assume now that $\|\alpha\| \neq \emptyset$. Suppose that $P_i \in P \odot \alpha$. By **(S1)** it follows that $P_i \in \|\alpha\|$. Suppose towards a contradiction that $P_i \notin Min(\|\alpha\|, \preceq_P)$. Let $P_j \in Min(\|\alpha\|, \preceq_P)$. Hence, $P_j \prec_P P_i$. It follows by definition of $\preceq_P$ that $P_i \notin P \odot \alpha_{P_i,P_j}$ and $P_j \in P \odot \alpha_{P_i,P_j}$. From which it follows by **(S1)** and **(S3)** that $P \odot \alpha_{P_i,P_j} = \{P_j\}$.

Assume that $\|\alpha\| = \{P_i, P_j\}$. It follows by **(S4)** that $P \odot \alpha = P \odot \alpha_{P_i,P_j}$. From which it follows that $P_i \in P \odot \alpha_{P_i,P_j}$. Contradiction.

Assume now that $\{P_i, P_j\} \subset \|\alpha\|$. Let α_1 be a formula such that $\|\alpha_1\| = \|\alpha\| \setminus \{P_i, P_j\}$. Hence $\vDash \alpha \leftrightarrow (\alpha_1 \vee \alpha_{P_i,P_j})$. By **(S4)** it follows that $P \odot \alpha = P \odot \alpha_1 \vee \alpha_{P_i,P_j}$. By **(S6)** it follows that $P \odot \alpha$ is either $P \odot \alpha_1$, $P \odot \alpha_{P_i,P_j}$ or $P \odot \alpha_1 \cup P \odot \alpha_{P_i,P_j}$. In the first case it follows from $P_i \in P \odot \alpha$ that $P_i \in P \odot \alpha_1$. This contradicts **(S1)**, since $P_i \notin \|\alpha_1\|$. The second case leads to a contradiction since $P_i \in P \odot \alpha$ and $P_i \notin P \odot \alpha_{P_i,P_j}$. For the third case, it follows from $P_i \in P \odot \alpha$ that either $P_i \in P \odot \alpha_1$ or $P_i \in P \odot \alpha_{P_i,P_j}$. Both cases, lead to a contradiction (reasoning analogously with the two previous cases).

Hence $P \odot \alpha \subseteq Min(\|\alpha\|, \preceq_P)$.

Let $P_i \in Min(\|\alpha\|, \preceq_P)$ and suppose towards a contradiction that $P_i \notin P \odot \alpha$. By **(S3)** and **(S1)** it follows that exists $P_j \in P \odot \alpha \subseteq \|\alpha\|$. It holds that $P_i \preceq_P P_j$. From which it follows, by definition of $\preceq_P$, that $P_i \in P \odot \alpha_{P_i,P_j}$. Assume that $\|\alpha\| = \{P_i, P_j\}$. It follows by **(S4)** that $P \odot \alpha = P \odot \alpha_{P_i,P_j}$. From which it follows that $P_i \in P \odot \alpha$. Contradiction. Consider now that $\{P_i, P_j\} \subset \|\alpha\|$.

Let α_1 be a formula such that $\|\alpha_1\| = \|\alpha\| \setminus \{P_i, P_j\}$. Hence $\vDash \alpha \leftrightarrow (\alpha_1 \vee \alpha_{P_i,P_j})$. By **(S4)** it follows that $P \odot \alpha = P \odot \alpha_1 \vee \alpha_{P_i,P_j}$. By **(S6)** it follows that $P \odot \alpha$ is either $P \odot \alpha_1$, $P \odot \alpha_{P_i,P_j}$ or $P \odot \alpha_1 \cup P \odot \alpha_{P_i,P_j}$. In the first case it follows from $P_j \in P \odot \alpha$ that $P_j \in P \odot \alpha_1$. This contradicts **(S1)**. The second and third cases lead to a contradiction since $P_i \notin P \odot \alpha$ and $P_i \in P \odot \alpha_{P_i,P_j}$. Hence $Min(\|\alpha\|, \preceq_P) \subseteq P \odot \alpha$, from which it follows that $P \odot \alpha = Min(\|\alpha\|, \preceq_P)$. $\square$

Proof of Observation 5.16. The proof is similar to the one presented for Observation 5.10 (basically it can be obtain from the proof presented for Observation 5.10 replacing postulates (Si) by (SPi) and instances of the form $P \odot \delta$ by $\Gamma \odot \delta$). $\square$

Proof of Theorem 5.17. (1 to 2) Assume that $\odot$ satisfies postulates **(SP1)** to **(SP5)** and **(SP7)** to **(SP9)**. By Observation 5.16 it holds that $\odot$ satisfies (SP10). For any profiles P_i and P_j we define a relation $\preceq_\Gamma$ as follows:

$$P_i \preceq_\Gamma P_j \text{ if and only if } \Gamma \odot \alpha_{P_i,P_j} = \{P_i\} \text{ or } P_i \in \Gamma.$$

We need to show that $\preceq_\Gamma$ is a pre-order.

(Reflexivity:) Let P_i be a profile. It holds that $\alpha_{P_i,P_i} \nvdash \bot$. It follows by **(SP1)** and **(SP3)** that $\Gamma \odot \alpha_{P_i,P_i} = \{P_i\}$. Therefore, by definition of $\preceq_\Gamma$, it follows that $P_i \preceq_\Gamma P_i$.

(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \preceq_\Gamma P_j$ and $P_j \preceq_\Gamma P_k$. We intend to prove that $P_i \preceq_\Gamma P_k$. It follows trivially if $P_i \in \Gamma$. Assume now that $P_i \notin \Gamma$. Hence $\Gamma \odot \alpha_{P_i,P_j} = \{P_i\}$. If $P_j \in \Gamma$, then $\Gamma \cap \|\alpha_{P_i,P_j}\| \neq \emptyset$. By **(SP5)** it follows that $P_j \in \Gamma \odot \alpha_{P_i,P_j}$. Thus $P_i = P_j \in \Gamma$. Contradiction. Hence $P_j \notin \Gamma$. Assuming that $P_k \in \Gamma$ leads to a contradiction (reasoning as above). Assume now that $P_k \notin \Gamma$.

It follows trivially that $P_i \preceq_\Gamma P_k$ if $P_i = P_j$ or $P_j = P_k$. It holds that $\preceq_\Gamma$ is reflexive (as shown above) hence it also follows that $P_i \preceq_\Gamma P_k$ if $P_i = P_k$. Consider now that $P_i \neq P_j$, $P_i \neq P_k$, $P_j \neq P_k$ and $\{P_i, P_j, P_k\} \cap \Gamma = \emptyset$. By definition of $\preceq_\Gamma$ it follows, from $P_i \preceq_\Gamma P_j$ and $P_j \preceq_\Gamma P_k$, that $\Gamma \odot \alpha_{P_i,P_j} = \{P_i\}$ and $\Gamma \odot \alpha_{P_j,P_k} = \{P_j\}$. The rest of the proof that $\preceq_\Gamma$ is transitive is very similar to the proof of the main case of the transitivity of $\preceq_P$ presented in the (1 to 2) part of Theorem 5.11 (basically by replacing postulates (Si) by (SPi), $\preceq_P$ by $\preceq_\Gamma$ and instances of the form $P \odot \beta$ by $\Gamma \odot \beta$).

Now we will prove that $\preceq_P$ is Γ-faithful.

If $P_i \in \Gamma$ and $P_j \in \Gamma$, then by definition of $\preceq_\Gamma$ it follows that $P_j \preceq_\Gamma P_i$. Thus $P_i \prec_\Gamma P_j$ does not hold.

Let $P_i \in \Gamma$ and $P_j \notin \Gamma$. It follows by definition of $\preceq_\Gamma$ that $P_i \preceq_\Gamma P_j$. Assume towards a contradiction that $P_j \preceq_\Gamma P_i$. Hence $\Gamma \odot \alpha_{P_i,P_j} =$

$\{P_j\}$. On the other hand it follows that $\Gamma \cap \|\alpha_{P_i,P_j}\| \neq \emptyset$. Thus, by **(SP5)** it follows that $\Gamma \odot \alpha_{P_i,P_j} = \Gamma \cap \|\alpha_{P_i,P_j}\|$. Hence $P_i \in \Gamma \odot \alpha_{P_i,P_j}$. Contradiction. Thus $P_j \npreceq_\Gamma P_i$ and consequently $P_i \prec_\Gamma P_j$.

It remains to prove that:

$$\Gamma \odot \alpha = \begin{cases} Min(\|\alpha\|, \preceq_\Gamma) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(SP2)** it follows that $\Gamma \odot \alpha = \Gamma$. Assume now that $\|\alpha\| \neq \emptyset$. Suppose that $P_i \in \Gamma \odot \alpha$. By **(SP1)** it follows that $P_i \in \|\alpha\|$. Suppose towards a contradiction that $P_i \notin Min(\|\alpha\|, \preceq_\Gamma)$. Hence there exists $P_j \in \|\alpha\|$ such that $P_j \prec_\Gamma P_i$. Thus $P_i \npreceq_\Gamma P_j$. Therefore $P_i \neq P_j$ (since $\preceq_\Gamma$ is reflexive).
From $P_j \prec_\Gamma P_i$ it follows, by definition of $\preceq_\Gamma$, that either $P_j \in \Gamma$ or $\Gamma \odot \alpha_{P_i,P_j} = \{P_j\}$. We will now show that in the former case it also follows that $\Gamma \odot \alpha_{P_i,P_j} = \{P_j\}$. Assume that $P_j \in \Gamma$. Assuming that $P_i \in \Gamma$ leads to a contradiction, since it would follow by definition of $\preceq_\Gamma$ that $P_i \preceq_\Gamma P_j$. Consider now that $P_i \notin \Gamma$. By **(SP5)** it follows from $\Gamma \cap \|\alpha_{P_i,P_j}\| = \{P_j\}$ that $\Gamma \odot \alpha_{P_i,P_j} = \{P_j\}$.
By **(SP10)** it follows that $\Gamma \odot \alpha \cap \|\alpha_{P_i,P_j}\| \subseteq \Gamma \odot (\alpha \wedge \alpha_{P_i,P_j})$. From which it follows, by **(SP4)**, that $\Gamma \odot \alpha \cap \{P_i, P_j\} \subseteq \Gamma \odot \alpha_{P_i,P_j}$. Hence $P_i \in \Gamma \odot \alpha_{P_i,P_j}$. Contradiction.
Hence $\Gamma \odot \alpha \subseteq Min(\|\alpha\|, \preceq_\Gamma)$.

The proof for the converse inclusion is similar to the one presented in the (1 to 2) part of Theorem 5.11 (having in mind that by **(SP5)** it follows that if $P_i \in \Gamma$, then $P_i \in \Gamma \odot \alpha_{P_i,P_j}$, for all $P_j \in \mathbb{P}_\mathbb{L}$).

(2 to 3) For a Γ-faithful pre-order $\preceq_\Gamma$, we define a relation $\preceq'_\Gamma$ as:

$P_i \preceq'_\Gamma P_j$ if and only if $P_i = P_j$ or $P_i \prec_\Gamma P_j$

The proof that $\preceq'_\Gamma$ is an order is analogous to the proof that $\preceq'_P$ is an order, presented in the (2 to 3) part of Theorem 5.11.

Now we will show that $\preceq'_\Gamma$ is Γ-faithful.
Let $P_i, P_j \in \Gamma$. Hence $P_i \not\prec_\Gamma P_j$. From which it follows by Lemma 1 that $P_i \not\prec'_\Gamma P_j$.
Let $P_i \in \Gamma$ and $P_j \in \mathbb{P}_\mathbb{L} \setminus \Gamma$. Hence $P_i \prec_\Gamma P_j$, from which it follows by Lemma 1 that $P_i \prec'_\Gamma P_j$.

The proof that the following equality holds is analogous to the one presented in the (2 to 3) part of Theorem 5.11.

$$\Gamma \odot \alpha = \begin{cases} Min(\|\alpha\|, \preceq'_\Gamma) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

(3 to 1) Assume that there exists a Γ-faithful order $\preceq_\Gamma$ on $\mathbb{P}_\mathbb{L}$ such that:

$$\Gamma \odot \alpha = \begin{cases} Min(\|\alpha\|, \preceq_\Gamma) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

We need to show that $\odot$ satisfies postulates (SP1) to (SP5) and (SP7) to (SP9).
The proofs for **(SP1)** to **(SP4)** are straightforward. The proofs that **(SP7)**, **(SP8)** and **(SP9)** hold are similar to the ones presented for, respectively, **(S7)**, **(S8)** and **(S9)** in the (3 to 1) part of Theorem 5.11.
(SP5) Let $\Gamma \cap \|\alpha\| \neq \emptyset$. Hence $\Gamma \odot \alpha = Min(\|\alpha\|, \preceq_\Gamma)$ and there exists $P_i \in \Gamma \cap \|\alpha\|$. We will prove that $\Gamma \odot \alpha = \Gamma \cap \|\alpha\|$ by double inclusion.
Let $P_j \in \Gamma \odot \alpha$. Hence $P_j \in Min(\|\alpha\|, \preceq_\Gamma)$. Hence $P_j \in \|\alpha\|$. It remains to prove that $P_j \in \Gamma$. Assume, towards a contradiction, that this is not the case. By condition 2 of Definition 4.7 it follows that $P_i \prec_\Gamma P_j$. It holds that $P_i \in \|\alpha\|$, from which it follows that $P_j \notin Min(\|\alpha\|, \preceq_\Gamma)$. Contradiction. Hence $\Gamma \odot \alpha \subseteq \Gamma \cap \|\alpha\|$.
Let $P_j \in \Gamma \cap \|\alpha\|$. Hence $P_j \in \Gamma$ and $P_j \in \|\alpha\|$. Let $P_k \in \|\alpha\|$. By conditions 1 and 2 of Definition 4.7 it follows that $P_k \not\prec_\Gamma P_j$. Thus $P_j \in Min(\|\alpha\|, \preceq_\Gamma) = \Gamma \odot \alpha$. Hence $\Gamma \cap \|\alpha\| \subseteq \Gamma \odot \alpha$. □

Proof of Theorem 5.18. **(2 to 1)** That $\odot$ satisfies postulates **(SP1)** to **(SP5)** follows from Theorem 5.17. The proof that $\odot$ satisfies **(SP6)** is similar to the one presented for **(S6)** in the (2 to 1) part of Theorem 5.12.

(1 to 2) Assume that $\odot$ satisfies **(SP1)** to **(SP6)**. We will define a total pre-order $\prec_\Gamma$ for each non-empty subset Γ of $\mathbb{P}_\mathbb{L}$ by using the operator $\odot$. For any such Γ we define a relation $\preceq_\Gamma$ as:

For any profiles P_i and P_j, $P_i \preceq_\Gamma P_j$ if and only if $P_i \in \Gamma \odot \alpha_{P_i,P_j}$.

The proof that $\preceq_\Gamma$ is a total pre-order is analogous to the proof that $\preceq_P$ is a total pre-order, presented in the (1 to 2) part of Theorem 5.12 (basically by replacing postulates (Si) by (SPi), $\preceq_P$ by $\preceq_\Gamma$ and instances of the form $P \odot \beta$ by $\Gamma \odot \beta$).

Now we will show that $\preceq_\Gamma$ is Γ-faithful.
(1) Let $P_i \in \Gamma$ and $P_j \in \Gamma$. It holds that $\|\alpha_{P_i,P_j}\| = \{P_i, P_j\}$. Hence $\Gamma \cap \|\alpha_{P_i,P_j}\| \neq \emptyset$. Thus by **(SP5)**, $\Gamma \odot \alpha_{P_i,P_j} = \Gamma \cap \|\alpha_{P_i,P_j}\| = \{P_i, P_j\}$. From which it follows, by definition of $\preceq$, that $P_j \preceq_\Gamma P_i$. Hence $P_i \not\prec_\Gamma P_j$.
(2) Let $P_i \in \Gamma$ and $P_j \in \mathbb{P}_\mathbb{L} \setminus \Gamma$. It holds that $P_i \in \Gamma \cap \|\alpha_{P_i,P_j}\|$. Thus by **(SP5)**, $\Gamma \odot \alpha_{P_i,P_j} = \Gamma \cap \|\alpha_{P_i,P_j}\|$. Hence $P_i \in \Gamma \odot \alpha_{P_i,P_j}$ and $P_j \notin \Gamma \odot \alpha_{P_i,P_j}$. From which it follows, by definition of $\preceq_\Gamma$, that $P_i \prec_\Gamma P_j$.

The proof that the following equality holds is similar to the proof presented in the last part of the (1 to 2) part of Theorem 5.12.

$$\Gamma \odot \alpha = \begin{cases} Min(\|\alpha\|, \preceq_\Gamma) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases} \quad \square$$

Proof of Observation 5.23. Let Γ, Γ_1 and Γ_2 be sets of profiles and $\diamond$ be a profile update operator.

1. Assume that $\diamond$ satisfies **(U8)** and $\Gamma_1 \subseteq \Gamma_2$. It follows from **(U8)** that $(\Gamma_1 \cup \Gamma_2) \diamond \alpha = \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$. Hence $\Gamma_2 \diamond \alpha = \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$. From which it follows that $\Gamma_1 \diamond \alpha \subseteq \Gamma_2 \diamond \alpha$.
2. Assume that $\diamond$ satisfies **(U2)** and **(U8)**. If $\|\alpha\| = \emptyset$, then $\Gamma \cap \|\alpha\| \subseteq \Gamma \diamond \alpha$. Assume now that $\|\alpha\| \neq \emptyset$ Let $P \in \Gamma \cap \|\alpha\|$. Hence $\{P\} \subseteq \Gamma$ and $\{P\} \subseteq \|\alpha\|$. From the former, by **(U10)** (that follows from **(U8)**), it follows that $\{P\} \diamond \alpha \subseteq \Gamma \diamond \alpha$. From $\{P\} \subseteq \|\alpha\|$, by **(U2)**, it follows that $\{P\} \diamond \alpha = \{P\}$. Hence $P \in \Gamma \diamond \alpha$. $\square$

Proof of Theorem 5.24. **(1 to 2)** Assume that the profile update operator $\diamond$ satisfies postulates **(U0)** to **(U8)**. For any profiles P_i and P_j we define a relation $\preceq_P$ as follows:

$P_i \preceq_P P_j$ if and only if either $P_i = P$ or $\{P\} \diamond \alpha_{P_i,P_j} = \{P_i\}$.

We start by showing that $\preceq_P$ is a pre-order.

(Reflexivity:) Let P_i be a profile. It follows by **(U1)** and **(U3)** that $\{P\} \diamond \alpha_{P_i,P_i} = \{P_i\}$. Therefore, by definition of $\preceq_P$, it follows that $P_i \preceq_P P_i$.

(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$. We intend to prove that $P_i \preceq_P P_k$. It follows immediately by definition of $\preceq_P$ if $P_i = P$.
If $P_j = P$, from $P_i \preceq_P P_j$, it follows, by definition of $\preceq_P$, that either $P_i = P$ or $\{P\} \diamond \alpha_{P_i,P} = \{P_i\}$. In the latter case, it follows that $\{P\} \subseteq \|\alpha_{P_i,P}\|$. From which it follows by **(U2)** that $\{P\} \diamond \alpha_{P_i,P} = \{P\}$. Hence in both cases it holds that $P_i = P$. By definition of $\preceq_P$, it follows that $P_i \preceq_P P_k$. If $P_k = P$, then reasoning as above it follows that $P_j = P = P_k$. Hence $P_i \preceq_P P_k$.
Assume now that $P_i \neq P$, $P_j \neq P$ and $P_k \neq P$.
It also follows trivially that $P_i \preceq_P P_k$ if $P_i = P_j$ or $P_j = P_k$. It holds that $\preceq$ is reflexive (as shown above) hence it also follows that $P_i \preceq_P P_k$ if $P_i = P_k$. Consider now that $P_i \neq P_j$, $P_i \neq P_k$ and $P_j \neq P_k$.

By definition of $\preceq_P$ it follows, from $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$, that $\{P\} \diamond \alpha_{P_i,P_j} = \{P_i\}$ and $\{P\} \diamond \alpha_{P_j,P_k} = \{P_j\}$.

By **(U5)** it follows that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_i,P_j}\| \subseteq \{P\} \diamond (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j})$.
Hence by **(U4)** it follows from $\vDash \alpha_{P_i,P_j} \leftrightarrow (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j})$, that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \{P_i, P_j\} \subseteq \{P\} \diamond \alpha_{P_i,P_j}$. Hence $P_j \notin \{P\} \diamond \alpha_{P_i,P_j,P_k}$. By **(U5)** it also follows that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_j,P_k}\| \subseteq \{P\} \diamond (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_j,P_k})$. And from $\vDash \alpha_{P_j,P_k} \leftrightarrow (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_j,P_k})$, by **(U4)**, it follows that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \{P_j, P_k\} \subseteq \{P\} \diamond \alpha_{P_j,P_k}$. Hence $P_k \notin \{P\} \diamond \alpha_{P_i,P_j,P_k}$. Therefore, by **(U1)** and **(U3)**, it follows that $\{P\} \diamond \alpha_{P_i,P_j,P_k} = \{P_i\}$. Hence $\{P\} \diamond \alpha_{P_i,P_j,P_k} \vDash \alpha_{P_i,P_k}$. By **(U1)** and **(U3)** it follows that $\emptyset \neq \{P\} \diamond \alpha_{P_i,P_k} \subseteq \{P_i, P_k\}$. Thus $\{P\} \diamond \alpha_{P_i,P_k} \vDash \alpha_{P_i,P_j,P_k}$. Hence by **(U6)** it follows that $\{P\} \diamond \alpha_{P_i,P_k} = \{P\} \diamond \alpha_{P_i,P_j,P_k} = \{P_i\}$. From which it follows by definition of $\preceq_P$ that $P_i \preceq_P P_k$.

Now we will show that the condition of Definition 4.9 holds.
Let $P' \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. It holds, by definition of $\preceq_P$ that $P \preceq_P P'$. On the other hand, it holds that $\{P\} \subseteq \|\alpha_{P,P'}\|$, from which it follows by **(U2)** that $\{P\} \diamond \alpha_{P,P'} = \{P\}$. Hence $P' \npreceq_P P$. Therefore $P \prec_P P'$.

It remains to prove that:

$$\Gamma \diamond \alpha = \begin{cases} \bigcup\limits_{P \in \Gamma} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(U0)** it follows that $\Gamma \diamond \alpha = \Gamma$. Assume now that $\|\alpha\| \neq \emptyset$.
Let $P_i \in \Gamma \diamond \alpha$ and suppose that $P_i \notin \bigcup\limits_{P \in \Gamma} Min(\|\alpha\|, \preceq_P)$. By **(U1)** it follows that $P_i \in \|\alpha\|$. Let $P \in \Gamma$. Hence there exists $P_j \in \|\alpha\|$ such that $P_j \prec_P P_i$ (as shown above $\preceq_P$ is reflexive, from which it follows that $P_i \neq P_j$). If $P_j = P$, then $\{P\} \subseteq \|\alpha_{P_i,P_j}\|$. Thus by **(U2)**, $\{P\} \diamond \alpha_{P_i,P_j} = \{P\} = \{P_j\}$. If $P_j \neq P$, then from $P_j \prec_P P_i$ it also follows by definition of $\preceq_P$ that $\{P\} \diamond \alpha_{P_i,P_j} = \{P_j\}$.
Hence for all $P \in \Gamma$ it follows that there exists P_j such that $P_i \notin \{P\} \diamond \alpha_{P_i,P_j} = \{P_j\}$. On the other hand, by **(U5)** it follows that $\{P\} \diamond \alpha \cap \|\alpha_{P_i,P_j}\| \subseteq \{P\} \diamond (\alpha \wedge \alpha_{P_i,P_j})$.
It holds that $\|\alpha_{P_i,P_j}\| \subseteq \|\alpha\|$. Hence, by **(U4)** it follows from $\vDash \alpha_{P_i,P_j} \leftrightarrow (\alpha \wedge \alpha_{P_i,P_j})$, that $\{P\} \diamond (\alpha \wedge \alpha_{P_i,P_j}) = \{P\} \diamond \alpha_{P_i,P_j}$. Hence $\{P\} \diamond \alpha \cap \{P_i, P_j\} \subseteq \{P\} \diamond \alpha_{P_i,P_j}$.
Hence $P_i \notin \{P\} \diamond \alpha$, for all $P \in \Gamma$. Let $\Gamma = \{P_1, ..., P_m\}$. By **(U8)** it follows that $\Gamma \diamond \alpha = (\{P_1\} \cup ... \cup \{P_m\}) \diamond \alpha = (\{P_1\} \diamond \alpha) \cup ... \cup (\{P_m\} \diamond \alpha)$. From which it follows that $P_i \notin \Gamma \diamond \alpha$. Contradiction.

Let $P_i \in \bigcup\limits_{P \in \Gamma} Min(\|\alpha\|, \preceq_P)$. Hence there exists some $P \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq_P)$. If $P_i = P$, then, by **(U11)**, $P_i \in \Gamma \diamond \alpha$ (according to Observation 5.23, **(U11)** follows from **(U2)** and **(U8)**). Assume now that $P_i \neq P$. Let $\|\alpha\| = \{P_1, ..., P_m\}$. For all $P_k \in \{P_1, ..., P_m\}$ it follows that $P_k \nprec_P P_i$. From which it follows, by definition of $\preceq_P$, that $\{P\} \diamond \alpha_{P_i,P_k} \neq \{P_k\}$, for all $P_k \in \{P_1, ..., P_m\}$.[11]

[11] Note that $P_k \neq P$, since as proven above, the condition of Definition 4.9 holds.

Hence by **(U1)** and **(U3)** it follows that $\{P\} \diamond \alpha_{P_i,P_k} = \{P_i\}$, for all $P_k \in \{P_1, ..., P_m\}$.
By repeated applications of **(U7)** it follows that $P_i \in \{P\} \diamond (\alpha_{P_i,P_1} \vee ... \vee \alpha_{P_i,P_m})$. It holds that $\models \alpha \leftrightarrow (\alpha_{P_i,P_1} \vee ... \vee \alpha_{P_i,P_m})$, from which it follows by **(U4)** that $P_i \in \{P\} \diamond \alpha$. Hence by **(U10)** it follows that $P_i \in \Gamma \diamond \alpha$ (according to Observation 5.23 **(U10)** follows from **(U8)**).

(2 to 3) For a pre-order $\preceq_P$, we define a relation $\preceq'_P$ as:

$$P_i \preceq'_P P_j \text{ if and only if } P_i = P_j \text{ or } P_i \prec_P P_j$$

We first show that $\preceq'_P$ is an order.

(Reflexivity:) Follows immediately from the definition of $\preceq'_P$.

(Transitivity:) Let $P_i \preceq'_P P_j$ and $P_j \preceq'_P P_k$. We intend to show that $P_i \preceq'_P P_k$. If $P_i = P_j$ or $P_j = P_k$ or $P_i = P_k$, then it follows trivially that $P_i \preceq'_P P_k$. Assume now that $P_i \neq P_j$, $P_j \neq P_k$ and $P_i \neq P_k$. It follows, from $P_i \preceq'_P P_j$ and $P_j \preceq'_P P_k$, by definition of $\preceq'_P$ that $P_i \prec_P P_j$ and $P_j \prec_P P_k$. It holds that $\preceq_P$ is transitive, hence its strict part is also transitive. Thus $P_i \prec_P P_k$. From which it follows by definition of $\preceq'_P$ that $P_i \preceq'_P P_k$.

(Antisymmetry) Let $P_i \preceq'_P P_j$ and $P_j \preceq'_P P_i$. Assume towards a contradiction that $P_i \neq P_j$. Hence, by definition of $\preceq'_P$ it follows that $P_i \prec_P P_j$ and $P_j \prec_P P_i$. Hence $P_i \preceq_P P_j$, $P_j \npreceq_P P_i$, $P_j \preceq_P P_i$ and $P_i \npreceq_P P_j$. Contradiction.

Now we will show that the condition of Definition 4.9 holds.
Let $P_i \in \mathbb{P}_{\mathbb{L}} \setminus \{P\}$. It holds, that $P \prec_P P_i$. From which follows by Lemma 1 that $P \prec'_P P_i$.

It remains to prove that:

$$\Gamma \diamond \alpha = \begin{cases} \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq'_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

We know that:

$$\Gamma \diamond \alpha = \begin{cases} \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

The proof follows trivially if we prove that:

$$\bigcup_{P\in\Gamma} Min(\|\alpha\|, \preceq'_P) = \bigcup_{P\in\Gamma} Min(\|\alpha\|, \preceq_P), \text{ whenever } \|\alpha\| \neq \emptyset.$$

We will prove by double inclusion. Assume that $\|\alpha\| \neq \emptyset$.
Let $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq'_P)$ and assume towards a contradiction that $P_i \notin \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$. From the latter, it follows that:
(1) $P_i \notin Min(\|\alpha\|, \preceq_P)$, for all $P \in \Gamma$.
From $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq'_P)$ it follows that there exists $P_j \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq'_{P_j})$. From (1) it follows that $P_i \notin Min(\|\alpha\|, \preceq_{P_j})$. Hence there exists $P_k \in \|\alpha\|$ such that $P_k \prec_{P_j} P_i$. Hence, by Lemma 1, there exists $P_k \in \|\alpha\|$ such that $P_k \prec'_{P_j} P_i$. Hence $P_i \notin Min(\|\alpha\|, \preceq'_{P_j})$. Contradiction.

Let $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$ and assume towards a contradiction that $P_i \notin \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq'_P)$. From the latter, it follows that:
(2) $P_i \notin Min(\|\alpha\|, \preceq'_P)$, for all $P \in \Gamma$.
From $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$ it follows that there exists $P_j \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq_{P_j})$. From (2) it follows that $P_i \notin Min(\|\alpha\|, \preceq'_{P_j})$. Hence there exists $P_k \in \|\alpha\|$ such that $P_k \prec'_{P_j} P_i$. Hence, by Lemma 1, there exists $P_k \in \|\alpha\|$ such that $P_k \prec_{P_j} P_i$. Hence $P_i \notin Min(\|\alpha\|, \preceq_{P_j})$. Contradiction.

(3 to 1) Assume that there is a UpdP-faithful assignment that maps each profile P to a partial pre-order $\preceq_P$. We define a profile update operator $\diamond$ by:

$$\Gamma \diamond \alpha = \begin{cases} \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

We need to show that $\diamond$ satisfies postulates (U0) to (U8).
The proofs for **(U0)**, **(U1)**, **(U3)** and **(U4)** are straightforward.
(U2) Let $\Gamma \subseteq \|\alpha\|$. We will prove that $\Gamma \diamond \alpha = \Gamma$ by double inclusion.
Let $P \in \Gamma$. Hence $P \in \|\alpha\|$. By Definition 4.9 it holds that $P' \nprec_P P$, for all $P' \in \mathbb{P}_{\mathbb{L}}$. From which it follows that $P \in Min(\|\alpha\|, \preceq_P)$. Hence $P \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P) = \Gamma \diamond \alpha$. Thus $\Gamma \subseteq \Gamma \diamond \alpha$.
Let $P_i \in \Gamma \diamond \alpha$. Hence $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$. Thus, there exists some $P \in \Gamma \subseteq \|\alpha\|$, such that $P_i \in Min(\|\alpha\|, \preceq_P)$. If $P_i \neq P$, then it follows by Definition 4.9 that $P \prec_P P_i$. Hence $P_i \notin Min(\|\alpha\|, \preceq_P)$. Contradiction. Hence $P_i = P \in \Gamma$. Thus $\Gamma \diamond \alpha \subseteq \Gamma$.
(U5) If $\|\alpha\| = \emptyset$, then $\|\alpha \wedge \beta\| = \emptyset$. By definition of $\diamond$ it follows that $\Gamma \diamond \alpha = \Gamma \diamond (\alpha \wedge \beta) = \Gamma$. From which it follows that $\Gamma \diamond \alpha \cap \|\beta\| \subseteq \Gamma \diamond (\alpha \wedge \beta)$.

Assume now that $\|\alpha\| \neq \emptyset$. Let $P_i \in \Gamma \diamond \alpha \cap \|\beta\|$. Hence $P_i \in \Gamma \diamond \alpha$ and $P_i \in \|\beta\|$. Thus $P_i \in \bigcup_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$ and $P_i \in \|\beta\|$. Hence there exists $P \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq_P)$. It holds that $P_i \in \|\alpha \wedge \beta\|$. Let P_j be an arbitrary element of $\|\alpha \wedge \beta\|$. Hence $P_j \in \|\alpha\|$. Thus $P_j \not\prec_P P_i$. Hence $P_i \in Min(\|\alpha \wedge \beta\|, \preceq_P)$. From which it follows that $P_i \in \bigcup_{P\in\Gamma} Min(\|\alpha \wedge \beta\|, \preceq_P) = \Gamma \diamond (\alpha \wedge \beta)$.

(U6) Let $\Gamma \diamond \alpha \vDash \beta$ and $\Gamma \diamond \beta \vDash \alpha$. It holds that $\|\alpha\| \neq \emptyset$ and $\|\beta\| \neq \emptyset$. Assume towards a contradiction that there exists $P_i \in \Gamma \diamond \alpha$ such that $P_i \notin \Gamma \diamond \beta$. Hence $P_i \in \|\beta\|$. From $P_i \notin \Gamma \diamond \beta$ it follows that $P_i \notin Min(\|\beta\|, \preceq_P)$, for all $P \in \Gamma$. Hence it holds that:
(1) for all $P \in \Gamma$, there exists $P_j \in Min(\|\beta\|, \preceq_P)$ such that $P_j \prec_P P_i$.
On the other hand it holds that $P_i \in \bigcup_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$. Hence there exists $P' \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq_{P'})$. It follows from (1) that there exists $P_j \in Min(\|\beta\|, \preceq_{P'})$ such that $P_j \prec_{P'} P_i$. Hence $P_j \in \Gamma \diamond \beta \subseteq \|\alpha\|$. Hence $P_i \notin Min(\|\alpha\|, \preceq_{P'})$. Contradiction. Hence $\Gamma \diamond \alpha \subseteq \Gamma \diamond \beta$. Similarly, we can obtain that $\Gamma \diamond \beta \subseteq \Gamma \diamond \alpha$.
(U7) Let P be a profile. It follows immediately by definition of $\diamond$ if either $\alpha \vDash \perp$ or $\beta \vDash \perp$. Assume now that $\alpha \nvDash \perp$ and $\beta \nvDash \perp$. Hence $\alpha \vee \beta \nvDash \perp$. Let $P_i \in \{P\} \diamond \alpha \cap \{P\} \diamond \beta$ and suppose towards a contradiction that $P_i \notin \{P\} \diamond (\alpha \vee \beta)$. Hence there exists $P_j \in \|\alpha \vee \beta\|$ such that $P_j \prec_P P_i$. It follows from $P_j \in \|\alpha \vee \beta\|$ that either $P_j \in \|\alpha\|$ or $P_j \in \|\beta\|$.
case 1) $P_j \in \|\alpha\|$. Hence $P_i \notin Min(\|\alpha\|, \preceq_P)$. From which it follows that $P_i \notin \{P\} \diamond \alpha$. Contradiction.
case 2) $P_j \in \|\beta\|$. This leads to a contradiction by reasoning as in the previous case.
(U8) Let Γ_1 and Γ_2 be sets of profiles. If $\alpha \vDash \perp$, then it follows immediately by definition of $\diamond$ that $(\Gamma_1 \cup \Gamma_2) \diamond \alpha = \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$. Consider now that $\alpha \nvDash \perp$. We will prove by double inclusion that $(\Gamma_1 \cup \Gamma_2) \diamond \alpha = \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$.
Let $P_i \in (\Gamma_1 \cup \Gamma_2) \diamond \alpha$. Hence $P_i \in \bigcup_{P\in(\Gamma_1\cup\Gamma_2)} Min(\|\alpha\|, \preceq_P)$. Hence there exists $P' \in \Gamma_1 \cup \Gamma_2$ such that $P_i \in Min(\|\alpha\|, \preceq_{P'})$. If $P' \in \Gamma_1$, then it follows that $P_i \in \Gamma_1 \diamond \alpha$. If $P' \in \Gamma_2$, then it follows that $P_i \in \Gamma_2 \diamond \alpha$. Thus $P_i \in \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$.
Assume now that $P_k \in \Gamma_1 \diamond \alpha \cup \Gamma_2 \diamond \alpha$. Hence $P_k \in \Gamma_1 \diamond \alpha$ or $P_k \in \Gamma_2 \diamond \alpha$.
case 1) $P_k \in \Gamma_1 \diamond \alpha$. Hence $P_k \in \bigcup_{P\in\Gamma_1} Min(\|\alpha\|, \preceq_P)$. From which it follows that there exists $P' \in \Gamma_1$ such that $P_k \in Min(\|\alpha\|, \preceq_{P'})$. It holds that $P' \in \Gamma_1 \cup \Gamma_2$. Thus $P_k \in \bigcup_{P\in(\Gamma_1\cup\Gamma_2)} Min(\|\alpha\|, \preceq_P) = (\Gamma_1 \cup \Gamma_2) \diamond \alpha$.
case 2) $P_k \in \Gamma_2 \diamond \alpha$. This case is symmetric with the previous one. □

Proof of Theorem 5.25. **(2 to 1)** By Theorem 5.24 it only remains to prove that **(U9)** holds. Assume that $\alpha \wedge \beta \nvDash \perp$, P is a profile and $(\{P\} \diamond \alpha) \cap \|\beta\| \neq \emptyset$. Let $P_i \in \{P\} \diamond (\alpha \wedge \beta)$, and assume towards a contradiction that $P_i \notin (\{P\} \diamond \alpha) \cap \|\beta\|$. It holds that $P_i \in \|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|$. Thus $P_i \notin \{P\} \diamond \alpha = Min(\|\alpha\|, \preceq_P)$. Let $P_j \in \{P\} \diamond \alpha) \cap \|\beta\|$ (P_j exists since we assumed that $(\{P\} \diamond \alpha) \cap \|\beta\| \neq \emptyset$). Thus $P_j \in \|\alpha \wedge \beta\|$. Hence $P_i \preceq_P P_j$ (since $\preceq_P$ is total). Let $P_k \in \|\alpha\|$. Hence $P_j \preceq_P P_k$. It holds that $\preceq_P$ satisfies transitivity, hence $P_i \preceq_P P_k$. From which it follows that $P_i \in Min(\|\alpha\|, \preceq_P) = \{P\} \diamond \alpha$. Contradiction.

(1 to 2) Assume that the profile update operator $\diamond$ satisfies postulates **(U0)** to **(U5)**, **(U8)** and **(U9)**. For any profiles P_i and P_j we define a relation $\preceq_P$ as follows:

$$P_i \preceq_P P_j \text{ if and only if either } P_i = P \text{ or } P_i \in \{P\} \diamond \alpha_{P_i,P_j}.$$

We first show that $\preceq_P$ is a total pre-order.
(Totality:) Let P_i and P_j be profiles. It holds that $\alpha_{P_i,P_j} \nvDash \perp$. By **(U1)** and **(U3)** it follows that $\emptyset \neq \{P\} \diamond \alpha_{P_i,P_j} \subseteq \{P_i, P_j\}$. Hence either $P_i \in \{P\} \diamond \alpha_{P_i,P_j}$ or $P_j \in \{P\} \diamond \alpha_{P_i,P_j}$. Hence, by the definition of $\preceq_P$, it holds that either $P_i \preceq_P P_j$ or $P_j \preceq_P P_i$.

(Reflexivity:) Let P_i be a profile. It follows by **(U1)** and **(U3)** that $\{P\} \diamond \alpha_{P_i,P_i} = \{P_i\}$. Therefore, by definition of $\preceq_P$, it follows that $P_i \preceq_P P_i$.

(Transitivity:) Let P_i, P_j and P_k be profiles such that $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$. We intend to prove that $P_i \preceq_P P_k$. The proof that $P_i \preceq_P P_k$ holds in the cases that $P_i = P_j$, $P_i = P_k$ or $P_j = P_k$ follows exactly as in the proof of transitivity presented in the (1 to 2) part of Theorem 5.24.
It follows immediately by definition of $\preceq_P$ if $P_i = P$.
If $P_j = P$, from $P_i \preceq_P P_j$, it follows, by definition of $\preceq_P$, that either $P_i = P$ or $P_i \in \{P\} \diamond \alpha_{P_i,P}$. In the first case we are done. On the other hand, it holds that $\{P\} \subseteq \|\alpha_{P_i,P}\|$. From which it follows by **(U2)** that $\{P\} \diamond \alpha_{P_i,P} = \{P\}$. Hence $P_i = P$. By definition of $\preceq_P$, it follows that $P_i \preceq_P P_k$. If $P_k = P$, then reasoning as above it follows that $P_j = P = P_k$. Hence $P_i \preceq_P P_k$.
Assume now that $P \notin \{P_i, P_j, P_k\}$, $P_i \neq P_j$, $P_i \neq P_k$ and $P_j \neq P_k$.

By definition of $\preceq_P$ it follows, from $P_i \preceq_P P_j$ and $P_j \preceq_P P_k$, that $P_i \in \{P\} \diamond \alpha_{P_i,P_j}$ and $P_j \in \{P\} \diamond \alpha_{P_j,P_k}$. Assume towards a contradiction that $P_i \npreceq_P P_k$. Hence $P_i \notin \{P\} \diamond \alpha_{P_i,P_k}$. From which it follows by **(U1)** and **(U3)** that $\{P\} \diamond \alpha_{P_i,P_k} = \{P_k\}$. By **(U5)** it follows that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_i,P_k}\| \subseteq \{P\} \diamond (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_k})$.
Hence by **(U4)** it follows from $\vDash \alpha_{P_i,P_k} \leftrightarrow (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_k})$, that $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \{P_i, P_k\} \subseteq \{P\} \diamond \alpha_{P_i,P_k}$. Hence $P_i \notin \{P\} \diamond \alpha_{P_i,P_j,P_k}$. By **(U1)** and **(U3)** it follows that either $\{P\} \diamond \alpha_{P_i,P_j,P_k} = \{P_k\}$ or $P_j \in \{P\} \diamond \alpha_{P_i,P_j,P_k}$. Assume first that $\{P\} \diamond \alpha_{P_i,P_j,P_k} = \{P_k\}$. Hence $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_j,P_k}\| \neq \emptyset$. Thus by **(U9)** it follows that $\{P\} \diamond (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_j,P_k}) \subseteq (\{P\} \diamond \alpha_{P_i,P_j,P_k}) \cap \|\alpha_{P_j,P_k}\|$. Hence by **(U4)** it follows that $\{P\} \diamond \alpha_{P_j,P_k} \subseteq (\{P\} \diamond \alpha_{P_i,P_j,P_k}) \cap \{P_j, P_k\}$. Hence $P_j \in \{P\} \diamond \alpha_{P_i,P_j,P_k}$. Contradiction.
Assume now that $P_j \in \{P\} \diamond \alpha_{P_i,P_j,P_k}$. Hence $\{P\} \diamond \alpha_{P_i,P_j,P_k} \cap \|\alpha_{P_i,P_j}\| \neq \emptyset$. Thus by **(U9)** it follows that $\{P\} \diamond (\alpha_{P_i,P_j,P_k} \wedge \alpha_{P_i,P_j}) \subseteq (\{P\} \diamond \alpha_{P_i,P_j,P_k}) \cap \|\alpha_{P_i,P_j}\|$. Hence by **(U4)** it follows that $\{P\} \diamond \alpha_{P_i,P_j} \subseteq (\{P\} \diamond \alpha_{P_i,P_j,P_k}) \cap \{P_i, P_j\}$. Hence $P_i \in \{P\} \diamond \alpha_{P_i,P_j,P_k}$. Contradiction.

The proof that the condition of Definition 4.9 holds follows exactly as in the proof present in the (1 to 2) part of Theorem 5.24.

It remains to prove that:

$$\Gamma \diamond \alpha = \begin{cases} \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P) & \text{if } \|\alpha\| \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$$

If $\|\alpha\| = \emptyset$, then by **(U0)** it follows that $\Gamma \diamond \alpha = \Gamma$.
Assume now that $\|\alpha\| \neq \emptyset$. We will prove buy double inclusion that $\Gamma \diamond \alpha = \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$.

Let $P_i \in \Gamma \diamond \alpha$ and suppose that $P_i \notin \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$. By **(U1)** it follows that $P_i \in \|\alpha\|$. Let $P \in \Gamma$. Hence there exists $P_j \in \|\alpha\|$ such that $P_i \npreceq_P P_j$ (as shown above $\preceq_P$ is reflexive, from which it follows that $P_i \neq P_j$). If $P_j = P$, then $\{P\} \subseteq \|\alpha_{P_i,P_j}\|$. Thus by **(U2)**, $\{P\} \diamond \alpha_{P_i,P_j} = \{P\} = \{P_j\}$. If $P_j \neq P$, then from $P_i \npreceq_P P_j$ it follows by definition of $\preceq_P$ that $P_i \notin \{P\} \diamond \alpha_{P_i,P_j}$. Hence, by **(U1)** and **(U3)** it follows that $\{P\} \diamond \alpha_{P_i,P_j} = \{P_j\}$. The rest of the proof for this inclusion follows exactly as in the proof presented in the (1 to 2) part of the Theorem 5.24 for the same inclusion.

For the other inclusion, let $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$ and assume towards a contradiction that $P_i \notin \Gamma \diamond \alpha$.
Let $\Gamma = \{P_1, \ldots, P_k\}$. By repeated applications of **(U8)** it follows that $\Gamma \diamond \alpha = \{P_1\} \diamond \alpha \cup \ldots \cup \{P_k\} \diamond \alpha$. Hence, from $P_i \notin \Gamma \diamond \alpha$ it follows that $P_i \notin \{P_j\} \diamond \alpha$, for all $j \in \{1, \ldots, k\}$. On the other hand, from $P_i \in \bigcup\limits_{P\in\Gamma} Min(\|\alpha\|, \preceq_P)$ it follows that there exists $P \in \Gamma$ such that $P_i \in Min(\|\alpha\|, \preceq_P)$. It holds that $P_i \notin \{P\} \diamond \alpha$ and $P_i \in \|\alpha\|$.

By **(U1)** and **(U3)** it follows that there exists $P_k \in \{P\} \diamond \alpha \subseteq \|\alpha\|$. Hence $P_i \preceq_P P_k$. By definition of $\preceq$ it follows that $P_i = P$ or $P_i \in \{P\} \diamond \alpha_{P_i,P_k}$. In both cases it holds that $P_i \in \{P\} \diamond \alpha_{P_i,P_k}$ (in the first case by **(U2)**).

On the other hand it holds that $P_k \in (\{P\} \diamond \alpha) \cap \|\alpha_{P_i,P_k}\|$.

By **(U9)** it follows that $\{P\} \diamond (\alpha \wedge \alpha_{P_i,P_k}) \subseteq (\{P\} \diamond \alpha) \cap \|\alpha_{P_i,P_k}\|$. It holds that $\models (\alpha \wedge \alpha_{P_i,P_k}) \leftrightarrow \alpha_{P_i,P_k}$. Hence by **(U4)** it follows that $\{P\} \diamond \alpha_{P_i,P_k} \subseteq (\{P\} \diamond \alpha) \cap \{P_i, P_k\}$. From $P_i \notin \{P\} \diamond \alpha$ it follows that $P_i \notin \{P\} \diamond \alpha_{P_i,P_k}$. Contradiction. □

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