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IID prophet inequality with a single data point ☆

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ABSTRACT

In this work, we study the single-choice prophet inequality problem, where a seller encounters a sequence of n online bids. These bids are modeled as independent and identically distributed (i.i.d.) random variables drawn from an unknown distribution. Upon the revelation of each bid's value, the seller must make an immediate and irrevocable decision on whether to accept the bid and sell the item to the bidder. The objective is to maximize the competitive ratio between the expected gain of the seller and that of the maximum bid. It is shown by Correa et al. [1] that when the distribution is unknown or only $o(n)$ uniform samples from the distribution are given, the best an algorithm can do is $1/e$-competitive. In contrast, when the distribution is known [2], or when $\Omega(n)$ uniform samples are given [3], the optimal competitive ratio of 0.7451 can be achieved. In this paper, we study the setting when the seller has access to a single point in the cumulative density function of the distribution, which can be learned from historical sales data. We investigate how effectively this data point can be used to design competitive algorithms. Motivated by the algorithm for the secretary problem, we propose the observe-and-accept algorithm that sets a threshold in the first phase using the data point and adopts the highest bid from the first phase as the threshold for the second phase. It can be viewed as a natural combination of the single-threshold algorithm for prophet inequality and the secretary problem algorithm. We show that our algorithm achieves a good competitive ratio for a wide range of data points, reaching up to 0.6785-competitive as $n \to \infty$ for certain data points. Additionally, we study an extension of the algorithm that utilizes more than two phases and show that the competitive ratio can be further improved to at least 0.6862.

1. Introduction

Prophet inequality is one of the most widely studied problems in optimal stopping theory. In the classic setting, a gambler faces a sequence of online non-negative random variables $x_1, \dots, x_n$ independently drawn from distributions $D_1, \dots, D_n$. The realizations of these variables represent potential rewards given to the gambler. After observing the realization of a variable, the gambler needs to

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make an irrevocable decision on whether to accept the reward and leave the game, or reject it and continue with the next variable. All rejected values cannot be collected anymore. A prophet knows all the realizations beforehand and thus can always select the highest reward. The expected reward of the prophet is then defined as $\mathsf{OPT} = \mathbb{E}_{x_1,\dots,x_n}[\max_i\{x_i\}]$. The gambler's goal is to choose a stopping rule so that her expected reward, which we refer to as ALG, is as close to that of the prophet as possible. The performance is measured by the *competitive ratio*, which is defined as the worst case of ALG/OPT over all possible distributions. The prophet inequality problem has received increasingly more attention due to its connections with Bayesian mechanism design, especially the posted price mechanism, where buyers arrive online and the seller provides each buyer with a take-it-or-leave-it offer [4–7].

It is proved in [8,9] and [10] that there exists a stopping rule such that the gambler's expected reward is at least $0.5 \cdot \mathsf{OPT}$, and this is the best possible competitive ratio when the distributions are distinct and the order of variables is adversarial. However, when the distributions are identical, i.e., $D_i = D$ for all i, much better competitive ratios can be achieved. Hill and Kertz [11] proved that there is an algorithm that achieves a competitive ratio of $1 - 1/e \approx 0.6321$ while Kertz [12] showed that no algorithms can do better than 0.7451. The best-known competitive ratio remained $1 - 1/e$ until Abolhassani et al. [13] improved it to 0.738. Later, Correa et al. [2] proposed a *blind quantile strategy* with a tight competitive ratio of 0.7451. Informally speaking, a blind quantile strategy defines a sequence of increasing probabilities $p_1 < \cdots < p_n$ (which depend on n and the distribution D, but are independent of the realizations), so that the acceptance probability for each variable x_i equals p_i. This is done by setting the i-th threshold θ_i such that $\Pr_{x\sim D}[x > \theta_i] = p_i$ and accepting the i-th variable if $x_i > \theta_i$. However, computing the probabilities $p_1, \dots, p_n$ that give the optimal ratio is highly non-trivial and requires complete information of the distribution D. As pointed out in a survey paper [14], an interesting research problem is to investigate the amount of information needed to achieve a good competitive ratio:

How much knowledge of the distributions is required to achieve a good competitive ratio?

To follow up this question, a line of recent works studies the prophet inequality problem on unknown distributions, most of which focus on the setting where uniform random samples from the distribution are given [1,3,15]. Particularly, it is shown in [3] that $O(n)$ samples are sufficient to achieve a competitive ratio arbitrarily close to 0.7451, and in [1] that no algorithm can use $o(n)$ samples to ensure a result better than $1/e$ (≈ 0.368)-competitive.

However, obtaining random samples can be challenging in some applications. Consider the following analogy to the prophet inequality problem: a seller sets a posted price v for a product, and a sequence of buyers with private values independently drawn from an unknown distribution decides whether to purchase the product. A buyer will buy the product if and only if the posted price is lower than their private value. In practice, the seller often lacks accurate knowledge of the buyers' private values, or buyers may choose not to reveal this information. This makes it difficult for the seller to observe the *complete* information on the samples (i.e., values drawn from the distribution) and thus challenging to infer information about the distribution directly from samples. However, the seller can observe *partial* information through sampling. Since a single price may be offered for months or even years, the seller can instead observe the sales records to establish a relationship between a historical price and the fraction of buyers who have accepted the price, which is precisely a *quantile* of the value distribution. Based on this discussion, we propose to study the prophet inequality problem with a single data point.

Prophet inequality with a single data point In this paper, we propose a new information model where the algorithm has access to a data point of the value distribution. Formally, a data point is a quantile-value pair (v, q) where v is the historical price and $q \in [0, 1]$ is a quantile such that $\Pr_{x\sim D}[x \le v] = q$, i.e., $(1 - q)$ fraction of the buyers have purchased the product at price v. We focus on developing competitive algorithms that make use of a single data point.

Our model aligns well with the study of query complexity in Bayesian auction design, a closely related area to prophet inequality. In particular, Allouah et al. [16] examined the revenue maximization problem using a single value-quantile pair. They demonstrated that, for example, given the value at 0.5 quantile, it is possible to achieve 85% of the optimal revenue obtained when the full distribution is known. To our knowledge, this type of model has not yet been studied in the context of prophet inequality.

Simple algorithms Besides algorithms that use little knowledge, *simple* algorithms are often much preferred due to their easy implementation in real-world scenarios. For example, in the *secretary problem* [17], a widely used simple strategy is to discard the first $1/e$ fraction of the variables and select the first variable whose realization is greater than all the discarded ones. We call this strategy the *observe-then-accept* algorithm. It was proved in Correa et al. [1] that this algorithm is $1/e$-competitive and is optimal when the distribution is unknown or $o(n)$ uniform samples are given. For the prophet inequality problem, the *single-threshold algorithm* simply uses a fixed threshold for all variables and accepts the first variable whose realization exceeds the threshold. It is shown in [18] that by setting an appropriate threshold, the single-threshold algorithm ensures a competitive ratio of $1 - 1/e$, which is the best possible ratio using just one threshold.[1] In this work, we prove that beating $1 - 1/e$ can also be achieved with a single data point via our *observe-and-accept algorithm*. Indeed, an intriguing takeaway from our work is that, with a specific data point, a straightforward combination of the observe-then-accept algorithm and the single-threshold algorithm outperforms the $(1 - 1/e)$-competitive benchmark.

[1] We remark that their result holds for the more general setting of the prophet secretary problem. When the distributions are discrete, randomization is required to achieve the $1 - 1/e$ competitive ratio.

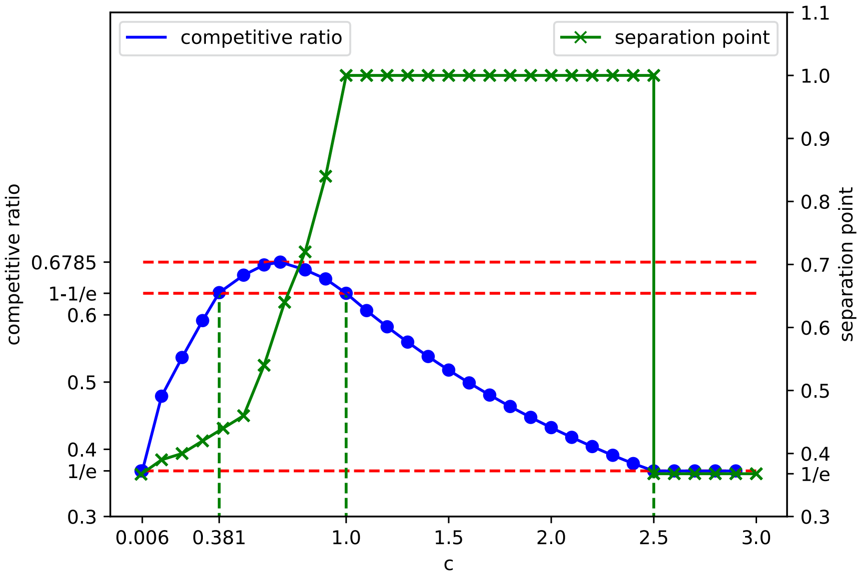

Fig. 1.1. Competitive ratio given data point $(v, 1 - c/n)$, where for $c \in (0.006, 2.5)$ the ratio is obtained by our observe-and-accept algorithm while for $c > 2.5$ or $c < 0.006$ the ratio is obtained using the optimal algorithm for the secretary problem.

1.1. Our contribution

We study the prophet inequality problem with unknown i.i.d. distributions with a single data point. The main contribution of this work is presenting a competitive algorithm that utilizes a single data point (v, q). This algorithm can be regarded as a natural combination of the single-threshold algorithm and the observe-then-accept algorithm. The algorithm partitions the time horizon into two phases. In the first phase, it uses the value v from the data point as a threshold to decide whether to accept the bids. In the second phase, if no bids are accepted in the first phase, the algorithm uses the highest bid from the first phase as the new threshold. We call this the *observe-and-accept* algorithm, as it is similar to the algorithm for the secretary problem but additionally allows the acceptance of bids during the observation phase.

We show that when $q = 1 - c/n$ for some $c \in (0.381, 1)$, our algorithm performs strictly better than $(1 - 1/e)$-competitive (see Fig. 1.1). In particular, when $c \approx 0.679$, our algorithm is at least 0.6785-competitive when $n \to \infty$. We can view c/n as c out of n buyers have private values higher than the historical price. When c is between 0.006 and 2.5, the quantile information is still beneficial in the sense that the algorithm has a competitive ratio better than $1/e$ – the best we can do when no information is given. For c larger than 2.5 or smaller than 0.006, our algorithm has small competitive ratios due to the algorithm and analysis framework.[2] However, we can still use the optimal algorithm for the secretary problem to recover a competitive ratio of $1/e$.

Our main technical contribution is a careful formulation of the lower bounds on the gain of the algorithm and the upper bounds on the expected maximum bid in terms of the parameters. The bounds then become the objective and constraints for a minimization linear program (LP) whose optimal solution provides a lower bound on the competitive ratio. However, it is challenging to bound the gain of the observe-and-accept algorithm since the second threshold is a random variable that depends on the realizations of the variables in the first phase, i.e., the algorithm is not a blind strategy algorithm. To overcome this major difficulty, we extend the LP by further dividing the domain of the second threshold into continuous segments, whose boundary points become additional variables of the LP. Via carefully choosing the separation point of the two phases, we show that the competitive ratio of our algorithm can be as good as 0.6785 when $n \to \infty$, with a certain data point. Moreover, we show that the competitive ratio can be improved to 0.6862 by introducing more phases to the algorithm, where the threshold of each phase, except for the first phase, is given by the maximum bid from the previous phase. In other words, we show that by leveraging the information from a single data point, we can guarantee $0.6862/0.7451 \approx 92\%$ of the revenue by optimal online algorithms when the full distribution information is known.

1.2. Related works

Prophet inequality with complete information The theory of prophet inequality is initiated by Gilbert and Mosteller [17] in the late sixties, and has been widely studied in the seventies and eighties; see, e.g., [8–12]. The problem regained significant interest in the last decade partly because of its application in posted price mechanisms that are widely adopted in (online) auctions [4–6]. For general (non-identical) distributions, the optimal competitive ratio is 0.5 when the ordering of the variables is adversarial [8–10]. In contrast, Chawla et al. [5] showed that when the algorithm can decide the arrival order of the variables, the competitive ratio

[2] It would be an interesting problem to explore the existence of algorithms that can beat $1/e$ for all c.

can be improved to $1 - 1/e$. The ratio was further improved to 0.634 and 0.669 by [19] and [20], respectively. These results also apply to the setting with random arrival orders, which is called the *prophet secretary problem* by Esfandiari et al. [21]. Recently, Peng and Tang [22] proposed a 0.7251-competitive algorithm that selects the arrival order of the variables based on a reduction to a continuous arrival time design problem. The ratio is further improved to 0.7258 by Bubna and Chiplunkar [23], who also provided a 0.7254 upper bound for the competitive ratio of algorithms for the prophet secretary problem, separating the best possible ratios for the two problems. Beyond the classic single-choice setting, recent works studied the setting where multiple variables can be selected subject to some combinatorial constraints [5,15,24–26]. In the aforementioned works, the objective is additive over the selected variables, and similar problems with non-additive objectives were investigated by Rubinstein and Singla [27] and Dütting et al. [28].

Prophet inequality with sampling With unknown i.i.d. distributions, Correa et al. [1] showed that the best an algorithm can do is $1/e$-competitive, and the competitive ratio cannot be improved even if it can observe $o(n)$ random samples. However, if the algorithm has n samples, it can achieve a competitive ratio $\alpha \in [0.6321, 0.6931]$. If $O(n^2)$ samples are given, the competitive ratio can be further improved to $(0.7451 - \epsilon)$, the best possible ratio even if the full distribution is known to the algorithm. Later, Rubinstein et al. [3] improved this result by showing that $O(n)$ samples suffice to achieve the optimal ratio. With non-i.i.d. distributions, Rubinstein et al. [3] proved that one random sample from each distribution is enough to define a 0.5-competitive algorithm. Nuti and Vondrák [29] studied the prophet secretary version of the same problem and showed that the probability of selecting the maximum value is 0.25, which is optimal. For multi-choice prophet inequality problems with random samples, constant competitive algorithms have been analyzed [15,30]. Beyond random samples, inaccurate prior distributions were considered by Dütting and Kesselheim [31], where the algorithm knows some estimations of the true distributions under various metrics. After the publication of a preliminary version of our work [32], a subsequent work [33] also considered algorithms with few thresholds and proved better lower bounds on the competitive ratios. Their algorithm achieves a competitive ratio of 0.70804 for 2 thresholds (which is optimal for two-threshold algorithms), 0.7233 for 3 thresholds, and 0.7321 for 4 thresholds.

Bayesian auctions with queries Designing mechanisms to generate approximately optimal revenue using random samples is a hot topic in Bayesian Auctions [34–37]. Similar to our motivation in the introduction, the literature also considered the partial information setting, where only a few number of points can be used, under the name of *query model*. For example, Allouah et al. [16] considered the single-item single-buyer revenue maximization problem in Bayesian auctions when the seller does not know the prior distributions but has one value-quantile pair. They showed that, for example, if we are given the value at quantile 0.01, 51% of the optimal revenue is guaranteed when the full distribution is known, and if we are given the value at quantile 0.5, the guarantee is improved to 85%. Chen et al. [38] studied a more general setting of multi-item multi-buyer auctions and gave an almost tight number of queries to ensure a constant fraction of the optimal revenue. Hu et al. [39] explored the middle ground between random samples and queries by allowing the algorithm to specify a quantile interval and sample from the prior distribution restricted to the interval. In a different flavor from the data points we study in this work, Leme et al. [40] defined *pricing queries* in that when the oracle is given a price, it generates a random sample and returns the sign of whether the sample is above the given price or not. Both query-related settings are of independent interest to be studied in prophet inequality.

1.3. Organization of the paper

The rest of this paper is organized as follows. We introduce the necessary notations and basic properties in Section 2. In Section 3, we begin by reviewing the analysis that establishes the $(1 - 1/e)$ competitive ratio of the single-threshold algorithm as a warm-up. Building on this foundation, Section 4 introduces the observe-and-accept algorithm using a single data point and provides a lower bound for its competitive ratio. In Section 5, we extend our algorithm to adopt multiple phases. We conclude our results and propose some open problems in Section 6.

2. Preliminaries

In our problem, there are n non-negative random variables $\{x_1, \ldots, x_n\}$ that are independently and identically drawn from an unknown distribution D. We slightly abuse the notation and use $x_i's$ to denote both random variables and their realizations. The realizations of the variables are revealed to the algorithm one by one. When the value of a variable is revealed, the algorithm needs to make an irrevocable decision on whether to *accept* the value and stop the algorithm, or *reject* this variable and proceed to the next one. The objective is to pick a realization that is as large as possible. For any integer t, we use $[t]$ to denote $\{1, 2, \ldots, t\}$. Throughout this paper, we use $\tau \in [n+1]$ to denote the stopping time of the algorithm. If the algorithm accepts variable x_i, then $\tau = i$; if the algorithm does not accept any variable, we set $\tau = n + 1$. We let ALG denote the expected gain of the algorithm. We compare the performance of the algorithm with a prophet, who sees all the realizations of variables before making a decision. In other words, the benchmark is

$$\mathsf{OPT} = \mathbb{E}_{(x_1,\ldots,x_n)\sim D^n}\left[\max_{i\in[n]}\{x_i\}\right].$$

The *competitive ratio* of the algorithm is defined as the minimum of ALG/OPT over all possible distributions. In this work, we consider the setting where the distribution is unknown but one data point of the value distribution is provided.

Data point of the value distribution We denote the CDF of the distribution by $F(\theta) = \Pr_{x\sim D}[x \le \theta]$ and the survival function by $G(\theta) = \Pr_{x\sim D}[x > \theta]$. Given any quantile $q \in [0,1]$, we define

$$v(q) := \inf_{\theta\in\mathbb{R}} \{F(\theta) \ge q\} = F^{-1}(q). \tag{2.1}$$

Consistent with common assumptions in prior research [33,41], we adopt the convention that the CDF of the distribution is strictly increasing so that $v(q)$ is always well-defined. If $v(q)$ is used as a threshold to decide the acceptances of variables, each variable is accepted with probability $1-q$. We note that $v(\cdot)$ is strictly increasing. In our model, we assume that a single data point (v,q) of the CDF is given, where $v = v(q)$ and $q = 1 - c/n$ for some constant c. For convenience, we define

$$\Delta(c) := n \cdot \mathbb{E}_{x\sim D}[(x - v(1-\tfrac{c}{n}))^+],$$

where $(\cdot)^+$ to denote $\max\{\cdot, 0\}$. Unless otherwise specified, all competitive ratios we achieve in this work are achieved under the assumption that $n\to\infty$. The numerical results for small n are presented in Appendix A.

No information We briefly discuss the case when no information about the distribution is known. The classic algorithm for the secretary problem [17] achieves the optimal competitive ratio of $1/e$, which also applies to our setting. Formally, we observe the first n/e variables without accepting any of them; for the following variables, we accept the first variable whose realization exceeds the maximum of the previous ones. The algorithm retains the $1/e$ competitive ratio since it can be equivalently described as independently drawing n realizations from the distribution and giving them a random arrival order. Interestingly, Correa et al. [1] showed that this is the best possible ratio for all algorithms that have no information about the distribution.

Theorem 2.1 *(No Information). There exists a $1/e$-competitive algorithm for the prophet inequality problem on unknown i.i.d. distributions, and the competitive ratio is optimal.*[3]

3. Warm-up analysis for single-threshold algorithms

In this section, we provide a short warm-up analysis for the single-threshold algorithms, which is the least we can do with a single data point (v,q) by using v as a threshold to accept the bids. We give the corresponding competitive ratio of the algorithm for different data points and show that the competitive ratio of the algorithm is $1-1/e$ when $q = 1-1/n$.

Single-threshold algorithms have been investigated by Ehsani et al. [18] and Correa et al. [20], where a $(1-1/e)$ competitive ratio was proved. Moreover, the competitive ratio is optimal for single-threshold algorithms [18]. For completeness, we give the formal analysis here, which will also serve as a warm-up for our proofs in later sections.

The single-threshold algorithm works as follows:

1. Fix $\theta = v(q)$. In other words, we have $\Pr_{x\sim D}[x > \theta] = 1-q$.
2. For the realization of each variable x_i, where $i = 1,2,\dots,n$, accept x_i and terminate if $x_i > \theta$; otherwise reject it and observe the next variable, if any.

We refer to θ as the *threshold* of the algorithm. Note that $F(\theta) = q$ and $G(\theta) = 1-q$. We give a lower bound of the competitive ratio of the algorithm by deriving an upper bound for OPT and a lower bound for ALG.

Let $x^* = \max_i\{x_i\}$ denote the maximum of the n i.i.d. random variables. Then we have $\mathsf{OPT} = \mathbb{E}[x^*]$. Given threshold θ, we have

$$\begin{aligned}\mathsf{OPT} = \mathbb{E}[x^*] &= \mathbb{E}[x^* \mid x^* \le \theta]\cdot\Pr[x^*\le\theta] + \mathbb{E}[x^* \mid x^* > \theta]\cdot\Pr[x^* > \theta]\\ &= \mathbb{E}[x^* \mid x^* \le \theta]\cdot\Pr[x^*\le\theta] + (\theta + \mathbb{E}[x^*-\theta \mid x^* > \theta])\cdot\Pr[x^* > \theta]\\ &= \mathbb{E}[x^* \mid x^* \le \theta]\cdot\Pr[x^*\le\theta] + \theta\cdot\Pr[x^*>\theta] + \mathbb{E}[x^*-\theta \mid x^* > \theta]\cdot\Pr[x^* > \theta].\end{aligned}$$

Note that the conditional expectation $\mathbb{E}[x^* \mid x^* \le \theta]$ is at most θ. Moreover,

$$\mathbb{E}[(x^*-\theta)^+] = 0\cdot\Pr[x^*\le\theta] + \mathbb{E}[x^*-\theta \mid x^*>\theta]\cdot\Pr[x^*>\theta]. \tag{3.1}$$

Thus we obtain the following upper bound on OPT,

$$\mathsf{OPT} \le \theta\cdot\Pr[x^*\le\theta] + \theta\cdot\Pr[x^*>\theta] + \mathbb{E}[(x^*-\theta)^+] = \theta + \mathbb{E}[(x^*-\theta)^+].$$

Since all x_i's are i.i.d. and x^* is the maximum, by defining $c = (1-q)\cdot n$, we have

$$\mathbb{E}[(x^*-\theta)^+] \le \sum_{i=1}^{n}\mathbb{E}[(x_i-\theta)^+] = n\cdot\mathbb{E}[(x-\theta)^+] = \Delta(c).$$

[3] Note that the result only holds for $n\to\infty$. For bounded n, better competitive ratios are possible, e.g., when $n=2$, we can do at least 0.5-competitive.

Overall, a valid upper bound for OPT is given by

$$\mathsf{OPT} \le \theta + \Delta(c). \tag{3.2}$$

Recall that $\tau \in [n+1]$ is the stopping time of the algorithm. The expected gain of the algorithm can then be written as

$$\mathsf{ALG} = \sum_{i=1}^{n} \Pr[\tau = i] \cdot \mathbb{E}[x_i \mid \tau = i]. \tag{3.3}$$

The algorithm accepts the i-th variable if and only if the first $i-1$ variables are smaller than the threshold θ and x_i is at least θ. In other words, we have

$$\Pr[\tau = i] = F(\theta)^{i-1} \cdot G(\theta). \tag{3.4}$$

Note that $\mathbb{E}[x_i \mid \tau = i] = \mathbb{E}[x_i \mid x_i > \theta]$. Conditioned on $x_i > \theta$, the algorithm receives θ and the expected difference between x_i and θ. That is,

$$\mathbb{E}[x_i \mid \tau = i] = \theta + \mathbb{E}[x_i - \theta \mid x_i > \theta] = \theta + \frac{\mathbb{E}[(x_i-\theta)^+]}{G(\theta)} = \theta + \frac{\Delta(c)/n}{G(\theta)}, \tag{3.5}$$

where the second last equality is derived from a similar argument to (3.1). By plugging Equations (3.4) and (3.5) into Equation (3.3), we have

$$\mathsf{ALG} = \sum_{i=1}^{n} F(\theta)^{i-1} \cdot G(\theta) \cdot \left(\theta + \frac{\Delta(c)/n}{G(\theta)}\right).$$

Finally, using Equation (3.2), we have

$$\begin{aligned}\frac{\mathsf{ALG}}{\mathsf{OPT}} &\ge \frac{\sum_{i=1}^{n} q^{i-1}\left((1-q)\theta + \frac{\Delta(c)}{n}\right)}{\theta + \Delta(c)} = \frac{(1-q^n)\left((1-q)\theta + \frac{\Delta(c)}{n}\right)}{(1-q)(\theta+\Delta(c))} \\ &= \frac{(1-(1-\frac{c}{n})^n)(c\cdot\theta + \Delta(c))}{c\cdot(\theta+\Delta(c))} \ge \frac{(1-e^{-c})(c\cdot\theta+\Delta(c))}{c\cdot(\theta+\Delta(c))}.\end{aligned}$$

This result covers the existing result of $1 - 1/e$ with a special data point.

Example 3.1. If the quantile of the data point is $q = 1 - 1/n$ (i.e., $c = 1$), the competitive ratio of the single-threshold algorithm is at least

$$\frac{\mathsf{ALG}}{\mathsf{OPT}} \ge \frac{(1-e^{-c})(c\cdot\theta+\Delta(c))}{c\cdot(\theta+\Delta(c))} = 1 - 1/e.$$

4. Observe-and-accept algorithm

In this section, we propose the observe-and-accept algorithm and show that its competitive ratio is larger than $1 - 1/e$ when the given data point (v, q) satisfies $q = 1 - c/n$ for some $c \in (0.381, 1)$. Our algorithm combines the single-threshold algorithm and the algorithm for the secretary problem. Let the phase-dividing point $\rho \in [0,1]$ be a parameter of the algorithm that depends on $c = (1-q)\cdot n$. For sufficiently large n we can assume w.l.o.g. that $\rho \cdot n$ is an integer. The algorithm, denoted by *observe-and-accept*, works as follows. Let $\theta_1 = v(1 - c/n)$. We have $\Pr_{x\sim D}[x > \theta_1] = G(\theta_1) = c/n$. In the first phase, for $i = 1, 2, \ldots, \rho\cdot n$, we accept variable x_i if and only if $x_i > \theta_1$. If no variables are accepted in the first phase, we enter the second phase and let $\theta_2 = \max_{i\le\rho\cdot n}\{x_i\}$. Note that θ_2 is a random variable and $\theta_2 \le \theta_1$. For $i = \rho\cdot n + 1, \ldots, n$, we accept variable x_i if and only if $x_i > \theta_2$ (see Algorithm 1).

Algorithm 1: Observe-and-Accept.

Input: Number of items n, data point $(v, 1 - c/n)$ and observation portion ρ.
Output: A bid selected by the seller.
Set $\theta_1 := v$;
for $i = 1, 2, \ldots, \rho\cdot n$ **do**
  if $x_i > \theta_1$ **then**
    return x_i and **terminate**; // phase 1
Set $\theta_2 := \max_{i\le\rho\cdot n}\{x_i\}$;
for $i = \rho\cdot n + 1, \ldots, n$ **do**
  if $x_i > \theta_2$ **then**
    return x_i and **terminate**; // phase 2

Theorem 4.1. *Algorithm 1 achieves a competitive ratio of* 0.6785 *when* $c = 0.6790$. *When* $c \in (0.381, 1)$, *the competitive ratio is larger than* $1 - 1/e$. *When* $c \in (0.006, 2.5)$, *the competitive ratio is larger than* $1/e$ *(see Fig. 1.1).*

We prove the theorem in the following two subsections using factor revealing LPs.

Remark. The technique of *factor revealing LPs* was first introduced by Jain et al. [42] and further extended by Mahdian and Yan [43]. Generally speaking, a family of LPs is called factor revealing if the infimum of the optimal values for these LPs serves as a lower bound on the competitive ratio for the original maximization problem. In our case, we construct factor revealing LPs for the adversary of the seller. After the seller has established an algorithm to select variables, the goal for the adversary is to find a distribution of the variables such that the seller's gain is minimized. We approximate the nonlinear optimization problem faced by the adversary with an LP whose objective will be a lower bound for the competitive ratio of the gambler. The objective function and the constraints of the factor revealing LP are formulated using inequalities similar to but more advanced than the ones derived in Section 3. We then leverage the power of LP solvers to find the best algorithm for the gambler such that the optimal value of the family of LPs is maximized. In the following subsections, we will show that by carefully choosing the parameters, the optimal value of the LPs (and thus the competitive ratio) for the observe-and-accept algorithm is 0.6785.

4.1. Bounding ALG and OPT

We first give lower bounds for ALG and upper bounds for OPT, which will then become the constraints of the factor revealing LP we study in the next subsection. Recall Equation (3.3) from Section 3, in which we have $\mathsf{ALG} = \sum_{i=1}^{n} \Pr[\tau = i] \cdot \mathbb{E}[x_i \mid \tau = i]$, where $\tau \in [n+1]$ is the stopping time of the algorithm. For $i \le \rho \cdot n$, the algorithm accepts variable x_i if its realization is at least θ_1, which happens with probability c/n. By a similar analysis seen in Equations (3.4) and (3.5), for all $i \le \rho \cdot n$ we have

$$\Pr[\tau = i] = \left(1 - \frac{c}{n}\right)^{i-1} \cdot \frac{c}{n}, \quad \text{and} \quad \mathbb{E}[x_i \mid \tau = i] = \theta_1 + \frac{\Delta(c)}{c}.$$

Therefore, the expected gain of the algorithm from the first phase is given by

$$\begin{aligned}
\Lambda_1 &:= \sum_{i=1}^{\rho n} \Pr[\tau = i] \cdot \mathbb{E}[x_i \mid \tau = i] = \sum_{i=1}^{\rho n} (1 - \frac{c}{n})^{i-1} \cdot \left(\frac{c}{n} \cdot \theta_1 + \frac{\Delta(c)}{n}\right) \\
&= \frac{1 - (1 - c/n)^{\rho n}}{c/n} \cdot \left(\frac{c}{n} \cdot \theta_1 + \frac{\Delta(c)}{n}\right) = \left(1 - (1 - \frac{c}{n})^{\rho n}\right) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right) \\
&\ge (1 - e^{-c\rho}) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right).
\end{aligned} \tag{4.1}$$

Next, we give a lower bound for the expected gain of the algorithm from the second phase, which is given by $\Lambda_2 := \sum_{i=\rho n+1}^{n} \Pr[\tau = i] \cdot \mathbb{E}[x_i \mid \tau = i]$. Unlike the first phase, the threshold $\theta_2 = \max_{i \le \rho n}\{x_i\}$ is a random variable and thus $F(\theta_2)$ is not a fixed probability. Our algorithm enters the second phase if and only if $\theta_2 \le \theta_1$, i.e., no variable passes the threshold θ_1 in the first phase. Note that

$$\Pr[\theta_2 \le \theta_1] = \Pr[\forall i \le \rho n, x_i \le \theta_1] = \left(1 - \frac{c}{n}\right)^{\rho n} \ge (1 - \epsilon) \cdot e^{-c\rho},$$

where $\epsilon > 0$ is an arbitrarily small constant given that n is sufficiently large. Moreover, given a realization of $\theta_2 \le \theta_1$, we can express the gain of the algorithm $h(\theta_2)$ in the second phase using a similar argument as above. Specifically, conditioned on a given $\theta_2 \le \theta_1$, we have

$$h(\theta_2) := \sum_{i=\rho n+1}^{n} \Pr[\tau = i \mid \tau > \rho n] \cdot \mathbb{E}[x_i \mid \tau = i] = \sum_{i=1}^{(1-\rho)n} F(\theta_2)^{i-1} \left(G(\theta_2)\theta_2 + \mathbb{E}[(x - \theta_2)^+]\right).$$

In summary, we can lower bound the expected total gain of the algorithm by

$$\begin{aligned}
\mathsf{ALG} &= \left(1 - (1 - \frac{c}{n})^{\rho n}\right) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right) + \left(1 - \frac{c}{n}\right)^{\rho n} \cdot \mathbb{E}_{\theta_2}[h(\theta_2) \mid \theta_2 \le \theta_1] \\
&\ge (1 - e^{-c\rho}) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right) + (1 - \epsilon) \cdot e^{-c\rho} \cdot \mathbb{E}_{\theta_2}[h(\theta_2) \mid \theta_2 \le \theta_1].
\end{aligned}$$

Unfortunately, it is challenging to derive a closed-form expression of the conditional expectation $\mathbb{E}_{\theta_2}[h(\theta_2) \mid \theta_2 \le \theta_1]$ in terms of c and ρ. To overcome this difficulty, we relax this term and lower bound it using a factor revealing LP. Specifically, consider the case when

$$v(1 - \frac{\beta_2}{n}) < \theta_2 \le v(1 - \frac{\beta_1}{n}),$$

where β_1, β_2 are constants satisfying $c < \beta_1 < \beta_2$. Recall that $F(v)$ and $v(q) = F^{-1}(q)$ are increasing functions of v and q, and $G(v)$ and $\mathbb{E}[(x - \theta)^+]$ are decreasing functions of v and θ. We can lower bound $h(\theta_2)$ by

$$
\begin{aligned}
h(\theta_2) &= \sum_{i=1}^{(1-\rho)n} F(\theta_2)^{i-1} \cdot \left(G(\theta_2)\cdot\theta_2 + \mathbb{E}[(x-\theta_2)^+]\right) \\
&\geq \sum_{i=1}^{(1-\rho)n} F\left(v(1-\frac{\beta_2}{n})\right)^{i-1} \cdot \left\{ G\left(v(1-\frac{\beta_1}{n})\right)\cdot v(1-\frac{\beta_2}{n}) + \mathbb{E}\left[\left(x - v(1-\frac{\beta_1}{n})\right)^+\right]\right\} \\
&\geq \sum_{i=1}^{(1-\rho)n} \left(1-\frac{\beta_2}{n}\right)^{i-1} \cdot \left(\frac{\beta_1}{n}\cdot v(1-\frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right) \\
&\geq \frac{1-\left(1-\frac{\beta_2}{n}\right)^{(1-\rho)n}}{\beta_2/n} \cdot \left(\frac{\beta_1}{n}\cdot v(1-\frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right) \\
&\geq \left(1-\left(1-\frac{\beta_2}{n}\right)^{(1-\rho)n}\right) \cdot \left(\frac{\beta_1}{\beta_2}\cdot v(1-\frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2}\right) \\
&\geq \left(1-e^{-\beta_2(1-\rho)}\right) \cdot \left(\frac{\beta_1}{\beta_2}\cdot v(1-\frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2}\right) := H(\beta_1,\beta_2).
\end{aligned}
$$

The important observation here is that $H(\beta_1,\beta_2)$ depends only on c, β_1, β_2, n and the distribution D, and is independent of θ_2. Moreover, it is linear in $v(1-\beta_2/n)$ and $\Delta(\beta_1)$. Note that conditioned on $\theta_2 \leq \theta_1$, the event of $v(1-\beta_2/n) < \theta_2 \leq v(1-\beta_1/n)$, where $c < \beta_1 < \beta_2$, happens with probability

$$
\begin{aligned}
&\Pr\left[v(1-\frac{\beta_2}{n}) < \theta_2 \leq v(1-\frac{\beta_1}{n}) \,\middle|\, \theta_2 \leq \theta_1\right] \\
&= \frac{\Pr[v(1-\frac{\beta_2}{n}) < \theta_2 \leq v(1-\frac{\beta_1}{n})]}{\Pr[\theta_2 \leq \theta_1]} = \frac{(1-\frac{\beta_1}{n})^{\rho n} - (1-\frac{\beta_2}{n})^{\rho n}}{(1-\frac{c}{n})^{\rho n}} := p(\beta_1,\beta_2).
\end{aligned}
$$

Therefore, for any sequence $c = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_k$ and sufficiently large n, we can lower bound the gain of the second phase by

$$
\begin{aligned}
\Lambda_2 &\geq \left(1-\frac{c}{n}\right)^{\rho n} \cdot \sum_{i=1}^{k} p(\beta_{i-1},\beta_i)\cdot H(\beta_{i-1},\beta_i) \\
&= \sum_{i=1}^{k} \left((1-\frac{\beta_{i-1}}{n})^{\rho n} - (1-\frac{\beta_i}{n})^{\rho n}\right) \cdot H(\beta_{i-1},\beta_i) \\
&\geq (1-\epsilon)\sum_{i=1}^{k}\left(e^{-\beta_{i-1}\rho} - e^{-\beta_i\rho}\right)\cdot\left(1-e^{-\beta_i(1-\rho)}\right)\cdot\left(\frac{\beta_{i-1}}{\beta_i}\cdot v(1-\frac{\beta_i}{n}) + \frac{\Delta(\beta_{i-1})}{\beta_i}\right). \qquad (4.2)
\end{aligned}
$$

So far, we have given a lower bound for ALG in Equations (4.1) and (4.2). In the following, we give upper bounds for OPT. We introduce some new variables δ_i defined as follows:

$$
\delta_i = \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x \geq t]\mathrm{d}t, \qquad \forall 0 \leq i \leq k-1.
$$

Recall that $x^* = \max_{i\in[n]}\{x_i\}$. We have the following upper bound for OPT:

$$
\begin{aligned}
\mathsf{OPT} &= \int_0^{v(1-\frac{\beta_k}{n})} \Pr[x^* \geq t]\mathrm{d}t + \sum_{i=0}^{k-1}\int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \geq t]\mathrm{d}t + \int_{v(1-\frac{\beta_0}{n})}^{\infty} \Pr[x^* \geq t]\mathrm{d}t \\
&\leq v(1-\frac{\beta_k}{n}) + \sum_{i=0}^{k-1}\delta_i + \Delta(\beta_0). \qquad (4.3)
\end{aligned}
$$

In the next three lemmas, we establish three upper bounds for δ_i, which, when combined with Equation (4.3), provide a tighter upper bound on OPT. Intuitively, the variable δ_i helps establish the connection between $v(1-\frac{\beta_i}{n})$ and $v(1-\frac{\beta_{i+1}}{n})$, and between $\Delta(\beta_{i+1})$ and $\Delta(\beta_i)$, which can be utilized to derive additional constraints on these variables.

Lemma 4.1. *For arbitrarily small $\epsilon > 0$ and sufficiently large n, we have*

$$\delta_i \le (1+\epsilon)\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)\cdot(1-e^{-\beta_{i+1}}). \tag{4.4}$$

Proof. By definition of δ_i, we have

$$\begin{aligned}\delta_i &= \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \ge t]\mathrm{d}t \le \Pr\left[x^* \ge v(1-\frac{\beta_{i+1}}{n})\right]\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)\\ &= \left(1-(1-\frac{\beta_{i+1}}{n})^n\right)\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)\\ &\le (1+\epsilon)\cdot(1-e^{-\beta_{i+1}})\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right),\end{aligned}$$

where the last inequality holds for all $n = \omega(\frac{1}{\epsilon})$. ■

Next, we observe a second upper bound for δ_i achieved by using the union bound.

Lemma 4.2. *We have the following inequality,*

$$\delta_i \le \Delta(\beta_{i+1}) - \Delta(\beta_i). \tag{4.5}$$

Proof. By definition of δ_i, we have

$$\begin{aligned}\delta_i &= \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \ge t]\mathrm{d}t = \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[(x_1 \ge t)\vee(x_2 \ge t)\vee\ldots\vee(x_n \ge t)]\mathrm{d}t\\ &\le n\cdot\int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr_{x\sim D}[x \ge t]\mathrm{d}t = n\cdot\int_{v(1-\frac{\beta_{i+1}}{n})}^{\infty} \Pr_{x\sim D}[x \ge t]\mathrm{d}t - n\cdot\int_{v(1-\frac{\beta_i}{n})}^{\infty} \Pr_{x\sim D}[x \ge t]\mathrm{d}t\\ &= \Delta(\beta_{i+1}) - \Delta(\beta_i),\end{aligned}$$

where the inequality holds by using the union bound. ■

In the following, we introduce a non-trivial third upper bound on δ_i.

Lemma 4.3. *For all $\zeta \in [\beta_i, \beta_{i+1}]$ and $\gamma = \frac{\zeta-\beta_i}{e^{-\zeta}-e^{-\beta_{i+1}}}$, we have*

$$\gamma\cdot\delta_i/(1+\epsilon) \le \Delta(\beta_{i+1}) - \Delta(\beta_i) - (\beta_i - \gamma(1-e^{-\zeta}))\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right). \tag{4.6}$$

Proof. Recall that $\Pr\left[x \ge v(1-\frac{\beta_i}{n})\right] = \beta_i/n$, $\Pr\left[x \ge v(1-\frac{\beta_{i+1}}{n})\right] = \beta_{i+1}/n$ and that $\Pr[x \ge t]$ is strictly decreasing in t. For all $\zeta \in [\beta_i, \beta_{i+1}]$, there must exist $t^* \in [v(1-\frac{\beta_{i+1}}{n}), v(1-\frac{\beta_i}{n})]$ such that $\Pr[x \ge t^*] = \zeta/n$. Thus we have,

$$\Pr[x \ge t] \in \begin{cases} [\frac{\zeta}{n}, \frac{\beta_{i+1}}{n}], & \forall t \in [v(1-\frac{\beta_{i+1}}{n}), t^*];\\ [\frac{\beta_i}{n}, \frac{\zeta}{n}), & \forall t \in (t^*, v(1-\frac{\beta_i}{n})].\end{cases}$$

Since $\Pr[x^* \ge t] = 1-(1-\Pr[x \ge t])^n$, for sufficiently large n, we have:

$$\begin{aligned}&\forall t \in [v(1-\frac{\beta_{i+1}}{n}), t^*], \quad \Pr[x^* \ge t] \le 1-\left(1-\frac{\beta_{i+1}}{n}\right)^n \le (1+\epsilon)\cdot(1-e^{-\beta_{i+1}});\\ &\forall t \in (t^*, v(1-\frac{\beta_i}{n})], \quad \Pr[x^* \ge t] < 1-\left(1-\frac{\zeta}{n}\right)^n \le (1+\epsilon)\cdot(1-e^{-\zeta}).\end{aligned}$$

For convenience, we define $\alpha := \frac{t^* - v(1-\frac{\beta_{i+1}}{n})}{v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})} \in [0,1]$. We have

$$\begin{aligned}\delta_i &= \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \geq t]\mathrm{d}t = \int_{v(1-\frac{\beta_{i+1}}{n})}^{t^*} \Pr[x^* \geq t]\mathrm{d}t + \int_{t^*}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \geq t]\mathrm{d}t \\ &\leq (1+\epsilon)\cdot\left(t^* - v(1-\frac{\beta_{i+1}}{n})\right)\cdot(1-e^{-\beta_{i+1}}) + (1+\epsilon)\cdot\left(v(1-\frac{\beta_i}{n}) - t^*\right)\cdot(1-e^{-\zeta}) \\ &= (1+\epsilon)\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)\cdot(1-e^{-\zeta} + \alpha\cdot(e^{-\zeta} - e^{\beta_{i+1}})).\end{aligned}$$

Rearranging the inequality gives

$$\alpha \geq \frac{1}{e^{-\zeta} - e^{-\beta_{i+1}}}\cdot\left(\frac{\delta_i}{(1+\epsilon)\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)} - (1-e^{-\zeta})\right). \tag{4.7}$$

On the other hand, with the definition of $\gamma = \frac{\zeta - \beta_i}{e^{-\zeta} - e^{-\beta_{i+1}}}$, we have

$$\begin{aligned}&n\cdot\int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x \geq t]\mathrm{d}t = n\cdot\int_{v(1-\frac{\beta_{i+1}}{n})}^{t^*} \Pr[x \geq t]\mathrm{d}t + n\cdot\int_{t^*}^{v(1-\frac{\beta_i}{n})} \Pr[x \geq t]\mathrm{d}t \\ &\geq \left(t^* - v(1-\frac{\beta_{i+1}}{n})\right)\cdot\zeta + \left(v(1-\frac{\beta_i}{n}) - t^*\right)\cdot\beta_i \\ &= \left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right)\cdot(\beta_i + \alpha\cdot(\zeta - \beta_i)) \\ &\geq \beta_i\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right) + \frac{\gamma}{1+\epsilon}\cdot\delta_i - \gamma(1-e^{-\zeta})\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right),\end{aligned}$$

where the last inequality holds from Inequality (4.7).

Recall that $n\cdot\int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x \geq t]\mathrm{d}t = \Delta(\beta_{i+1}) - \Delta(\beta_i)$. The above lower bound implies

$$\Delta(\beta_{i+1}) - \Delta(\beta_i) \geq (\beta_i - \gamma(1-e^{-\zeta}))\cdot\left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right) + \frac{\gamma}{1+\epsilon}\cdot\delta_i.$$

Rearranging the inequality completes the proof. ■

4.2. Factor revealing LP

We finish the proof of Theorem 4.1 by formulating the lower and upper bounds into a factor revealing LP and choosing appropriate parameters to obtain a lower bound on the competitive ratio. In the following, we fix $\beta_i = c + 0.01\cdot i$ for all $i = 0, 1, \ldots, k$. For ease of notation, we use $v_i = v(1-\beta_i/n)$ and $\Delta_i = \Delta(\beta_i)$, for all $i = 0, 1, \ldots, k$. Note that $c = \beta_0$, $\theta_1 = v(1-\beta_0/n) = v_0$ and $\Delta(c) = \Delta_0$. Recall that $v_0 \geq v_1 \geq \cdots \geq v_k$ and $\Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_k$. By fixing c and k, we uniquely define a sequence $\beta_0, \beta_1, \ldots, \beta_k$. However, the values of $\{v_i, \Delta_i, \delta_i\}_{0\leq i\leq k}$ depend on the unknown distribution D. Recall that we have established a lower bound for ALG and several upper bounds for OPT in Section 4.1. More importantly, these bounds are linear in $\{v_i, \Delta_i\}_{0\leq i\leq k}$ and $\{\delta_i\}_{0\leq i<k}$. By taking the lower bound on ALG as the objective and upper bounds on OPT as constraints, we construct an LP whose optimal value provides a lower bound for the competitive ratio of the algorithm. The LP variables are $\{v_i, \Delta_i\}_{0\leq i\leq k}$ and $\{\delta_i\}_{0\leq i<k}$.

Recall that we have lower bounded $\mathsf{ALG} = \Lambda_1 + \Lambda_2$, where

$$\begin{aligned}\Lambda_1 &\geq \left(1 - e^{-\beta_0\rho}\right)\cdot(v_0 + \Delta_0/\beta_0), \\ \Lambda_2 &\geq (1-\epsilon)\sum_{i=1}^{k}\left(e^{-\beta_{i-1}\rho} - e^{-\beta_i\rho}\right)\cdot\left(1 - e^{-\beta_i(1-\rho)}\right)\cdot((\beta_{i-1}/\beta_i)\cdot v_i + \Delta_{i-1}/\beta_i),\end{aligned}$$

are both linear in $\{v_i, \Delta_i\}_{0\leq i\leq k}$. We have also upper bounded OPT by

$$\mathsf{OPT} \leq v(1-\frac{\beta_k}{n}) + \sum_{i=0}^{k-1}\delta_i + \Delta_0,$$

and each δ_i, where $i \in \{0, 1, \dots, k-1\}$, by (recall that $\gamma = \frac{\zeta - \beta_i}{e^{-\zeta} - e^{-\beta_{i+1}}}$)

$$\delta_i \le (1+\epsilon) \cdot (1 - e^{-\beta_{i+1}}) \cdot (v_i - v_{i+1}),$$
$$\delta_i \le \Delta(\beta_{i+1}) - \Delta(\beta_i),$$
$$\gamma \cdot \delta_i / (1+\epsilon) \le \Delta_{i+1} - \Delta_i - (\beta_i - \gamma(1 - e^{-\zeta})) \cdot (v_i - v_{i+1}), \quad \forall \zeta \in [\beta_i, \beta_{i+1}].$$

Since $\epsilon > 0$ can be arbitrarily small when $n \to \infty$, the terms $(1-\epsilon)$ and $(1+\epsilon)$ can be removed. The factor revealing LP is as follows.

$$\begin{aligned}
\text{minimize} \quad & \Lambda_1 + \Lambda_2 \\
\text{subject to} \quad & \Lambda_1 \ge (1 - e^{-\beta_0 \rho}) \cdot (v_0 + \Delta_0/\beta_0), \\
& \Lambda_2 \ge \textstyle\sum_{i=1}^{k} \left(e^{-\beta_{i-1}\rho} - e^{-\beta_i \rho}\right) \cdot \left(1 - e^{-\beta_i(1-\rho)}\right) \cdot ((\beta_{i-1}/\beta_i) \cdot v_i + \Delta_{i-1}/\beta_i), \\
& 1 \le v_k + \textstyle\sum_{i=0}^{k-1} \delta_i + \Delta_0, \\
& \delta_i \le \left(1 - e^{-\beta_{i+1}}\right) \cdot (v_i - v_{i+1}), \qquad \forall 0 \le i \le k-1, \\
& \delta_i \le \Delta_{i+1} - \Delta_i, \qquad \forall 0 \le i \le k-1, \\
& \gamma \cdot \delta_i \le \Delta_{i+1} - \Delta_i - (\beta_i - \gamma(1 - e^{-\zeta})) \cdot (v_i - v_{i+1}), \ \forall 0 \le i \le k-1, \zeta \in [\beta_i, \beta_{i+1}], \\
& v_0 \ge v_1 \ge \cdots \ge v_k \ge 0, \\
& \Delta_0, \delta_0, \dots, \delta_{k-1} \ge 0,
\end{aligned}$$

where ρ, $\{\beta_i\}_{0 \le i \le k}$ and $\gamma = \frac{\zeta - \beta_i}{e^{-\zeta} - e^{-\beta_{i+1}}}$ are constants; Λ_1, Λ_2, $\{v_i, \Delta_i\}_{0 \le i \le k}$, $\{\delta_i\}_{0 \le i < k}$ are variables of the LP.

Theorem 4.2. *Given parameters c and ρ of the obverse-and-accept algorithm, the optimal value of the above LP provides a lower bound for the competitive ratio of the obverse-and-accept algorithm when $n \to \infty$.*

Proof. The objective of the LP comes from Equations (4.1) and (4.2), the lower bound for ALG, where we omit the $(1-\epsilon)$ term since we have $\epsilon \to 0$ when $n \to \infty$. We will also omit the $(1+\epsilon)$ terms in the following for the same reason. By scaling we can assume w.l.o.g. that $\mathsf{OPT} = 1$. Note that our algorithm does not need to know the value of OPT to decide the parameters. Therefore, Equation (4.3) gives the first constraint. The three proceeding sets of constraints follow from Equations (4.4), (4.5) and (4.6), the three upper bounds on δ_i that we have established in Section 4.1. The last two sets of constraints follow straightforwardly from the definitions of the parameters.

Observe that every distribution D induces a set of variables $\{v_i, \Delta_i\}_{0 \le i \le k}$ and $\{\delta_i\}_{0 \le i < k}$, which will form a feasible solution to the LP as they must abide by the corresponding constraints. Since the objective of any feasible solution induced by distribution D provides a lower bound on ALG/OPT, the optimal (minimum) objective of the LP provides a lower bound on the competitive ratio, i.e., the worst-case performance against all distributions. ■

With the above result, it remains to set the appropriate parameters of the obverse-and-accept algorithm such that the optimal value of the LP is as large as possible.

The following claim is verified by the GNU Linear Programming Kit (GLPK),[4] which completes the proof of Theorem 4.1.

Claim 4.1. *By fixing constants $c = 0.6790, \rho = 0.6181$, $k = 2500$, $\beta_i = c + 0.01 \cdot i$ for all $0 \le i \le k$, the optimal solution to the above LP has an objective of at least 0.6785. For other values of $c \in (0.006, 2.5)$, by appropriately setting the other parameters (depending on c), we can lower bound the competitive ratio as shown in Fig. 1.1.*

4.3. Upper bound on the competitive ratio of the algorithm

To complement our lower bound on the competitive ratio of the algorithm, we also explore upper bounds for this type of algorithm. In particular, we show the following.

Lemma 4.4. *Algorithm 1 (with any c and ρ) is at most 0.6915-competitive.*

Proof. Consider any observe-and-accept algorithm with parameters c and ρ. In the following, we construct two distributions D_1 and D_2, and show that the performance of the algorithm on one of the two distributions is at most 0.6915 and satisfy $\mathsf{OPT} = 1$.

[4] GLPK: https://www.gnu.org/software/glpk/.

Let the first distribution D_1 be a uniform distribution over $[1-\epsilon, 1+\epsilon]$, for some arbitrarily small $\epsilon > 0$. Then we have

$$\begin{aligned}\mathsf{ALG} &\le 1+\epsilon - \Pr[\tau > n] = 1+\epsilon - \Pr[\forall i \le \rho n, x_i \le \theta_1 \text{ and } \forall i > \rho n, x_i < \max_{j\le \rho n}\{x_j\}] \\ &= 1+\epsilon-(1-\frac{c}{n})^n\cdot\rho \le 1+\epsilon-\rho\cdot e^{-c}, \end{aligned}\tag{4.8}$$

where the last equality holds because $\tau > n$ happens if and only if all variables are below θ_1 and the maximum of them appears in the first $\rho \cdot n$ variables.

Let the second distribution D_2 be defined as follows. Let $M \gg n$ be an arbitrarily large number and $\epsilon > 0$ be arbitrarily small. With probability $\frac{1}{n\cdot M}$, x is uniformly distributed over $[M-\epsilon, M+\epsilon]$; with probability $1-\frac{1}{n\cdot M}$, x is uniformly distributed over $[0,\epsilon]$.[5] Note that when $M \to \infty$ and $\epsilon \to 0$,

$$\mathsf{OPT} \approx M\cdot\left(1-(1-\frac{1}{n\cdot M})^n\right) = M\cdot\frac{1}{M} = 1.$$

Observe that for distribution D_2, we have $\theta_1 \le \epsilon$ and $\Delta(c) = n\cdot\mathbb{E}[(x-\theta_1)^+] \approx 1$. Since θ_1 is close to 0, the gain of the algorithm is determined by how likely the algorithm accepts a variable with a value close to M. Therefore, we can upper bound ALG by

$$\mathsf{ALG} < \epsilon + M\cdot\sum_{i=1}^{n}\Pr[\tau > i-1]\cdot\Pr[x_i \ge M-\epsilon].$$

Note that $\Pr[x_i \ge M-\epsilon] = \frac{1}{n\cdot M}$ for all i. For $i \le \rho n$, we have

$$\Pr[\tau > i-1] = (1-\frac{c}{n})^{i-1}.$$

For $i > \rho n$, the event $\tau > i-1$ happens if and only if all variables $x_1,\dots,x_{i-1}$ are below θ_1, and the maximum of them appears before index ρn. Hence,

$$\Pr[\tau > i-1] = (1-\frac{c}{n})^{i-1}\cdot\frac{\rho n}{i-1}.$$

Putting everything together, as $n \to \infty$, we get

$$\begin{aligned}\mathsf{ALG} &< \epsilon + \frac{1}{n}\cdot\sum_{i=1}^{n}\Pr[\tau > i-1] = \epsilon + \frac{1}{n}\cdot\sum_{i=1}^{\rho n}(1-\frac{c}{n})^{i-1} + \frac{1}{n}\cdot\sum_{i=\rho n+1}^{n}(1-\frac{c}{n})^{i-1}\cdot\frac{\rho n}{i-1} \\ &= \epsilon + \frac{1-(1-\frac{c}{n})^{\rho n}}{c} + \sum_{i=\rho n+1}^{n}(1-\frac{c}{n})^{i-1}\cdot\frac{\rho}{i-1} \approx \epsilon + \frac{1-e^{-c\rho}}{c} + \rho\cdot\int_{\rho}^{1}\frac{e^{-ct}}{t}\mathrm{d}t. \end{aligned}\tag{4.9}$$

It is not difficult to check that when $c > 1$ the above upper bound is less than $1-1/e$. Thus any observe-and-accept algorithm with a competitive ratio $> 1-1/e$ must have $c < 1$. It remains to show that when $\epsilon \to 0$, $n \to \infty$ and $M \to \infty$, for any $c, \rho \in [0,1]$, at least one of the two upper bounds given in Equations (4.8) and (4.9) has a value less than 0.6915:

$$\max_{c,\rho\in[0,1]}\left\{\min\left\{1-\rho\cdot e^{-c}, \frac{1-e^{-c\rho}}{c} + \rho\cdot\int_{\rho}^{1}\frac{e^{-ct}}{t}\mathrm{d}t\right\}\right\} \le 0.6915. \tag{4.10}$$

We prove Equation (4.10) by discretizing the domain for c and ρ and using computational tools. For example, letting $I = \{\frac{i}{10^5} : 1 \le i \le 10^5\}$, we can check that the LHS of Equation (4.10) can be approximated by replacing "$c, \rho \in [0,1]$" with "$c, \rho \in I$", with an additive error of at most 3×10^{-5}. By enumerating all $c, \rho \in I$ and recording the maximum, we get an upper bound of 0.69145, which is achieved when $c = 0.38007$ and $\rho = 0.45123$. Combining the above discretized upper bound with the additive error of 3×10^{-5} gives an upper bound of $0.69148 < 0.6915$. ■

5. Multi-phase observe-and-accept algorithm

In this section, we further leverage the power of a single data point by letting the algorithm adopt more phases. Suppose that a data point $(v, 1-c/n)$ is given. We consider the m-phase ($m = 3,\dots,n$) algorithm with a single data point: in the first phase, we use the threshold $\theta_1 = v$ from the given data point; in the subsequent phase $t \in \{2,\dots,m\}$, we use the maximum realization of the previous phase as the threshold. In each phase, we accept the first variable whose realization is at least the threshold and terminate the algorithm; otherwise, we go to the next phase. Our analysis of the gain of the algorithm in the first two phases is similar to that

[5] Recall that we require that the CDF of the distribution is strictly increasing while D_2 is not. However, this can be easily fixed by assuming that with an arbitrarily small probability, x is uniformly distributed over $(\epsilon, M-\epsilon)$, which has a negligible effect on the distribution and our analysis.

from the previous section. However, for the later phases, the distribution of the threshold is more difficult to characterize, which introduces some complications to the formulation of the lower bounds.

5.1. Three-phase observe-and-accept algorithm

We first consider the 3-phase observe-and-accept algorithm with a single data point. The steps of the algorithm are summarized in Algorithm 2.

Algorithm 2: 3-Phase Observe-and-Accept.

Input: Number of items n, parameters c, ρ_1 and ρ_2.
Output: A variable selected by the seller.
Set $\theta_1 := v(1-c/n)$;
for $i = 1, 2, \ldots, \rho_1 \cdot n$ **do**
  if $x_i > \theta_1$ **then**
    return x_i and **terminate**; // phase 1
Set $\theta_2 := \max_{i \le \rho_1 \cdot n}\{x_i\}$;
for $i = \rho_1 \cdot n + 1, \ldots, \rho_2 \cdot n$ **do**
  if $x_i > \theta_2$ **then**
    return x_i and **terminate**; // phase 2
Set $\theta_3 := \max_{\rho_1 \cdot n+1 \le i \le (\rho_1+\rho_2)\cdot n}\{x_i\}$;
for $i = (\rho_1+\rho_2) \cdot n + 1, \ldots, n$ **do**
  if $x_i > \theta_3$ **then**
    return x_i and **terminate**; // phase 3

Theorem 5.1. *Algorithm 2 achieves a competitive ratio of* 0.6843 *when* $c = 0.7148$.

We use $\Lambda_1, \Lambda_2, \Lambda_3$ to denote the expected gain of the algorithm in phases 1, 2, and 3, respectively. Now we lower bound $\Lambda_1, \Lambda_2, \Lambda_3$ one-by-one. For phase 1,

$$\begin{aligned}\Lambda_1 &= \sum_{i=1}^{\rho_1 \cdot n} \Pr[\tau = i] \cdot \mathbb{E}[x_i | \tau = i] \\ &= \sum_{i=1}^{\rho_1 \cdot n} \left(1 - \frac{c}{n}\right)^{i-1} \cdot \left(\frac{c}{n} \cdot \theta_1 + \frac{\Delta(c)}{n}\right) \approx (1 - e^{-c\rho_1}) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right).\end{aligned}$$

For phase 2, we abuse the notion and use $h(\theta_2)$ to denote the expected gain of the algorithm in phase 2, conditioned on $\tau > \rho_1 n$ and the threshold being θ_2. Note that only when $\theta_2 \le \theta_1$, the algorithm enters phase 2. Hence,

$$\Lambda_2 = \Pr[\tau > \rho_1 n] \cdot \mathbb{E}_{\theta_2}[h(\theta_2)].$$

Moreover, given a realization of θ_2 ($\theta_2 \le \theta_1$), we can express the expected gain of the algorithm in the second phase using a similar argument as before:

$$\begin{aligned}h(\theta_2) &= \sum_{i=\rho_1 n+1}^{(\rho_1+\rho_2)n} \Pr[\tau = i \mid \tau > \rho_1 n] \cdot \mathbb{E}[x_i \mid \tau = i] \\ &= \sum_{i=\rho_1 n+1}^{(\rho_1+\rho_2)n} \frac{\Pr[\tau = i]}{\Pr[\tau > \rho_1 n]} \cdot \mathbb{E}[x_i \mid \tau = i] \\ &= \sum_{i=1}^{\rho_2 n} F(\theta_2)^{i-1} \cdot (G(\theta_2) \cdot \theta_2 + \mathbb{E}[(x - \theta_2)^+]).\end{aligned}$$

Given any constants β_1 and β_2 satisfying $c < \beta_1 < \beta_2$, if $v(1 - \frac{\beta_2}{n}) < \theta_2 \le v(1 - \frac{\beta_1}{n})$, we can lower bound $h(\theta_2)$ by

$$\begin{aligned}h(\theta_2) &\ge \sum_{i=1}^{\rho_2 n} \left(1 - \frac{\beta_2}{n}\right)^{i-1} \cdot \left(\frac{\beta_1}{n} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right) \\ &= \frac{1 - (1 - \frac{\beta_2}{n})^{\rho_2 n}}{\frac{\beta_2}{n}} \cdot \left(\frac{\beta_1}{n} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right)\end{aligned}$$

$$= \left(1 - \left(1 - \frac{\beta_2}{n}\right)^{\rho_2 n}\right) \cdot \left(\frac{\beta_1}{\beta_2} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2}\right)$$
$$\approx \left(1 - e^{-\beta_2\rho_2}\right) \cdot \left(\frac{\beta_1}{\beta_2} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2}\right) := H(\beta_1, \beta_2).$$

Notice that $H(\beta_1,\beta_2)$ is linear in $v(1-\frac{\beta_2}{n})$ and $\Delta(\beta_1)$. Conditioned on $\theta_2 \le \theta_1$, the probability that $v(1-\frac{\beta_2}{n}) < \theta_2 \le v(1-\frac{\beta_1}{n}) \le \theta_1$ is

$$\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_2 \le v(1 - \frac{\beta_1}{n}) \mid \tau > \rho_1 n\right] = \frac{\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_2 \le v(1 - \frac{\beta_1}{n}) \wedge \tau > \rho_1 n\right]}{\Pr[\tau > \rho_1 n]}$$
$$= \frac{(1 - \frac{\beta_1}{n})^{\rho_1 n} - (1 - \frac{\beta_2}{n})^{\rho_1 n}}{\Pr[\tau > \rho_1 n]} := p(\beta_1, \beta_2).$$

Given above, for any sequence of constants $c = \beta_0 < \beta_1 < \cdots < \beta_k$, we have

$$\Lambda_2 \ge \Pr[\tau > \rho_1 n] \cdot \sum_{i=1}^{k} \left(p(\beta_{i-1}, \beta_i) \cdot H(\beta_{i-1}, \beta_i)\right)$$
$$\approx \sum_{i=1}^{k} \left(e^{-\beta_{i-1}\rho_1} - e^{-\beta_i\rho_1}\right) \cdot \left(1 - e^{-\beta_i\rho_2}\right) \cdot \left(\frac{\beta_{i-1}}{\beta_i} \cdot v(1 - \frac{\beta_i}{n}) + \frac{\Delta(\beta_{i-1})}{\beta_i}\right).$$

We use $w(\theta_3)$ to denote the expected gain of the algorithm in phase 3, conditioned on $\tau > (\rho_1 + \rho_2)n$ and the threshold of phase 3 being θ_3. We have

$$\Lambda_3 = \Pr[\tau > (\rho_1 + \rho_2)n] \cdot \mathbb{E}_{\theta_3}[w(\theta_3)],$$

and

$$w(\theta_3) = \sum_{i=(\rho_1+\rho_2)n+1}^{n} \Pr[\tau = i \mid \tau > (\rho_1 + \rho_2)n] \cdot \mathbb{E}[x_i \mid \tau = i]$$
$$= \sum_{i=1}^{(1-\rho_1-\rho_2)n} F(\theta_3)^{i-1} \cdot (G(\theta_3) \cdot \theta_3 + \mathbb{E}[(x - \theta_3)^+]).$$

Given any constants β_1 and β_2 satisfying $c < \beta_1 < \beta_2$, if we have $v(1-\frac{\beta_2}{n}) < \theta_3 \le v(1-\frac{\beta_1}{n})$, then we can lower bound $w(\theta_3)$ by

$$w(\theta_3) \ge \sum_{i=1}^{(1-\rho_1-\rho_2)n} \left(1 - \frac{\beta_2}{n}\right)^{i-1} \cdot \left(\frac{\beta_1}{n} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right)$$
$$\approx \left(1 - e^{-\beta_2(1-\rho_1-\rho_2)}\right) \cdot \left(\frac{\beta_1}{\beta_2} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2}\right) := W(\beta_1, \beta_2).$$

Notice that $W(\beta_1,\beta_2)$ is linear in $v(1-\frac{\beta_2}{n})$ and $\Delta(\beta_1)$. Conditioned on $\tau > (\rho_1+\rho_2)n$, the probability that $v(1-\frac{\beta_2}{n}) < \theta_3 \le v(1-\frac{\beta_1}{n})$ is

$$\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_3 \le v(1 - \frac{\beta_1}{n}) \mid \tau > (\rho_1 + \rho_2)n\right]$$
$$= \frac{\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_3 \le v(1 - \frac{\beta_1}{n}) \wedge \tau > (\rho_1 + \rho_2)n\right]}{\Pr[\tau > (\rho_1 + \rho_2)n]}$$
$$= \frac{\Pr\left[\theta_3 \le v(1 - \frac{\beta_1}{n}) \wedge \tau > (\rho_1 + \rho_2)n\right] - \Pr\left[\theta_3 \le v(1 - \frac{\beta_2}{n}) \wedge \tau > (\rho_1 + \rho_2)n\right]}{\Pr[\tau > (\rho_1 + \rho_2)n]}.$$

For notational convenience, we use $q(\beta_i)$ to denote $\Pr\left[\theta_3 \le v(1-\frac{\beta_i}{n}) \wedge \tau > (\rho_1+\rho_2)n\right]$. We can rewrite the above equation as

$$\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_3 \le v(1 - \frac{\beta_1}{n}) \mid \tau > (\rho_1 + \rho_2)n\right] = \frac{q(\beta_1) - q(\beta_2)}{\Pr[\tau > (\rho_1 + \rho_2)n]}.$$

Focusing on $q(\beta_1)$, we have

$$\Pr\left[\theta_3 \le v(1 - \frac{\beta_1}{n}) \wedge \tau > (\rho_1 + \rho_2)n\right]$$

$$
\begin{aligned}
&= \Pr\left[\max_{i\le\rho_1 n}\{x_i\} \le v(1-\frac{\beta_1}{n}) \wedge \max_{i\in(\rho_1 n,(\rho_1+\rho_2)n]}\{x_i\} \le \max_{i\le\rho_1 n}\{x_i\}\right] \\
&\quad + \Pr\left[v(1-\frac{\beta_1}{n}) < \max_{i\le\rho_1 n}\{x_i\} \le \theta_1 \wedge \max_{i\in(\rho_1 n,(\rho_1+\rho_2)n]}\{x_i\} \le v(1-\frac{\beta_1}{n})\right] \\
&= \left(1-\frac{\beta_1}{n}\right)^{(\rho_1+\rho_2)n} \cdot \frac{\rho_1}{\rho_1+\rho_2} + \left(\left(1-\frac{c}{n}\right)^{\rho_1 n} - \left(1-\frac{\beta_1}{n}\right)^{\rho_1 n}\right) \cdot \left(1-\frac{\beta_1}{n}\right)^{\rho_2 n} \\
&\approx e^{-(\rho_1+\rho_2)\beta_1} \cdot \frac{\rho_1}{\rho_1+\rho_2} + \left(e^{-c\rho_1} - e^{-\beta_1\rho_1}\right) \cdot e^{-\beta_1\rho_2}.
\end{aligned}
$$

Via a similar method, we can get the expression of $q(\beta_i)$, for all $i = 2, \dots, k$. Consequently, for any sequence of constants $c = \beta_0 < \beta_1 < \beta_2 < \dots < \beta_k$, we have

$$
\begin{aligned}
\Lambda_3 &\ge \Pr[\tau > (\rho_1+\rho_2)n] \cdot \sum_{i=1}^{k} W(\beta_{i-1},\beta_i) \cdot \left(\frac{q(\beta_{i-1}) - q(\beta_i)}{\Pr[\tau > (\rho_1+\rho_2)n]}\right) \\
&= \sum_{i=1}^{k} W(\beta_{i-1},\beta_i) \cdot \left(q(\beta_{i-1}) - q(\beta_i)\right).
\end{aligned}
$$

Plugging in the expressions of $W(\beta_{i-1},\beta_i)$ and $q(\beta_i)$, we can lower bound Λ_3 by

$$
\begin{aligned}
\Lambda_3 \ge \sum_{i=1}^{k} &\left(1 - e^{-(1-\rho_1-\rho_2)\beta_i}\right) \cdot \left(\frac{\beta_{i-1}}{\beta_i} \cdot v(1-\frac{\beta_i}{n}) + \frac{\Delta(\beta_{i-1})}{\beta_i}\right) \\
&\cdot \left(e^{-\rho_1 c}\left(e^{-\rho_2\beta_{i-1}} - e^{-\rho_2\beta_i}\right) - \frac{\rho_2}{\rho_1+\rho_2}\left(e^{-(\rho_1+\rho_2)\beta_{i-1}} - e^{-(\rho_1+\rho_2)\beta_i}\right)\right).
\end{aligned}
$$

So far, we have provided a lower bound for ALG. In the following, we give upper bounds for OPT. Following similar analyses as in Section 4, we define

$$
\delta_i = \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x \ge t]\mathrm{d}t, \qquad \forall 0 \le i \le k-1.
$$

Then we have

$$
\begin{aligned}
\mathsf{OPT} &= \int_{0}^{v(1-\frac{\beta_k}{n})} \Pr[x^* \ge t]\mathrm{d}t + \sum_{i=0}^{k-1} \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \ge t]\mathrm{d}t + \int_{v(1-\frac{\beta_0}{n})}^{\infty} \Pr[x^* \ge t]\mathrm{d}t \\
&\le v(1-\frac{\beta_k}{n}) + \sum_{i=0}^{k-1} \delta_i + \Delta(\beta_0).
\end{aligned}
$$

As in Section 4, for arbitrarily small $\epsilon > 0$, sufficiently large n and $i \in \{0, 1, \dots, k-1\}$, we have

$$
\begin{aligned}
&\delta_i \le (1+\epsilon) \cdot (1 - e^{-\beta_{i+1}}) \cdot \left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right), \\
&\delta_i \le \Delta(\beta_{i+1}) - \Delta(\beta_i).
\end{aligned}
$$

For all $\zeta \in [\beta_i, \beta_{i+1}]$ and $\gamma = \frac{\zeta-\beta_i}{e^{-\zeta}-e^{-\beta_{i+1}}}$, it holds that

$$
\gamma \cdot \delta_i/(1+\epsilon) \le \Delta(\beta_{i+1}) - \Delta(\beta_i) - (\beta_i - \gamma(1-e^{-\zeta})) \cdot \left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right).
$$

As before, we construct a factor revealing LP to lower bound the competitive ratio. The next claim is verified by the GLPK, which completes the proof of Theorem 5.1.

Claim 5.1. *By fixing constants* $c = 0.7148, \rho_1 = 0.6969, \rho_2 = 0.2361,$ $k = 2500,$ $\beta_i = c + 0.010 \cdot i$ *for all* $0 \le i \le k$*, the optimal objective to the corresponding LP is at least* 0.6843.

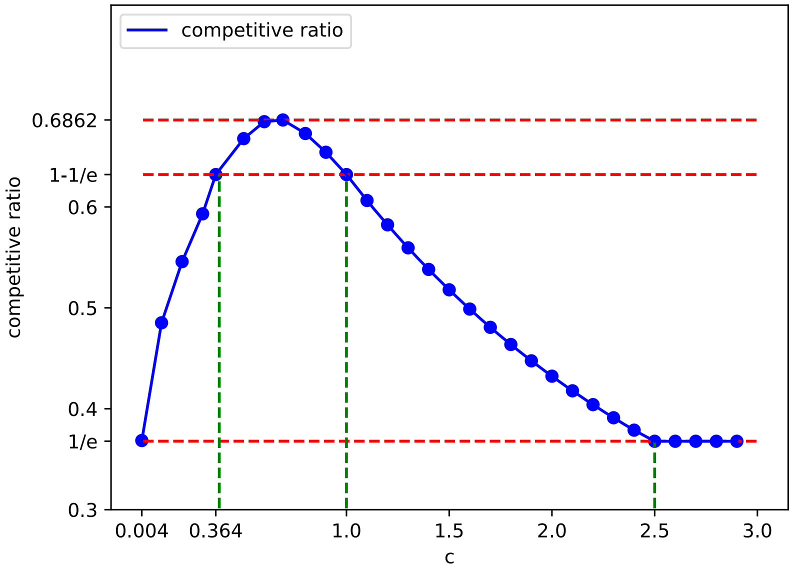

Fig. 5.1. Competitive ratio given data point $(v, 1 - c/n)$, where for $c \in (0.004, 2.5)$ the ratio is obtained by our observe-and-accept algorithm while for $c > 2.5$ and $c < 0.004$ the ratio is obtained using the optimal algorithm for the secretary problem.

5.2. Multi-phase observe-and-accept algorithm

In this section, we analyze the general multi-phase observe-and-accept algorithm ($m \geq 4$). While our analysis framework applies to all $m \geq 4$, we observe that the improvement in the competitive ratio becomes marginal for $m \geq 5$. Therefore, in the following, we focus on the result obtained from the 4-phase algorithm.

Theorem 5.2. *The* 4*-phase observe-and-accept algorithm achieves a competitive ratio of* 0.6862 *when* $c = 0.6906$*. For other values of c, the lower bounds for the competitive ratios are shown in Fig. 5.1.*

We remark that the median of the maximum random variable $\max_{1\leq i\leq n}\{X_i\}$ corresponds to a data point $(v, 1 - c/n)$ with $c = \ln(2) \approx 0.693$, which is very close to the peak of Fig. 5.1. In other words, given the median of the maximum random variable, we can achieve a competitive ratio of at least 0.68.

In the following two subsections, we prove Theorem 5.2.

5.2.1. Bounding ALG and OPT

We use Λ_i to denote the expected gain of the algorithm in phase i. Similar to the 3-phase algorithm, we have

$$\Lambda_1 \approx (1 - e^{-c\rho_1}) \cdot \left(\theta_1 + \frac{\Delta(c)}{c}\right),$$

$$\Lambda_2 \geq \sum_{i=1}^{k} \left(e^{-\beta_{i-1}\rho_1} - e^{-\beta_i\rho_1}\right) \cdot \left(1 - e^{-\beta_i\rho_2}\right) \cdot \left(\frac{\beta_{i-1}}{\beta_i} \cdot v(1 - \frac{\beta_i}{n}) + \frac{\Delta(\beta_{i-1})}{\beta_i}\right).$$

In the following, we lower bound Λ_t for $t \in \{3, \dots, m\}$. We define $\rho_{\leq t-1} = \sum_{i=1}^{t-1} \rho_i$ and $\theta_t = \max_{i\in[\rho_{t-2}\cdot n+1, \rho_{t-1}\cdot n]}\{x_i\}$, for the ease of notation. We use $d_t(\theta_t)$ to denote the expected gain of the algorithm in phase t, conditioned on $\tau > \rho_{\leq t-1} n$ and the threshold of phase t being θ_t. Note that only when $\theta_t \leq \theta_{t-1}$, the algorithm enters phase t, therefore we have

$$\Lambda_t = \Pr[\tau > (\rho_{\leq t-1})n] \cdot \mathbb{E}_{\theta_t}[d_t(\theta_t)].$$

Moreover, given a realization of $\theta_t \leq \theta_{t-1}$, we can express the gain of the algorithm in phase t using a similar argument as before:

$$d_t(\theta_t) = \sum_{i=1}^{\rho_t n} F(\theta_t)^{i-1} \cdot (G(\theta_t) \cdot \theta_t + \mathbb{E}[(x - \theta_t)^+]).$$

Given any constants β_1 and β_2 satisfying $c < \beta_1 < \beta_2$, if we have $v(1 - \frac{\beta_2}{n}) < \theta_t \leq v(1 - \frac{\beta_1}{n})$, then we can lower bound $d_t(\theta_t)$ by

$$d_t(\theta_t) \geq \sum_{i=1}^{\rho_t n} \left(1 - \frac{\beta_2}{n}\right)^{i-1} \cdot \left(\frac{\beta_1}{n} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{n}\right)$$

$$\approx \left(1 - e^{-\beta_2 \rho_t}\right) \cdot \left(\frac{\beta_1}{\beta_2} \cdot v(1 - \frac{\beta_2}{n}) + \frac{\Delta(\beta_1)}{\beta_2} \right) := D_t(\beta_1, \beta_2).$$

Conditioned on $\tau > \rho_{\le t-1} n$, the probability that $v(1 - \frac{\beta_2}{n}) < \theta_t \le v(1 - \frac{\beta_1}{n})$ is

$$\begin{aligned}
&\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_t \le v(1 - \frac{\beta_1}{n}) | \tau > \rho_{t-1} n \right] \\
=& \frac{\Pr\left[\theta_t \le v(1 - \frac{\beta_1}{n}) \wedge \tau > \rho_{\le t-1} n\right] - \Pr\left[\theta_t \le v(1 - \frac{\beta_2}{n}) \wedge \tau > \rho_{\le t-1} n\right]}{\Pr[\tau > \rho_{\le t-1} n]}.
\end{aligned}$$

With a slight abuse of notion, we use $q(\beta_i)$ to denote $\Pr\left[\theta_t \le v(1 - \frac{\beta_i}{n}) \mid \tau > \rho_{\le t-1} n\right]$. We can rewrite the above equality as

$$\Pr\left[v(1 - \frac{\beta_2}{n}) < \theta_t \le v(1 - \frac{\beta_1}{n}) \mid \tau > \rho_{t-1} n \right] = \frac{q(\beta_1) - q(\beta_2)}{\Pr[\tau > \rho_{\le t-1} n]}.$$

For $q(\beta_1)$, we have

$$\begin{aligned}
&\Pr\left[\theta_t \le v(1 - \frac{\beta_1}{n}) \wedge \tau > \rho_{\le t-1} n\right] \\
=& \Bigg(\Pr\left[\theta_t \le v(1 - \frac{\beta_1}{n}) \le \theta_{t-1} \le \cdots \le \theta_1\right] + \Pr\left[\theta_t \le \theta_{t-1} < v(1 - \frac{\beta_1}{n}) \le \cdots \le \theta_1\right] \\
&+ \cdots + \Pr\left[\theta_t \le \theta_{t-1} < \cdots < v(1 - \frac{\beta_1}{n}) \le \theta_2 \le \theta_1\right] \Bigg) \\
&+ \Pr\left[\theta_t \le \theta_{t-1} \le \cdots \le \theta_2 \le v(1 - \frac{\beta_1}{n}) \le \theta_1\right] \\
=& \sum_{i=1}^{t-2} \underbrace{\Pr\left[v(1 - \frac{\beta_1}{n}) \le \theta_{t-i} \le \cdots \le \theta_1\right] \cdot \Pr\left[\theta_t \le \cdots \le \theta_{t+1-i} < v(1 - \frac{\beta_1}{n})\right]}_{\text{(Complex Part)}} \\
&+ \underbrace{\Pr\left[\theta_t \le \cdots \le \theta_2 \le v(1 - \frac{\beta_1}{n}) \le \theta_1\right]}_{\text{(Simple Part)}}.
\end{aligned}$$

Complex part This part can be written as

$$\begin{aligned}
&\left((1 - \frac{c}{n})^{\rho_{\le t-1-i} n} - (1 - \frac{\beta_1}{n})^{\rho_{\le t-1-i} n} \right) \cdot \frac{\rho_1}{\rho_1 + \rho_2 + \cdots + \rho_{t-1-i}} \\
&\cdot \frac{\rho_2}{\rho_2 + \cdots + \rho_{t-1-i}} \cdot \ldots \cdot \frac{\rho_{t-2-i}}{\rho_{t-2-i} + \rho_{t-1-i}} \cdot (1 - \frac{\beta_1}{n})^{(\rho_{t-i} + \rho_{t-i+1} + \cdots + \rho_{t-1}) n} \\
&\cdot \frac{\rho_{t-i}}{\rho_{t-i} + \cdots + \rho_{t-1}} \cdot \ldots \cdot \frac{\rho_{t-2}}{\rho_{t-1} + \rho_{t-2}} \\
=& \left((1 - \frac{c}{n})^{\rho_{\le t-1-i} n} - (1 - \frac{\beta_1}{n})^{\rho_{\le t-1-i} n} \right) \cdot \prod_{j=1}^{t-1-i} \frac{\rho_j}{\rho_{\le t-1-i} - \rho_{\le j-1}} \\
&\cdot \left(1 - \frac{\beta_1}{n}\right)^{(\rho_{\le t-1} - \rho_{\le t-1-i}) n} \cdot \prod_{j=t-i}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}}.
\end{aligned}$$

Simple part We write this part as follows:

$$\Pr\left[\theta_t \le \theta_{t-1} \le \cdots \le \theta_2 \le v(1 - \frac{\beta_1}{n}) \le \theta_1\right] = \left(1 - \frac{\beta_1}{n}\right)^{\rho_{\le t-1} n} \cdot \prod_{j=1}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}}.$$

Putting things together We can rewrite $q(\beta_1)$ as

$$\begin{aligned}
&\Pr\left[\theta_t \le v(1 - \frac{\beta_1}{n}) \wedge \tau > \rho_{\le t-1} n\right] \\
=& (1 - \frac{\beta_1}{n})^{\rho_{\le t-1} n} \cdot \prod_{j=1}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} + \sum_{i=1}^{t-2} \Bigg(\left[(1 - \frac{c}{n})^{\rho_{\le t-1-i} n} - (1 - \frac{\beta_1}{n})^{\rho_{\le t-1-i} n} \right]
\end{aligned}$$

$$\cdot \prod_{j=1}^{t-1-i} \frac{\rho_j}{\rho_{\le t-1-i} - \rho_{\le j-1}} \cdot (1 - \frac{\beta_1}{n})^{(\rho_{\le t-1} - \rho_{\le t-1-i})n} \cdot \prod_{j=t-i}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} \Bigg)$$

$$\approx e^{-\beta_1 \rho_{\le (t-1)}} \cdot \prod_{j=1}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} + \sum_{i=1}^{t-2} \Bigg(\left(e^{-c\rho_{\le t-1-i}} - e^{-\beta_1 \rho_{\le t-1-i}} \right)$$

$$\cdot \prod_{j=1}^{t-1-i} \frac{\rho_j}{\rho_{\le t-1-i} - \rho_{\le j-1}} \cdot e^{-\beta_1(\rho_{\le t-1} - \rho_{\le t-1-i})} \cdot \prod_{j=t-i}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} \Bigg).$$

By similar arguments, we can get the expression of $q(\beta_i)$ for all $i = 1, 2, \ldots, k$. Hence, for any sequence of constants $c = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_k$, we can lower bound Λ_t by

$$\Lambda_t \ge \Pr[\tau > \rho_{\le t-1} n] \cdot \sum_{z=1}^{k} D_t(\beta_{z-1}, \beta_z) \cdot \frac{q(\beta_{z-1}) - q(\beta_z))}{\Pr[\tau > \rho_{\le t-1} n]}$$

$$= \sum_{z=1}^{k} D_t(\beta_{z-1}, \beta_z) \cdot (q(\beta_{z-1}) - q(\beta_z)).$$

Plugging in the expressions of $D_t(\beta_{z-1}, \beta_z)$ and $q(\beta_z)$, we have

$$\Lambda_t \ge \sum_{z=1}^{k} (1 - e^{-\beta_z \rho_t}) \cdot \left(\frac{\beta_{z-1}}{\beta_z} \cdot v(1 - \frac{\beta_z}{n}) + \frac{\Delta(\beta_{z-1})}{\beta_z} \right) \cdot \left(\sum_{i=1}^{t-2} \prod_{j=1}^{t-1-i} \frac{\rho_j}{\rho_{\le t-1-i} - \rho_{\le j-1}} \right.$$

$$\cdot \prod_{j=t-i}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} \cdot \left(e^{-c\rho_{\le t-1-i}} \cdot \left(e^{-\beta_{z-1}(\rho_{\le t-1} - \rho_{\le t-1-i})} - e^{-\beta_z(\rho_{\le t-1} - \rho_{\le t-1-i})} \right) \right.$$

$$\left. \left. - \left(e^{-\beta_{z-1}\rho_{\le t-1}} - e^{-\beta_z \rho_{\le t-1}} \right) \right) + \prod_{j=1}^{t-1} \frac{\rho_j}{\rho_{\le t-1} - \rho_{\le j-1}} \cdot \left(e^{-\beta_{z-1}\rho_{\le t-1}} - e^{-\beta_z \rho_{\le t-1}} \right) \right).$$

The same upper bounds for OPT from the previous sections are also applied.

5.2.2. Lower bound on competitive ratio of the algorithm

Similar to previous sections, we construct a factor revealing LP whose feasible solution is the lower bound on the competitive ratio of our algorithm. The following claim is verified by the GLPK, which completes the proof of Theorem 5.2.

Claim 5.2. *By fixing constants $c = 0.6906, \rho_1 = 0.6831, \rho_2 = 0.2342, \rho_3 = 0.0711, k = 2500$ and $\beta_i = c + 0.01 \cdot i$ for all i, the optimal solution to the factor revealing LP of the 4-phase observe-and-accept algorithm is at least 0.6862. For other values of $c \in (0.004, 2.5)$, by appropriately setting other parameters (depending on c), we can lower bound the competitive ratio as shown in Fig. 5.1.*

6. Conclusion

In this work, we study the single-choice prophet inequality problem with unknown i.i.d. distributions and propose a novel model in which an algorithm has access to a single data point of the value distribution. We show that for a wide range of data points, the proposed algorithm achieves a competitive ratio strictly larger than $1 - 1/e$, and can be as good as 0.6862-competitive.

Our investigation uncovers many intriguing open problems for further exploration. A natural progression involves analyzing the full potential of leveraging a single data point. While our work demonstrates the existence of 0.6862-competitive algorithms with a single data point, the extent of possible improvements remains to be fully understood. Ideal outcomes would include algorithms achieving competitive ratios surpassing $1/e$ for any data point (v, q) where $q \in (0, 1)$. Additionally, deriving non-trivial upper bounds on the competitive ratio of algorithms with a single data point can provide insight into the limitations of using one data point. It is interesting to investigate the prophet secretary problem under the proposed model. That is, the buyers can have different distributions in their private values but arrive in a uniformly-at-random order. The problem generalizes the prophet inequality problem on i.i.d. distributions. It can be shown that a single data point of the distribution D^* of $x^* = \max_{1 \le i \le n}\{x_i\}$ suffices to achieve a competitive ratio of $1 - 1/e$. Whether we can beat $1 - 1/e$ with a single data point on D^* would be an interesting open problem to investigate. Finally, we believe that it would be an interesting topic to investigate the power of other information about D^*, e.g., the mean or median.

CRediT authorship contribution statement

Yilong Feng: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Bo Li:** Writing – review & editing, Writing – original draft, Supervision, Project administration, Funding acquisition, Formal analysis, Conceptualization. **Haolong Li:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Xiaowei**

Wu: Writing – review & editing, Writing – original draft, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Yutong Wu:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Numerical results for small *n*

In this section, we provide the numerical results for the competitive ratio of the two-phase algorithm when $n \in \{2, 3, \dots, 100\}$.[6] We remark that there are a few key differences in the analysis when n is a small constant and when $n \to \infty$. First, for small n, the optimization of $\rho \in (0,1)$ must follow the constraint that $\rho \cdot n$ is a constant. Therefore we use t to denote the number of variables in the first phase and optimize $t \in \{1, 2, \dots, n-1\}$. Second, the gap between $(1-c/n)^n$ and e^{-c} is no longer negligible when n is small. Therefore we keep the form of $(1-c/n)^n$ throughout the whole analysis and computation when n is small.

We further remark that for small n, setting a threshold of 0 for the last variable, i.e., accepting the last variable for any realization, is crucial for achieving a good competitive ratio. However, in order to be consistent with our previous sections, here we only analyze the algorithm that uses a fixed threshold (determined by the maximum realization of variables in the first phase) for all variables in the second phase.

A.1. Analysis

Following similar analysis as in Section 4, we express $\mathsf{ALG} = \Lambda_1 + \Lambda_2$, and lower bound the two terms in the RHS as follows. Similar to Equation (4.1), we lower bound Λ_1 by

$$\Lambda_1 := \sum_{i=1}^{t} \Pr[\tau = i] \cdot \mathbb{E}[x_i \mid \tau = i] = \sum_{i=1}^{t} (1 - \frac{c}{n})^{i-1} \cdot \left(\frac{c}{n} \cdot \theta_1 + \frac{\Delta(c)}{n} \right)$$
$$= \frac{1-(1-c/n)^t}{c/n} \cdot \left(\frac{c}{n} \cdot \theta_1 + \frac{\Delta(c)}{n} \right) = \left(1 - \left(1 - \frac{c}{n} \right)^t \right) \cdot \left(\theta_1 + \frac{\Delta(c)}{c} \right). \tag{A.1}$$

Similar to Equation (4.2), we fix $c = \beta_0 < \beta_1 < \dots < \beta_k \le n$ and lower bound Λ_2 by

$$\Lambda_2 \ge \left(1 - \frac{c}{n} \right)^t \cdot \sum_{i=1}^{k} p(\beta_{i-1}, \beta_i) \cdot H(\beta_{i-1}, \beta_i)$$
$$= \sum_{i=1}^{k} \left((1 - \frac{\beta_{i-1}}{n})^t - (1 - \frac{\beta_i}{n})^t \right) \left(1 - (1 - \frac{\beta_i}{n})^{n-t} \right) \left(\frac{\beta_{i-1}}{\beta_i} \cdot v(1 - \frac{\beta_i}{n}) + \frac{\Delta(\beta_{i-1})}{\beta_i} \right). \tag{A.2}$$

For the upper bound of OPT, as before, we define δ_i as:

$$\delta_i = \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x \ge t] \mathrm{d}t, \qquad \forall 0 \le i \le k-1,$$

and upper bound OPT by:

$$\mathsf{OPT} = \int_{0}^{v(1-\frac{\beta_k}{n})} \Pr[x^* \ge t] \mathrm{d}t + \sum_{i=0}^{k-1} \int_{v(1-\frac{\beta_{i+1}}{n})}^{v(1-\frac{\beta_i}{n})} \Pr[x^* \ge t] \mathrm{d}t + \int_{v(1-\frac{\beta_0}{n})}^{\infty} \Pr[x^* \ge t] \mathrm{d}t$$
$$\le v(1 - \frac{\beta_k}{n}) + \sum_{i=0}^{k-1} \delta_i + \Delta(\beta_0).$$

Similar to Lemmas 4.1, 4.2 and 4.3, we derive three upper bounds for δ_i as follows.

$$\delta_i \le \left(v(1 - \frac{\beta_i}{n}) - v(1 - \frac{\beta_{i+1}}{n}) \right) \cdot (1 - (1 - \frac{\beta_{i+1}}{n})^n). \tag{A.3}$$

[6] The computation time (for solving LPs) is too slow for larger n.

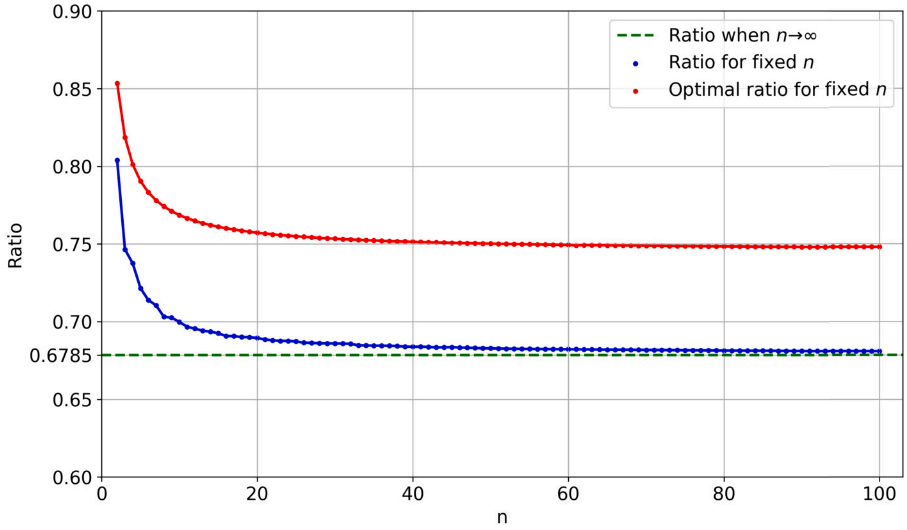

Fig. A.1. Lower Bounds of the Ratio for Small n.

$$\delta_i \le \Delta(\beta_{i+1}) - \Delta(\beta_i). \tag{A.4}$$

For all $\zeta \in [\beta_i, \beta_{i+1}]$ and $\gamma = \frac{\zeta-\beta_i}{(1-\frac{\zeta}{n})^n-(1-\frac{\beta_{i+1}}{n})^n}$, we have

$$\gamma \cdot \delta_i \le \Delta(\beta_{i+1}) - \Delta(\beta_i) - (\beta_i - \gamma\left(1-(1-\frac{\zeta}{n})^n\right)) \cdot \left(v(1-\frac{\beta_i}{n}) - v(1-\frac{\beta_{i+1}}{n})\right). \tag{A.5}$$

Putting the lower bounds for ALG and upper bounds on OPT together, we formulate an LP whose optimal objective serves as a lower bound for the competitive ratio.

A.2. Numerical results for solving LPs

Recall that in Section 4.2, we set $\beta_i = c + 0.01 \cdot i$ and $k = 2500$. Therefore we have $\beta_i \in (0, 26)$. This is sufficient for deriving a good lower bound for the competitive ratio because $\lim_{n\to\infty}(1-26/n)^n = e^{-26} \approx 0$. However, for small n, we need to guarantee that $\beta_k \le n$ while making sure that $(1-\beta_k/n)^n$ is sufficiently small. Therefore, we set $\beta_i = c + 0.01 \cdot i$ for all $0 \le i \le k$ and let $k \le \lfloor 100(n-c)\rfloor$ be sufficiently large such that $(1-\beta_k/n)^n$ is small enough (e.g., $\le 10^{-5}$).[7]

In Fig. A.1, we plot the lower bounds of the ratio for $n = 2, 3, \ldots, 100$ (with optimized parameters c and t). As a comparison, we also plot the ratio of the optimal online algorithm (for small n, from Hill and Kertz [11]), and the ratio we obtained in Section 4 for $n \to \infty$. As we can observe from the result, the ratio gradually decreases when n increases, and is already very close to the ratio 0.6785 we obtained for $n \to \infty$ when $n = 100$.

Data availability

No data was used for the research described in the article.

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