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On trivalent logics, probabilistic weak deduction theorems, and a general import-export principle

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A R T I C L E I N F O

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A B S T R A C T

In this paper we first recall some results for conditional events, compound conditionals, conditional random quantities, p-consistency, and p-entailment. We discuss the equivalence between conditional bets and bets on conditionals, and review de Finetti's trivalent analysis of conditionals. But we go beyond de Finetti's early trivalent logical analysis and his later ideas, aiming to take his proposals to a higher level. We examine two recent articles that explore trivalent logics for conditionals and their definitions of logical validity and compare them with the approach to compound conditionals introduced by Gilio and Sanfilippo within the framework of conditional random quantities. As we use the notion of p-entailment, the full deduction theorem does not hold. We prove a Probabilistic Weak Deduction Theorem for conditional events. After that we study some variants of it, with further results, and we present several examples. Moreover, we illustrate how to derive new inference rules related to selected Aristotelian syllogisms. We focus on iterated conditionals and the invalidity of the Import-Export principle in the light of our Probabilistic Weak Deduction Theorem. We use the inference from a disjunction, *A or B*, to the conditional, *if not-A then B*, as an example to show the invalidity of this principle. We introduce a General Import-Export principle by examining examples and counterexamples. In particular, when considering the inference rules of System P, we find that a General Import-Export principle is satisfied, even if the assumptions of the Probabilistic Weak Deduction Theorem do not hold. We also deepen further aspects related to p-entailment and p-consistency. Finally, we briefly discuss some related work relevant to AI.

1. Introduction

Conditionals are important in human reasoning under uncertainty because they allow people to make decisions and inferences based on incomplete or uncertain information. Thus, the interpretation and evaluation of conditionals is a key challenge for artificial and human reasoning under uncertainty. Conditionals are relevant in AI because they are often used to implement if-then rules and are

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a crucial tool in a wide range of tasks and applications, from language processing to decision making. In order to manage uncertainty, we adopt the coherence-based subjective theory of de Finetti. This approach is realistic because no algebraic structure of events is required; moreover, conditioning events with zero probability can be properly managed. Based on the relevant information, we can make coherent probabilistic assessments on the events of interest; moreover, by de Finetti's fundamental theorem of probability, we can coherently extend the initial assessment to any further event. [3] There has recently been increasing interest in de Finetti's analysis of the conditional, *if A then B*, as what he called a *conditional event* ([37,39]) and symbolized as $B|A$. Some logicians and psychologists of reasoning have focused on his *trivalent* classification of conditionals as true, false, or void ([5,6,47,94,95,113]). This classification has been also adopted in the field of artificial intelligence and probabilistic nonmonotonic reasoning (see, e.g., [9,10,19,29,30,57,61,106,109]). Research on trivalent logics and compounds of conditionals has been presented in many papers (see, e.g., [1,8,13,16,27,22,23,46,78,83,100,101,103,121]).

A number of authors argue that the set of values {*true, false, void*} of a conditional event $B|A$ should be represented in numerical terms as $\{1, 0, P(B|A)\}$, where the (subjective) probability $P(B|A)$ may assume any value in the interval between 0 and 1, $[0, 1]$ (see, e.g., [56,60,69,82,87,91,108,116]). Based on this representation, the geometrical approach for coherence checking has been extended to conditional probability assessments ([56]).

In this paper, we give our reasons for adopting this proposal, and we explain how to use it as the basis of an account of some compound conditionals as conditional random quantities with values in the interval $[0, 1]$. We follow a principled line in our analysis at the logical, cognitive, and psychological levels and discuss our work in the light of some selected recent literature. In particular, we examine two recent articles that adopt the trivalent view of conditionals and compare trivalent definitions of validity with the notion of probabilistic validity, p-validity. We apply the coherence-based probability interpretation of conditionals, and compositions of conditionals introduced in [66,69], by studying their connections to the consequence relation (i.e., the probabilistic relation between conclusions and premises). We give, in our account of conditional reasoning, some results on several probabilistic weak versions of the deduction theorem. Moreover, we study properties of the behaviour of iterated conditionals in the context of a General Import-Export principle, by examining some well known inference rules of nonmonotonic reasoning. Compared to other approaches in the literature, the compound and iterated conditionals utilized in this paper, besides preserving basic logical and probabilistic properties, allow us to properly interpret some intuitively acceptable complex sentences involving conditionals ([74]). In particular, they allow us to suitably characterize the p-validity of inference rules in nonmonotonic reasoning (see, e.g., [62,70,73]).

We follow de Finetti in closely comparing an indicative conditional (IC), with a *conditional bet* (CB). When we make this link, we do not look at an IC, *if A then B*, or a CB, informally of the form *if A then I bet on B*, as a material conditional, which is logically equivalent to $\overline{A} \vee B$ (not-A or B), or a Stalnaker/Lewis conditional (see [118] for the differences in detail). A CB only comes into effect when A holds. We will show formally later (see Section 2.2) that such a conditional bet can be interpreted as a bet on the conditional event $B|A$. For an informal example, let the IC be *If the coin is spun then it will come up heads*, and the CB be *If the coin is spun, I bet it comes up heads*. There is this parallel relation between these conditionals. The IC is true, and the CB is won, when the conjunction $A \wedge B$ holds, the IC is false, and the CB is lost, when $A \wedge \overline{B}$ holds. When $\overline{A}$ holds, the IC is void, and a counterfactual, *If the coin had been spun, then it would have come up heads*, might be used in its place, and the CB will also be void in the sense of being neither won nor lost.

At this stage, it might appear that we have described only a trivalent account of the conditional, yielding a three-valued truth table, with the numerical values 1, 0, and v for void. But de Finetti did not stop at this early trivalent analysis ([39]): he eventually extended his theory to another level.

In [38, pp. 1164-1165], de Finetti distinguished three levels of analysis: Level 0, Level 1, and Level 2. In particular, Level 0 is that of classical logic, where we assume that every statement is true or false, i.e., in numerical terms 1 or 0. This topic is also discussed in [6,104]. A complete analysis of the levels of knowledge is beyond the scope of this paper. Here, we briefly illustrate the three levels of an individual's knowledge regarding any event E, providing some details (for a related discussion see also [32]).

- Level 0. From a logical point of view, E is false, "0", or true, "1".
- Level 1. From a cognitive point of view, E can be false, "0", or uncertain, "?", or true, "1".
- Level 2. From a psychological (subjective) point of view, in case of certainty, E is false, "0", or true, "1". In case of uncertainty, E is given a (subjective) probability "$P(E)$".

In other words, Level 0 specifies that events are two-valued logical entities (independently from the opinion of individuals).

Level 1 clarifies that individuals can be certain (i.e., they know that E is false, or that it is true), or uncertain about E (i.e., they do not know whether E is false, or true).

Level 2 specifies that people who are uncertain about E represent their uncertainty by a (subjective) probability, which is a measure of their degree of belief on E being true.

By considering now a conditional event $B|A$, at Level 0 we know that $B|A$ is true, or false, or void. At Level 1, we are uncertain about the truth-value of $B|A$. At Level 2, for instance, when we assess $P(B|A)$ we represent the *(conditional) uncertainty* of $A \wedge B$ with respect to the two alternatives: $A \wedge B$ or $A \wedge \overline{B}$.

[3] Notice that, by exploiting de Finetti's fundamental theorem of prevision and linear programming techniques, a critical discussion of the quantum violation of Bell's inequality has been presented in [93].

When the coin is not spun, we are uncertain whether it would have landed heads. At Level 2, we can refine our uncertainty into different degrees of belief, i.e., of subjective probability, and at this level, we can make precise bets on outcomes. For example, we may know that the coin has a worn edge on one side and a certain tendency to come up heads. At this level of de Finetti's analysis, v is what we are willing to pay, in money or epistemic utility, in a fair bet, which will pay us 1 unit, of money or epistemic utility, when $A \wedge B$ holds, and 0 units, when $A \wedge \overline{B}$ holds, and which will be returned to us when $\overline{A}$ holds. In the simplest case, we may believe that the coin is fair, and then we will pay, 0.5 of a euro, 50 cents, to win 1 euro provided that the coin is spun and comes up heads. We will receive 0 euros when the coin is spun and comes up tails, and we will get our 50 cents back when the coin is not spun.

We can derive what v is from the expected value, or prevision, of the conditional bet when this bet is fair, i.e., its expected value is 0. We have

$$0 = P(A \wedge B)(1 - v) + P(A \wedge \overline{B})(0 - v) + P(\overline{A})(v - v),$$

or equivalently: $v = P(A \wedge B) + P(\overline{A})v$, and so $P(A)v = P(A \wedge B)$ and $v = P(A \wedge B)/P(A)$ when $P(A) > 0$.

We see, then, that v is the conditional probability of B given A, $P(B|A)$, and we identify v with $P(B|A)$ for both IC and CB. Further aspects will be examined in Section 2.2.

Our full de Finetti analysis, at Level 2, is like an "interval-valued" account, because $P(B|A)$ can have any value from 0 to 1. If we believe the coin is double-headed, we will say $P(B|A) = 1$. If we think that the coin is slightly biased to heads, we may judge $P(B|A) = 0.55$.

There are certainly reasons to study the partial three-valued analysis at Level 1. It is important, for instance, to study uncertainty in the psychology of reasoning, without presupposing that this state of mind can always be made more precise with probability judgements. People could sometimes just be uncertain about X and unable to refine this to a probability judgement, $P(X)$. Nevertheless, there are many differences in a full Level 2 de Finetti analysis. For instance, a "logical truth", such as most simply *if A then A*, is never different from the value 1 at Level 2, because $P(A|A) = 1$.

Another advantage is that the full analysis allows us to define a logically valid inference in a more intuitive and natural way than can be done at Level 1, as we will point out in the formal development below, where we compare different definitions of logical validity (see further [33], on logically valid inference).

Notice that the compound conditionals, such as conjunctions and disjunctions, introduced by Gilio and Sanfilippo ([65,66,69]), were defined (not as three-valued objects, but) as suitable conditional random quantities, where some of their values are (coherent) probability values.

For instance, given two conditional events $A|H$ and $B|K$, if (for a given individual) $P(A|H) = x$ and $P(B|K) = y$, then the possible values of the conjunction $(A|H) \wedge (B|K)$ are: $1, 0, x, y, z$, where z is (for the same individual) the prevision of $(A|H) \wedge (B|K)$. Then, we directly define compound conditionals at Level 2. In what follows this aspect will be implied, and it will be clear from the context at what level we are examining the objects.

The deduction theorem is a fundamental metalogical theorem in classical logic.[4] It explains the relation between the material conditional, $\overline{A} \vee B$, and logical consequence. More formally, in classical propositional logic, the (full) deduction theorem states that the premise set $\Gamma \cup \{A\}$, where Γ is a set of propositional formulas, logically implies the conclusion B if and only if Γ implies the material conditional $\overline{A} \vee B$ (see, e.g., [53]). However, the deduction theorem is not generally valid in our probability logic, where Γ is a (p-consistent) set of conditional events, the consequence relation is based on p-entailment ($\Rightarrow_p$) and the material conditional $\overline{A} \vee B$ is replaced by the conditional event $B|A$. Some closely related metalogical theorems for the material conditional, particularly monotonicity, contraposition, and transitivity, usually do not hold in probability logic for the conditional event. In this paper, we will investigate the conditions under which a (restricted) version of the deduction theorem, the Probabilistic Weak Deduction Theorem, holds. We will also compare the approach of this paper with other proposals for a probabilistic logic and what these imply about the deduction theorem. We believe that the interpretation and evaluation of simple and compound conditionals are a key challenge for reasoning about conditionals in AI. In particular, we extend trivalent logics based on compound and nested conditionals defined as conditional random quantities in the setting of coherence. In this way all the basic logical and probabilistic properties are preserved, providing a significant advantage compared to other approaches (see, e.g., [21,74,75]). Our novel contributions include proving, for the first time, probabilistic weak deduction theorems, new inference rules, and studying a General Export-Import principle. More precisely:

- we make some comparisons with selected trivalent semantics;
- we give probabilistic weak versions of the deduction theorem;
- we give new inference rules related to Aristotelian syllogisms;
- we introduce a General Import-Export principle;

[4] The presence of the deduction theorem allows proofs in logic which are "much more natural, simple and convenient" compared to its absence [120, p. 80]. Historically, although the distinction between object- and meta-language was introduced much later, the admissibility of the deduction theorem, however, was already "taken for granted by Aristotle and explicitly by the Stoics" [88, p. 320]. According to Kleene, "the deduction theorem as an informal theorem proved about particular systems like the propositional calculus and the predicate calculus [. . .] first appears explicitly in Herbrand 1930 [*Recherches sur la théorie de la démonstration*] (and without proof in Herbrand 1928 [*Sur la théorie de la démonstration*]); and as a general methodological principle for axiomatic-deductive systems in Tarski 1930 [*Über einige fundamentale Begriffe der Metamathematik*]. According to Tarski 1956 [*Logic, semantics, metamathematics*] footnote to p. 32, it was known and applied by Tarski since 1921" [86, p. 39]. Surma ([120]) traces the deduction theorem back to Bolzano's *Wissenschaftslehre* (1837). Bolzano's discovery was rediscovered by Tarski in the 1920ies. Surma also mentions that the deduction theorem's name was coined by David Hilbert [120, p. 79].

- we show that the General Import-Export principle is satisfied for some well-known inference rules of System P.

The paper is organized in the following way. In Section 2, we recall some results for conditional events, conditional random quantities, and the equivalence, when evaluating a conditional probability, of the scheme of betting on a conditional, the scheme of a conditional bet, and a third scheme. We also recall the notions of p-consistency and p-entailment. Then, we illustrate some notions of and results on compound and iterated conditionals in the framework of conditional random quantities. Moreover, we introduce the generalized Equation or conditional prevision hypothesis. In Section 3, we examine two recent articles that explore trivalent logics for conditionals and their definitions of logical validity, by making a comparison with the compound and iterated conditionals adopted in this paper. In particular, we discuss the notions of assertability for conditionals and $SS \cap TT$-validity. In Section 4, we show that the deduction theorem does not follow for conditional events when using p-entailment, and then we obtain some probabilistic weak versions of the deduction theorem, with further results and examples. Moreover, we give new inference rules related to some Aristotelian syllogisms. In Section 5 we first focus on iterated conditionals and the invalidity of the Import-Export principle for the conditional event. In particular, we consider the inference from a disjunction, A *or* B, to a conditional, *if not-*A *then* B, as an example and explain how the invalidity of the classical deduction theorem is related to the invalidity of the Import-Export principle for the conditional event. Then, we introduce a *General Import-Export principle* in relation to iterated conditionals, and we give a result which relates it to p-consistency and p-entailment. We also illustrate the validity of the General Import-Export principle for some inference rules of System P where the Probabilistic Weak Deduction Theorem is not applicable. We make some further comments on p-consistency and on p-entailment for a family of conditional events $\mathcal{F}$ and a conditional event $E|H$, as well as on the p-entailment from $\mathcal{F} \cup \{H\}$ to the event E. In Section 6 we briefly discuss some related work on conditionals. Finally, in Section 7 we give some concluding remarks. In Appendix A we illustrate and expand some aspects of de Finetti's trivalent logic. In Appendix B we introduce and discuss a variant of the notion of $SS \cap TT$-validity.

2. Preliminary notions, results, and discussions

In this section we recall some general notions and results. In particular, we recall the notion of coherence for conditional probability and prevision assessments. We discuss conditional bets, bets on conditionals, and a further equivalent scheme which we implicitly use when defining compound conditionals. We recall the notions of p-consistency and p-entailment for conditional random quantities with values in the unit interval. Then, we illustrate the notions of compound and iterated conditionals, we cover basic properties of compound and iterated conditionals, and we introduce the generalized Equation.

2.1. *Coherent probabilities and coherent previsions*

Uncertainty about objective facts will be formalized here by judgements about events. In formal terms, an event A is a two-valued logical entity: *true*, or *false*. The *indicator* of A, denoted by the same symbol, is 1, or 0, according to whether A is true, or false, respectively. We denote by Ω the sure event and by $\emptyset$ the impossible one. We denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, the conjunction (resp., disjunction) of A and B. By $\bar{A}$ we denote the negation of A. Given two events A and B we say that A logically implies B, denoted by $A \subseteq B$, when $A\bar{B} = \emptyset$, or equivalently $\bar{A} \vee B = \Omega$. Notice that the symbol $\subseteq$ will be also used to denote the inclusion relation between two sets. Therefore the interpretation of $\subseteq$ will be context-dependent. Moreover, we use the symbol $\Longleftrightarrow$ to denote the logical equivalence *if and only if*.

Given two events A and H, with $H \neq \emptyset$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively. The notion of logical implication between two events has been generalized to the case of two conditional events by Goodman and Nguyen in [77]. Given two conditional events $A|H$ and $B|K$, $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$, that is ([77, formula (3.18)])

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } \bar{B}K \subseteq \bar{A}H. \tag{1}$$

In the betting framework, to assess $P(A|H) = x$ amounts to saying that, for every real number s, you are willing to pay an amount sx and to receive s, or 0, or sx, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true (in this case the bet is called off), respectively. Hence, for the random gain $G = sH(A - x)$, the possible values are $s(1 - x)$, or $-sx$, or 0, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively.

Moreover in general, a conditional probability assessment P on an arbitrary family $\mathcal{K}$ of conditional events can be looked at as a (real) function defined on $\mathcal{K}$. Given a function P on $\mathcal{K}$, consider a finite subfamily $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{P} = (x_1, \dots, x_n)$, where $x_i = P(E_i|H_i)$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(E_i - x_i)$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, by the *betting scheme* of de Finetti, coherence is defined as:

Definition 1. A conditional probability assessment P on the family $\mathcal{K}$ is coherent if and only if, $\forall n \geqslant 1$, $\forall s_1, \dots, s_n$, $\forall \mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{K}$, it holds that: $min\, \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant max\, \mathcal{G}_{\mathcal{H}_n}$.

In other words, the *coherence* of P means that *in any finite combination of n bets, after discarding the case where all the bets are called off, it cannot happen that the values of the random gain are all positive, or all negative*. If you are incoherent a Dutch book can be made against you, i.e. there exists a finite combination of n bets, ensuring that you suffer a loss in all the cases where at least a bet is not called off.

We denote by X a *random quantity*, that is an uncertain real quantity, which has a well determined but unknown value. We assume that X has a finite set of possible values. Given any event $H \neq \emptyset$, to assess μ as your prevision of "*X conditional on H*" (or short: "*X given H*"), which we denote by $\mathbb{P}(X|H)$, means that for any given real number s you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. The associated random gain is $G = sH(X - \mu)$. A conditional prevision assessment $\mathbb{P}$ on an arbitrary family $\mathcal{K}$ of finite conditional random quantities is a (real) function defined on $\mathcal{K}$. Given a function $\mathbb{P}$ on $\mathcal{K}$, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(X_i - \mu_i)$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, the notion of coherence for $\mathbb{P}$ is defined as:

Definition 2. A conditional prevision assessment $\mathbb{P}$ on a family $\mathcal{K}$ is coherent if and only if, $\forall n \geqslant 1$, $\forall s_1, \dots, s_n$, $\forall \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $min\, \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant max\, \mathcal{G}_{\mathcal{H}_n}$.

In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. Given a conditional event $A|H$ with $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1 - H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{2}$$

Notice that $\mathbb{P}(AH + x\overline{H}) = xP(H) + xP(\overline{H}) = x$. The third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = x$. When $H \subseteq A$ (i.e., $AH = H$), it holds that $P(A|H) = 1$; then, for the indicator $A|H$ it holds that

$$A|H = AH + x\overline{H} = H + \overline{H} = 1, \ \text{(when } H \subseteq A\text{)}. \tag{3}$$

Likewise, if $AH = \emptyset$ (and $H \neq \emptyset$), it holds that $P(A|H) = 0$; then

$$A|H = 0 + 0\overline{H} = 0, \ \text{(when } AH = \emptyset\text{)}. \tag{4}$$

The negation $\overline{A|H}$ of the conditional event $A|H$ is defined as

$$\overline{A|H} = \overline{A}|H. \tag{5}$$

Then, as $A|H + \overline{A}|H = 1$, for the indicator of $\overline{A|H}$ it holds that $\overline{A|H} = \overline{A}|H = 1 - A|H$. We recall that, given two conditional events $A|H$ and $B|K$, it holds that ([74, Equation (15)])

$$A|H \leqslant B|K \iff A|H \subseteq B|K, \ \text{or } AH = \emptyset, \ \text{or } K \subseteq B. \tag{6}$$

Given a random quantity X and an event $H \neq \emptyset$, with a conditional prevision assessment $\mathbb{P}(X|H) = \mu$, likewise formula (2), the conditional random quantity $X|H$, in agreement with [92], is defined as

$$X|H = XH + \mu\overline{H}. \tag{7}$$

Notice that the prevision of the conditional random quantity $X|H$ coincides with the conditional prevision μ, indeed

$$\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(XH) + \mu P(\overline{H}) = \mu P(H) + \mu P(\overline{H}) = \mu. \tag{8}$$

For a discussion on this extended notion of a conditional random quantity and on the notion of coherence of a conditional prevision assessment see, e.g., [69,71,116].

Remark 1. Given a random quantity X and an event $H \neq \emptyset$, if $XH = cH$ for a suitable constant c, then $\mathbb{P}(X|H) = c$ and hence $X|H = cH + c\overline{H} = c$. In particular, for any event $A \neq \emptyset$, $A|A = 1$ and $\overline{A}|A = 0$. Moreover, given a constant a and a conditional random quantity $X|H$, with $\mathbb{P}(X|H) = \mu$, it holds that

$$(aX)|H = aXH + \mathbb{P}[(aX)|H]\overline{H} = a(XH + \mu\overline{H}) = a \cdot X|H.$$

More in general, given any n random quantities $X_1, \dots, X_n$, n real quantities $a_1, \dots, a_n$, and n events $H_1 \neq \emptyset, \dots, H_n \neq \emptyset$, by taking into account (7) and by the linearity property of unconditional previsions, a coherent conditional prevision $\mathbb{P}$ satisfies the following property of linearity (see, e.g., [65])

$$\mathbb{P}(\sum_{i=1}^{n}(a_i X_i)|H_i) = \mathbb{P}[\sum_{i=1}^{n} a_i(X_i|H_i)] = \sum_{i=1}^{n} a_i \mathbb{P}(X_i|H_i).$$

Remark 2. We recall the set of constituents generated by a family of conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ which will be exploited in Section 3.2.2. By expanding the expression

$$\Omega = \bigwedge_{i=1}^{n}(E_i H_i \vee \bar{E}_i H_i \vee \overline{H}_i)$$

we obtain a disjunction of 3^n conjunctions $A_1 \cdots A_n$, where $A_i \in \{E_i H_i, \overline{E}_i H_i, \overline{H}_i\}$. By discarding the conjunctions which coincide with the impossible event $\emptyset$, the sure event can be represented as a suitable disjunction $\Omega = C_0 \vee C_1 \vee \cdots C_m$, $m \leqslant 3^n - 1$, where $C_0 = \overline{H}_1 \cdots \overline{H}_n$ and $C_1 \vee \cdots \vee C_m = H_1 \vee \cdots \vee H_n$. The set of conjunctions $\{C_0, C_1, \dots, C_m\}$ is the set of constituents generated by $\mathcal{F}$. Under logical independence of $E_1, H_1, \dots, E_n, H_n$, it holds that $m = 3^n - 1$.

2.2. Conditional bets and bets on conditionals

In this section, we illustrate that the assessment $P(B|A) = x$ can equivalently be made by a *bet on a conditional* or by a *conditional bet*. Then, we examine a further equivalent scheme for such probability assessment. A conditional *if A then B* is represented by the conditional event $B|A$, which in a pioneering paper by de Finetti ([39]) was defined as a three-valued object with possible values: *true, false, void*; thus we adopt the Equation, or conditional probability hypothesis (CPH), that is $P(\textit{if } A \textit{ then } B) = P(B|A)$. We point out that the approach of de Finetti has been developed in many respects. In numerical terms $B|A$ has possible values 1, 0, and $P(B|A)$, which is useful, for instance, in order to extend the geometrical approach for coherence checking to the case of conditional events. Moreover, the coherence-based approach allows us to study in full generality the notion of p-entailment. We next consider three equivalent betting schemes, (S_1), (S_2), and (S_3), in order to evaluate $P(B|A)$. Although the three schemes are equivalent, they differ in interpretation: (S_1) pertains to a *conditional bet*, (S_2) to a *bet on a conditional*, and (S_3) to a *bet on a random quantity*. In the paper of de Finetti ([39, p. 186]) the assessment $P(B|A) = x$ is operatively based on the following *conditional bet*:

Scheme (S_1) You evaluate $P(B|A)$ when A is uncertain. After A is verified, the bet comes into effect and you accept to pay x in order to bet on B, by receiving 1 if B is true, or 0 if B is false. In the case where A is not verified the bet has no effect, because there is no bet. Within the scheme (S_1) we can say that *if A, then I bet that B* is a bet on B conditionally on A being true. However, if $\overline{A}$, then there is *no bet* within the scheme (S_1).

Then, by definition, the coherence of the assessment $P(B|A) = x$ is checked by (only) considering the cases where the bet is effective, that is when A is verified.

In equivalent terms, scheme (S_1) can be expressed as a bet on a conditional by the following scheme ([37, p. 145]):

Scheme (S_2) If you assess $P(B|A) = x$, then (before knowing the truth value of A) you accept to pay x, to receive 1 if AB is true, or 0 if $A\overline{B}$ is true, or x if $\overline{A}$ is true (the bet is called off). For the checking of coherence only the cases in which the bet is not called off are considered (that is, the case when A is *false*, in which the bet is called off, is discarded).

Based on the scheme (S_2), the indicator of $B|A$, denoted by the same symbol, is defined as (see, e.g. [72, Section 2.2])

$$B|A = AB + x\overline{A} = \begin{cases} 1, & \text{if } AB \text{ is true,} \\ 0, & \text{if } A\overline{B} \text{ is true,} \\ x, & \text{if } \overline{A} \text{ is true.} \end{cases} \tag{9}$$

Then, when you assess $P(B|A) = x$, you agree to pay the amount x, to receive the random quantity $B|A$. Of course, for the prevision of the indicator it holds that

$$\mathbb{P}(B|A) = \mathbb{P}(AB + x\overline{A}) = P(AB) + xP(\overline{A}) = x[P(A) + P(\overline{A})] = x = P(B|A).$$

Within the scheme (S_2) we are considering a bet on the conditional event $B|A$, or on the conditional *if A then B*, and this bet is equivalent to the conditional bet on B, supposing that A is true (and nothing more). In other words the notions of conditional bets and bets on conditionals coincide, in agreement with the equivalence of (S_1) and (S_2).

Now we will examine a further scheme (S_3), in which we consider a bet on the random quantity $Y = AB + y\overline{A}$, where by definition y is the prevision of Y. In the betting framework y is the amount to be paid in order to receive Y. You must assess y, remembering that for checking coherence, you *discard all cases where you receive back the amount you paid, whatever the amount be*. Specifically, you discard the case where $\overline{A}$ is true.

Question: in what way does the quantity y (subjectively) depend on the events A and B?

Answer: it can be shown that $y = P(B|A)$.

Scheme (S_3) You have to assess the value y, which represents your prevision of the random quantity $AB + y\overline{A}$; then, by the betting scheme, you agree to pay y with the proviso to receive the random quantity $AB + y\overline{A}$.

Coherence condition for the scheme (S_3)*:* in order to check the coherence of y, you must discard all the cases where you receive back y, whatever y be.

For an individual who wants to assess $P(B|A)$ are the schemes (S_2) and (S_3) equivalent? In other words, is it the case that $y = x = P(B|A)$?

We show below that the answer is "Yes", i.e. (S_2) and (S_3) are equivalent.
We observe that, within the scheme (S_2), you assess $P(B|A) = x$; then you pay x and you receive $AB + x\bar{A}$. Within the scheme (S_3), you assess the prevision y of $AB + y\bar{A}$; then you pay y with the proviso that you receive $AB + y\bar{A}$. As a consequence, if we consider a bet on the random quantity $(AB + x\bar{A}) - (AB + y\bar{A})$, you agree, for instance, to pay $x - y$ by receiving $(AB + x\bar{A}) - (AB + y\bar{A}) = (x - y)\bar{A}$. The random quantity $(x - y)\bar{A}$ is equal to $x - y$, or 0, according to whether $\bar{A}$ is true, or false, respectively. By the coherence condition in the scheme (S_3), the case where $\bar{A}$ is true must be discarded because in this case you receive back the paid amount $x - y$ (whatever $x - y$ may be). Therefore, for checking coherence, we only consider the value 0 that you receive when $\bar{A}$ is false. Then, in order for the assessment $x - y$ on $(AB + x\bar{A}) - (AB + y\bar{A})$ to be coherent, it must be that $x - y = 0$, that is $x = y$.

In conclusion, for the same individual, it is equivalent to evaluate $P(B|A)$ by the scheme (S_1), or (S_2), or (S_3). The previous reasoning can be applied to a conditional random quantity $Z|H$ when evaluating the conditional prevision $\mathbb{P}(Z|H)$; see also [49, Section 2.1].

2.3. Conditional random quantities and the notions of p-consistency and p-entailment

We now consider the notions of p-consistency and p-entailment for conditional random quantities which take values in a finite subset of $[0, 1]$ ([118]).

Definition 3. Let $\mathcal{F}_n = \{X_i|H_i\,,\, i = 1, \dots, n\}$ be a family of n conditional random quantities which take values in a finite subset of $[0, 1]$. Then, $\mathcal{F}_n$ is *p-consistent* if and only if, the (prevision) assessment $(\mu_1, \mu_2, \dots, \mu_n) = (1, 1, \dots, 1)$ on $\mathcal{F}_n$ is coherent.

Definition 4. A p-consistent family $\mathcal{F}_n = \{X_i|H_i\,,\, i = 1, \dots, n\}$ *p-entails* a conditional random quantity $X|H$ which takes values in a finite subset of $[0, 1]$, denoted by $\mathcal{F}_n \Rightarrow_p X|H$, if and only if for any coherent (prevision) assessment $(\mu_1, \dots, \mu_n, z)$ on $\mathcal{F}_n \cup \{X|H\}$: if $\mu_1 = \cdots = \mu_n = 1$, then $z = 1$.

We say that the inference from a p-consistent family $\mathcal{F}_n$ to $X|H$ is *p-valid* if and only if $\mathcal{F}_n \Rightarrow_p X|H$.

Remark 3. Notice that, if we consider conditional events instead of conditional random quantities, we obtain the notions of p-consistency, p-entailment, and p-validity given in the setting of coherence (see, e.g., [9,57,63,64,67]). We recall that the notion of p-entailment in the setting of coherence ([57, Definition 6], [9, Theorem 4.9]) is based on Adams's theory ([1]). However, our analysis is different from Adams's with respect to the treatment of zero-probability antecedents. We recall that in his theory Adams by convention defines $P(E|H) = 1$ when $P(H) = 0$. From this convention, however, problematic consequences follow. For example, we would obtain that the following condition

$$P(C|A) = 1 \iff P(\bar{A}|\bar{C}) = 1 \tag{10}$$

is satisfied, as shown below.

($\Rightarrow$) If $P(C|A) = 1$, it holds that $P(A) = P(C|A)P(A) = P(AC)$ and hence $P(A\bar{C}) = 0$. Then, $P(\bar{C}) = P(\bar{A}\bar{C})$ and, in the case $P(\bar{C}) > 0$, it follows $P(\bar{A}|\bar{C}) = \frac{P(\bar{A}\bar{C})}{P(\bar{C})} = 1$. In the case $P(\bar{C}) = 0$, by the convention of Adams, it still follows $P(\bar{A}|\bar{C}) = 1$.

($\Leftarrow$) If $P(\bar{A}|\bar{C}) = 1$, by a symmetrical reasoning, it follows $P(C|A) = 1$.

Adams could not accept Definition 4, because his convention would then make Contraposition valid, and he knew that it should be invalid in a probabilistic approach, what he showed by counterexamples [1, p. 14f]. Note that, in the coherence-based analysis, the assessment (x, y) on $\{C|A, \bar{A}|\bar{C}\}$ is coherent, for every $(x, y) \in [0, 1]^2$; thus, the condition (10) does not hold and hence Contraposition is not p-valid. Of course, equation (10) is satisfied under the restriction $P(A) > 0$ and $P(\bar{C}) > 0$, that is under the assumption that the antecedents have positive probability. However, the notion of p-entailment given in the setting of coherence does not include these restrictions and allows zero-probability antecedents.

The quasi conjunction $QC(\mathcal{F})$ of a family of conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ is defined as the following conditional event

$$QC(\mathcal{F}) = \bigwedge_{i=1}^{n}(\bar{H}_i \vee E_iH_i) \,\Big|\, \bigvee_{i=1}^{n} H_i. \tag{11}$$

A characterization of p-entailment by means of the quasi conjunction is given below ([67, Theorem 6], see also [45, Definition 1]).

Theorem 1. *Let be given a p-consistent family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$ and a conditional event $E|H$. The following assertions are equivalent:*

1. *$\mathcal{F}_n$ p-entails $E|H$;*
2. *The assessment $\mathcal{P} = (1, \dots, 1, z)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \dots, n$, $P(E|H) = z$, is coherent if and only if $z = 1$;*
3. *The assessment $\mathcal{P} = (1, \dots, 1, 0)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \dots, n$, $P(E|H) = 0$, is not coherent;*

4. *There exists a nonempty subset S of $\mathcal{F}_n$ such that $\mathcal{QC}(S) \subseteq E|H$, or $H \subseteq E$.*
5. *There exists a nonempty subset S of $\mathcal{F}_n$ such that $\mathcal{QC}(S)$ p-entails $E|H$.*

We observe that, given a p-consistent family of n conditional events $\mathcal{F}_n$ and a further conditional event $E|H$, if the assessment $(1,\dots,1,z)$ on $\mathcal{F}_n \cup \{E|H\}$ is coherent for some $z \in (0,1)$, then it can be proved that $(1,\dots,1,z)$ is coherent for every $z \in [0,1]$ (see [67, Theorem 8]).

2.4. Compound and iterated conditionals

In this section we cover the notions of compound and iterated conditionals, and we note some basic properties. We start with the notion of conjunction of two conditional events ([69]).

Definition 5. Given a coherent probability assessment $P(A|H)=x$, $P(B|K)=y$, the conjunction of $A|H$ and $B|K$ is defined as

$$(A|H)\wedge(B|K)=\begin{cases}1, & \text{if } AHBK \text{ is true,}\\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,}\\ x, & \text{if } \overline{H}BK \text{ is true,}\\ y, & \text{if } AH\overline{K} \text{ is true,}\\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true,}\end{cases}$$

that is,

$$(A|H)\wedge(B|K)=AHBK+x\overline{H}BK+yAH\overline{K}+z\overline{H}\,\overline{K}, \tag{12}$$

where z is the prevision of $(A|H)\wedge(B|K)$.

We require that (x,y,z) be coherent (and hence z is a coherent extension of (x,y)). Notice that, unlike for conditional events which are three-valued objects, the conjunction $(A|H)\wedge(B|K)$ is no longer a three-valued object, but a five-valued object with values in $[0,1]$. As for the conditional event, the values 1, 0, and z, in a conditional bet where you pay z, correspond to the cases *win*, *lose*, and *money back*, when both conjuncts are true, at least one is false, or both conjuncts are void, respectively. The additional two values x and y stem from the fact that we need also to consider the two cases of *partial win*, where one conjunct is true and the other one is void. In these last two cases, one can also say that the conjunction is *partially true* ([18]). As shown below, the conjunction $(A|H)\wedge(B|K)$ coincides with the conditional random quantity $(AHBK+x\overline{H}BK+yAH\overline{K})|(H\vee K)$.

Remark 4. Based on Section 2.2, we can verify that the conjunction

$$(A|H)\wedge(B|K)=AHBK+x\overline{H}BK+yAH\overline{K}+z\overline{H}\,\overline{K},$$

where $x=P(A|H)$, $y=P(B|K)$, and $z=\mathbb{P}[(A|H)\wedge(B|K)]$, coincides with the conditional random quantity $Z|(H\vee K)=(AHBK+x\overline{H}BK+yAH\overline{K})|(H\vee K)$. In the framework of the betting scheme (see scheme (S_3)) z is the amount to be paid (resp., to be received) in order to receive (resp., to pay) the random amount $(A|H)\wedge(B|K)$. Moreover, in the case when $\overline{H}\,\overline{K}$ is true you receive back (resp., give back) z, whatever it be; thus, in order to check coherence, the case $\overline{H}\,\overline{K}$ must be discarded. By setting $\mu=\mathbb{P}(Z|(H\vee K))$, from (7), it holds that

$$\begin{aligned}Z|(H\vee K)&=(AHBK+x\overline{H}BK+yAH\overline{K})(H\vee K)+\mu\overline{H}\,\overline{K}=\\ &=AHBK+x\overline{H}BK+yAH\overline{K}+\mu\overline{H}\,\overline{K}.\end{aligned}\tag{13}$$

Moreover, for the random quantity

$$D=(A|H)\wedge(B|K)-Z|(H\vee K)=(z-\mu)\overline{H}\,\overline{K},$$

it holds that $\mathbb{P}(D)=z-\mu$. Then, in a bet on D one should pay, for instance, $z-\mu$ by receiving the random amount D, with the bet called off when $\overline{H}\,\overline{K}$ is true (indeed, in this case one would receive back the paid amount $z-\mu$, whatever it be). We observe that D is 0, or $z-\mu$, according to whether $H\vee K$ is true, or $\overline{H}\,\overline{K}$ is true, respectively. Therefore, when the bet is not called off it holds that $D=0$ and, by coherence, $\mathbb{P}(D)=z-\mu=0$, that is $\mathbb{P}[(A|H)\wedge(B|K)]=z=\mu=\mathbb{P}[Z|(H\vee K)]$. Thus, from (12) and (13) we obtain that

$$(A|H)\wedge(B|K)=(AHBK+x\overline{H}BK+yAH\overline{K})|(H\vee K). \tag{14}$$

By the previous comments the conjunction can be also defined as the conditional random quantity in (14) (as done in [72, Definition 2]), that is, by noting (8),

$$(A|H)\wedge(B|K)=AHBK+x\overline{H}BK+yAH\overline{K}+z\overline{H}\,\overline{K},$$

where $x=P(A|H)$, $y=P(B|K)$, $z=\mathbb{P}[(AHBK+x\overline{H}BK+yAH\overline{K})|(H\vee K)]$.

We now come to the notion of the disjunction of two conditional events ([69]).

Definition 6. Given a coherent probability assessment $P(A|H)=x$, $P(B|K)=y$, the disjunction of $A|H$ and $B|K$ is defined as

$$(A|H)\vee(B|K)=AH\vee BK+x\overline{H}\,\overline{B}K+y\overline{A}H\overline{K}+w\overline{H}\,\overline{K}, \tag{15}$$

where w is the prevision of $(A|H)\vee(B|K)$.

It can be verified that

$$(A|H)\vee(B|K)=(AH\vee BK+x\overline{H}\,\overline{B}K+y\overline{A}H\overline{K})|(H\vee K). \tag{16}$$

We recall below the notion of conjunction of n conditional events.

Definition 7. Let n conditional events $E_1|H_1,\dots,E_n|H_n$ be given. For each nonempty strict subset S of $\{1,\dots,n\}$, let x_S be a prevision assessment on $\bigwedge_{i\in S}(E_i|H_i)$. Then, the conjunction $(E_1|H_1)\wedge\cdots\wedge(E_n|H_n)$ is the conditional random quantity $C_{1\cdots n}$ defined as

$$C_{1\cdots n}=\begin{cases}1, & \text{if } \bigwedge_{i=1}^n E_iH_i \text{ is true},\\ 0, & \text{if } \bigvee_{i=1}^n \overline{E}_iH_i \text{ is true},\\ x_S, & \text{if } (\bigwedge_{i\in S}\overline{H}_i)\wedge(\bigwedge_{i\notin S}E_iH_i) \text{ is true, } \emptyset\neq S\subset\{1,2,\dots,n\},\\ x_{1\cdots n}, & \text{if } \bigwedge_{i=1}^n \overline{H}_i \text{ is true},\end{cases} \tag{17}$$

where $x_{1\cdots n}=x_{\{1,\dots,n\}}=\mathbb{P}(C_{1\cdots n})$.

For $n=1$ we obtain $C_1=E_1|H_1$. In Definition 7 each possible value x_S of $C_{1\cdots n}$, $\emptyset\neq S\subset\{1,\dots,n\}$, is evaluated when defining (in a previous step) the conjunction $C_S=\bigwedge_{i\in S}(E_i|H_i)$. Then, after the conditional prevision $x_{1\cdots n}$ is evaluated, $C_{1\cdots n}$ is completely specified. Of course, we require coherence for the prevision assessment $(x_S,\emptyset\neq S\subseteq\{1,\dots,n\})$, so that $C_{1\cdots n}\in[0,1]$. In the framework of the betting scheme, $x_{1\cdots n}$ is the amount that you agree to pay with the proviso that you will receive:

- the amount 1, if all conditional events are true;
- the amount 0, if at least one of the conditional events is false;
- the amount x_S equal to the prevision of the conjunction of that conditional events which are void, otherwise. In particular you receive back $x_{1\cdots n}$ when all conditional events are void.

As we can see from (17), the conjunction $C_{1\cdots n}$ assumes values in the interval $[0,1]$ and is (in general) a (2^n+1)-valued object because the number of nonempty subsets S, and hence the number of possible values x_S, is 2^n-1.

Remark 5. It can be verified that

$$x_{1\cdots n}=\mathbb{P}\Big[\Big(\bigwedge_{i=1}^n E_iH_i+\sum_{\emptyset\neq S\subset\{1,2\dots,n\}}x_S(\bigwedge_{i\in S}\overline{H}_i)\wedge(\bigwedge_{i\notin S}E_iH_i)\Big)\,\Big|\,\Big(\bigvee_{i=1}^n H_i\Big)\Big] \tag{18}$$

and

$$C_{1\cdots n}=\Big(\bigwedge_{i=1}^n E_iH_i+\sum_{\emptyset\neq S\subset\{1,2\dots,n\}}x_S(\bigwedge_{i\in S}\overline{H}_i)\wedge(\bigwedge_{i\notin S}E_iH_i)\Big)\,\Big|\,\Big(\bigvee_{i=1}^n H_i\Big).$$

A similar comment can be done for x_S and C_S, for each nonempty subset $S\subset\{1,2\dots,n\}$. In particular

$$(A|H)\wedge(B|K)=(AHBK+x\overline{H}BK+yAH\overline{K})|(H\vee K)=AHBK|(H\vee K)+x\overline{H}BK|(H\vee K)+yAH\overline{K}|(H\vee K), \tag{19}$$

where $x=P(A|H)$, $y=P(B|K)$. Then,

$$\mathbb{P}[(A|H)\wedge(B|K)]=P[AHBK|(H\vee K)]+P(A|H)P[\overline{H}BK|(H\vee K)]+P(B|K)P[AH\overline{K}|(H\vee K)]. \tag{20}$$

Notice that, when $P(H\vee K)>0$, formula (20) becomes the well known formula of McGee ([100]) and Kaufmann ([83])

$$\mathbb{P}[(A|H)\wedge(B|K)]=\frac{P(AHBK)+P(A|H)P(\overline{H}BK)+P(B|K)P(AH\overline{K})}{P(H\vee K)}.$$

A critical examination of claimed counterexamples to this notion of the conjunction of conditionals is given in [18].

We recall a result which shows that the prevision of the conjunction on n conditional events satisfies the Fréchet-Hoeffding bounds ([70, Theorem 13]).

Theorem 2. *Let n conditional events $E_1|H_1, \dots, E_n|H_n$ be given, with $x_i = P(E_i|H_i)$, $i = 1, \dots, n$ and $x_{1\cdots n} = \mathbb{P}(\mathcal{C}_{1\cdots n})$. Then*

$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \leqslant x_{1\cdots n} \leqslant \min\{x_1, \dots, x_n\}.$$

In [72, Theorem 10] we have shown, under logical independence, the sharpness of the Fréchet-Hoeffding bounds.

Remark 6. Given a finite family $\mathcal{F}$ of conditional events, their conjunction is also denoted by $\mathcal{C}(\mathcal{F})$. We recall that in [70], given two finite families of conditional events $\mathcal{F}_1$ and $\mathcal{F}_2$, the object $\mathcal{C}(\mathcal{F}_1) \wedge \mathcal{C}(\mathcal{F}_2)$ is defined as $\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2)$. Then, conjunction satisfies the commutativity and associativity properties ([70, Propositions 1 and 2]). Moreover, the operation of conjunction satisfies the monotonicity property ([70, Theorem 7]), that is $\mathcal{C}_{1\cdots n+1} \leqslant \mathcal{C}_{1\cdots n}$. Then,

$$\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \leqslant \mathcal{C}(\mathcal{F}_1), \;\; \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \leqslant \mathcal{C}(\mathcal{F}_2). \tag{21}$$

The next two results characterize p-consistency and p-entailment by exploiting the notion of conjunction ([70, Theorems 17 and 18]).

Theorem 3. *A family of n conditional events $\mathcal{F}$ is p-consistent if and only if the prevision assessment $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$ is coherent.*

Theorem 4. *Let be given a p-consistent family of n conditional events $\mathcal{F}$ and a further conditional event $E|H$. Then, the following assertions are equivalent:*

(i) *$\mathcal{F}$ p-entails $E|H$;*
(ii) *the conjunction $\mathcal{C}(\mathcal{F} \cup \{E|H\})$ coincides with the conjunction $\mathcal{C}(\mathcal{F})$;*
(iii) *the inequality $\mathcal{C}(\mathcal{F}) \leqslant E|H$ is satisfied.*

In the inequality (iii) of Theorem (4) the symbol $E|H$ is the indicator of the corresponding conditional event.

Remark 7. In order to show the relationship between Theorems 3 and 4 we observe that by Theorem 1 the condition $\mathcal{F}$ p-entails $E|H$ is equivalent to the incoherence of the assessment $(1, \cdots, 1, 0)$ on $\mathcal{F} \cup \{E|H\}$, that is the incoherence of $(1, \cdots, 1, 1)$ on $\mathcal{F} \cup \{\bar{E}|H\}$, or in other words the incoherence of the assessment $\mathbb{P}[\mathcal{C}(\mathcal{F} \cup \{\bar{E}|H\})] = 1$. Indeed, by Theorem 2, it holds that

$$\mathbb{P}[\mathcal{C}(\mathcal{F} \cup \{\bar{E}|H\})] = 1 \text{ is coherent} \iff P(E_1|H_1) = \cdots = P(E_n|H_n) = P(\bar{E}|H) = 1 \text{ is coherent.}$$

Thus, under the p-consistency of $\mathcal{F}$,

$$\mathcal{F} \text{ p-entails } E|H \iff \mathcal{F} \cup \{\bar{E}|H\} \text{ is not p-consistent.}$$

Notice that $\mathcal{C}(\mathcal{F}) \leqslant \mathcal{QC}(\mathcal{F})$ ([71, Formula (22)]), and hence $\mathcal{C}(\mathcal{F}) \leqslant E|H$ does not imply $\mathcal{QC}(\mathcal{F}) \leqslant E|H$. We also observe that the Assertion (iii) in Theorem 4, that is $\mathcal{C}(\mathcal{F}) \leqslant E|H$, is equivalent to the Assertion 4 in Theorem 1, that is to the existence of a nonempty subset $\mathcal{S} \subseteq \mathcal{F}$ such that $\mathcal{QC}(\mathcal{S}) \subseteq E|H$, or $H \subseteq E$, that is by (6) $\mathcal{QC}(\mathcal{S}) \leqslant E|H$. Finally, the p-entailment of $E|H$ from $\mathcal{F}$, that is $\mathcal{C}(\mathcal{F}) \leqslant E|H$, by Definition 4 amounts to say that $\mathcal{C}(\mathcal{F})$ p-entails $E|H$. This is the counterpart of Assertion 5 in Theorem 1 for the conjunction.

We now consider the notion of iterated conditional given in [66] (see also [69]). Such notion has the structure $\Box|\bigcirc = (\Box \wedge \bigcirc) + \mathbb{P}(\Box|\bigcirc) \cdot \bar{\bigcirc}$, where $\mathbb{P}$ denotes the prevision, which reduces to formula (2) when $\Box = A$ and $\bigcirc = H$.

Definition 8 *(Iterated conditioning)*. Given any pair of conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, the iterated conditional $(B|K)|(A|H)$ is defined as

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu \cdot \bar{A}|H, \tag{22}$$

where $\mu = \mathbb{P}[(B|K)|(A|H)]$.

In what follows we simply denote by $\mu\bar{A}|H$ the quantity $\mu \cdot \bar{A}|H$, which by Remark 1 also coincides with $(\mu \cdot \bar{A})|H)$.

Remark 8. Notice that we assumed that $AH \neq \emptyset$ to give a nontrivial meaning to the notion of iterated conditional. Indeed, if AH were equal to $\emptyset$, then $A|H = (B|K) \wedge (A|H) = 0$ and $\bar{A}|H = 1$, from which it would follow $(B|K)|(A|H) = (B|K)|0 = (B|K) \wedge (A|H) + \mu\bar{A}|H = \mu$; that is, $(B|K)|(A|H)$ would coincide with the (indeterminate) value μ. Similarly to the case of a conditional event $A|H$, which is of no interest when $H = \emptyset$ (in numerical terms $H = 0$), the iterated conditional $(B|K)|(A|H)$ is not considered when $AH = \emptyset$ (and $H \neq \emptyset$), in which case, from (4), $A|H$ is constant and coincides with 0. Of course, we do not consider the iterated conditional also when $H = \emptyset$, because in this case $A|H$ is not defined.

Definition 8 has been generalized in [70] (see also [73]) to the case where the antecedent is the conjunction of n conditional events.

Definition 9. Let $n+1$ conditional events $E_1|H_1,\dots,E_{n+1}|H_{n+1}$ be given, with $\mathcal{C}_{1\cdots n}=(E_1|H_1)\wedge\cdots\wedge(E_n|H_n)\neq 0$. We denote by $(E_{n+1}|H_{n+1})|\mathcal{C}_{1\cdots n}$ the random quantity

$$(E_1|H_1)\wedge\cdots\wedge(E_{n+1}|H_{n+1})+\mu(1-(E_1|H_1)\wedge\cdots\wedge(E_n|H_n)) = \mathcal{C}_{1\cdots n+1}+\mu(1-\mathcal{C}_{1\cdots n}),$$

where $\mu=\mathbb{P}[(E_{n+1}|H_{n+1})|\mathcal{C}_{1\cdots n}]$.

We observe that, based on the betting analysis, the quantity μ is the amount to be paid in order to receive the amount $\mathcal{C}_{1\cdots n+1}+\mu(1-\mathcal{C}_{1\cdots n})$. We also observe that, defining $\mathbb{P}(\mathcal{C}_{1\cdots n})=x_{1\cdots n}$ and $\mathbb{P}(\mathcal{C}_{1\cdots n+1})=x_{1\cdots n+1}$, by the linearity of prevision it holds that $\mu=x_{1\cdots n+1}+\mu(1-x_{1\cdots n})$; then, $x_{1\cdots n+1}=\mu x_{1\cdots n}$, that is (*compound prevision theorem*)

$$\mathbb{P}(\mathcal{C}_{1\cdots n+1})=\mathbb{P}[(E_{n+1}|H_{n+1})|\mathcal{C}_{1\cdots n}]\mathbb{P}(\mathcal{C}_{1\cdots n}). \tag{23}$$

We recall a result where it is shown that the p-entailment of a conditional event $E|H$ from a p-consistent family $\mathcal{F}$ is equivalent to condition $(E|H)|\mathcal{C}(\mathcal{F})=1$ ([73, Theorem 7]).

Theorem 5. *A p-consistent family $\mathcal{F}$ p-entails $E|H$ if and only if the iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ is equal to 1.*

In particular, given two (p-consistent) conditional events $A|H$ and $B|K$, it holds that ([62, Theorem 4])

$$A|H \text{ p-entails } B|K \iff (B|K)|(A|H)=1. \tag{24}$$

We observe that, as compound and iterated conditionals are conditional random quantities, uncertainties about them are quantified (not by conditional probabilities, but) by conditional previsions. At our Level 2 compound and iterated conditionals satisfy the basic logical and probabilistic properties relative to unconditional events ([74]), as illustrated by the list below:

(a) the Fréchet-Hoeffding lower and upper prevision bounds for the conjunction of conditional events still hold;
(b) the inequalities $\max\{A+B-1,0\}\leqslant AB\leqslant\min\{A,B\}$ become

$$\max\{A|H+B|K-1,0\}\leqslant(A|H)\wedge(B|K)\leqslant\min\{A|H,B|K\},$$

and the inequalities $\max\{A,B\}\leqslant A\vee B\leqslant\min\{A+B,1\}$ become

$$\max\{A|H,B|K\}\leqslant(A|H)\vee(B|K)\leqslant\min\{A|H+B|K,1\};$$

(c) by defining $\overline{(A|H)\wedge(B|K)}=1-(A|H)\wedge(B|K)$ and $\overline{(A|H)\vee(B|K)}=1-(A|H)\vee(B|K)$, De Morgan's Laws are satisfied ([70]), that is

$$\overline{(A|H)\wedge(B|K)}=(\bar{A}|H)\vee(\bar{B}|K) \text{ and } \overline{(A|H)\vee(B|K)}=(\bar{A}|H)\wedge(\bar{B}|K);$$

(d) the inclusion-exclusion formula for the disjunction of conditional events is valid ([71]); for instance, the equalities $A\vee B=A+B-AB$ and $P(A\vee B)=P(A)+P(B)-P(AB)$ become $(A|H)\vee(B|K)=(A|H)+(B|K)-(A|H)\wedge(B|K)$ and (prevision sum rule)

$$\mathbb{P}[(A|H)\vee(B|K)]=P(A|H)+P(B|K)-\mathbb{P}[(A|H)\wedge(B|K)], \tag{25}$$

respectively;
(e) the set of (conditional) constituents can be introduced, with properties analogous to the case of unconditional events ([71]);
(f) as shown by Theorems 4 and 5, conjunctions and iterated conditionals provide two characterizations of the probabilistic entailment of Adams for conditionals;
(g) a suitable generalization of probabilistic modus ponens to conditional events ([21,117]) and the study of one-premise and two-premise centering inferences ([59,118]) can be given;
(h) we explain some intuitive probabilistic assessments discussed in [43] by making some implicit background information explicit ([116]);
(i) finally, the compound probability theorem, that is $P(AB)=P(B|A)P(A)$, is generalized by the compound prevision theorem (a particular case of (23))

$$\mathbb{P}[(B|K)\wedge(A|H)]=\mathbb{P}[(B|K)|(A|H)]P(A|H). \tag{26}$$

2.5. The generalized equation for iterated conditionals

The hypothesis that the probability of a conditional in natural language is the conditional probability has so many deep consequences for understanding reasoning in natural language that it has been called the *Equation* ([46]):

$$P(\text{if } A, \text{ then } B) = P(B|A). \tag{27}$$

The Equation becomes the *conditional probability hypothesis* (CPH) in the psychology of reasoning, where it has been highly confirmed as descriptive of people's probability judgements about a wide range of conditionals (see, e.g., [105], see also [87]).

We observe that $P(B|A) = P(AB|(AB \vee A\overline{B}))$, then from (27) the probability of the conditional is the probability that it is true, given that it is true or false. Moreover, when $P(A) > 0$ it holds that

$$P(\text{If } A, \text{ then } B) = P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(AB)}{P(AB) + P(A\overline{B})}, \tag{28}$$

that is, the probability of the conditional is the probability that it is true, divided by the probability that it is true or false.

Given two conditionals *if H, then A* and *if K, then B*, we represent them by the conditional events $A|H$ and $B|K$, respectively. Then, we write the iterated conditional *if (A given H), then (B given K)* as *if $A|H$, then $B|K$*, and we represent it by the iterated conditional $(B|K)|(A|H)$. Based on this representation, we identify the "probability" of *if $A|H$, then $B|K$* with the prevision of $(B|K)|(A|H)$, that is

$$P(\textit{if } A|H, \textit{ then } B|K) = \mathbb{P}[(B|K)|(A|H)], \tag{29}$$

which can be called *generalized Equation*, or *conditional prevision hypothesis*. Now, by the reasoning below, we generalize formula (28) to the iterated conditional $(B|K)|(A|H)$. We set $\mu = \mathbb{P}[(B|K)|(A|H)]$ and $\nu = \mathbb{P}[(\overline{B}|K)|(A|H)]$, then

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\overline{A}|H, \quad (\overline{B}|K)|(A|H) = (\overline{B}|K) \wedge (A|H) + \nu\overline{A}|H,$$

and hence

$$(B|K)|(A|H) + (\overline{B}|K)|(A|H) = (B|K) \wedge (A|H) + (\overline{B}|K) \wedge (A|H) + (\mu + \nu)\overline{A}|H.$$

Moreover, we set $P(A|H) = x$, $P(B|K) = y$, $\mathbb{P}[(B|K) \wedge (A|H)] = z$, $\mathbb{P}[(\overline{B}|K) \wedge (A|H)] = \eta$. Then, by [117, Proposition 1] (see also the decomposition formula in [71]), it holds that

$$(B|K) \wedge (A|H) + (\overline{B}|K) \wedge (A|H) = A|H$$

and

$$\mathbb{P}[(B|K) \wedge (A|H)] + \mathbb{P}[(\overline{B}|K) \wedge (A|H)] = P(A|H). \tag{30}$$

Therefore, a formula like (28) also holds for the iterated conditional $(B|K)|(A|H)$, that is when $P(A|H) > 0$ from (26) and (30) it holds that

$$\mathbb{P}[(B|K)|(A|H)] = \frac{\mathbb{P}[(B|K) \wedge (A|H)]}{P(A|H)} = \frac{\mathbb{P}[(B|K) \wedge (A|H)]}{\mathbb{P}[(B|K) \wedge (A|H)] + \mathbb{P}[(\overline{B}|K) \wedge (A|H)]}. \tag{31}$$

Finally, defining the negation of $(B|K)|(A|H)$ as $\overline{(B|K)|(A|H)} = (\overline{B|K})|(A|H) = (\overline{B}|K)|(A|H)$, it follows that [111, Remark 4]

$$\overline{(B|K)|(A|H)} = 1 - (B|K)|(A|H), \tag{32}$$

which reduces to the well known relation $\overline{B|A} = 1 - B|A$ when $H = K = \Omega$.

3. Some notes on trivalent logics and conditional random quantities

In this section we discuss two recent papers on trivalent logics and show why selected properties hold in our Level 2 framework, which do not hold in these other approaches ([47,95]).

3.1. Comments on a paper by Lassiter and Baratgin

In this section we make some comparisons with a recent paper by Lassiter and Baratgin ([95]), where a trivalent semantics of nested conditionals is adopted. Some comments on [95] are:

- the authors argue that the counterexamples to and criticisms of the trivalent logic of de Finetti are resolved by taking into account the notion of genericity and the basic aspect that in de Finetti's theory events are "singular".
- the validity of the Import-Export principle is accepted;
- no particular analysis and/or proposal is given for defining compound and/or iterated conditionals, under the constraint of preserving the basic probabilistic properties.

As clearly stated by de Finetti ([36]), events are indeed singular, not generic (see also [29,51,40]). However, the Import-Export principle for consequent-nested conditionals, which we also call *consequent-nested conditional reduction*, is not valid in our analysis; that is, by Definition 8, it holds that $(D|C)|A \neq D|AC$. The invalidity of this principle is illustrated, for instance, by the following example.

Example 1. Given two events H and A, let us consider the sentence

$$\text{“if } H\text{, then (if } A \text{ then } H\text{)”}, \tag{33}$$

which we represent by the iterated conditional $(H|A)|H$. If the Import-Export principle were valid, the sentence (33) would be equivalent to the sentence

$$\text{“if (} H \text{ and } A\text{), then } H\text{” }. \tag{34}$$

Then, it would be $\mathbb{P}[(H|A)|H)] = P(H|AH) = 1$. However, based on Definition 8, by setting $P(A|H) = x$, $P(H|A) = y$ and $\mathbb{P}[(H|A)|H)] = \mu$, it holds that

$$(H|A)|H = (H|A) \wedge H + \mu\,\overline{H} = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ y, \text{ if } \overline{A}H \text{ is true,} \\ \mu, \text{ if } \overline{H} \text{ is true.} \end{cases}$$

By coherence $\mu \in [y, 1]$, hence $\mathbb{P}[(H|A)|H] \geqslant P(H|A)$, with necessarily $\mathbb{P}[(H|A)|H] = 1$ when $P(H|A) = 1$. More precisely it holds that

$$\mu = P(A|H) + yP(\overline{A}|H) = x + y(1-x).$$

Another example on the invalidity of the Import-Export principle for consequent-nested conditionals is given in Section 5, where it is shown that $(B|\overline{A})|(A \vee B) \neq B|(\overline{A} \wedge (A \vee B)) = B|\overline{A}B$. In the next example we observe that an Import-Export principle for antecedent-nested conditionals, which we call *antecedent-nested conditional reduction*, is invalid too ([116, Remark 7]).

Example 2. Consider two urns U and V. The urn U contains 99 white balls and 1 black ball and V contains 50 white balls and 50 black balls. Imagine you randomly draw a ball without knowing from which of the two urns. Let A and H be the following events:
$A =$“The drawn ball is white”;
$H =$“The ball is drawn from U”.
In the absence of particular information, the natural evaluation is that $P(A|H) = 0.99$. We denote by y the probability that the ball is drawn from U, that is $P(H) = y$. It seems clear that $P(H)$ does not depend in any way on $P(A|H)$. Indeed, it can be verified that the assessment $P(A|H) = x$, $P(H) = y$ is coherent for every $(x, y) \in [0, 1]^2$. Now, let us consider the sentence “if A when H, then H”, that is the sentence

$$\text{“if (if } H \text{ then } A\text{), then } H\text{”}, \tag{35}$$

which we represent by the iterated conditional $H|(A|H)$. If antecedent-nested conditional reduction were valid, the sentence (35) would be equivalent to the sentence

$$\text{“if (} H \text{ and } A\text{), then } H\text{” }, \tag{36}$$

which we represent by the conditional event $H|AH$. As $P(H|AH) = 1$, it holds that $H|AH$ coincides with the constant 1. As a consequence, independently from the values x and y, the probability of the sentence in (35) should be 1, that is P(“if A when H, then H”) = 1. Instead, we represent the sentence (35) by the iterated conditional $H|(A|H)$, which we discuss below.
In general, by setting $P(A|H) = x$ and $\mathbb{P}[H|(A|H)] = \mu$, it holds that

$$H|(A|H) = H \wedge (A|H) + \mu\,\overline{A}|H = AH + \mu\overline{A}|H = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ \mu, \text{ if } \overline{A}H \text{ is true,} \\ \mu(1-x), \text{ if } \overline{H} \text{ is true.} \end{cases} \tag{37}$$

By linearity of prevision it holds that

$$\mu = P(AH) + \mu P(\overline{A}|H) = xP(H) + \mu(1-x),$$

that is $\mu x = xP(H)$, and when $x > 0$ it follows that $\mu = P(H)$. In our case $x = 0.99$, $\mu = P(H) \in [0, 1]$ and

$$H|(A|H) = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ P(H), \text{ if } \overline{A}H \text{ is true,} \\ 0.01P(H), \text{ if } \overline{H} \text{ is true,} \end{cases}$$

which does not coincide with the constant 1, that is $H|(A|H) \neq H|AH$. Thus, antecedent-nested conditional reduction is invalid. In the extreme case where $x = 0$, as $A|H = AH + x\overline{H} = AH$, it follows that $H|(A|H) = H|AH = 1$ and hence $\mu = 1$; thus, when $x = 0$, antecedent-nested conditional reduction is valid.
Interestingly, in classical logic (where a conditional $B|A$ is interpreted as the material conditional $\overline{A} \vee B$) antecedent-nested conditional reduction is invalid because $H|AH = \overline{AH} \vee H = \Omega$ and

$$H|(A|H) = \overline{A|H} \vee H = (\overline{\overline{H} \vee A}) \vee H = \overline{A}H \vee H = H. \tag{38}$$

Thus, in classical logic $H|(A|H)$ coincides with H, while for the iterated conditional in (37) (under the assumption $x > 0$) it only holds that its prevision μ coincides with $P(H)$.

The following example shows that antecedent-nested conditional reduction is invalid even in the case where H is logically equivalent to A.

Example 3. Given any event $H \neq \emptyset$, let us consider the iterated sentence "if (if H then H), then H", which we represent by the iterated conditional $H|(H|H)$. If antecedent-nested conditional reduction were valid, we would obtain that $H|(H|H) = H|HH = 1$, and $\mathbb{P}[H|(H|H)] = 1$. But, by Definition 8, the equality $H|(H|H) = 1$ is not valid. Indeed, if it were $H|(H|H) = 1$, then from (24) it would follow that $H|H$ would p-entail H, in which case the unique coherent extension $P(H) = x$ of the (unique coherent) assessment $P(H|H) = 1$ should be $x = 1$. On the contrary, the assessment $(1, x)$ on $\{H|H, H\}$, with $H \neq \Omega$, is coherent for every $x \in [0, 1]$. Moreover, in agreement with the intuition, we can show that $H|(H|H) = H$. Indeed, by defining $\mathbb{P}[H|(H|H)] = \mu$ and by observing that $H \wedge (H|H) = H$ and $\overline{H}|H = 0$, it holds that

$$H|(H|H) = H \wedge (H|H) + \mu(\overline{H}|H) = H \quad \text{and} \quad \mathbb{P}[H|(H|H)] = P(H). \tag{39}$$

For instance, by considering the event H = *Sue passes the exam*, under antecedent-nested conditional reduction the probability that *Sue passes the exam, conditionally on Sue passes the exam, given that she passes the exam*, should be 1, which is a very strange result. Instead, as a natural result, the prevision of the iterated sentence *if Sue passes the exam when she passes the exam, then she passes the exam* simply coincides with the probability that *Sue passes the exam.*

3.2. *Trivalent semantics and validity: notes on a paper by Égré, Rossi, and Sprenger*

In this section we make a comparison with some results given in [47]. With respect to [47], we adopt a restricted approach where basic events, like A and C, are two-valued objects, which are true or false. Then, we construct on them conditional events which are true, or false, or void. We do not consider the case where basic events can be uncertain, because this uncertainty results from individual judgements ([38]).

In [47] the truth values of a conditional *if A then C*, true, or false, or void, are represented numerically by 1, or 0, or $\frac{1}{2}$, respectively. In particular, $\frac{1}{2}$ is a kind of objective representation of the logical value *void*. The same representation is used for compound conditionals which are still interpreted as three-valued objects. Égré et al. ([47]) remark, among many other aspects, that the choice of how "grouping indeterminate values with false antecedents... is a classical point of contention between trivalent logics of conditionals". Moreover, they observe that de Finetti does not handle conjunction and disjunction à la Bochvar/Weak Kleene ([12]), but that he is in line with the strong Kleene scheme ([7,84,85]).

3.2.1. *On the assertability of conditionals*

Égré et al. ([47]) note that "de Finettian conditionals keep the epistemic notion of assertability and the semantic notion of truth separate, while allowing for a fruitful interaction: degrees of assertability can be defined directly in terms of the truth conditions". Given a simple conditional X, its degree of assertability is defined as: $\text{Ast}(X) = P(X \text{ is true}|X \text{ has a classical truth value})$. The notion of assertion has a central importance in our approach too. Indeed, given a conditional *if A then C*, in our analysis when for instance $P(C|A) = \frac{1}{2}$ we will neither be inclined to assert the conditional as true nor as false. As $P(C|A)$ goes up to $0.6, 0.7, \ldots$, we'll become more and more inclined to assert *if A then C*. But what does *if A then C* having a value of $\frac{1}{2}$ mean for assertion in [47]? When do we become more inclined to assert *if A then C*, according to them? Probably, their answer would be that *if A then C* having a value of $\frac{1}{2}$ only means that the conditional is 'void', while its assertability is represented by $P(C|A)$. We can then observe that approach in this paper is more natural because by $P(C|A)$ we represent both the logical value 'void' and the assertability of the conditional. Moreover, like the case of unconditional events where you pay $P(A)$ by receiving $A \in \{1, 0\}$, if in a conditional bet you pay the amount $P(C|A)$ you will receive the random amount $C|A \in \{1, 0, P(C|A)\}$. Also the close relationship between truth and assertability can be better explained in our framework. Let us consider, for instance, the following sentence, examined in [47],

$$\text{if } \overbrace{\textit{Paul is in Paris}}^{A}, \text{ then } \overbrace{\textit{Paul is in France}}^{C}, \tag{40}$$

which typically is judged as true, even if Paul is for instance in Berlin, in which case the value of (40) is *void*. In [47] it is pointed out that "when we call sentences such as (40) *true*, what we really mean is that they are *maximally assertable*", that is $P(C|A) = 1$. Indeed, as A logically implies C, it holds that the value 0 becomes impossible and $P(C|A) = 1$; then, $C|A \in \{1, P(C|A)\} = \{1\}$, that is $C|A = 1$, which is the natural way to explain the intuitive judgement that the sentence (40) is true. As another example (see [47, Conditional (3)]), suppose Mary believes the following conditional:

$$\text{If } \overbrace{\textit{the Church is East of the City Hall}}^{A}, \text{ then } \overbrace{\textit{the City Hall is West of the Church}}^{C}. \tag{41}$$

As claimed in [47] "the proposition that Mary believes appears analytically true. Nonetheless, on the de Finettian analysis its truth value depends on the position of the City Hall with respect to the Church: the conditional may be evaluated either as true or as indeterminate." The apparent analyticity of (41) is explained because it is maximally assertable, regardless of its actual truth value. The sentences A and C in (41) are equivalent, and the conditional *if A then C* is maximally assertable because $P(C|A)=P(A|A)=1$.

Remark 9. We add a further comment about the notion of assertability. For [47], given any event $A \neq \emptyset$, the conditional *if A then A* is not a logical truth, for it fails to be *true* for them when A is *false* (and in the case where A is a conditional event, also when A has the *middle*, or $\frac{1}{2}$, indeterminate truth value). Nevertheless, *if A then A* is always *maximally assertable* for them because $P(A|A)=1$. They make essentially this point in the example examined in [47, p. 194, sentence (1)]. But, in their interpretation of de Finetti's three-valued logic, we can derive maximum assertability also for the following nested conditionals:

$$(a)\ \ \textit{if (if } A \textit{ then } A\textit{) then } A\ ; \qquad (b)\ \ \textit{if (if } \overline{A} \textit{ then } \overline{A}\textit{) then } \overline{A}\,.$$

Notice that (a) and (b) do not make the de Finetti three-valued logic inconsistent. That is because *if A then A* and *if $\overline{A}$ then $\overline{A}$* are not always *true* in that system, and so we cannot infer, by MP, that both A and $\overline{A}$ are always *true* in the system. But by what [47] say about *assertability*, (a) and (b), and *if A then A* and *if $\overline{A}$ then $\overline{A}$*, should always be *assertable* for them. According to [47]'s account of assertability ([47, p. 193]), the probability of a conditional, *if A then C*, is the probability that it is true given that it is non-void, i.e. it has a *classical* truth value. This is given by the cases in which AC is true out of the cases in which A is true. Now, considering for instance (a), (*if A then A*) *and* A is true whenever *if A then A* is true, and so (a) must have maximum assertability. By what [47] hold, (a) is assertable when A is not. To fix the assertability of (a), not-A cases are irrelevant and are ignored, giving (a) maximum assertability. As we see it, their position implies this result. Why then are not both A and $\overline{A}$ always assertable for them? It should always be the case that C is assertable when *if A then C* and A are both assertable. Otherwise, we have to follow [100] in claiming that MP is invalid, and [47] appear disinclined to do this.
In formal terms, in de Finetti trivalent logic, by (A.5), it holds that $(B|K)|_{df}(A|H)=B|AHK$, and hence

$$A|_{df}(A|A) \;=\; A|A,\;\; \overline{A}|_{df}(\overline{A}|\overline{A}) \;=\; \overline{A}|\overline{A}.$$

Therefore, both (a) and (b) are maximally assertable.
This problem can be avoided when the iterated conditional sentence *if* (*if H then A*), *then* (*if K then B*) is represented by the iterated conditional (see Definition 8)

$$(B|K)|(A|H)=(A|H)\wedge(B|K)+\mu\overline{A}|H,$$

where μ is the prevision of $(B|K)|(A|H)$, which in the betting framework represents the amount to be paid in order to receive the random quantity $(B|K)|(A|H)$. We observe that, when $B=H=A$ and $K=\Omega$, it follows that $(A|H)\wedge(B|K)=(A|A)\wedge(A|\Omega)=A$ and $\overline{A}|H=\overline{A}|A=0$. Then the iterated conditional sentence becomes the sentence (a), which we represent by the iterated conditional

$$A|(A|A)=(A|A)\wedge A+\mu\overline{A}|A=A\,,\ \text{ with }\ \mu=P(A)\,,$$

and similarly the sentence (b) is represented by the iterated conditional

$$\overline{A}|(\overline{A}|\overline{A})=(\overline{A}|\overline{A})\wedge\overline{A}+\eta A|\overline{A}=\overline{A}\,,\ \text{ with }\ \eta=P(\overline{A})\,.$$

As we can see, both (a) and (b) are not maximally assertable.

We differ, in particular, from the approach of [47] in the notion of validity, which for us is given by the notion of p-validity of Adams, except for his convention about conditionals whose antecedents have 0 probability. By considering, for instance, just one-premise inferences, we can define p-validity by saying that such an inference is p-valid if and only if the conclusion has probability 1 whenever the premise has probability 1. This definition does not imply, for us, that it is p-valid to infer AC from $C|A$, as $P(AC)$ will be 0 when $P(A)=0$, while $P(C|A)$ could be 1 for us. In [47] the notion of validity where a one-premise inference is valid if and only if its conclusion has value 1 whenever the premise has value 1 is called SS-validity (*strict-to-strict* validity in [24]). This definition would imply for them that it was "valid" to infer AC from their conditional "if A then C". When A has value 0, for them "if A then C" must have value $\frac{1}{2}$. By recalling (9), when A is false it holds that $C|A$ has value $P(C|A)$. Moreover, to infer AC from $C|A$ is not p-valid because $P(C|A)=1$ does not imply $P(AC)=1$; indeed the assessment $(1,y)$ on $\{C|A, AC\}$ is coherent for every $y \in [0,1]$. Another definition of "validity" considered in [47] is the *tolerant* one (TT-validity), which implies that an inference is invalid when its premise can have value 1/2 while its conclusion has value 0. We observe that to infer "if A then C" from "not-A or C" is TT-valid, but not p-valid because

$$P(\overline{A}\vee C)=1 \;\not\Rightarrow\; P(C|A)=1. \tag{42}$$

Indeed $P(\overline{A}\vee C)=P(AC)+P(\overline{A})=P(C|A)P(A)+P(\overline{A})$, so that $P(\overline{A}\vee C)=1$ amounts to $P(C|A)P(A)=P(A)$. Then, from $P(\overline{A}\vee C)=1$ it follows that $P(C|A)=1$ only if $P(A)>0$. However, when $P(A)=0$ and $P(\overline{A}\vee C)=1$, the extension $P(C|A)=z$ is coherent for every $z\in[0,1]$.
In this paper compound conditionals are no longer three-valued conditionals, but they are introduced at Level 2 as suitable conditional

random quantities with values in the unit interval $[0,1]$. We continue to represent a conditional "if A then C" by the conditional event $C|A$, which is a three-valued object; but, compound conditionals have an increasing number of numerical values. In particular, under logical independence of basic events, the conjunction of n conditional events has 2^n+1 possible values. For instance, given any logically independent events A, H, B, K, the conjunction of the two conditionals "if H then A" and "if K then B" is a conditional random quantity, denoted by $(A|H) \wedge (B|K)$, with a set of $2^2+1=5$ possible values $\{1,0,x,y,z\} \subset [0,1]$, where $x=P(A|H)$, $y=P(B|K)$, and z is the prevision of $(A|H) \wedge (B|K)$. The values x,y,z are coherently assessed in a subjective way and hence, when specifying them, there are infinitely many ways of defining the conjunction. In particular, for $A|H$ there are infinitely many ways of (subjectively) assessing x and hence, as a numerical object (the so-called indicator), we can (subjectively) define $A|H$ in infinitely many ways.

We can also briefly examine the Linearity principle, which says that the disjunction $(A|B) \vee (B|A)$ cannot be false. This principle is validated by the two main logics (CC/TT and DF/TT), studied in [47], and hence appears as a limitation of these logics. For instance, neither of "if John is red-haired, then John is a doctor" and "if John is a doctor, then he is red-haired" is accepted in ordinary reasoning ([99]).
Based on Definition (6), we make a probabilistic analysis of the disjunction $(A|B) \vee (B|A)$.

Theorem 6. *Let A and B be two logically independent events, with $P(A|B)=x$ and $P(B|A)=y$. Then, $(A|B) \vee (B|A) = (AB + xA\overline{B} + y\overline{A}B)|(A \vee B)$ and*

$$\mathbb{P}[(A|B) \vee (B|A)] = \begin{cases} x+y-\frac{xy}{x+y-xy}, & \textit{if } x+y>0, \\ 0, & \textit{if } x=y=0. \end{cases}$$

Hence, $(A|B) \vee (B|A)$ is not maximally assertable.

Proof. By applying (16) with $H=B$ and $K=A$, we obtain $(A|B) \vee (B|A) = (AB + xA\overline{B} + y\overline{A}B)|(A \vee B)$. We recall that ([68]):

$$(A|B) \wedge (B|A) = AB|(A \vee B), \quad \mathbb{P}[(A|B) \wedge (B|A)] = \begin{cases} \frac{xy}{x+y-xy}, & \text{if } x+y>0, \\ 0, & \text{if } x=y=0. \end{cases}$$

Then, by recalling (25), it holds that

$$\mathbb{P}[(A|B) \vee (B|A)] = P(A|B) + P(B|A) - \mathbb{P}[(A|B) \wedge (B|A)] = \begin{cases} x+y-\frac{xy}{x+y-xy}, & \text{if } x+y>0, \\ 0, & \text{if } x=y=0. \end{cases}$$

Finally, $(A|B) \vee (B|A)$ is not "maximally assertable" because its prevision is not necessarily equal to 1. Indeed, $(A|B) \vee (B|A)$ is "maximally assertable" only if $\max\{x,y\}=1$, because in this case $\mathbb{P}[(A|B) \vee (B|A)]=1$. However, when $\max\{x,y\}<1$, it holds that $\mathbb{P}[(A|B) \vee (B|A)]<1$. □

In [47] are also examined some conjunctive sentences which can never be true on DF/TT or CC/TT logics, because one of the conjuncts will always be indeterminate. Given any event A, with $A \neq \emptyset$, and $A \neq \Omega$, an instance of "obvious truth" is obtained when considering the conjoined conditional $(A|A) \wedge (\overline{A}|\overline{A})$, which is always classified as indeterminate. However, in [47, p. 208] it is observed that

> "a sentence such as:
> *If the sun shines tomorrow, John goes to the beach; and if it rains, he goes to the museum*
> seems to be true (with hindsight) if the sun shines tomorrow and John goes indeed to the beach. ... How can intuitively plausible compound sentences have positive degree of assertability if they can never be true?"

We can easily verify that the conjunction $(A|A) \wedge (\overline{A}|\overline{A})$ is maximally assertable. Indeed, $P(A|A) = P(\overline{A}|\overline{A}) = 1$ and $A|A = \overline{A}|\overline{A} = 1$; then, by recalling that

$$(A|H) \wedge (B|K) = (AHBK + P(A|H)\overline{H}BK + P(B|K)AH\overline{K})|(H \vee K),$$

when $H=A$ and $B=K=\overline{A}$ it follows that

$$(A|A) \wedge (\overline{A}|\overline{A}) = (P(A|A)\overline{A} + P(\overline{A}|\overline{A})A)|(A \vee \overline{A}) = \overline{A} + A = 1,$$

and hence $\mathbb{P}[(A|A) \wedge (\overline{A}|\overline{A})] = 1$, that is $(A|A) \wedge (\overline{A}|\overline{A})$ is maximally assertable.

3.2.2. On $SS \cap TT$ validity

Based on [47, Definition 3.2], a conditional sentence *if H then E* is said to be S-true (resp., T-true) when EH is true (resp., when $\overline{H} \vee EH$ is true). Moreover, the inference of *if A then B* from *if H then E* is said to be SS-valid (denoted *if H then E* $\vDash_{SS}$ *if A then B*) if $B|A$ is S-true when $E|H$ is S-true, that is, AB is true when EH is true. The same inference is said to be TT-valid (denoted *if H then E* $\vDash_{TT}$ *if A then B*) if $B|A$ is T-true when $E|H$ is T-true, that is, $\overline{A} \vee AB$ is true when $\overline{H} \vee EH$ is true (or equivalently, $H\overline{E}$

is true when $A\bar{B}$ is true). Based on [47, Definition 3.3], an inference from a set of conditionals Γ to a conclusion $B|A$ is SS-valid if and only if when each conditional $E|H$ in Γ is S-true then $B|A$ is S-true. Moreover, an inference from Γ to $B|A$ is TT-valid if and only if when each conditional $E|H$ in Γ is T-true then $B|A$ is T-true.

The symbol $\models_{SS\cap TT}$ denotes that the inference is both SS-valid and TT-valid. Then, an inference from a set Γ of conditionals to a conclusion $B|A$ is $SS \cap TT$-valid if and only if when each conditional $E|H$ in Γ is S-true then $B|A$ is S-true and when each conditional $E|H$ in Γ is T-true then $B|A$ is T-true. In particular, when $\Gamma = \{E|H\}$, by the Goodman and Nguyen inclusion relation $\subseteq$ given in formula (1), it holds that

$$\textit{if } H \textit{ then } E \ \models_{SS\cap TT} \ \textit{if } A \textit{ then } B \iff E|H \subseteq B|A. \tag{43}$$

Given a family of conditionals $\Gamma = \{E_1|H_1, \dots, E_n|H_n\}$, in the trivalent logic of de Finetti the conjunction (see Appendix A) $C_{df}(\Gamma) = (E_1|H_1) \wedge_{df} \cdots \wedge_{df} (E_n|H_n)$ is given by

$$C_{df}(\Gamma) = E_1H_1 \cdots E_nH_n|(E_1H_1 \cdots E_nH_n \vee \bar{E}_1H_1 \vee \cdots \vee \bar{E}_nH_n). \tag{44}$$

Theorem 7. *Given a family of conditionals $\Gamma = \{E_1|H_1, \dots, E_n|H_n\}$ and a further conditional $B|A$, it holds that*

$$\Gamma \models_{SS\cap TT} B|A \iff C_{df}(\Gamma) \subseteq B|A \iff C_{df}(\Gamma \cup \{B|A\}) = C_{df}(\Gamma). \tag{45}$$

Proof. By (44) we observe that the conditional event $C_{df}(\Gamma)$ is true if and only if all the conditionals in Γ are true, that is all the conditionals in Γ are S-true; $C_{df}(\Gamma)$ is not false if and only if all the conditional events in Γ are not false, that is all the conditional events in Γ are T-true. Then, given a further conditional event $B|A$, it holds that

$$\Gamma \models_{SS\cap TT} B|A \iff C_{df}(\Gamma) \models_{SS\cap TT} B|A, \tag{46}$$

which, by recalling (43) and (A.2), amounts to

$$\Gamma \models_{SS\cap TT} B|A \iff C_{df}(\Gamma) \subseteq B|A \iff C_{df}(\Gamma \cup \{B|A\}) = C_{df}(\Gamma). \quad \square \tag{47}$$

Example 4. In this example we show that $SS \cap TT$-validity implies transitivity. We set $\Gamma = \{C|B, B|A\}$ and we consider the conclusion $C|A$. We observe that

$$C_{df}(\Gamma) = ABC|(ABC \vee A\bar{B} \vee B\bar{C}) \subseteq C|A,$$

because $ABC \subseteq AC$ and $A\bar{C} = AB\bar{C} \vee A\bar{B}\bar{C} \subseteq A\bar{B} \vee B\bar{C} = \overline{ABC} \wedge (ABC \vee A\bar{B} \vee B\bar{C})$. Therefore, $\{C|B, B|A\} \models_{SS\cap TT} C|A$, that is transitivity is $SS \cap TT$-valid. However, $SS \cap TT$ validity from a set of premises does not imply that the set p-entails the conclusion, because transitivity is not p-valid ([61]). For instance, let us consider the events A =“Peter will win millions in a lottery”, B =“Peter quits his job”, C =“Peter will have money troubles”. Peter might be confident that, if he quits his job, he will have money troubles (i.e. $P(C|B)$ high), and that, if he wins millions in a lottery, he will quit his job (i.e. $P(B|A)$ high). But Peter will have no confidence at all that, if he wins millions in the lottery, he will have money troubles (i.e. $P(C|A)$ low). We also recall that Transitivity is a non-theorem of any nonmonotonic logic [90]. Notice that $SS \cap TT$-validity does not imply transitivity, when the basic objects, that is A, B, and C, are (not events, but) three-valued objects in de Finetti's trivalent logic.[5]

Now we show that the property of p-entailment (see Definition 4) does not imply $SS \cap TT$-validity. We observe that ([74, Equation (15)])

$$E|H \leqslant B|A \iff E|H \subseteq B|A, \text{ or } EH = \emptyset, \text{ or } A \subseteq B, \tag{48}$$

and by assuming $E|H$ p-consistent (i.e., $EH \neq \emptyset$), from ([68, Theorem 7]) and (43) it holds that

$$E|H \Rightarrow_p B|A \iff E|H \subseteq B|A, \textit{ or } A \subseteq B \iff E|H \models_{SS\cap TT} B|A, \textit{ or } A \subseteq B \iff E|H \leqslant B|A. \tag{49}$$

We observe that, in the particular case where $A \subseteq B$ it holds that $P(B|A) = 1$, thus $B|A = 1$; then $E|H \leqslant B|A$ and hence $E|H \Rightarrow_p B|A$. However, in this case one has that *if H then E* $\not\models_{SS\cap TT}$ *if A then B*, that is the inference is not $SS \cap TT$-valid. Indeed, when $\bar{A}EH$ is true, it holds that *if H then E* is true, but *if A then B* is void, and hence the inference is not SS-valid. Thus, p-entailment does not imply $SS \cap TT$-validity.

To deepen the differences among p-entailment and $SS \cap TT$-validity, let us consider a premise set $\Gamma = \{E_1|H_1, E_2|H_2\}$ and a conclusion $E_3|H_3$, with $H_3 \nsubseteq E_3$. By Remark 2, under logical independence the 27 constituents generated by $\{E_1|H_1, E_2|H_2, E_3|H_3\}$ are illustrated in Table 1. We observe that, in order the inference from Γ to $E_3|H_3$ be $SS \cap TT$-valid, it should be $C_2 = C_3 = C_8 = C_{20} = C_{26} = \emptyset$. Moreover, in order the quasi conjunction $QC(\Gamma)$ of the premises in Γ satisfies the condition $QC(\Gamma) \subseteq E_3|H_3$, it should be $C_2 = C_3 = C_8 = C_9 = C_{20} = C_{21} = C_{26} = \emptyset$. Then

[5] We thank P. Égré for this remark.

Table 1
Constituents C_h's associated with the family $\{E_1|H_1, E_2|H_2, E_3|H_3\}$.

C_1	$E_1H_1E_2H_2E_3H_3$	C_{10}	$\overline{E}_1H_1E_2H_2E_3H_3$	C_{19}	$\overline{H}_1E_2H_2E_3H_3$
C_2	$E_1H_1E_2H_2\overline{E}_3H_3$	C_{11}	$\overline{E}_1H_1E_2H_2\overline{E}_3H_3$	C_{20}	$\overline{H}_1E_2H_2\overline{E}_3H_3$
C_3	$E_1H_1E_2H_2\overline{H}_3$	C_{12}	$\overline{E}_1H_1E_2H_2\overline{H}_3$	C_{21}	$\overline{H}_1E_2H_2\overline{H}_3$
C_4	$E_1H_1\overline{E}_2H_2E_3H_3$	C_{13}	$\overline{E}_1H_1\overline{E}_2H_2E_3H_3$	C_{22}	$\overline{H}_1\overline{E}_2H_2E_3H_3$
C_5	$E_1H_1\overline{E}_2H_2\overline{E}_3H_3$	C_{14}	$\overline{E}_1H_1\overline{E}_2H_2\overline{E}_3H_3$	C_{23}	$\overline{H}_1\overline{E}_2H_2\overline{E}_3H_3$
C_6	$E_1H_1\overline{E}_2H_2\overline{H}_3$	C_{15}	$\overline{E}_1H_1\overline{E}_2H_2\overline{H}_3$	C_{24}	$\overline{H}_1\overline{E}_2H_2\overline{H}_3$
C_7	$E_1H_1\overline{H}_2E_3H_3$	C_{16}	$\overline{E}_1H_1\overline{H}_2E_3H_3$	C_{25}	$\overline{H}_1\overline{H}_2E_3H_3$
C_8	$E_1H_1\overline{H}_2\overline{E}_3H_3$	C_{17}	$\overline{E}_1H_1\overline{H}_2\overline{E}_3H_3$	C_{26}	$\overline{H}_1\overline{H}_2\overline{E}_3H_3$
C_9	$E_1H_1\overline{H}_2\overline{H}_3$	C_{18}	$\overline{E}_1H_1\overline{H}_2\overline{H}_3$	C_0	$\overline{H}_1\overline{H}_2\overline{H}_3$

Table 2
Values of the Jeffrey conditional $\phi \to \psi$, where $d_i \in \{1, \frac{1}{2}\}$, $i = 1, 2, 3, 4$.

ϕ \ ψ	1	$\frac{1}{2}$	0
1	1	d_1	0
$\frac{1}{2}$	d_2	d_3	0
0	$\frac{1}{2}$	d_4	$\frac{1}{2}$

$$QC(\Gamma) \subseteq E_3|H_3 \implies \Gamma \vDash_{SS\cap TT} E_3|H_3,$$

but

$$\Gamma \vDash_{SS\cap TT} E_3|H_3 \;\;\not\Longrightarrow\;\; QC(\Gamma) \subseteq E_3|H_3\,.$$

In addition, the inference from Γ to $E_3|H_3$ may be $SS\cap TT$-valid, but not p-valid, i.e. it may be that $\Gamma \vDash_{SS\cap TT} E_3|H_3$, but $\Gamma \not\Rightarrow_p E_3|H_3$. In order to illustrate this aspect let us assume that $C_2 = C_3 = C_8 = C_{20} = C_{26} = \emptyset$ and, so that the inference is $SS\cap TT$-valid, with $C_9 \neq \emptyset$ and $C_{21} \neq \emptyset$. Then, we can verify that $\Gamma \not\Rightarrow_p E_3|H_3$. Indeed, if C_9 is true, then $E_1|H_1$ is true and $E_3|H_3$ is void, and hence $E_1|H_1 \nsubseteq E_3|H_3$. Likewise, if C_{21} is true, then $E_2|H_2$ is true and $E_3|H_3$ is void, and hence $E_2|H_2 \nsubseteq E_3|H_3$. Moreover, if $C_9 \vee C_{21}$ is true, then $QC(\Gamma)$ is true and $E_3|H_3$ is void, and hence $QC(\Gamma) \nsubseteq E_3|H_3$. Therefore, by Theorem 1, $\Gamma \not\Rightarrow_p E_3|H_3$, and the assessment $(1, 1, 0)$ on $\{E_1|H_1, E_2|H_2, E_3|H_3\}$ is coherent.

For a variant of $SS\cap TT$-validity, which is equivalent to $QC(\Gamma) \subseteq E_3|H_3$, see Appendix B.

4. P-entailment and probabilistic weak deduction theorems

In this section we consider the notion of TT-validity illustrated in Section 3.2.2 and Jeffrey conditionals examined in [47] in the framework of trivalent logics. Then, we recall a full deduction theorem which is TT-valid for Jeffrey conditionals. Going beyond trivalent logics, we use the compound and iterated conditionals, defined in Section 2.4, alongside the notion of p-validity. We show that the classical deduction theorem does not hold when using p-validity. Then, we obtain some probabilistic weak deduction theorems and related results. In addition, we give some examples.

Égré et al. ([47]) remark that their notion of a de Finetti conditional operator is not adequate in some respects (for instance, Modus Ponens and the Identity Law are not satisfied); in the same paper the Jeffrey conditionals ([81]) are discussed as a class of trivalent conditionals that support Modus Ponens and are adequate for TT-validity.

Given two conditionals ϕ and ψ, with values in $\{1, \frac{1}{2}, 0\}$, the (trivalent) Jeffrey conditional $\phi \to \psi$ is defined in Table 2, where $d_i \in \{1, \frac{1}{2}\}$, $i = 1, 2, 3, 4$. Of course, we get a different table for different values of the d_i's. Indeed, there are $2^4 = 16$ possible tables.

Then, in [47, Proposition 5.5], it is shown that, given a set Γ of conditionals and two further conditionals ϕ and ψ, for Jeffrey conditionals and TT-validity the full classical deduction theorem holds, that is

$$\Gamma, \phi \vDash_{J/TT} \psi \iff \Gamma \vDash_{J/TT} \phi \to \psi\,. \tag{50}$$

Recall that, in our analysis, ϕ and ψ are two events A and B (i.e., with values in $\{1, 0\}$) and hence $\phi \to \psi$ is the conditional $A \to B$, that is the conditional event $B|A$. Moreover, when we employ the notion of p-validity, the classical deduction theorem is not valid, as shown by the counterexample below.

Example 5. Given two logically independent events A and B, consider the *or-to-if* inference

from "*A or B*" to "*if* $\overline{A}$ *then B*", (51)

or equivalently

from "$\bar{A}$ *or* B" to "*if* A *then* B".

The inference (51) is valid under the material conditional interpretation, where "*if* ○ *then* □" is interpreted as "$\bar{○} \vee □$". In addition, under the conditional event interpretation, where "*if* ○ *then* □" is interpreted as "□|○", the inference (51) is J/TT-valid. Indeed, by setting $\Gamma = \{A \vee B\}$, it holds that $\Gamma \vDash_{J/TT} B|\bar{A}$, because $A \vee B$ is false when $B|\bar{A}$ is false. From (50) it also holds that $\Gamma \cup \{\bar{A}\} \vDash_{J/TT} B$; indeed, when $A \vee B$ and $\bar{A}$ are both T-true (i.e., both true, or equivalently $\bar{A}B$ true), then B is T-true (i.e., B is true).

However, by observing that $\Gamma \cup \{\bar{A}\}$ is p-consistent, it holds that

$$\Gamma \cup \{\bar{A}\} \ \Rightarrow_p \ B, \ \text{ but } \Gamma \ \not\Rightarrow_p \ B|\bar{A}, \tag{52}$$

and hence (50) is not satisfied when TT-validity is replaced by p-validity. In order to verify (52) we observe that if $P(A \vee B) = 1$ and $P(\bar{A}) = 1$, then $P(AB) = P(A\bar{B}) = 0$ and $P(B) = P(A \vee B) - P(A\bar{B}) = P(A \vee B) = 1$, thus $\Gamma \cup \{\bar{A}\} \ \Rightarrow_p \ B$. But, recalling (42), $P(A \vee B) = 1$ does not imply $P(B|\bar{A}) = 1$ and hence $\Gamma \not\Rightarrow_p B|\bar{A}$. The same conclusion also follows from (49) by observing that $A \vee B \nleqslant B|\bar{A}$. Indeed, when $x < 1$, if A is true, then $A \vee B = 1 > x = B|\bar{A}$. Note that some instances of (51) in natural language are highly intuitive, and that, if (51) were valid, then the paradoxical inference from "A" to "if $\bar{A}$ then B" would also be valid. For "A or B" would follow from "A", and then "if $\bar{A}$ then B" would follow by (51) (see [60] for a full treatment of when this inference is, and is not, intuitive; see also [34]).

Remark 10 *(The Classical Deduction Theorem).* Analogous comments can be made for the full deduction theorem derivation of "B entails (if A then B)" in the trivalent analysis, making the conditional "close to" the material conditional and its paradoxes. Indeed, by the full deduction theorem it holds that

$$B, A \ \vDash_{J/TT} \ B \iff B \ \vDash_{J/TT} \ B|A.$$

We do not have this problem either with p-entailment. Indeed $\{B, A\} \Rightarrow_p B$, but $B \not\Rightarrow_p B|A$. As observed above, the full deduction theorem states that given a set Γ of conditionals and two further conditionals A and B, $\Gamma \cup \{A\}$ entails B if and only if Γ entails the conditional *if* A *then* B; we call this result *The Classical Deduction Theorem*. It holds when a conditional *if* H *then* E is looked at as the event $\bar{H} \vee E$ (the material conditional) and, as we observed before, it also holds for Jeffrey's conditionals and TT-validity.

4.1. The probabilistic weak deduction theorem

Recalling that compound conditionals are conditional random quantities, we now study the deduction problem under p-entailment. As shown by Example 5 under p-entailment the Classical Deduction Theorem fails. More precisely, given two events $A \neq \emptyset$, $B \neq \emptyset$ and a p-consistent family Γ of conditional events, from $\Gamma \cup \{A\} \Rightarrow_p B$ it does not follow that $\Gamma \Rightarrow_p B|A$. Interestingly, under the assumption that $\Gamma \cup \{A\}$ is p-consistent, the converse holds, as shown by the result below.

Theorem 8. *Let $A \neq \emptyset$, $B \neq \emptyset$ be any events and Γ be any finite family of conditional events such that $\Gamma \Rightarrow_p B|A$. If $\Gamma \cup \{A\}$ is p-consistent, then $\Gamma \cup \{A\} \Rightarrow_p B$.*

Proof. As $\Gamma \Rightarrow_p B|A$, it holds that $\mathcal{C}(\Gamma) \leqslant B|A$. Then, as $\mathcal{C}(\Gamma) \leqslant B|A$ and $(B|A) \wedge A = AB$, it follows that

$$\mathcal{C}(\Gamma) \wedge A \leqslant (B|A) \wedge A = AB \leqslant B,$$

which, under the assumption that $\Gamma \cup \{A\}$ is p-consistent, amounts to the p-entailment of B from $\Gamma \cup \{A\}$, that is $\Gamma \cup \{A\} \Rightarrow_p B$. □

Remark 11. We observe that even if Γ is p-consistent and $\Gamma \Rightarrow_p B|A$, it could be that $\Gamma \cup \{A\}$ is not p-consistent. For instance, let $\Gamma = \{\bar{A}\}$, where $A \neq \emptyset$ and $A \neq \Omega$. Γ is p-consistent because $\bar{A} \neq \emptyset$, and $\Gamma \Rightarrow_p A|A$ because $A|A = 1$; but $\Gamma \cup \{A\} = \{\bar{A}, A\}$ is not p-consistent because $P(A) = P(\bar{A}) = 1$ is not coherent.

By recalling Theorem 4, the property $\mathcal{F} \Rightarrow_p E|H$, with $\mathcal{F}$ p-consistent, can be equivalently written as $\mathcal{C}(\mathcal{F}) \leqslant E|H$, or $\mathcal{C}(\mathcal{F}) \wedge (E|H) = \mathcal{C}(\mathcal{F})$, where by recalling Remark 6, $\mathcal{C}(\mathcal{F}) \wedge (E|H) = \mathcal{C}(\mathcal{F} \cup \{E|H\})$.

Theorem 9. *Let $A \neq \emptyset$, $B \neq \emptyset$ be any events and Γ be any finite family of conditional events, with $\Gamma \cup \{A\}$ p-consistent. Then, the following assertions are equivalent:*

(i) $\Gamma \Rightarrow_p A$ *and* $\Gamma \cup \{A\} \Rightarrow_p B$;
(ii) $\Gamma \Rightarrow_p AB$;
(iii) $\Gamma \Rightarrow_p A$ *and* $\Gamma \Rightarrow_p B$.

Proof. We will prove the theorem by verifying that

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i).$$

$(i) \Longrightarrow (ii)$. By Theorem 4, as $\Gamma \Rightarrow_p A$ and $\Gamma \cup \{A\} \Rightarrow_p B$, it holds that

$$\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma \cup \{A\}) = \mathcal{C}(\Gamma \cup \{A\} \cup \{B\}) = \mathcal{C}(\Gamma) \wedge AB.$$

Then, still by Theorem 4, $\Gamma \Rightarrow_p AB$.

$(ii) \Longrightarrow (iii)$. The condition $\Gamma \Rightarrow_p AB$ is equivalent to the condition $\mathcal{C}(\Gamma) \leqslant AB$. Thus, $\mathcal{C}(\Gamma) \leqslant A$ and $\mathcal{C}(\Gamma) \leqslant B$; hence $\Gamma \Rightarrow_p A$ and $\Gamma \Rightarrow_p B$.

$(iii) \Longrightarrow (i)$. We only need to prove that $\Gamma \cup \{A\} \Rightarrow_p B$. As $\Gamma \Rightarrow_p A$ and $\Gamma \Rightarrow_p B$, by Theorem 4, it follows that $\mathcal{C}(\Gamma) \wedge A = \mathcal{C}(\Gamma \cup \{A\}) = \mathcal{C}(\Gamma) \leqslant B$. Then, $\Gamma \cup \{A\} \Rightarrow_p B$. □

Remark 12. We observe that Theorem 9 still holds if we exchange the roles of A and B. Therefore,

$$\Gamma \Rightarrow_p A \ \text{ and } \ \Gamma \cup \{A\} \Rightarrow_p B \quad \Longleftrightarrow \quad \Gamma \Rightarrow_p B \ \text{ and } \ \Gamma \cup \{B\} \Rightarrow_p A.$$

Notice that, if $\Gamma \cup \{A\} \Rightarrow_p B$, then the assessment $P(E|H) = 1, \forall E|H \in \Gamma, P(A) = P(B) = 1$ is coherent, and hence $\Gamma \cup \{B\}$ is p-consistent.

Based on Theorem 9, we now show that, by adding the condition $\Gamma \Rightarrow_p A$ to the assumption $\Gamma \cup \{A\} \Rightarrow_p B$, we obtain that $\Gamma \Rightarrow_p B|A$ (and $\Gamma \Rightarrow_p A|B$), which we call *Probabilistic Weak Deduction Theorem.*

Theorem 10 *(The Probabilistic Weak Deduction Theorem). Let $A \neq \emptyset$, $B \neq \emptyset$ be any events and Γ be any finite family of conditional events, with $\Gamma \cup \{A\}$ p-consistent. If $\Gamma \Rightarrow_p A$ and $\Gamma \cup \{A\} \Rightarrow_p B$, then: (a) $\Gamma \Rightarrow_p B|A$; (b) $\Gamma \Rightarrow_p A|B$.*

Proof. By Theorem 9, as $\Gamma \Rightarrow_p A$ and $\Gamma \cup \{A\} \Rightarrow_p B$, it holds that $\Gamma \Rightarrow_p AB$, or equivalently $\mathcal{C}(\Gamma) \leqslant AB$. By defining $P(A|B) = x$ and $P(B|A) = y$, it holds that $AB \leqslant AB + x\overline{A} = A|B$ and $AB \leqslant AB + y\overline{B} = B|A$. Therefore $\mathcal{C}(\Gamma) \leqslant A|B$ and $\mathcal{C}(\Gamma) \leqslant B|A$. Then, both conditions (a) and (b) are satisfied. □

The following result shows how, given appropriate hypotheses, we can generate a p-valid inference from a not p-valid one.

Corollary 1. *Let $A \neq \emptyset$, $B \neq \emptyset$ be any events and Γ be any finite family of conditional events such that $\Gamma \not\Rightarrow_p B|A$. If $\Gamma \cup \{A\}$ is p-consistent and $\Gamma \cup \{A\} \Rightarrow_p B$, then $\Gamma \cup \{A\} \Rightarrow_p B|A$ and $\Gamma \cup \{A\} \Rightarrow_p A|B$.*

Proof. We first observe that $\Gamma \cup \{A\} \Rightarrow_p A$ because $A \in \Gamma \cup \{A\}$. Then, the proof follows by applying Theorem 10 where Γ is replaced by $\Gamma \cup \{A\}$. □

The next result, which concerns the biconditional event $(A|B) \wedge (B|A)$, suggests an equivalent formulation of the Probabilistic Weak Deduction Theorem.

Theorem 11. *Let $A \neq \emptyset$, $B \neq \emptyset$ be any events and Γ be any finite p-consistent family of conditional events. Then, $\Gamma \Rightarrow_p B|A$ and $\Gamma \Rightarrow_p A|B$ if and only if $\Gamma \Rightarrow_p (A|B) \wedge (B|A)$.*

Proof. The proof immediately follows by recalling Definition 4 and by observing that $P(B|A) = P(A|B) = 1$ if and only if $\mathbb{P}[(A|B) \wedge (B|A)] = 1$. □

4.2. Some examples

In order to illustrate the previous theorems, we examine some examples, with a further result.

Example 6. *Transitivity:* $\{C|B, B|A\} \not\Rightarrow_p C|A$. As it is well known, given three logically independent events A, B, C, the inference from $\Gamma = \{C|B, B|A\}$ to $C|A$, that is transitivity is not p-valid ([61], see also [70, Section 10.1]). However, it can be verified that $\Gamma \cup \{A\}$ is p-consistent and $\Gamma \cup \{A\} \Rightarrow_p C$; see also [2, p. 132, Exercise 1]. Indeed, as $(B|A) \wedge A = BA$, it follows that

$$\mathcal{C}(\Gamma \cup \{A\}) = (C|B) \wedge (B|A) \wedge A = (C|B) \wedge BA = ABC. \tag{53}$$

As $\mathcal{C}(\Gamma \cup \{A\}) = ABC \leqslant C$, by Theorem 4, $\Gamma \cup \{A\} \Rightarrow_p C$. Like Example 5, this example shows that, differently from the Classical Deduction Theorem, $\Gamma \cup \{A\} \Rightarrow_p C$, but $\Gamma \not\Rightarrow_p C|A$.

Example 7. $\{C|B, B|A, A\} \Rightarrow_p (C|A) \wedge (A|C)$. Let us consider the non p-valid inference from $\Gamma = \{C|B, B|A\}$ to $C|A$, as in Example 6. By Corollary 1, it holds that $\Gamma \cup \{A\} \Rightarrow_p \{C|A\}$ and $\Gamma \cup \{A\} \Rightarrow_p \{A|C\}$. Moreover, by Theorem 11, it holds that $\Gamma \cup \{A\} \Rightarrow_p (A|C) \wedge (C|A)$.

The next theorem gives a result related with the previous examples.

Theorem 12. *Let $A_1, \dots, A_n$ be logically independent events. Then,*

$$\{A_1, A_2|A_1, \dots, A_n|A_{n-1}\} \Rightarrow_p A_i|A_j, \quad \forall\, i, j = 1, \dots, n,$$

and

$$\{A_1, A_2|A_1, \dots, A_n|A_{n-1}\} \Rightarrow_p A_i, \quad \forall\, i = 1, \dots, n.$$

Proof. We set $\Gamma = \{A_1, A_2|A_1, \dots, A_n|A_{n-1}\}$. Then, by observing that $A_iA_j \leqslant A_i|A_j$, for all i, j, and that

$$A_1 \cdots A_{k-1} \wedge (A_k|A_{k-1}) = A_1 \cdots A_k, \;\; k = 2, \dots, n,$$

we obtain

$$\begin{aligned}&\mathcal{C}(\Gamma) = A_1 \wedge (A_2|A_1) \wedge \cdots \wedge (A_n|A_{n-1}) = A_1A_2 \wedge (A_3|A_2) \wedge \cdots \wedge (A_n|A_{n-1}) = \\ &= \cdots = A_1 \cdots A_{n-1} \wedge (A_n|A_{n-1}) = A_1 \cdots A_n.\end{aligned}$$

As the assessment $P(A_1 \cdots A_n) = 1$ is coherent, Γ is p-consistent. Moreover $A_1 \cdots A_n \leqslant A_iA_j \leqslant A_i|A_j$, $i, j = 1, 2, ..., n$. Thus, $\Gamma \Rightarrow_p A_i|A_j$, for every $i, j = 1, \dots, n$. Finally, as $\mathcal{C}(\Gamma) = A_1 \cdots A_n \leqslant A_i$, $i = 1, \dots, n$, it holds that $\{A_1, A_2|A_1, \dots, A_n|A_{n-1}\} \Rightarrow_p A_i$, $i = 1, \dots, n$. □

Remark 13. *Multiple Modus Ponens.* We observe that, in particular, by Theorem 12, it holds that $\{A_1, A_2|A_1, \dots, A_n|A_{n-1}\} \Rightarrow_p A_n$.

Example 8. *On combining evidence, an example from Boole:* $\{C|B, C|A\} \not\Rightarrow_p C|AB$. Let A, B, C be logically independent events. We recall that the inference from $\Gamma = \{C|B, C|A\}$ to $C|AB$ is not p-valid ([70, Section 10.1]). Moreover, $\Gamma \cup \{AB\} \Rightarrow_p C$, because

$$\mathcal{C}(\Gamma \cup \{AB\}) = (C|B) \wedge (C|A) \wedge AB = (C|B) \wedge B \wedge (C|A) \wedge A = ABC \leqslant C. \tag{54}$$

However, differently from the Classical Deduction Theorem, $\Gamma \not\Rightarrow_p C|AB$. Based on Corollary 1, it holds that $\Gamma \cup \{AB\} \Rightarrow_p C|AB$ and $\Gamma \cup \{AB\} \Rightarrow_p AB|C$. Moreover, by Theorem 11 $\Gamma \cup \{AB\} \Rightarrow_p (AB|C) \wedge (C|AB)$. In other words, as $\Gamma \cup \{AB\} = \{C|B, C|A, AB\}|$ and $(AB|C) \wedge (C|AB) = ABC|(AB \vee C)$, it holds that

$$\{C|B, C|A, AB\} \Rightarrow_p ABC|(AB \vee C).$$

Example 9. *Contraposition:* $C|A \not\Rightarrow_p \bar{A}|\bar{C}$. As shown in Remark 3, the inference from $\Gamma = \{C|A\}$ to $\bar{A}|\bar{C}$ is not p-valid. Moreover, $\Gamma \cup \{\bar{C}\} \Rightarrow_p \bar{A}$ because

$$\mathcal{C}(\Gamma \cup \{\bar{C}\}) = (C|A) \wedge \bar{C} = 0(A + \bar{A}C) + P(C|A)\bar{A}\,\bar{C} = P(C|A)\bar{A}\,\bar{C} \leqslant \bar{A}. \tag{55}$$

Based on Corollary 1, $\Gamma \cup \{\bar{C}\} \Rightarrow_p \bar{A}|\bar{C}$ and $\Gamma \cup \{\bar{C}\} \Rightarrow_p \bar{C}|\bar{A}$. Moreover, by Theorem 11 it holds that $\Gamma \cup \{\bar{C}\} \Rightarrow_p (\bar{A}|\bar{C}) \wedge (\bar{C}|\bar{A})$. In other words, as $\Gamma \cup \{\bar{C}\} = \{C|A, \bar{C}\}$ and $(\bar{A}|\bar{C}) \wedge (\bar{C}|\bar{A}) = \bar{A}\,\bar{C}|(\bar{A} \vee \bar{C})$, it holds that

$$\{C|A, \bar{C}\} \Rightarrow_p \bar{A}\,\bar{C}|(\bar{A} \vee \bar{C}). \tag{56}$$

Remark 14. Notice that in Example 9, as $\Gamma \cup \{\bar{C}\} = \{C|A, \bar{C}\}$, the inference $\Gamma \cup \{\bar{C}\} \Rightarrow_p \bar{A}$ coincides with Modus Tollens. Then, from (24), it holds that $\bar{A}|(\bar{C} \wedge (C|A)) = 1$ (see also [62, Section 4.1]). However, as Contraposition is not p-valid, still from (24) it holds that $(\bar{A}|\bar{C})|(C|A) \neq 1$, that is the "export" of $C|A$ from $\bar{A}|(\bar{C} \wedge (C|A))$ to $(\bar{A}|\bar{C})|(C|A)$ does not hold. Moreover, as $\{C|A, \bar{C}\} \Rightarrow_p \bar{A}|\bar{C}$, in terms of iterated conditionals, by Theorem 5, it holds that $(\bar{A}|\bar{C})|(\bar{C} \wedge (C|A)) = 1$.

Example 10. *Weak monotonicity:* $\{C|A, AB\} \Rightarrow_p C|AB$. Monotonicity, that is the inference from $\Gamma = \{C|A\}$ to $C|AB$, is not p-valid. Indeed, the assessment $(1, z)$ on $\{C|A, C|AB\}$ is coherent for every $z \in [0, 1]$. Moreover, $\Gamma \cup \{AB\} \Rightarrow_p C$ because

$$\mathcal{C}(\Gamma \cup \{AB\}) = (C|A) \wedge A \wedge B = ABC \leqslant C. \tag{57}$$

Based on Corollary 1, $\Gamma \cup \{AB\} \Rightarrow_p C|AB$, that is

$$\{C|A, AB\} \Rightarrow_p C|AB \quad \text{(Weak monotonicity)}, \tag{58}$$

and $\Gamma \cup \{AB\} \Rightarrow_p AB|C$. Moreover, by Theorem 11 it holds that $\Gamma \cup \{AB\} \Rightarrow_p (AB|C) \wedge (C|AB)$. In other words, as $(AB|C) \wedge (C|AB) = (ABC)|(AB \vee C)$, it holds that

$$\{C|A, AB\} \Rightarrow_p ABC|(AB \vee C). \tag{59}$$

4.3. The generalized probabilistic weak deduction theorem

We give two results: Theorem 13, which generalizes Theorem 9, and Theorem 14 (Generalized Probabilistic Weak Deduction Theorem), which generalizes Theorem 10. In the statements of these new results, the events A and B are replaced by the conditional events $A|H$ and $B|K$, respectively. Then, the conjunction AB becomes $(A|H)\wedge(B|K)$; moreover, $B|A$ and $A|B$ become the iterated conditionals $(B|K)|(A|H)$ and $(A|H)|(B|K)$, respectively. We also give some further related results and examples.

Theorem 13. *Let $A|H,\ B|K$ be two conditional events, with $AH\neq\emptyset$ and $BK\neq\emptyset$, and Γ any finite family of conditional events, with $\Gamma\cup\{A|H\}$ p-consistent. Then, the following assertions are equivalent:*

(i) $\Gamma\Rightarrow_p A|H$ *and* $\Gamma\cup\{A|H\}\Rightarrow_p B|K$;
(ii) $\Gamma\Rightarrow_p (A|H)\wedge(B|K)$;
(iii) $\Gamma\Rightarrow_p A|H$ *and* $\Gamma\Rightarrow_p B|K$.

Proof. We will prove the theorem by verifying that

$$(i)\Longrightarrow(ii)\Longrightarrow(iii)\Longrightarrow(i).$$

$(i)\Longrightarrow(ii)$. By Theorem 4, as $\Gamma\Rightarrow_p A|H$ and $\Gamma\cup\{A|H\}\Rightarrow_p B|K$, it holds that

$$\mathcal{C}(\Gamma)=\mathcal{C}(\Gamma)\wedge(A|H)=\mathcal{C}(\Gamma)\wedge(A|H)\wedge(B|K)\leqslant(A|H)\wedge(B|K). \tag{60}$$

As Γ is p-consistent, the assessment $P(E|H)=1$, for every $E|H\in\Gamma$, is coherent. Moreover, $P(E|H)=1$, $\forall E|H\in\Gamma$, by Theorem 2, implies that $\mathbb{P}[\mathcal{C}(\Gamma)]=1$ and from (60) $\mathbb{P}[(A|H)\wedge(B|K)]=1$. Thus, by Definition 4, $\Gamma\Rightarrow_p(A|H)\wedge(B|K)$.

$(ii)\Longrightarrow(iii)$. As $\Gamma\Rightarrow_p(A|H)\wedge(B|K)$, when $P(E|H)=1$, for every $E|H\in\Gamma$, it follows that $\mathbb{P}[(A|H)\wedge(B|K)]=1$ and hence $P(A|H)=P(B|K)=1$, because $(A|H)\wedge(B|K)\leqslant\min\{(A|H),(B|K)\}$. Then $\Gamma\Rightarrow_p A|H$ and $\Gamma\Rightarrow_p B|K$.

$(iii)\Longrightarrow(i)$. We only need to prove that $\Gamma\cup\{A|H\}\Rightarrow_p B|K$. As $\Gamma\Rightarrow_p A|H$ and $\Gamma\Rightarrow_p B|K$, by Theorem 4, it follows that $\mathcal{C}(\Gamma)\wedge(A|H)=\mathcal{C}(\Gamma\cup\{A|H\})=\mathcal{C}(\Gamma)\leqslant B|K$. Then, $\Gamma\cup\{A|H\}\Rightarrow_p B|K$. $\square$

Remark 15. Theorem 13 still holds if we exchange the roles of $A|H$ and $B|K$. Therefore,

$$\Gamma\Rightarrow_p A|H \ \text{ and } \ \Gamma\cup\{A|H\}\Rightarrow_p B|K \iff \Gamma\Rightarrow_p B|K \ \text{ and } \ \Gamma\cup\{B|K\}\Rightarrow_p A|H.$$

Theorem 14 *(The Generalized Probabilistic Weak Deduction Theorem). Let $A|H,\ B|K$ be two conditional events, with $AH\neq\emptyset$ and $BK\neq\emptyset$, and Γ any p-consistent finite family of conditional events, with $\Gamma\cup\{A|H\}$ p-consistent. If $\Gamma\Rightarrow_p A|H$ and $\Gamma\cup\{A|H\}\Rightarrow_p B|K$, then: (a) $\Gamma\Rightarrow_p(B|K)|(A|H)$; (b) $\Gamma\Rightarrow_p(A|H)|(B|K)$.*

Proof. By Theorem 13, as $\Gamma\Rightarrow_p A|H$ and $\Gamma\cup\{A|H\}\Rightarrow_p B|K$, it holds that $\Gamma\Rightarrow_p(A|H)\wedge(B|K)$. Moreover, by defining $\mu=\mathbb{P}[(B|K)|(A|H)]$, it holds that $(A|H)\wedge(B|K)\leqslant(A|H)\wedge(B|K)+\mu(1-(A|H))=(B|K)|(A|H)$. Therefore, when $P(E|H)=1$, for every $E|H\in\Gamma$, it follows that $\mathbb{P}[(A|H)\wedge(B|K)]=P(A|H)=1$ and hence

$$\mathbb{P}[(B|K)|(A|H)]=\mu=\mathbb{P}[(A|H)\wedge(B|K)]+\mu(1-P(A|H))=1.$$

Then, condition (a) is satisfied. Likewise, the condition (b) is satisfied too. $\square$

Remark 16. We can verify that, if

$$\Gamma\ \Rightarrow_p\ A \ \text{ and } \ \Gamma\cup\{A\}\ \Rightarrow_p\ B, \tag{61}$$

then, for every event H^*, it holds that

$$\Gamma\ \Rightarrow_p\ A|(A\vee H^*) \ \text{ and } \ \Gamma\cup\{A|(A\vee H^*)\}\ \Rightarrow_p\ B|(A\vee H^*), \tag{62}$$

and

$$\Gamma\ \Rightarrow_p\ A|(A\vee B\vee H^*) \ \text{ and } \ \Gamma\cup\{A|(A\vee B\vee H^*)\}\ \Rightarrow_p\ B|(A\vee B\vee H^*). \tag{63}$$

Indeed, by Theorem 9 the conditions in (61) are equivalent to $\Gamma\Rightarrow_p AB$. Then, by observing that $AB\leqslant AB|(A\vee H^*)$ and $AB\leqslant AB|(A\vee B\vee H^*)$, when the conditions in (61) are satisfied it holds that

$$\Gamma\ \Rightarrow_p\ AB|(A\vee H^*) \ \text{ and } \ \Gamma\ \Rightarrow_p\ AB|(A\vee B\vee H^*),$$

which, by Theorem 13, are equivalent to the conditions in (62) and (63), respectively.

We give a result related with Theorem 10, where the set of assumptions $\Gamma \Rightarrow_p A$ and $\Gamma \cup \{A\} \Rightarrow_p B$ are replaced by the assumptions $\Gamma \Rightarrow_p A|(A \vee H^*)$ and $\Gamma \cup \{A|(A \vee H^*)\} \Rightarrow_p B|(A \vee H^*)$.

Theorem 15. *Let $A \neq \emptyset$, $B \neq \emptyset$, and H^* be any events and Γ be any finite family of n conditional events, with $\Gamma \cup \{A|(A \vee H^*)\}$ p-consistent. If $\Gamma \Rightarrow_p A|(A \vee H^*)$ and $\Gamma \cup \{A|(A \vee H^*)\} \Rightarrow_p B|(A \vee H^*)$, then $\Gamma \Rightarrow_p B|A$.*

Proof. By applying Theorem 13, with $H = K = A \vee H^*$, we obtain that $\Gamma \Rightarrow_p AB|(A \vee H^*)$. Moreover, as $AB|(A \vee H^*) \leqslant B|A$, it follows that $\Gamma \Rightarrow_p B|A$. $\square$

When $B \nsubseteq H^*$, it holds that $AB|(A \vee H^*) \nleqslant A|B$; thus, from $\Gamma \Rightarrow_p AB|(A \vee H^*)$, it does not follow in general that $\Gamma \Rightarrow_p A|B$, that is Theorem 15 is not symmetric with respect to A and B.

We also observe that, when $H^* = \overline{A}$, the assumptions of Theorem 15 coincide with the assumptions of Theorem 10, and hence in this case $\Gamma \Rightarrow_p A|B$ as well.

Remark 17. Given two conditional events $A|H$ and $B|K$, it holds that ([71, Section 9])

$$(A|H) \wedge (B|K) = AHBK|(H \vee K), \text{ when } AH\overline{K} = \overline{H}BK = \emptyset. \tag{64}$$

In other words, the conjunction $(A|H) \wedge (B|K)$ reduces to the conditional event $AHBK|(H \vee K)$, when are impossible the constituents such that a conditional event is true and the other one is void. In this case, $(A|H) \wedge (B|K)$ coincides with $(A|H) \wedge_{df} (B|K)$.

Example 11. *A weak version of transitivity rule:* $\{C|B, B|A, A|(A \vee B)\} \Rightarrow_p C|A$. We illustrate an example of p-entailment, related to the Aristotelian syllogism *Barbara* studied in [112], where the assumptions of Theorem 15 are satisfied, while the assumptions of Theorem 10 are not. Let A, B, C be three logically independent events and $\Gamma = \{C|B, B|A, A|(A \vee B)\}$. We consider the inference from Γ to the conditional event $C|A$ ([54]). With respect to Example 6 we added to Γ the event $A|(A \vee B)$. The assumptions of Theorem 15 are satisfied with $H^* = B$; thus $\Gamma \Rightarrow_p C|A$ (see also [70, Section 10.2], or [61, Section 4]). However, the assumptions of Theorem 10 are not satisfied because $\Gamma \cup \{A\} \Rightarrow_p C$, but $\Gamma \not\Rightarrow_p A$. Indeed, based on (64), it holds that $(B|A) \wedge (A|(A \vee B)) = AB|(A \vee B)$. Then, $\mathcal{C}(\Gamma) = (C|B) \wedge (B|A) \wedge (A|(A \vee B)) = (C|B) \wedge (AB|(A \vee B))$ and still from (64) it follows that $\mathcal{C}(\Gamma) = ABC|(A \vee B)$. Moreover, $\mathcal{C}(\Gamma) \nleqslant A$ because, under the assumption $P(ABC|(A \vee B)) > 0$, if $\overline{A}\,\overline{B}$ is true, then $\mathcal{C}(\Gamma) > 0$, while $A = 0$. We also observe that $\Gamma \not\Rightarrow_p A|C$, because $\mathcal{C}(\Gamma) = ABC|(A \vee B) \nleqslant A|C$.

Example 12. *CM rule:* $\{C|A, B|A\} \Rightarrow_p C|AB$. In this example, by recalling that the inference from $\{C|A\}$ to $C|AB$ is not p-valid, we show that by applying Theorem 15 with A, B, and H^* replaced by AB, C, and A, respectively, and with $\Gamma = \{C|A, B|A\}$, it holds that $A|(A \vee H^*)$ and $B|(A \vee H^*)$ become $B|A$ and $C|A$, respectively. Moreover, as $\Gamma \Rightarrow_p B|A$ and $\Gamma \cup \{C|A\} \Rightarrow_p C|A$, it follows that $\Gamma \Rightarrow_p C|AB$, which is the CM rule. We observe that the premise AB, added to $\{C|A\}$, for obtaining weak monotonicity (formula (58)), and the premise $B|A$ added to $\{C|A\}$, for obtaining CM rule, satisfy the relation $AB \leqslant B|A$. Then, CM rule is "weaker" (or more "cautious") than weak monotonicity, indeed $P(AB) = 1$ implies $P(B|A) = 1$, while the converse does not hold. Finally, in this example $\Gamma \not\Rightarrow_p AB|C$ because

$$\mathcal{C}(\Gamma) = (C|A) \wedge (B|A) = BC|A \nleqslant AB|C.$$

Another result, more general than Theorem 10, is given below, where the set of assumptions $\Gamma \Rightarrow_p A$ and $\Gamma \cup \{A\} \Rightarrow_p B$ are replaced by the assumptions

$$\Gamma \Rightarrow_p A|(A \vee B \vee H^*) \text{ and } \Gamma \cup \{A|(A \vee B \vee H^*)\} \Rightarrow_p B|(A \vee B \vee H^*).$$

Theorem 16. *Let $A \neq \emptyset$, $B \neq \emptyset$, and H^* be any events and Γ be any finite family of conditional events, with $\Gamma \cup \{A|(A \vee B \vee H^*)\}$ p-consistent. If $\Gamma \Rightarrow_p A|(A \vee B \vee H^*)$ and $\Gamma \cup \{A|(A \vee B \vee H^*)\} \Rightarrow_p B|(A \vee B \vee H^*)$, then: (a) $\Gamma \Rightarrow_p B|A$; (b) $\Gamma \Rightarrow_p A|B$.*

Proof. By applying Theorem 13, with $H = K = A \vee B \vee H^*$, we obtain that $\Gamma \Rightarrow_p AB|(A \vee B \vee H^*)$. Moreover, as $AB|(A \vee B \vee H^*) \leqslant B|A$ and $AB|(A \vee B \vee H^*) \leqslant A|B$, it follows that the conditions (a) and (b) are satisfied, i.e., $\Gamma \Rightarrow_p B|A$ and $\Gamma \Rightarrow_p A|B$. $\square$

Notice that, when $H^* = \overline{A \vee B}$, Theorem 16 reduces to Theorem 10.

Example 13. *A different weak version of transitivity rule:* $\{C|B, B|A, A|(A \vee C)\} \Rightarrow_p C|A$. By considering Example 6, we add to the premises the event $A|(A \vee C)$. We can show that, by defining $\Gamma = \{C|B, B|A, A|(A \vee C)\}$, it holds that $\Gamma \Rightarrow_p C|A$ and $\Gamma \Rightarrow_p A|C$. Indeed, based on (64), it holds that $(B|A) \wedge (A|(A \vee C)) = AB|(A \vee C)$. Then, $\mathcal{C}(\Gamma) = (C|B) \wedge (B|A) \wedge (A|(A \vee C)) = (C|B) \wedge (AB|(A \vee C))$ and still from (64) it holds that $\mathcal{C}(\Gamma) = ABC|(A \vee B \vee C)$. As $ABC|(A \vee B \vee C) \leqslant C|A$ and $ABC|(A \vee B \vee C) \leqslant A|C$, it follows that $\Gamma \Rightarrow_p C|A$ and $\Gamma \Rightarrow_p A|C$. We observe that the same conclusions can be obtained by applying Theorem 16, with $H^* = \emptyset$, by verifying that $\Gamma \Rightarrow_p A|(A \vee C)$ and $\Gamma \cup \{A|(A \vee C)\} \Rightarrow_p C|(A \vee C)$. Indeed, $\Gamma \Rightarrow_p A|(A \vee C)$ because $A|(A \vee C) \in \Gamma$. Moreover, $\Gamma \cup \{A|(A \vee C)\} = \Gamma$, with $\mathcal{C}(\Gamma) = ABC|(A \vee B \vee C) \leqslant C|(A \vee C)$.

In the next example we show that some counterintuitive results can depend on some strange assessments of the premises and not on the wrongness of the inference rule. In other words it can be the case that the inference rule is valid, but the counterintuitive results of the conclusion only depend on the "extravagant" assessment of the premises.

Example 14. $\{A \vee B, \overline{A}\} \Rightarrow_p B|\overline{A}$. Let A, B be logically independent events, with $\overline{A} \neq \emptyset$, and $\Gamma = \{A \vee B\}$. As shown in Example 5, the inference from Γ to the conditional event $B|\overline{A}$ is not p-valid. We observe that $\Gamma \cup \{\overline{A}\}$ is p-consistent. Moreover, $\mathcal{C}(\Gamma \cup \{\overline{A}\}) = (A \vee B) \wedge \overline{A} = \overline{A}B \leqslant B$ and hence $\Gamma \cup \{\overline{A}\} \Rightarrow_p B$, that is

$$\{A \vee B, \overline{A}\} \Rightarrow_p B, \tag{65}$$

which is the probabilistic version of what in logic is often called "disjunctive syllogism". Moreover, based on Corollary 1, it holds that $\Gamma \cup \{\overline{A}\} \Rightarrow_p B|\overline{A}$, that is

$$\{A \vee B, \overline{A}\} \Rightarrow_p B|\overline{A}. \tag{66}$$

We now consider the case where A = "Smoking is unhealthy"; B = "strawberries are blue". We observe that the assessment $P(A \vee B) = P(\overline{A}) = 1$ is coherent, even if in this case $P(\overline{A}) = 1$ is counterintuitive. Moreover, if we assess $P(A \vee B) = 1$, $P(\overline{A}) = 1$, from (66) it follows that $P(B) = P(B|\overline{A}) = 1$, which is counterintuitive too. Of course the inference in (66) is p-valid. Actually, "what is strange in this inference" is the probabilistic assessment of the premise $\overline{A}$, i.e. $P(\overline{A}) = 1$, from which the strange conclusion follows, i.e. $P(B) = P(B|\overline{A}) = 1$. In a classical approach, we can reject an inference because it is not classically valid, or because we believe that it has a false premise. In a probabilistic approach, we can reject an inference because it is not p-valid, or because we disagree with the probabilistic assessment of one of the premises.

4.4. On Aristotelian syllogisms and new inference rules

In this section we show how to derive new inference rules by exploiting the probabilistic analysis of the Aristotelian syllogisms studied in [112]. Below, we consider some inference rules related to the so-called Camestres syllogism. We first recall that the inference from $\Gamma = \{B|C, \overline{B}|A\}$ to $\overline{C}|A$, where A, B, C are three logically independent events, is not p-valid. Indeed, the assessment (x, y, z) on $\{B|C, \overline{B}|A, \overline{C}|A\}$ is coherent for every $(x, y, z) \in [0, 1]^3$ ([112, Proposition 3]), that is the imprecise assessment $[0, 1]^3$ on $\{B|C, \overline{B}|A, \overline{C}|A\}$ is totally coherent ([58]). However, the family $\Gamma \cup \{A\} = \{B|C, \overline{B}|A, A\}$ is p-consistent because the assessment $(1, 1, 1)$ on $\{B|C, \overline{B}|A, A\}$ is coherent. Indeed, the probability assessment

$$P(ABC) = P(AB\overline{C}) = P(A\overline{B}C) = P(\overline{A}BC) = P(\overline{A}\,\overline{B}C) = P(\overline{A}\,\overline{C}) = 0,\ P(A\overline{B}\overline{C}) = 1,$$

on the set of constituents $\{ABC,\ AB\overline{C},\ A\overline{B}C,\ \overline{A}BC,\ \overline{A}\overline{B}C,\ \overline{A}\overline{C},\ A\overline{B}\overline{C}\}$, generated by the procedure in Remark 2, is coherent and implies that

$$P(B|C) = P(\overline{B}|A) = P(A) = 1.$$

Thus, the assessment $(1, 1, 1)$ on $\Gamma \cup \{A\}$ is coherent and hence $\Gamma \cup \{A\}$ is p-consistent.
Moreover, $\Gamma \cup \{A\} \Rightarrow_p \overline{C}$. Indeed, as $(\overline{B}|A) \wedge A = \overline{B}A$, it follows that

$$\mathcal{C}(\Gamma \cup \{A\}) = (B|C) \wedge (\overline{B}|A) \wedge A = (B|C) \wedge \overline{B}A. \tag{67}$$

We observe that

$$\begin{aligned}(B|C) \wedge \overline{B}A &= (B|C) \wedge \overline{B}A \wedge (C \vee \overline{C}) = [(B|C) \wedge \overline{B}A \wedge C] \vee [(B|C) \wedge \overline{B}A \wedge \overline{C}] = \\ &= BC\overline{B}A \vee [(B|C) \wedge \overline{B}A \wedge \overline{C}] = (B|C) \wedge \overline{B}A \wedge \overline{C} \leqslant \overline{C},\end{aligned}$$

that is, $\mathcal{C}(\Gamma \cup \{A\}) \leqslant \overline{C}$. Then, by Theorem 4, $\Gamma \cup \{A\} \Rightarrow_p \overline{C}$. Differently from the Classical Deduction Theorem, $\Gamma \cup \{A\} \Rightarrow_p \overline{C}$, but $\Gamma \not\Rightarrow_p \overline{C}|A$. Now, a new inference rule associated with the Camestres Aristotelian syllogism can be given by adding the event A to the premise set $\{B|C, \overline{B}|A\}$. Then, we examine the p-validity of the inference from $\Gamma = \{B|C, \overline{B}|A, A\}$ to $\overline{C}|A$.
As shown above, $\Gamma \cup \{A\} = \{B|C, \overline{B}|A, A\}$ is p-consistent and $\Gamma \cup \{A\} \Rightarrow_p \overline{C}$. Then, by Corollary 1, it holds that $\Gamma \cup \{A\} \Rightarrow_p \{\overline{C}|A\}$. Hence, a new inference rule related to the Camestres syllogism is the following:

$$\{B|C, \overline{B}|A, A\} \Rightarrow_p \overline{C}|A. \tag{68}$$

We remark that, still by Corollary 1, the following inference rule also follows:

$$\{B|C, \overline{B}|A, A\} \Rightarrow_p A|\overline{C}. \tag{69}$$

Moreover, by recalling Theorem 11, a further inference rule is:

$$\{B|C, \overline{B}|A, A\} \Rightarrow_p (\overline{C}|A) \wedge (A|\overline{C}). \tag{70}$$

By reasoning as in Example 11, and applying Theorem 15 with $\Gamma = \{B|C, \overline{B}|A\}$ and $H^* = C$, the following inference rule can be obtained:

$$\{B|C, \overline{B}|A, A|(A \vee C)\} \Rightarrow_p \overline{C}|A. \tag{71}$$

Further inference rules could be obtained based on the probabilistic interpretation of other Aristotelian syllogisms, such as *Celarent* and *Cesare* ([112]).

5. Iterated conditionals and a general import-export principle

In this section, we first recall the notion of the Import-Export principle and we illustrate an example. We define the notion of the *General Import-Export principle*, which generalizes Import-Export by replacing the antecedent (an unconditional event) with a conjunction of conditionals. Then, we give a result which shows the relation between the General Import-Export principle and p-entailment. Moreover, we examine, in the light of the probabilistic weak deduction theorems, selected p-valid inference rules by showing that the General Import-Export principle is satisfied. Finally, we deepen some aspects related to p-entailment and p-consistency.

We recall below the Import-Export principle for the case of unconditional events.

Definition 10. Given three events E, H, K, the Import-Export principle is satisfied if the iterated conditional $(E|H)|K$ coincides with the conditional event $E|HK$.

Based on *the or-to-if inference* examined in Example 5, we show that the invalidity of the Classical Deduction Theorem can be related to the invalidity of the Import-Export principle.

Example 15. Given two events A and B, with $\overline{A}B \neq \emptyset$, the export of $(A \vee B)$ from $B|((A \vee B)\overline{A})$ to the iterated conditional $(B|\overline{A})|(A \vee B)$ does not hold. Indeed, as shown in [74, Section 4.2], it holds that

$$(B|\overline{A})|(A \vee B) \neq B|((A \vee B)\overline{A}) = B|\overline{A}B = 1\,. \tag{72}$$

Notice that, if the Import-Export principle were valid, then from (72) it would follow $(B|\overline{A})|(A \vee B) = 1$ and, by Theorem 5, the inference from $A \vee B$ to $B|\overline{A}$ would be p-valid. This is clearly unacceptable as shown, for instance, by the following example ([3, p. 1], see also [107]). Imagine that Jones "is about to be dealt a five card poker hand from a shuffled deck of 52 cards". We set A ="Jones's first card is not an ace" and B ="Jones's second card is an ace". It holds that $P(A \vee B) = P(A) + P(B|\overline{A})P(\overline{A}) = \frac{48}{52} + \frac{3}{51}\frac{4}{52} = \frac{205}{221} \simeq 0.928$, $P(B|\overline{A}) = \frac{3}{51} \simeq 0.059$. Then, the inference from $A \vee B$ to $B|\overline{A}$ is weak, because it is constructive, indeed $P(A) = \frac{48}{52} \simeq 0.923$ ([60]). Actually, as $P(A \vee B) > 0$, by recalling (31) it holds that

$$\mathbb{P}[(B|\overline{A})|(A \vee B)] = \frac{P(B|\overline{A})}{P(A \vee B)} = \frac{\frac{3}{51}}{\frac{48}{52} + \frac{3}{51}\frac{4}{52}} = \frac{13}{205} \simeq 0.063,$$

which is close to $P(B|\overline{A})$, in agreement with the intuition, and very different from the value $P(B|\overline{A}B) = 1$ obtained under the Import-Export assumption.

5.1. The general import-export principle

Given a family of conditional events $\mathcal{F}$, and a further conditional event $E|H$, by recalling Definition 9, let us consider the iterated conditional

$$(E|H)|\mathcal{C}(\mathcal{F}) = (E|H) \wedge \mathcal{C}(\mathcal{F}) + \mu\,[1 - \mathcal{C}(\mathcal{F})]$$

where $\mu = \mathbb{P}[(E|H)|\mathcal{C}(\mathcal{F})]$, and the iterated conditional

$$E\,|\,(H \wedge \mathcal{C}(\mathcal{F})) = EH \wedge \mathcal{C}(\mathcal{F}) + \eta\,[1 - H \wedge \mathcal{C}(\mathcal{F})],$$

where $\eta = \mathbb{P}[E\,|\,(H \wedge \mathcal{C}(\mathcal{F}))]$. By Definition 9, in order to consider the previous two iterated conditionals, we must assume that $\mathcal{C}(\mathcal{F}) \neq 0$ and $H \wedge \mathcal{C}(\mathcal{F}) \neq 0$. Of course, as $H \wedge \mathcal{C}(\mathcal{F}) \leqslant \mathcal{C}(\mathcal{F})$, it is enough to assume that $H \wedge \mathcal{C}(\mathcal{F}) \neq 0$.

Definition 11. The iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ satisfies the General Import-Export principle if it holds that

$$(E|H)|\mathcal{C}(\mathcal{F}) = E\,|\,(H \wedge \mathcal{C}(\mathcal{F})). \tag{73}$$

Remark 18. Notice that the General Import-Export principle may be valid, that is the equality (73) may be satisfied even if the iterated conditionals are different from the constant 1. For instance, given any events E, H and K, with $\emptyset \neq K \subseteq H$, $\mathcal{F} = \{K\}$, and E logically independent from both H and K, it holds that

$$(E|H)|C(\mathcal{F}) = (E|H)|K = (E|H) \wedge K + \mu\overline{K} = \begin{cases} 1, & \text{if } EK \text{ is true,} \\ 0, & \text{if } \overline{E}K \text{ is true,} \\ \mu, & \text{if } \overline{K} \text{ is true,} \end{cases}$$

where $\mu = \mathbb{P}[(E|H)|K]$. Moreover,

$$E|(H \wedge C(\mathcal{F})) = E|HK = E|K = \begin{cases} 1, & \text{if } EK \text{ is true,} \\ 0, & \text{if } \overline{E}K \text{ is true,} \\ y, & \text{if } \overline{K} \text{ is true,} \end{cases}$$

where $y = P(E|K)$. We observe that $(E|H)|K - E|K = (\mu - y)\overline{K} \in \{0, \mu - y\}$. Then, by coherence, $\mu = y$ and hence

$$(E|H)|C(\mathcal{F}) = (E|H)|K = E|K = E|(H \wedge C(\mathcal{F})).$$

Finally, as E and K are logically independent, it holds that $E|K$ does not coincide with the constant 1.

The next result shows that, under a suitable p-consistency hypothesis, the p-entailment of a conditional $E|H$ from a family $\mathcal{F}$ of conditional events implies the satisfaction of the General Import-Export principle (with value 1 in the expression (73)).

Theorem 17. *Let a p-consistent family of conditional events $\mathcal{F}$ and a further conditional event $E|H$ be given. If $\mathcal{F}$ p-entails $E|H$ and $\mathcal{F} \cup \{H\}$ is p-consistent, then $(E|H)|C(\mathcal{F}) = E|(H \wedge C(\mathcal{F})) = 1$, and hence the General Import-Export principle is satisfied.*

Proof. As $\mathcal{F}$ p-entails $E|H$, by Theorem 5 it holds that $(E|H)|C(\mathcal{F}) = 1$. Moreover, by Theorem 8, as $\mathcal{F}$ p-entails $E|H$ and $\mathcal{F} \cup \{H\}$ is p-consistent, it holds that $\mathcal{F} \cup \{H\}$ p-entails E and hence, by Theorem 5, $E|(H \wedge C(\mathcal{F})) = 1$. Thus, $(E|H)|C(\mathcal{F}) = E|(H \wedge C(\mathcal{F})) = 1$. □

The next result shows that, under the assumptions of the Probabilistic Weak Deduction Theorem (i.e., Theorem 10), the General Import-Export principle is satisfied.

Theorem 18. *Let a family of conditional events $\mathcal{F}$ and a further conditional event $E|H$ be given, with $\mathcal{F} \cup \{H\}$ p-consistent. If $\mathcal{F} \Rightarrow_p H$ and $\mathcal{F} \cup \{H\} \Rightarrow_p E$, then $(E|H)|C(\mathcal{F}) = E|(H \wedge C(\mathcal{F})) = 1$.*

Proof. By Theorem 10 it holds that $\mathcal{F} \Rightarrow_p E|H$. As $\mathcal{F} \cup \{H\}$ is p-consistent, from Theorem 17, it follows that $(E|H)|C(\mathcal{F}) = E|(H \wedge C(\mathcal{F})) = 1$, and hence the General Import-Export principle is satisfied. □

In the next section we show by some examples that, even if the assumptions of Theorem 10 do not hold, the General Import-Export principle is satisfied because $(E|H)|C(\mathcal{F}) = E|(H \wedge C(\mathcal{F})) = 1$.

5.2. General import-export principle and p-validity

We examine the (p-valid) Cautious Monotonicity (CM), Cumulative Transitivity (Cut), and OR inference rules of System P. For each inference rule, we show that the General Import-Export principle is satisfied by verifying the p-consistency of the family $\mathcal{F} \cup \{H\}$, where $\mathcal{F}$ is the set of premises and H is the antecedent of the conclusion.

CM rule: $\{C|A,\ B|A\} \Rightarrow_p C|AB$ In this case $\mathcal{F} = \{C|A,\ B|A\}$ and $E|H = C|AB$. We set $P(C|A) = x, P(B|A) = y, P(C|AB) = z$, and we observe that, as $\mathcal{F}$ p-entails $C|AB$, the assessment $(x, y, z) = (1, 1, 1)$ on $\{C|A, B|A, C|AB\}$ is coherent. In particular, the assessment $z = P(C|AB) = 1$ is coherent and hence $ABC \neq \emptyset$. The CM rule has been characterized in terms of the iterated conditional $(C|AB)|((C|A) \wedge (B|A))$ in [62], by showing that

$$(C|AB)|((C|A) \wedge (B|A)) = 1. \tag{74}$$

We observe that

$$(C|A) \wedge (B|A) \wedge AB = BC|A \wedge A \wedge B = ABC \neq \emptyset.$$

Thus, the assessment $\mathbb{P}[(C|A) \wedge (B|A) \wedge AB] = 1$, that is $P(ABC) = 1$, is coherent because $ABC \neq \emptyset$ and, by Theorem 3, the family $\{C|A, B|A, AB\}$ is p-consistent. Then, by Theorem 17, the General Import-Export principle is satisfied, with

$$(C|AB)|[(C|A) \wedge (B|A)] = C|[AB \wedge (C|A) \wedge (B|A)] = 1.$$

We also observe that $\{C|A, B|A\} \not\Rightarrow_p AB$ because

$$(C|A) \wedge (B|A) \wedge AB = ABC \neq BC|A = (C|A) \wedge (B|A).$$

Then, CM rule provides an example where the p-entailment of $C|AB$ from $\{C|A, B|A\}$ does not follow from Theorem 10 (Probabilistic Weak Deduction Theorem). Indeed, $\{C|A, B|A\} \cup \{AB\} \Rightarrow_p C$, but $\{C|A, B|A\} \not\Rightarrow_p AB$; thus not all assumptions of Theorem 10 are satisfied.

Cut rule: $\{C|AB,\ B|A\} \ \Rightarrow_p \ C|A$ In this case $\mathcal{F} = \{C|AB, B|A\}$ and $E|H = C|A$. We set $P(C|AB) = x, P(B|A) = y, P(C|A) = z$, and we observe that, as $\mathcal{F}$ p-entails $C|A$, the assessment $(x, y, z) = (1, 1, 1)$ on $\{C|AB, B|A, C|A\}$ is coherent. In particular, the assessment $x = P(C|AB) = 1$ is coherent and hence $ABC \neq \emptyset$. The Cut rule has been characterized in terms of the iterated conditional $(C|A)|((C|AB) \wedge (B|A))$ in [62], by showing that

$$(C|A)|((C|AB) \wedge (B|A)) = 1. \tag{75}$$

We observe that

$$(C|AB) \wedge (B|A) \wedge A = (C|AB) \wedge AB = ABC \neq \emptyset.$$

Thus, the assessment $\mathbb{P}[(C|AB) \wedge (B|A) \wedge A] = 1$, that is $P(ABC) = 1$, is coherent because $ABC \neq \emptyset$, and, by Theorem 3, the family $\{C|AB, B|A, C|A\}$ is p-consistent. Then, by Theorem 17, the General Import-Export principle is satisfied, with

$$(C|A)|[(C|AB) \wedge (B|A)] = C|[A \wedge (C|AB) \wedge (B|A)] = 1\,.$$

We also observe that $\{C|AB, B|A\} \not\Rightarrow_p A$ because

$$(C|AB) \wedge (B|A) \wedge A = ABC \ \neq \ BC|A = (C|AB) \wedge (B|A)\,.$$

Therefore $\{C|AB, B|A\} \Rightarrow_p C|A$, and $\{C|AB, B|A\} \cup \{A\} \Rightarrow_p C$, but $\{C|AB, B|A\} \not\Rightarrow_p A$. In other words, the Cut rule cannot be obtained by applying Theorem 10.

Or rule: $\{C|A,\ C|B\} \ \Rightarrow_p \ C|(A \vee B)$ In this case $\mathcal{F} = \{C|A, C|B\}$ and $E|H = C|(A \vee B)$.

The Or rule has been characterized in ([62]) in terms of the iterated conditional $(C|(A \vee B))|((C|A) \wedge (C|B))$, by showing that

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = 1. \tag{76}$$

As $\mathcal{F}$ p-entails $C|(A \vee B)$, the assessment $P(C|A) = P(C|B) = P[C|(A \vee B)] = 1$ is coherent, and this implies that

$$AC \neq \emptyset, \quad BC \neq \emptyset, \quad C \wedge (A \vee B) = ABC \vee A\overline{B}C \vee \overline{A}BC \neq \emptyset\,.$$

Then, at least one of the following conditions must be satisfied:

$$(i) \quad ABC \neq \emptyset, \qquad (ii) \quad A\overline{B}C \neq \emptyset \text{ and } \overline{A}BC \neq \emptyset\,.$$

By considering the partition $\{ABC, \overline{A}BC, A\overline{B}C, (A \vee B) \wedge \overline{C}, \overline{A}\,\overline{B}\}$, in the case (i) we assess

$$\mathcal{P}_1 : \qquad P(ABC) = 1\,,\ P(\overline{A}BC) = P(A\overline{B}C) = P[(A \vee B) \wedge \overline{C}] = P(\overline{A}\,\overline{B}) = 0\,;$$

in the case (ii) we assess

$$\mathcal{P}_2 : \qquad P(ABC) = 0\,,\ P(\overline{A}BC) = P(A\overline{B}C) = \frac{1}{2}\,,\ P[(A \vee B) \wedge \overline{C}] = P(\overline{A}\,\overline{B}) = 0\,.$$

For both assessments $\mathcal{P}_1$ and $\mathcal{P}_2$, the (unique and coherent) propagation to the family $\{C|A, C|B, A \vee B\}$ is given by $P(C|A) = P(C|B) = P(A \vee B) = 1$. Therefore, the family $\{C|A, C|B, A \vee B\}$ is p-consistent and, by Theorem 17, the General Import-Export principle is satisfied, with

$$(C|(A \vee B))\,|\,(C|A) \wedge (C|B)) = C|((A \vee B) \wedge (C|A) \wedge (C|B)) = 1. \tag{77}$$

We also observe that $\{C|A, C|B\} \not\Rightarrow_p A \vee B$ because;

$$(C|A) \wedge (C|B) \nleq (A \vee B);$$

indeed when $A \vee B = 0$ it holds that $(C|A) \wedge (C|B) = \mathbb{P}[(C|A) \wedge (C|B)]$, where $\mathbb{P}[(C|A) \wedge (C|B)]$ is not necessarily zero. Therefore $\{C|A, C|B\} \Rightarrow_p C|(A \vee B)$ and $\{C|A, C|B\} \cup \{A \vee B\} \Rightarrow_p C$, but $\{C|A, C|B\} \not\Rightarrow_p A \vee B$. In other words, Or rule cannot be obtained by applying Theorem 10.

5.3. Some deepenings of p-consistency and p-entailment

In this subsection, by examining suitable examples, we deepen some aspects related to the p-entailment of $E|H$ from $\mathcal{F}$, and of E from $\mathcal{F} \cup \{H\}$. As it is shown by the results in the previous sections, given a p-consistent family of conditional events $\mathcal{F}$ and a further conditional event $E|H$, with $H \neq \emptyset$, we distinguish the following cases:

(a) $\mathcal{F} \cup \{H\}$ is not p-consistent, with the sub-cases:
 $(a.1)$ $\mathcal{F} \not\Rightarrow_p E|H$;
 $(a.2)$ $\mathcal{F} \Rightarrow_p E|H$.
(b) $\mathcal{F} \cup \{H\}$ is p-consistent, with the sub-cases:
 $(b.1)$ $\mathcal{F} \not\Rightarrow_p E|H$ and $\mathcal{F} \cup \{H\} \not\Rightarrow_p E$;
 $(b.2)$ $\mathcal{F} \not\Rightarrow_p E|H$ and $\mathcal{F} \cup \{H\} \Rightarrow_p E$;
 $(b.3)$ $\mathcal{F} \Rightarrow_p E|H$ and $\mathcal{F} \cup \{H\} \Rightarrow_p E$.

Notice that the hypothetical sub-case $(b.4)$, that is $\mathcal{F} \Rightarrow_p E|H$ and $\mathcal{F} \cup \{H\} \not\Rightarrow_p E$, is impossible. Indeed, in the case (b), where $\mathcal{F} \cup \{H\}$ is p-consistent, if $\mathcal{F} \Rightarrow_p E|H$, then by Theorem 8 it follows that $\mathcal{F} \cup \{H\} \Rightarrow_p E$. In what follows we illustrate the previous cases by suitable examples.

$(a.1)$. An example is given by considering $\mathcal{F} = \{\overline{H}\}$ and a conditional event $E|H$, with E and H logically independent. Indeed, $\mathcal{F} = \{\overline{H}\}$ is p-consistent because $P(\overline{H}) = 1$ is coherent. $\mathcal{F} \cup \{H\} = \{\overline{H}, H\}$ is not p-consistent because the assessment $P(H) = P(\overline{H}) = 1$ is not coherent. Finally $\mathcal{F} = \{\overline{H}\} \not\Rightarrow_p E|H$ because $\overline{H} \nsubseteq E|H$, or equivalently because $P(\overline{H}) = 1$ does not imply $P(E|H) = 1$.

$(a.2)$. An example is given by considering $\mathcal{F} = \{AB|(A \vee B)\}$ and $E|H = (\overline{A} \vee \overline{B})|A\overline{B}$, with A, B logically independent. Of course, $\mathcal{F} = \{AB|(A \vee B)\}$ is p-consistent, because the assessment $P(AB|(A \vee B)) = 1$ is coherent. The family $\mathcal{F} \cup \{H\} = \{AB|(A \vee B), A\overline{B}\}$ is not p-consistent; indeed, $(AB|(A \vee B)) \wedge A\overline{B} = 0$ and hence $\mathbb{P}((AB|(A \vee B)) \wedge A\overline{B}) = 0$. Thus, by Theorem 3 the family $\{AB|(A \vee B), A\overline{B}\}$ is not p-consistent. Moreover, $E|H = (\overline{A} \vee \overline{B})|A\overline{B} = A\overline{B}|A\overline{B} = 1$ and hence $\mathcal{F} \Rightarrow_p E|H$.

$(b.1)$. An example is given by considering a family $\mathcal{F} = \{A|B\}$ and $E|H$, where the events A, B, E, H are logically independent. In this case the assessment

$$P(A|B) = x, P(E|H) = y, P(H) = z, P(E) = t$$

is coherent for every $(x, y, z) \in [0,1]^3$ and $yz \leqslant t \leqslant 1$. Then, $\mathcal{F}$ and $\mathcal{F} \cup \{H\}$ are p-consistent. But, from $x = 1$ it does not necessarily follow that $y = 1$, and hence $\mathcal{F} \not\Rightarrow_p E|H$. Moreover, from $(x, z) = (1, 1)$ it does not necessarily follow that $t = 1$, and hence $\mathcal{F} \cup \{H\} \not\Rightarrow_p E$.

$(b.2)$. Instances of this case are given in Examples 6, 8, 9, and 10.

$(b.3)$. This case arises when $\mathcal{F} \Rightarrow_p E|H$ and $\mathcal{F} \cup \{H\}$ is p-consistent, from which, by Theorem 8, it follows that $\mathcal{F} \cup \{H\} \Rightarrow_p E$, and by Theorem 17 the General Import-Export principle is satisfied. For instance (see Section 5), in the CM rule, where $\mathcal{F} = \{C|A, C|B\}$ and $E|H = C|AB$, it holds that $\{C|A, C|B\} \Rightarrow_p C|AB$ and $\{C|A, C|B, AB\}$ is p-consistent; thus $\{C|A, C|B, AB\} \Rightarrow_p C$. Similar reasoning applies to the Cut and the Or rules.

6. Related work

In this section we briefly make a comparison with related work, especially in the field of AI. The contributions of this paper are summarized as follows:

(i) we discuss two related papers on trivalent logics by examining selected properties which do not hold within these conditional logics;
(ii) based on the theory of compound and iterated conditionals, we give probabilistic versions of the classical deduction theorem;
(iii) we apply our probabilistic weak deduction theorems to checking the p-validity of some inference rules and for the construction of some new rules;
(iv) we introduce a General Import-Export principle, by relating it to p-consistency, p-entailment, and to Probabilistic Weak Deduction Theorem.

Many works on conditionals existing in literature may have connections or implications with this paper, see e.g., [8,20,23,41,42,44,45,47,49,52,54,76,79,80,90,96,98,101,102,114].

For instance, in [90] and [8] nonmonotonic systems for reasoning with conditional statements have been studied. In particular, in [90] several families of nonmonotonic consequence relations have been examined. Moreover, the well known System P, which is related to the Adams' logic of infinitesimal probabilities, has been introduced in [90]. In [8] the basic tools for the study of nonmonotonic systems are three-valued conditional objects, which are a qualitative counterpart to conditional probabilities within the framework of preferential entailment.

In [95], a trivalent semantics of nested conditionals is utilized. The authors argue that counterexamples to and criticisms of de Finetti's trivalent logic are addressed by considering the concept of genericity and the singular nature of events in de Finetti's theory. They accept the validity of the Import-Export principle but do not provide a specific analysis or proposal for defining compound and/or iterated conditionals, while preserving some probabilistic properties.

As a further instance, starting with de Finetti's trivalent logic, [47] examines the trivalent logics proposed by various authors (see, e.g., [15,17,31,45,48,55,81]). The trivalent notion of conjunction of de Finetti, which coincides with the Kleene-Lukasiewicz-Heyting, or KLH, conjunction (see, e.g., [6,21,23]), has also been utilized in [79] for examining the connections among belief revision, defeasible conditionals, and nonmonotonic inference. A significant difference from these trivalent logics is that the results of this paper rely on compound and iterated conditionals, which are many-valued objects defined at Level 2 of knowledge. By means of them we compare some notions of validity for inference rules, in selected trivalent logics, with the notion of p-validity of Adams in the setting of coherence.

In [52] a Boolean algebra of conditionals has been given, where some probabilistic results have been obtained based on a suitable canonical extension of a probability measure. Further developments have been given in [49,50] in agreement with previous results obtained in [69,71]. Notice that in [52] conditionals are logical objects and their interpretation remains at the symbolic level, without a numerical counterpart. In [80] a formalization of conditional probabilistic argumentation based on probabilistic conditional logic is provided. We note that in such a paper the conditional probability $P(B|A)$, as commonly defined in the literature, is given by the ratio $\frac{P(AB)}{P(A)}$, which leaves $P(B|A)$ undefined when $P(A)=0$.

We employ de Finetti's coherence-based theory, wherein conditional probabilities are directly evaluated, avoiding the need to represent them as ratios of unconditional probabilities. This approach allows us to properly handle cases where some conditioning events have zero probability. A general approach to conditional probability logic in the setting of coherence has been developed in [29]. In this book the role of zero probabilities for conditional events has been throughly discussed, in connection with zero-layers, stochastic independence, probabilistic nonmonotonic reasoning, and fuzzy sets. We note that the notions of union and intersection studied in [29] do not coincide with compound conditionals used here.

The relationship between probabilistic reasoning under coherence and model-theoretic probabilistic reasoning, with concepts from default reasoning, has been studied in [9,10].

For similar approaches to conditional logics in the setting of coherence see for instance [4,28,30].

The Classical Deduction Theorem recalled in Remark 10, which is a basic tool in conditional logics (see, e.g., [47]), in our framework is not valid because we use the notions of p-entailment and p-validity. The paper compares various notions of validity for inference rules in trivalent logics with Adams' notion of p-validity in coherence settings, highlighting differences. In this paper we go beyond the Classical Deduction Theorem by presenting probabilistic versions of it and by providing a probabilistic semantics for more general conditional inference rules.

In [21], different notions of iterated conditioning within the framework of the trivalent logics explored in [14,31,39,48] have been examined, showing that some related basic probabilistic properties are not preserved. In the same paper is also discussed the Import-Export principle ([55,100]), recalled in Definition 10).

Such a principle, adopted in many trivalent logics, jointly with the classical probabilistic properties, leads to the Lewis' triviality results ([97,119]); some criticisms are given in [1,83].

This principle fails for the notions of compound and iterated conditionals, utilized in this paper, which preserve all the basic logical and probabilistic properties (see, e.g., [69,74,116]). In the paper, we introduce a General Import-Export principle and studying cases where it is satisfied. Moreover, we relate it to the property of p-validity by examining selected inference rules of System P: CM, Cut, and Or rules.

7. Conclusions

There has been increasing interest in recent years in de Finetti's analysis of conditional events. In his early work, he classified a conditional event, $B|A$, as a three-valued entity, which is true when A and B are true, false when A is true and B is false, and "void" when A is false. But we have pointed out that, in later research, a higher level of analysis is distinguished, in which the void value becomes the conditional probability, $P(B|A)$. We have acknowledged that there are some benefits in continuing to make the early simple three-valued distinction. But we have also argued that there are significant advantages to adopting the later, higher-level analysis. At this higher level, we can give an account of certain compound conditionals, and we can provide an intuitive definition of probabilistic entailment, p-entailment. By this definition, a set of premises p-entails a conclusion if and only if the conclusion has a probability of 1 when the premises have a probability of 1. We have specified exactly how this definition of p-entailment is related to less straightforward definitions of entailment using just the three values. We have done this by examining two recent articles that are restricted to de Finetti's trivalent analysis. We showed that the Classical Deduction Theorem fails for p-entailment. Then in a novel step, we proved for the first time a Probabilistic Weak Deduction Theorem, with some variants of it, and further related results. We also analyzed many examples; in particular, we used the or-to-if inference, from *A or B* to *if not-A then B*, to illustrate the invalidity of the Classical Deduction Theorem, and the invalidity of the Import-Export principle, for p-entailment. By exploiting our probabilistic deduction theorems, we derived new inference rules related to selected Aristotelian syllogisms. As another novelty, we introduced a General Import-Export principle for iterated conditionals by relating it to p-consistency and p-entailment. Finally we illustrated the validity of the General Import-Export principle in some p-valid inference rules of System P: CM, Cut, Or. We also observed that in these examples the Probabilistic Weak Deduction Theorem is not applicable. We plan to extend these results to further relevant cases, such as other Aristotelian syllogisms, further nonmonotonic inference schemes, knowledge representation in AI, and the psychology of human reasoning under uncertainty.

CRediT authorship contribution statement

Angelo Gilio: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **David E. Over:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Niki Pfeifer:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Giuseppe Sanfilippo: Writing – review & editing, Writing – original draft, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. The trivalent logic of de Finetti

In this appendix we deepen (some aspects of) the trivalent logic of de Finetti, which is discussed by many authors. By the trivalent truth tables in de Finetti ([39]) the conjunction of two conditional events $A|H$ and $B|K$, here denoted by $(A|H)\wedge_{df}(B|K)$, is the logical product between tri-events, which is *true* when $AHBK$ is true, *false* when $\bar{A}H \vee \bar{B}K$, and *void* otherwise. In other words, $(A|H)\wedge_{df}(B|K)$ is true when all conditional events are true, is false when at least a conditional event is false, and is void otherwise. Thus

$$(A|H)\wedge_{df}(B|K) = AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K), \tag{A.1}$$

and hence $P[(A|H)\wedge_{df}(B|K)]$, that is $P[AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K)]$, is the probability that the conjunction is true, given that it is true or false. In particular, when $P(AHBK \vee \bar{A}H \vee \bar{B}K) > 0$, it holds that

$$P[(A|H)\wedge_{df}(B|K)] = \frac{P(AHBK)}{P(AHBK \vee \bar{A}H \vee \bar{B}K)},$$

which is the probability that the conjunction is true divided by the probability that it is true or false.

Remark 19. When $A|H \subseteq B|K$, it holds that $P(A|H) \leqslant P(B|K)$, for every conditional probability P (see, e.g. [68, Theorem 6]). This means that the set of coherent assessments (x, y) on $\{A|H, B|K\}$, where $x = P(A|H), y = P(B|K)$, is characterized by the property that $x \leqslant y$; that is to say, there does not exist any coherent assessment (x, y) such that $x > y$. In terms of comparative, or qualitative, probabilities (see, e.g., [11,25,26,35,37,89]), if $A|H \subseteq B|K$, then "A given H is no more probable than B given K". A computational analysis of comparative conditional probabilities in the framework of coherence has been given in [102]. In [41] a logic of qualitative probability has been developed, where the binary operator $\preceq$ describing the relation "is no more probable than" is regarded as an operator on finite sequences of formulas.

We observe that the conjunction of Goodman and Nguyen ([77, formula (3.4)]) coincides with $\wedge_{df}$. Then, (see [77, formula (3.17)], see also [115, Remark 1])

$$(A|H)\wedge_{df}(B|K) = A|H \iff A|H \subseteq B|K. \tag{A.2}$$

This trivalent notion of conjunction of de Finetti coincides with the Kleene-Lukasiewicz-Heyting, or KLH, conjunction (see, e.g., [6]). Such trivalent logic has been used for studying the connections among belief revision, defeasible conditionals and nonmonotonic inference in [79]. We also observe that the Fréchet-Hoeffding bounds are not preserved by $\wedge_{df}$. Indeed, under logical independence of the basic events, the lower and upper bounds for $z = P((A|H)\wedge_{df}(B|K))$ are $z' = 0$ and $z'' = \min\{P(A|H), P(B|K)\}$, respectively ([115, Theorem 3]), while the Fréchet-Hoeffding lower bound is $\max\{P(A|H) + P(B|K) - 1, 0\}$. We observe that, if the Fréchet-Hoeffding bounds were valid, the *And* rule would be also satisfied, that is $\{A|H, B|K\}$ would p-entail $(A|H)\wedge_{df}(B|K)$. On the contrary, with de Finetti's trivalent conjunction, when there are some logical dependencies, it may happen that the only coherent extension on the conjunction of conditional events is zero, even if we assign probability 1 to each conditional event. That the And rule is invalid is counterintuitive as its validity is also required by the basic nonmonotonic reasoning System P ([57,90]). We now illustrate the invalidity of the And rule, when using the conjunction $\wedge_{df}$, by two examples.

Example 16. We show a counterintuitive aspect of de Finetti's trivalent conjunction. A double-headed coin is going to be either tossed or spun. Consider the events H ="the coin is tossed", $\overline{H}$ ="the coin is spun", and A ="the coin comes up heads". Of course $P(A|H) = P(A|\overline{H}) = 1$, but as $H\overline{H} = \emptyset$ it holds that $(A|H) \wedge_{df} (A|\overline{H}) = AH\overline{H}|(AH\overline{H} \vee \overline{A}H \vee \overline{A}\,\overline{H}) = \emptyset|\overline{A}$, and hence $P[(A|H) \wedge_{df} (A|\overline{H})] = P(\emptyset|\overline{A}) = 0$. Thus, with de Finetti's conjunction, as the coin cannot be both tossed and spun at the same time, the compound "if the coin is tossed it will come up heads and if the coin is spun it will come up heads" has, counterintuitively, a probability of 0, even if both conjuncts, "if the coin is tossed it will come up heads" and "if the coin is spun it will come up heads", have probability 1. Notice that this problem is avoided when using Definition 5. Indeed, by Theorem 2 Fréchet-Hoeffding bounds are satisfied, and hence $\mathbb{P}[(A|H) \wedge (A|\overline{H})] = 1$ when $P(A|H) = P(A|\overline{H}) = 1$.

Example 16 is interesting for connexive logic, and its probabilistic semantics ([110]), as it concerns a special case of what is negated in Aristotle's Second Thesis ([111]).

Example 17. In [67, Example 2], given any logically independent events A, B, C, D, it has been shown that the family $\{C|B, B|A, A|(A \vee B), B|(A \vee B), D|\overline{A}\}$ is p-consistent and p-entails $C|A$. In this example we consider the sub-family $\mathcal{F} = \{C|B, B|A, A|(A \vee B), D|\overline{A}\}$, which of course is p-consistent too. Then, the assessment

$$P(C|B) = P(B|A) = P(A|(A \vee B)) = P(D|\overline{A}) = 1 \tag{A.3}$$

is coherent. We observe that

$$(C|B) \wedge_{df} (B|A) \wedge_{df} (A|(A \vee B)) = ABC|(ABC \vee B\overline{C} \vee A\overline{B} \vee \overline{A}B) = ABC|(A \vee B)$$

and

$$\begin{aligned}&(C|B) \wedge_{df} (B|A) \wedge_{df} (A|(A \vee B)) \wedge_{df} (D|\overline{A}) = (ABC|(A \vee B)) \wedge_{df} (D|\overline{A}) = \\ &= ABC\overline{A}D|(ABC\overline{A}D \vee AB\overline{C} \vee A\overline{B} \vee \overline{A}B \vee \overline{A}\,\overline{D}) = \emptyset|(AB\overline{C} \vee A\overline{B} \vee \overline{A}B \vee \overline{A}\,\overline{D}).\end{aligned}$$

Then, the unique coherent extension z of the assessment (A.3) on $(C|B) \wedge_{df} (B|A) \wedge_{df} (A|(A \vee B)) \wedge_{df} (D|\overline{A})$ is $z = 0$. As we can see, with the trivalent notion of conjunction of de Finetti, the well known And rule does not hold. On the contrary, by using Definition 5, the And rule is valid ([118, Remark 4]).

As another counterintuitive aspect, given any event A, with $A \neq \emptyset$ and $A \neq \Omega$, coherence requires that both conditional events $A|A$, $(\overline{A}|\overline{A})$ have probability 1 and hence, if the And rule were valid, their conjunction $(A|A) \wedge_{df} (\overline{A}|\overline{A})$ would have probability 1. Actually, this conjunction does not exist because $(A|A) \wedge_{df} (\overline{A}|\overline{A}) = \emptyset|\emptyset$. Notice that, by using Definition 5, in agreement with the intuition, it holds that $(A|A) \wedge (\overline{A}|\overline{A}) = 1$. This example will be also examined at the end of Section 3.2.

Remark 20. We observe that, unlike our Level 2 approach based on the later development in de Finetti's thought, de Finetti's early trivalent notions of conjunction and disjunction do not satisfy the probabilistic sum rule. Indeed, as

$$(A|H) \vee_{df} (B|K) = (AH \vee BK)|(\overline{A}H\overline{B}K \vee AH \vee BK), \tag{A.4}$$

De Morgan Laws are satisfied, then

$$(A|H) \vee_{df} (B|K) = 1 - (\overline{A}|H) \wedge_{df} (\overline{B}|K).$$

Assuming $P(A|H) = P(B|K) = 1$, the extension $P((A|H) \wedge_{df} (B|K)) = 0$ is coherent ([115, Theorem 3]). Moreover, $P(\overline{A}|H) = P(\overline{B}|K) = 0$ and the unique coherent extension on $(\overline{A}|H) \wedge_{df} (\overline{B}|K)$ is $P((\overline{A}|H) \wedge_{df} (\overline{B}|K)) = 0$ ([115, Theorem 3]). As a consequence, the unique coherent extension on $(A|H) \vee_{df} (B|K)$ is $P((A|H) \vee_{df} (B|K)) = 1$. Then, the assessment $P(A|H) = P(B|K) = 1$, $P((A|H) \wedge_{df} (B|K)) = 0$ and $P((A|H) \vee_{df} (B|K)) = 1$, is coherent, with

$$P((A|H) \vee_{df} (B|K)) = 1 \neq 2 = P(A|H) + P(B|K) - P((A|H) \wedge_{df} (B|K)).$$

Concerning the iterated conditional $(B|K)|_{df}(A|H)$, de Finetti's trivalent approach implies that the Import-Export principle is valid both for antecedent-nested and consequent-nested conditionals, that is it holds that $B|_{df}(A|H) = B|AH$ and $(B|K)|_{df}A = B|AK$, respectively. Indeed, de Finetti defines the iterated conditional as the conditional event

$$(B|K)|_{df}(A|H) = B|AHK; \tag{A.5}$$

thus $B|_{df}(A|H) = B|AH$, when $K = \Omega$, and $(B|K)|_{df}A = B|AK$, when $H = \Omega$.

We remark that the quantity $P((A|H) \wedge_{df} (B|K))$ in general does not coincide with the product $= P[(B|K)|_{df}(A|H)]P(A|H)$, that is the compound probability theorem does not hold in the trivalent logic of de Finetti (see also [21]). Indeed, in general

$$P((A|H) \wedge_{df} (B|K)) = P[AB|(AHBK \vee \overline{A}H \vee \overline{B}K)] \neq P(B|AHK)P(A|H) = P[(B|K)|_{df}(A|H)]P(A|H).$$

For instance, by assuming that A, H, B, K are stochastically independent, with $P(AHBK \vee \bar{A}H \vee \bar{B}K) > 0$, $P(AHK) > 0$, it holds that

$$P((A|H) \wedge_{df} (B|K)) = \frac{P(ABHK)}{P(AHBK \vee \bar{A}H \vee \bar{B}K)} = P(A)P(B)\frac{P(H)P(K)}{P(AHBK \vee \bar{A}H \vee \bar{B}K)},$$

while

$$P[(B|K)|_{df}(A|H)]P(A|H) = P(B|AHK)P(A|H) = P(A)P(B),$$

because $P(B|AHK) = P(B)$ and $P(A|H) = P(A)$.

Appendix B. A variant of $SS \cap TT$-validity

In this appendix we discuss a variant of the $SS \cap TT$-validity (Section 3.2.2), which can be related to the logic of conditionals of Adams. Given a set of premises Γ and a further conditional $B|A$, the notion of $SS \cap TT$-validity would become equivalent to the condition $QC(\Gamma) \subseteq B|A$ if we slightly modify it in the following way:

An inference from a set Γ to a conclusion $B|A$ is $(SS \cap TT)^*$-valid, denoted by $\Gamma \vDash_{(SS \cap TT)^*} B|A$, if and only if the following conditions are satisfied: (i) when at least one conditional in Γ is true and all the other ones are void it holds that $B|A$ is true; (ii) when $B|A$ is false it holds that at least a conditional in Γ is false.

Notice that, in the case where $\Gamma = \{E|H\}$, the condition $E|H \vDash_{(SS \cap TT)^*} B|A$ is equivalent to $E|H \vDash_{SS \cap TT} B|A$, that is $E|H \subseteq B|A$, as shown in (43).

Under the notion of $(SS \cap TT)^*$-validity, we can show that the concepts of p-validity (in the setting of coherence), delta-epsilon validity and "yielding" of Adams, and $(SS \cap TT)^*$-validity are all deeply related to each other. These relationships are illustrated by the following steps:

(a) $(SS \cap TT)^*$-validity of an inference from a set of premises Γ to a conclusion $B|A$ means that when $QC(\Gamma)$ is true then $B|A$ is true and when $B|A$ is false then $QC(\Gamma)$ is false;
(b) $(SS \cap TT)^*$-validity of the inference from Γ to $B|A$ is equivalent to the condition $QC(\Gamma) \subseteq B|A$;
(c) as defined by Adams ([2, p. 168]), Γ "yields" $B|A$ if and only if $QC(\Gamma) \subseteq B|A$ and hence $(SS \cap TT)^*$-validity coincides with the notion of "yielding";
(d) a p-consistent family $\mathcal{F}$ p-entails $B|A$ if and only if there exists $\Gamma \subseteq F$ such that Γ "yields" $B|A$, with $\Gamma = \emptyset$ when $B|A$ is a logical truth, i.e. $A \subseteq B$ ([2, p. 187]);
(e) analogously, in the setting of coherence, by Theorem 1, it holds that

$$\mathcal{F} \Rightarrow_p B|A \quad \Longleftrightarrow \quad QC(S) \subseteq B|A, \text{ for some nonempty } S \subseteq \mathcal{F}, \text{ or } A \subseteq B,$$

or, in other terms, the p-entailment of $B|A$ from $\mathcal{F}$ means that, given the coherent assessment $P(E|H) = 1, \forall E|H \in \mathcal{F}$, its unique coherent extension on $B|A$ is $P(B|A) = 1$;
(f) equivalently, denoting by $\mathcal{C}(\mathcal{F})$ the conjunction of the conditional events in a p-consistent family $\mathcal{F}$, p-entailment has been characterized in the following way (see Theorem 4)

$$\mathcal{F} \Rightarrow_p B|A \iff \mathcal{C}(\mathcal{F}) \leqslant B|A \iff \mathcal{C}(\mathcal{F}) \wedge B|A = \mathcal{C}(\mathcal{F}). \tag{B.1}$$

Note that p-entailment is a monotonic inference relation. This means that, if a p-consistent family $\mathcal{F}$ p-entails $B|A$, then any p-consistent family $\mathcal{F}'$, such that $\mathcal{F} \subset \mathcal{F}'$, p-entails $B|A$ as well. This is because $\mathcal{C}(\mathcal{F}') \leqslant \mathcal{C}(\mathcal{F}) \leqslant B|A$.

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