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Truthful aggregation of budget proposals with proportionality guarantees ☆

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ABSTRACT

We study a participatory budgeting problem, where a set of strategic agents wish to split a divisible budget among different projects, by aggregating their proposals on a single division. Unfortunately, the straightforward rule that divides the budget proportionally is susceptible to manipulation. Recently, a class of truthful mechanisms has been proposed, namely the moving phantom mechanisms. One such mechanism satisfies the proportionality property, in the sense that in the extreme case where all agents prefer a single project to receive the whole amount, the budget is assigned proportionally.

While proportionality is a naturally desired property, it is defined over a limited type of preference profiles. To address this, we expand the notion of proportionality, by proposing a quantitative framework that evaluates a budget aggregation mechanism according to its worst-case distance from the proportional allocation. Crucially, this is defined for every preference profile. We study this measure on the class of moving phantom mechanisms, and we provide approximation guarantees. For two projects, we show that the Uniform Phantom mechanism is optimal among all truthful mechanisms. For three projects, we propose a new, proportional mechanism that is virtually optimal among all moving phantom mechanisms. Finally, we provide impossibility results regarding the approximability of moving phantom mechanisms.

1. Introduction

Participatory budgeting is an emerging democratic process that engages community members with public decision-making, particularly when public expenditure should be allocated to various public projects. Since its initial adoption in the Brazilian city of Porto Alegre in the late 1980s [10], its usage has been spread in various cities across the world. Madrid, Paris, San Francisco, and Toronto provide an indicative, but far from exhaustive, list of cities that have adopted participatory budgeting procedures. See the survey by Aziz and Shah [3] for more examples.

In this paper, we follow the model of Freeman et al. in [17], where voters are tasked to split an exogenously given amount of money among various projects. As an illustrative example, consider a city council inquiring the residents on how to divide the

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upcoming year's budget on education, among a list of publicly funded schools. Each citizen proposes her preferred allocation of the budget and the city council uses a suitable aggregation mechanism to allocate the budget among the schools.

A natural way to aggregate the proposals is to compute the arithmetic mean for each project and assign to each project exactly that proportion of the budget. Following the above example, consider a community of 80 thousand residences, wishing to split a budget over 3 schools. If 20 thousand residents propose a $(20\%, 0\%, 80\%)$ division of the budget, other 20 thousand residents propose a $(40\%, 40\%, 20\%)$ division and the remaining 40 thousand residents propose a $(100\%, 0\%, 0\%)$ division, the budget should be split as $(65\%, 10\%, 25\%)$ over the 3 schools. This method, (or variations[1] of it) is used in practice in economics and sports. See [30,32,33] for some applications. Assigning the budget proportionally comes with some perks: it can be easily described, it is calculated efficiently and it scales naturally to any number of projects.

Unfortunately, allocating the budget proportionally comes also with a serious drawback; namely, it is susceptible to manipulation. Indeed, consider a simple example with two projects and one hundred voters. Fifty voters propose a $(50\%, 50\%)$ allocation, while the other fifty voters propose a $(100\%, 0\%)$ allocation. Hence, the proportional allocation is $(75\%, 25\%)$. Assume now, that one voter changes her $(50\%, 50\%)$ proposal to $(0\%, 100\%)$. This turns the aggregated division to $(74.5\%, 25.5\%)$, a division which is closer to the $(50\%, 50\%)$ proposal that she prefers. Hence, she may have an incentive to misreport her most preferred allocation to obtain a better outcome, according to her preference.

Truthful mechanisms, i.e. mechanisms nullifying the incentives for strategic manipulation have already been proposed in the literature, for voters with ℓ_1 preferences. Under ℓ_1 preferences (See [17]), a voter has an ideal division in mind and suffers a disutility equal to the ℓ_1 distance from her ideal division. Goel et al. [19] and Lindner et al. [23] propose truthful budget aggregation mechanisms that minimize the sum of disutilities for the voters, a quantity known in the literature as the *utilitarian social welfare*.

Recently, Freeman et al. in [17] observed that these mechanisms are disproportionately biased towards the opinion of the majority.[2] This led them to propose the *proportionality* property. A mechanism is said to be *proportional* if, in any input consisting only of *single-minded* voters (voters which fully assign the budget to a single project), each project receives the proportion of the voters supporting that project. They also proposed a truthful and proportional mechanism, called the *Independent Markets* mechanism.

The Independent Markets mechanism belongs to a broader class of truthful mechanisms, called *moving phantom mechanisms*. A moving phantom mechanism for n voters and m projects, allocates to each project the median between the voters' proposals for that project and $n+1$ carefully selected *phantom values*. The selection of the phantom values is crucial: it ensures both the strategy-proofness of the mechanism, as well as its ability to return a feasible aggregated division, i.e. that the portions sum up to 1. For example, the Independent Markets mechanism, places the $n+1$ phantom values uniformly in the interval $[0, x]$, for some $x \in [0, 1]$, that guarantees feasibility.

While proportionality is a natural fairness property, it is defined only under a limited scope: A proportional mechanism guarantees to provide the proportional division *only when all voters are single-minded*, and provides no guarantee for all other inputs. In this paper, we move one step further and we address the question: "How far from the proportional division can the outcome of a truthful mechanism be?"

Building on the work of Freeman et al. [17], we propose a more robust measure, and we extend the notion of proportionality as follows: Given any input of budget proposals, we define the *proportional division* as the coordinate-wise mean of the proposals and then we measure the ℓ_1 distance between the outcome of any mechanism and the proportional division. We call this metric the *ℓ_1-loss*. We say that a mechanism is α-approximate if the maximum ℓ_1-loss, computed for all preference profiles, is upper bounded by $\alpha \in [0, 2]$. So, in the one extreme, $\alpha = 0$ implies that the mechanism always achieves the proportional solution, while in the other extreme, $\alpha = 2$ provides no useful implication.

1.1. Our contribution

In this paper, we expand the notion of proportionality due to Freeman et al. [17], by proposing a quantitative worst-case measure that compares the outcome of a mechanism with the proportional division. We evaluate this measure on truthful mechanisms, focusing on the important class of moving phantom mechanisms [17]. Our main objective is to design truthful mechanisms with small α-approximation. We are able to provide effectively optimal mechanisms for the cases of two and three projects.

For the case of two projects, we show that the Uniform Phantom mechanism from [17] is $1/2$-approximate. Then, for the case of three projects, we first examine the Independent Markets mechanism and we show that this mechanism cannot be better than 0.6862-approximate. We then propose a new, proportional moving phantom mechanism which we call the *Piecewise Uniform* mechanism which is $(2/3 + \epsilon)$-approximate, where ϵ is a small constant.[3] The analysis of this mechanism is substantially more involved than the case of two projects and en route to proving the approximation guarantee we characterize the instances bearing the maximum ℓ_1-loss, for any moving phantom mechanism.

We complement our results by showing matching impossibility results: First, we show that there exists no α-approximate moving phantom mechanism for any $\alpha < 1 - 1/m$. This implies that our results for two and three projects are essentially best possible, within

[1] A usual variation is the *trimmed mean* mechanism, where some of the extreme bids are discarded. This is done to discourage a single voter to heavily influence a particular alternative.

[2] As an illustrative example consider an instance where $2k+1$ voters propose budget divisions over two projects. $k+1$ voters assign the whole budget to the first project, while k voters assign the whole budget to the second project. For this instance, $(100\%, 0\%)$ is the unique division that minimizes the sum of disutilities for the voters.

[3] This constant is at most 10^{-5}, and arises as a computational error.

the family of moving phantom mechanisms. Furthermore, we show that no α-approximate truthful mechanism exists, for $\alpha < 1/2$, implying that the Uniform Phantom mechanism is the best possible among all truthful mechanisms.

Finally, we turn our attention to cases with a large number of projects. We show that our proposed Piecewise Uniform mechanism, as well as the Independent Markets mechanism, are at least $(2 - \Theta(m^{-1/3}))$-approximate. In addition, we show that any utilitarian social welfare maximizing mechanism is $(2 - \Theta(m^{-1}))$-approximate.

1.2. Further related work

Arguably the work closest to our work is the work by Freeman et al. [17]. Apart from the Independent Markets mechanism they propose and analyze another moving phantom mechanism, which maximizes the social welfare and turned out to be equivalent to the truthful mechanisms of Goel et al. [19] and Lindner et al. [23], at least up to tie-breaking rules. They also showed that this mechanism is the only mechanism that guarantees Pareto optimal budget divisions.

Both the Uniform Phantom mechanism and the class of moving phantom mechanisms are conceptually related to the *generalized median rules*, from Moulin's seminal work [26]. The generalized median rules are defined on a single-dimensional domain (e.g. the $[0, 1]$ line) and are proven to characterize all truthful mechanisms under the mild assumptions of anonymity and continuity. These rules use $n + 1$ *phantom values*, where n is the number of voters. For the case of two projects, generalized median rules can be used directly. In fact, the Uniform Phantom mechanism, analyzed in Section 3.1, is a generalized median rule, tailored to our setting. We also note that variants of the Uniform Phantom mechanism appear under different contexts in various papers [12,30,32,6].

Generalized median rules cannot be used directly for more than two projects, since there exists no guarantee that the outcome would be a valid division (i.e. the total budget is allocated to the projects). For example, for more than two projects, the Uniform Phantom mechanism returns an outcome that sums to a value at least equal to 1. The family of moving phantom mechanisms extends the idea of the generalized median rules, by carefully selecting a coordinated set of phantom values, imposing an outcome to be a valid division, and retaining truthfulness for any number of projects. This family is broad, and it is an open question whether it includes all truthful mechanisms, under the mild assumptions of anonymity, neutrality, and continuity. See also the related discussion and references in [17].

Voters with ℓ_1 preferences are a special case of the well-studied *single-peaked* preferences [26] and have some precedence in public policy literature [19]. Recently, a natural utility model, equivalent to ℓ_1 preferences has been proposed in [27]. We note here that the family of moving phantom mechanisms fails to remain strategyproof when the ℓ_1 preferences assumption is dropped, and replaced with the more general assumption of single-peaked preferences (see the related discussion in Varloot and Laraki [37]).

The non-truthful mechanism which assigns a budget proportionally has been studied from a strategic point of view in single-dimensional domains (i.e., for two projects). Renault and Trannoy [30,31] explore the equilibria imposed by this mechanism. Rosar [33] compares allocations based on the mean and the median in a single-dimensional domain in a model with incomplete information. Finally, in a recent work, Puppe and Rollmann [29] conducted an experimental study between a normalized median rule and the aggregated mean in a setting similar to ours. Their findings show that under the normalized-median rule, which is not truthful, people were frequently sincere, while under the aggregated median rule, the voters' behavior was mainly polarized towards the extremes.

Our work falls squarely under the Participatory Budgeting paradigm in terms of applications. For a survey on Participatory Budgeting, from a mechanism design perspective, the reader is referred to [3]. According to the taxonomy therein, our model belongs to the Divisible Participatory Budgeting class. Other examples under the same category include [16,18,5,1,8,15,25,36]. Among others, these works analyze mechanisms with various fairness notions, some of which are in the spirit of proportionality. As an example Michorzewski et al. [25] quantify the effect of imposing fairness guarantees versus the maximum social possible welfare under the utility model of dichotomous preferences. A large part of the literature concerning Participatory Budgeting covers a model where projects cannot be funded partially but instead are either fully funded or not funded at all. Part of the work of Goel et al. [19] is dedicated to this model. Other notable examples include [7,4,24]. The problem we tackle here is also related to facility location, which has attracted substantial interest from the computational social choice and mechanism design community. For a recent survey, see Chan et al. [14].

Finally, our work follows the agenda of approximate mechanism design without money, firstly promoted in Procaccia and Tennenholtz [28]. We use an additive approximation measure. Such measures have been used both in the approximation mechanism [2,20] and the mechanism design [11,35,34] literature.

1.3. Roadmap

The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. In Section 3 we present mechanisms with small approximation guarantees for two (Section 3.1) and three (Section 3.2) projects. In Section 4 we present impossibility results. Complementary results and additional examples appear in Appendix.

2. Preliminaries

Let $[k] = \{1, ..., k\}$ for any $k \in \mathbb{N}$. Let $[n]$ be a set of voters and $[m]$ be a set of projects, for $n \geq 2$ and $m \geq 2$. Let $\mathcal{D}(m) = \{\mathbf{x} \in [0, 1]^m : \sum_{j \in [m]} x_j = 1\}$. This set is also known as the *standard simplex* [9].

We call a *division* among m projects any tuple $\mathbf{x} \in \mathcal{D}(m)$. Let $d(\mathbf{x},\mathbf{y}) = \sum_{j\in[m]} \left|x_j - y_j\right|$ denote the ℓ_1 distance between the divisions $\mathbf{x}$ and $\mathbf{y}$. Voters have structured preferences over budget divisions. Each voter $i \in [n]$ has a most preferred division, her *peak*, $\mathbf{v}_i^*$, and for each division $\mathbf{x}$, she suffers a disutility equal to $d(\mathbf{v}_i^*, \mathbf{x})$, i.e. the ℓ_1 distance between her peak $\mathbf{v}^*$ and $\mathbf{x}$.

Each voter $i \in [n]$ reports a division $\mathbf{v}_i$. These divisions form a *preference profile* $\mathbf{V} = (\mathbf{v}_i)_{i\in[n]}$. A *budget aggregation mechanism* $f : \mathcal{D}(m)^n \to \mathcal{D}(m)$ uses the proposed divisions to decide an aggregate division $f(\mathbf{V})$. A mechanism f is *continuous* when the function $f : \mathcal{D}(m)^n \to \mathcal{D}(m)$ is continuous. A mechanism is *anonymous* if the output is independent of any voters' permutation and *neutral* if any permutation of the projects (in the voters' proposals) permutes the outcome accordingly.

In this paper, we focus on *truthful* mechanisms, i.e. mechanisms where no voter can alter the aggregated division to her favor, by misreporting her preference.

Definition 1 *(Truthfulness).* [17] A budget aggregation mechanism f is truthful if, for all preference profiles $\mathbf{V}$, voters i, and divisions $\mathbf{v}_i^*$ and $\mathbf{v}_i$, $d(f(\mathbf{V}_{-i}, \mathbf{v}_i)) \geq d(f(\mathbf{V}_{-i}, \mathbf{v}_i^*))$.

A large part of this work is concerned with the class of *moving phantom mechanisms*, proposed in [17].

Definition 2 *(Moving phantom mechanisms).* [17] Let $\mathcal{Y} = \{y_k : k \in \{0..n\}\}$ be a family of functions such that, for every $k \in \{0..n\}$, $y_k : [0,1] \to [0,1]$ is a continuous, weakly increasing function with $y_k(0) = 0$ and $y_k(1) = 1$. In addition, $y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t)$ for every $t \in [0,1]$. The set $\mathcal{Y}$ is called a *phantom system*. For any valid phantom system, a *moving phantom mechanism* $f^{\mathcal{Y}}$, is defined as follows: For any profile $\mathbf{V}$ and any project $j \in [m]$,

$$f_j^{\mathcal{Y}}(\mathbf{V}) = \operatorname{med}\left(\mathbf{V}_{i\in[n],j}, (y_k(t^*))_{k\in\{0..n\}}\right) \tag{1}$$

for some

$$t^* \in \left\{ t : \sum_{j\in[m]} \operatorname{med}\left(\mathbf{V}_{i\in[n],j}, (y_k(t))_{k\in\{0..n\}}\right) = 1 \right\}. \tag{2}$$

Each function $y_k(t)$ represents a phantom. At each time t, the phantom system returns the values $(y_0(t), \ldots, y_n(t))$. For the particular time t^*, these values are sufficient for the sum of the medians to be equal to 1 (see equation (2)). Freeman et al. in [17] show that one such t^* always exists and, in case of multiple candidate values, the specific choice of t^* does not affect the outcome.

The following theorem from [17] states that every budget aggregation mechanism following Definition 2 is truthful.

Theorem 1. *[17] Every moving phantom mechanism is truthful.*

Each median in Definition 2 can be computed using a sorted array with $2n+1$ slots, numbered from 1 to $2n+1$. The median value is located in slot $n+1$. Throughout this paper we refer to the slots 1 to n as the *lower* slots, and $n+2$ to $2n+1$ as the *upper* slots.

For a given preference profile $\mathbf{V}$, let

$$\bar{\mathbf{V}} = \left(\frac{1}{n}\sum_{i\in[n]} v_{i,j}\right)_{j\in[m]}$$

be the *proportional division*. A *single-minded* voter is a voter $i \in [n]$ such that $v_{i,j} = 1$ for some project $j \in [m]$. A budget aggregation mechanism is called *proportional*, if for any preference profile $\mathbf{V}$ consisted solely of single-minded voters, it holds $f(\mathbf{V}) = \bar{\mathbf{V}}$.

For a given budget aggregation mechanism f and a preference profile $\mathbf{V}$, we define the ℓ_1-loss as the ℓ_1 distance between the outcome $f(\mathbf{V})$ and the proportional division $\bar{\mathbf{V}}$, i.e.

$$\ell(\mathbf{V}) = d(f(\mathbf{V}), \bar{\mathbf{V}}) = \sum_{j\in[m]} \left|f_j(\mathbf{V}) - \bar{\mathbf{V}}_j\right|. \tag{3}$$

We say that a budget aggregation mechanism is α-approximate when the ℓ_1-loss for any preference profile is no larger than α. We note that no mechanism can be more than 2-approximate, as the ℓ_1 distance between any two arbitrary divisions is at most 2.

3. Upper bounds

In this section, we present mechanisms with small approximation guarantees for $m = 2$ and $m = 3$. As we will see later on, the guarantee for 2 projects is optimal for any truthful budget aggregation mechanism. For the case of 3 projects, our guarantee is practically optimal, when we are confined to the family of moving phantom mechanisms.

3.1. Two projects

For the case of two projects, we focus on the *Uniform Phantom* mechanism [17], for which we show a 1/2-approximation. This mechanism is proven to be the unique truthful and proportional mechanism, when aggregating a budget between 2 projects.

The Uniform Phantom mechanism places $n+1$ phantoms uniformly over the $[0,1]$ line, i.e.

$$f_j = \operatorname{med}\left(\mathbf{V}_{i\in[n],j}, (k/n)_{k\in\{0..n\}}\right),$$

for $j \in \{1,2\}$. Later, in Theorem 6, we show that $1/2$ is the best approximation which can be achieved by any truthful mechanism.

Theorem 2. *For $m=2$, the Uniform Phantom mechanism is $1/2$-approximate.*

Proof. Let f be the Uniform Phantom mechanism, and let $\mathbf{V}$ be a preference profile. Let $f(\mathbf{V}) = (x, 1-x)$ and $\bar{\mathbf{V}} = (\bar{v}, 1-\bar{v})$ for some $x \in [0,1]$ and $\bar{v} \in [0,1]$. The loss of the mechanism for $\mathbf{V}$ is

$$\ell(\mathbf{V}) = 2\,|x - \bar{v}|\,. \tag{4}$$

Let $k \in \{0..n\}$ be the minimum phantom index such that $x \le \frac{k}{n}$. This implies that the phantoms with indices $k, \dots, n$ are located in the slots $n+1$ to $2n+1$. These phantoms are exactly $n+1-k$ i.e. exactly k voters' reports are located in the same area. Since all values in these slots are at least equal to the median we get that

$$\frac{k}{n}\cdot x \le \bar{v} \le \frac{n-k}{n}\cdot x + \frac{k-1}{n} + \frac{1}{n}\cdot \mathbb{1}\{x = k/n\} + \frac{x}{n}\cdot \mathbb{1}\{x < k/n\} \tag{5}$$

The first inequality holds, since exactly k voters' reports have value at least equal to the median x. For the second inequality, we note that exactly $n-k$ voters' reports have value at most x, while at least $k-1$ voters' reports can have value at most 1. If the median is equal to k/n, we can safely assume that this is a phantom value, and there should be exactly k values upper bounded by 1. Otherwise, if the median is strictly smaller than k/n, then x should be a voter's report and exactly $k-1$ voters' reports are located in the upper slots.

By removing x from both inequalities in (5) we get:

$$\frac{k}{n}\cdot x - x \le \bar{v} - x \le \frac{k-1}{n} + \frac{1}{n}\cdot \mathbb{1}\{x = k/n\} + \frac{x}{n}\cdot \mathbb{1}\{x < k/n\} - \frac{k}{n}\cdot x. \tag{6}$$

When the median is a phantom value, i.e. $x = \frac{k}{n}$, inequalities (6) imply that

$$|\bar{v} - x| \le \max\left\{x\left(1-\frac{k}{n}\right), \frac{k}{n}(1-x)\right\} = \frac{k}{n}\left(1-\frac{k}{n}\right), \tag{7}$$

which is maximized to for $k = n/2$ to a value no greater than $1/4$. When x is a voter's report, i.e. $\frac{k-1}{n} < x < \frac{k}{n}$, inequalities (6) imply that

$$\begin{aligned}|\bar{v} - x| &\le \max\left\{x\left(1-\frac{k}{n}\right), \frac{k-1}{n}(1-x)\right\}\\ &< \max\left\{\frac{k}{n}\left(1-\frac{k}{n}\right), \frac{k-1}{n}\left(1-\frac{k-1}{n}\right)\right\}.\end{aligned} \tag{8}$$

Equations (7) and (8) are both upper bounded by $1/4$. The theorem follows. □

3.2. Three projects

In this subsection we provide a $(2/3+\epsilon)$-approximate truthful mechanism for some $\epsilon \le 10^{-5}$. This mechanism belongs to the family of moving phantom mechanisms, and it is also proportional. In the following, we describe the mechanism, and then we prove the approximation guarantee. Later, in Theorem 7, we show that $2/3$ is the best possible guarantee among the class of moving phantom mechanisms.

3.2.1. The Piecewise Uniform mechanism

The Piecewise Uniform mechanism uses the phantom system $\mathcal{Y}^{\mathrm{PU}} = \{y_k(t) : k \in \{0..n\}\}$, for which

$$y_k(t) = \begin{cases} 0 & \frac{k}{n} < \frac{1}{2} \\ \frac{4tk}{n} - 2t & \frac{k}{n} \ge \frac{1}{2} \end{cases} \tag{9}$$

for $t < 1/2$, while

$$y_k(t) = \begin{cases} \frac{k(2t-1)}{n} & \frac{k}{n} < \frac{1}{2} \\ \frac{k(3-2t)}{n} - 2 + 2t & \frac{k}{n} \ge \frac{1}{2} \end{cases} \tag{10}$$

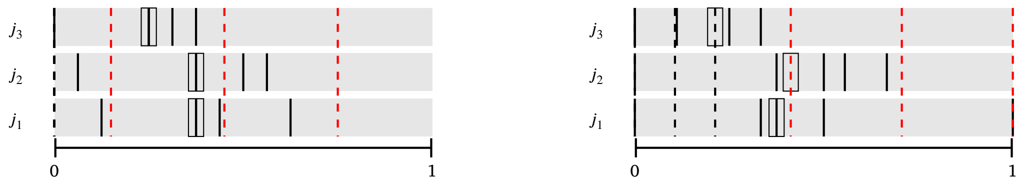

Fig. 1. Examples of the Piecewise Uniform mechanism, with 5 voters. The dashed lines correspond to the phantom values, the small rectangles correspond to the medians, and the thick lines correspond to the voters' reports. In the first example $\mathbf{v}_1 = \mathbf{v}_2 = (3/8, 3/8, 1/4)$, $\mathbf{v}_3 = (1/8, 1/2, 3/8)$, $\mathbf{v}_4 = (7/16, 9/16, 0)$ and $\mathbf{v}_5 = (5/8, 1/16, 5/16)$ and $t^* = 3/8$, while the final outcome is $(3/8, 3/8, 1/4)$. In the second example, $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 0)$, $\mathbf{v}_3 = (0, 2/3, 1/3)$, $\mathbf{v}_4 = (1/3, 5/9, 1/9)$, $\mathbf{v}_5 = (3/8, 3/8, 1/4)$ and $t^* = 49/64$.

for $t \geq 1/2$. This mechanism belongs to the family of moving phantom mechanisms[4]: each $y_k(t)$ is a continuous, weakly increasing function, and $y_k(t) \geq y_{k-1}(t)$ for $k \in [n]$ and any $t \in [0, 1]$. To distinguish between the two types of phantoms, we call a phantom with index $k < n/2$ a *black* phantom, and a phantom with index $k \geq n/2$, a *red* phantom (irrespectively of the value of t).

This mechanism can be seen as a combination of two different mechanisms: For $t < 1/2$, the mechanism uses $n/2$ phantom values equal to 0, and the rest are uniformly located in $[0, y_n(t)]$. For $t \geq 1/2$, the mechanism assigns half of the phantoms uniformly in $[0, y_{\lfloor n/2 \rfloor}(t)]$, while the rest are uniformly distributed in $[y_{\lceil n/2 \rceil}, 1]$. See the examples of Fig. 1, for an illustration.

We emphasize here that the Piecewise Uniform mechanism admits polynomial-time computation using a binary search algorithm, since $\mathcal{Y}^{\text{PU}}$ is a *piecewise linear phantom system* (see Theorem 4.7 from [17]).

We continue by showing that this mechanism is proportional. Note that this does not necessarily need to hold to show the desired approximation guarantee, but it is a nice extra feature of our mechanism.

Theorem 3. *The Piecewice Uniform mechanism is proportional.*

Proof. Consider any preference profile which consists exclusively of single-minded voters. Note that by using $t = 1$, the phantom with index k has the value k/n, for any $k \in \{0..n\}$. Let that $a_j \in \{0..n\}$ be the number of 1-valued proposals on project j. Consequently, $n - a_j$ is the number of 0-valued proposals. Then the median in each project is exactly the phantom value a_j/n, i.e. the proportional allocation. □

3.2.2. Upper bound

Overview The analysis for the upper bound is substantially more involved than the analysis for the case of two projects. Here we present an outline of the proof. To help the analysis, we assume at this point that no zero values exist in the aggregated division. We analyze the case where zeros exist in the outcome in Section Appendix B in the Appendix.

We first provide a characterization of the worst-case preference profiles (i.e. profiles that may yield the maximum loss) in Theorem 4. This characterization states that essentially all worst-case preference profiles belong to a specific family, which we call *three-type* profiles (see Definition 3). The family of three-type profiles depends crucially on the moving phantom mechanism used. Given a moving phantom mechanism, Lemma 2 characterizes further the family of three-type for that mechanism.

We combine Theorem 4 and Lemma 2 to build a Non-Linear Program (NLP; see Fig. 5) which explores the space created by the worst-case instances. Finally we present the optimal solution of the NLP in Theorem 5.

Characterization of worst-case instances We concentrate on a family of preference profiles which are maximal (with respect to the loss) in a local sense: A preference profile $\mathbf{V}$ is *locally maximal* if, for all voters $i \in [n]$, it holds that $\ell(\mathbf{V}) \geq \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$ for any division $\mathbf{v}'_i$. In other words, in such profiles, any single change in the voting divisions cannot increase the ℓ_1-loss. Inevitably, any profile which may yield the maximum loss belongs to this family, and we can focus our analysis on such profiles. Our characterization shows that the class of locally maximal preference profiles and the class of three-type profiles are equivalent with respect to ℓ_1-loss, for any moving phantom mechanism.

Definition 3 *(three-type profiles)*. For a given moving phantom mechanism f, a preference profile $\mathbf{V}$ is called a *three-type* profile if every voter $i \in [n]$ belongs to one of the following classes:

1. *fully-satisfied* voters, where voter i proposes a division equal to the outcome of the mechanism, i.e. $f(\mathbf{V}) = \mathbf{v}_i$,
2. *double-minded* voters, where voter i agrees with the outcome in one project, i.e. $v_{i,j} = f_j$ for some $j \in [3]$, while $v_{i,j'} = 1 - f_j$ for some different project j', and
3. *single-minded* voters, where $v_{i,j} = 1$ for some project $j \in [3]$.

To build intuition, we provide the following example:

[4] Note that, as presented here, this mechanism does not entirely fit Definition 2 since $y_k(t) < 1$, for all $k \in \{0..n-1\}$. This can be fixed easily however, with an alternative definition, where all phantom functions are shifted slightly to the left and a third set of linear functions is added, such that and $y_k(1) = 1$ for all $k \in \{0..n\}$. See Section Appendix A in the Appendix for a detailed explanation.

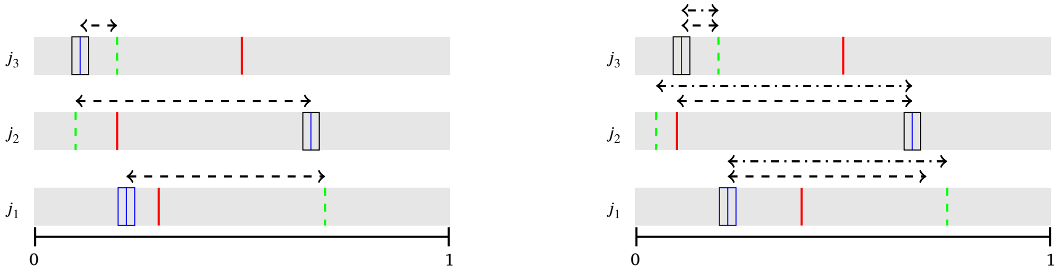

Fig. 2. An example (Fig. 2a) where the loss can be increased by a single change in voter's i proposed division. Note that voter i is neither fully-satisfied, single-minded nor double-minded. Voter's i proposals are depicted with solid vertical lines, the mean with dashed vertical lines and the outcome of the mechanism is depicted with vertical solid lines inside a rectangle. By moving the proposals (Fig. 2b) of voter i away for the median, the loss strictly increases.

Example 1 *(three-type profile).* Consider a moving phantom mechanism f, and the preference profile $\mathbf{V}$ with 5 voters: $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,0,1)$, which are single-minded voters, $\mathbf{v}_3 = (1/2, 1/2, 0)$, $\mathbf{v}_4 = (0, 1/4, 3/4)$ and $\mathbf{v}_5 = (1/2, 1/4, 1/4)$. Then, if $f(\mathbf{V}) = \mathbf{v}_5$, the preference profile $\mathbf{V}$ is a three-type profile for mechanism f. Voter 5 is a fully-satisfied voter, while voters 3 and 4 are double-minded voters.

In Theorem 4 which follows, we show that for any locally maximal preference profile $\mathbf{V}$, there exists a three-type profile $\hat{\mathbf{V}}$ (not necessarily different than $\mathbf{V}$) for which $\ell(\hat{\mathbf{V}}) \geq \ell(\mathbf{V})$. Therefore, we can search for the maximum ℓ_1-loss by focusing only on the profiles described in Definition 3. The following lemma is an important stepping stone for the proof of Theorem 4.

Lemma 1. *Let f be a moving phantom mechanism for $m = 3$, $\mathbf{V}$ a preference profile, and $i \in [n]$, a voter which is neither single-minded, double-minded nor fully-satisfied. Let $\mathbf{v}_i$ be voter's i proposal. Then there exists a division $\mathbf{v}'_i$ such that $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) \geq \ell(\mathbf{V})$. Furthermore, when $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) = \ell(\mathbf{V})$ the division $\mathbf{v}'_i$ is double-minded, single-minded or fully-satisfied, and $f(\mathbf{V}) = f(\mathbf{V}_{-i}, \mathbf{v}'_i)$.*

Proof. Let $\bar{v}_j = \frac{1}{n}\sum_{i=1}^{n} v_{i,j}$ for $j \in [3]$ and $\mathbf{f} = f(\mathbf{V})$. We will prove the lemma by constructing the division $\mathbf{v}'_i$. For that, we alter the proposals only on two projects, and we keep the proposal for the third project invariant, in such a way that $\mathbf{v}'_i$ is a valid division. Our first attempt is to strictly increase the loss. When we fail to do that, we create $\mathbf{v}'_i$ in such a way that the loss is not decreasing (comparing $\mathbf{V}$ to $(\mathbf{V}_{-i}, \mathbf{v}'_i)$).

The following claim, allows us to focus our analysis only on two cases.

Claim 1. *Let that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be valid divisions over m projects, and let that $\mathbf{x} \neq \mathbf{y}$. Then there exists a pair j, j' of projects such that either i) $x_j \leq \min\{y_j, z_j\}$ and $x_{j'} \geq \max\{y_{j'}, z_{j'}\}$ or ii) $y_j \leq x_j \leq z_j$ and $z_{j'} \leq x_{j'} \leq y_{j'}$.*

Proof. We firstly notice that there exists two projects, say 1 and 2, such that $x_1 < y_1$ and $y_2 < x_2$, since $\mathbf{x} \neq \mathbf{y}$. There exist four possible relations between x_1, x_2 and z_1, z_2: If $x_1 \leq z_1$ and $z_2 \leq x_2$ then condition (i) is satisfied. Similarly, if $z_1 \leq x_1$ and $x_2 \leq z_2$, condition (ii) is satisfied. If $x_1 \leq z_1$ and $x_2 \leq z_2$, we notice that there exists a different project, say project 3, such that $z_3 \leq x_3$; otherwise $\sum_{j\in[m]} z_j > 1$. We have now two cases, according to the relation between y_3 and x_3: when $y_3 \leq x_3$, condition (i) holds between projects 1 and 3; when $x_3 \leq y_3$, condition (ii) holds between projects 2 and 3. Similar arguments hold for the case $z_1 \leq x_1$ and $z_2 \leq x_2$. $\square$

By Claim 1, we can assume without loss of generality that either i) $f_1 \leq \min\{v_{i,1}, \bar{v}_1\}$ and $f_2 \geq \max\{v_{i,2}, \bar{v}_2\}$ or ii) $\bar{v}_1 \leq f_1 \leq v_{i,1}$ and $v_{i,2} \leq f_2 \leq \bar{v}_2$.

Case (i): When $v_{i,2} > 0$, we can increase the loss as follows: we move $v_{i,1}$ to $v'_{i,1} = v_{i,1} + \epsilon$ and $v_{i,2}$ to $v'_{i,2} = v_{i,2} - \epsilon$, for some $0 < \epsilon \leq \min\{v_{i,2}, 1 - v_{i,1}\}$. This increases $\bar{v}_1$ to $\bar{v}'_1 = \bar{v}_j + \epsilon/n$ and decreases $\bar{v}_2$ to $\bar{v}'_2 = \bar{v}_2 - \epsilon/n$. Note that these moves do not affect the outcome of f, since no voters' reports move from the lower to the upper slots or vice versa. Also, $\mathbf{v}'_{i,3} = \mathbf{v}_{i,3}$, hence $\bar{v}'_3 = \bar{v}_3$. Thus $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) > \ell(\mathbf{V})$. See also Fig. 2.

If $v_{i,2} = 0$, i can be transformed to a single or double-minded voter, without decreasing the loss. Note that $v_{i,1} = 1 - v_{i,3}$. When $v_{i,3} \leq f_3$, $v_{i,3}$ moves to $v'_{i,3} = 0$ and $v_{i,1}$ moves to $v'_{i,1} = 1$ to create a single-minded division. When $v_{i,3} > f_3$, we can move $v_{i,3}$ to $v'_{i,3} = f_3$ and $v_{i,1}$ to $v'_{i,1} = 1 - f_3$ (note that $1 - v_{i,3} < 1 - f_3$) to create a double-minded division $\mathbf{v}'_i$. In any case, $v'_{i,1} = v_{i,1} + \epsilon$ and $v'_{i,3} = v_{i,3} - \epsilon$, where $\epsilon = v_{i,3} \cdot \mathbb{1}\left\{v_{i,3} \leq f_3\right\} + (v_{i,3} - f_3) \cdot \mathbb{1}\left\{v_{i,3} > f_3\right\}$. Also, $f(\mathbf{V}_{-i}, \mathbf{v}'_i) = f(\mathbf{V})$ and $\bar{v}'_2 = \bar{v}_2$. Thus,

$$
\begin{aligned}
\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) &= \bar{v}_1 + \epsilon - f_1 + f_2 - \bar{v}_2 + \left|\bar{v}_3 - \epsilon - f_3\right| \\
&\geq \ell(\mathbf{V}).
\end{aligned}
$$

The inequality holds due to $|x| \geq x$ for $x \in \mathbb{R}$. See also Fig. 3.

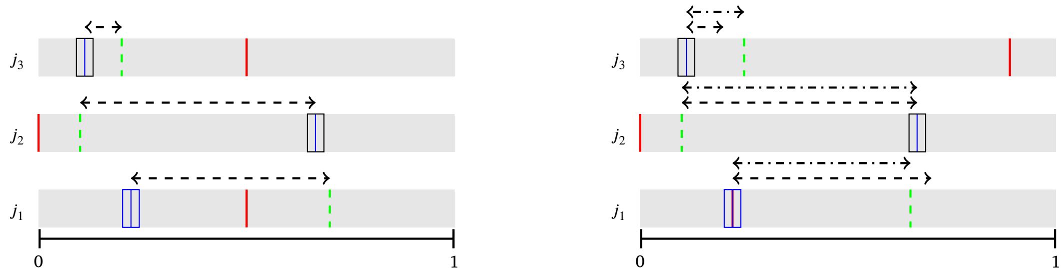

Fig. 3. An example where the loss cannot be decreased by a single change in voter's i proposed division, without changing the outcome of the mechanism. Note that voter i is neither fully-satisfied, single-minded nor double-minded. Voter's i proposals are depicted with solid vertical lines, the mean with dashed vertical lines and the outcome of the mechanism is depicted with vertical solid lines inside a rectangle. We transform the voter to a double-minded voter, while the loss is preserved.

Case (ii): Recall that $\bar{v}_1 \le f_1 \le v_{i,1}$ and $v_{i,2} \le f_2 \le \bar{v}_2$. If there exists some $0 < \epsilon \le \min\{v_{i,1} - f_1, f_2 - v_{i,2}\}$ we can strictly increase the loss by $2\epsilon/n$ by moving $v_{i,1}$ to $v'_{i,j} - \epsilon$ and $v'_{i,2} = v_{i,2} + \epsilon$. When no such ϵ exists, either $f_1 = v_{i,1}$ or $f_2 = v_{i,2}$ (note that this cannot happen for both projects; this would lead to $\mathbf{v}_i = \mathbf{f}$). Assume that $v_{i,1} = f_1$. We firstly note that for project 3, $v_{i,3} \ge f_3$; otherwise, $\sum_{j\in[3]} v_{i,j} < 1$. We also note that $v_{i,2} < f_2$ and $v_{i,3} > f_3$, otherwise $\mathbf{v}_i = \mathbf{f}$. We will transform $\mathbf{v}_i$ to a fully-satisfied voter, proposing $\mathbf{v}'_i$, without decreasing the loss. This is done by increasing $v_{i,2}$ to $v'_{i,2} = f_2$ and decreasing $v'_{i,3}$ to $v'_{i,3} = f_3$. Hence $\bar{v}'_2 = \bar{v}_2 + \frac{f_2 - v_{i,2}}{n}$ and $\bar{v}'_3 = \bar{v}_3 - \frac{v_{i,3} - f_3}{n} = \bar{v}_3 - \frac{f_3 - v_{i,2}}{n}$ (recall that $v_{i,1} = f1$). Note that $f(\mathbf{V}_{-i}, \mathbf{v}'_i) = f(\mathbf{V})$ and $\bar{v}'_1 = \bar{v}_1$. Let $\epsilon = \bar{v}_2 + \frac{f_2 - v_{i,2}}{n}$ and

$$\begin{aligned}\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) &= \bar{v}_1 - f_1 + f_2 + \epsilon - \bar{v}_2 + |\bar{v}_3 - \epsilon - f_3| \\ &\ge \ell(\mathbf{V}).\end{aligned}$$

A symmetric argument holds when $v_{i,2} = f_2$. □

Theorem 4. *Let f be a moving phantom mechanism for $m = 3$ and let $\mathbf{V}$ be a locally maximal preference profile, i.e. $\ell(\mathbf{V}) \ge \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$, for any $i \in [n]$ and any division $\mathbf{v}'_i$. Then, there exists a three-type profile $\hat{\mathbf{V}}$ such that $\ell(\hat{\mathbf{V}}) \ge \ell(\mathbf{V})$.*

Proof. Let S denote the set of single-minded, double-minded or fully-satisfied voters (for mechanism f and for profile $\mathbf{V}$) and let $\bar{S} = [n] \setminus S$.

If $\bar{S} = \emptyset$, $\mathbf{V}$ is a three-type profile, hence $\hat{\mathbf{V}} = \mathbf{V}$ and the theorem holds trivially. Otherwise, let $i \in \bar{S}$. By Lemma 1, we know that we can transform $\mathbf{v}_i$ to $\mathbf{v}'_i$ such that either (a) $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) > \ell(\mathbf{V})$ or (b) i becomes a double-minded, single-minded or fully-satisfied voter, $f(\mathbf{V}) = f(\mathbf{V}_{-i}, \mathbf{v}'_i)$ and $\ell(\mathbf{V}) = \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$. When (a) holds, clearly profile $\mathbf{V}$ is not locally maximal. Hence, we can assume that (b) holds for all voters in $\bar{S}$ and we can create $\hat{\mathbf{V}}$ by transforming all voters in $\bar{S}$ to single-minded, double-minded or fully satisfied, one-by-one. By Lemma 1, both the outcome and the loss stay invariant in each transformation. Hence, $f(\hat{\mathbf{V}}) = f(\mathbf{V})$ and $\ell(\hat{\mathbf{V}}) = \ell(\mathbf{V})$. The theorem follows. □

From now on, we turn our attention on three-type profiles, and in the following we define variables to describe them. A three-type profile $\mathbf{V}$ can be presented using 13 independent variables:

- $\mathbf{x} = (x_1, x_2, x_3)$, the division of the fully satisfied voters,
- a_1, a_2, a_3, three integer variables counting the single-minded voter towards each project,
- $b_{1,2}, b_{1,3}, b_{2,1}, b_{2,3}, b_{3,1}, b_{3,2}$, six integer variables counting the double-minded voters (e.g. $b_{2,1}$ counts the voters proposing $(1 - x_2, x_2, 0)$),
- and the total number of voters n.

We also use $A = \sum_{j\in[3]} a_j$ and $B = \sum_{j,k\in[3],k\neq j} b_{k,j}$ to count the single-minded and the double-minded voters, respectively. Consequently, the number of fully satisfied voters is $C = n - A - B$. These profiles can have at most 8 distinct voters' reports: values x_1, x_2 and x_3, from fully-satisfied and double-minded voters, values $1 - x_1$, $1 - x_2$ and $1 - x_3$ which we call *complementary values* from the double-minded voters and, reports with values equal to 1 and 0. Note that, apart from values 0 and 1, in project 1 we can find values x_1, $1 - x_2$ and $1 - x_3$, in project 2 values x_2, $1 - x_1$ and $1 - x_3$ and finally in project 3 the values x_3, $1 - x_1$ and $1 - x_2$.

Recall that Definition 3 demands that $f(\mathbf{V}) = \mathbf{x}$. To ensure this, we prove the following lemma. Note that we assume that $x_j > 0$, for all $j \in [3]$ at this point.

Lemma 2. *Let that $x_j > 0$, for all $j \in [3]$. Let $z_j = a_j + \sum_{k\in[3]\setminus\{j\}} b_{k,j}$ and $q_j = \sum_{k\in[3]\setminus\{j\}} b_{j,k}$. For any moving phantom mechanism f, defined by the phantom system $\mathcal{Y} = \{y_k(t) : k \in \{0..n\}\}$, and any three-type profile $\mathbf{V}$, then $f(\mathbf{V}) = \mathbf{x}$ if and only if*

$$y_{z_j}(t^*) \le x_j \le y_{z_j+q_j+C}(t^*) \tag{11}$$

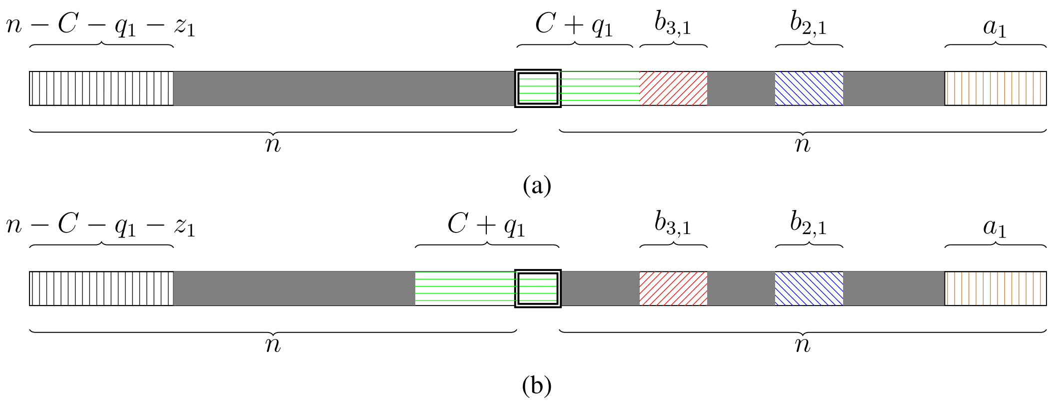

Fig. 4. Example for the positioning of phantom values and voters' reports in project 1, for a given three-type profile. The 5 patterned intervals represent the voters' reports. The solid, dark intervals represent the phantom values. The double-lined rectangle in the middle represents the median. Fig. 4a illustrates the case where the $C + b_{1,2} + b_{1,3} = C + q_1$ voters' reports with values x_1 are located in the upper $n+1$ slots. The $z_1 = a_1 + b_{2,1} + b_{3,1}$ voters' reports with values 1 and $1 - x_2$ and $1 - x_3$ must also be located in the top $n+1$ slots. Fig. 4b illustrates the other extreme, where $C + q_1$ values equal to the median are located in the lower $n+1$ slots.

for any

$$t^* \in \left\{ t : \sum_{j \in [m]} \text{med}\left(\mathbf{V}_{i\in[n],j}, (y_k(t))_{k\in\{0..n\}}\right) = 1 \right\}.$$

Proof. First note that for $x_j > 0$ for all $j \in [3]$, all complementary values $1 - x_1$, $1 - x_2$ and $1 - x_3$ are located in the upper slots. Assume otherwise, that there exists some complementary value, say $1 - x_2$ such that $1 - x_2 \le x_1$. Then $1 \le x_1 + x_2$, which is not possible when $x_3 > 0$. In addition, all 1-valued voters' reports should be located in the upper slots, while all 0-valued voter reports should be located in the lower slots. Note also that $z_j = a_j + \sum_{k\in[3]\setminus\{j\}} b_{k,j}$ counts exactly the voters' reports located in the upper slots.

(if direction) Let $\mathbf{V}$ be a three-type profile and let $f(\mathbf{V}) = \mathbf{x}$ for some $t^* \in [0,1]$. Assume, for the sake of contradiction that $y_{z_j}(t^*) > x_j$. This implies that the $n + 1 - z_j$ phantoms with indices $z_j, \dots, n$ are located in the upper slots (i.e. the n higher slots). Since at least z_j voters' reports are also be located in the upper slots there exists at least $n+1$ values for n slots. A contradiction. Suppose now that $y_{z_j+q_j+C}(t^*) < f_j(\mathbf{V})$. This implies that $z_j + C + 1$ phantom values (those with indices $0, \dots, z_j + C$)) are located in the lower slots (i.e. the n lower slots). The voters' reports with value 0 must be also located in the lower slots, since $0 < x_j$ for any $\in [3]$. There are exactly $A + B - q_j - z_j = n - C - q_j - z_j$ such values. Hence at least $n+1$ values should be located in the lower slots. A contradiction.

(only if direction) Let that inequalities (11) hold and let $\mathbf{V}$ be a three-type profile. Assume for the sake of contradiction that there exists a project $j \in [3]$ such that $f_j(\mathbf{V}) < x_j$. Hence, the C values of the possibly fully satisfied voters, plus the q_j values equal to x_j should be located in the upper slots. The 1-valued voters' reports, which count to a_1 and the complementary values, which count to $\sum_{k\in[3]\setminus\{j\}} b_{k,j}$ are also located in the upper slots. These count to $z_j + q_j + C$ values. Furthermore, since $x_j \le y_{z_j+q_j+C}(t^*)$, another $n - C - z_j - q_j + 1$ phantom values should be located in the upper slots. A contradiction.

Similarly, assume for the sake of contradiction that there exists a profile $j \in [3]$ such that $f_j(\mathbf{V}) > x_j$. Then the $C + q_j$ values should be located in the lower slots, along with the $n - C - q_j - z_j$ voters' reports equal to 0. Since $y_{z_j}(t^*) \le x_j$, $z_j + 1$ phantom values are also located in the lower slots. These are in total $n+1$ values, a contradiction (see Fig. 4 for an illustration of the extreme possible positioning of phantom values and voters' reports in a single project). □

We note that the only-if direction is not required for the proof of Theorem 5, but it is a nice feature that we include for the sake of completeness.

A non-linear program We show that the Piecewise Uniform mechanism is $(2/3 + \epsilon)$-approximate using a Non-Linear Program. The feasible region of this program is defined by the class of three-type profiles, and we search for the maximum ℓ_1-loss among them. For simplicity, we firstly normalize some of our variables with n. We introduce new variables $\hat{a}_j = a_j/n$ for $j \in [3]$ and $\hat{b}_{j,j'} = b_{j,j'}/n$ for $j, j' \in [3]$, $j \ne j$, and $\hat{C} = C/n$. We also use a relaxed version of the Piecewise Uniform mechanism: For every $x \in [0,1]$:

$$\hat{y}(x,t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \textbf{ and } x \le \frac{1}{2} \\ 4tx - 2t & 0 < t < \frac{1}{2} \textbf{ and } x > \frac{1}{2} \\ x(3-2t) - 2 + 2t & \frac{1}{2} \le t \le 1 \textbf{ and } x > \frac{1}{2} \\ x(2t-1) & \frac{1}{2} \le t \le 1 \textbf{ and } x \le \frac{1}{2}. \end{cases}$$

To help the presentation, we introduce also variables for the mean for each project $j \in [3]$:

$$\text{maximize} \sum_{j=1}^{3} \left| \bar{v}_j - x_j \right| \tag{12}$$

subject to

$$\sum_{j=1}^{3} x_j = 1,$$

$$\hat{A} = \sum_{j=1}^{3} \hat{a}_j,$$

$$\hat{B} = \sum_{j,k\in[3],j\neq k} \hat{b}_{k,j},$$

$$\hat{z}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{k,j}, \qquad \forall j \in [3]$$

$$\hat{q}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k}, \qquad \forall j \in [3]$$

$$x_j \geq \hat{y}\left(\hat{z}_j, t^*\right), \qquad \forall j \in [3] \tag{13}$$

$$x_j \leq \hat{y}\left(\hat{C} + \hat{q}_j + \hat{z}_j, t^*\right), \qquad \forall j \in [3] \tag{14}$$

$$\hat{A} + \hat{B} \leq 1,$$

$$x_j \geq 0, \hat{a}_j \geq 0, \qquad \forall j \in [3]$$

$$\hat{b}_{k,j} \geq 0, \qquad \forall j,k \in [3], j \neq k$$

$$0 \leq t^* \leq 1.$$

Fig. 5. The Non-Linear Program used to upper bound the maximum ℓ_1-loss for the Piecewise Uniform mechanism.

$$\bar{v}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} (1 - x_k)\hat{b}_{k,j} + x_j\left(\hat{C} + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k}\right).$$

The Non-Linear Program is presented in Fig. 5. Inequalities (13) and (14) ensure that we are searching over all three-type profiles for the Piecewise Uniform mechanism. Crucially, any profile which does not meet these two conditions cannot have **x** as the outcome (see Lemma 2). Finally, we let the program optimize over any $t^* \in [0, 1]$. Lemma 2 ensures that any value t^* that satisfies inequalities (13) and (14) will return a valid outcome.

Maximum loss computation The Non-Linear Program in Fig. 5 computes the ℓ_1-loss for the Piecewise Uniform mechanism for three-type profiles. As we have already seen in Theorem 4, these are the profiles where the ℓ_1-loss gets its highest value. Hence, bounding from above the objective function in equation (12) will give us an upper bound for the mechanism's worst-case approximation.

To analyze this complex Non-Linear Program we will break it into a set of simpler, still non-linear, but more manageable programs. The simplification process is based on three axes: (i) according to the signs of the $\bar{v}_j - x_j$ terms on the objective function, (ii) according to the types of the phantoms enclosing the medians, and (iii) depending on whether $t^* \leq 1/2$ or not. This way we are left with a set of programs with only linear and bilinear expressions (in inequalities (13) and (14)). These alterations transform the original program into a set of Quadratic Programs with Quadratic Constraints (QPQC). These programs are still non-convex, yet their special form allows us to use industrial solvers to compute their global maxima.

To deal with the signs of the $\bar{v}_j - x_j$ terms, we define *sign patterns*, as tuples in $\{+, -\}^3$. For example the sign pattern $(+, +, -)$ shows that $\bar{v}_1 \geq x_1$ and $\bar{v}_2 \geq x_2$, while $x_3 \geq \bar{v}_3$. We note that we cannot have the same sign in all projects, unless the loss is equal to 0.[5] Hence, we only need to check the patterns $(+, -, -)$ and $(+, +, -)$.

To address the discontinuities in function $\hat{y}(x, t)$ (with respect to x) we use the tuple (r, r) to distinguish whether the median lies between two red phantoms, the tuple (b, b) to distinguish if the median is located between two black phantoms, and finally, the tuple (b, r) to distinguish whether the median is between a black and a red phantom. By noting that $\hat{z}_j > 1/2$ implies that $\hat{C} + \hat{q}_j + \hat{z}_j > 1/2$, we can safely assume that no median is upper bounded by a black phantom and lower bounded by a red phantom, and we define *phantom patterns*, as tuples in $\{(b, b), (b, r), (r, r)\}^3$. We build a quadratic program for each phantom pattern.

In total, we end up with $2 \times 2 \times 27 = 108$ Quadratic Programs with Quadratic Constraints. To build intuition, in Fig. C.10 in Appendix C we present in more detail one of these programs. Specifically, we present the case where $t > 1/2$, the sign pattern is $(+, -, -)$ and the phantom pattern is $((r, r), (b, b), (b, b))$. This quadratic program corresponds to a case yielding the highest upper bound for the ℓ_1-loss.

To prove Theorem 5 which follows, we first compute the maxima for 27 programs corresponding to the case $t^* > 1/2$ and the sign pattern $(+, -, -)$. For the remaining QPQCs, i.e. for $t \geq 1/2$ and the sign pattern $(+, +, -)$ and for $t^* < 1/2$ for both sign patterns, we check whether any of these cases can yield loss greater than $2/3$. We present one example in Fig. C.11 in Appendix C. All these

[5] Assume otherwise, that there exists a sign pattern with the same sign in all projects, say $(+, +, +)$. Then, $\sum_{j=1}^{3}(\bar{v}_j - x_j) = \sum_{j=1}^{3} \bar{v}_j - \sum_{j=1}^{3} x_j = 0$.

Table 1
The bounds computed by the QPQCs, for $t > 1/2$ and the sign pattern $(+, -, -)$. The programs without feasible solutions are not presented, as well as symmetric cases. The lower bound corresponds to the largest loss for a feasible solution computed by the solver. The upper bound corresponds to the smaller non-feasible lower bound computed be the solver. The last column shows the gap between them. Gaps smaller than 10^{-5} are insignificant due to the tolerance of the solver.

Phantoms	Status	Loss (lower bound)	Loss (upper bound)	Gap
$(b,b),(b,b),(b,b)$	OPTIMAL	0.333332	0.333333	1.83e − 6
$(b,b),(b,b),(b,r)$	OPTIMAL	0.357003	0.357007	3.69e − 6
$(b,b),(b,b),(r,r)$	OPTIMAL	0.357003	0.357010	7.38e − 6
$(b,b),(b,r),(b,r)$	OPTIMAL	0.499998	0.500014	1.62e − 5
$(b,b),(b,r),(r,r)$	OPTIMAL	0.499999	0.500010	1.11e − 5
$(b,b),(r,r),(b,b)$	OPTIMAL	0.357004	0.357008	4.27e − 6
$(b,b),(r,r),(r,r)$	OPTIMAL	0.000000	0.000000	0.00
$(b,r),(b,b),(b,b)$	OPTIMAL	0.666667	0.666667	−2.22e − 16
$(b,r),(b,b),(b,r)$	OPTIMAL	0.666666	0.666668	1.92e − 6
$(b,r),(b,b),(r,r)$	OPTIMAL	0.529134	0.529141	7.08e − 6
$(b,r),(b,r),(b,r)$	OPTIMAL	0.666667	0.666672	5.37e − 6
$(b,r),(b,r),(r,r)$	OPTIMAL	0.500000	0.500006	5.67e − 6
$(b,r),(r,r),(b,b)$	OPTIMAL	0.529134	0.529139	5.40e − 6
$(b,r),(r,r),(r,r)$	OPTIMAL	0.000000	0.000000	0.00
$(r,r),(b,b),(b,b)$	OPTIMAL	0.666667	0.666669	1.86e − 6
$(r,r),(b,b),(b,r)$	OPTIMAL	0.666666	0.666667	5.70e − 7
$(r,r),(b,b),(r,r)$	OPTIMAL	0.527863	0.527866	3.25e − 6
$(r,r),(b,r),(b,r)$	OPTIMAL	0.666666	0.666667	1.12e − 6
$(r,r),(b,r),(r,r)$	OPTIMAL	0.500000	0.500004	3.87e − 6
$(r,r),(r,r),(b,b)$	OPTIMAL	0.527864	0.527867	2.55e − 6

programs are infeasible, i.e. no one of these cases yield an ℓ_1-loss greater than $2/3$ plus a computational error term which is upper bounded by 10^{-5}.

We solve these programs using the Gurobi optimization software [21]. The solver models our programs as Mixed Integer Quadratic Programs and uses the *spatial Branch and Bound Method* (see [22]) with various heuristics, to return a global maximum, when the program is feasible. The solver computes arithmetic solutions with 10^{-5} error tolerance.

Theorem 5. *The Piecewise Uniform mechanism is* $(2/3 + \epsilon)$*-approximate, for some* $\epsilon \in [0, 10^{-5}]$.

Proof. Theorem 4 states that the maximum loss for any moving phantom mechanism happens in a three-type profile. The Non-Linear Program in Fig. 5 searches for the profile with maximum loss, over all three-type profiles. The latter is guaranteed by Lemma 2. We first solve 27 QPQCs, corresponding to the sign pattern $(+, -, -)$ and $t^* > 1/2$. The maximum value is no higher than $2/3 + \epsilon$, where ϵ is due to the error tolerance of the solver. Table 1 shows analytically the upper bounds computed for each one of the 27 QPQCs (excluding some symmetric cases).

For the other 81 QPQCs we check whether any of them yields loss at least $2/3$. For that, we add the constraint $\sum_{j=1}^{3} s(j)(\bar{v}_j - x_j) \geq 2/3$, where $s(j)$ denotes the sign for project j according to the sign pattern and we search for any feasible solution. Eventually, no feasible solution exists, i.e. there exists no other preference profile with loss at least $2/3$ plus the computational error imposed by the solver.

To complete our analysis we need to address the case where the outcome includes at least one 0 value. We analyze this case in Section Appendix B in the Appendix and we show that the ℓ_1-loss cannot be higher than $1/2 + \epsilon$ in this case. □

4. Lower bounds

In this section, we provide impossibility results for our proposed measure. Theorem 6 shows that no truthful mechanism can be less than $1/2$-approximate. Theorem 7 focuses on the class of moving phantom mechanisms and shows that no such mechanism can be less than $(1 - 1/m)$-approximate. Theorem 8 shows that the Independent Markets mechanism from [17] is at least 0.6862-approximate. Finally, we present lower bounds for large m: A combined lower bound of $2 - \frac{8}{m^{1/3}}$ for both the Independent Markets mechanism and the Piecewise Uniform mechanism, and a lower bound of $2 - \frac{4}{m+1}$ for any mechanism which maximizes the social welfare.

4.1. A lower bound for any truthful mechanism

In the following, we show that truthfulness inevitably admits ℓ_1-loss at least $1/2$ in the worst case. We recall that the Uniform Phantom mechanism achieves this bound for $m = 2$.

Theorem 6. *No truthful mechanism can achieve* ℓ_1*-loss less than* $1/2$.

Proof. Let f be a truthful mechanism over m projects. Consider a profile with 2 voters $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2)$, such that $\mathbf{v}_1 = (1, 0, ..., 0)$ and $\mathbf{v}_2 = (0, 1, 0, ..., 0)$ and let that $f(\mathbf{V}) = (x_1, ..., x_m) = \mathbf{x}$ for some $x \in \mathcal{D}(m)$. Consider also the profile $\mathbf{V}' = (\mathbf{x}, \mathbf{v}_2)$. Due to truthfulness, then $f(\mathbf{V}') = \mathbf{x}$. Assume otherwise, that $f(\mathbf{V}) = \mathbf{x}' \neq \mathbf{x}$; when voter's 1 peak is at $\mathbf{x}$, i.e. $\mathbf{v}_1^* = \mathbf{x}$, then the disutility for voter 1 when proposing $\mathbf{v}_1$ is:

$$d(f(\mathbf{V}), \mathbf{v}_1^*) = d(\mathbf{x}, \mathbf{x}) = 0 \tag{15}$$

while the disutility for voter 1 when proposing $\mathbf{v}_1^*$ is

$$d(f(\mathbf{V}'), \mathbf{v}_1^*) = d(\mathbf{x}', \mathbf{x}) > 0, \tag{16}$$

a contradiction. With a similar argument we can show that for $\mathbf{V}'' = (\mathbf{v}_1, \mathbf{x})$, $f(\mathbf{V}'') = \mathbf{x}$. Hence, the ℓ_1-loss for these two preference profiles is:

$$\ell(\mathbf{V}') = \left|x_1 - \frac{x_1}{2}\right| + \left|x_2 - \frac{1 + x_2}{2}\right| + \sum_{j=3}^{m} \left|x_j - \frac{x_j}{2}\right| = 1 - x_2, \text{ and}$$

$$\ell(\mathbf{V}'') = \left|x_1 - \frac{1 + x_1}{2}\right| + \left|x_2 - \frac{x_2}{2}\right| + \sum_{j=3}^{m} \left|x_j - \frac{x_j}{2}\right| = 1 - x_1.$$

The optimal mechanism should minimize the quantity $\max\{1 - x_1, 1 - x_2\}$, given that $x_1 + x_2 \leq 1$ and $x_1 \geq 0$, $x_2 \geq 0$. Note that $x_1 \leq 1 - x_2$, hence $\max\{1 - x_1, 1 - x_2\} \geq \max\{1 - x_1, x_1\}$ which is minimized for $x_1 = 1/2$ to a value at least $1/2$. □

4.2. A lower bound for any moving phantom mechanism

In this subsection we present a preference profile where any phantom mechanism yields loss equal to $1 - 1/m$. We recall that the Piecewise Uniform mechanism achieves this bound for $m = 3$. We note that the counter example construction we present in the proof holds for any even number of voters.

Theorem 7. *No moving phantom mechanism can achieve ℓ_1-loss less than $1 - 1/m$, for any $m \geq 2$.*

Proof. Let $n \geq 2$ and even, and let $S = \{1, ..., n/2\}$, $Q = \{n/2 + 1, ..., n\}$ be two sets of voters. Let f be a moving phantom mechanism defined over m projects and consider the preference profile $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_{n/2}, \mathbf{v}_{n/2+1}, ..., \mathbf{v}_n)$. All voters $i \in S$ propose the divisions $\mathbf{v}_i = (1, 0, ..., 0)$ while all voters $k \in Q$ propose the division $\mathbf{v}_k = (1/m, ..., 1/m)$. Let y_i, for $\in \{0, .., n\}$ denote the i-th phantom value and let that $(y_i)_{i \in \{0, ..., n\}}$ induce a valid outcome for the moving phantom mechanism. Hence, the outcome of the mechanism is equal to:

$$f_1(\mathbf{V}) = \text{med}\,\Big(\overbrace{\tfrac{1}{m}, ..., \tfrac{1}{m}}^{n/2}, y_0, ..., y_n, \overbrace{1, ..., 1}^{n/2}\Big) = x$$

while, for $j \in \{2, ..., m\}$

$$f_j(\mathbf{V}) = \text{med}\,\Big(\overbrace{0, ..., 0}^{n/2}, y_0, ..., y_n, \overbrace{\tfrac{1}{m}, ..., \tfrac{1}{m}}^{n/2}\Big) = z.$$

We will show that both $x \leq 1/m$ and $z \leq 1/m$, which implies that $x = z = 1/m$ for the outcome to sum up to 1, and thus being a valid outcome to moving phantom mechanism. Assume, for the sake of contradiction, that either $x > 1/m$ or $z > 1/m$. Starting from the case $z > 1/m$, note that $z \leq x$. For the outcome to sum up to 1, i.e. $x + (m - 1)z = 1$, it must be $x < 1/m$, a contradiction.

We continue with the case $x > 1/m$. Then, $n/2$ voters' reports with value equal to $1/m$ should be located in the n lower slots, in the computation of the median for the first project. Hence, at most $n/2$ phantoms can be located in the n lower slots, and the phantom with index $n/2$ should be located in one of the $n + 1$ slots higher than the median, i.e. $y_{n/2} \geq x > 1/m$. Observe also that $z < 1/m$, otherwise $x + (1 - m)z > 1$. This implies that the $n/2$ voters' reports with value equal to $1/m$ should be located in the n upper slots, in the computation of the medians for the projects 2 to m. Hence, at most $n/2$ phantom values should be located in the n lower slots, and the phantom with index $n/2$ should be located in one of the $n + 1$ slots lower than the median, i.e. $y_{n/2} \leq z < 1/m$. A contradiction.

Eventually, for a valid outcome of the mechanism it must be that $x = z = 1/m$ and the ℓ_1-loss becomes:

$$\ell(\mathbf{V}) = \left|\frac{1}{2} - \frac{1}{m}\right| + (m - 1)\left|\frac{1}{2 \cdot m}\right| = 1 - \frac{1}{m}. \quad \square \tag{17}$$

4.3. A lower bound for the Independent Markets mechanism

In this subsection, we present a class of instances where the Independent Markets mechanism from [17] yields an ℓ_1-loss of at least 0.6862, for three projects and a large enough number of voters n. Formally, the Independent Markets mechanism is the phantom mechanism utilizing the phantom system $\mathcal{Y}^{\text{IM}} = \{y_k(t) : k \in \{0..n\}\}$ for which

$$
\begin{array}{l}
\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
1 & 0 & 0 \\
0.4142 & 0.2929 & 0.2929 \\
\vdots & \vdots & \vdots \\
0.4142 & 0.2929 & 0.2929
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \lfloor n\rho \rfloor \text{ voters} \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \lceil n(1-\rho) \rceil \text{ voters}
\end{array}
\end{array}
$$

Fig. 6. The preference profile which yields a loss at least 0.6862 for the Independent Markets mechanism.

$$y_k(t) = \min\{k \cdot t, 1\}. \tag{18}$$

Theorem 8. *The Independent Markets mechanism is at least* 0.6862*-approximate for three projects.*

Proof. Let f be the Independent Markets mechanism and let $\rho = 2 - \sqrt{2} \approx 0.5858$. Consider a preference profile $\mathbf{V}$ with n voters, where $\lfloor n\rho \rfloor$ voters propose the division $(1,0,0)$ while $\lceil n(1-\rho) \rceil$ voters propose the division $\mathbf{x} = (\sqrt{2}-1, 1-\sqrt{2}/2, 1-\sqrt{2}/2)$. See also Fig. 6. Let that $t = \frac{\sqrt{2}}{2n}$. Then, $x_1 = \sqrt{2} - 1 = n\rho t \geq \lfloor n\rho \rfloor\ t$, i.e. the smallest $\lfloor n\rho \rfloor + 1$ phantom values, with indexes 0 to $\lfloor n\rho \rfloor$, are at most equal to x_1. Hence, there exists $n+1$ values (phantoms and voters' reports) at most equal to x_1, thus $f_1(\mathbf{V}) = x_1$. Similarly, $x_j = n(1-\rho)t \leq \lceil n(1-\rho) \rceil\ t$ for $j \in \{1,2\}$, i.e. the $n+1-\lfloor n\rho \rfloor$ phantom values with indices $\lfloor n\rho \rfloor$ to n are at least equal to x_j. Hence there exists $n+1$ values at least equal to x_j, thus $f_j(\mathbf{V}) = x_j$ for $j \in \{2,3\}$.

The ℓ_1-loss for the preference profile $\mathbf{V}$ is

$$\ell(\mathbf{V}) = \left(3 - 2\sqrt{2}\right)\left(1 - \frac{\lceil n(1-\rho) \rceil}{n}\right) + \frac{\lfloor n\rho \rfloor}{n} \geq 0.6862.$$

The inequality holds for $n \geq 2 \cdot 10^4$. $\square$

4.4. Lower bounds for many projects

In this section, we provide two impossibility results for large m. These results show that the Independent Markets mechanism, the Piecewise Uniform mechanism, and any utilitarian mechanism may yield a loss that approximates 2, the worst possible ℓ_1-loss, as the number of projects m grows large. For the two proportional mechanisms, Independent Markets and Piecewise Uniform, we use the same construction to show that the ℓ_1-loss can be as large as $2 - \frac{8}{m^{1/3}}$, for large m. Then, we focus on the mechanisms that maximize social welfare and show an even higher lower bound, at $2 - \frac{4}{m+1}$ for every $m \geq 3$.

4.4.1. Proportional mechanisms

In this section, we show that both the Piecewice Uniform and the Independent Markets mechanisms yield an ℓ_1-loss which is arbitrarily close to 2, for a large enough number of projects. An interesting open question regarding this is whether this holds for any proportional mechanism.

Theorem 9. *Both the Piecewise Uniform mechanism and the Independent Markets mechanism yield loss at least* $2 - \frac{8}{m^{1/3}}$ *for* $m \geq 14$.

Proof. We will construct a preference profile with m projects and m voters. In this profile, a supermajority of the voters (denoted by the integer variable z) are single-minded, towards a unique project each. The rest $m - z$ voters are fully-satisfied.

Let $z = \lfloor m - m^{2/3} \rfloor$ and $a = (m-z)^2$ for some $m \geq 14$. Let $\mathbf{s}^j$ be the division such that $s_i^j = 1$ when $i = j$ and $s_i^j = 0$ when $i \neq j$, for all $j \in [m]$, i.e., $\mathbf{s}^j$ is the division where the j-th voter assigns the whole budget to the j-th project. Also, let $\mathbf{x}$ be the division such that $x_j = \frac{1}{a+z}$ for $j \in [z]$ and $x_j = \frac{m-z}{a+z}$ for $j \in \{z+1, \dots, m\}$. Given that, we can design the instance $\mathbf{V}$ where $\mathbf{v}_i = \mathbf{s}^i$ for each voter $i \in [z]$ while for each voter $i \in \{z+1, \dots, m\}$, $\mathbf{v}_i = \mathbf{x}$. Fig. 7 displays this instance.

We will first show that the Piecewise Uniform mechanism returns the division $\mathbf{x}$ for any $m \geq 14$, by setting $t^* = \frac{1}{2} + \frac{m}{2(a+z)}$. Indeed, observe that in this case $y(1, t^*) = \frac{1}{a+z}$. Hence, for each project $j \in [z]$, $f_j(\mathbf{V}) = \frac{1}{a+z}$, since there exist exactly $m+1$ values (phantom values and voters' proposals) at most equal to $\frac{1}{a+z}$. Also, observe that $z \geq m - m^{2/3} - 1 \geq m/2$ for $m \geq 14$, hence $y(m-z, t^*) = \frac{m-z}{a+z}$. This implies that for all $j \in \{z+1, \dots, m\}$, $f_j(\mathbf{V}) = \frac{m-z}{a+z}$, since there exist exactly $m+1$ values at

$$
\begin{array}{c}
\overbrace{\hphantom{1 \quad 0 \quad 0 \quad \cdots \quad 0}}^{z} \;\; \overbrace{\hphantom{0 \quad \cdots \quad 0}}^{m-z}
\end{array}
$$

$$
\left.\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0\\
\hline
\frac{1}{a+z} & \frac{1}{a+z} & \frac{1}{a+z} & \cdots & \frac{1}{a+z} & \frac{m-z}{a+z} & \cdots & \frac{m-z}{a+z}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
\frac{1}{a+z} & \frac{1}{a+z} & \frac{1}{a+z} & \cdots & \frac{1}{a+z} & \frac{m-z}{a+z} & \cdots & \frac{m-z}{a+z}
\end{pmatrix}\right.
\begin{array}{l}
\left.\vphantom{\begin{matrix}1\\0\\0\\\vdots\\0\end{matrix}}\right\} z\\
\left.\vphantom{\begin{matrix}\frac{1}{a+z}\\\vdots\\\frac{1}{a+z}\end{matrix}}\right\} m-z
\end{array}
$$

Fig. 7. The preference profile which yields a loss at least $2-\frac{8}{m^{1/3}}$ for both the Piecewise Uniform and the Independent Markets mechanisms.

least equal to $\frac{m-z}{a+z}$. The same analysis, using $t^*=\frac{1}{a+z}$, shows that the Independent Markets mechanism returns the same outcome.

To compute the loss for $\mathbf{V}$ for the outcome $f(\mathbf{V})=\mathbf{x}$, first notice that $\left|\bar{v}_j-x_j\right|=\frac{1}{m}+\frac{m-z}{m}\frac{1}{a+z}-\frac{1}{a+z}=\frac{a}{m(a+z)}$ for all $j\in[z]$ and $\left|\bar{v}_j-x_j\right|=\frac{m-z}{a+z}-\frac{m-z}{m}\cdot\frac{m-z}{a+z}=\frac{(m-z)z}{m(a+z)}$. Note that $\frac{1}{m}+\frac{m-z}{m}\frac{1}{a+z}\geq\frac{1}{a+z}$ and $\frac{m-z}{a+z}\leq\frac{m-z}{m}\cdot\frac{m-z}{a+z}$. Hence the loss is equal to

$$
\begin{aligned}
\ell(\mathbf{V}) &= z\cdot\frac{a}{m(a+z)}+(m-z)\cdot\frac{(m-z)z}{m(a+z)}\\
&=2\cdot\frac{a}{a+z}\cdot\frac{z}{m}\\
&>2\left(\frac{m^{4/3}}{m^{4/3}+2m}\right)\left(\frac{m-m^{2/3}-1}{m}\right)\\
&\geq 2\left(1-\frac{2}{m^{1/3}}\right)^2\\
&\geq 2-\frac{8}{m^{1/3}}.
\end{aligned}\tag{19}
$$

For the first inequality, we have used the following facts: $z>m-m^{2/3}-1$, $a\geq m^{4/3}$ and $a+z\leq m^{4/3}+2m$ for $m\geq 5$. The first one is due to $x\geq\lfloor x\rfloor>x-1$ for any $x\in\mathbb{R}$. This implies also that $a=(m-\lfloor m-m^{2/3}\rfloor)^2\geq m^{4/3}$. Also, notice that $a+z=(m-\lfloor m-m^{2/3}\rfloor)^2+\lfloor m-m^{2/3}\rfloor\leq(m^{2/3}+1)^2+m-m^{2/3}+1\leq m^{4/3}+2m$. Second inequality is due to $\left(\frac{m^{4/3}}{m^{4/3}+2m}\right)\geq 1-\frac{2}{m^{1/3}}$ for $m\geq 0$ and $\left(\frac{m-m^{2/3}-1}{m}\right)\geq 1-\frac{2}{m^{1/3}}$ for $m\geq 1$. The last inequality is due to $\left(1-\frac{c}{x}\right)^2\geq\left(1-\frac{2c}{x}\right)$ for $c\geq 0$ and $x>0$. $\square$

4.4.2. Utilitarian mechanisms

In this part, we focus on a family of budget aggregation mechanisms, which we call *utilitarian* mechanisms. A utilitarian mechanism returns an aggregated division maximizing the social welfare. This is done by minimizing the ℓ_1 distance between the outcome and each voter's proposal, i.e. for a utilitarian mechanism f and a preference profile $\mathbf{V}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)$, then

$$
f(\mathbf{V})\in\operatorname*{argmin}_{\mathbf{x}\in\mathcal{D}(m)}\sum_{i\in[n]}d(\mathbf{v}_i,\mathbf{x}).\tag{20}
$$

Mechanisms that maximize social welfare have been examined in the literature (see Section 1). One of them is a moving phantom mechanism and is proven to be the unique Pareto optimal mechanism in this family, providing a dichotomy between Proportional the Pareto optimal moving phantom mechanisms. Here we show that any utilitarian mechanism inevitably yields very high loss. In contrast to the two proportional mechanisms, these mechanisms yield quite large loss even for small m, e.g., for three projects, the ℓ_1-loss is at least 1.

Theorem 10. *For any utilitarian mechanism, there exists an instance with ℓ_1-loss equal to $2-\frac{4}{m+1}$.*

Proof. Let $\mathbf{s}^j$ be the division such that $s^j_j=1$ for $j\in[m]$. Consider a preference profile with m projects and $m+1$ voters. Each voter $i\in[m+1]$ is single-minded, and $\mathbf{v}_i=\mathbf{s}^i$ when $i \mod m=0$. Hence, project 1 is fully supported by two voters, while the other projects are supported by one voter each. See Fig. 8 for an illustrative example with 5 voters.

Let f be a utilitarian mechanism and let $f(\mathbf{V})=\mathbf{x}$. Let $x_1=1-\epsilon$ and $\sum_{j=2}^m x_j=\epsilon$ for some $\epsilon\in[0,1]$. We call the quantity $\sum_{i\in[n]}d(\mathbf{v}_i,\mathbf{x})$ the *social cost* (SC). Recall that a utilitarian mechanism minimizes the social cost.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 8. An example of the construction used in Theorem 10, with 5 voters and 4 projects.

$$\begin{aligned} \mathrm{SC} &= \sum_{i=1}^{m+1} d(\mathbf{v}_i, \mathbf{x}) \\ &= \sum_{i=1}^{m+1} \sum_{j=1}^{m} \left| v_{i,j} - x_j \right| \\ &= 2(1-x_1) + 2\sum_{j=2}^{m} x_j + \sum_{i=2}^{m} \left(2 - 2x_i\right) \\ &= 2\epsilon + 2(m-1) \end{aligned} \tag{21}$$

The social cost is minimized for $\epsilon = 0$, i.e. the only possible outcome of a utilitarian mechanism for this preference profile is $x_1 = 1$ and $x_j = 0$ for $j \in \{2, \dots, m\}$.

On the other hand, the proportional division assigns $\bar{\mathbf{V}}_1 = \frac{2}{m+1}$ and $\bar{\mathbf{V}}_j = \frac{1}{n+1}$ for any $j \in \{2, \dots, m\}$. Eventually, the ℓ_1 loss is:

$$\begin{aligned} \ell(\mathbf{V}) &= \left(1 - \frac{2}{m+1}\right) + \sum_{j=2}^{m} \frac{1}{m+1} \\ &= 2 - \frac{4}{m+1}. \quad \square \end{aligned} \tag{22}$$

5. Conclusion

This paper proposes an approximation framework that rates budget aggregation mechanisms according to the worst-case distance from the proportional allocation, a natural fairness desideratum. We propose essentially optimal mechanisms within the class of moving phantom mechanisms for the cases of two and three projects. The most interesting open question is whether there exists any $(2-\epsilon)$-approximate mechanism, for some constant $\epsilon > 0$, with an arbitrary number of projects. Our constructions in Section 4 show that our mechanism, and two important mechanisms already explored in the literature cannot yield a bound asymptotically better than 2, the least informative bound.

Moving beyond moving phantom mechanisms, one can ask the questions: Is there any mechanism with worst-case ℓ_1-loss smaller than $2/3$ for the case of three projects? Our lower bound for any truthful mechanism is just $1/2$. While this lower bound is very simple, and probably a more sophisticated construction may answer the question negatively, we should note that until now the existence of one such mechanism is still possible. Even more, one could ask for mechanisms with improved approximation guarantees (e.g. even below $1/2$ by using a relaxed version of truthfulness).

Our approximation framework can be expanded for other desirable properties. For example, it is natural to study moving phantom mechanisms with good approximation guarantees with respect to the *egalitarian* or the *Nash* social welfare (see [3]). We also note that similar notions of approximation can be defined for other participatory budgeting problems. Consider for example the case where the voters divide the budget using approval voting, and each voter's utility is the proportion of the budget given to the projects she approves (see [8] for this line of work). A well-known fairness notion under this model is the *fair share*, which demands that each voter has a utility of at least $1/n$, where n is the number of voters. For this example, one can use as an approximation the worst-case distance between the outcome of the mechanism and all the divisions which satisfy the fair share property.

CRediT authorship contribution statement

Ioannis Caragiannis: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **George Christodoulou:** Conceptualization, Formal analysis, Supervision, Writing – original draft, Writing – review & editing. **Nicos Protopapas:** Conceptualization, Formal analysis, Project administration, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. The Piecewise Uniform mechanism is a Moving Phantom mechanism

In this section we show that the Piecewise Uniform mechanism is a moving phantom mechanism. For that, we will present an alternative phantom system $\mathcal{Y}^{\mathrm{PU}'}$ which satisfies Definition 2. Then, we show that the Piecewise Uniform mechanism simulates this new alternative definition.

The alternative phantom system is $\mathcal{Y}^{\mathrm{PU}'} = \{y'_k(t) : k \in \{0..n\}\}$, for which

$$y'_k(t) = \begin{cases} 0 & \frac{k}{n} < \frac{1}{2} \\ \frac{4tk}{n\left(\frac{1}{2}-\epsilon\right)} - 2\frac{t}{\frac{1}{2}-\epsilon} & \frac{k}{n} >\geq \frac{1}{2} \end{cases} \tag{A.1}$$

for $t < 1/2 - \epsilon$,

$$y'_k(t) = \begin{cases} \frac{k(2t-1)}{n} + \frac{2k\epsilon}{n} + \frac{2k\epsilon}{n} & \frac{k}{n} < \frac{1}{2} \\ \frac{k(3-2t)}{n} - 2 + 2t + 2\epsilon - \frac{2k\epsilon}{n} & \frac{k}{n} \geq \frac{1}{2}, \end{cases} \tag{A.2}$$

for $1/2 - \epsilon \leq t < 1 - \epsilon$, and

$$y'_k(t) = \frac{k}{n}\left(\frac{1-t}{\epsilon}\right) + \frac{t-1}{\epsilon} + 1, \tag{A.3}$$

for $t > 1 - \epsilon$, for some $0 < \epsilon < 1/2$.

Observe that this alternative phantom system satisfies Definition 2: All phantom functions are continuous, $y_k(0) = 0$ and $y_k(1) = 1$ for all $k \in \{0..n\}$. Also, $y_{k+1}(t) \geq y_k(t)$ for all $k \in \{0..n-1\}$.

We notice also that there exists no feasible solution for $t > 1 - \epsilon$, for this phantom system. For $t = 1 - \epsilon$ the phantoms of the mechanism described by the phantom system $\mathcal{Y}^{\mathrm{PU}'}$ are exactly the phantoms used by the Uniform Phantom mechanism. In the following lemma we show that the sum of the medians of the Uniform Phantom mechanism is at least 1. Hence, any phantom returned by the phantom system $\mathcal{Y}^{\mathrm{PU}'}$ with $t > 1 - \epsilon$, would return an outcome that sums to a value strictly larger than 1.

Lemma 3. *Let* $\mathbf{x}$ *be the outcome of the Uniform Phantom mechanism on an arbitrary preference profile for some* $m \geq 3$. *Then* $\sum_{j\in[m]} x_j \geq 1$.

Proof. Let k_j be the largest index such that $\frac{k_j}{n} \leq x_j$. Assume for the sake of contradiction that $\sum_{j\in[m]} x_j < 1$, i.e. $\sum_{j\in[m]} \frac{k_j}{n} < 1$. In the slots 1 to $n+1$, there exist exactly k_j phantom values, for each project $j \in [m]$. As a result, there exist exactly $n - k_j$ voters' reports in the same slots. In total, $mn - \sum_{j\in[m]} k_j > n(m-1)$ voters' reports are located in the slots 1 to $n+1$. Similarly, there exist exactly k_j voters' reports in each project j in the upper slots (slot $n+2$ to slot $2n+1$). Hence in total there exist exactly $\sum_{j\in[m]} k_j < n$ upper slots filled by phantom values, out of mn slots. Since there are mn slots in the upper phantoms, there should be at least $n(m-1)+1$ voters' reports, but we already know that at least $n(m-1)$ out of nm voters' reports are located either in the lower slots or in the slots of the medians. A contradiction. □

We are ready now to show that the two phantom systems $\mathcal{Y}^{\mathrm{PU}}$ and $\mathcal{Y}^{\mathrm{PU}'}$ describe the same moving phantom mechanism.

Lemma 4. *The phantom systems* $\mathcal{Y}^{PU}$ *and* $\mathcal{Y}^{PU'}$ *implement the same moving phantom mechanism.*

Proof. We use $y_k(t)$ to denote the functions from the phantom system $\mathcal{Y}^{\mathrm{PU}}$ and $y'_k(t)$ to denote the functions for the phantom system $\mathcal{Y}^{\mathrm{PU}'}$. Consider any preference profile $\mathbf{V}$ over m projects. Let $f_j(\mathbf{V}) = \mathrm{med}\left(\mathbf{V}_{i\in[n],j}, (y_k(t))_{k\in\{0..n\}}\right)$ and $f'_j(\mathbf{V}) = \mathrm{med}\left(\mathbf{V}_{i\in[n],j}, (y'_k(t'))_{k\in\{0..n\}}\right)$, for suitable $t \in [0,1]$ and $t' \in [0,1]$ such that $\sum_{j\in m} f_j(\mathbf{V}) = 1$ and $\sum_{j\in[m]} f'_j(\mathbf{V}) = 1$. We will show that the Piecewise Uniform implements the mechanism described by $\mathcal{Y}^{\mathrm{PU}'}$.

We consider the phantom system $\mathcal{Y}^{\text{PU}'}$. Assume that $t' > 1-\epsilon$. By Lemma 3, $\sum_{j\in[m]} f'_j(\mathbf{V}) > 1$, i.e. no feasible outcome is possible with $t' > 1-\epsilon$. Assume now that $1/2-\epsilon < t' \leq 1-\epsilon$. Then, by using $t = t'+\epsilon$, the tuples $(y'_k(t'))_{k\in\{0..n\}}$ and $(y_k(t))_{k\in\{0..n\}}$ are equivalent, hence $f'(\mathbf{V}) = f(\mathbf{V})$. Finally, assume that $f(\mathbf{V})$ uses some $t \leq 1/2-\epsilon$ and returns a feasible solution. Then, by using $t' = \frac{t}{1-2\epsilon}$, the tuples $(y'_k(t'))_{k\in\{0..n\}}$ and $(y_k(t))_{k\in\{0..n\}}$ are equivalent, hence $f'(\mathbf{V}) = f(\mathbf{V})$. □

Appendix B. Completeness of Theorem 5

In this section we tackle the remaining case for the proof of Theorem 5, where zero values exist in the outcome. Our technique is similar to Section 3.2.2.

Due to Theorem 4 we can focus on three-type profiles to upper bound the ℓ_1-loss. Recall that the division $\mathbf{x} = (x_1, x_2, x_3)$ represents the outcome of the mechanism on a three-type profile. First, note that when the outcome includes two 0 values, i.e. $\mathbf{x} = (1,0,0)$, any three-type profile $\mathbf{V}$ contains only single-minded voters. Since the mechanism is proportional, the loss is 0, and the mechanism is optimal for such profiles. We turn now our attention to the case where there exists only one 0 value in the outcome, say $x_3 = 0$. For that, we need to check two possibilities. If $t > 1/2$, there can be only one phantom value equal to 0, by the definition of the mechanism, and for $x_3 = 0$, at least n voters' reports should be equal to 0. Thus all voters propose 0 for project 3. This can be reduced to the case of 2 projects. The Piecewise Uniform mechanism can ensure a feasible solution by using the phantoms $(k/n)_{k\in\{0..n\}}$, which we can enforce by setting $t^* = 1$, i.e. the mechanism simulates the Uniform Phantom mechanism for this case. By Theorem 2, the ℓ_1-loss in this case cannot be higher than $1/2$.

Finally we tackle the remaining case where $x_3 = 0$, $x_1 > 0$, $x_2 > 0$ and $t \leq 1/2$. We will build a different Non-Linear Program to show that this case cannot yield a loss higher than $1/2+\epsilon$ for some $\epsilon \leq 10^{-5}$. Recall first that $b_{1,2}$ and $b_{2,1}$ counts divisions $(x_1, x_2, 0)$, $b_{1,3}$ counts divisions $(1,0,0)$, $b_{2,3}$ counts divisions $(0, x_2, x_1)$, $b_{3,1}$ counts divisions $(x_1, 0, x_2)$ and $b_{3,2}$ counts divisions $(0,1,0)$. Notice that for each project $j \in \{1,2\}$ the only possible voters' reports are $0, x_j$ and 1. Hence, for project 1 there exists $a_1 + b_{1,3}$ voters' reports with value 1, $b_{2,1}+b_{1,2}+b_{3,1}$ voters' reports with values x_1 and $a_2+a_3+b_{2,3}+b_{3,2}$ voters' reports with value 0. Similarly, there exists $a_2+b_{3,2}$ voters' reports with value 1, $b_{2,1}+b_{1,2}+b_{2,3}$ voters' reports with value x_2 and $a_1+a_3+b_{3,1}+b_{1,3}$ voters' reports with value 0. For project 3, there exists a_3 voters' reports with value 1, $b_{2,3}$ voters' reports with value x_1 and $b_{3,1}$ voters reports with value x_2. For project 3, all complementary values are positive, i.e. they are located in the upper slots. This is not always the case for projects 1 and 2, where we can only guarantee that a_1 and a_2 voters' reports (i.e. the 1 valued reports) are located in the upper slots, respectively. On the other side, there exists $C + b_{2,3} + b_{3,2} + a_2 + a_3$ zero values in project 1 and $C + b_{1,3} + b_{3,1} + a_1 + a_3$ from the double-minded, single-minded and the zeros of the fully-satisfied voters. In the following lemma, we use this information to impose sufficient and necessary conditions (similar to Lemma 2) for the mechanism to return $\mathbf{x}$ as an outcome.

Lemma 5. *Let that $x_1 > 0$, $x_2 > 0$, $x_3 = 0$. For any moving phantom mechanism f, defined by the phantom system $\mathcal{Y} = \{y_k(t) : k \in \{0..n\}\}$, and a three-type profile $\mathbf{V}$ then $f(\mathbf{V}) = \mathbf{x}$ if and only if:*

$$y_{a_1+b_{1,3}}(t^*) \leq x_1 \leq y_{n-a_2-a_3-b_{2,3}-b_{3,2}}(t^*) \tag{B.1}$$

$$y_{a_2+b_{3,2}}(t^*) \leq x_2 \leq y_{n-a_1-a_3-b_{1,3}-b_{3,1}}(t^*) \tag{B.2}$$

for any

$$t^* \in \left\{ t : \sum_{j\in[m]} \text{med}\left(\mathbf{V}_{i\in[n],j}, (y_k(t))_{k\in\{0..n\}}\right) = 1 \right\}.$$

Proof. We focus on inequality (B.1). Similar arguments can be used to show inequality (B.2).

(if direction) Let $\mathbf{V}$ be a three-type profile and $f(\mathbf{V}) = \mathbf{x}$ for some $t^* \in [0,1]$. Assume for the sake of contradiction that $y_{a_1+b_{1,3}}(t^*) > x_1$. This implies that the $n - a_1 - b_{1,3} + 1$ phantoms with indices $a_1 + b_{1,3}, \dots, n$ are located in the upper slots. Since at least $a_1 + b_{1,3}$ voters' reports should be located in the upper slots (since they have the value 1), at least $n+1$ values should be located in the lower slots. A contradiction. Suppose now, that $y_{n-a_2-a_3+b_{2,3}+b_{3,2}}(t^*) < f_1(\mathbf{V})$. Then, $n-a_2-a_3+b_{2,3}+b_{3,2}+1$ phantoms are located in the lower slots. Also, at least $a_3+a_2+b_{2,3}+b_{3,2}$ zero values should be located in the lower slots. These are $n+1$ values, while the lower slots are n. A contradiction.

(only if direction) Let that inequalities (B.1) hold, and consider a three-type profile $\mathbf{V}$, where $f_1(\mathbf{V}) < x_1$. Hence the C voters reports from the fully-satisfied voters plus the $b_{1,2} + b_{2,1} + b_{3,1}$ reports equal to x_1 from the double-minded voters and the $a_1 + b_{1,3}$ 1-valued reports, should be located in the upper slots. From inequality (B.1), there exists $a_2 + a_3 + b_{2,3} + b_{3,2} + 1$ phantoms in the upper slots. These are $n+1$ values, which cannot fit in the n upper slots. Suppose now, that $x_1 < f_1(\mathbf{V})$. Then the C voters reports from the fully-satisfied voters plus the $b_{1,2} + b_{2,1} + b_{3,1}$ reports equal to x_1 from the double-minded voters and the $a_2 + a_3 + b_{2,3} + b_{3,2}$ voters' reports equal to 0 should be located in the lower slots. Hence, $C + b_{1,2} + b_{1,3} + b_{2,1} + a_1 + b_{3,1} + a_2 + b_{2,3} + b_{3,2}$ voters' reports are located in the lower slots. From inequality (B.1), at least $a_1 + b_{1,3} + 1$ voter reports are located in the lower slots, hence at least $n+1$ values are located in the lower slots. A contradiction. □

Table B.2
The upper bounds computed by the QPQCs, for $t < 1/2$ and $x_3 = 0$. The lower bound corresponds to the largest loss for a feasible solution computed by the solver. The upper bound corresponds to the smaller non-feasible lower bound computed be the solver. The last column shows the gap between them. Gaps smaller than 10^{-5} are insignificant due to the tolerance of the solver.

signs	phantoms	Status	Loss (lower bound)	Loss (upper bound)	Gap
$(-,+,+)$	$(r,r),(r,r)$	INFEASIBLE	–	–	–
$(-,+,+)$	$(b,r),(r,r)$	OPTIMAL	0.000000	0.000000	0.00
$(-,+,+)$	$(r,r),(b,r)$	OPTIMAL	0.000000	0.000000	0.00
$(-,+,+)$	$(b,r),(b,r)$	OPTIMAL	0.250002	0.250009	7.49e − 6
$(-,-,+)$	$(r,r),(r,r)$	INFEASIBLE	–	–	–
$(-,-,+)$	$(b,r),(r,r)$	OPTIMAL	0.000000	0.000000	0.00
$(-,-,+)$	$(r,r),(b,r)$	OPTIMAL	0.000000	0.000000	0.00
$(-,-,+)$	$(b,r),(b,r)$	OPTIMAL	0.499999	0.500003	3.43e − 6

$$\text{maximize} \quad \sum_{j=1}^{3} \left| \bar{v}_j - x_j \right| \tag{B.3}$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{3} x_j = 1 \\
&\hat{A} = \sum_{j=1}^{3} \hat{a}_j \\
&\hat{B} = \sum_{j,k\in[3],j\neq k} \hat{b}_{k,j} \\
&\hat{y}_{\hat{a}_1}(t^*) \le x_1 \le \hat{y}_{1-\hat{a}_2-\hat{a}_3-\hat{b}_{2,3}-\hat{b}_{3,2}}(t^*) \\
&\hat{y}_{\hat{a}_2}(t^*) \le x_2 \le \hat{y}_{1-\hat{a}_1-\hat{a}_3-\hat{b}_{1,3}-\hat{b}_{3,1}}(t^*) \\
&x_3 = 0, \\
&\hat{A} + \hat{B} \le 1 \\
&x_j \ge 0, \hat{a}_j \ge 0, && \forall j \in [3] \\
&\hat{b}_{k,j} \ge 0, && \forall j,k \in [3] \\
&0 \le t^* \le 1/2.
\end{aligned}$$

Fig. B.9. The Non-Linear Program used to upper bound the maximum ℓ_1-loss for the Piecewise Uniform mechanism for the special case $x_3 = 0$ and $t < 1/2$.

Using Lemma 5, we build a NLP to upper bound the ℓ_1-loss for this case, described in detail in Fig. B.9. Following our techniques from Section 3.2.2, we solve this NLP using a set of simpler, Quadratic Programs with Quadratic Constraints.

Since $x_3 = 0$, note that $\bar{v}_3 - x_3 \ge 0$ and the only sign patterns we need to tackle are $(-,-,+)$ and $(-,+,+)$. Recall that $(+,+,+)$ is only possible when the ℓ_1-loss is equal to 0.

For the phantom patterns, note (using Lemma 5) that $n - a_2 - a_3 - b_{2,3} - b_{3,2} > 1/2$ and $n - a_1 - a_3 - b_{1,3} - b_{3,1} > 1/2$; otherwise at least one of the x_1 or x_2 is equal to 0, a case we have already covered. Hence, the upper bounds in inequalities (B.1) and (B.2) refer to red phantoms. We don't have any guarantee for the lower phantoms, hence we use a phantom pattern in $\{(b,r),(r,r)\}^2$ for these cases. We check all 4 possible sign patterns, for projects 1 and 2. We don't need to examine project 3, since $x_3 \le y_k(t)$ for any $k \in \{0..n\}$ and any $t \in [0,1/2]$. Eventually, we need to check 2×4 QPQCs. By using the solver, we provide upper bounds for each program, which do not exceed $1/2 + \epsilon$, for some ϵ no larger than 10^{-5}. The detailed computed upper bounds are depicted in Table B.2.

Appendix C. Examples of QPQCs

In this section we provide some detailed examples of the Quadratic Programs with Quadratic Constraints we have used for the proof of Theorem 5. Fig. C.10 depicts an example of the first 27 programs Quadratic Programs we solve to get an initial upper bound, slightly higher than 2/3. Fig. C.11 depicts one of the 81 Quadratic Programs we solve to check if any other case may yield loss higher than 2/3. Finally, in Fig. C.12 we present a specific Quadratic Program for the special case where there exists a single zero value in the outcome.

$$
\begin{aligned}
&\text{maximize} \quad \bar{v}_1 - x_1 + \sum_{j=2}^{3} x_j - \bar{v}_j, \\
&\text{subject to,} \\
&\sum_{j=1}^{3} x_j = 1, \\
&\hat{A} = \sum_{j=1}^{3} \hat{a}_j, \\
&\hat{B} = \sum_{j,k\in[3],j\neq k} \hat{b}_{k,j}, \\
&\hat{z}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{k,j}, && \forall j \in [3] \\
&\hat{q}_j = \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k}, && \forall j \in [3] \\
&x_1 \geq \hat{z}_1(3-2t^*) - 2 + 2t^*, \\
&\hat{z}_1 \geq 1/2, \\
&1/2 \leq \hat{C} + \hat{z}_1 + \hat{q}_1, \\
&x_1 \leq \left(\hat{C} + \hat{z}_1 + \hat{q}_2\right)(3-2t^*) - 2 + 2t^*, \\
&x_j \geq \hat{z}_j(2t^*-1), && \forall j \in \{2,3\} \\
&x_j \leq \left(\hat{C} + \hat{z}_j + \hat{q}_j\right)(2t^*-1), && \forall j \in \{2,3\} \\
&\hat{z}_j \leq 1/2, && \forall j \in \{2,3\} \\
&1/2 \geq \hat{C} + \hat{z}_j + \hat{q}_j, && \forall j \in \{2,3\} \\
&\hat{A} + \hat{B} \leq 1, \\
&x_j \geq 0, \hat{a}_j \geq 0, && \forall j \in [3] \\
&\hat{b}_{k,j} \geq 0, && \forall j,k \in [3] \\
&1/2 \leq t^* \leq 1.
\end{aligned}
$$

Fig. C.10. The Quadratic Program with Quadratic Constraints for the maximum loss computation for the case $t > 1/2$, $(+,-,-)$, $((r,r),(b,b),(b,b))$. The inequalities referring to function $\hat{y}$ in Fig. 5 are now replaced by 8 new inequalities, for the specified phantom pattern.

$$
\begin{aligned}
&\text{maximize} \quad 1 \\
&\text{subject to} \\
&2/3 \leq \bar{v}_1 - x_1 + \sum_{j=2}^{3} x_j - \bar{v}_j, \\
&\sum_{j=1}^{3} x_j = 1, \\
&\hat{A} = \sum_{j=1}^{3} \hat{a}_j, \\
&\hat{B} = \sum_{j,k\in[3],j\neq k} \hat{b}_{k,j}, \\
&\hat{z}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{k,j}, && \forall j \in [3] \\
&\hat{q}_j = \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k}, && \forall j \in [3] \\
&x_j \geq \hat{z}_j(2t^*-1), && \forall j \in [3] \\
&x_j \leq \left(\hat{C} + \hat{z}_j + \hat{q}_j\right)(2t^*-1), && \forall j \in [3] \\
&\hat{z}_j \leq 1/2, && \forall j \in [3] \\
&\hat{C} + \hat{z}_j + \hat{q}_j \leq 1/2, && \forall j \in [3] \\
&\hat{A} + \hat{B} \leq 1, \\
&x_j \geq 0, \hat{a}_j \geq 0, && \forall j \in [3] \\
&\hat{b}_{k,j} \geq 0, && \forall j,k \in [3] \\
&0 \leq t^* \leq 1/2.
\end{aligned}
$$

Fig. C.11. The Quadratic Program with Quadratic Constraints which checks whether the case $t \leq 1/2$, $(+,+,-)$, $((b,b),(b,b),(b,b))$ yields ℓ_1-loss greater than $2/3$. The inequalities referring to function $\hat{y}$ in Fig. 5 are now replaced by 4 new inequalities, for the specified phantom pattern. This program has no feasible solutions.

$$
\begin{aligned}
&\text{maximize} \quad \sum_{j=1}^{2} x_j - \bar{v}_j + \bar{v}_3 \\
&\text{subject to} \\
&\sum_{j=1}^{2} x_j = 1, \\
&\quad \hat{A} = \sum_{j=1}^{3} \hat{a}_j, \\
&\quad \hat{B} = \sum_{j,k\in[3],j\neq k} \hat{b}_{k,j}, \\
&\quad x_1 \le (1 - \hat{a}_2 - \hat{a}_3 - \hat{b}_{2,3} - \hat{b}_{3,2}) \frac{4t^*}{n} - 2t^*, \\
&\quad x_2 \le (1 - \hat{a}_1 - \hat{a}_3 - \hat{b}_{1,3} - \hat{b}_{3,1}) \frac{4t^*}{n} - 2t^*, \\
&\quad x_3 = 0, \\
&\quad \hat{a}_1 \le 1/2, \\
&\quad \hat{a}_2 \le 1/2, \\
&\quad 1/2 \le 1 - \hat{a}_2 - \hat{a}_3 - \hat{b}_{2,3} - \hat{b}_{3,2}, \\
&\quad 1/2 \le 1 - \hat{a}_1 - \hat{a}_3 - \hat{b}_{1,3} - \hat{b}_{3,1}, \\
&\hat{A} + \hat{B} \le 1, \\
&\quad x_j \ge 0, \hat{a}_j \ge 0, && \forall j \in [3] \\
&\quad \hat{b}_{k,j} \ge 0, && \forall j,k \in [3] \\
&\quad 0 \le t^* \le 1/2.
\end{aligned}
$$

Fig. C.12. The Quadratic Program with Quadratic Constraints to compute an upper bound for the ℓ_1-loss for the case $t \le 1/2$, $(-,-,+)$, $((b,r),(b,r))$. The inequalities referring to function $\hat{y}$ in Fig. B.9 are now replaced by 6 new inequalities, for the specified phantom pattern.

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