S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Individual

(Please select 5 problems to solve)

1. Let V be a finite dimensional complex vector space. Let A, B be two linear endomorphisms of V satisfying $AB - BA = B$. Prove that there is a common eigenvector for A and B.

2. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Its group of multiplicative units is $GL_2(\mathbb{R})$, consisting of invertible matrices in $M_2(\mathbb{R})$.

(a) Find an injective homomorphism from the field $\mathbb{C}$ of complex numbers into the ring $M_2(\mathbb{R})$.

(b) Show that if ϕ_1 and ϕ_2 are two such homomorphisms, then there exists a $g \in GL_2(\mathbb{R})$ such that $\phi_2(x) = g\phi_1(x)g^{-1}$ for all $x \in \mathbb{C}$.

(c) Let h be an element in $GL_2(\mathbb{R})$ whose characteristic polynomial $f(x)$ is irreducible over $\mathbb{R}$. Let $F \subset M_2(\mathbb{R})$ be the subring generated by h and $a \cdot I$ for all $a \in \mathbb{R}$, where I is the idenity matrix. Show that F is isomorphic to $\mathbb{C}$.

(d) Let h' be any element in $GL_2(\mathbb{R})$ with the same characteristic polynomial $f(x)$ as h in (c). Show that h and h' are conjugate in $GL_2(\mathbb{R})$.

(e) If $f(x)$ in (c) and (d) is reducible over $\mathbb{R}$, will the same conclusion on h and h' hold? Give reasons.

3. Let G be a non-abelian finite group. Let $c(G)$ be the number of conjugacy classes in G Define $\overline{c}(G) := c(G)/|G|$, ($|G| = \mathrm{Card}(G)$).

(a) Prove that $\overline{c}(G) \leq \frac{5}{8}$.

(b) Is there a finite group H with $\overline{c}(H) = \frac{5}{8}$?

(c) (open ended question) Suppose that there exists a prime number p and an element $x \in G$ such that the cardinality of the conjugacy class of x is divisible by p. Find a good/sharp upper bound for $\overline{c}(G)$.

4. Let F be a splitting field over $\mathbb{Q}$ the polynomial $x^8 - 5 \in \mathbb{Q}[x]$. Recall that F is the subfield of $\mathbb{C}$ generated by all roots of this polynomial.

(a) Find the degree $[F : \mathbb{Q}]$ of the number field F.
(b) Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$.

5. Let $T \subset \mathbb{N}_{>0}$ be a finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences $(t_1, \ldots, t_m)$ with $m \leq n$, $t_i \in T$ for all $i = 1, \ldots, m$ and $t_1 + \ldots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geq 1} a_n \, z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z, and find this rational function.

6. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).

S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Team

(Please select 5 problems to solve)

1. For a real number r, let $[r]$ denote the maximal integer less or equal than r. Let a and b be two positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Show that the two sequences of integers $[ax]$, $[bx]$ for $x = 1, 2, 3, \cdots$ contain all natural numbers without repetition.

2. Let $n \geq 2$ be an integer and consider the Fermat equation

$$X^n + Y^n = Z^n, \qquad X, Y, Z \in \mathbb{C}[t].$$

Find all nontrivial solution (X, Y, Z) of the above equation in the sense that X, Y, Z have no common zero and are not all constant.

3. Let $p \geq 7$ be an odd prime number.

(a) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

4. For a positive integer a, consider the polynomial

$$f_a = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1.$$

Show that it is irreducible. Let F be the splitting field of f_a. Show that its Galois group is solvable.

5. Prove that a group of order 150 is not simple.

6. Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n-th symmetric power $V_n = \operatorname{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?

S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Individual

(Please select 5 problems to solve)

1. a) Let x_k, $k = 1, \ldots, n$ be real numbers from the interval $(0, \pi)$ and define $x = \dfrac{\sum_{i=1}^{n} x_i}{n}$. Show that

$$\prod_{k=1}^{n} \frac{\sin x_k}{x_k} \leq \left(\frac{\sin x}{x}\right)^n.$$

b) From

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin(x^2) dx$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points x in $\mathbb{R}$ where f is continuous is a countable intersection of open sets.

3. Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ $(z \in D)$. If z_0 is a point in D such that both z_0 and $-z_0$ are zeros of order m of $f(z)$ and $0 < |z_0| \leq \dfrac{m-1}{m}$, then $|f(0)| < e^{-2}$.

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y-axis, the function has limit 1, while when approaching any point in the negative half of the y-axis, the function has limit -1.

5. Let $K(x, y) \in L^1([0, 1] \times [0, 1])$. For all $f \in C^0[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y) f(y) dy$$

Prove that $Tf \in C^0([0, 1])$. Moreover $\Omega = \{Tf |\ ||f||_{sup} \leq 1\}$ is pre-compact in $C^0([0, 1])$, i.e. every sequence in Ω has a converging subsequence, here $||f||_{sup} = \sup\{|f(x)||x \in [0, 1]\}$. (Hint: Every Lebesgue integrable function over the square can be approximated by polynomial functions in the L^1 norm.)

6. Consider the equation $\dot{x} = -x + f(t, x)$, where $|f(t, x)| \leq \phi(t)|x|$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\int^{\infty} \phi(t)dt < \infty$. Prove that every solution approaches zero as $t \to \infty$.

S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Team

(Please select 5 problems to solve)

1. a) Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ ($z \in D$). Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}. \qquad (z \in D)$$

b) For any finite complex value a, prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{i\theta}| d\theta = \max\{\log |a|, 0\}.$$

2. Let $f \in C^1(\mathbf{R})$, $f(x+1) = f(x)$, for all x, then we have

$$||f||_\infty \leq \int_0^1 |f(t)|dt + \int_0^1 |f^{'}(t)|dt.$$

3. Consider the equation

$$\ddot{x} + (1 + f(t))x = 0.$$

We assume that $\int^\infty |f(t)|dt < \infty$. Study the Lyapunov stability of the solution $(x, \dot{x}) = (0, 0)$.

4. Suppose $f : [a, b] \to \mathbf{R}$ be a L^1-integrable function. Extend f to be 0 outside the interval $[a, b]$. Let

$$\phi(x) = \frac{1}{2h} \int_{x-h}^{x+h} f$$

Show that

$$\int_a^b |\phi| \leq \int_a^b |f|.$$

5. Suppose $f \in L^1[0, 2\pi]$, $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx$, prove that

1) $\sum_{|n|=0}^{\infty} |\hat{f}(n)|^2 < \infty$ implies $f \in L^2[0, 2\pi]$,

2) $\sum_n |n\hat{f}(n)| < \infty$ implies that $f = f_0, a.e., f_0 \in C^1[0, 2\pi]$,

where $C^1[0, 2\pi]$ is the space of functions f over $[0, 1]$ such that both f and its derivative $f^{'}$ are continuous functions.

6. Let Ω be a bounded domain of $\mathbf{R}^n$ and let f be a smooth function defined in $[0, +\infty)$ such that $f(t)/t$ is strictly decreasing. Assume that u_1 and u_2 are positive solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

Show that $u_1 = u_2$. (Hint: Calculate $\Delta \log \frac{u_2}{u_1}$ and consider the maximum principle.)

S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Individual
(Please select 5 problems to solve)

1. Let $Z_1, \cdots, Z_n$ be i.i.d. random variables with $Z_i \sim N(\mu, \sigma^2)$. Find

$$E(\sum_{i=1}^{n} Z_i | Z_1 - Z_2 + Z_3).$$

2. Let $X_1, \cdots, X_n$ be pairwise independent. Further, assume that $EX_i = 1 + i^{-1}$ and that $\max_{1\le i\le n} E|X_i|^{1+\epsilon} < \infty$ for some $\epsilon > 0$. Show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} 1.$$

3. Let $Z_1, \cdots, Z_6$ be i.i.d. random variables with $Z_i \sim N(0,1)$. Set

$$U^2 = \frac{(Z_1Z_2 + Z_3Z_4 + Z_5Z_6)^2}{Z_2^2 + Z_4^2 + Z_6^2}, \quad V^2 = \frac{U^2(Z_2^2 + Z_4^2)}{U^2 + Z_6^2}.$$

Find and identify the densities of U^2 and V^2.

4. Suppose that three characteristics in a large propulation can be observed according to the following frequencies

$$p_1 = \theta^3, \quad p_2 = 3\theta(1-\theta), \quad p_3 = (1-\theta)^3,$$

where $\theta \in (0,1)$. Let $N_j,\ j = 1, 2, 3$ be the observed frequencies of characteristic j in a random sample of size n.

(a) Construct the approximate level $(1-\alpha)$ maximum likelihood confidence set for θ.

(b) Derive the asymptotic distribution for the frequency substitution estimator $\hat{\theta}_2 = 1 - (N_3/n)^{1/3}$.

5. (1) Suppose

$$S = \begin{bmatrix} \sigma & \mathbf{u}^T \\ 0 & S_c \end{bmatrix}, \quad T = \begin{bmatrix} \tau & \mathbf{v}^T \\ 0 & T_c \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \beta \\ \mathbf{b}_c \end{bmatrix},$$

where σ, τ and β are scalars, S_c and T_c are n-by-n matrices, and $\mathbf{b}_c$ is an n-vector. Show that if there exists a vector $\mathbf{x}_c$ such that

$$(S_cT_c - \lambda I)\mathbf{x}_c = \mathbf{b}_c$$

and $\mathbf{w}_c = T_c\mathbf{x}_c$ is available, then

$$\mathbf{x} = \begin{bmatrix} \gamma \\ \mathbf{x}_c \end{bmatrix}, \quad \gamma = \frac{\beta - \sigma\mathbf{v}^T\mathbf{x}_c - \mathbf{u}^T\mathbf{w}_c}{\sigma\tau - \lambda}$$

solves $(ST - \lambda I)\mathbf{x} = \mathbf{b}$.

(2) Hence or otherwise, derive an $O(n^2)$ algorithm for solving the linear system $(U_1U_2 - \lambda I)\mathbf{x} = \mathbf{b}$ where U_1 and U_2 are n-by-n upper triangular matrices, and $(U_1U_2 - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(n^2)$ complexity.

(3) Hence or otherwise, derive an $O(pn^2)$ algorithm for solving the linear system $(U_1U_2 \cdots U_p - \lambda I)\mathbf{x} = \mathbf{b}$ where $\{U_i\}_{i=1}^p$ are all n-by-n upper triangular matrices, and $(U_1U_2 \cdots U_p - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(pn^2)$ complexity.

6. (1) Let $A \in \mathbb{R}^{m\times n}$, i.e. A is an m-by-n real matrix. Show that there exists an m-by-m orthogonal matrix U and an n-by-n orthogonal matrix V such that

$$U^TAV = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p),$$

where $p = \min\{m, n\}$ and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0.$$

(2) Let rank$(A)= r$. Show that for any positive integer $k < r$,

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \sigma_{k+1}.$$

(*Hint:* Consider the matrix $A_k = \sum_{i=1}^{k} \sigma_i\mathbf{u}_i\mathbf{v}_i^T$, where $\mathbf{u}_i$ and $\mathbf{v}_i$ are columns of U and V respectively.)

S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Team

(Please select 5 problems to solve)

1. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with continuous distribution functions $F(x_1), \cdots, F(x_n)$, respectively. Let $Y_1 < \cdots < Y_n$ be the order statistics of $X_1, \cdots, X_n$. Prove that $Z_j = F(Y_j)$ has the beta $(j, n-j+1)$ distribution ($j = 1, \cdots, n$).

2. Let $X_1, \cdots, X_n$ be i.i.d. random variable with a continuous density f at point 0. Let

$$Y_{n,i} = \frac{3}{4b_n}(1 - X_i^2/b_n^2)I(|X_i| \leq b_n).$$

Show that

$$\frac{\sum_{i=1}^n (Y_{n,i} - EY_{n,i})}{(b_n \sum_{i=1}^n Y_{n,i})^1/2} \xrightarrow{L} N(0, 3/5),$$

provided $b_n \to 0$ and $nb_n \to \infty$.

3. Let $X_1, \cdots, X_n$ be independently and indentically distributed random variables with $X_i \sim N(\theta, 1)$. Suppose that it is known that $|\theta| \leq \tau$, where τ is given. Show

$$\min_{a_1, \cdots, a_{n+1}} \sup_{|\theta| \leq \tau} E(\sum_{i=1}^n a_i X_i + a_{n+1} - \theta)^2 = \frac{\tau^2 n^{-1}}{\tau^2 + n^{-1}}.$$

Hint: Carefully use the sufficiency principle.

4. The rules for "1 and 1" foul shooting in basketball are as follows. The shooter gets to try to make a basket from the foul line. If he succeeds, he gets another try. More precisely, he make 0 baskets by missing the first time, 1 basket by making the first shot and xsmissing the second one, or 2 baskets by making both shots.

Let n be a fixed integer, and suppose a player gets n tries at " 1 and 1" shooting. Let N_0, N_1, and N_2 be the random variables recording the number of times he makes 0, 1, or 2 baskets, respectively. Note that $N_0 + N_1 + N_2 = n$. Suppose that shots are independent Bernoulli trails with probability p for making a basket.

(a) Write down the likelihood for (N_0, N_1, N_2).

(b) Show that the maximum likelihood estimator of p is

$$\hat{p} = \frac{N_1 + 2N_2}{N_0 + 2N_1 + 2N_2}.$$

(c) Is $\hat{p}$ an unbiased estimator for p? Prove or disprove. (Hint: $E\hat{p}$ is a polynomial in p, whose order is higher than 1 for $p \in (0, 1)$.)

(d) Find the asymptotic distribution of $\hat{p}$ as n tends to ∞.

5. When considering finite difference schemes approximating partial differential equations (PDEs), for example, the scheme

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n) \tag{1}$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$, approximating the PDE

$$u_t + u_x = 0, \tag{2}$$

we are often interested in stability, namely

$$||u^n|| \leq C||u^0||, \qquad n\Delta t \leq T \tag{3}$$

for a constant $C = C(T)$ independent of the time step Δt and the spatial mesh size Δx. Here $||\cdot||$ is a given norm, for example the L^2 norm or the L^∞ norm, of the numerical solution vector $u^n = (u_1^n, u_2^n, \cdots, u_N^n)$. The mesh points are $x_j = j\Delta x$, $t^n = n\Delta t$, and the numerical solution u_j^n approximates the exact solution $u(x_j, t^n)$ of the PDE (2) with a periodic boundary condition.

(i) Prove that the scheme (1) is stable in the sense of (3) for both the L^2 norm and the L^∞ norm under the time step restriction $\lambda \leq 1$.

(ii) Since the numerical solution u^n is in a finite dimensional space, Student A argues that the stability (3), once proved for a specific norm $|| \cdot ||_a$, would also automatically hold for any other norm $|| \cdot ||_b$. His argument is based on the equivalency of all norms in a finite dimensional space, namely for any two norms $|| \cdot ||_a$ and $|| \cdot ||_b$ on a finite dimensional space W, there exists a constant $\delta > 0$ such that

$$\delta||u||_b \leq ||u||_a \leq \frac{1}{\delta}||u||_b.$$

Do you agree with his argument? If yes, please give a detailed proof of the following theorem: If a scheme is stable, namely (3) holds for one particular norm (e.g. the L^2 norm), then it is also stable for any other norm. If not, please explain the mistake made by Student A.

6. We have the following 3 PDEs

$$u_t + Au_x = 0, \tag{4}$$

$$u_t + Bu_x = 0, \tag{5}$$

$$(6) \qquad u_t + Cu_x = 0, \qquad C = A + B.$$

Here u is a vector of size m and A and B are $m \times m$ real matrices. We assume $m \geq 2$ and both A and B are diagonalizable with only real eigenvalues. We also assume periodic initial condition for these PDEs.

(i) Prove that (4) and (5) are both well-posed in the L^2-norm. Recall that a PDE is well-posed if its solution satisfies

$$||u(\cdot,t)|| \leq C(T)||u(\cdot,0)||, \qquad 0 \leq t \leq T$$

for a constant $C(T)$ which depends only on T.

(ii) Is (6) guaranteed to be well-posed as well? If yes, give a proof; if not, give a counter example.

(iii) Suppose we have a finite difference scheme

$$u^{n+1} = A_h u^n$$

for approximating (4) and another scheme

$$u^{n+1} = B_h u^n$$

for approximating (5). Suppose both schemes are stable in the L^2-norm, namely (3) holds for both schemes. If we now form the splitting scheme

$$u^{n+1} = B_h\, A_h u^n$$

which is a consistent scheme for solving (6), is this scheme guaranteed to be L^2 stable as well? If yes, give a proof; if not, give a counter example.

S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Individual

(Please select 5 problems to solve)

1. Let $D^* = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ be the punctured unit disc in the Euclidean plane. Let g be the complete Riemannian metric on D^* with contsant curvature -1. Find the disctance under the metric between the points $(e^{-2\pi}, 0)$ and $(-e^{-\pi}, 0)$.

2. Show that every closed hypersurface in $\mathbb{R}^n$ has a point at which the second fundamental form is positive definite.

3. Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. Suppose $\pi : M_1 \longrightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \longrightarrow T_{\pi(p)}M_2$ is a vector space isomorphism.
(a). Show that if M_1 is connected, then π is a covering space projection.
(b). Given an example where M_2 is compact but $\pi : M_1 \longrightarrow M_2$ is not a covering space (but has the π_* isomorphism property).

5. Let Σ_g be the closed orientable surface of genus g. Show that if $g > 1$, then Σ_g is a covering space of Σ_2.

6. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
(a). Construct a symplectic form on $\mathbb{R}^4$.
(b). Show that there are no symplectic forms on S^4.

S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Team

(Please select 5 problems to solve)

1. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, and $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ the equator n-plane through the center of S^n. Let N be the north pole of S^n. Define a mapping $\pi : S^n \backslash \{N\} \to \mathbb{R}^n$ called the stereographic projection that takes $A \in S^n \backslash \{N\}$ into the intersection $A' \in \mathbb{R}^n$ of the equator n-plane $\mathbb{R}^n$ with the line which passes through A and N. Prove that the stereographic projection is a conformal change, and derive the standard metric of S^n by the stereographic projection.

2. Let M be a (connected) Riemannian manifold of dimension 2. Let f be a smooth non-constant function on M such that f is bounded from above and $\Delta f \geq 0$ everywhere on M. Show that there *does not* exist any point $p \in M$ such that $f(p) = \sup\{f(x) : x \in M\}$.

3. Let M be a compact smooth manifold of dimension d. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be regularly embedded in the Euclidean space $\mathbb{R}^n$.

4. Show that any C^∞ function f on a compact smooth manifold M (without boundary) must have at least two critical points. When M is the 2-torus, show that f must have more than two critical points.

5. Construct a space X with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H_2(X) = \mathbb{Z}$, and all other homology groups of X vanishing.

6. (a). Define the degree $\deg f$ of a C^∞ map $f : S^2 \longrightarrow S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.
(b). Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : S^2 \longrightarrow S^2$ of degree k.

S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Individual

2:30–5:00 pm, July 10, 2011
(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let $K = \mathbb{Q}(\sqrt{-3})$, an imaginary quadratic field.

(a) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$? (Here S_3 is the symmetric group in 3 letters.)

(b) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$?

(c) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong Q$? Here Q is the quaternion group with 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$, a finite subgroup of the group of units $\mathbb{H}^\times$ of the ring $\mathbb{H}$ of all Hamiltonian quaternions.

2. Let f be a two-dimensional (complex) representation of a finite group G such that 1 is an eigenvalue of $f(\sigma)$ for every $\sigma \in G$. Prove that f is a direct sum of two one-dimensional representations of G

3. Let $F \subset \mathbb{R}$ be the subset of all real numbers that are roots of monic polynomials $f(X) \in \mathbb{Q}[X]$.

(1) Show that F is a field.

(2) Show that the only field automorphisms of F are the identity automorphism $\alpha(x) = x$ for all $x \in F$.

4. Let V be a finite-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation such that

(1) the minimal polynomial of T is irreducible;

(2) there exists a vector $v \in V$ such that $\{T^i v \mid i \geq 0\}$ spans V.

Show that V contains no non-trivial proper T-invariant subspace.

5. Given a commutative diagram

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

of Abelian groups, such that (i) both rows are exact sequences and (ii) every vertical map, except the middle one, is an isomorphism. Show that the middle map $C \to C'$ is also an isomorphism.

6. Prove that a group of order 150 is not simple.

S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Team

9:00–12:00 pm, July 9, 2011

(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let F be a field and $\bar{F}$ the algebraic closure of F. Let $f(x,y)$ and $g(x,y)$ be polynomials in $F[x,y]$ such that g.c.d.$(f,g)=1$ in $F[x,y]$. Show that there are only finitely many $(a,b)\in \bar{F}^{\times 2}$ such that $f(a,b)=g(a,b)=0$. Can you generalize this to the cases of more than two-variables?

2. Let D be a PID, and D^n the free module of rank n over D. Then any submodule of D^n is a free module of rank $m\le n$.

3. Identify pairs of integers $n\neq m\in\mathbb{Z}_+$ such that the quotient rings $\mathbb{Z}[x,y]/(x^2-y^n)\cong\mathbb{Z}[x,y]/(x^2-y^m)$; and identify pairs of integers $n\neq m\in\mathbb{Z}_+$ such that $\mathbb{Z}[x,y]/(x^2-y^n)\ncong\mathbb{Z}[x,y]/(x^2-y^m)$.

4. Is it possible to find an integer $n>1$ such that the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$$

is an integer?

5. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(a) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.

(b) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).

(c) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

6. For a ring R, let $SL_2(R)$ denote the group of invertible 2×2 matrices. Show that $SL_2(\mathbb{Z})$ is generated by $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. What about $SL_2(\mathbb{R})$?

S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Individual
2:30–5:00 pm, July 9, 2011
(Please select 5 problems to solve)

1. a) Compute the integral: $\int_{-\infty}^{\infty} \frac{x \cos x dx}{(x^2+1)(x^2+2)}$,

b) Show that there is a continuous function $f : [0, +\infty) \to (-\infty, +\infty)$ such that $f \not\equiv 0$ and $f(4x) = f(2x) + f(x)$.

2. Solve the following problem:

$$\begin{cases} \frac{d^2u}{dx^2} - u(x) = 4e^{-x}, & x \in (0,1), \\ u(0) = 0, & \frac{du}{dx}(0) = 0. \end{cases}$$

3. Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \setminus (-\infty, -1]$ to the unit disc.

4. Assume $f \in C^2[a,b]$ satisfying $|f(x)| \le A, |f^{''}(x)| \le B$ for each $x \in [a,b]$ and there exists $x_0 \in [a,b]$ such that $|f^{'}(x_0)| \le D$, then $|f^{'}(x)| \le 2\sqrt{AB} + D, \forall x \in [a,b]$.

5. Let $C([0,1])$ denote the Banach space of real valued continuous functions on [0, 1] with the sup norm, and suppose that $X \subset C([0,1])$ is a dense linear subspace. Suppose $l : X \to \mathbb{R}$ is a linear map (not assumed to be continuous in any sense) such that $l(f) \ge 0$ if $f \in X$ and $f \ge 0$. Show that there is a unique Borel measure μ on [0, 1] such that $l(f) = \int f d\mu$ for all $f \in X$.

6. For $s \ge 0$, let $H^s(T)$ be the space of L^2 functions f on the circle $T = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int_0^{2\pi} e^{-inx} f(x) dx$ satisfy $\Sigma(1+n^2)^s ||\hat{f}_n|^2 < \infty$, with norm $||f||_s^2 = (2\pi)^{-1} \Sigma(1+n^2)^s |\hat{f}_n|^2$.

a. Show that for $r > s \ge 0$, the inclusion map $i : H^r(T) \to H^s(T)$ is compact.

b. Show that if $s > 1/2$, then $H^s(T)$ includes continuously into $C(T)$, the space of continuous functions on T, and the inclusion map is compact.

S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Team

9:00–12:00 am, July 9, 2011

(Please select 5 problems to solve)

1. Let $H^2(\Delta)$ be the space of holomorphic functions in the unit disk $\Delta = \{|z| < 1\}$ such that $\int_\Delta |f|^2|dz|^2 < \infty$. Prove that $H^2(\Delta)$ is a Hilbert space and that for any $r < 1$, the map $T : H^2(\Delta) \to H^2(\Delta)$ given by $Tf(z) := f(rz)$ is a compact operator.

2. For any continuous function $f(z)$ of period 1, show that the equation

$$\frac{d\varphi}{dt} = 2\pi\varphi + f(t)$$

has a unique solution of period 1.

3. Let $h(x)$ be a C^∞ function on the real line $\mathbb{R}$. Find a C^∞ function $u(x, y)$ on an open subset of $\mathbb{R}^2$ containing the x-axis such that $u_x + 2u_y = u^2$ and $u(x, 0) = h(x)$.

4. Let $S = \{x \in \mathbb{R} \mid |x - \frac{p}{q}| \le c/q^3, \text{for infinitely many relatively prime } p, q \in \mathbb{Z}, q > 0, \text{for a fixed } c > 1\}$, show that S is uncountable and its measure is zero.

5. Let $sl(n)$ denote the set of all $n \times n$ real matrices with trace equal to zero and let $SL(n)$ be the set of all $n \times n$ real matrices with determinant equal to one. Let $\varphi(z)$ be a real analytic function defined in a neighborhood of $z = 0$ of the complex plane $\mathbb{C}$ satisfying the conditions $\varphi(0) = 1$ and $\varphi'(0) = 1$.
(a) If φ maps any near zero matrix in $sl(n)$ into $SL(n)$ for some $n \ge 3$, show that $\varphi(z) = \exp(z)$.
(b)Is the conclusion of (a) still true in the case $n = 2$? If it is true, prove it. If not, give a counterexample.

6. Use mathematical analysis to show that:
(a) e and π are irrational numbers;
(b) e and π are also transcendental numbers.

S.-T. Yau College Student Mathematics Contests 2011

Applied Math., Computational Math., Probability and Statistics

Individual

6:30–9:00 pm, July 9, 2011

(Please select 5 problems to solve)

1. Given a weight function $\rho(x) > 0$, let the inner-product corresponding to $\rho(x)$ be defined as follows:

$$(f, g) := \int_a^b \rho(x) f(x) g(x) \mathrm{d}x,$$

and let $\|f\| := (f, f)$.

(1) Define a sequence of polynomials as follows:

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x - a_1, \\ p_n(x) &= (x - a_n) p_{n-1}(x) - b_n p_{n-2}(x), \quad n = 2, 3, \cdots \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{(x p_{n-1}, p_{n-1})}{(p_{n-1}, p_{n-1})}, \quad n = 1, 2, \cdots \\ b_n &= \frac{(x p_{n-1}, p_{n-2})}{(p_{n-2}, p_{n-2})}, \quad n = 2, 3, \cdots . \end{aligned}$$

Show that $\{p_n(x)\}$ is an orthogonal sequence of monic polynomials.

(2) Let $\{q_n(x)\}$ be an orthogonal sequence of monic polynomials corresponding to the ρ inner product. (A polynomial is called *monic* if its leading coefficient is 1.) Show that $\{q_n(x)\}$ is unique and it minimizes $\|q_n\|$ amongst all monic polynomials of degree n.

(3) Hence or otherwise, show that if $\rho(x) = 1/\sqrt{1 - x^2}$ and $[a, b] = [-1, 1]$, then the corresponding orthogonal sequence is the Chebyshev polynomials:

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, 2, \cdots .$$

and the following recurrent formula holds:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \quad n = 1, 2, \cdots .$$

(4) Find the best quadratic approximation to $f(x) = x^3$ on $[-1, 1]$ using $\rho(x) = 1/\sqrt{1 - x^2}$.

2. If two polynomials $p(x)$ and $q(x)$, both of fifth degree, satisfy

$$p(i) = q(i) = \frac{1}{i}, \qquad i = 2, 3, 4, 5, 6,$$

and

$$p(1) = 1, \qquad q(1) = 2,$$

find $p(0) - q(0)$.

3. Lay aside m black balls and n red balls in a jug. Supposes $1 \le r \le k \le n$. Each time one draws a ball from the jug at random.

1) If each time one draws a ball without return, what is the probability that in the k-th time of drawing one obtains exactly the r-th red ball?
2) If each time one draws a ball with return, what is the probability that in the first k times of drawings one obtained totally an odd number of red balls?

4. Let X and Y be independent and identically distributed random variables. Show that

$$E[|X + Y|] \ge E[|X|].$$

Hint: Consider separately two cases: $E[X^+] \ge E[X^-]$ and $E[X^+] < E[X^-]$.

5. Suppose that $X_1, \cdots, X_n$ are a random sample from the Bernoulli distribution with probability of success p_1 and $Y_1, \cdots, Y_n$ be an independent random sample from the Bernoulli distribution with probability of success p_2.

(a) Give a minimum sufficient statistic and the UMVU (uniformly minimum variance unbiased) estimator for $\theta = p_1 - p_2$.

(b) Give the Cramer-Rao bound for the variance of the unbiased estimators for $v(p_1) = p_1(1 - p_1)$ or the UMVU estimator for $v(p_1)$.

(c) Compute the asymptotic power of the test with critical region

$$|\sqrt{n}(\hat{p}_1 - \hat{p}_2)/\sqrt{2\hat{p}\hat{q}}| \ge z_{1-\alpha}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2}\Delta$, where $\hat{p} = 0.5\hat{p}_1 + 0.5\hat{p}_2$.

6. Suppose that an experiment is conducted to measure a constant θ. Independent unbiased measurements y of θ can be made with either of two instruments, both of which measure with normal errors: for $i = 1, 2$, instrument i produces independent errors with a $N(0, \sigma_i^2)$ distribution. The two error variances σ_1^2 and σ_2^2 are known. When a measurement y is made, a record is kept of the instrument used so that after n measurements the data is $(a_1, y_1), \ldots, (a_n, y_n)$, where $a_m = i$ if y_m is obtained using instrument i. The choice between instruments is made independently for each observation in such a way that

$$P(a_m = 1) = P(a_m = 2) = 0.5, \quad 1 \le m \le n.$$

Let x denote the entire set of data available to the statistician, in this case $(a_1, y_1), \ldots, (a_n, y_n)$, and let $l_\theta(x)$ denote the corresponding log likelihood function for θ. Let $a = \sum_{m=1}^{n}(2 - a_m)$.

(a) Show that the maximum likelihood estimate of θ is given by

$$\hat{\theta} = \left(\sum_{m=1}^{n} 1/\sigma_{a_m}^2\right)^{-1} \left(\sum_{m=1}^{n} y_m/\sigma_{a_m}^2\right).$$

(b) Express the expected Fisher information I_θ and the observed Fisher information I_x in terms of n, σ_1^2, σ_2^2, and a. What happens to the quantity I_θ/I_x as $n \to \infty$?

(c) Show that a is an ancillary statistic, and that the conditional variance of $\hat{\theta}$ given a equals $1/I_x$. Of the two approximations

$$\hat{\theta} \dot{\sim} N(\theta, 1/I_\theta)$$

and

$$\hat{\theta} \dot{\sim} N(\theta, 1/I_x),$$

which (if either) would you use for the purposes of inference, and why?

S.-T. Yau College Student Mathematics Contests 2011

Applied Math., Computational Math., Probability and Statistics

Team

9:00–12:00 am, July 9, 2011

(Please select 5 problems to solve)

1. Let A be an N-by-N symmetric positive definite matrix. The conjugate gradient method can be described as follows:

$$\begin{aligned}
&\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0, \mathbf{p}_0 = \mathbf{r}_0, \mathbf{x}_0 = 0\\
&\text{FOR } n = 0, 1, \ldots\\
&\qquad \alpha_n = \|\mathbf{r}_n\|_2^2/(\mathbf{p}_n^T A\mathbf{p}_n)\\
&\qquad \mathbf{x}_{n+1} = \mathbf{x}_n + \alpha_n \mathbf{p}_n\\
&\qquad \mathbf{r}_{n+1} = \mathbf{r}_n - \alpha_n A\mathbf{p}_n\\
&\qquad \beta_n = -\mathbf{r}_{k+1}^T A\mathbf{p}_k/\mathbf{p}_k^T A\mathbf{p}_k\\
&\qquad \mathbf{p}_{n+1} = \mathbf{r}_{n+1} + \beta_n \mathbf{p}_n\\
&\text{END FOR}
\end{aligned}$$

Show

(a) α_n minimizes $f(\mathbf{x}_n + \alpha\mathbf{p}_n)$ for all $\alpha \in \mathbb{R}$ where

$$f(\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}.$$

(b) $\mathbf{p}_i^T \mathbf{r}_n = 0$ for $i < n$ and $\mathbf{p}_i^T A\mathbf{p}_j = 0$ if $i \neq j$.

(c) $\mathrm{Span}\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}\} = \mathrm{Span}\{\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}\} \equiv K_n$.

(d) $\mathbf{r}_n$ is orthogonal to K_n.

2. We use the following scheme to solve the PDE $u_t + u_x = 0$:

$$u_j^{n+1} = au_{j-2}^n + bu_{j-1}^n + cu_j^n$$

where a, b, c are constants which may depend on the CFL number $\lambda = \frac{\Delta t}{\Delta x}$. Here $x_j = j\Delta x$, $t^n = n\Delta t$ and u_j^n is the numerical approximation to the exact solution $u(x_j, t^n)$, with periodic boundary conditions.

(i) Find a, b, c so that the scheme is second order accurate.

(ii) Verify that the scheme you derived in Part (i) is exact (i.e. $u_j^n = u(x_j, t^n)$) if $\lambda = 1$ or $\lambda = 2$. Does this imply that the scheme is stable for $\lambda \leq 2$? If not, find λ_0 such that the scheme is stable for $\lambda \leq \lambda_0$. Recall that a scheme is stable if there exist constants M and C, which are independent of the mesh sizes Δx and Δt, such that

$$\|u^n\| \leq Me^{CT}\|u^0\|$$

for all Δx, Δt and n such that $t^n \leq T$. You can use either the L^∞ norm or the L^2 norm to prove stability.

3. Let X and Y be independent random variables, identically distributed according to the Normal distribution with mean 0 and variance 1, $N(0,1)$.

(a) Find the joint probability density function of (R,θ), where

$$R=(X^2+Y^2)^{1/2} \quad \text{and} \quad \theta=\arctan(Y/X).$$

(b) Are R and θ independent? Why, or why not?

(c) Find a function U of R which has the uniform distribution on (0, 1), Unif(0, 1).

(d) Find a function V of θ which is distributed as Unif(0,1).

(e) Show how to transform two independent observations U and V from Unif(0,1) into two independent observations X, Y from $N(0,1)$.

4. Let X be a random variable such that $E[|X|]<\infty$. Show that

$$E[|X-a|]=\inf_{x\in R} E[|X-x|],$$

if and only if a is a median of X.

5. Let $Y_1,\ldots,Y_n$ be iid observations from the distribution $f(x-\theta)$, where θ is unknown and $f(\ )$ is probability density function symmetric about zero.

Suppose *a priori* that θ has the improper prior $\theta\sim$ Lebesgue (flat) on $(-\infty,\infty)$. Write down the posterior distribution of θ.

Provides some arguments to show that this flat prior is noninformative.

Show that with the posterior distribution in (a), a 95% probability interval is also a 95% confidence interval.

6. Suppose we have two independent random samples $\{Y_1,\ i=1,\ldots,n\}$ from Poisson with (unknown) mean λ_1 and $\{Y_i,\ i=n+1,\ldots,2n\}$ from Poisson with (unknown) mean λ_2 Let $\theta=\lambda_1/(\lambda_1+\lambda_2)$.

(a) Find an unbiased estimator of θ

(b) Does your estimator have the minimum variance among all unbiased estimators? If yes, prove it. If not, find one that has the minimum variance (and prove it).

(c) Does the unbiased minimum variance estimator you found attain the Fisher information bound? If yes, show it. If no, why not?

S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Individual

9:30–12:00 am, July 10, 2011
(Please select 5 problems to solve)

1. Suppose M is a closed smooth n-manifold.
a) Does there always exist a smooth map $f : M \to S^n$ from M into the n-sphere, such that f is essential (i.e. f is not homotopic to a constant map)? Justify your answer.
b) Same question, replacing S^n by the n-torus T^n.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, y in X. Show that f is an onto map.

3. Let C_1, C_2 be two linked circles in $\mathbb{R}^3$. Show that C_1 cannot be homotopic to a point in $\mathbb{R}^3 \backslash C_2$.

4. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

5. A regular curve C in $\mathbb{R}^3$ is called a *Bertrand Curve,* if there exists a diffeomorphism $f : C \to D$ from C onto a different regular curve D in $\mathbb{R}^3$ such that $N_x C = N_{f(x)} D$ for any $x \in C$. Here $N_x C$ denotes the principal normal line of the curve C passing through x, and $T_x C$ will denote the tangent line of C at x. Prove that:
a) The distance $|x - f(x)|$ is constant for $x \in C$; and the angle made between the directions of the two tangent lines $T_x C$ and $T_{f(x)} D$ is also constant.
b) If the curvature k and torsion τ of C are nowhere zero, then there must be constants λ and μ such that $\lambda k + \mu \tau = 1$

6. Let M be the closed surface generated by carrying a small circle with radius r around a closed curve C embedded in $\mathbb{R}^3$ such that the center moves along C and the circle is in the normal plane to C at each point. Prove that

$$\int_M H^2 d\sigma \geq 2\pi^2,$$

and the equality holds if and only if C is a circle with radius $\sqrt{2}r$. Here H is the mean curvature of M and $d\sigma$ is the area element of M.

S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Team
9:00–12:00 am, July 9, 2011
(Please select 5 problems to solve)

1. Suppose K is a finite connected simplicial complex. True or false:
a) If $\pi_1(K)$ is finite, then the universal cover of K is compact.
b) If the universal cover of K is compact then $\pi_1(K)$ is finite.

2. Compute all homology groups of the the m-skeleton of an n-simplex, $0 \leq m \leq n$.

3. Let M be an n-dimensional compact oriented Riemannian manifold with boundary and X a smooth vector field on M. If $\mathbf{n}$ is the inward unit normal vector of the boundary, show that

$$\int_M div(X)\ dV_M = \int_{\partial M} X \cdot \mathbf{n}\ dV_{\partial M}.$$

4. Let $\mathcal{F}^k(M)$ be the space of all C^∞ k-forms on a differentiable manifold M. Suppose U and V are open subsets of M.
a) Explain carefully how the usual exact sequence

$$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}V) \longrightarrow \mathcal{F}(U \cap V) \longrightarrow 0$$

arises.
b) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (a) and describe explicitly how the map

$$H^k_{deR}(U \cap V) \longrightarrow H^{k+1}_{deR}(U \cup V)$$

arises.

5. Let M be a Riemannian n-manifold. Show that the scalar curvature $R(p)$ at $p \in M$ is given by

$$R(p) = \frac{1}{vol(S^{n-1})} \int_{S^{n-1}} Ric_p(x) dS^{n-1},$$

where $Ric_p(x)$ is the Ricci curvature in direction $x \in S^{n-1} \subset T_pM$, $vol(S^{n-1})$ is the volume of S^{n-1} and dS^{n-1} is the volume element of S^{n-1}.

6. Prove the Schur's Lemma: If on a Riemannian manifold of dimension at least three, the Ricci curvature depends only on the base point but not on the tangent direction, then the Ricci curvature must be constant everywhere, i.e., the manifold is Einstein.

INDIVIDUAL TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Prove that the polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ cannot be written as a product of two polynomials of rational coefficients and positive degrees.

2. Let $\mathbb{F}_p$ be the field of p-elements and $GL_n(\mathbb{F}_p)$ the group of invertible n by n matrices.

(1) Compute the order of $GL_n(\mathbb{F}_p)$.
(2) Find a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.
(3) Compute the number of Sylow p-subgroups.

3. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a nondegenerate symmetric bilinear form $(,)$. Let

$$O(V) = \{g \in GL(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group. Prove that fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is 1-dimensional.

4. Let $\mathfrak{D}$ be the ring consisting of all linear differential operators of finite order on $\mathbb{R}$ with polynomial coefficients, of the form

$$D = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$$

for some natural number $n \in \mathbb{N}$ and polynomials $a_0(x), \cdots, a_n(x) \in \mathbb{R}[x]$. This ring $\mathfrak{D}$ operates naturally on $M := \mathbb{R}[x]$, making M a left $\mathfrak{D}$-module.

(1) (to warm up) Suppose that $b(x) \in \mathbb{R}[x]$ is a non-zero polynomial in M, and let $c(x)$ be any element in M. Show that there is an element $D \in \mathfrak{D}$ such that $D(b(x)) = c(x)$.
(2) Suppose that m is a positive integer, $b_1(x), \cdots, b_m(x)$ are m polynomials in M linearly independent over $\mathbb{R}$ and $c_1(x), \cdots, c_m(x)$ are m polynomials in M. Prove that there exists an element $D \in \mathfrak{D}$ such that $D(b_i(x)) = c_i(x)$ for $i = 1, \cdots, m$.

5. Let Λ be a lattice of $\mathbb{C}$, i.e., a subgroup generated by two $\mathbb{R}$-linear independent elements. Let R be the subring of $\mathbb{C}$ consists of elements α such that $\alpha\Lambda \subset \Lambda$. Let $R^\times$ denote the group of invertible elements in R.

(1) Show that either $R = \mathbb{Z}$ or have rank 2 over $\mathbb{Z}$.

(2) Let $n \geq 3$ be a positive integer and $(R/nR)^\times$ the group of invertible elements in the quotient R/nR. Show that the canonical group homomorphism
$$R^\times \to (R/nR)^\times$$
is injective.

(3) Find maximal size of $R^\times$.

6. Let V be a (possible) infinite dimensional vector space over $\mathbb{R}$ with a positive definite quadratic norm $\|\cdot\|$. Let A be an additive subgroup of V with following properties:

(1) $A/2A$ is finite;

(2) for any real number c the set
$$\{a \in A : \qquad \|a\| < c\}$$
is finite.

Prove that A is of finite rank over $\mathbb{Z}$.

GROUP TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems.

1. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be complex numbers such that $a_i+b_j \neq 0$ for all $i, j = 1, \cdots, n$. Define $c_{ij} := \dfrac{1}{a_i + b_j}$ for all $i, j = 1, \cdots, n$, and let C be the $n \times n$ determinant with entries c_{ij}. Prove that

$$det(C) = \frac{\prod_{1 \leq i < j \leq n} (a_i - a_j)(b_i - b_j)}{\prod_{1 \leq i, j \leq n} (a_i + b_j)}.$$

2. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(1) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.
(2) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).
(3) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

3. Let V be a finite dimensional vector space with a positive definite quadratic form $(-,-)$. Let $O(V)$ denote the orthogonal group:

$$O(V) = \{g \in GL(V) : \qquad (gx, gy) = (x, y), \quad \forall x, y \in V\}.$$

For any non-zero $v \in V$, let s_v denote the reflection on V:

$$s_v(w) = w - 2\frac{(v, w)}{(v, v)} v.$$

(1) Show that $s_v \in O(V)$;
(2) Show that if v and w are vectors in V with $\|v\| = \|w\|$,then there is either a reflection or product of two reflections that takes v into w;
(3) Deduce that every element of the orthogonal group of V can be written as the product of at most $2 \dim V$ reflections.

4. Consider the real Lie group $SL_2(\mathbb{R})$ of 2 by 2 matrices of determinant one. Compute the fundamental group of $SL_2(\mathbb{R})$ and describe the Lie group structure on the universal covering

$$\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

5. Let $f \in \mathbb{C}[x, y, z]$ be an irreducible homogenous polynomial of degree $d > 0$. For each integer $n \geq d$, define

$$P(n) = \dim_{\mathbb{C}} \mathbb{C}[x, y, z]_n / f \cdot \mathbb{C}[x, y, z]_{n-d}$$

where $\mathbb{C}[x, y, z]_d$ is the subspace of homogenous polynomials of degree n. Show there are constants c such that for n sufficiently large,

$$P(n) = dn + c.$$

6. Let p be an odd prime and $\mathbb{Z}_p$ the p-adic integer which can be defined as the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$ and let $\mathbb{Q}_p$ be its fractional field. Let $\mathbb{Z}_p^\times$ denote the group of invertible elements in $\mathbb{Z}_p$ which is also the projective limit of $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

(1) For any integer a is not divisible by p, show that the sequence $(a^{p^n})_n$ convergent to an element $\omega(a) \in \mathbb{Z}_p$ satisfying

$$\omega(a)^{p-1} = 1, \qquad \omega(a) \equiv a \pmod p.$$

Moreover, $\omega(a)$ depends only on $a \mod p$.

(2) Define a logarithmic function log on $1 + p\mathbb{Z}_p$ by usual formula:

$$\log(1 + px) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{p^n}{n} x^n.$$

Show that the logarithmic function is convergent and define an isomorphism

$$1 + p\mathbb{Z}_p \to p\mathbb{Z}_p.$$

Moreover, on the dense subgroup $\log(1 + p)\mathbb{Z}$, the inverse is given by

$$\log(1 + p) \cdot x \mapsto (1 + p)^x, \qquad \forall x \in \mathbb{Z}.$$

(3) Deduce from above that $\mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/(p - 1)\mathbb{Z}$.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Compute the integral
$$\int_0^\infty \frac{x^p}{1+x^2}dx, -1 < p < 1.$$

2. Construct a one to one conformal mapping from the region
$$U = \{z \in \mathbb{C} || z - \frac{i}{2}| < \frac{1}{2}\} / \{z || z - \frac{i}{4}| < \frac{1}{4}\}$$
onto the unit disk.

3. Let $f(x)$ be a nonlinear C^2 function on $\mathbb{R}$. Show that
$$\sup |f'(x)|^2 \leq 4 \sup |f(x)| \sup |f''(x)|.$$

4. Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$ is a measurable function on $\mathbb{R}$.

5. For $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, let g in L^q. Consider the linear functional F on L^p given by: $F(f)$ is equal to the integral of fg. Show that $||F|| = ||g||_q$.

6. Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$. Show that the formula
$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy, x \in \mathbb{R}^n_+$$
is a solution of the problem
$$\Delta u = 0, \text{ in } \mathbb{R}^n_+, u = g \text{ on } \partial \mathbb{R}^n_+,$$
where α_n is the volume of the unit n dimensional sphere.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems.

1. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \cdots, x_n) = \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j)$. Prove that f is in $L^1(\mathbb{R}^n)$ if and only if the matrix A is positive definite.

Compute $\int_{\mathbb{R}^n} \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j + \Sigma_{i=1}^n b_ix_i)dx$ when A is positive definite.

2. Let V be a simply connected region in the complex plane and $V \neq \mathbb{C}$. Let a, b be two distinct points in V. Let ϕ_1, ϕ_2 be two one-to-one holomorphic maps of V onto itself. If $\phi_1(a) = \phi_2(a)$ and $\phi_1(b) = \phi_2(b)$, show that $\phi_1(z) = \phi_2(z)$ for all $z \in V$.

3. In the unit interval [0, 1] consider a subset
$E = \{x|$ in the decimal expansion of x there is no $4\}$,
show that E is measurable and calculate its measure.

4. Let $1 < p < \infty, L^p([0,1], dm)$ be the completion of $C[0,1]$ with the norm: $||f||p = (\int_0^1 |f(x)|^p dm)^{\frac{1}{p}}$, where dm is the Lebesgue measure. Show that $\lim_{\lambda\to\infty} \lambda^p m(x||f(x)| > \lambda) = 0$.

5. Let $\mathfrak{F} = \{e_\nu\}, \nu = 1, 2, ..., n$ or $\nu = 1, 2, ...$ is an orthonormal basis in an inner product space H. Let E be the closed linear subspace spanned by $\mathfrak{F}$. For any $x \in H$ show that the following are equivalent: 1) $x \in E$; 2) $||x||^2 = \Sigma_\nu|(x, e_\nu)|^2$; 3) $x = \Sigma_\nu(x, e_\nu)e_\nu$.

Let $H = L^2[0, 2\pi]$ with the inner product $< f, g >= \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx$,
$\mathfrak{F} = \{\frac{1}{\sqrt{2}}, \cos x, \sin x, ..., \cos nx, \sin nx, ...\}$
be an orthonormal basis. Show that the closed linear sub-space E spanned by $\mathfrak{F}$ is H.

6. Let $\mathcal{H} = L^2[0, 1]$ relative to the Lebesgue measure and define $(Kf)(s) = \int_0^s f(t)dt$ for each f in $\mathcal{H}$. Show that K is a compact operator without eigenvalues.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied Math. and Computational Math.

Please solve 4 out of the following 5 problems,
or highest scores of 4 problems will be counted.

1. In the numerical integration formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(c), \tag{1}$$

if the constants a, b, c can be chosen arbitrarily, what is the highest degree k such that the formula is exact for all polynomials of degree up to k? Find the constants a, b, c for which the formula is exact for all polynomials of degree up to this k.

2. Here is the definition of a moving least square approximation of a function $f(x)$ near a point $\overline{x}$ given K points x_k around $\overline{x}$ in $\mathbb{R}$, $k \in [1, \cdots, K]$.

$$\min_{P_{\overline{x}} \in \Pi_m} \sum_{k=1}^{K} |P_{\overline{x}}(x_k) - f_k|^2 \tag{2}$$

where $f_k = f(x_k)$, Π_m is the space of polynomials of degree less or equal to m, i.e.

$$P_{\overline{x}}(x) = \mathbf{b}_{\overline{x}}(x)^T \mathbf{c}(\overline{x}),$$

$\mathbf{c}(\overline{x}) = [c_0, c_1, \cdots, c_m]^T$ is the coefficient vector to be determined by (2),
$\mathbf{b}_{\overline{x}}(x)$ is the polynomial basis vector, $\mathbf{b}_{\overline{x}}(x) = [1, x - \overline{x}, (x - \overline{x})^2, \ldots, (x - \overline{x})^m]^T$.
Assume that there are $K > m$ different points x_k and $f(x)$ is smooth,
(a) prove that there is a unique solution $\overline{P}_{\overline{x}}(x)$ to (2)
(b) denote $h = \max_k \ |x_k - \overline{x}|$, prove

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+1-i}, \ i = 0, 1, \ldots, m,$$

where $f^{(i)}(\cdot)$ is the i-th derivative of f and $C(f, i)$ denote some constant depending on f, i.

(c) if $S = \{x_k | k = 1, 2, \ldots, K\}$ are symmetrically distributed around $\overline{x}$, that is, if $x_k \in S$ then $2\overline{x} - x_k \in S$, prove that

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+2-i}, \; i = 0, 1, \ldots, m,$$

for i ($\in \{0, 1, \cdots, m\}$) with the same parity of m.

3. Describe the forward-in-time and center-in-space finite difference scheme for the one-wave wave equation:

$$u_t + u_x = 0.$$

(i). Conduct the von Neumann stability analysis and comment on their stability property.

(ii). Under what condition on Δt and Δx would this scheme be stable and convergent?

(iii). How many ways you can modify this scheme to make it stable when the CFL condition is satisfied.

4. Let C and D in $\mathbb{C}^{n \times n}$ be Hermitian matrices. Denote their eigenvalues by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,$$

respectively. Then it is known that

$$\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 \leq \|C - D\|_F^2.$$

1) Let A and B be in $\mathbb{C}^{n \times n}$. Denote their singular values by

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \quad \text{and} \quad \tau_1 \geq \tau_2 \geq \cdots \geq \tau_n,$$

respectively. Prove that the following inequality holds:

$$\sum_{i=1}^{n} (\sigma_i - \tau_i)^2 \leq \|A - B\|_F^2.$$

2) Given $A \in \mathbb{R}^{n \times n}$ and its SVD is $A = U \Sigma V^T$, where $U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$, $V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are orthogonal matrices, and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

Suppose $\operatorname{rank}(A) > k$ and denote by

$$U_k = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k), \quad V_k = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \quad \Sigma_k = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k),$$

and

$$A_k = U_k \Sigma_k V_k^T = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$

Prove that

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F^2 = \|A - A_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2.$$

3) Let the vectors $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$, be in the space $\mathcal{W}$ with dimension d, where $d \ll n$. Let the orthonormal basis of $\mathcal{W}$ be $W \in \mathbb{R}^{n \times d}$. Then we can represent $\mathbf{x}_i$ by

$$\mathbf{x}_i = \mathbf{c} + W\mathbf{r}_i + \mathbf{e}_i, \ i = 1, 2, \ldots, n,$$

where $\mathbf{c} \in \mathbb{R}^n$ is a constant vector, $\mathbf{r}_i \in \mathbb{R}^d$ is the coordinate of the point $\mathbf{x}_i$ in the space $\mathcal{W}$, and $\mathbf{e}_i$ is the error. Denote $R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ and $E = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$. Find W, R and $\mathbf{c}$ such that the error $\|E\|_F$ is minimized.
(*Hint*: write $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \mathbf{c}(1, 1, \ldots, 1) + WR + E$.)

5. Two primes p and q are called *twin primes* if $q = p + 2$. For example, 5 and 7, 11 and 13, 29 and 31 are twin primes. There is a still unproven (but extensively numerically verified) conjecture that there are infinitely many twin primes and that they are rather common. Show how to factor an integer N which is a product of two twin primes.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied Math. and Computational Math.

Please solve 4 out of the following 5 problems.

1. If the function $u(x)$ is in C^{k+1} (has continuous $(k+1)$-th derivative) on the interval $[0,2]$, and a sequence of polynomials $p_n(x)$ $(n = 1,2,3,...)$ of degree at most k satisfies

$$(1) \qquad |u(x) - p_n(x)| \le \frac{C}{n^{k+1}} \qquad \forall\ 0 \le x \le \frac{1}{n},$$

where the constant C is independent of n, prove

$$|u(x) - p_n(x)| \le \frac{\tilde{C}}{n^{k+1}} \qquad \forall\ \frac{1}{n} \le x \le \frac{2}{n},$$

with another constant $\tilde{C}$ which is also independent of n.

2. Consider the one-dimensional elliptic equation

$$-\frac{d^2}{dx^2}u(x) = f(x), \quad 0 < x < 1,$$

with homogeneous boundary condition, $u(0) = 0$ and $u(1) = 0$, $f \in L^2(0,1)$.

(i) Describe the standard piecewise linear finite element method for this boundary value problem.

(ii) Is this method stable and convergent? If so, what is the order of convergence?

(iii). In this case, the linear finite element method has a super convergence property at the nodal point x_j $(j = 1,2,...,N)$, i.e. $u_h(x_j) = u(x_j)$, here u_h is the finite element solution and u is the exact solution. Could you explain why?

3. Let $A = (a_{ij}) \in M_{N\times N}(\mathbb{C})$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{j=1, j\neq i}^{N} |a_{ij}| \text{ for all } 1 \le i \le N,$$

Assume that $A = I + L + U$ where I is the identity matrix, L and U are the lower and upper triangular matrices with zero diagonal entries.

Now, we consider solving the linear system $Ax = b$ by the following iterative scheme:

$$(*) \quad x^{k+1} = (I + \alpha\Omega L)^{-1}[(I - \Omega) - (1 - \alpha)\Omega L - \Omega U]x^k + (I + \alpha\Omega L)^{-1}b$$

where $\Omega := \mathbf{diag}(\omega_1, ...\omega_N)$ and $0 \leq \alpha \leq 1$. (When $\alpha = 1$, it gives the SOR method.)

(1) Prove that the linear system $Ax = b$ has a unique solution.
(2) Prove that the necessary condition for the convergence of (*) is

$$\prod_{i=1}^{N} |1 - \omega_i| < 1$$

(3) Let $M = (I + \alpha\Omega L)^{-1}[(I - \Omega) - (1 - \alpha)\Omega L - \Omega U)]$. Prove that the spectral radius $\rho(M)$ of M is bounded by:

$$\rho(M) \leq \max_i \frac{|1 - \omega_i| + |\omega_i|(|1 - \alpha|l_i + u_i)}{1 - |\omega_i \alpha| l_i}$$

whenever $|\omega_i \alpha| l_i < 1$ for all $1 \leq i \leq N$ where $l_i = \sum_{j<i} |a_{ij}|$ and $u_i = \sum_{j>i} |a_{ij}|$.
(4) Using (c), prove that the sufficient condition for the convergence of (*) is

$$0 < \omega_i < \frac{2}{1 + l_i + u_i} \quad \text{for all } 1 \leq i \leq N$$

4. The famous *RSA cryptosystem* is based on the assumed difficulty of factoring integers $N = pq$ (called RSA integers) which are products of two large primes p and q which should be kept secret. Currently p and q are chosen to be about 500 bits long, that is,

$$p, q \approx 2^{500}.$$

Assume someone uses the following algorithm to find secret n-bit primes p and q to form an RSA integer $N = pq$:

- Choose a random odd 500-bit integer s.
- Test the odd numbers s, $s+2$, $s+4$, etc. for primality until the first prime p is found (note the primality testing is very easy nowdays).
- Continue testing $p + 2$, $p + 4$, $p + 6$, etc. for primality until the second prime q is found.
- Compute and publish $N = pq$, but keep p and q secret.

How secure is this procedure? Can you suggest an algorithm to factor an RSA integer $N = pq$ generated this way?

Note that there are about $x/\log x$ primes up to x, where $\log x$ is the natural logarithm. This means that the expected gap between two consecutive n-bit primes is

$$\log 2^n = n \log 2 \approx 0.69 \cdot n.$$

5. The solution $h(r,t)$ of the following Boussinesq equation describes the hight of a circular drop of fluid spreading on a dry surface $h=0$:

$$\frac{\partial h}{\partial t} = \Delta_r(h^2) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(h^2)}{\partial r}\right), \quad r>0, \quad t>1$$

with

$$\left.\frac{\partial h}{\partial r}\right|_{r=0} = 0, \quad \int_0^\infty h(r,t)rdr \equiv \frac{1}{64}$$

The solution is positive on a finite range $0 \le r \le r_*(t)$ with $h(r_*(t),t) = 0$ defining a moving "edge" position with no fluid outside of the droplet. For $r > r_*(t)$ truncate the solution beyond the edge to be zero ($h \equiv 0$ for $r > r_*(t)$).

(a): Show that this problem is scale invariant by finding relations $h(r,t) = H(T)\tilde{h}(\tilde{r},\tilde{t})$, $r = R(T)\tilde{r}$, $t = T\tilde{t}$ so that the problem for $\tilde{h}(\tilde{r},\tilde{t})$ is identical to the original problem.

(b): Determine the ODE for the similarity function $\Phi(\eta)$ with $h(r,t) = t^\alpha\Phi(\eta)$, $r = \eta t^\beta$.

(c): Determine the explicit solution for $\Phi(\eta)$ and then use $h(r,t) = t^\alpha\Phi(\eta)$ to find $r_*(t)$ for $t \ge 1$.
Hint $\int_0^\infty hrdr = \int_0^{r_*} hrdr$.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Show that $\pi_3(S^2) \neq 0$.

2. Let M be a smooth manifold of dimension n, and $X_1, \cdots, X_k$ be k everywhere linearly independent smooth vector fields on an open set $U \subset M$ satisfying that $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Prove that for any point $p \in U$ there is a coordinate chart (V, y^i) with $p \in V \subseteq U$ and coordinates $\{y^1, \cdots, y^n\}$ such that $X_i = \frac{\partial}{\partial y^i}$ on V for each $1 \leq i \leq k$.

3. Show that any self homeomorphism of $\mathbb{CP}^2$ is orientation preserving.

4. Prove the following version of the isoperimetric inequality: Suppose C is a simple (that is, without self-intersection), smooth, closed curve in the Euclidean plane, with length L. Show that the area enclosed by C is less than or equal to $\frac{L^2}{4\pi}$, and the equality occurs when and only when C is a round circle.

5. Let $x : M \to \mathbb{R}^3$ be a closed surface in 3-dimensional Euclidean space. Its Gaussian curvature and mean curvature are denoted by K and H respectively. Prove that:

$$\iint\limits_M H dA + \iint\limits_M pK dA = 0, \quad \iint\limits_M pH dA + \iint\limits_M dA = 0,$$

where $p = \vec{x} \cdot \vec{n}$ is the support function of M, $\vec{x}$ denotes the position vector of M, $\vec{n}$ denotes the unit normal to M, and dA is the area element of M.

6. Write the structure equation of an orthonormal frame on a Riemannian manifold. Prove the following Riemannian metric g has constant sectional curvature c using the structure equation:

$$g = \frac{\sum_{i=1}^n (dx^i)^2}{[1 + \frac{c}{4} \sum_{i=1}^n (x^i)^2]^2}$$

where $(x^1, \ldots, x^n)$ is a local coordinate system.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems.

1. Prove that the real projective space $\mathbb{RP}^n$ is a differentiable manifold of dimension n.

2. Let M, N be n-dimensional smooth, compact, connected manifolds, and $f : M \to N$ a smooth map with rank equals to n everywhere. Show that f is a covering map.

3. Given any Riemannian manifold (M^n, g), show that there exists a unique Riemannian connection on M^n.

4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $f : S^n \to S^n$ a continuous map. Assume that the degree of f is an odd integer. Show that there exists $x_0 \in S^n$ such that $f(-x_0) = -f(x_0)$.

5. State and prove the Stokes theorem for oriented compact manifolds.

6. Let M be a surface in $\mathbb{R}^3$. Let D be a simply-connected domain in M such that the boundary ∂D is compact and consists of a finite number of smooth curves. Prove the Gauss-Bonnet Formula:

$$\int_{\partial D} k_g \, ds \; + \sum_j (\pi - \alpha_j) + \iint_D K \, dA \; = \; 2\pi,$$

where k_g is the geodesic curvature of the boundary curve. Each α_j is the interior angle at a vertex of the boundary, K is the Gaussian curvature of M, and the 2-form dA is the area element of M.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Probability and Statistics

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Solve the following two problems:

1) An urn contains b black balls and r red balls. One of the balls was drawn at random, and putted back in the urn with a additional balls of the same color. Now suppose that the second ball drawn at random is red. What is the probability that the first ball drawn was black?

2) Let (X_n) be a sequence of random variables satisfying

$$\lim_{a\to\infty}\sup_{n\geq 1} P(|X_n|>a)=0.$$

Assume that sequence of random variables (Y_n) converges to 0 in probability. Prove that (X_nY_n) converges to 0 in probability.

2. Solve the following two problems:

1) Let $(\Omega,\mathcal{F},P)$ be a probability space, $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Assume that X is a non-negative integrable random variable. Set $Y=E[X|\mathcal{G}]$. Prove that

(a)$[X>0]\subset[Y>0]$,a.s.;

(b)$[Y>0]=\text{ess.inf}\{A : A\in\mathcal{G},\ [X>0]\subset A\}$.

2) Let X and Y have a bivariate normal distribution with zero means, variances σ^2 and τ^2, respectively, and correlation ρ. Find the conditional expectation $E(X|X+Y)$.

3. Suppose that $\{p(i,j) : i=1,2,\cdots,m; j=1,2,\cdots,n\}$ is a finite bivariate joint probability distribution, that is,

$$p(i,j)>0,\ \sum_{i=1}^{m}\sum_{j=1}^{n}p(i,j)=1.$$

(i) Can $\{p(i,j)\}$ be always expressed as

$$p(i,j)=\sum_k \lambda_k a_k(i) b_k(j)$$

for some finite $\lambda_k\geq 0,\ \sum_k\lambda_k=1,\ a_k(i)\geq 0,\ \sum_{i=1}^m a_k(i)=1,\ b_k(j)\geq 0,\ \sum_{j=1}^n b_k(j)=1$?

(ii) Prove or disprove the above relation by use of conditional probability.

4. Let $X_1, \cdots, X_m$ be an independent and identically distributed (i.i.d.) random sample from a cumulative distribution function (CDF) F, and $Y_1, \cdots, Y_n$ an i.i.d. random sample from a CDF G. We want to test $H_0 : F = G$ versus $H_1 : F \neq G$. The total sample size is $N = m + n$. Consider the following two nonparametric tests.

- The Wilcoxon rank sum tests. The test proceeds by first ranking the pooled X and Y samples and then taking the sum of the ranks associated with the Y sample. Let $R_{y_1}, \cdots, R_{y_n}$ be the rankings of the sample $y_1 < \cdots < y_n$ from the pooled sample in increasing order. The Wilcoxon rank sum statistic is defined as $W = \sum_{j=1}^{n} R_{y_j}$.
- The Mann-Whitney U-test. Let $U_{ij} = 1$ if $X_i < Y_j$, and $U_{ij} = 0$ otherwise. The Mann-Whitney U-statistic is defined as $U = \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij}$. The probability $\gamma = P(X < Y)$ can be estimated as $U/(mn)$. The decision rule is based on assessing if $\gamma = 0.5$.

Assume that there are no tied data values.

(a) Show that $W = U + \frac{1}{2}n(n+1)$, which shows that the two test statistics differ only by a constant and yield exactly the same p-values.

(b) Using the central limit theorem, the Wilcoxon rank sum statistic W can be converted to a Z-variable, which provides an easy-to-use approximation. The transformation is

$$Z_W = \frac{W - \mu_W}{\sigma_W},$$

where μ_W and σ_W^2 are the mean and variance of W under H_0. Show that $\mu_W = \frac{1}{2}n(N+1)$ and $\sigma_W^2 = \frac{1}{12}mn(N+1)$.

5. Let X be a random variable with $EX^2 < \infty$, and $Y = |X|$. Assume that X has a Lebesgue density symmetric about 0. Show that random variables X and Y are uncorrelated, but they are not independent.

6. Let $E_1, \cdots, E_n$ be i.i.d. random variables with $E_i \sim$ Exponential(1). Let $U_1, \cdots, U_n$ be i.i.d. uniformly (on [0,1]) distributed random variables. Further, assume that $E_1, \cdots, E_n$ and $U_1, \cdots, U_n$ are independent.

(a) Find the density of $X = (E_1 + \cdots + E_m)/(E_1 + \cdots + E_n)$, where $m < n$.

(b) Show that $Y = \frac{(n-m)X}{m(1-X)}$ is distributed as the F-distribution with degrees of freedom $(2m, 2(n-m))$

(c) Find the density of $(U_1 \cdots U_n)^{-X}$.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Probability and Statistics

Please solve 5 out of the following 6 problems.

1. Let (X_n) be a sequence of i.i.d. random variables.

1) Assume that each X_n satisfies the exponential distribution with parameter 1 (i.e. $P(X_n \geq x) = e^{-x}, x \geq 0$). Prove that

(a) $P(X_n > \alpha \log n, i.o.) = 0$, if $\alpha > 1$; $P(X_n > \alpha \log n, i.o.) = 1$, if $\alpha \leq 1$.

Here "i.o" stands for "infinitely often", and $A_n, i.o.$ stands $\limsup_{n\to\infty} A_n$.

(b) Let $L = \limsup_{n\to\infty}(X_n/\log n)$, then $P(L = 1) = 1$.

2) Assume that each X_n satisfies the Poisson distribution with parameter λ (i.e. $P(X_n = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, \cdots$.) Put

$$L = \limsup_{n\to\infty}(X_n \log\log n/\log n).$$

Prove that $P(L = 1) = 1$.

2. Let X_i be i.i.d exponential r.v with rate one, $i \geq 1$. Let N be a geometric random variable with success probability p, $0 < p < 1$, i.e. $P(N = k) = (1-p)^{k-1}p, k = 1, 2, \cdots$, and independent of all X_i, $i \geq 1$. Find the distribution of $\sum_{i=1}^{N} X_i$.

3. Let X and Y be i.i.d real valued r.v's. Prove that $P(|X+Y| < 1) \leq 3P(|X - Y| < 1)$.

4. Suppose $S = X_1 + X_2 + \cdots + X_n$, a sum of independent random variables with X_i distributed Binomial(1, p_i). Show that $\mathbb{P}(S\ even) = 1/2$ if and only if at least one p_i equals $1/2$.

5. Let B_θ denote the closed unit ball in $\mathbb{R}^2$ with center θ. Suppose $X_1, X_2, \cdots, X_n$ are independently and uniformly distributed on B_θ, for an unknown θ in $\mathbb{R}^2$. Denote that maximum likelihood estimator by $\hat{\theta}$. Show that $|\hat{\theta} - \theta| = O_p(1/n)$.

6. Suppose that $X_1, \cdots, X_n$ are a random sample from the Bernoulli distribution with probability of success p_1 and $Y_1, \cdots, Y_n$ be an independent random sample from the Bernoulli distribution with probability of success p_2.

(a) Derive the maximum likelihood ratio test statistic for

$$H_0 : p_1 = p_2 \longleftrightarrow H_1 : p_1 \neq p_2.$$

(Note: No simplification of the resulting test statistic is required. However, you need to give the asymptotic null.)

(b) Compute the asymptotic power of the test with critical region

$$|\sqrt{n}(\hat{p}_1 - \hat{p}_2)/\sqrt{2\hat{p}\hat{q}}| \geqslant z_{1-\alpha}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2}\Delta$, where $\hat{p} = 0.5.\hat{p}_1 + 0.5\hat{p}_2$.

S.-T. Yau College Student Mathematics Contests 2013

Algebra and Number Theory

Individual

This exam of 160 points is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

1. (20pt)

1.1 (15 pt) Classify finite groups of order 26 up to isomorphisms.

1.2 (5 pt) For each finite group G of order 26, describe the group $Aut(G)$ of automorphisms of G.

2. (20 pt) Consider $f \in \mathbb{Z}_{>0}$ and nonzero vector spaces V_i indexed by $i \in \mathbb{Z}/f\mathbb{Z}$. Suppose that there are linear maps $\phi_i : V_i \to V_{i+1}$ and $\psi_i : V_i \to V_{i-1}$ such that

$$\phi_{i-1} \circ \psi_i = 0, \quad \psi_{i+1} \circ \phi_i = 0.$$

(We may think of a circular graph with oriented edges such that the "Orpheus condition" holds: *Whenever you turn back while traveling through the graph you are killed*.)

Prove that there exists lines $\ell_i \subset V_i$ for every $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\phi_i(\ell_i) \subset \ell_{i+1}, \quad \psi_i(\ell_i) \subset \ell_{i-1}$$

under one of the following two conditions:

2.1 (10 pt) all $\psi_i = 0$, or

2.2 (10 pt) $\dim V_i$ are equal to each other.

3. (20pt) For a parameter $t = (t_0, t_1, \cdots t_5) \in \mathbb{F}_5^6$ with $t_0 \neq 0$ and $\{t_i, i > 0\}$ an ordering of elements in $\mathbb{F}_5$, define a polynomial

$$P_t(x) = (x - t_1)(x - t_2)(x - t_3) + t_0(x - t_4)(x - t_5).$$

3.1 (7 pt) Show that $P_t(x)$ is irreducible in $\mathbb{F}_5[x]$;

3.2 (6 pt) Show that two parameters t, t' give the same polynomial if and only if $t_0 = t'_0$ and $\{t_4, t_5\} = \{t'_4, t'_5\}$.

3.3 (7 pt) Show that every irreducible cubic monic polynomial over $\mathbb{F}_5$ is obtained by this way.

4. (40 pt) For k non-negative integer, let $V_k := \mathbb{R}[x]_{\leq k}$ be the vector space of real polynomials of degree at most k with an action by $\mathrm{SL}_2(\mathbb{R})$ by

$$\gamma \cdot P(x) = (cx+d)^k P\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

4.1 (20 pt) Show that V_k is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$;

4.2 (15 pt) For non-negative integers m, n, consider $V_{m,n} := V_m \otimes V_n$ as a subspace of $\mathbb{C}[x, y]$ of polynomials with both x, y-degrees at most k, with diagonal action of $\mathrm{SL}_2(\mathbb{R})$. Assume $m \geq n \geq 1$. Show that following exact sequence is exact and split as representations of $\mathrm{SL}_2(\mathbb{R})$.

$$0 \longrightarrow V_{m-1,n-1} \xrightarrow{\cdot(y-x)} V_{m,n} \xrightarrow{y=x} V_{m+n} \longrightarrow 0.$$

This implies the following decomposition of representations:

$$V_m \otimes V_n = \oplus_{i=0}^{n} V_{m+n-2i}.$$

4.3 (5 pt) For non-negative integers $\ell \geq m \geq n$ consider the space of invariants $(V_\ell \otimes V_m \otimes V_n)^{\mathrm{SL}_2(\mathbb{R})}$. Show that this space is either trivial or one-dimensional; it is non-trivial if and only if

$$\ell + m + n \equiv 0 \mod 2, \qquad \ell + m \geq n.$$

5. (60 pt)

5.1 (20pt) Find a polynomial $f(x)$ with integer coefficients which has a root over $\mathbb{F}_p$ for each prime p but has not root over $\mathbb{Q}$.

5.2 (20pt) Can you find f irreducible?

5.3 (20pt) What is the smallest possible degree of f?

S.-T. Yau College Student Mathematics Contests 2013

Algebra and Number Theory

Team

The exam contains 6 problems. Please choose 5 of them to work on.

1. (20pt) Let A be an $n \times n$ skew symmetric real matrix.

1.1 (10 pt) Prove that all eigenvalues of A are imaginary or zero and that e^A is orthogonal.

1.2 (10 pt) Find conditions on an orthogonal B such that $B = e^A$ is solvable for some skew symmetric and real matrix A.

2. (20pt) Let E/F be a field extension. Let A be an $m \times m$ matrix with entries in E such that $tr(A^n)$ belongs to F for every $n \geq 2$. Show that $tr(A)$ belongs to F by following steps.

2.1 (5pt) Show that there is a polynomial $P(x) = \sum_i a_i x^i \in \bar{E}[x]$ with $a_0 = 1$ such that

$$\sum_i a_i tr(A^{i+k}) = 0, \qquad \forall k \geq 1.$$

2.2 (5pt) Show that we have a polynomial $Q = \sum_i b_i x^i \in F[x]$ with $b_0 = 1$ such that

$$\sum_i b_i tr(A^{i+k}) = 0, \qquad \forall k \geq 2.$$

2.3 (5pt) Let $t \in \bar{E}$ be an eigenvalue of A with multiplicity m invertible in F. Show that $Q(t) = 0$.

2.4 (5pt) Show that $tr(A)$ belongs to F.

Hint: Let $t_i \in \bar{E}$ be all distinct non-zero eigen values of A with multiplicity m_i invertible in F. Then

$$tr(A^n) = \sum_i m_i t_i^n.$$

3. (20pt) Let p be a prime and $G = \mathrm{SL}_2(\mathbb{F}_p)$.

3.1 (10pt) Find the order of G.

3.2 (10pt) Show that the order of every element of G divides either (p^2-1) or $2p$.

4. (20pt) Let S_4 be the symmetric group of 4 letters.

4.1 (10pt) Classify all complex irreducible representations of S_4;

4.2 (10pt) Find the character table of S_4.

5. (20pt) Let $\mathbb{F}_2$ be the finite field of two elements.

5.1 (10pt) Find all irreducible polynomials of degree 2 and 3 over $\mathbb{F}_2$;

5.2 (10pt) What is the number of irreducible polynomials of degree 6 over $\mathbb{F}_2$?

6. (20pt) Let F be the splitting field of $x^4 - 2$.

6.1 (10pt) Describe the field F and the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$.

6.2 (10pt) Describe all subfields K of F and corresponding Galois subgroups $G_K = \mathrm{Gal}(F/K)$.

S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that f is an integrable function on $\mathbf{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x||f(x)| > \alpha\}$. Prove that:

$$\int_{\mathbf{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

2. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $a_1, a_2, \cdots, a_d$. Show that

$$\sum_{i=1}^{d} \frac{1}{p'(a_i)} = 0.$$

3. Let α be a number such that α/π is not a rational number. Show that:

1) $\lim_{N\to\infty} \Sigma_{n=1}^N e^{ik(x+n\alpha)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt.$

2) For every continuous periodic function $f : \mathbf{R} \to \mathbf{C}$ of period 2π, we have

$$\lim_{N\to\infty} \frac{1}{N}\Sigma_{n=1}^N f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

4. Let u be a positive harmonic function over the punctured complex plane $\mathbf{C}/\{0\}$. Show that u must be a constant function.

5. Suppose $H = L^2(B), B$ is the unit ball in $\mathbf{R}^d$. Let $K(x,y)$ be a measurable function on $B \times B$ that satisfies

$$|K(x,y)| \leq A|x-y|^{-d+\alpha}$$

for some $\alpha > 0$, whenever $x, y \in B$. Define

$$Tf(x) = \int_B K(x,y)f(y)dy$$

(a) Prove that T is a bounded operator on H.

(b) Prove that T is compact.

6. Let A be a $n\times n$ real non-degenerate symmetric matrix. For $\lambda \in \mathbf{R}^+$, we define: $\int_{\mathbf{R}} \exp(i\lambda x^2)dx = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} \exp(i\lambda x^2 - \frac{1}{2}\epsilon x^2)dx$. Show that:

$$\int_{\mathbf{R}^n} \exp(i\frac{\lambda}{2} < Ax, x > -i < x, \xi >)dx$$

$$= (\frac{2\pi}{\lambda})^{n/2} |\det(A)|^{-1/2} \exp(-\frac{i}{2\lambda} < A^{-1}\xi, \xi >) \exp(\frac{i\pi}{4} sgnA).$$

Here $\lambda \in \mathbf{R}^+, \xi \in \mathbf{R}^n, sgn(A) = \nu_+(A) - \nu_-(A), \nu_+(A)(\nu_-(A))$ is the number of positive (negative) eigenvalues of A.

S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Suppose $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ is the open unit disk in the complex plane. Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \tag{1}$$

for all z in Δ. If equality holds in (1) for some $z_0 \in \Delta$, show that $f \in \mathrm{Aut}(\Delta)$, and that

$$\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$$

for all $z \in \Delta$.

2. Let f be a function of bounded variation on $[a, b]$, f_1 its generalized derivative as a measure, i.e. $f(x) - f(a) = \int_a^x f_1(y)dy$ for every $x \in [a, b]$ and $f_1(x)$ is an integrable function on $[a, b]$. Let f' be its weak derivative as a generalized function, i.e. $\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx$, for any smooth function $g(x)$ on $[a, b]$, $g(a) = g(b) = 0$. Show that:

a) If f is absolutely continuous, then $f' = f_1$.

b) If the weak derivative f' of f is an integrable function on $[a, b]$, then $f(x)$ is equal to an absolutely continuous function outside a set of measure zero.

3. Show that the convex hull of the roots of any polynomial contains all its critical points as well as all the zeros of higher derivatives of the polynomial. Here the convex hull of a given bounded set in the plane is the smallest convex set containing the given set in the plane.

4. Let $D \subset \mathbf{R}^3$ be an open domain. Show that every smooth vector field $\mathbf{F} = (P, Q, R)$ over D can be written as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ such that $rot(\mathbf{F}_1) = 0, div(\mathbf{F}_2) = 0$, where $rot(\mathbf{F}) = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}), div(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

5. Let $\mathbf{H}$ be a Hilbert space and $\mathbf{A}$ a compact self-adjoint linear operator over $\mathbf{H}$. Show that there exists an orthor-normal basis of $\mathbf{H}$ consisting of eigenvectors φ_n of $\mathbf{A}$ with non-zero eigenvalues λ_n such that every vector $\xi \in \mathbf{H}$ can be written as: $\xi = \Sigma_k c_k \varphi_k + \xi'$, where $\xi' \in Ker\mathbf{A}, i.e., \mathbf{A}\xi' = 0$. We also have $\mathbf{A}\xi = \Sigma_k \lambda_k c_k \varphi_k$.

If there are infinitely many eigenvectors then $\lim_{n\to\infty} \lambda_n = 0$.

6. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)x') \le \lambda f(x) + (1-\lambda) f(x')$$

for $0 \le \lambda \le 1$ and each $x, x' \in \mathbb{R}$, and is called strictly convex if

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$

for $0 < \lambda < 1$. We assume that $|f(x)| < \infty$ whenever $|x| < \infty$.

(a) Show that a convex function f is continuous and the function

$$g(y) = \max_{x \in \mathbf{R}} (xy - f(x))$$

is a well-defined convex function over $\mathbf{R}$.

(b) Show that a convex function f is differentiable except at most countably many points.

(c) f is differentiable everywhere if both f and g are strictly convex.

S.-T. Yau College Student Mathematics Contests 2013

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. We consider the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times \mathbb{R}_+$.

(a): (5 pts) A right going pulse with speed 1

$u(x, y, z, t) = 1$ for $t < x < t + 1$; $\quad u(x, y, z, t) = 0$ else

is clearly a solution to the wave equation. However, it is a discontinuous solution, explain in which sense it is a solution to the equation.

(b): (5 pts) Surprisingly, one can construct smooth progressive wave solutions with speed larger than 1. In astronomy this kind of wave known as superluminal wave. Try a solution of the form

$$u(x, y, z, t) = v(\frac{x - ct}{\sqrt{c^2 - 1}}, y, z), \quad c \in \mathbb{R}^3, \quad |c| > 1.$$

Derive an equation for v and show that there is a nontrivial solution with compact support in (y, z) for any fixed x, t.

(c): (5 pts) For any $R > 0$, $0 < t < R$, show that energy

$$E(t) := \int_{|\vec{x}| \leq R - t} \left(|u_t(\cdot, t)|^2 + |\nabla u(\cdot, t)|^2 \right) d\vec{x}$$

is a decreasing function.

(c): (10 pts) Show that smooth superluminal progressive wave solutions of the form

$$u(\vec{x}, t) = v(\vec{x} - \vec{c}t), \vec{c} \in \mathbb{R}^3, \quad |\vec{c}| > 1.$$

cannot have a finite energy.

Hint: Using (c) and look at the energy of the solution in various balls.

2. Finite time extinction and hyper-contractiveity are important properties in modeling of some physical and biology systems. The essence of estimates is given by the following problem for ODE.

Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then

(a) (10 points) For $a > 1$, $y(t)$ has the following hyper-contractive property

$$y(t) \leq (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t} \right]^{\frac{1}{a-1}}, \quad \text{for } t > 0.$$

(b) (2 points) For $a = 1$, $y(t)$ decays exponentially

$$y(t) \le \alpha/\beta + y(0)e^{-\beta t}.$$

(c) (10 points) For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, which means that there exists T_{ext} such that $0 < T_{ext} \le \frac{y^{1-a}(0)}{\beta(1-a)}$ and that $y(t) = 0$ for all $t > T_{ext}$.

3. Consider the speed v of a ball (density ρ, radius R) falling through a viscous fluid (density ρ_f, viscosity μ) with drag coefficient given by Stokes' law $\zeta = 6\pi R\mu$:

$$\frac{4}{3}\pi R^3 \rho \frac{dv}{dt} = \frac{4}{3}\pi R^3(\rho - \rho_f)g - \zeta v, \quad v(0) = v_0$$

(a): (5 points) Nondimensionalize the equation by writing, $v(t) = V\tilde{v}(\tilde{t})$ with $t = T\tilde{t}$. Select V, T (characteristic scales known as terminal velocity and settling time respectively) so that all coefficients in the ODE but one are equal to 1. Your equation will have a single dimensionless parameter given by the ratio of the initial speed v_0 to the characteristic speed V.

(b): (2 points) Solve the nondimensional problem for $\tilde{v}(\tilde{t})$.

(c): (8 points) Describe the behavior of the solution if the initial speed v_0 is (i) faster than and (ii) slower than the characteristic speed V. Compute the time to reach $(v_0 + V)/2$.

4. Let

$$V_h = \{v : v|_{I_j} \in P^k(I_j) \qquad 1 \le j \le N\}$$

where

$$I_j = (x_{j-1}, x_j), \qquad 1 \le j \le N$$

with

$$x_j = jh, \qquad h = \frac{1}{N}.$$

Here $P^k(I_j)$ denotes the set of polynomials of degree at most k in the interval I_j.

Recall the L^2 projection of a function $u(x)$ into the space V_h is defined by the unique function $u_h \in V_h$ which satisfies

$$||u - u_h|| \le ||u - v|| \qquad \forall v \in V_h$$

where the norm is the usual L^2 norm. We assume $u(x)$ has at least $k + 2$ continuous derivatives.

(1) (5 points) Prove the error estimate

$$||u - u_h|| \le Ch^{k+1}$$

Explain how the constant C depends on the derivatives of $u(x)$.

(2) (10 points) If another function $\varphi(x)$ also has at least $k+2$ continuous derivatives, prove

$$|\int_0^1 (u(x) - u_h(x))\varphi(x)dx| \leq Ch^{2k+2}$$

Explain how the constant C depends on the derivatives of $u(x)$ and $\varphi(x)$.

5. (15 points) Let $G(V, E)$ be a simple graph of order n and δ the minimum degree of vertices. Suppose that the degree sum of any pair of nonadjacent vertices is at least n and $F \subset E$ with $|F| \leq \lfloor \frac{\delta-2}{2} \rfloor$. Let $G - F$ be the graph obtained from G by deleting the edges in F. Prove that

(1) $G - F$ is connected and
(2) $G - F$ is Hamiltonian.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1 = 1, \ldots, F_{n+2} = F_{n+1} + F_n$.

Establish a relation between $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m/2}\right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.

S.-T. Yau College Student Mathematics Contests 2013

Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. Scaling behavior is one of the most important phenomena in scientific modeling and mathematical analysis. The following problem shows the universality and rigidity of scaling limits.

(a): (10 points) Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is a function $0 < \psi(x) < \infty$ for $x > 0$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x > 0$$

Then $\psi(x) = x^\alpha$ for some $\alpha \geq 0$.

(b): (10 points) The above problem can be generalized as:

Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is an extended function $0 \leq \psi(x) \leq \infty$ and a set A dense in $[0, \infty)$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x \in A$$

Then $\psi(x) = x^\alpha$ for some $\alpha \in [0, \infty]$.

(c): (15 points) (Warming: this part is hard).

A function $L : (0, \infty) \to (0, \infty)$ is called slowly varying at ∞ if

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \qquad \text{for all } x \in A \text{ dense in } (0, \infty)$$

The function U in **(a)** and **(b)** can be recast as $U(x) = c\,x^\alpha L(x)$ for some $c \geq 0$. Now we can extend **(b)** to an even more general setting:

Suppose $U > 0$ is increasing on $(0, \infty)$, set A dense in $[0, \infty)$ and

$$\lim_{n\to\infty} a_n U(b_n x) = \psi(x) \leq \infty, \text{ for all } x \in A.$$

where $b_n \to \infty$ and $\frac{a_{n+1}}{a_n} \to 1$ for some interval. Then there is a real number $\alpha \in [0, \infty]$, constant $c \geq 0$, and a function L slowly varying at ∞ such that $\psi(x) = x^\alpha$ and $U(x) = c\,x^\alpha L(x)$.

2. The following three operators are important for many mathematics and physics problems. Let $\phi(x)$ be a smooth periodic function in $\mathbb{T}^n$, Δ, ∇, $\nabla\cdot$ be the standard Laplacian, gradient and divergence operators.

(i): Fokker-Planck operator: $\mathcal{F}u = -\Delta u - \nabla \cdot (u \nabla \phi)$

(ii): Witten Laplacian operator: $\mathcal{W}u = -\Delta u + \nabla\phi \cdot \nabla u$
(iii): Schrödinger operator: $\mathcal{S}u = -\Delta u + \left(\frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi\right) u$

Show that

(a): (5 points) The Fokker-Planck operator can be recast as $\mathcal{F}u = -\nabla \cdot (e^{-\phi}\nabla(e^{\phi}u)$.
(b): (10 points) These three operators have same eigenvalues.
(c): (5 points) Find all equilibrium solutions for these three operators.

3. (15 points) Let $f(x)$ defined on $[0,\ 1]$ be a smooth function with sufficiently many derivatives. $x_i = ih$, where $h = \frac{1}{N}$ and $i = 0, 1, \cdots, N$ are uniformly distributed points in $[0,\ 1]$. What is the highest integer k such that the numerical integration formula

$$I_N = \frac{1}{N}\left(a_0(f(x_0) + f(x_N)) + a_1(f(x_1) + f(x_{N-1})) + \sum_{i=2}^{N-2} f(x_i)\right)$$

is k-th order accurate, namely

$$\left| I_N - \int_0^1 f(x)dx \right| \le Ch^k$$

for a constant C independent of h? Please describe the procedure to obtain the two constants a_0 and a_1 for this k.

4. The wave guide problem is defined as

$$u_t + u_x = 0, \qquad v_t - v_x = 0$$

with the boundary condition

$$u(-1,t) = v(-1,t), \qquad v(1,t) = u(1,t)$$

and the initial condition

$$u(x,0) = f(x), \qquad v(x,0) = g(x).$$

The upwind scheme for the guide problem is defined as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \qquad j = -N+1, \cdots, N;$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{v_{j+1}^n - v_j^n}{\Delta x} = 0, \qquad j = -N, \cdots, N-1;$$

with the boundary condition

$$u_{-N}^{n+1} = v_{-N}^{n+1}, \qquad v_N^{n+1} = u_N^{n+1}$$

where u_j^n and v_j^n approximate $u(x_j, t^n)$ and $v(x_j, t^n)$ respectively at the grid point (x_j, t^n), with $x_j = j\Delta x$, $t^n = n\Delta t$, $\Delta x = \frac{1}{N}$.

(1) (5 points) For the solution to the wave guide problem with the above boundary condition, prove the energy conservation

$$\frac{d}{dt}\int_{-1}^{1}(u^2+v^2)dx = 0.$$

(2) (5 points) For the numerical solution of the the upwind scheme, if we define the discrete energy as

$$E^n = \sum_{j=-N+1}^{N}(u_j^n)^2 + \sum_{j=-N}^{N-1}(v_j^n)^2,$$

prove the discrete energy stability

$$E^{n+1} \le E^n$$

under a suitable time step restriction $\frac{\Delta t}{\Delta x} \le \lambda_0$. You should first find λ_0.

(3) (10 points) Under the same time step restriction, is the numerical solution stable in the maximum norm? That is, can you prove

$$\max_{-N\le j\le N}\max(|u_j^{n+1}|,|v_j^{n+1}|) \le \max_{-N\le j\le N}\max(|u_j^n|,|v_j^n|)?$$

5. (15 points) Let $G=(V,E)$ be a graph of order n. Let X_1, X_2, $\dots$, X_q with $2\le q\le \kappa(X)$ be subsets of the vertex set V such that $X = X_1\cup X_2\cup\ldots\cup X_q$. If for each i, $i=1,2,\ldots,q$, and for any pair of nonadjacent vertices $x,y\in X_i$, we have

$$d(x)+d(y)\ge n,$$

then X is cyclable in G (*i.e.*, there is a cycle containing all vertices of X.).

Where $d(x)$ is the degree of x and $\kappa(X)$ is the smallest number of vertices separating two vertices of X if X does not induce a complete subgraph of G, otherwise we put $\kappa(X)=|X|-1$ if $|X|\ge 2$ and $\kappa(X)=1$ if $|X|=1$.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1 = 1, \ldots, F_{n+2} = F_{n+1}+F_n$.

Establish a relation between $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^m = \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{m/2}\right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.

S.-T. Yau College Student Mathematics Contests 2013

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Find the homology and fundamental group of the space $X = S^1 \times S^1/\{p, q\}$ obtained from the torus by identifying two distinct points p, q to one point.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, $y \in X$. Show that f is an onto map.

3. Let M^2 ba a complete regular surface and K be the Gaussian curvature. Suppose $\sigma : [0, \infty) \to M$ is a geodesic such that $K(\sigma(t)) \leq f(t)$, where f is a differentiable function on $[0, \infty)$. Prove that any solution $u(t)$ of the equation

$$u''(t) + f(t)u(t) = 0$$

has a zero on $[0, t_0]$, where $\sigma(t_0)$ is the first conjugate point to $\sigma(0)$ along σ.

4. Let g_1, g_2 be Riemannian metrics on a differentiable manifold M, and denote by R_1 and R_2 their respective Riemannian curvature tensor. Suppose that $R_1(X, Y, Y, X) = R_2(X, Y, Y, X)$ holds for any tangent vectors X, $Y \in T_pM$. Show that $R_1(X, Y, Z, W) = R_2(X, Y, Z, W)$ for any X, Y, Z, $W \in T_pM$.

5. Let M^n be an even dimensional, orientable Riemannian manifold with positive sectional curvature. Let $\sigma : [0, l] \to M$ be a closed geodesic, namely, σ is a geodesic with $\sigma(0) = \sigma(l)$ and $\sigma'(0) = \sigma'(l)$. Show that there exist an $\epsilon > 0$ and a smooth map $F : [0, l] \times (-\epsilon, \epsilon) \to M$, such that $F(t, 0) = \sigma(t)$, and for any fixed $s \neq 0$ in $(-\epsilon, \epsilon)$, $\sigma_s(t) = F(t, s)$ is a closed smooth curve with length less than that of σ.

6. Let (M^2, ds^2) be a minimal surface in $\mathbb{R}^3$, where ds^2 is the restriction of the Euclidean metric. Assume that the Gaussian curvature K of (M^2, ds^2) is negative. Denote by $\widetilde{K}$ the Gaussian curvature of the metric $\widetilde{ds^2} = -K ds^2$. Show that $\widetilde{K} = 1$.

S.-T. Yau College Student Mathematics Contests 2013

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let X be the space

$$\{(x, y, 0) \mid x^2 + y^2 = 1\} \cup \{(x, 0, z) \mid x^2 + z^2 = 1\}$$

Find the fundamental group $\pi_1(\mathbb{R}^3 \setminus X)$.

2. Let M be a smooth connected manifold and $f : M \to M$ be an injective smooth map such that $f \circ f = f$. Show that the image set $f(M)$ is a smooth submanifold in M.

3. Let $T^2 = \{(z, w) \in \mathbb{C}^2 \mid \ |z| = 1, |w| = 1\}$ be the torus. Define a map $f : T^2 \to T^2$ by $f(z, w) = (zw^3, w)$. Prove that f is a diffeomorphism.

4. Prove: Any 3-dimensional Einstein manifold has constant curvature.

5. State and prove the Myers theorem for complete Riemannian manifolds.

6. Let C be a regular closed curve in $\mathbb{R}^3$. Its torsion is τ. The integral $\frac{1}{2\pi}\int_C \tau ds$ is called the total torsion of C, where s is the arc length parameter. Prove: Given a smooth surface M in $\mathbb{R}^3$, if for any regular closed curve C on M, the total torsion of C is always an integer, then M is a part of a sphere or a plane.

S.-T. Yau College Student Mathematics Contests 2013

Probability and Statistics Problems

Individual

Please solve 5 out of the following 6 problems.

Problem 1. Let (X_n) be a sequence of random variables.

(1) Assume that $\sum_{n=0}^{\infty} P(|X_n| > n) < \infty$. Prove that $\limsup_{n\to\infty} \frac{|X_n|}{n} \leq 1$.

(2) Prove that (X_n) converges in probability to 0 if and only if for certain $r > 0$, $E\left[\frac{|X_n|^r}{1+|X_n|^r}\right] \to 0$.

Problem 2. Let X and Y be independent $N(0,1)$ random variables.

(1) Find $E[X+Y|X \geq 0, Y \geq 0]$;

(2) Find the distribution function of $X+Y$ given that $X \geq 0$ and $Y \geq 0$.

(Hint: For b) using the fact that $U = (X+Y)/\sqrt{2}$ and $V = (X-Y)/\sqrt{2}$ are independent and $N(0,1)$ distributed.)

Problem 3. Let $\{X_n\}$ be a sequence of independent and identically distributed continuous real valued random variables, and regard n as time. Let A_n be the following event:

$$A_n = \{X_n = \max\{X_1, X_2, \cdots, X_n\}\}.$$

We say that a maximum record occurs at n in such an event.

(1) Evaluate the probability $P(A_n)$.

(2) Denote by Y_n the number of maximum records occurred until time n, i.e.,

$$Y_n = \text{the number of } \ \{1 \leq k \leq n : X_k = \max\{X_1, X_2, \cdots, X_k\}\}.$$

Evaluate the expectation EY_n and the variance DY_n.

Problem 4. Let $X = (X_1, \cdots, X_n)$ be an iid sample from an exponential density with mean θ. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$. Let $P(X)$ = your p-value for an appropriate test.

(a) What is $E_{\theta_0}(P(X))$? Derive your answer explicitly.

(b) Derive $E_\theta(P(X))$ for $\theta \neq \theta_0$. Specifically, assuming only one sample, i.e. $n = 1$, calculate $E_\theta(P(X))$ as explicitly as possible for $\theta \neq \theta_0$.

(c) When there is only one sample, is $E_\theta(P(X))$ a decreasing function of θ? In general, can you prove your result for an arbitrary MLR family?

Problem 5. Let X_1, X_2 be iid uniform on $\theta - \frac{1}{2}$ to $\theta + \frac{1}{2}$.

(a) Show that for any given $0 < \alpha < 1$, you can find $c > 0$ such that

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c\} = 1 - \alpha,$$

where $\bar{X}$ is the sample mean.

(b) Show that for ϵ positive and sufficiently small

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c \bigm| |X_2 - X_1| \geq 1 - \epsilon\} = 1$$

(c) The statement in (a) is used to assert that $\bar{X} \pm c$ is a $100(1-\alpha)\%$ confidence interval for θ. Does the assertion in (b) contradict this? If your sample observations are $X_1 = 1, X_2 = 2$, would you use the confidence interval in (a)?

Problem 6. Suppose you want to estimate the total number of enemy tanks in a war on the basis of the captured tanks. Assume without loss of generality that the tank serial numbers are $1, 2, \cdots, N$, where N is the unknown total number of enemy tanks. Also assume the serial numbers of the n captured tanks are iid uniform on $1, 2, \cdots, N$. (This is a simplified assumption which provides a good approximation if $n << N$).

(a) Find the complete sufficient statistic.

(b) Suggest how you may find the minimum variance unbiased estimate of N.

S.-T. Yau College Student Mathematics Contests 2013

Probability and Statistics Problems

Team

Please solve 5 out of the following 6 problems.

Problem 1. The characteristic function f of a probability distribution function F is defined by

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

Show that $f_1(t) = (\cos t)^2$ is a characteristic function and $f_2(t) = |\cos t|$ is not a characteristic function.

Problem 2. Let $I = [0, 1]$ be the unit interval and $\mathcal{B}$ the σ-algebra of Borel sets on I. Let P be the Lebesgue measure on I. Show that on the probability space $(I, \mathcal{B}, P)$ the set of points of x with the following property has probability 1: for all but finitely many rational numbers $p/q \in (0, 1)$,

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{(q \log q)^2}.$$

Problem 3. Let X be an integrable random variable, $\mathcal{G}$ a $\sigma-$algebra, and $Y = E[X|\mathcal{G}]$. Assume that X and Y have the same distribution.

(1) Prove that if X is square-integrable, then $X = Y$, a.s. (i.e. X must be $\mathcal{G}$ measurable) ;

(2) Using a) to prove that for any pair of real numbers a, b with $a < b$, we have $\min\{\max\{X, a\}, b\} = \min\{\max\{Y, a\}, b\}$, and consequently, $X = Y$, a.s.

Problem 4. Let $X_1, \cdots, X_n$ be iid $N(\theta, \sigma^2)$, σ^2 known, and let θ have a double exponential distribution, that is, $\pi(\theta) = e^{-|\theta|/a}/(2a)$, a known. A Bayesian test of the hypothesis $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ will decide in favor of H_1 if its posterior probability is large.

(a) For a given constant K, calculate the posterior probability that $\theta > K$, that is, $P(\theta > K \mid x_1, \cdots, x_n, a)$.

(b) Find an expression for $\lim_{a\to\infty} P(\theta > K \mid x_1, \cdots, x_n, a)$.

(c) Compare your answer in part (b) to the p-value associated with the classical hypothesis test.

Problem 5. Two sets of interesting ideas emerging in the 1990's are the proposal of model selection with L^1-penalty (e.g., lasso) and the proposal of soft thresholding in simultaneous inferences. Consider a linear model

$$Y = X\beta + \varepsilon,$$

where the set up is as usual (i.e., X is a non-random n by p matrix with $1 < p < n$, $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ with I_n being the identity matrix). The lasso procedure is to obtain an estimate of the parameter vector β through minimizing

$$(L): \qquad \frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1,$$

which we denote by β^*_λ; here $\lambda > 0$ is a tuning parameter, $||\cdot||_2$ denote the usual L^2 vector norm, and $||\cdot||_1$ denotes the usual L^1 vector norm.

Denote the ordinary least square estimate of β by $\hat\beta$, we have

$$\frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1 = \frac{1}{2}||Y - X\hat\beta||_2^2 + \frac{1}{2}||X(\beta - \hat\beta)||_2^2 + \lambda \cdot |\beta||_1. \qquad (0.1)$$

Furthermore, if X has orthonormal columns, e.g.,

$$X'X = I_p,$$

then it can be shown that

$$\beta^*_{\lambda,i} = \begin{cases} \hat\beta_i - \gamma, & \hat\beta_i > \gamma, \\ 0, & |\hat\beta_i| \le \gamma, \\ \hat\beta_i + \gamma, & \beta_i < -\gamma. \end{cases} \qquad (0.2)$$

(0.2) is called the *soft thresholding* of $\hat\beta_i$'s. This says that with orthonormal design, lasso solution is equivalent to applying soft thresholding to the ordinary least square solution.

(a) Prove equation (0.1) without assuming X is orthogonal.

(b) Show that the lasso estimator is obtained by (0.2) under the assumption that X is orthogonal, and find the relationship between λ and γ.

Problem 6. Consider a usual linear model $Y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ and X has n rows and p columns where $1 < p < n$. Consider a p-dimensional column vector $a \neq 0$.

(a) Show that, if $Xa = 0$, then $a'\beta$ is not estimable.

(b) Prove or disprove that, if $a'\beta$ is not estimable, then $Xa = 0$.

(c) Show that X is full rank if and only if $a'\beta$ are estimable for all a.

S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Individual

This exam of 6 problems is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

Problem 1. Let G be a finite subgroup of $\mathrm{GL}(V)$ where V is an n-dimensional complex vector space.

(a) (5 points) Let
$$H = \{h \in G : hv = \eta(h)v \text{ for some } \eta(h) \in \mathbb{C}^\times \text{ and all } v \in V\}.$$
Prove that H is a normal subgroup of G and that the map $h \mapsto \eta(h)$ is an isomorphism between H and its image in $\mathbb{C}^\times$.

(b) (5 points) Let χ_V be the character function of G acting on V, i.e., $\chi_V(g) = \mathrm{tr}(g)$ with g viewed as an automorphism of V. Prove $|\chi_V(g)| \le n$ for all $g \in G$, and the equality holds if and only if $g \in H$.

(c) (10 points) Let W be an irreducible representation of G. Then W is isomorphic to a direct summand of $V^{\otimes m}$ for some m (as representations of G).

Problem 2. Let $a_1, \ldots, a_n$ be nonnegative real numbers.

(a) (6 points) Prove that the $n \times n$ matrix $A = (t^{a_i + a_j})$ is positive semi-definite for every real number $t > 0$. Find the rank of A.

(b) (7 points) Let $B = (c_{ij})_{n \times n}$ be an $n \times n$-matrix with $c_{ij} = \frac{1}{1 + a_i + a_j}$. Prove that A is a positive semi-definite matrix.

(c) (7 points) Prove that B is positive definite if and only if a_i are all distinct.

Problem 3. Consider the equations
$$X^2 - 82Y^2 = \pm 2$$

(a) (5 points) Show that if (x, y) is a solution for $X^2 - 82Y^2 = \pm 2$, then $(9x - 82y, x - 9y)$ is a solution for $X^2 - 82Y^2 = \mp 2$.

(b) (7 points) Show that the equations have solutions over $\mathbb{Z}/p^n\mathbb{Z}$ for any n and odd prime p.

(c) (8 points) Show that the equations have no solutions over $\mathbb{Z}$.

Problem 4. Let S and T be nonabelian finite simple groups, and write $G = S \times T$.

(a) (7 points) Show that the total number of normal subgroups of G is four.

(b) (6 points) If S and T are isomorphic, show that G has a maximal proper subgroup not containing either direct factor.

(c) (7 points) If G has a maximal proper subgroup that contains neither of the direct factors of G, show that S and T are isomorphic.

Problem 5. (20 points) Let $\mathbb{F}$ be a finite field and $f_i \in \mathbb{F}[X_1, X_2, ..., X_n]$ be polynomials of degree d_i, where $1 \leq i \leq r$, such that $f_i(0, ..., 0) = 0$ for all i. Show that if

$$n > \sum_{i=1}^{r} d_i,$$

then there exists nonzero solution to the system of equations: $f_i = 0$, for all $1 \leq i \leq r$. (Hint: you may first verify that the number of integral solutions is congruent to the following number modulo p

$$\sum_{X \in \mathbb{F}^n} \prod_{i=1}^{r} \left(1 - f_i(X)^{q-1}\right).$$

)

Problem 6.

(a) (5 points) Let A and B be two real $n \times n$ matrices such that $AB = BA$. Show that $\det(A^2 + B^2) \geq 0$.

(b) (15 points) Generalize this to the case of k pairwise commuting matrices.

S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Team

Solve 5 out of 6 problems, or the highest 5 scores will be counted.

Problem 1. Let the special linear group (of order 2)

$$\mathrm{SL}_2(\mathbb{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \det g = 1\}$$

act on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ linear fractionally:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

(a) (5 points) Prove that the action is transitive, i.e., for any two $z_1, z_2 \in \mathbb{H}$, there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gz_1 = z_2$.

(b) (5 points) For a fixed $z \in \mathbb{H}$, prove that its stabilizer $G_z = \{g \in \mathrm{SL}_2(\mathbb{R}) : gz = z\}$ is isomorphic to $\mathrm{SO}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : gg^t = 1\}$, where g^t is the transpose of g.

(c)(10 points) Let $\mathbb{Z}$ be the set of integers and let

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}, \quad a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod 2 \right\}$$

be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (no need to prove this), and let it act on $\mathbb{Q} \cup \{\infty\}$ linearly fractionally as above. How many orbits does this action have? Give a representative for each orbit.

Problem 2. Let $p \geq 7$ be an odd prime number.

(a) (5 points) (to warm up) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) (15 points) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

Problem 3. (20 points, 10 points each) For any 3×3 matrix $A \in M_3(\mathbb{Q})$, let A^{db} be the 6×6 matrix

$$A^{db} := \begin{pmatrix} 0 & I_3 \\ A & 0 \end{pmatrix}$$

(a) Express the characteristic and minimal polynomials of A^{db} over $\mathbb{Q}$ in terms of the characteristic and minimal polynomial of A.

(b) Suppose that $A, B \in M_3(\mathbb{Q})$ are such that A^{db} and B^{db} are conjugate in the sense that there exists an element $C \in GL_6(\mathbb{Q})$ such that $C \cdot A^{db} \cdot C^{-1} = B^{db}$. Are A and B conjugate? (Either prove this statement or give a counterexample.)

Problem 4. (20 points) Classify all groups of order 8.

Problem 5. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let

$$O(V) = \{g \in \mathrm{GL}(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group.

(a) (10 points) Prove that

$$(V \otimes_{\mathbb{C}} V)^{O(V)} \cong \mathrm{End}_{O(V)}(V),$$

and construct one such isomorphism. Here $O(V)$ acts on $V \otimes_{\mathbb{C}} V$ via $g(a \otimes b) = ga \otimes gb$, and $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is the fixed point subspace of $V \otimes V$.

(b) (10 points) Prove that the fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is 1-dimensional.

Problem 6. (20 points) Let c be a non-zero rational integer.

(a) (6 points) Factorize the three variable polynomial

$$f(x, y, z) = x^3 + cy^3 + c^2z^3 - 3cxyz$$

over $\mathbb{C}$ (you may assume $c = \theta^3$ for some $\theta \in \mathbb{C}$).

(b) (7 points) When $c = \theta^3$ is a cube for some rational integer θ, prove that there are only finitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

(c) (7 points) When c is not a cube of any rational integers, prove that there infinitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies

$$\sup_{x,y\in\mathbb{R}} |f(x+y) - f(x) - f(y)| < \infty.$$

If we have $\lim\limits_{n\to\infty, n\in\mathbb{N}} \dfrac{f(n)}{n} = 2014$, prove $\sup\limits_{x\in\mathbb{R}} |f(x) - 2014\,x| < \infty$.

2. Let $f_1, ..., f_n$ are analytic functions on $D = \{z||z| < 1\}$ and continuous on $\bar{D}$, prove that $\phi(z) = |f_1(z)| + |f_2(z)| + ... + |f_n(z)|$ achieves maximum values at the boundary ∂D.

3. Prove that if there is a conformal mapping between the annulus $\{z|r_1 < |z| < r_2\}$ and the annulus $\{z|\rho_1 < |z| < \rho_2\}$, then $\frac{r_2}{r_1} = \frac{\rho_2}{\rho_1}$.

4. Let $U(\xi)$ be a bounded function on $\mathbb{R}$ with finitely many points of discontinuity, prove that

$$P_U(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

is a harmonic function on the upper half plane $\{z \in \mathbb{C}|Imz > 0\}$ and it converges to $U(\xi)$ as $z \to \xi$ at a point ξ where $U(\xi)$ is continuous.

5. Let $f \in L^2(\mathbb{R})$ and let $\hat{f}$ be its Fourier transform. Prove that

$$\int_{-\infty}^{\infty} x^2|f(x)|^2 dx \int_{-\infty}^{\infty} \xi^2|\hat{f}(\xi)|^2 d\xi \geq \frac{(\int_{-\infty}^{\infty} |f(x)|^2 dx)^2}{16\pi^2},$$

under the condition that the two integrals on the left are bounded.

(Hint: Operators $f(x) \to xf(x)$ and $\hat{f}(\xi) \to \xi\hat{f}(\xi)$ after Fourier transform are non-commuting operators. The inequality is a version of the uncertainty principle.)

6. Let Ω be an open domain in the complex plane $\mathbb{C}$. Let $\mathbb{H}$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω.

a) Show that $\mathbb{H}$ is a closed subspace of $L^2(\Omega)$, and hence is a Hilbert space with inner product

$$(f, g) = \int_{\Omega} f(z)\bar{g}(z) dxdy, \text{ where } z = x + iy.$$

b) If $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathbb{H}$, then

$$\Sigma_{n=0}^{\infty}|\phi_n(z)|^2 \leq \frac{c^2}{d(z,\Omega^c)}, \text{ for } z \in \Omega.$$

c) The sum

$$B(z,w) = \Sigma_{n=0}^{\infty}\phi_n(z)\bar{\phi}_n(w)$$

converges absolutely for $(z,w) \in \Omega \times \Omega$, and is independent of the choice of the orthonormal basis.

S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Calculate the integral:

$$\int_0^\infty \frac{\log x}{1+x^2} dx.$$

2. Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

3. Prove that any bounded analytic function F over $\{z|r < |z| < R\}$ can be written as $F(z) = z^\alpha f(z)$, where f is an analytic function over the disk $\{z||z| < R\}$ and α is a constant.

4. Let $D \subset \mathbb{R}^n$ be a bounded open set, $f : \bar{D} \to \bar{D}$ is a smooth map such that its Jacobian $\left|\dfrac{\partial f}{\partial x}\right| \equiv 1$, where $\bar{D}$ denotes the closure of D. Prove

(a) for each small ball $B_\epsilon(x)$, there exists a positive integer k such that $f^k(B_\epsilon(x)) \cap B_\epsilon(x) \neq \varnothing$, where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ;

(b) there exists $x \in \bar{D}$ and a sequence $k_1, k_2, \cdots k_j, \cdots$ such that $f^{k_j}(x) \to x$ as $k_j \to \infty$.

5. Let u be a subharmonic function over a domain $\Omega \subset \mathbf{C}$, i.e., it is twice differentiable and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Prove that u achieves its maximum in the interior of Ω only when u is a constant.

6. Suppose that $\phi \in C_0^\infty(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} \phi dx = 1$. Let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon), x \in \mathbf{R}^n, \epsilon > 0$. Prove that if $f \in L^p(\mathbf{R}^n), 1 \leq p < \infty$, then $f * \phi_\epsilon \to f$ in $L^p(\mathbf{R}^n)$, as $\epsilon \to 0$. It is not true for $p = \infty$.

S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. (20 pts) Ming Antu (1692-1763) is one of the greatest Chinese/Mongolian mathematicians. In the 1730s, he first established and used what was later to be known as Catalan numbers (Euler (1707-1763) rediscovered them around 1756; Belgian mathematician Eugene Catalan (1814-1894) "rediscovered" them again in 1838),

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \cdots$$

and Ming Antu derived the following half-angle formula in 1730:

$$\sin^2\frac{\theta}{2} = \sum_{n=1}^{\infty} c_{n-1}\left(\frac{\sin\theta}{2}\right)^{2n}$$

Prove this formula.

Hint: you may use generating function

$$F(z) = \sum_{n=0}^{\infty} c_n z^n$$

and show that $\sum_{m+k=n} c_m c_k = c_{n+1}$ *and then show* $zF(z)^2 = F(z) - 1$.

2. *Many algorithms, including polynomial factorisation in finite fields, require to compute* $\gcd(f(X), X^N - 1)$ *for a polynomial* f *of reasonably small degree* n *and a binomial* $X^N - 1$ *of very large degree* N. *Since* N *is very large the direct application of the Euclid algorithm is very inefficient.*

Questions:

(i) (10 pts) Suggest a more efficient approach the direct computation of $\gcd(f(X), X^N - 1)$ via the Euclid algorithm.

(ii) (10 pts) Generalise it to $\gcd(f(X), A_1X^{N_1} + \ldots + A_mX^{N_m} + A_{m+1})$.

Hint: If for three polynomials f, g *and* h *we have* $g \equiv h \pmod f$ *then*

$$\gcd(f, g) = \gcd(f, h).$$

3. For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \le x \le 1 \tag{1}$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

$$\frac{d}{dt}u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N, \tag{2}$$

with periodic boundary condition

$$u_0 = u_N, \tag{3}$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(i) (15 pts) Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \sum_{j=1}^{N} |u_j|^2 \Delta x$.

(ii) (15 pts) Do you believe (4) is true for $E(t) = \sum_{j=1}^{N} |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counter example.

4. Let A be an $n \times n$ matrix with real and positive eigenvalues and b be a given vector. Consider the solution of $Ax = b$ by the following Richardson's iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

where ω is a damping coefficient. Let λ_1 and λ_n be the smallest and the largest eigenvalues of A. Let $G_\omega = I - \omega A$.

(i) (4 points) Prove that the Richardson's iteration converges if and only if

$$0 < \omega < \frac{2}{\lambda_n}.$$

(ii) (8 points) Prove that the optimal choice of ω is given by

$$\omega_{\text{opt}} = \frac{2}{\lambda_1 + \lambda_n}.$$

Prove also that

$$\rho(G_\omega) = \begin{cases} 1 - \omega\lambda_1 & \omega \le \omega_{\text{opt}} \\ (\lambda_n - \lambda_1)/(\lambda_n + \lambda_1) & \omega = \omega_{\text{opt}} \\ \omega\lambda_n - 1 & \omega \ge \omega_{\text{opt}} \end{cases}$$

where $\rho(G_\omega)$ is the spectral radius of G_ω.

(iii) (8 points) Prove that, if A is symmetric and positive definite, then

$$\rho(G_{\omega_{\text{opt}}}) = \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}$$

where $\kappa_2(A)$ is the spectral condition number of A.

5. (10 pts) For solving the following heat equation on interval

$$u_t = u_{xx}, \qquad 0 \le x \le 1 \tag{5}$$

with boundary condition

$$u(0) = u_0, \quad u(1) = u_1, \tag{6}$$

we first discretize the interval $[0, 1]$ into N subintervals uniformly, that is, the mesh size $h = 1/N$. We choose a temporal step size k and approximate the solution $u(jh, nk)$ by U_j^n, $j = 1, ..., N-1$, $n = 0, 1, 2, ...$. Using the backward Euler method in time and central finite difference in space, the discrete function U_j^n satisfies:

$$U_j^{n+1} - U_j^n = \lambda(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}), \quad j = 1, ..., N-1, \tag{7}$$

where $\lambda = k/h^2$, and

$$U_0^{n+1} = u_0, \ U_N^{n+1} = u_1.$$

Show that

$$\frac{1}{2}\sum_{j=1}^{N-1}\left((U_j^{n+1})^2 - (U_j^n)^2\right) \le -\lambda\sum_{j=1}^{N-2}(U_{j+1}^{n+1} - U_j^{n+1})^2$$
$$- \frac{\lambda}{2}((U_1^{n+1})^2 + (U_{N-1}^{n+1})^2) + \frac{\lambda}{2}(u_0^2 + u_1^2) \tag{8}$$

S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. (15 pts)
Given a finite positive (Borel) measure $d\mu$ on $[0,1]$, define its sequence of moments as follows

$$c_j = \int_0^1 x^j \, d\mu(x) \,, \quad j = 0, 1, \ldots .$$

Show that the sequence is *completely monotone* in the sense that that

$$(I - S)^k c_j \geq 0 \quad \text{ for all } j, k \geq 0,$$

where S denotes the backshift operator given by $Sc_j = c_{j+1}$ for $j \geq 0$.

2. (20 pts)
We recall that a polynomial

$$f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

is called an Eisenstein polynomial if for some prime p we have

(i) $p \mid a_i$ for $i = 0, \ldots, d-1$,
(ii) $p^2 \nmid a_0$,
(iii) $p \nmid a_d$.

Eisenstein polynomials are well-know to be irreducible over $\mathbb{Z}$, so they can be used to construct explicit examples of irreducible polynomials.

Questions:

(i) Prove that a composition $f(g(X))$ of two Eisenstein polynomials f and g is an Eisenstein polynomial again.

(ii) Suggest a multivariate generalisation of the Eisenstein polynomials. That is, describe a class polynomials $F(X_1, \ldots, X_m)$ in terms of the divisibility properties of their coefficients that are guaranteed to be irreducible.

3. (20 pts) For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \leq x \leq 1 \tag{1}$$

where $f'(u) \geq 0$, with periodic boundary condition, we can use the following semi-discrete discontinuous Galerkin method: Find $u_h(\cdot, t) \in V_h$ such that, for all $v \in V_h$ and $j = 1, 2, \cdots, N$,

$$\int_{I_j} (u_h)_t v dx - \int_{I_j} f(u_h) v_x dx + f((u_h)^-_{j+1/2}) v^-_{j+1/2} - f((u_h)^-_{j-1/2}) v^+_{j-1/2} = 0, \tag{2}$$

with periodic boundary condition

$$(u_h)^-_{1/2} = (u_h)^-_{N+1/2}; \quad (u_h)^+_{N+1/2} = (u_h)^+_{1/2}, \tag{3}$$

where $I_j = (x_{j-1/2}, x_{j+1/2})$, $0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = 1$, $h = \max_j(x_{j+1/2} - x_{j-1/2})$, $v^{\pm}_{j+1/2} = v(x^{\pm}_{j+1/2}, t)$, and

$$V_h = \{v : v|_{I_j} \text{ is a polynomial of degree at most } k \text{ for } 1 \le j \le N\}.$$

Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \int_0^1 (u_h(x,t))^2 dx$.

4. Consider the linear system $Ax = b$. The GMRES method is a projection method which obtains a solution in the m-th Krylov subspace K_m so that the residual is orthogonal to AK_m. Let r_0 be the initial residual and let $v_0 = r_0$. The Arnoldi process is applied to build an orthonormal system $v_1, v_2, \cdots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|$. The approximate solution is obtained from the following space

$$K_m = \text{span}\{v_0, v_1, \cdots, v_{m-1}\}.$$

(i) (5 points) Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

(ii) (5 points) Prove that the residual r_k is orthogonal to $\{v_1, v_2, \cdots, v_{k-1}\}$.

(iii) (5 points) Find a formula for the residual norm.

(iv) (5 points) Derive the complete algorithm.

5. (10 pts)

(i) Set $x_0 = 0$. Write the recurrence

$$x_k = 2x_{k-1} + b_k, \quad k = 1, 2, \cdots, n,$$

in a matrix form $A\vec{x} = \vec{b}$. For $b_1 = -1/3$, $b_k = (-1)^k, k = 2, 3, \cdots, n$, verify that $x_k = (-1)^k/3, k = 1, 2, \cdots, n$ is the exact solution.

(ii) Find A^{-1} and compute condition number of A in L^1 norm.

S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_n(X)$. Do the same for S^3 with antipodal points of the equator $S^2 \subset S^3$ identified.

2. Let $M \to \mathbb{R}^3$ be a graph defined by $z = f(u, v)$ where $\{u, v, z\}$ is a Descartes coordinate system in $\mathbb{R}^3$. Suppose that M is a minimal surface. Prove that:
(a) The Gauss curvature K of M can be expressed as

$$K = \Delta \log \left(1 + \frac{1}{W}\right), \quad W := \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2},$$

where Δ denotes the Laplacian with respect to the induce metric on M (i.e., the first fundamental form of M).
(b) If f is defined on the whole uv-plane, then f is a linear function (Bernstein theorem).

3. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

4. Let $p : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. This is a submersion $p : \tilde{M} \to M$ such that for each $x \in \tilde{M}$, $Dp : \ker^{\perp}(Dp) \to T_{p(x)}(M)$ is a linear isometry.

(a) Show that p shortens distances.
(b) If $(\tilde{M}, \tilde{g})$ is complete, so is (M, g).
(c) Show by example that if (M, g) is complete, $(\tilde{M}, \tilde{g})$ may not be complete.

5. Let $\Psi : M \to \mathbb{R}^3$ be an isometric immersion of a compact surface M into $\mathbb{R}^3$. Prove that $\int_M H^2 d\sigma \geq 4\pi$, where H is the mean curvature of M and $d\sigma$ is the area element of M.

6. The unit tangent bundle of S^2 is the subset

$$T^1(S^2) = \{(p, v) \in \mathbb{R}^3 \mid \| p \| = 1,\ (p, v) = 0 \text{ and } \| v \| = 1\}.$$

Show that it is a smooth submanifold of the tangent bundle $T(S^2)$ of S^2 and $T^1(S^2)$ is diffeomorphic to $\mathbb{R}P^3$.

S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Compute the fundamental and homology groups of the wedge sum of a circle S^1 and a torus $T = S^1 \times S^1$.

2. Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold M, show that M/G is orientable if and only if M is orientable and $F(g, \cdot)$ preserves the orientation of M. Use this statement to show that the Möbius band is not orientable and that $\mathbb{R}P^n$ is orientable if and only if n is odd.

3. (a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

4. Prove that a bi-invariant metric on a Lie group G has nonnegative sectional curvature.

5. Let M be the upper half-plane $\mathbb{R}^2_+$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^k}.$$

For which values of k is M complete?

6. Given any nonorientable manifold M show the existence of a smooth orientable manifold $\overline{M}$ which is a double covering of M. Find $\overline{M}$ when M is $\mathbb{R}P^2$ or the Möbius band.

S.-T. Yau College Student Mathematics Contests 2014

Probability and Statistics Problems

Individual

Please solve the following 5 problems.

Problem 1. Let X be a real valued random variable such that for all smooth functions $f : R \to R$ with compact support we have $E[Xf(X)] = E[f'(X)]$. Show that X has the standard normal distribution.

Problem 2. Let (X_n) be a sequence of uncorrelated random variables of mean zero such that

$$\sum_{n=1}^{\infty} nE|X_n|^2 < \infty.$$

Show that $S_n = \sum_{i=1}^n X_i$ converges almost surely.

Problem 3. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let P and Q be two probabilities which are mutually absolutely continuous on $\mathcal{F}$. We denote by X_0 the Radon-Nikodym density of Q with respect to P on $\mathcal{F}$. Show that the following two properties are satisfied:

(a) $0 < E_P[X_0|\mathcal{G}] < +\infty$, P-a.s.;

(b) for every $\mathcal{F}$-measurable non-negative random variable f,

$$E_P[fX_0|\mathcal{G}] = E_Q[f|\mathcal{G}]E_P[X_0|\mathcal{G}].$$

Problem 4. Suppose $X_1, \ldots, X_n, \ldots$ is a sequence of random numbers drawn from the uniform distribution $U(0,1)$. One observes these numbers sequentially. At time n, one keeps a record of $Y_n \overset{def}{=} X_{(n)} = \max_{i=1}^n X_i = \max\{Y_{n-1}, X_n\}$ and $Z_n \overset{def}{=} \bar{X}_n = \sum_{i=1}^n X_i/n = (n-1)/nZ_{n-1} + 1/nX_n$ and discards all previous recordings.

(a) What is the best guess of X_1 if one only observes Y_n?

(b) What is the best guess of X_1 if one only observes Z_n?

(c) Comparing the two guesses of X_1, which one is better (and in what sense)?

Give good reasoning to justify your answers.

Problem 5. Suppose we take a random sample of size n from a bag of colored balls (red, blue and yellow balls) with replacement. Let X_1 denote the number of red balls, X_2 denote the number of blue balls, and X_3 denote the number of yellow balls in the sample. Assuming we know that the total number of yellow balls is triple the total number of red balls in the bag. Or in other words, the red, blue and yellow balls occur with probability p_1, p_2 and $p_3 = 3p_1$, respectively in the bag.

1. Find the aymptotic distribution (after appropriate normalization) for the MLE of p_2.

2. Construct the likelihood ratio test statistic for the null hypothesis that $p_1 = p_2 = p_3/3$ (the alternative is that $p_1 = p_2 = p_3/3$ is not true). What is the asymptotic distribution of your test statistic under null?

S.-T. Yau College Student Mathematics Contests 2014

Probability and Statistics Problems

Team

Please solve the following 5 problems.

Problem 1. Suppose that X_n converges to X in distribution and Y_n converges to a constant c in distribution. Show that

(a) Y_n converges to c in probability;

(b) $X_n Y_n$ converges to cX in distribution.

Problem 2. Let X and Y be two random variables with $|Y| > 0$, a.s.. Let $Z = X/Y$.

(a) Assume the distribution function of (X, Y) has the density $p(x, y)$. What is the density function of Z?

(b) Assume X and Y are independent and X is $N(0, 1)$ distributed, Y has the uniform distribution on $(0, 1)$. Give the density function of Z.

Problem 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and $\Gamma \in \mathcal{F}$. Prove that the following properties are equivalent:

(i) Γ is independent of $\mathcal{G}$ under P,

(ii) for every probability Q on $(\Omega, \mathcal{F})$, equivalent to P, with dQ/dP being $\mathcal{G}$ measurable, we have $Q(\Gamma) = P(\Gamma)$.

(b) Let X, Y, Z be random variables and Y is integrable. Show that if (X, Y) and Z are independent, then $E[Y|X, Z] = E[Y|X]$.

Problem 4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$, and let M be the mean of $|X_1|, ..., |X_n|$.

1. Find $c \in R$ so that $\hat{\sigma} = cM$ is a consistent estimator of σ.
2. Determine the limiting distribution for $\sqrt{n}(\hat{\sigma} - \sigma)$.
3. Identify an approximate $(1 - \alpha)\%$ confidence interval for σ.
4. Is $\hat{\sigma} = cM$ asymptotically efficient? Please justify your answer.

Problem 5. The shifted exponential distribution has the density function

$$f(y;\phi,\theta) = 1/\theta \exp\{-(u-\phi)/\theta\}, \qquad y > \phi, \theta > 0.$$

Let $Y_1,\ldots,Y_n$ be a random sample from this distribution. Find the maximum likelihood estimator (MLE) of ϕ and θ and the limiting distribution of the MLE.

You may use the following Rényi representation of the order statistics: Let $E_1,\ldots,E_n$, be a random sample from the standard exponential distribution (i.e., the above distribution with $\phi = 0, \theta = 1$). Let $E_{(r)}$ denote the r-th order statistics. According to the Rényi representation,

$$E_{(r)} \stackrel{D}{=} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \qquad r = 1,\ldots,n.$$

Here, the symbol $\stackrel{D}{=}$ denotes equal in distribution.

S.-T. Yau College Student Mathematics Contests 2015

Algebra and Number Theory

Individual (5 problems)

This exam of 160 points is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

Problem 1. (20 pt) Let G be an finite $\mathbb{Z}$-module (i.e., a finite abelian group with additive group law) with a bilinear, (strongly) alternative, and non-degenerate pairing

$$\ell : G \times G \to \mathbb{Q}/\mathbb{Z}.$$

Here "(strongly) alternating" means for every $a \in G$, $\ell(a, a) = 0$; "non-degenerate" means for every nonzero $a \in G$ there is a $b \in G$ such that $\ell(a, b) \neq 0$. Show in steps the following statement:

(S) : *G is isomorphic to $H_1 \oplus H_2$ for some finite abelian groups $H_1 \simeq H_2$ such that* $\ell|_{H_i \times H_i} = 0$.

(1.1) (5pt) For every $a \in G$, write $o(a)$ for the order of a and $\ell_a : G \longrightarrow \mathbb{Q}/\mathbb{Z}$ for the homomorphism $\ell_a(b) = \ell(a, b)$. Show that the image of ℓ_a is $o(a)^{-1}\mathbb{Z}/\mathbb{Z}$.

(1.2) (5pt) Show that G has a pair of elements a, b with the following properties:

(a) $o(a)$ is maximal in the sense that for any $x \in G$, $o(x) \mid o(a)$;

(b) $\ell(a, b) = o(a)^{-1} \mod \mathbb{Z}$.

(c) $o(a) = o(b)$

We call the subgroup $< a, b >:= \mathbb{Z}a + \mathbb{Z}b$ a *maximal hyperbolic subgroup* of G.

(1.3) (5pt) Let $< a, b >$ be a maximal hyperbolic subgroup of G. Let G' be the orthogonal complement of $< a, b >$ consisting of elements $x \in G$ such that $\ell(x, c) = 0$ for all $c \in< a, b >$. Show that G is a direct sum as follows:

$$G = \mathbb{Z}a \oplus \mathbb{Z}b \oplus G'.$$

(1.4) (5pt) Finish the proof of (S) by induction.

Problem 2 (40pt)**.** Let $\mathrm{O}_n(\mathbb{C})$ denote the group of $n \times n$ orthogonal complex matrices, and $M_{n\times k}(\mathbb{C})$ the space of $n \times k$ complex matrices, where n and k are two positive integers. For $i = 0, 1$, let F_i be the space of rational function f on $\mathrm{M}_{n\times k}(\mathbb{C})$ such that

$$(*) \qquad f(gx) = \det(g)^i f(x) \quad \text{for all } g \in \mathrm{O}_n(\mathbb{C}) \text{ and } x \in M_{n\times k}(\mathbb{C}).$$

We want to study in steps the structures of F_0 and F_1.

(2.1) (10pt) For each $x \in M_{n\times k}$, let V_x denote the subspace of $\mathbb{C}^n$ generated by columns of x, and let $Q(x) = x^t \cdot x \in M_{k\times k}(\mathbb{C})$. Show the following are equivalent:

(a) the space V_x has dimension k, and the Euclidean inner product $(\cdot,\cdot)$ is non-degenerate on V_x in the sense that $V_x^{\perp} \cap V_x = 0$.

(b) $\det Q(x) \neq 0$.

(2.2) (10pt) Show that F_0 is a field generated by entries of $Q(x)$.

(2.3) (10pt) Assuem $k < n$ and let $f \in F_1$. Show that $f = 0$ by the following two steps:

(a) for any $x \in M_{n\times k}(\mathbb{C})$ with $\det Q(x) \neq 0$, construct a $g \in O_n(\mathbb{C})$ such that $g|_{V_x} = 1$ and $\det g = -1$.

(b) Show that f vanishes on a general point $x \in M_{n\times k}(\mathbb{C})$ with $\det Q(x) \neq 0$, thus $f \equiv 0$.

(2.4) (10pt) Assume $k \geq n$. Show that F_1 is a free vector space of rank 1 over F_0.

Problem 3. (40pt) Consider the equation $f(x) := x^3 + x + 1 = 0$. We want to show in steps that

for any prime p, if $\left(\frac{31}{p}\right) = -1$, then $x^3 + x + 1$ is solvable mod p.

Let x_1, x_2, x_3 be three roots of $f(x) := x^3 + x + 1 = 0$. Let $F = \mathbb{Q}(x_1)$, and $L = \mathbb{Q}(x_1, x_2, x_3)$, and $K = \mathbb{Q}(\sqrt{\Delta})$ where Δ is the discriminant of $f(x)$:

$$\Delta = [(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)]^2.$$

(3.1) (10pt) Show that f is irreducible, that $\Delta = -31$, and that F is not Galois over $\mathbb{Q}$;

(3.2) (10pt) Show that $\mathrm{Gal}(L/\mathbb{Q}) \simeq S_3$, the permutation group of three letters, that $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/3\mathbb{Z}$, and that $\mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z}$;

(3.3) (20pt) Let O_F, O_K, O_L = be rings of integers of F, K, L respectively. Let p be a prime such that $x^3 + x + 1 = 0$ is not soluble in $\mathbb{Z}/p\mathbb{Z}$. Show the following:

(a) (5pt) pO_F is still a prime ideal in O_F,

(b) (5pt) pO_L is product of two prime ideals in O_L, and

(c) (5pt) pO_K is product of two primes ideals in O_K, and

(d) (5pt) $x^2 + 31 = 0$ is soluble in $\mathbb{F}_p$.

Problem 4. (40pt) Let p be a prime and $\mathbb{Z}_p$ the ring of p-adic integers with a p-adic norm normalized by $|p| = p^{-1}$. Let $\phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be a map defined by a power series of the form

$$\phi(x) = x^p + p \sum a_n x^n, \qquad a_n \in \mathbb{Z}_p, \quad |a_n| \longrightarrow 0.$$

Let E be a field, and F the E-vector space of locally constant E-valued functions on $\mathbb{Z}_p$ with an operator ϕ^* defined by $\phi^* f = f \circ \phi$. We want to show in steps the following statement:

The set of eigenvalues of ϕ^* on F is $\{0, 1\}$.

(4.1) (10pt) Show that ϕ is a contraction map on each residue class $R \in \mathbb{Z}_p/p\mathbb{Z}_p$:

$$|\phi(x) - \phi(y)| \leq p^{-1}|x - y|, \qquad \forall x, y \in R.$$

(4.2) (10pt) Show that there is a $\epsilon_R \in R$ for each residue class R such that

$$\lim_n \phi^n(x) = \epsilon_R, \qquad \forall x \in R.$$

Here ϕ^n is defined inductively by $\phi^1 = \phi$, $\phi^n = \phi^{n-1} \circ \phi$.

(4.3) (10pt) Let F_0 (resp. F_1) be the subspace of functions f vanishing on each ϵ_R (resp. constant on R) for all residue class R. Show that $\phi^* = 1$ on F_1, and that for each $f \in F_0$ $\phi^{*n} f = 0$ for some $n \in \mathbb{N}$.

(4.4) (10pt) Show that for any $a \in E$, $a \neq 0, 1$, the operator $\phi^* - a$ is invertible on F.

Problem 5 (20pt)**.** Check if the following rings are UFD (unique factorization domain).

(5.1) (5pt) $R_1 = \mathbb{Z}[\sqrt{6}]$;

(5.2) (5pt) $R_2 = \mathbb{Z}[(1 + \sqrt{-11})/2]$;

(5.3) (5pt) $R_3 = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$;

(5.4) (5pt) $R_4 = \mathbb{C}[x, y]/(x^3 + y^3 - 1)$.

S.-T. Yau College Student Mathematics Contests 2015

Algebra and Number Theory

Team

This exam contains 6 problems. Please choose 5 of them to work on.

Problem 1. (20pt) Let $V = \mathbb{R}^n$ be an Euclidean space equipped with usual inner product, and g an orthogonal matrix acting on V. For $a \in V$, let s_a denote the reflection

$$s_a(x) := x - 2\frac{(x,a)}{(a,a)}a, \qquad \forall x \in V.$$

(1.1) (10pt) For $a = (g-1)b \neq 0$, show that

$$\ker(s_a g - 1) = \ker(g-1) \oplus \mathbb{R}b.$$

(1.2) (10pt) Show that g is a product of $\dim[(g-1)V]$ reflections.

Problem 2. (20pt) Let p and q be two distinct prime numbers. Let G be a non-abelian finite group satisfying the following conditions:

1. all nontrivial elements have order either p or q;
2. The q-Sylow subgroup H_q is normal and is a nontrivial abelian group.

Show in steps the following statement:

The group G is of the form $(\mathbb{Z}/p\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^n$, where the action of $1 \in \mathbb{Z}/p\mathbb{Z}$ on $(\mathbb{Z}/q\mathbb{Z})^n \simeq \mathbb{F}_q^n$ is given by a matrix $M(1) \in \mathrm{GL}_n(\mathbb{F}_q)$ whose eigenvalues are all primitive *p-th roots of unities.*

(2.1) (5pt) Let H_p denote a p-Sylow subgroup of G. Show that its inclusion into G induces an isomorphism $H_p \cong G/H_q$, and that $G \simeq H_p \ltimes H_q$.

(2.2) (5pt) Let $M : H_p \longrightarrow \mathrm{Aut}(H_q) \simeq \mathrm{GL}_n(\mathbb{F}_q)$ be the homomorphism induced from the conjugations. Show that for each $1 \neq a \in H_p$, $M(a)$ is semisimple whose eigenvalues are all *primitive* p-th roots of unities. In particular M is injective.

(2.3) (5pt) Show that if two nontrivial elements $a, b \in H_p$ commute with each other, then $a = b^n$ for some $n \in \mathbb{N}$, and that $H_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(2.4) (5pt) Complete the solution of the problem.

Problem 3. (20pt) Let ζ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geq 3$ and an algebraic integer η. Show that $\zeta = 1$.

Problem 4. (20pt) Let G be a finite group and (π, V) a finite dimensional $\mathbb{C}G$-module. For $n \geq 0$, let $\mathbb{C}[V]_n$ be the space of homogeneous polynomial functions on V of degree n. For a simple G-representation ρ, denote by $a_n(\rho)$ the multiplicity of ρ in $\mathbb{C}[V]_n$. Show that

$$\sum_{n\geq 0} a_n(\rho)t^n = \frac{1}{|G|}\sum_{g\in G} \frac{\overline{\chi_\rho(g)}}{\det(\mathrm{id}_V - \pi(g)t)}.$$

Problem 5. (20pt) Let A be an $n \times n$ complex matrix considered as an operator on $V = (\mathbb{C}^n, (\cdot,\cdot))$ with standard hermitian form. Let $A^* = \bar{A}^t$ be the hermitian transpose of A:

$$(Ax, y) = (x, A^*y), \qquad \forall x, y \in \mathbb{C}^n.$$

(5.1) (5pt) For any $\lambda \in \mathbb{C}$, show the identity:

$$\ker(A - \lambda)^\perp = (A^* - \bar{\lambda})V.$$

(5.2) (15pt) Show the equivalence of the following two statements:

(a) A commutes with A^*;

(b) there is a unitary matrix U (in the sense $U^* = U^{-1}$), such that UAU^{-1} is diagonal.

Problem 6. (20pt) Consider the polynomial $f(x) = x^5 - 80x + 5$.

(6.1) (5pt) Show that f is irreducible over $\mathbb{Q}$;

(6.2) (15 pt) Show in steps that the split field K of f has Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to S_5, the symmetric group of 5 letters.

(a) (5pt) $f = 0$ has exactly two complex roots;

(b) (5pt) G can be embedded into S_5 with image containing cycles (12345) and (12);

(c) (5pt) $G \simeq S_5$.

S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f_n \in L^2(R)$ be a sequence of measurable functions over the line, $f_n \to f$ almost everywhere. Let $||f_n||_{L^2} \to ||f||_{L^2}$, prove that $||f_n - f||_{L^2} \to 0$.

2. Let f be a continuous function on $[a, b]$, define $M_n = \int_a^b f(x)x^n dx$. Suppose that $M_n = 0$ for all integers $n \geq 0$, show that $f(x) = 0$ for all x.

3. Determine all entire functions f that satisfying the inequality

$$|f(z)| \leq |z|^2 |Im(z)|^2$$

for z sufficiently large.

4. Describe all functions that are holomorphic over the unit disk $D = \{z||z| < 1\}$, continuous on $\bar{D}$ and map the boundary of the disk into the boundary of the disk.

5. Let $T : H_1 \to H_2, Q : H_2 \to H_1$ be bounded linear operators of Hilbert spaces H_1, H_2. Let $QT = Id - S_1, TQ = Id - S_2$ where S_1 and S_2 are compact operators. Prove $KerT = \{v \in H_1, Tv = 0\}, CokerT = H_2/\overline{Im(T)}$ are finite dimensional and $Im(T) = \{Tv \in H_2, v \in H_1\}$ is closed in H_2.

Note: S is compact means for every bounded sequence $x_n \in H_1, Sx_n$ has a converging subsequence.

6. Let H_1 be the Sobolev space on the unit interval $[0, 1]$, i.e. the Hilbert space consisting of functions $f \in L^2([0, 1])$ such that

$$||f||_1^2 = \Sigma_{n=-\infty}^{\infty}(1 + n^2)|\hat{f}(n)|^2 < \infty;$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^1 f(x)e^{-2\pi inx} dx$$

are Fourier coefficients of f. Show that there exists constant $C > 0$ such that

$$||f||_{L^\infty} \leq C||f||_1$$

for all $f \in H_1$, where $||.||_{L^\infty}$ stands for the usual supremum norm. (Hint: Use Fourier series.)

S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $\phi \in C([a,b],R)$. Suppose for every function $h \in C^1([a,b],R), h(a) = h(b) = 0$, we have

$$\int_a^b \phi(x)h(x)dx = 0.$$

Prove that $\phi(x) = 0$.

2. Let f be a Lebesgue integrable function over $[a, b+\delta], \delta > 0$, prove that

$$\lim_{h \to 0+} \int_a^b |f(x+h) - f(x)|dx \to 0.$$

3. Let $L(q, q', t)$ be a function of $(q, q', t) \in TU \times R, U$ is an open domain in R^n. Let $\gamma : [a,b] \to U$ be a curve in U. Define a functional $S(\gamma) = \int_a^b L(\gamma(t), \gamma'(t), t)dt$. We say that γ is an extremal if for every smooth variation of $\gamma, \phi(t,s), s \in (-\delta, \delta), \phi(t,0) = \gamma(t), \phi_s = \phi(t,s)$, we have $\frac{dS(\phi_s)}{ds}|_{s=0} = 0$. Prove that every extremal γ satisfies the Euler-Lagrange equation: $\frac{d}{dt}(\frac{\partial L}{\partial q'}) = \frac{\partial L}{\partial q}$.

4. Let $f : U \to U$ be a holomorphic function with U a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f'(0) = 1$, prove that $f(z) = z$.

5. Let $T : H_1 \to H_2$ be a bounded operator of Hilbert spaces H_1, H_2. Let $S : H_1 \to H_2$ be a compact operator, that is, for every bounded sequence $\{v_n\} \in H_1, Sv_n$ has a converging subsequence. Show that $Coker(T+S) = H_2/\overline{Im(T+S)}$ is finite dimensional and $Im(T+S)$ is closed in H_2. (Hint: Consider equivalent statements in terms of adjoint operators.)

6. Let $u \in C^2(\bar{\Omega}), \Omega \subset R^d$ is a bounded domain with a smooth boundary.

1) Let u be a solution of the equation $\Delta u = f, u|_{\partial\Omega} = 0, f \in L^2(\Omega)$. Prove that there is a constant C depends only Ω such that

$$\int_\Omega (\Sigma_{j=1}^n (\frac{\partial u}{\partial x_j})^2 + u^2)dx \leq C \int_\Omega f^2(x)dx.$$

2) Let $\{u_n\}$ be a sequence of harmonic functions on Ω, such that $||u_n||_{L^2(\Omega)} \leq M < \infty$, for a constant M independent of n. Prove that there is a converging subsequence $\{u_{n_k}\}$ in $L^2(\Omega)$.

S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Individual (5 problems)

Problem 1. Let r and s be relatively prime positive integers. Prove that the number of lattice paths from $(0,0)$ to (r,s), which consists of steps $(1,0)$ and $(0,1)$ and never go above the line $ry = sx$ is given by

$$\frac{1}{r+s}\binom{r+s}{s}.$$

Problem 2. The following 2×2 block matrix

$$C(\alpha) = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}$$

plays a key role in an augmented system method to solve linear least squares problem, a fundamental numerical linear algebra problem for fitting a linear model to observations subject to errors in science, where $A \in \mathbf{R}^{m\times n}$ is of full rank $n \le m$, I is a $m\times m$ identity matrix, and $\alpha \ge 0$. Prove the following results which address the question of optimal choice of scaling α for stabiltiy of the augmented system method.

(a) The eigenvalues of $C(\alpha)$ are

$$\frac{\alpha}{2} \pm \left(\frac{\alpha^2}{4} + \sigma_i^2\right)^{1/2} \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \alpha \quad (m-n \text{ times}),$$

where σ_i for $i = 1, 2, \ldots, n$ are the singular values of A, arranged in the decreasing order, i.e., $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$.

(b) The condition number $\kappa_2(C(\alpha)) = \|C(\alpha)\|_2\, \|[C(\alpha)]^{-1}\|_2$ has the following bounds:

$$\sqrt{2}\kappa_2(A) \le \min_{\alpha} \kappa_2(C(\alpha)) \le 2\kappa_2(A),$$

with $\min_\alpha \kappa_2(C(\alpha))$ being achieved for $\alpha = \sigma_n/\sqrt{2}$, and

$$\max_{\alpha} \kappa_2(C(\alpha)) > \kappa_2(A)^2,$$

where $\|\cdot\|$ is the spectral norm of a matrix.

Recall that any matrix $A \in \mathbf{R}^{m\times n}$ has a singular value decomposition (SVD):

$$A = U\Sigma V^T, \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \in \mathbf{R}^{m\times n}, \quad p = \min(m,n),$$

where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_p \ge 0$, and $U \in \mathbf{R}^{m\times m}$, $V \in \mathbf{R}^{n\times n}$ are both orthogonal. The σ_i are the singular values of A and the columns of U and V are the left and right singular vectors of A, respectively.

Problem 3. Solve the following linear hyperbolic partial differential equation

$$u_t + au_x = 0, \quad t \geq 0, \tag{1}$$

where a is a constant. Using the finite difference approximation, we can obtain the forward-time central-space scheme as follows,

$$\frac{u_m^{n+1} - u_m^n}{k} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0, \tag{2}$$

where k and h are temporal and spatial mesh sizes.

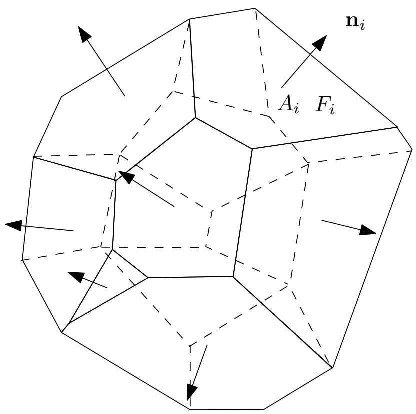
(a) Show that when we fix $\lambda = k/h$ as a positive constant, the forward-time central-space scheme (2) is consistent with equation (1).
(b) Analyze the stability of this method. Is the method stable with $\lambda = k/h$ being fixed as a constant?
(c) How would the answer change if you are allowed to make $\lambda = k/h$ small?
(d) Would this is a good scheme to use even if you can make it stable by making λ small? If not, please provide a simple modification to make this scheme stable by keeping λ fixed.

Problem 4. Let $A, H, Q \in \mathbb{C}^{n\times n}$ and Q is non-singular. Assume that $H = Q^{-1}AQ$ and H is properly upper Hessenberg. Show that

$$\text{span}\{q_1, q_2, \ldots, q_j\} = \mathcal{K}_j(A, q_1), \qquad j = 1, 2, \ldots, n$$

where q_j is the j-th column of Q, and $\mathcal{K}_j(A, q_1) = \text{span}\{q_1, Aq_1, \ldots, A^{j-1}q_1\}$.

Problem 5. Minkowski Problem.



Assume P is a convex polyhedron embedded in $\mathbb{R}^3$, the faces are $\{F_1, F_2, \cdots, F_k\}$, the unit normal vector to the face F_i is $\mathbf{n}_i$, the area of F_i is A_i, $1 \leq i \leq k$.

- Show that

$$A_1\mathbf{n}_1 + A_2\mathbf{n}_2 + \cdots A_k\mathbf{n}_k = \mathbf{0}, \tag{3}$$

- Given k unit vectors $\{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_k\}$ which can not be contained in any half space, and k real positive numbers $\{A_1, A_2, \cdots, A_k\}$, $A_i > 0$, and satisfying the condition (3), show that there exists a convex polyhedron P, whose face normals are $\mathbf{n}_i$'s, face areas are A_i's.

S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Team (5 problems)

Problem 1. Consider the elliptic interface problem

$$(a(x)u_x)_x = f,\ x \in (0,1)$$

with the Dirichlet boundary condition

$$u(0) = u(1) = 0.$$

Here, f is a smooth function, the elliptic coefficient $a(x)$ is discontinuous across an interface point ξ, that is,

$$a(x) = \begin{cases} a_0 & \text{for } 0 < x < \xi \\ a_1 & \text{for } \xi < x < 1, \end{cases}$$

$a_0, a_1 > 0$ are positive constants, and $0 < \xi < 1$ is an interface point. Across the interface, we need to impose two jump conditions

$$u(\xi-) = u(\xi+),\ a(\xi-)u_x(\xi-) = a(\xi+)u_x(\xi+).$$

Question:

1. (25%) Design a numerical method to solve this problem. The method should be at least first order. It is better to be high order (if your method is first order, you get 20% points).
2. (75%) Prove your accuracy and convergence arguments (if your method is first order, you get 60% points).

Problem 2. Let G be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one.

An unbalanced cycle in G is a a cycle which have odd number of negative edges. Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself!

A resigning in G at a vertex v of G is to switch the type (positive or negative) of all edges incident to v.

Question: Show that one can switch all edge of G into positive edges using a sequence resigning if and only if there is no unbalanced cycle in G.

Problem 3. We consider particles which are able to produce new particles of like kind. A single particle forms the original, or zero, generation. Every particle has probability p_k ($k = 0, 1, 2, \dots$) of creating exactly k new particles; the direct descendants of the nth generation form the $(n + 1)$st generation. The particles of each generation act independently of each other.

Assume $0 < p_0 < 1$. Let $P(x) = \sum_{k\geq 0} p_k x^k$ and $\mu = P'(1) = \sum_{k\geq 0} kp_k$ be the expected number of direct descendants of one particle. Prove that if $\mu > 1$, then the probability x_n that the process terminates at or before the nth generation tends to the unique root $\sigma \in (0,1)$ of equation $\sigma = P(\sigma)$.

Problem 4. (Isopermetric inequality). Consider a closed plane curve described by a parametric equation $(x(t), y(t)), 0 \leq t \leq T$ with parameter t oriented counterclockwise and $(x(0), y(0)) = (x(T), y(T))$.

(a): Show that the total length of the curve is given by

$$L = \int_0^T \sqrt{(x'(t))^2 + (y'(t))^2)}\, dt$$

(b): Show that the total area enclosed by the curve is given by

$$A = \frac{1}{2}\int_0^T \big(x(t)y'(t) - y(t)x'(t)\big)\, dt$$

(c): The classical iso-perimetric inequality states that for closed plane curves with a fixed length L, circles have the largest enclosed area A. Formulate this question into a variational problem.

(d): Derive the Euler-Lagrange equation for the variational problem in **(c)**.

(e): Show that there are two constants x_0 and y_0 such that

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \equiv r^2$$

where $r = L/(2\pi)$. Explain your result.

Problem 5. Let $A \in \mathbb{R}^{n\times m}$ with rank $r < \min(m, n)$. Let $A = U\Sigma V^T$ be the SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(a) Show that, for every $\epsilon > 0$, there is a full rank matrix $A_\epsilon \in \mathbb{R}^{n\times m}$ such that $||A - A_\epsilon||_2 = \epsilon$.

(b) Let $A_k = U\Sigma_k V^T$ where $\Sigma_k = \mathrm{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$ and $1 \leq k \leq r-1$. Show that $\mathrm{rank}(A_k) = k$ and

$$\sigma_{k+1} = ||A - A_k||_2 = \min\{||A - B||_2 \quad | \quad \mathrm{rank}(B) \leq k\}$$

(c) Assume that $r = \min(m, n)$. Let $B \in \mathbb{R}^{n\times m}$ and assume that $||A - B||_2 < \sigma_r$. Show that $\mathrm{rank}(B) = r$.

S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let n, m be positive integers. Show that the product of spheres $S^n \times S^m$ has trivial tangent bundle if and only if n or m is odd.

2. Show that there does not exist a compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

3. Let M^n be a smooth manifold without boundary and X a smooth vector field on M. If X does not vanish at $p \in M$, show that there exists a local coordinate chart $(U; x_1, \ldots, x_n)$ centered at p such that in U the vector field X takes the form $X = \frac{\partial}{\partial x_1}$.

4. Let $M \to \mathbb{R}^3$ be a compact simply-connected closed surface. Prove that if M has constant mean curvature, then M is a standard sphere.

5. Let M be an n-dimensional compact Riemannian manifold with diameter π/c and Ricci curvature $\geq (n-1)c^2 > 0$. Show that M is isometric to the standard n-sphere in $\mathbb{R}^{n+1}$ with radius $1/c$.

6. Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{k,l} R_{iklj} x_k x_l + O(|x|^3).$$

S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let $SO(3)$ be the set of all 3×3 real matrices A with determinant 1 and satisfying ${}^tAA = I$, where I is the identity matrix and ${}^tA$ is the transpose of A. Show that $SO(3)$ is a smooth manifold, and find its fundamental group. You need to prove your claims.

2. Let X be a topological space. The *suspension* $S(X)$ of X is the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

3. Let $F : M \to N$ be a smooth map between two manifolds. Let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Prove that if $Y_1 = F_*X_1$ and $Y_2 = F_*X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[\ ,\ ]$ is the Lie bracket.

4. Let M_1 and M_2 be two compact convex closed surfaces in $\mathbb{R}^3$, and $f : M_1 \to M_2$ a diffeomerphism such that M_1 and M_2 have the same inner normal vectors and Gauss curvatures at the corresponding points. Prove that f is a translation.

5. Prove the second Bianchi identity:

$$R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$$

6. Let M_1, M_2 be two complete n-dimensional Riemannian manifolds and $\gamma_i : [0, a] \to M_i$ are two arc length parametrized geodesics. Let ρ_i be the distance function to $\gamma_i(0)$ on M_i. Assume that $\gamma_i(a)$ is within the cut locus of $\gamma_i(0)$ and for any $0 \leq t \leq a$ we have the inequality of sectional curvatures

$$K_1(X_1, \frac{\partial}{\partial \gamma_1}) \geq K_2(X_2, \frac{\partial}{\partial \gamma_2}),$$

where $X_i \in T_{\gamma_i(t)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}$.

Then

$$Hess(\rho_1)(\widetilde{X}_1, \widetilde{X}_1) \leq Hess(\rho_2)(\widetilde{X}_2, \widetilde{X}_2),$$

where $\widetilde{X}_i \in T_{\gamma_i(a)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}(a)$.

S.-T. Yau College Student Mathematics Contests 2015

Probability and Statistics

Individual (5 problems)

Problem 1. (a) Let X and Y be two random variables with zero means, variance 1, and correlation ρ. Prove that

$$\mathbb{E}[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1-\rho^2}.$$

(b) Let X and Y have a bivariate normal distribution with zero means, variances σ^2 and τ^2, respectively, and correlation ρ. Find the conditional expectation $\mathbb{E}(X|Y)$.

Problem 2. We flip a fair coin until heads turns out twice consecutively. What is the expected number of flips?

Problem 3. Let $(X_n, n \geq 1)$ be a sequence of independent Gaussian variables, with respective mean μ_n, and variance σ_n^2.

(a) Prove that if $\sum_n X_n^2$ converges in L^1, then $\sum_n X_n^2$ converges in L^p, for every $p \in [1, \infty)$.

(b) Assume that $\mu_n = 0$, for every n. Prove that if $\sum_n \sigma_n^2 = \infty$, then

$$\mathbb{P}(\sum_n X_n^2 = \infty) = 1.$$

Problem 4. Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution with pdf $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$ for $x, \theta > 0$, zero elsewhere. Let $Y_1 = \min\{X_1, \ldots, X_n\}$. Consider an estimator nY_1.

(a) Show this estimate is unbiased.

(b) Prove or disprove: This estimate is a consistent estimator.

(c) Prove or disprove: This estimate is an efficient estimator.

Problem 5. Let the independent normal random variables $Y_1, \ldots, Y_n$ have, respectively, the probability density functions $N(\mu, \gamma^2 x_i^2)$, $i = 1, \ldots, n$, where the given $x_1, \ldots, x_n$ are not all equal and no one of which is zero.

(a) Construct a confidence interval for γ with significance level $1 - \alpha$.

(b) Discuss the test of the hypothesis $H_0 : \gamma = 1, \mu$ unspecified, against all alternatives $H_1 : \gamma \neq 1, \mu$ unspecified.

S.-T. Yau College Student Mathematics Contests 2015

Probability and Statistics

Team (5 problems)

Problem 1. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

Problem 2. Assume a sequence of random variables X_n converges in distribution to a random variable X. Let $\{N_t, t \geq 0\}$ be a set of positive integer-valued random variables, which is independent of (X_n) and converges in probability to ∞ as $t \to \infty$. Prove that X_{N_t} converges in distribution to X as $t \to \infty$.

Problem 3. Suppose $T_1, T_2, \ldots, T_n$ is a sequence of independent, identically distributed random variables with the exponential distribution of the density function

$$p(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Let $S_n = T_1 + T_2 + \cdots + T_n$. Find the distribution of the random vector

$$V_n = \left\{ \frac{T_1}{S_n}, \frac{T_2}{S_n}, \cdots, \frac{T_n}{S_n} \right\}.$$

Problem 4. Suppose that X and Z are jointly normal with mean zero and standard deviation 1. For a strictly monotonic function $f(\cdot)$, $\mathrm{cov}(X, Z) = 0$ if and only if $\mathrm{cov}(X, f(Z)) = 0$, provided the latter covariance exists. **Hint:** Z can be expressed as $Z = \rho X + \varepsilon$ where X and ε are independent and $\varepsilon \sim N(0, \sqrt{1-\rho^2})$.

Problem 5. Consider the following penalized least-squares problem (Lasso):

$$\frac{1}{2}\|\mathbf{Y} - \mathbf{X}\beta\|^2 + \lambda\|\beta\|_1$$

Let $\widehat{\beta}$ be a minimizer and $\boldsymbol{\Delta} = \widehat{\beta} - \beta^*$ for any given β^*. If $\lambda > 2\|\mathbf{X}^T(\mathbf{Y} - \mathbf{X}\beta^*)\|_\infty$, show that

1. $\|\mathbf{Y} - \mathbf{X}^T\widehat{\beta}\|^2 - \|\mathbf{Y} - \mathbf{X}^T\beta^*\|^2 > -\lambda\|\boldsymbol{\Delta}\|_1$.

2. $\|\mathbf{\Delta}_{S^c}\|_1 \leq 3\|\mathbf{\Delta}_S\|_1$, where $S = \{j : \beta_j^* \neq 0\}$ is the support of the vector β^*, S^c is its complement set, $\mathbf{\Delta}_S$ is the subvector of $\mathbf{\Delta}$ restricted on the set S, and $\|\mathbf{\Delta}_S\|_1$ is its L_1-norm.

S.-T. Yau College Student Mathematics Contests 2016

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Let E be a linear space over $\mathbb{R}$, of finite dimension $n \geq 2$, equipped with a positive definite symmetric bilinear form $\langle \cdot , \cdot \rangle$. Let $u_1, u_2, \ldots, u_n$ be a basis of E. Let $v_1, v_2, \ldots, v_n$ be the dual basis, that is,

$$\langle u_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, 2, \ldots, n$.

(a) (8 points) Assume that $\langle u_i, u_j \rangle \leq 0$ for all $1 \leq i < j \leq n$. Show that there is an orthogonal basis $u'_1, u'_2, \ldots, u'_n$ of E such that u'_i is a non-negative linear combination of $u_1, u_2, \ldots, u_i$, for all $i = 1, 2, \ldots, n$.

(b) (6 points) With the same assumption as in Part (a), show that $\langle v_i, v_j \rangle \geq 0$ for all $1 \leq i < j \leq n$.

(c) (6 points) Assume that $n \geq 3$. Show that the condition $\langle u_i, u_j \rangle \geq 0$ for all $1 \leq i < j \leq n$ does not imply that $\langle v_i, v_j \rangle \leq 0$ for all $1 \leq i < j \leq n$.

Problem 2 (20 points)**.** Let $d \geq 1$ and $n \geq 1$ be integers.

(a) (5 points) Show that there are only finitely many subgroups $G \subseteq \mathbb{Z}^d$ of index n. Let $f_d(n)$ denote the number of such subgroups.

(b) (5 points) Let $g_d(n)$ denote the number of subgroups $H \subseteq \mathbb{Z}^d$ of index n such that the quotient group is cyclic. Show that $f_d(mn) = f_d(m)f_d(n)$ and $g_d(mn) = g_d(m)g_d(n)$ for coprime positive integers m and n.

(c) (5 points) Compute $g_d(p^r)$ for every prime power p^r, $r \geq 1$.

(d) (5 points) Compute $f_2(20)$.

Problem 3 (20 points)**.** Let A be a complex $m \times m$ matrix. Assume that there exists an integer $N \geq 0$ such that $t_n = \operatorname{tr}(A^n)$ is an algebraic integer for all $n \geq N$. The goal of this problem is to show that the eigenvalues $a_1, \ldots, a_m$ of A are algebraic integers.

(a) (10 points) Show that there exist algebraic numbers $b_{ij} \in \mathbb{C}$, $1 \leq i, j \leq m$ such that

$$a_i^n = \sum_{j=1}^{m} b_{ij} t_{n+j-1}$$

for all $n \geq 0$ and all $1 \leq i \leq m$. In particular, $a_1, \ldots, a_m$ are algebraic numbers.

(b) (8 points) Let R be the ring of all algebraic integers in $\mathbb{C}$ and let K be the field of all algebraic numbers in $\mathbb{C}$. Show that for $a \in K$, if $R[a]$ is contained in a finitely-generated R-submodule of K, then $a \in R$.

(c) (2 points) Conclude that $a_1, \ldots, a_m$ are algebraic integers.

Problem 4 (20 points)**.** Let E be a Euclidean plane. For each line l in E, write $s_l \in \mathrm{Iso}(E)$ for the reflection with respect to l, where $\mathrm{Iso}(E)$ denotes the group of distance-preserving bijections from E to itself.

(a) (6 points) Let l_1 and l_2 be two distinct lines in E. Find the necessary and sufficient condition that s_{l_1} and s_{l_2} generate a finite group.

(b) (7 points) Let l_1, l_2 and l_3 be three pairwise distinct lines in E. Assume that s_{l_1}, s_{l_2} and s_{l_3} generate a finite group. Show that l_1, l_2, l_3 intersect at a point.

(c) (7 points) Let G be a finite subgroup of $\mathrm{Iso}(E)$ generated by reflections. Show that G is generated by at most two reflections.

Problem 5 (20 points)**.** Let G be a finite group of order $2^n m$ where $n \geq 1$ and m is an odd integer. Assume that G has an element of order 2^n. The goal of this problem is to show that G has a normal subgroup of order m.

(a) (5 points) Show that if M is a normal subgroup of G of order m, then M is the only subgroup of G of order m.

(b) (5 points) Let N be a normal subgroup of G and let P be a 2-Sylow subgroup of G. Show that $P \cap N$ is a 2-Sylow subgroup of N.

(c) (5 points) Show that the homomorphism $G \to \{\pm 1\}$ carrying g to $\mathrm{sgn}(l_g)$ is surjective. Here $\mathrm{sgn}(l_g)$ denotes the sign of the permutation $l_g \colon G \to G$ given by left multiplication by g.

(d) (5 points) Deduce by induction on n that G has a normal subgroup of order m.

S.-T. Yau College Student Mathematics Contests 2016

Algebra and Number Theory

Team

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Find all real orthogonal 2×2 matrices k with the following property: There is an upper triangular 2×2 real matrix b with all diagonal entries being positive numbers such that kb is a positive definite symmetric matrix.

Problem 2 (20 points)**.** For $x \in \mathbb{Z}$ and $k \geq 0$, define the binomial coefficients

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \quad \binom{x}{0} = 1.$$

(a) (6 points) Show that $x \in \mathbb{Z} \implies \binom{x}{k} \in \mathbb{Z}$.

(b) (6 points) Show that every function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ can be expressed as $f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}$, where $a_k \in \mathbb{Z}$ are uniquely determined by f.

(c) (8 points) Define

$$\phi_k(x) = \binom{x + \lfloor k/2 \rfloor}{k}.$$

Show that every function $f\colon \mathbb{Z} \to \mathbb{Z}$ can be expressed as $f(x) = \sum_{k=0}^{\infty} a_k \phi_k(x)$, where $a_k \in \mathbb{Z}$ are uniquely determined by f.

Problem 3 (20 points)**.** Let K be the splitting field of the polynomial

$$x^4 - x^2 - 1.$$

(a) (10 points) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_4. Here we adopt the convention that D_4 is the group of symmetries of a square and has order 8.

(b) (10 points) Determine the lattice of subfields of K: Find all subfields of K and describe the partial order induced by inclusion.

Problem 4 (20 points)**.** Let G be a (not necessarily finite) group and let F be a field of characteristic $\neq 2$. Let $V \neq 0$ be an **indecomposable** finite-dimensional linear representation of G over F. Let $R = \mathrm{End}_F(V)^G$ be the ring of G-equivariant endomorphisms of V.

(a) (5 points) Prove the following form of Fitting's lemma: Every element of R is either invertible or nilpotent.

(b) (5 points) Deduce that the set $I \subseteq R$ of non-invertible elements is a two-sided ideal and the quotient R/I is a division algebra over F.

(c) (5 points) We say that V is *orthogonal* if there **exists** a G-invariant nondegenerate symmetric bilinear form on V. We say that V is *symplectic* if there **exists** a G-invariant nondegenerate alternating bilinear form on V. Deduce that if there exists a G-invariant nondegenerate bilinear form on V, then V is orthogonal or symplectic.

(d) (5 points) Assume that F is algebraically closed. Deduce from (b) that V cannot be both orthogonal and symplectic.

Problem 5 (20 points)**.**

(a) (5 points) Let G be a finite group. Let $x_1, \ldots, x_h$ be representatives of the conjugacy classes of G. Let $n_i = \#\mathrm{Cent}_G(x_i)$ be the cardinality of the centralizer of x_i. Prove the identity

$$1 = \sum_{i=1}^{h} \frac{1}{n_i}.$$

(b) (10 points) Deduce that for any integer $h \geq 1$, there exist only finitely many isomorphism classes of finite groups with exactly h conjugacy classes.

(c) (5 points) Find all the finite groups with exactly 3 conjugacy classes.

S.-T. Yau College Student Mathematics Contests 2016

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that F is continuous on $[a,b], F^{'}(x)$ exists for every $x \in (a,b), F^{'}(x)$ is integrable. Prove that F is absolutely continuous and

$$F(b) - F(a) = \int_a^b F^{'}(x)dx.$$

2. Suppose that f is integrable on $\mathbf{R}^n$, let $K_\delta(x) = \delta^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{\delta}}$ for each $\delta > 0$. Prove that the convolution

$$(f * K_\delta)(x) = \int_{\mathbf{R}^n} f(x-y)K_\delta(y)dy$$

is integrable and $||(f * K_\delta) - f||_{L^1(\mathbf{R}^n)} \to 0$, as $\delta \to 0$.

3. Prove that a bounded function on interval $I = [a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. You may prove this by the following steps.

Define $I(c,r) = (c-r, c+r), osc(f,c,r) = \sup_{x,y \in J \cap I(c,r)} |f(x) - f(y)|, osc(f,c) = \lim_{r \to 0} osc(f,r,c)$.

1) f is continuous at $c \in J$ if and only if $osc(f,c) = 0$.

2) For arbitrary $\epsilon > 0, \{c \in J | osc(f,c) \geq \epsilon\}$ is compact.

3) If the set of discontinuities of f has measure 0, then f is Riemann integrable.

4. 1) Let f be the Rukowski map: $w = \frac{1}{2}(z + \frac{1}{z})$. Show that it maps $\{z \in \bar{\mathbf{C}} || z| > 1\}$ to $\bar{\mathbf{C}}/[-1,1], \bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

2) Compute the integral:

$$\int_0^\infty \frac{\log x}{x^2 - 1} dx.$$

5. Let f be a doubly periodic meromorphic function over the complex plane, i.e. $f(z + 1) = f(z), f(z+i) = f(z), z \in \mathbf{C}$, prove that the number of zeros and the number of poles are equal.

6. Let A be a bounded self-adjoint operator over a complex Hilbert space. Prove that the spectrum of A is a bounded closed subset of the real line $\mathbf{R}$.

S.-T. Yau College Student Mathematics Contests 2016

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $D \subset \mathbf{R}^d, d \geq 2$ be a compact convex set with smooth boundary ∂D so that the origin belongs to the interior of D. For every $x \in \partial D$ let $\alpha(x) \in (0, \infty)$ be the angle between the position vector x of the outer normal vector $n(x)$. Let ω_d be the surface area of the unit sphere in $\mathbf{R}^d$. Compute:

$$\frac{1}{\omega_d} \int_{\partial D} \frac{\cos(\alpha(x))}{|x|^{d-1}} d\sigma(x)$$

where $d\sigma$ denotes the surface measure on ∂D.

2. Let $p > 0$ and suppose $f_n, f \in L^p[0,1]$ and $||f_n - f||_p = (\int_0^1 |f_n(x) - f(x)|^p dx)^{\frac{1}{p}} \to 0$ as $n \to \infty$.

a) Show that for every $\epsilon > 0$,

$$\lim_{n \to \infty} m(\{x \in [0,1] || f_n(x) - f(x)| > \epsilon\}) = 0.$$

Here m is the Lebesgue measure.

b) Show that there exists a subsequence f_{n_j} such that $f_{n_j}(x) \to f(x)$ for almost every $x \in [0,1]$.

3. 1) Let f be a holomorphic function on the unit disk $D = \{z \in \mathbf{C} || z| < 1\}$ except 0. Assume $f \in L^2(D), i.e. \int_D |f(z)|^2 dz d\bar{z} < \infty$, then 0 is a removable singularity.

2) Let f_n be a sequence of holomorphic functions over a domain $\Omega \subset C$ converging to f uniformly on any compact subset of Ω, does the sequence of its derivatives f_n' also have this property?

4. Consider the torus $T^2 = \mathbf{C}/\Lambda, \Lambda = \{m + in | m, n \in \mathbf{Z}\}$, i.e. $z_1, z_2 \in \mathbf{C}$ are equivalent if and only if there are integers m, n such that $z_2 = z_1 + m + in$ and T^2 are the space of equivalent classes. Show that the group of holomorphic automorphisms of T^2 is $SL(2, \mathbf{Z})$ of 2 x 2 integer matrices of determinant 1.

5. Let $\{e_n\}$ be an orth-normal basis of l_2 of square integrable functions over a circle. Let $A : l_2 \to l_2, Ae_1 = 0, Ae_n = \frac{e_{n-1}}{n-1}, n > 1$ be a linear operator. Show that A is an compact operator and A has no eigenvectors. What are the spectrum of A?

6. If $M = [0,1]$ is the unit interval, the heat kernel on M can be written

$$p(x,y,t) = \Sigma_k \phi_k(x) \phi_k(y) e^{\lambda_k t},$$

where $\{\lambda_k\}$ is an enumeration of the eigenvalues of the $\Delta = \frac{d^2}{dx^2}$ on M and $\{\phi_k\}$ are the corresponding eigenfunctions which vanish on ∂M.

i) Calculate $\{\lambda_k\}$ and the corresponding eigenfunctions.
ii) Prove that $|p(x,y,t)| \le Ct^{-1/2}$, for all x, y, and $0 < t < 1$.
iii) What is the exponential rate of decay of $p(x,y,t)$ as $t \to \infty$, i.e. compute:

$$\lim_{t\to\infty} \log(p(x,y,t)).$$

S.-T. Yau College Student Mathematics Contests 2016

Applied Math. and Computational Math.

Individual (5 problems)

Problem 1. Consider the implicit leapfrog scheme

$$\frac{u_m^{n+1} - u_m^{n-1}}{2k} + a\left(1 + \frac{h^2}{6}\delta^2\right)^{-1}\delta_0 u_m^n = f_m^n$$

for the one-way wave equation

$$u_t + au_x = f.$$

Here δ^2 is the central second difference operator, and δ_0 is the central first difference operator.

(1) show that the scheme is of order $(2,4)$.
(2) show that the scheme is stable if and only if $|\frac{ak}{h}| < \frac{1}{\sqrt{3}}$.

Problem 2. A simple version of an enzyme-mediate chemical reaction process is given by

$$\mathrm{S} + \mathrm{E} \underset{k_2}{\overset{k_1}{\longleftrightarrow}} \mathrm{C} \overset{k_3}{\longrightarrow} \mathrm{P} + \mathrm{E}$$

where S is the substrate reactant and P is the concentration of the desired product. An enzyme (or catalyst) E is a compound whose special property is that it allows for intermediate reaction steps that lead to a the overall reaction,

$$\mathrm{S} \longrightarrow \mathrm{P}.$$

Assume the initial conditions

$$S(0) = S_0, \quad E(0) = E_0, \quad C(0) = 0, \quad P(0) = 0;$$

k_1, k_2, k_3 are reaction rate constants.

(a) Convert the chemical reaction equation into a system of rate equations (ODEs) for $S(T)$, $E(T)$, $C(T)$, and $P(T)$ where T is the dimensional time. Nondimensionalize the equations using the scalings

$$T = t/(k_1 E_0), \quad S(T) = S_0 s(t), \quad P(T) = S_0 p(t), \quad E(T) = E_0 s(t), \quad C(T) = E_0 c(t),$$

$$\epsilon = \frac{E_0}{S_0} \ll 1, \quad \lambda = \frac{k_2}{k_1 S_0}, \quad \mu = \frac{k_2 + k_3}{k_1 S_0}.$$

(b) Use the expansions $s(t) = s_0(t) + \epsilon s_1(t) + O(\epsilon^2)$, $c(t) = c_0(t) + \epsilon c_1(t) + O(\epsilon^2)$, etc to determine the equations for the leading order slow solution. Show that $s_0(t)$ and $p_0(t)$ satisfies the following Michaelis-Menten equations

$$\dot{s}_0(t) = -(\mu - \lambda)\frac{s_0}{\mu + s_0}, \quad \dot{p}_0(t) = (\mu - \lambda)\frac{s_0}{\mu + s_0}.$$

Problem 3. We say that a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ is *multiplicatively dependent* if there is a non-zero vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ for which

$$u_1^{k_1} \cdots u_n^{k_n} = 1. \tag{1}$$

This notion plays a very important role in many number theoretic algorithms, such as *factorisation* and *primality testing*. It also (in a more general form) appears in some questions in *algebraic dynamics*. However the algorithm to decide whether $\mathbf{u}$ is multiplicatively dependent is not immediately obvious. The following statement *informally* means that if $\mathbf{u}$ is multiplicatively dependent the exponents $k_1, \ldots, k_n$ can be chosen to be reasonably small. Prove that if $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ is multiplicatively dependent with $\|\mathbf{u}\|_\infty \le H$ where $\|\mathbf{u}\|_\infty = \max_{1 \le i \le n} |u_i|$, then there is a non-zero vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with

$$\|\mathbf{k}\|_\infty \le \left(\frac{2n \log H}{\log 2}\right)^{n-1}$$

(and hence for a fixed n it can be found in polynomial time of order $(\log H)^{n(n-1)}$).
Comment: To solve this problem, you can use the following statement (without proof) which *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \le n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \le H, \quad 1 \le i \le m, \ 1 \le j \le n.$$

Then there is an integer **non-zero** vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \le (2nH)^{n-1}$$

where $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$.

Problem 4. Consider a symmetric matrix $A_{n \times n}$, and let λ_i be a simple eigenvalue of A with

$$|\lambda_j - \lambda_i| = O(1), \quad j \ne i.$$

In inverse iteration of compute eigenvalue and eigenvector, one needs to solve the following linear system

$$(A - \mu I) y_{k+1} = x_k,$$

where μ is an approximation of eigenvalue λ_i, $\|x_k\| = 1$and obtain

$$x_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}.$$

However, for μ close to λ_i, $A - \mu I$ has a very small eigenvalue and the linear system will be ill-conditioned. So there may be large error in the numerical solution to the linear system, denoted by $\tilde{y}_{k+1}$. Even though we may get large error in $\tilde{y}_{k+1}$, the $\tilde{x}_{k+1}$ we get from $\tilde{x}_{k+1} = \frac{\tilde{y}_{k+1}}{\|\tilde{y}_{k+1}\|}$ is accurate.

(1) $\tilde{y}_{k+1}$ satisfies
$$(A - \mu I + \delta A)\tilde{y}_{k+1} = x_k,$$
where $\|\delta A\| = O(\epsilon)$ and ϵ is the machine precision. Show that
$$(A - \lambda_i)\frac{\tilde{y}_{k+1}}{\|\tilde{y}_{k+1}\|}\| \leq |\mu - \lambda_i| + \|\delta A\| + \frac{1}{\|\tilde{y}_{k+1}\|}.$$
(2) Let $\alpha_i = x_k^t q_i$, where q_i is the normalized eigenvector corresponding to λ_i. Show that
$$\|\tilde{y}_{k+1}\| \geq \frac{|\alpha_i|}{|\mu - \lambda_i| + \|\delta A\|}.$$
(3) Conclude that
$$\|x_{k+1} - (\pm)q_i\| = O(|\lambda_i - \mu| + \epsilon).$$

Problem 5. A function $f : \mathbb{R}^n \to \mathbb{R}$ in C^2 is called strongly convex if its Hessian matrix satisfies $\nabla^2 f \succeq mI$ for some $m > 0$. Show that the following statements are equivalent:

(a) f is strongly convex, i.e. $\nabla^2 f(x) \succeq mI$ for all $x \in \mathbb{R}^n$;
(b) For any $t \in [0, 1]$, any $x, y \in \mathbb{R}$,
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{m}{2}t(1-t)\|x - y\|^2;$$
(c) $\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq m\|x - y\|^2$ for any $x, y \in \mathbb{R}^n$.

S.-T. Yau College Student Mathematics Contests 2016

Applied Math. and Computational Math.

Team (5 problems)

Problem 1. For solving the following partial differential equation

$$u_t + u_x = 0, \qquad -\infty \le x \le \infty \tag{1}$$

with compactly supported initial condition, we consider the following one-step, three-point scheme on a uniform mesh $x_j = j\Delta x$ with spatial mesh size Δx:

$$u_j^{n+1} = au_j^n + bu_{j-1}^n + cu_{j-2}^n, \qquad j = \cdots, -1, 0, 1, \cdots \tag{2}$$

where a, b, c are constants which may depend on the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$. Here Δt is the time step, and u_j^n approximates the exact solution at $u(x_j, t^n)$ with $t^n = n\Delta t$.

(1) Find the constants a, b, c such that the scheme (2) is second order accurate.
(2) Find the CFL number λ_0 such that the scheme (2), with the constants determined by the step above, is stable in L^2 under the time step restriction $\lambda \le \lambda_0$.
(3) If the PDE (1) is defined on $(0, \infty)$ with an initial condition compactly supported in $(0, \infty)$ and a boundary condition $u(0, t) = g(t)$, how would you modify the scheme (2) so that it can be applied? Can you prove the stability and accuracy of your modified scheme?

Problem 2. Inverse problem. Answer the famous Mark Kac's equation: "can you hear the shape of drum?" for the special case.

Consider the one-dimensional oscillator $\ddot{x} = -u'(x)$ with symmetric potential $u(-x) = u(x)$, $u(0) = u'(0) = 0$, $u'(x) > 0$ for $x > 0$, $\lim_{x\to\infty} u(x) = \infty$. Denote the inverse function of $y = u(x)$, $x \ge 0$ as $x = u^{-1}(y) = \phi(y)$.

(a) For any solution $x(t)$, show there is a conservation of energy

$$\frac{\dot{x}^2(t)}{2} + u(x(t)) \equiv e$$

where e is a constant.

(b) For any energy $e > 0$, find a periodic solution with total energy e. Show that the period is given by

$$P(e) = 2\sqrt{2}\int_0^{x_{max}} \frac{dx}{\sqrt{e - u(x)}}, \quad x_{max} = \phi(e) > 0\,.$$

(c) Show that

$$\phi(z) = \frac{1}{2\pi\sqrt{2}}\int_0^z \frac{P(e)\,de}{\sqrt{z-e}}\,.$$

(d) In the case of iso-chronous $P(e) \equiv 2\pi$, show that $\phi(z) = \sqrt{2z}$. Then you have $u(x) = \frac{1}{2}x^2$, $x(t) = a\cos(t) + b\sin(t)$, the famous harmonic oscillator.

Problem 3. The following statement *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \leq n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \leq H, \quad 1 \leq i \leq m, \ 1 \leq j \leq n.$$

Prove that there is an integer **non-zero** vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \leq (2nH)^{n-1}$$

where $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

Problem 4. This problem considers an iterative scheme

$$x_{k+1} = x_k + \beta_k p_k$$

for the linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is a given $n \times n$ non-singular matrix and $b \in \mathbb{R}^n$ is a given vector. In the above scheme, x_k denotes the approximate solution at the k-th iteration, β_k is a scalar and $p_k \in \mathbb{R}^n$ is a search direction. If x_k is given, the above scheme will determine x_{k+1} so that the residual $r_{k+1} := b - Ax_{k+1}$ is the smallest possible with respect to the 2-norm.

(1) Determine β_k.
(2) Prove that the residual r_{k+1} is orthogonal to Ap_k with respect to the usual inner-product.
(3) Prove that the residuals satisfy
$$\|r_{k+1}\| \leq \|r_k\| \sin(\alpha)$$
where α is the angle between r_k and Ap_k, and $\|\cdot\|$ denotes the 2-norm.
(4) Assume that the inner product of r_k and Ap_k is non-zero. Will the above scheme always converge?
(5) Assume that A is positive definite. We take the search direction $p_k = r_k$. Show that the above scheme converges for any initial guess x_0.

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and in C^1. Suppose f has a local minimum x^*.

(1) Must this local minimum x^* be a global minimum?
(2) Consider the following backward gradient method: starting from any $x^0 \in \mathbb{R}^n$, define
$$x^k = x^{k-1} - t\nabla f(x^k), \quad k \geq 1,$$
where $t > 0$ is a fixed step size. Do you need any condition on t to guarantee $\{f(x^k)\}$ converge? Prove your convergence argument, if $\{f(x^k)\}$ converges.
(3) Suppose f is strongly convex, that is, $\exists m > 0$ such that $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq m\|x - y\|^2$. Under this additional condition, show that $\{x^k\}$ converges.

S.-T. Yau College Student Mathematics Contests 2016

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a compact odd-dimensional manifold with boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

2. Compute the de Rham cohomology of a punctured two-dimensional torus $T^2 - \{p\}$, where $p \in T^2$. If $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates $(x, y) \in \mathbb{R}^2$, then is the volume form $\omega = dx \wedge dy$ exact?

3. Let $M^n \to \mathbb{R}^{n+1}$ be a closed oriented hypersurface. The r-th mean curvature of M^n is defined by

$$H_r := \frac{1}{\binom{n}{r}} \sum_{i_1 < i_2 \cdots < i_r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}, \quad (1 \le r \le n)$$

where λ_i $(i = 1, \cdots, n)$ are principal curvatures of M^n. Prove that if all of λ_i are positive and H_r =constant for a certain r, then M^n is a hypersphere in $\mathbb{R}^{n+1}$.

4. State and prove the cut-off function lemma on a differentiable manifold.

5. Let M be a compact Riemannian manifold without boundary. Show that if M has positive Ricci curvature, then $H^1(M, \mathbb{R}) = 0$.

6. Let M be an orientable, closed and embedded minimal hypersurface in S^{n+1}. Denote by λ_1 the first eigenvalue for the Laplace-Beltrami operator on M. Prove that $\lambda_1 \ge n/2$.

S.-T. Yau College Student Mathematics Contests 2016

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Show that $\mathbb{CP}^{2n}$ does not cover any manifold except itself.

2. Let X be a topological space and $p \in X$. The *reduced suspension* ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and ΣX.

3. State and prove the Frobenius Theorem on a differentiable manifold.

4. Show that all geodesics on the sphere S^n are precisely the great circles.

5. Let M be an n-dimensional Riemannian manifold. Denote by R and K_M the curvature tensor and sectional curvature of M. If $a \le K_M \le b$ at a point $x \in M$, then, at this point,

$$R(e_1, e_2, e_3, e_4) \le \frac{2}{3}(b-a)$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_xM$.

6. Let M be a closed minimal hypersurface with constant scalar curvature in S^{n+1}. Denote by S the squared length of the second fundamental form of M. Show that $S = 0$, or $S \ge n$.

S.-T. Yau College Student Mathematics Contests 2016

Probability and Statistics

Individual (5 problems)

Problem 1. A random walker moves on the lattice $\mathbb{Z}^2$ according to the following rule: in the first step it moves to one of its neighbors with probability 1/4, and then in step $n > 1$ it moves to one of the neighbors that it didn't visit in the step $n-1$ with equal probability. Let T be the time when the random walker steps on a site that it already visited. Please show that the expectation of T is less than 35.

Problem 2. Let X be a $N \times N$ random matrix with i.i.d. random entries, and

$$\mathbb{P}(X_{11} = 1) = \mathbb{P}(X_{11} = -1) = 1/2$$

Define

$$\|X\|_{op} = \sup_{\mathbf{v} \in \mathbb{C}^N : \|\mathbf{v}\|_2 = 1} \|X\mathbf{v}\|_2$$

Please show that for any fixed $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}(\|X\|_{op} \geq N^{1/2+\delta}) = 0$$

Hint: $\|X\|_{op}^2 \leq \mathrm{tr}|X|^2$

Problem 3. Suppose that 2016 balls are put into 2016 boxes with each ball independently being put into box i with probability $\frac{1}{3\times 1008}$ for $i \leq 1008$ and $\frac{2}{3\times 1008}$ for $i > 1008$. Let T be the number of boxes containing exactly 2 balls. Please find the variance of T.

Problem 4. Let $b > a > 0$ be real numbers. Let X be a random variable taking values in $[a, b]$, and let $Y = \frac{1}{X}$. Determine the set of all possible values of $\mathbb{E}(X) \times \mathbb{E}(Y)$.

Problem 5. Let X_1, X_2, ... be independent and identically distributed real-valued random variables such that $\mathbb{E}(X_1) = -1$. Let $S_n = X_1 + \cdots + X_n$ for all $n \geq 1$, and let T be the total number of $n \geq 1$ satisfying $S_n \geq 0$. Compute $P(T = \infty)$.

S.-T. Yau College Student Mathematics Contests 2016

Probability and Statistics

Team (5 problems)

Problem 1. For a random walk process on the complete infinite binary tree (see Fig 1.) starting from root (i.e. level 0), we assume that the object moves to the neighbor nodes with equal probability. Let X_n denote the level number at time $= n$. Please prove that

$$\mathbb{E}X_n \leq 1/3n + 4/3$$

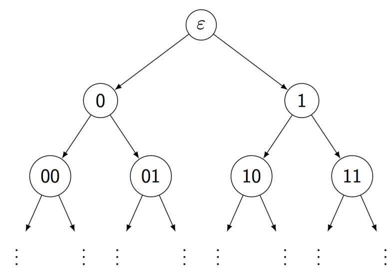


Fig 1.

Problem 2. The goal is to show the concentration inequality for the median of mean estimator. We divide the problem into three simple steps.

1. Let X be a random variable with $\mathbb{E}X = \mu < \infty$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Suppose we have m i.i.d. random samples $\{X_i\}_{i=1}^m$. Let $\hat{\mu}_m = \frac{1}{m}\sum_{i=1}^m X_i$ from X. Show that
$$P\big(|\widehat{\mu}_m - \mu| \geq 2\sigma\sqrt{\frac{1}{m}}\big) \leq \frac{1}{4}.$$

2. Given k i.i.d. Bernoulli random variables $\{B_j\}_{j=1}^k$ with $\mathbb{E}B_j = p < \frac{1}{2}$. Use the moment generating function of B_j, i.e., $\mathbb{E}(\exp(tB_j))$, to show that
$$P\big(\frac{1}{k}\sum_{j=1}^k B_j \geq \frac{1}{2}\big) \leq (4p(1-p))^{\frac{k}{2}}.$$

3. Suppose we have n i.i.d. random samples $\{X_i\}_{i=1}^n$ from a population with mean μ and variance σ^2. For any positive integer k, we randomly and uniformly divide all the samples into k subsamples, each having size $m = n/k$ (for simplicity, we assume n is always divisible by k). Let $\widehat{\mu}_j$ be the sample average of the j^{th}

subsample and $\widetilde{m}$ be the median of $\{\widehat{\mu}_j\}_{j=1}^k$. Apply the previous two results to show that

$$P\Big(|\widetilde{m}-\mu| \geq 2\sigma\sqrt{\frac{k}{n}}\Big) \leq \big(\frac{\sqrt{3}}{2}\big)^k.$$

Hint: Consider the Bernoulli random variable $B_j = \mathbb{1}\{|\widehat{\mu}_j - \mu| \geq 2\sigma\sqrt{\frac{k}{n}}\}$ for $j = 1, ..., k$.

Problem 3. (a) Let $N \geq 2$ be an integer, and let X be a random variable taking values in $\{0, 1, 2, \ldots\}$ such that $P\{X \equiv k \,(\operatorname{mod} N)\} = \frac{1}{N}$ for all $k \in \{0, 1, \ldots, N-1\}$. Compute $\mathbb{E}(e^{i(2\pi m)X/N})$ (with $i = \sqrt{-1}$) for all integers $m \geq 1$.

(b) A game for N players (numbered as 0, 1, 2, ..., $N-1$) is as follows: Each player independently shows a random number of fingers (uniformly chosen from $\{0, 1, 2, 3, 4, 5\}$); if S denotes the total number of fingers shown, then the player number $S \bmod N$ is declared to be the winner of the game.

Find all N such that the players have equal chance to win the game.

Problem 4. Let X_1, X_2, ... be independent and identically distributed real-valued random variables. Prove or disprove: If $\limsup_{n\to\infty} \frac{|X_n|}{n} \leq 1$ almost surely, then $\sum_{n=1}^{\infty} P(|X_n| \geq n) < \infty$.

Problem 5. Choose, at random, 2016 points on the circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into 2016 arcs. Compute the expected length of the arc that contains the point $(1, 0)$. How about the variance.

S.-T. Yau College Student Mathematics Contests 2017

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Let $\mathbb{Q}_p$ denote the field of p-adic numbers and let $\mathbb{Z}_p$ denote the ring of p-adic integers (p is a prime number).

(a) (5 points) Show that for every integer $k \geq 0$, $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$ as abelian groups.

(b) (5 points) Determine the endomorphism ring of the abelian group $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p$ ($k \geq 0$).

(c) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$.

(d) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}/\mathbb{Z}$.

Problem 2 (20 points)**.** Let A be a finite abelian group and let $\phi : A \to A$ be an endomorphism. Put

$$A_{\mathrm{nil}} := \{x \in A \mid \phi^k(x) = 0 \text{ for some } k \geq 1\}.$$

(a) (15 points) Show that there is a subgroup A_0 of A such that ϕ restricts to an automorphism of A_0 and $A = A_0 \oplus A_{\mathrm{nil}}$.

(b) (5 points) Show that such a subgroup is unique.

Problem 3 (20 points)**.** Let L/F be a Galois field extension, not necessarily finite. Let $x \in L$.

(a) (6 points) Show that the set $\mathcal{P}$ of subextensions of L/F not containing x has a maximal element E. Let K/E be a nontrivial finite extension contained in L. Show that $x \in K$.

(b) (6 points) Let K' be the Galois closure of K/E in L. Show that there exists $g \in G = \mathrm{Gal}(K'/E)$ such that $gx \neq x$.

(c) (8 points) Deduce that K/E is a cyclic Galois extension.

Problem 4 (20 points)**.** The goal of this problem is to prove the Chevalley–Warning theorem. Let p be a prime number and q a power of p.

(a) (8 points) Let $0 \leq a < q-1$ be an integer. Show that $S(X^a) := \sum_{x \in \mathbb{F}_q} x^a$ equals 0. Here we adopt the convention $x^0 = 1$ in $\mathbb{F}_q$ even for $x = 0$.

(b) (12 points) Let $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ be polynomials in n variables satisfying

$$\sum_{i=1}^{m} \deg(f_i) < n.$$

Show that $P = \prod_{i=1}^{m}(1 - f_i^{q-1})$ satisfies

$$S(P) := \sum_{(x_1,\ldots,x_n) \in \mathbb{F}_q^n} P(x_1, \ldots, x_n) = 0.$$

Deduce that p divides the cardinality of the set

$$V = \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid f_i(x_1, \ldots, x_n) = 0 \quad \forall i\}.$$

Problem 5 (20 points)**.** In this problem, all matrices are $n \times n$ with complex entries. Let U and V be matrices such that $UV \neq VU$. Assume that U is diagonalizable and commutes with VUV^{-1}.

(a) (10 points) For $\lambda, \mu \in \mathbb{C}$, let

$$E_{\lambda,\mu} = \{x \in \mathbb{C}^n \mid Ux = \lambda x, \quad VUV^{-1}x = \mu x\}.$$

Show that there exist couples $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$, satisfying $\lambda_i \neq \mu_i$ and $E_{\lambda_i,\mu_i} \neq 0$ for $i = 1, 2$.

(b) (10 points) For a matrix A, we define $\mathrm{N}(A) := \mathrm{tr}(A^*A)$, where $A^* = \bar{A}^T$ is the conjugate transpose of A. Assume that U and V are unitary (namely, $U^*U = V^*V$ is the identity matrix). Deduce that $\mathrm{N}(1 + V) \geq 4$.

S.-T. Yau College Student Mathematics Contests 2017

Algebra and Number Theory

Team

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Let G be a group and let $g \in G$ be an element of finite order n. Suppose that n is the product of two positive integers r and s which are coprime to each other.

(a) (10 points) Show that there is a pair (g_1, g_2) of elements of G such that $g_1^r = 1$, $g_2^s = 1$ and $g_1 g_2 = g_2 g_1 = g$.

(b) (10 points) Show that such a pair is unique.

Problem 2 (20 points)**.** Let $p > 2$ be a prime number and let T be a linear operator of order p on an n-dimensional vector space V over $\mathbb{Q}$.

(a) (8 points) Show that the trace $\mathrm{tr}(T)$ of T on V is an integer in $\mathbb{Z}$.

(b) (12 points) Show that $\mathrm{tr}(T)$ is congruent to n modulo p.

Problem 3 (20 points)**.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. For each $x \in \mathfrak{g}$, define a linear map

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y].$$

Put

$$n(x) := \text{the dimension of the kernel of the operator } (\mathrm{ad}_x)^{\dim \mathfrak{g}}.$$

The rank of $\mathfrak{g}$ is defined to be

$$\mathrm{rank}(\mathfrak{g}) := \min\{n(x) \mid x \in \mathfrak{g}\}.$$

Show that

$$\{x \in \mathfrak{g} \mid n(x) > \mathrm{rank}(\mathfrak{g})\}$$

is the zero set of a polynomial function on $\mathfrak{g}$.

Problem 4 (20 points)**.** Let p be a prime number and let $K = \mathbb{F}_p(T)$ be the field of rational functions over $\mathbb{F}_p$. Consider the polynomials

$$f(X) = X^p - TX - T, \quad g(X) = X^{p-1} - T.$$

(a) (5 points) Show that f and g are irreducible and separable over K.

(b) (5 points) Let M be the splitting field of g over K. Show that $\mathrm{Gal}(M/K)$ is isomorphic to $\mathbb{F}_p^\times$.

(c) (10 points) Let L be the splitting field of f over K. Show that g splits in L and $\mathrm{Gal}(L/K)$ is isomorphic to the semidirect product $G = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$, where $\mathbb{F}_p^\times$ acts on $\mathbb{F}_p$ by homotheties.

Problem 5 (20 points)**.** Let a and b be a pair of coprime positive integers. Let $\mathbb{N}(a,b)$ be the set of nonnegative integral linear combinations of a and b:

$$\mathbb{N}(a,b) := \{N \in \mathbb{Z} \mid N = ax + by \text{ for some } x, y \in \mathbb{Z}_{\geq 0}\}.$$

(a) (7 points) Show that every integer N satisfying $N \geq a(b-1)$ belongs to $\mathbb{N}(a,b)$.

(b) (13 points) Show that there are exactly $(a-1)(b-1)/2$ positive integers **not** belonging to $\mathbb{N}(a,b)$.

S.-T. Yau College Student Mathematics Contests 2017

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose f is integrable on $[-\pi, \pi]$, prove that $\Sigma_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}$ tends to $f(x)$ for a.e. x, as $r \to 1, r < 1$. Here $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

2. Let H be a Hilbert space equipped with an inner product $(.,.)$ and a norm $||.|| = (.,.)^{\frac{1}{2}}$. A sequence $\{f_k\}$ is converge to $f \in H$ if $||f_k - f|| \to 0$. A sequence $\{f_k\} \subset H$ is said converge weakly to $f \in H$ if $(f_k, g) \to (f, g)$ for any $g \in H$. Prove the following statements:

a) $\{f_k\}$ converges to f if and only if $||f_k|| \to ||f||$ and $\{f_k\}$ converges weakly to f.

b) If H is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.

3. Let $f : \mathbf{C}/\{0\} \to \mathbf{C}$ be a holomorphic function and

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{1/2}},$$

for z near 0. Determine all such functions.

4. Find a conformal mapping which maps the region $\{z||z - i| < \sqrt{2}, |z + i| < \sqrt{2}\}$ onto the unit disk.

5. If E is a compact set in a region Ω, prove that there exists a constant $M > 0$, depending only on E and Ω, such that every positive harmonic function $u(z)$ in Ω satisfies $u(z_2) \leq M u(z_1)$ for any two points $z_1, z_2 \in E$.

6. 1) For any bounded domain $\Omega \subset \mathbf{R}^n$, there exists a smallest constant $C(\Omega)$, such that

$$\int_{\Omega} |u|^2 dx \leq C(\Omega) \int_{\Omega} \Sigma_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^2 dx$$

for every function $u \in H_0^1(\Omega) = \overline{C_0^{\infty}(\Omega)} \subset H^1(\Omega)$, where $C_0^{\infty}(\Omega)$ is the space of smooth functions over Ω and vanishing on boundary of Ω and $H^1(\Omega)$ is the Banach space of functions $u \in L^2(\Omega), \nabla u \in L^2(\Omega)^{\otimes n}$ with the norm:

$$||u||_{H^1(\Omega)}^2 = ||u||_{L^2(\Omega)}^2 + ||\nabla u||_{L^2(\Omega)^{\otimes n}}^2$$

$$= \int_{\Omega} (|u|^2 + \Sigma_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^2) dx.$$

$H_0^1(\Omega)$ is the completion of $C^{\infty}(\Omega)$ in $H^1(\Omega)$ with the above norm.

2) Let $\Pi = \{(x, y)|0 < x < a, 0 < y < b\}$, show that $C(\Pi) \geq \frac{a^2 b^2}{\pi^2 (a^2 + b^2)}$.

S.-T. Yau College Student Mathematics Contests 2017

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Suppose that f is an integrable function on the circle with $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = 0$, for all $n \in \mathbf{Z}$. Then $f(\theta) = 0$ whenever f is continuous at the point θ.

2. Let F be continuous on $[a, b]$. Show that:

1) Suppose $(D^+F)(x) = \limsup_{h\to 0, h>0} \frac{F(x+h)-F(x)}{h} > 0$ for every $x \in [a, b)$. Then F is increasing on $[a, b]$.

2) If $F^{'}(x)$ exists for every $x \in (a, b)$ and $|F^{'}(x)| \leq M$, then $|F(x) - F(y)| \leq M|x - y|$ and F is absolutely continuous.

3. Let $f(z)$ be holomorphic for $|z| < 1$ and assume $|f(z)| < 1$, if $|z| < 1$. If α and β are such that $|\alpha| < 1, |\beta| < 1, \alpha \neq \beta$, and $f(\alpha) = f(\beta) = 0$, then

$$|f(z)| \leq |\frac{z-\alpha}{1-\bar{\alpha}z}\frac{z-\beta}{1-\bar{\beta}z}|, |z| < 1.$$

4. Show that a single-valued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of $\int \frac{dz}{\sqrt{1-z^2}}$ over a closed curve in the region?

5. In $\mathbf{R}^n$, consider the Laplace equation

$$u_{11} + u_{22} + ... + u_{nn} = 0.$$

Show that the equation is invariant under orthogonal transformation. Find all rotationally symmetric solutions to this equation.

6. Let A be an operator over the function space $C[a, b]$ defined as:

$$(Af)(x) = \int_a^b K(x, y)f(y)dy,$$

where $K(x, y)$ is continuous over $[a, b] \times [a, b]$ except at several curves defined by $y = \phi_k(x), x \in [a, b], k = 1, ..., n$ with ϕ_k continuous. Show that A is a compact operator.

S.-T. Yau College Student Mathematics Contests 2017

Applied Math. and Computational Math.

Individual (5 problems)

1. The Chebyshev polynomial of the first kind is defined on $[-1,1]$ by

$$T_n(x) = \cos(n \arccos x).$$

Prove: The envelope for the extremals of $T_{n+1}(x) - T_{n-1}(x)$ forms an ellipse.

2. Consider a fixed point iteration

$$x_n = g(x_{n-1}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose this fixed point method does converge to a fixed point x^*. The Steffensen algorithm is an acceleration method to find x^* which reads

$$\hat{x}_n = x_{n-2} - \frac{(x_{n-1} - x_{n-2})^2}{x_n - 2x_{n-1} + x_{n-2}}.$$

or

$$x_{n+1} = G(x_n)$$

where

$$G(x) = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}.$$

(a) Show that the Steffensen algorithm $\{x_k\}$ converges quadratically.

(b) Can you extend this method to two dimensions?

3. We consider a piecewise smooth function

$$f(x) = \begin{cases} f_1(x), & x \leq 0, \\ f_2(x), & x > 0 \end{cases}$$

where $f_1(x)$ is a C^∞ function on $(-\infty, 0]$ and $f_2(x)$ is a C^∞ function on $[0, \infty)$, but $f_1(0) \neq f_2(0)$. Suppose $p(x)$ is a k-th degree polynomial ($k \geq 1$) interpolating $f(x)$ at $k+1$ equally-spaced grid points x_j, $j = 0, 1, 2, \cdots, k$ with $x_i < 0 < x_{i+1}$ for some i between 0 and $k-1$. Prove that, when the grid size $h = x_{j+1} - x_j$ is small enough, $p'(x) \neq 0$ for $x_i \leq 0 \leq x_{i+1}$, that is, $p(x)$ is monotone in the interval $[x_i, x_{i+1}]$. (**Hint: first prove the case when $f_1(x) = c_1$, $f_2(x) = c_2$ and $c_1 \neq c_2$ are two constants.**)

4. Let $b \in \mathbb{R}^n$. Suppose $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$ are two $n \times n$ matrices. Let A to be non-singular.

(a) Consider the iterative scheme: $Ax^{k+1} = b - Bx^k$.
State and prove the necessary and sufficient condition for the iterative scheme to converge.

(b) Suppose the spectral radius of $A^{-1}B$ satisfies $\rho(A^{-1}B) = 0$. Prove that the iterative scheme converges in n iterations.

(c) Consider the following iterative scheme:

$$x^{(k+1)} = \omega_1 x^{(k)} + \omega_2(c_1 - Mx^{(k)}) + \omega_3(c_2 - Mx^{(k)}) + ... + \omega_k(c_{k-1} - Mx^{(k)})$$

where M is symmetric and positive definite, $\omega_1 > 1$, $\omega_2, ..., \omega_k > 0$ and $c_1, ..., c_{k-1} \in \mathbb{R}^n$. Deduce from (a) that the iterative scheme converges if and only if all eigenvalues of M (denote it as $\lambda(M)$) satisfies:

$$(\omega_1 - 1)/(\sum_{i=2}^{k} \omega_i) < \lambda(M) < (\omega_1 + 1)/(\sum_{i=2}^{k} \omega_i).$$

(d) Let A be non-singular. Now, consider the following system of iterative scheme (*):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (*) to converge.

For the iterative scheme (**):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k+1)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (**) to converge. Compare the rate of convergence of the iterative schemes (*) and (**).

5. Consider the differential equation

$$-u'' + \alpha u = f, \; x \in (0, 1).$$

Here, prime denotes for d/dx and α is a constant. We consider a mixed boundary condition

$$u(0) = 0, \; u'(1) - bu(0) = 0.$$

This equation is approximated by a standard finite difference method:

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{h^2} + \alpha U_j = f_j, j = 1, ..., N - 1.$$

Here, N is the number of grid points, $h = 1/N$ is the mesh size, U_j is the approximate solution at $x_j := jh$, and $f_j = f(x_j)$. The noundary condition is approximated by

$$U_0 = 0, \; \frac{U_N - U_{N-1}}{h} - bU_N = 0.$$

The resulting linear system is $AU = F$ with

$$\begin{bmatrix} \beta & -1 & 0 & \cdots & & & \\ -1 & \beta & -1 & \cdots & & & \\ & & & \ddots & & & \\ & & & & -1 & \beta & -1 \\ & & & & 0 & -1 & 1-bh \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix} = \begin{bmatrix} h^2 f_1 \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-1} \\ 0 \end{bmatrix}$$

where $\beta = 2 + \alpha h^2$.

$$u_t + a u_x = 0, \ \ a > 0.$$

We discretize this PDE by For solving the following partial differential equation

(1) $$u_t + f(u)_x = 0, \qquad 0 \le x \le 1$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

(2) $$\frac{d}{dt}u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N,$$

with periodic boundary condition

(3) $$u_0 = u_N,$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(a) Prove the following L^2 stability of the scheme

(4) $$\frac{d}{dt}E(t) \le 0$$

where $E(t) = \sum_{j=1}^{N} |u_j|^2 \Delta x$.

(b) Do you believe (4) is true for $E(t) = \sum_{j=1}^{N} |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counterexample.

S.-T. Yau College Student Mathematics Contests 2017

Applied Math. and Computational Math.

Team (5 problems)

1. Given an integer parameter K, one can test whether for a vector $\vec{u} = (u_1, \ldots, u_n) \in \mathbf{N}^n$ there is non-zero vector $\vec{k} = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ with

$$\|\vec{k}\|_\infty \leq K \text{ and } u_1^{k_1} \cdots u_n^{k_n} = 1,$$

where $\|\vec{k}\|_\infty = \max_{1 \leq i \leq n} |k_i|$ in about $O((2K+1)^n)$ arithmetic operations with integers having about $O(nK \log(\|\vec{u}\|_{\infty+1}))$ bits via testing all possible combinations of the exponents (ordered lexicographically) and direct computation.
Assuming that the memory is essentially unlimited, suggest a better algorithm which uses about $O((2K+1)^{n/2})$ arithmetic operations with integers of the same size as above.
Hints: (i) Use the divide-and-conquer strategy; (ii) Recall that a list L of M real numbers can be sorted via $O(M \log M)$ comparisons; (iii) A sorted list L of M real numbers can be searched for $x \in L$ via $O(\log M)$ comparisons; (iv) To decide whether $a/b > c/d$ for two rational numbers with $b, d > 0$ we simply compare the products ad and bc.

2. We have the following partial differential equation

$$u_t = H(u)_{xx}, \qquad 0 \leq x < 1 \tag{1}$$

with an initial condition $u(x, 0) = f(x)$ and periodic boundary condition. Here $0 \leq H'(u) \leq d$. Consider the following one-step, three-point scheme on a uniform mesh $x_j = j\Delta x$ with spatial mesh size Δx:

$$u_j^{n+1} = u_j^n + aH(u_{j-1}^n) + bH(u_j^n) + cH(u_{j+1}^n), \tag{2}$$

where a, b, c are constants which may depend on the mesh ratio $\mu = \frac{\Delta t}{\Delta x^2}$, Δt is the time step, and u_j^n approximates the exact solution at $u(x_j, t^n)$ with $t^n = n\Delta t$.

(1) Find the constants a, b, c such that the scheme (2) is second order accurate.
(2) Find the CFL number μ_0 such that the scheme (2), with the constants determined by Step 1 above, is stable under the time step restriction $\mu \leq \mu_0$. Please specify which norm you are using for stability, and prove this stability result.

3. Consider the problem describing projectile motion on the surface of the Earth, written in physical variables as follows:

$$\frac{d^2y}{dt^2} = -\frac{GM}{(R+y)^2}, \quad y(0) = 3\text{m}, \quad y'(0) = -V \text{ m/sec}$$

Let $y(t) = L\tilde{y}(\tilde{t})$ and $t = T\tilde{t}$. Consider two out of the following four cases:

(a): $R = O(1)$, $V \to \infty$, $M = O(1)$: the fast projectile limit
(b): $R = O(1)$, $V = O(1)$, $M \to \infty$: the dense Earth limit
(c): $R = O(1)$, $V = O(1)$, $M \to 0$: the light Earth limit
(d): $R \to 0$, $V = O(1)$, $M = O(1)$: the small Earth limit (two possible scalings, determine both)

In each case:

- Choose your scalings for L, T to normalize as many terms as possible. Pick your scalings so that the time it takes for the projectile to fall to $\tilde{y} = 0$ is $\tilde{t} = O(1)$.
- Write the scaled (normalized) problem, identify all remaining dimensionless parameters.
- Identify a limiting small parameter and the leading order problem.

Note: DO NOT solve-out the problems, just write them!
Hint: If any scaled coefficients blow-up in the leading order problem, the scaling is not good.

4. Let f be an arbitrary function in $C^n(\mathbb{R})$. Given n distinct points $x_1, x_2, \cdots, x_n \in \mathbb{R}$ and an extra point $x_0 \in \mathbb{R}$, we want to approximate $f'(x_0)$ using a linear combination of the function values at $x_1, x_2, \cdots, x_n$, i.e. we want to compute the coefficients $c_1, c_2, \cdots, c_n$ such that

$$f'(x_0) \approx c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n)$$

in some sense.

(a) Consider the undetermined coefficients method. You use Taylor expansion to expand each $f(x_i), i = 1, 2, \cdots, n$ about the point x_0,

$$f(x_i) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x_0)(x_i - x_0)^k + \frac{1}{n!} f^{(n)}(\xi_i)(x_i - x_0)^n, \quad \xi_i \in [x_0, x_i] \text{ or } [x_i, x_0],$$

and choose the coefficients so that the resulting approximation is as accurate as possible. This gives you the linear system

$$\frac{1}{k!} \sum_{i=1}^{n} c_i (x_i - x_0)^k = \delta_{k,1}, \quad k = 0, 1, \cdots, n-1,$$

where $\delta_{k,1} = 1$ if $k = 1$; otherwise $\delta_{k,1} = 0$. Explain why this linear system is nonsingular. Then use this method to solve the case when $n = 3$, $x_1 = x_0$, $x_2 = x_0 + h$, $x_3 = x_0 + 2h$ for some constant h.

(b) You can also make use of interpolation method. Consider the n-point interpolating polynomial

$$p(x) = \sum_{i=1}^{n} \Big(\prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \Big) f(x_i),$$

and the approximation is given by

$$f'(x_0) \approx p'(x_0).$$

This gives the coefficients as

$$c_i = \Big(\prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \Big)'\Big|_{x=x_0}.$$

Show that the approximation of this method is exact if f is a polynomial of degree no more than $n-1$.

(c) Show that the two methods given in (a) and (b) are essentially the same, i.e. the coefficients obtained in (a) and (b) are the same.

5. Let

$$\mathcal{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & \gamma \end{bmatrix}$$

Find $\mathcal{A}^{-1}$ explicitly. Show that all entries of $\mathcal{A}^{-1}$ are nonnegative if and only if $\gamma \geq 1$.

S.-T. Yau College Student Mathematics Contests 2017

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero. Show that M admits a nowhere vanishing vector field.

2. Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f : S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f : S^2 \to S^2$ is a map of degree d_i. Compute the integral homology groups of $(S^2 \vee S^2) \cup_f D^3$. Here D^3 is the unit solid ball with boundary S^2.

3. Let X and Y be smooth vector fields on a smooth manifold. Prove that the Lie derivative satisfies the identity

$$L_X Y = [X, Y].$$

4. State and prove the Liouville formula for the geodesic curvature κ_g along a regular curve on a smooth surface in $\mathbb{R}^3$.

5. On a Riemannian manifold, let F be the set of smooth functions f on M with $|\mathrm{grad} f| \leq 1$. For any x, y in the manifold, show that

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in F\}.$$

6. Let M be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by K_M the sectional curvature of M. Prove that if $K_M > \dfrac{p-1}{2p-1}$, then M is the great sphere S^n.

S.-T. Yau College Student Mathematics Contests 2017

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1] \times [-1,1])/((1,y) \sim (-1,-y))$. Determine the fundamental group of X.

2. If $f : X \to X$ is a self-map, then the "mapping torus of f" is the quotient

$$T_f := (X \times [0,1])/(x,0) \sim (f(x),1), \quad \forall x \in X.$$

For $n \in \mathbb{Z}$, let f_n be a degree n map on S^3. Compute the integral homology groups of T_{f_n}.

3. Let C be a regular curve on a smooth surface S in R^3. Denote by $I = Edu^2 + 2Fdudv + Gdv^2$ and $II = Ldu^2 + 2Mdudv + Ndv^2$ the first and second fundamental forms of S, respectively. Assume that the equation of C is given by $u = u(s)$, $v = v(s)$, where s is the arc-length parameter of C. Show that geodesic torsion along the curve C satisfies

$$\tau_g = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \left(\dfrac{dv}{ds}\right)^2 & -\dfrac{du}{ds}\dfrac{dv}{ds} & \left(\dfrac{du}{ds}\right)^2 \\ E & F & G \\ L & M & N \end{vmatrix}.$$

4. Let $\{e_i\}_{i=1,\ldots,n}$ be a basis of a vector space V. Denote by $\{\omega^i\}_{i=1,\ldots,n}$ the dual basis of $\{e_i\}_{i=1,\ldots,n}$. Show that the set

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$

is a basis of $\bigwedge^r V^*$, where r is a positive integer and $r \le n$.

5. Let $M \to R^{n+1}$ be a compact closed hypersurface in the $(n+1)$-dimensional Euclidean space R^{n+1}. Prove that M is a hypersphere if M has constant scalar curvature and nonnegative Ricci curvature.

6. On a Riemannian manifold, if f is a smooth function such that $|\mathrm{grad} f| = 1$. Show that the integral curves of $\mathrm{grad} f$ are geodesics.

S.-T. Yau College Student Mathematics Contests 2017

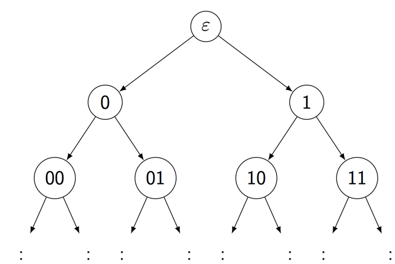
Probability and Statistics

Individual (5 problems)

Problem 1. A box contains 750 red balls and 250 blue balls. Repeatedly pick a ball uniformly at random from the box and remove it until all remaining balls have a single color. (Note: no replacement).

Please find integer m such that the expectation value for the total number of the remaining balls $\in [m, m+1]$

Problem 2. Suppose a number $X_0 \in \{1, -1\}$ at the root of a binary tree



is propagated away from the root as follows. The root is the node at level 0. After obtaining the 2^h numbers at the nodes at level h, each number at level $h+1$ is obtained from the number adjacent to it (at level h) by flipping its sign with probability $p \in (0, 1/2)$ independently.

Let X_h be the average of the 2^h values received at the nodes at level h. Define the *signal-to-noise ratio* at level h to be

$$R_h := \frac{\big(\mathbb{E}[X_h \mid X_0 = 1] - \mathbb{E}[X_h \mid X_0 = -1]\big)^2}{Var[X_h | X_0 = 1]}.$$

Find the threshold number p_c such that R_h converges to 0 if $p \in (p_c\,,\ 1/2)$ and diverges if $p \in (0,\ p_c)$, as $h \to \infty$.

Problem 3. Consider the space representing an infinite sequence of coin flips, namely $\Omega := \{H, T\}^\infty$, (H: head, T: tail) with the associated σ-field $\mathcal{F}$ generated by finite dimensional rectangles. For $0 \leq p \leq 1$, denote by $\mathbb{P}_p$ the probability measure on $(\Omega, \mathcal{F})$ corresponding to flipping a coin an infinite number of times with probability of H being p and probability of T being $q = 1 - p$ at each flip.

Show that for each $p \in [0, 1]$, there exists A_p such that

$$\mathbb{P}_p(A_p) > 1/2$$

and for any $p' \neq p$, $p' \in [0,1]$

$$\mathbb{P}_{p'}(A_p) < 1/2$$

Problem 4. Let $G := G(n,p)$ be a random graph with n vertices where each possible edge has probability p of existing. The existence of the edges are independent to each other. With G, we say $A \subset \{1, 2, \cdots, n\}$ is a fully connected set if and only if

$$i, j \in A \implies i-th \text{ and } j-th \text{ vertices are (directly) connected with an edge in } G$$

Define T as the size of the largest fully connected set

$$T := \max\{|A| : \text{ A is a fully connected set}\}$$

Let's fix $p \in (0,1)$, please prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T}{2\log_{\frac{1}{p}} n} \leq 1+\epsilon\right) = 1, \quad \forall \epsilon > 0,$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T}{\sqrt{2\log_{\frac{1}{p}} n}} \geq 1-\epsilon\right) = 1, \quad \forall \epsilon > 0,$$

Hint:

$$\mathbb{P}(T = n) = p^{\binom{n}{2}} = p^{n(n-1)/2}$$

Problem 5. Consider a population of constant size $N+1$ that is suffering from an infectious disease. We can model that spread of the disease as Markov process. Let $X(t)$ be the number of healthy individuals at time t and suppose that $X(0) = N$. We assume that if $X(t)$

$$\lim_{h\to 0} \frac{1}{h}\mathbb{P}\left(X(t+h) = n-1 | X(t) = n)\right) = \lambda n \left(N+1-n\right)$$

For $0 \leq s \leq 1$, $0 \leq t$, define

$$G(s,t) := \mathbb{E}(s^{X(t)})$$

Please find a non-trivial partial differential equation for $G(s,t)$, which involves $\partial_t G$.

S.-T. Yau College Student Mathematics Contests 2017

Probability and Statistics

Team (5 problems)

Problem 1. Let μ^n be the uniform probability measure on the n-dimensional cube $[-1,1]^n$. Let $H \in \mathbb{R}^n$ be the hyperplane orthogonal to the principal diagonal, i.e., $H = (1, \cdots, 1)^{\perp}$. For any $r > 0$, we further define

$$A_{H,r} := \{x \in [-1,1]^n, \mathrm{dist}(x, H) \leq r\},$$

where $\mathrm{dist}(x, H)$ represents the distance from the point x to the hyperplane H. Show that for any constant $\varepsilon > 0$, the following two estimates hold for all sufficiently large n

$$1: \quad \mu^n\big(A_{H,n^\varepsilon}\big) \geq 1 - n^{-2\varepsilon}, \qquad\qquad 2: \quad \mu^n\big(A_{H,n^\varepsilon}\big) \geq 1 - e^{-n^{\varepsilon/2}}$$

Problem 2. Let X_1, X_2, $\ldots$ be positive random variables. We assume that X_n converges to 0 in probability, and that $\lim_{n\to\infty} E(X_n) = 2$. Prove that $\lim_{n\to\infty} E(|X_n - 1|)$ exists and compute its value.

Problem 3. There are n people playing a game. Initially everybody had one dollar at hand. During each round of the game, we randomly pick two people and they will toss a fair coin, to decide who wins this round of the game. The loser will submit one dollar (note: just one, not all of his money) to the winner. Assume that a person who had no money at hand will be immediately driven out of the game. The game stops until all money is at the hand of only one person. Calculate the average number of rounds that the game plays.

Note: In each round only two players are involved.

Problem 4.

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{X'_n\}_{n\in\mathbb{N}}$ be two independent simple random walks on $\mathbb{Z}^d$ such that $X_0 = X'_0 = 0$. Here simple walk means if $x, y \in \mathbb{Z}^d$ and $\|x - y\| = 1$, then

$$\mathbb{P}(X_{n+1} = y | X_n = x) = (2d)^{-1}$$

Let $\mathcal{I} = \{(s,t) : X_s = X'_t\}$. Prove that $|\mathcal{I}| < \infty$ a.s.

Hint: You can first prove that

$$\mathbb{P}(X_n = 0) = O(n^{-d/2}), \quad n \to \infty$$

Problem 5. Suppose $X_1, \ldots, X_n$ are i.i.d. Poisson variables with mean λ and we are interested in estimating $p = P_\lambda(X_i = 0) = e^{-\lambda}$.

(a) One estimator for p is the proportion of zeros in the sample, $\tilde{p} = \#\{i \leq n : X_i = 0\}/n$. Determine limiting distribution for $\sqrt{n}(\tilde{p} - p)$.

(b) Another estimator would be the maximum likelihood estimator $\hat{p}$. Give a formula for $\hat{p}$ and determine limiting distribution for $\sqrt{n}(\hat{p} - p)$.

(c) Find the asymptotic relative efficiency of $\tilde{p}$ with respect to $\hat{p}$.

S.-T. Yau College Student Mathematics Contests 2018

Algebra and Number Theory
Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.**

(a) (6 points) Show that if $2^k - 1$ is a prime for some integer $k \geq 1$, then k is a prime.

(b) (6 points) Show that if $2^k + 1$ is a prime for some integer $k \geq 1$, then k is a power of 2.

(c) (8 points) Prove the following theorem of Goldbach: for integers $i, j \geq 0$ with $i \neq j$, the integers $2^{2^i} + 1$ and $2^{2^j} + 1$ are coprime.

Problem 2 (20 points)**.** Let $K = \mathbb{Q}(\sqrt[3]{5})$ and let L be the Galois closure of K.

(a) (6 points) Prove that L has a unique subfield M satisfying $[M : \mathbb{Q}] = 2$. Prove that every prime number $p \equiv 1 \pmod 3$ splits in M.

(b) (6 points) Determine all prime numbers which are *ramified* in L.

(c) (8 points) Let $p \geq 7$ be a prime number. Let f_p be the inertia degree of a prime ideal of the ring of integers $\mathcal{O}_L$ of L above p. Recall that 5 is called a *cubic residue* mod p if $x^3 \equiv 5 \pmod p$ has a solution in $\mathbb{F}_p$. Prove the following decomposition law in L.

  (i) If $p \equiv 1 \pmod 3$ and 5 is a cubic residue mod p, then p splits completely in L.

  (ii) If $p \equiv 1 \pmod 3$ and 5 is *not* a cubic residue mod p, then $f_p = 3$.

  (iii) If $p \equiv 2 \pmod 3$, then 5 is a cubic residue and $f_p = 2$.

Problem 3 (20 points)**.** Prove that every group of order 99 is abelian.

Problem 4 (20 points)**.** Let K be a field and let V be a finite-dimensional K-vector space.

(a) (6 points) Assume that K is infinite. Show that V is not the union of finitely many proper linear K-subspaces.

(b) (6 points) Assume that K is finite and V is non-zero. Let S be the set of affine hyperplanes of V. Let $g\colon V \to \mathbb{R}$ be a function. The Radon transform $Rg\colon S \to \mathbb{R}$ is defined by $(Rg)(\xi) = \sum_{x\in\xi} g(x)$ for $\xi \in S$. Show that $Rg = 0$ implies $g = 0$.

(c) (8 points) Let $v_1, \ldots, v_n, w_1, \ldots, w_n \in V$. Assume that for every K-linear map $f\colon V \to K$, $(f(v_1), \ldots, f(v_n))$ and $(f(w_1), \ldots, f(w_n))$ coincide up to permutation of the indices. Deduce that $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ coincide up to permutation of the indices. Here we make no assumptions on K.

Problem 5 (20 points)**.** Let p be a prime number and let $v_p(\cdot)$ denote the p-adic valuation on $\mathbb{Q}_p$. Let $A = (a_{ij})_{1\le i,j\le n} \in \mathrm{M}_n(\mathbb{Q}_p)$ be an $n \times n$ matrix with entries in $\mathbb{Q}_p$. Assume that we know the following:

(1) $A^2 = p^{n+1} \cdot I_{n\times n}$;

(2) $v_p(a_{ij}) \ge i$ for all i, j.

Prove that $v_p(a_{ij}) \ge \max\{i, n+1-j\}$ and $a_{i,n+1-i} \in p^i\mathbb{Z}_p^\times$, i.e.

$$A \in \begin{pmatrix} p^n\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p & \cdots & p^3\mathbb{Z}_p & p^2\mathbb{Z}_p & p\mathbb{Z}_p^\times \\ p^n\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p & \cdots & p^3\mathbb{Z}_p & p^2\mathbb{Z}_p^\times & p^2\mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p & \cdots & p^3\mathbb{Z}_p^\times & p^3\mathbb{Z}_p & p^3\mathbb{Z}_p \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ p^n\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p^\times & \cdots & p^{n-2}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p & p^{n-2}\mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p^\times & p^{n-1}\mathbb{Z}_p & \cdots & p^{n-1}\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p & p^{n-1}\mathbb{Z}_p \\ p^n\mathbb{Z}_p^\times & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \cdots & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \end{pmatrix}.$$

Hint. Consider the antidiagonal matrix

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & p \\ 0 & 0 & \cdots & p^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & p^{n-1} & \cdots & 0 & 0 \\ p^n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

S.-T. Yau College Student Mathematics Contests 2018

Algebra and Number Theory

Team

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Recall that a ring E is said to be *local* if for every $u \in E$, at least one of the elements u and $1-u$ is invertible. Let R be a ring and let M be an R-module.

(a) (8 points) Show that if $\mathrm{End}_R(M)$ is a local ring, then M is indecomposable.

(b) (12 points) Assume M indecomposable and of finite length. Prove the Fitting lemma: Every endomorphism u of M is either invertible or nilpotent. Deduce that $\mathrm{End}_R(M)$ is a local ring.

Problem 2 (20 points)**.**

(a) (6 points) Let $n \geq 2$ be an integer. Show that there exists an integer m with $1 \leq m \leq n-1$ such that the binomial coefficient $\binom{n}{m}$ satisfies $\binom{n}{m} \geq 2^n/n$.

(b) (6 points) Let $0 \leq m \leq n$ be integers with $n \geq 1$. Show that for every prime number p,

$$v_p\left(\binom{n}{m}\right) \leq \log_p(n)$$

Here v_p is the p-adic valuation: $v_p(p^a b) = a$ for integers b prime to p and $a \geq 0$.

(c) (8 points) Let $n \geq 2$ be an integer and let $\pi(n)$ denote the number of prime numbers $p \leq n$. Deduce the following inequality of Chebyshev:

$$\pi(n) \geq \frac{n}{\log_2 n} - 1.$$

Problem 3 (20 points)**.** Let $n \geq 1$ be an integer and let $\Phi_n(X) \in \mathbb{Q}[X]$ denote the n-th cyclotomic polynomial, i.e.

$$\Phi_n(X) := \prod_{\xi}(X - \xi),$$

where ξ runs through primitive n-th roots of unity in $\mathbb{C}$. Recall that $X^n - 1 = \prod_{d|n} \Phi_d(X)$ and $\Phi_n(X)$ belongs to $\mathbb{Z}[X]$. Let p be a prime number such that $p \nmid n$. Denote by $\overline{\Phi}_n$ the residue class of Φ_n in $\mathbb{F}_p[X]$. Prove the following statements:

(a) (8 points) The roots of $\overline{\Phi}_n = 0$ in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ are exactly the *primitive* n-th roots of 1 in $\overline{\mathbb{F}}_p$.

(b) (12 points) $\overline{\Phi}_n$ is irreducible in $\mathbb{F}_p[X]$ if and only if $(\mathbb{Z}/n\mathbb{Z})^\times$ is a cyclic group generated by the class of p.

Problem 4 (20 points)**.** Let G be a finite group. Let V be a finite-dimensional complex representation of G and let $\chi\colon V \to \mathbb{C}$ be the associated character.

(a) (8 points) Show that there exists a subfield $L \subseteq \mathbb{C}$ containing the image of χ such that $L/\mathbb{Q}$ is a finite Galois extension. Show moreover that

$$B(\chi) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} \prod_{g \in G} \sigma(\chi(g))$$

belongs to $\mathbb{Z}$.

(b) (12 points) Suppose that χ is irreducible and $\dim(V) \geq 2$. Show that there exists $g \in G$ with $\chi(g) = 0$. (*Hint.* One may apply the inequality of arithmetic and geometric means to $|B(\chi)|^2$.)

Problem 5 (20 points)**.** Let F be a field, V an F-vector space of dimension d and $W \subseteq V$ a subspace. Let $f\colon W \to V$ be an F-linear map. Assume that the only subspace $W' \subseteq W$ such that $f(W') \subseteq W'$ is $\{0\}$.

(a) (6 points) Let $v \in V$ be a non-zero vector. Show that there exists a unique integer $k(v) \geq 0$ such that $v, f(v), f^2(v), \dots, f^{k(v)-1}(v) \in W$ but $f^{k(v)}(v) \notin W$. Show moreover that $v, f(v), \dots, f^{k(v)}(v)$ are linearly independent over F.

(b) (14 points) Prove that given $\lambda_1, \dots, \lambda_d \in F$, there exists an F-linear extension of f to $\tilde{f}\colon V \to V$ such that the characteristic polynomial of $\tilde{f}$ is $\prod_{i=1}^{d}(\lambda - \lambda_i)$. (*Hint.* You may first treat the special case $V = \bigoplus_{i=0}^{k(v)} F f^i(v)$. For the general case, consider the subset $W_n \subseteq V$ of vectors $v \in V$ with $k(v) \geq n$ or $v = 0$.)

S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Individual

Please solve the following 6 problems.

1. Let the sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $L^2(\mathbf{R}^d)$ satisfy that $\|f_n\|_{L^2} = 1$.

(1)Show that there exists a subsequence of function $\{f_{n_j}\}_{j=1}^{\infty}$ such that f_{n_j} converges weakly to some function f in $L^2(\mathbf{R}^d)$, i.e.

$$(f_{n_j}, g) \to (f, g)$$

for all $g \in L^2(\mathbf{R}^d)$.

(2)If $f_n \to f$ weakly in $L^2(\mathbf{R}^d)$, and $\|f_n\|_{L^2} \to \|f\|_{L^2}$ as $n \to \infty$. Show that $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.

2. Let $f : U \to \mathbf{C}$ be a non-constant holomorphic function where $U \subset \mathbf{C}$ is the open set containing the closure $\overline{D}$ of the unit disk $D = \{z \in \mathbf{C} | |z| < 1\}$.

If $|f(z)| = 1$, for all $z \in \partial D$, Prove that $D \subset f(\overline{D})$.

3. Prove that if a sequence of harmonic function on the open disk converges uniformly on compact subset of the disk, then the limit is harmonic.

4. Let μ be a Borel measure on $\mathbf{R}^n$. Let $\rho > 0$, a fixed positive number, and $B_\rho(x) = \{y \in \mathbf{R}^n | d(x, y) < \rho\}$. For $x \in \mathbf{R}^n$, define a function:

$$\theta(x) : x \to \mu(\overline{B_\rho(x)})$$

1) Show that θ is upper semi-continuous, i.e. for every $x \in \mathbf{R}^n, \theta(x) \geq \limsup_{y \to x} \theta(y)$.

2) Give an example of a Borel measure μ, such that the function θ is not continuous.

5. Let g denote a smooth function on $\mathbf{R}^n$ with compact support. Let f denote the function given by the formula

$$f(x) = \frac{1}{n(n-1)\alpha(n)} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-2}} g(y) dy.$$

Here $\alpha(n)$ is volume of the unit ball in $\mathbf{R}^n$.

(a) Prove that the integral that defines f converges for each $x \in \mathbf{R}^n$.

(b) Prove that f is differentiable and that the gradient of f if given by the formula

$$\nabla f|_x = \frac{1}{n(n-1)\alpha(n)} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-2}} (\nabla g)|_y dy.$$

(c) Prove the f obeys $-\Delta f = g$ with Δ denoting $\dfrac{\partial^2}{\partial {x_1}^2} + \cdots + \dfrac{\partial^2}{\partial {x_n}^2}$.

6. If u is a positive harmonic function on $\mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$, then exist constants $a \geq 0, b \geq 0$ such that

$$u(x) = a + b|x|^{2-n}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Team

Please solve the following 5 problems.

1. Suppose $\{f_n\}_{n=1}^{\infty} \in L^2(\mathbf{R})$ is a sequence that converges to 0 in the L^2 norm .
Prove that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

2. Let $\hat{f}(\xi) = \int e^{-ix\xi} f(x)dx$ be the Fourier transform on Schwartz function $f \in S(\mathbf{R})$. Suppose $f \in S(\mathbf{R})$ satisfies $f(2\pi n) = 0$ and $\hat{f}(n) = 0$ for all integers n. Prove that $f = 0$.

3. If f is integrable on $\mathbf{R}^d$, then

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)dy = f(x),$$

for a.e. x, B is an open ball centered at x.

4. Let $C[0,1] = \{f : [0,1] \to \mathbf{R} | \ f \text{ is continuous}\}$ be the space of continuous function on $[0,1]$. Let $\rho(f,g) = \int_0^1 |f(x) - g(x)|dx$ be a metric on $[0,1]$.
Show that $(C[0,1], \rho)$ is not a complete metric vector space.
Construct a complete metric vector space $(W, \widetilde{\rho})$ such that $i : (C[0,1], \rho) \hookrightarrow (W, \widetilde{\rho})$ is an isometric embedding such that $\widetilde{\rho}|_{C[0,1]} = \rho$, $\overline{C[0,1]} = W$.

5. Let Ω be a simply connected domain in $\mathbf{C}$. Consider a point $z_0 \in \Omega$ and solve the Dirichlet problem in Ω with the boundary values $\log|\zeta - z_0|$. The solution is denoted by $G(z, z_0)$ and let $g(z, z_0) = G(z, z_0) - \log|z - z_0|$. Let $w = f(z) : \Omega \to D_1 = \{z||z| < 1\}$ be the one to one surjective conformal mapping with $f(z_0) = 0$. Show that
1) $g(z, z_0) = -\log|f(z)|$.
2) $g(z, z_0) = g(z_0, z)$. (Hint: Let $g(z, z_1) = g_1, g(z, z_2) = g_2$, calculate the integral $g_1 * dg_2 - g_2 * dg_1$ over the cycle $\partial\Omega - c_1 - c_2$, where c_1, c_2 are small circles around $z_1, z_2, du = u_x dx + u_y dy, *du = -u_y dx + u_x dy$.)

S.-T. Yau College Student Mathematics Contests 2018

Applied Math. and Computational Math.

Individual (5 problems)

1. We consider the following convection-diffusion equation

$$u_t + au_x = bu_{xx}, \qquad 0 \le x < 1 \tag{1}$$

with an initial condition $u(x,0) = f(x)$ and periodic boundary condition, where a and $b > 0$ are constants. The first order IMEX (implicit-explicit) time discretization and second order central spatial discretization are used to give the following scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \tag{2}$$

with a uniform mesh $x_j = j\Delta x$ with spatial mesh size Δx and time step Δt. Here u_j^n is the numerical solution approximating the exact solution of (1) at $x = x_j$ and $t = n\Delta t$. Prove that the scheme is L^2 stable under the very mild time step restriction

$$\Delta t \le c \tag{3}$$

with a constant c which is independent of Δx. Can you determine the dependency of c on the two constants a and b in (1)?

2. Velocity-Verlet method.

(a) Recast the following Newtonian formula for the acceleration and potential force

$$q''(t) = -\nabla V(q),$$

into a Hamiltonian system and show that the corresponding map on the phase space is symplectic.

(b) Show that the velocity-Verlet (recovered many times: Delambre 1791, Størmer in 1907, Cowell & Crommelin 1909, Verlet 1960s) method

$$\begin{aligned} p_{n+1/2} &= p_n - \frac{\Delta t}{2}\nabla V(q_n); \\ q_{n+1} &= q_n + \Delta t p_{n+1/2}; \\ p_{n+1} &= p_{n+1/2} - \frac{\Delta t}{2}\nabla V(q_{n+1}) \end{aligned}$$

is symplectic and is second order accurate.

Hint: Let $u(t) = (p(t), q(t))$ be a solution of the Hamiltonian system with initial data $u_0 = (p_0, q_0)$ and we view the solution $u(t)$ as a map map on the phase space $\varphi_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ $\varphi_t(u_0) = u(t)$. We call the flow map is symplectic *if its Jacobian*

$$\Phi_t(u_0) = \frac{\partial \varphi_t(u_0)}{\partial u_0} = \begin{pmatrix} \frac{\partial p(t)}{\partial p_0} & \frac{\partial p(t)}{\partial q_0} \\ \frac{\partial q(t)}{\partial p_0} & \frac{\partial q(t)}{\partial q_0} \end{pmatrix}$$

satisfies $\Phi_t(u_0)^{\mathrm{T}} J \Phi_t(u_0) = J$ for any $u_0 \in \mathbb{R}^d \times \mathbb{R}^d$. Here $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

A scheme $\varphi_n(u_0), n = 1, 2 \ldots,$ is symplectic if the map $\varphi_n(u_0)$ is symplectic.

3. We begin with some definitions.

(1) A graph G is a pair $G = (V, E)$ where V is a finite set, called the vertices of G, and E is a subset of $P_2(V)$ (*i.e.*, a set E of (unordered) two-element subsets of V), called the edges of G. A simple graph G is a graph without loops (edge that connects a vertex to itself) or multiple edges between any pair of vertices. The order of the graph is $|V|$. We often put $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_iv_j |\ v_i \text{ and } v_j \text{ are adjacent}\}$.

(2) Two vertices x and y are adjacent if $xy \in E$. The neighborhood of a vertex x, denoted by $N_G(x)$ or $N(x)$, is the set of vertices that is adjacent to x. The degree of a vertex x, denoted by $d_G(x)$ or $d(x)$, is $|N(x)|$ (*i.e.* the number of vertices that is adjacent to x).

(3) A path is a collection of distinct vertices $v_{i_1}v_{i_2}\cdots v_{i_k}$ such that $v_{i_j}v_{i_{j+1}} \in E$ for all j, $1 \le j < k$. v_{i_1} and v_{i_k} are the ends of the path. A Hamiltonian path P is a path containing all vertices of the graph. A cycle is a closed path with $v_{i_1} = v_{i_k}$. A Hamiltonian cycle is a cycle containing all vertices of the graph. A graph is called Hamiltonian if it has a Hamiltonian cycle.

(4) A graph G is (Hamilton) connected, if for every pair of vertices there is a (Hamiltonian) path between them.

An example of a simple graph: $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3,\ v_3v_4, v_2v_4\}$. In this graph, the order of the graph is 4, $N(v_1) = \{v_2\}$, $N(v_4) = \{v_2, v_3\}$, $d(v_3) = 2$, $d(v_2) = 3$ and $v_1v_2v_4v_3$ is a Hamiltonian path with ends v_1 and v_3.

Let G be a simple graph of order n. Suppose that the degree sum of any pair of nonadjacent vertices is at least $n+1$. Show that G is Hamilton-connected (*i.e.* between any pair of vertices x and y, there is a Hamiltonian path in which x and y are the ends).

4. Define the Hermite polynomials as

$$H_n(x) = (-1)^n \exp(\frac{x^2}{2})\frac{d^n}{dx^n}[\exp(-\frac{x^2}{2})], \quad x \in (-\infty, +\infty),\ n = 0, 1, 2, \cdots . \tag{4}$$

(a) Prove the weighted orthogonality of the Hermite polynomials:

$$\langle H_n(x), H_m(x)\rangle_\rho \triangleq \int_{-\infty}^{+\infty} \rho(x)H_n(x)H_m(x)dx = n!\sqrt{2\pi}\delta_{n,m}, \tag{5}$$

where $\rho(x) = \exp(-\frac{x^2}{2})$.

(b) Prove the three recurrence formula:

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \quad n \ge 1, \tag{6}$$

and then show that for all $n \ge 1$, $H_n(x)$ and $H_{n-1}(x)$ share no common roots.

(c) Use the recurrence formula and induction to prove the differential relation:

$$\frac{d}{dx}H_n(x) = nH_{n-1}(x), \quad n \ge 1, \tag{7}$$

and then prove that H_n is an eigenfunction of the following eigenvalue problem

$$xu'(x) - u''(x) = \lambda u. \tag{8}$$

You need to find the eigenvalue λ_n corresponding to $H_n(x)$.

5. Take $\sigma_i(A)$ to be the i-th singular value of the square matrix $A \in \mathbb{R}^{n\times n}$. Define the *nuclear norm* of A to be

$$\|A\|_* \equiv \sum_{i=1}^{n} \sigma_i(A).$$

(1) Show that $\|A\|_* = \operatorname{tr}(\sqrt{A^T A})$.
(2) Show that $\|A\|_* = \max_{X^T X = I} \operatorname{tr}(AX)$.
(3) Show that $\|A + B\|_* \leq \|A\|_* + \|B\|_*$
(4) Explain informally why minimizing $\|A - A_0\|_{\mathrm{F}}^2 + \|A\|_*$ over A for a fixed $A_0 \in \mathbb{R}^{n\times n}$ might yield a low-rank approximation of A_0.

Notation: The trace of a matrix $\operatorname{tr}(A)$ is the sum $\sum_i a_{ii}$ of its diagonal elements. We define the square root of a symmetric positive semidefinite matrix M to be $\sqrt{M} \equiv UD^{1/2}U^T$, where $D^{1/2}$ is the diagonal matrix containing (nonnegative) square roots of the eigenvalues of M and U contains the eigenvectors of $M = UDU^T$.

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Applied Math. and Computational Math.

Team (5 problems)

1. Let H be a bipartite graph with the bipartition $V = V_1 \cup V_2$, where $|V_1| = |V_2| = n$. We say that H satisfies the (p, q)-condition if (i) for all subsets $I \subseteq V_1$ of cardinality at most p, the inequality $|I| \le |N(I)|$ holds, and (ii) for all subsets $J \subseteq V_2$ of cardinality at most q, the inequality $|J| \le |N(J)|$ holds. Note that the $(n, 0)$-condition is Hall's original condition in his marriage theorem.

Prove that if H satisfies the (p, q)-condition with $n \le p + q$, then H contains a matching of size n.

2. Let C_n be the n dimensional hypercube, i.e., the graph whose vertex set V is $\{0,1\}^n$, and whose edges are defined by: two vertices $u = u_1u_2 \ldots u_n$ and $v = v_1v_2 \ldots v_n$ are adjacent iff $u_i \ne v_i$ for exactly one $i \in [n]$. Let $\mathbb{R}[V]$ be the vector space of all the functions $f : V \to \mathbb{R}$. The space $\mathbb{R}[V]$ has a natural inner product. For $f, g \in \mathbb{R}[V]$,

$$< f, g >= \sum_{u \in \{0,1\}^n} f(u)g(u).$$

The standard basis of $\mathbb{R}[V]$ is the set $\{f_u : u \in \{0,1\}^n\}$ where $f_u(v) = \delta_{u,v}$, the Kronecker delta, for $u, v \in \{0,1\}^n$. Denote by B_1 the standard basis.

(1) For any two vertices $u, v \in \{0,1\}^n$, $u \cdot v$ is defined to be $\sum_i u_i v_i$. For each $u \in \{0,1\}^n$, define a function $\chi_u \in \mathbb{R}[V]$ by letting

$$\chi_u(v) = (-1)^{u \cdot v}.$$

Prove that the set $\{\chi_u : u \in \{0,1\}^n\}$ is orthogonal with respect to the inner product of $\mathbb{R}[V]$, i.e.,

$$< \chi_u, \chi_v >= \delta_{u,v} 2^n,$$

for all $u, v \in \{0,1\}^n$.

(2) Prove that the set $\{\chi_u : u \in \{0,1\}^n\}$ forms a basis of the vector space $\mathbb{R}[V]$. Denoted by B_2 this basis.

(3) For $1 \le i \le n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0,1\}^n$ where the only 1 occurs in position i. Let $S = \{e_1, e_2, \ldots, e_n\}$.

Define a linear transformation $\Phi : \mathbb{R}[V] \to \mathbb{R}[V]$ as follows. For $f \in \mathbb{R}[V]$, Φf is the element in $\mathbb{R}[V]$ which is given by

$$(\Phi f)(v) = \sum_{e_i \in S} f(v + e_i)$$

where $v + e_i$ is the usual vector addition modulo 2.

Prove that the matrix of Φ with respect to the standard basis B_1 is just $A(C_n)$, the adjacency matrix of the hypercube C_n.

(4) Prove that $\Phi\chi_u = \lambda_u \chi_u$ for each $u \in \{0,1\}^n$, where

$$\lambda_u = \sum_{e\in S} (-1)^{u\cdot e} = n - 2|u|,$$

where $|u|$ is the number of 1's in $u = u_1 u_2 \dots u_n$.

(5) Compute the eigenvalues of the matrix $A(C_n)$.

3. Let $A \in \mathbb{R}^{n\times n}$, and assume that there are unitary matrix Q and diagonal matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ such that $A = QDQ^*$. Let E_k be the space spanned by the first k columns of Q. We let

$$\widehat{P} = \begin{pmatrix} I_k & \\ & 0 \end{pmatrix}, \quad P = Q\widehat{P}Q^*$$

where I_k is the $k \times k$ identity matrix.

(1) Show that P is an orthogonal projection onto E_k.

(2) Assume that

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|.$$

Let $X^{(0)} \in \mathbb{R}^{n\times k}$ and assume $PX^{(0)}$ is injective. We define the iterations

$$X^{(m+1)} = AX^{(m)}.$$

Show that there is a matrix $\Lambda \in \mathbb{R}^{k\times k}$ such that

$$\frac{\|(AX^{(m)} - X^{(m)}\Lambda)y\|}{\|PX^{(m)}y\|} \leq \Big(\frac{|\lambda_{k+1}|}{|\lambda_k|}\Big)^m \frac{\|(AX^{(0)} - X^{(0)}\Lambda)y\|}{\|PX^{(0)}y\|}, \quad \forall y \in \mathbb{R}^k\backslash\{0\}.$$

4. For the one-way equation

$$u_t + a u_x = f, \tag{1}$$

consider the multistep scheme given by

$$\frac{3u_m^{n+1} - 4u_m^n + u_m^{n-1}}{2k} + a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = f_m^{n+1}. \tag{2}$$

(1) Show that the scheme is second order accurate.

(2) Show that the scheme is unconditionally stable.
(Hint: (i) apply von Neumann analysis to the scheme with $f \equiv 0$ and find the characteristic polynomial. (ii) show that for all k, h, the characteristic polynomial satisfies the root condition: all roots reside in the unit disk, and all roots on the unit circle are simple. (iii) for a root r of the characteristic polynomial, it would be more convenient to study the form $\frac{1}{r} = X + iY$ and prove that $X^2 + Y^2 \geq 1$.)

5. For a convex function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is convex and open, define a subgradient of f at $x_0 \in D$ to be any vector $s \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) \geq s \cdot (x - x_0)$$

for all $x \in D$. The subgradient is a plausible choice for generalizing the notion of a gradient at a point where f is not differentiable. The subdifferential $\partial f(x_0)$ is the set of all subgradients of f at x_0.

(1) What is $\partial f(0)$ for the function $f(x) = |x|$.

(2) Suppose we wish to minimize a convex and continuous function $f : \mathbb{R}^n \to \mathbb{R}$, which may not differentiable everywhere. Propose an optimality condition involving subdifferential for a point x_* to be a minimizer of f. Show that your condition holds if and only if x_* is a globally minimizer f.

(3) The *subgradient method* extends the gradient descent to a wider class of functions. Analogously to the gradient descent, the subgradient method performs the iteration

$$x_{k+1} = x_k - \alpha g_k,$$

where $\alpha > 0$ is small stepsize that is known as the learning rate, and g_k is *any* subgradient of f at x_k. This method might not decrease f in each iteration, so instead we keep track of the best iterate we have seen so far, x_k^{best}.

In the following parts, assume that f is Lipschitz continuous with constant $L > 0$, $\|x_1 - x_*\|_2 \leq B$ for some $B > 0$. Under these assumptions we will show that

$$\lim_{k\to\infty} f(x_k^{\text{best}}) \leq f(x_*) + \frac{L^2}{2}\alpha, \tag{3}$$

a bound characterizing convergence of the subgradient method.

(a) Derive an upper bound for the error $\|x_{k+1} - x_*\|_2^2$ of x_{k+1} in terms of $\|x_k - x_*\|_2^2$, g_k, α, $f(x_k)$ and $f(x_*)$.

(b) By recursively applying the result from Problem 3a, provide an upper bound for $\|x_{k+1} - x_*\|_2^2$.

(c) Incorporate $f(x_k^{\text{best}})$ into your upper bound in Problem 3b, and take a limit as $k \to \infty$ to obtain the desired convergence result (3).

(d) Suggest a best choice of the learning rate α.

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Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$, and let $M\#N$ denote their connect sum.

(a) Compute the fundamental group of $M\#N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).

(b) Compute the homology groups of $M\#N$.

(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

2. Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

3. Classify all vector bundles over the circle S^1 up to isomorphism.

4. Suppose C is a regular curve in the unit sphere S^2. For any point $W \in S^2$, there exists the only oriented great circle S_W (determined by the right hand rule) in S^2 such that W is the pole of S_W. Denote by $n(W)$ the number of points at which the oriented great circle S_W and C intersect. Prove the Crofton formula

$$\iint_{S^2} n(W)dW = 4L,$$

where dW and L is the area element of S^2 and the length of C, respectively.

5. Let M be an n-dimensional closed submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Prove the following inequality

$$\int_M H^n dV \geq vol(S^n),$$

where H and dV is the mean curvature (i.e., norm of the mean curvature vector) and the volume element of M, and S^n is the standard unit sphere of dimension n.

6. Let M be an even dimensional compact and oriented Riemannian manifold with positive sectional curvature. Show that M is simply connected.

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Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map

$$S^2 \to S^2 \vee S^2$$

which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

2. (a) Let A be a single circle in $\mathbb{R}^3$. Compute the fundamental group $\pi_1(\mathbb{R}^3 - A)$.

(b) Let A and B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half space, respectively. Compute $\pi_1(\mathbb{R}^3 - (A \cup B))$.

3. Consider the differential 1-form $\omega = xdy - ydx + dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

4. Show that

$$Q^n := \{(x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} (x^i)^4 = 1\}$$

is a differentiable manifold.

5. Let M be a closed surface in $\mathbb{R}^3$. Prove that

$$\int_M |K| d\sigma \geq 4\pi(1 + g),$$

where K, g and $d\sigma$ is the Gaussian curvature, the genus and the area element of M, respectively.

6. Let M be an n-dimensional compact and simply connnected Riemannian manifold. If the sectional curvature K_M of M satisfies

$$\frac{1}{4} < K_M \leq 1,$$

then M is homeomorphic to S^n.

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Probability and Statistics

Individual (5 problems)

Problem 1. A submarine is lost in some ocean. There are two (and only two) possible regions: A and B. Experts estimate the probability of being lost in A is 70%. On the other hand, for each search, the probability of finding this submarine is 40% if it is lost in A. This number is 80% for region B. Now we have independently searched region A 4 times and region B once, but still have not found the submarine yet. Now based on these informations, which region we should search next? And why?

Problem 2. A teacher and 12 students sit around a circle. In the beginning the teach holds a gift, he will randomly pass it to the left person or right person next to him, so as the other students each time. (For the gift, It is like a random walk between these people) The rule is that the gift will be eventually given to some student (not teacher) if he/she

is the last student who ever touches the gift.

Which student(s) have the highest probability to get this gift (i.e., win) ?

Problem 3. In a party, N people attend, each of them brings k gifts. When they leave, each of them randomly picks k gifts. Let X be the total number of gifts which are taken back by their owners. Let's fix k, please find the limiting distribution of X when $N \to \infty$.

Problem 4. Suppose that a random vector $\mathbf{x} = (x_1, ..., x_n)' \in R^n (n \geq 2)$ is distributed as a multivariate normal distribution $N(\mathbf{0}, \Sigma)$ with the following joint probability density function

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} det(\Sigma)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}\right\}, \ \mathbf{x} \in R^n,$$

where Σ is an $n \times n$ positive definite matrix. Let the (i, j) element of $\Omega = \Sigma^{-1}$ be ω_{ij} $(1 \leq i, j \leq n)$. For $1 \leq i \neq j \leq n$, show that if $\omega_{ij} = 0$, then x_i and x_j are conditionally independent when the other elements of $\mathbf{x}$ are given.

Problem 5. Let$\mathbf{x}, \mathbf{y}$ be two independent random vectors in R^n $(n \geq 3)$. Assume that $P(\mathbf{y} = \mathbf{0}) = 0$ and $\mathbf{x}$ has a standard multivariate normal distribution, i.e., $\mathbf{x} \sim N(0, I_n)$.

(a) For any nonzero constant vector $\mathbf{a} \in R^n$ satisfying $||\mathbf{a}|| = (\mathbf{a}'\mathbf{a})^{1/2} = 1$, prove that

$$\sqrt{n-1}\frac{\mathbf{a}'\mathbf{x}}{\sqrt{||\mathbf{x}||^2 - (\mathbf{a}'\mathbf{x})^2}} \sim t_{n-1},$$

here t_{n-1} stands for a t distribution with $n - 1$ degrees of freedom.

(b) The sample correlation coefficient between $\mathbf{x} = (x_1, ..., x_n)'$ and $\mathbf{y} = (y_1, ..., y_n)'$ is defined as

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}.$$

where $\bar{x} = \sum_{i=1}^n x_i/n, \bar{y} = \sum_{i=1}^n y_i/n$. Show that $\sqrt{n-2}\frac{r}{\sqrt{1-r^2}} \sim t_{n-2}$.

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Probability and Statistics

Team (5 problems)

Problem 1. Let X_i, $1 \leq i \leq N$ be i.i.d. random variables. Here X_1 is uniformly distributed on $[0,1]$. We reorder them as

$$\widetilde{X}_1 \leq \widetilde{X}_2 \leq \cdots \widetilde{X}_N$$

a) Let $N = 2m-1$, and $Y = \widetilde{X}_m$, please find the A and B such that

$$\frac{Y-A}{N^B}$$

has nontrivial distribution, and please find this distribution.

b) Let $N = 2m$, and $Y = \widetilde{X}_m - \widetilde{X}_{m-1}$, please find the A and B such that

$$\frac{Y-A}{N^B}$$

has nontrivial distribution, and please find this distribution.

Problem 2. Let $\mathbf{X} = (\mathbb{Z}_2)^{\mathbb{N}}$, i.e., $\mathbf{X} = (X_1, X_2 \cdots, X_N \cdots)$, $X_i \in (0,1)$. It can be considered as countable lightbulbs. 0 means off, 1 means on. We start with $\mathbf{X}_0 = \mathbf{0}$. Keep generating independent geometric random variables, whose distribution are $geom(1/2)$. Denote them as K_1, $K_2 \cdots$. Now let $\mathbf{X}_m$ (for $m \geq 1$) be as follows

$$(\mathbf{X}_m - \mathbf{X}_{m-1})_k = \mathbf{1}(k = K_m), \qquad \mathbb{Z}_2$$

i.e, in the $m-th$ turn, we only change the status of the K_m−th light bulb. Then what is the probability of all lights being off again, i.e.,

$$\mathbb{P}\left(\exists m > 1, \quad \mathbf{X}_m = \mathbf{0}\right)$$

Problem 3. Let $x_1, x_2, \ldots, x_n$ be d-dimensional vectors of real numbers with n sufficiently large but the exact value is not of importance.

A function of μ is defined to be

$$\ell(\mu) = \sup\{\sum_{i=1}^{n} \log p_i : \sum_{i=1}^{n} p_i x_i = \mu; \sum_{i=1}^{n} p_i = 1, p_1 > 0, \ldots, p_n > 0\}$$

on the space of the interior of the convex hull of $x_1, \ldots, x_n$.

(a) Show that this is a concave function of μ on the convex hull.

(b) Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$. Let $\mathbf{a}$ be a vector of length d. Prove that $\ell(\bar{x} + t\mathbf{a})$ is a decreasing function of t when $t > 0$.

Problem 4. Consider the histogram estimator, defined as follows. We observe *iid* random variables $X_1, \ldots, X_n$, taking values in $[0,1]$ according to the distribution with PDF f (assuming it is sufficiently smooth). Define bins

$$B_1 = \left[0, \frac{1}{m}\right), B_2 = \left[\frac{1}{m}, \frac{2}{m}\right), \ldots, B_m = \left[\frac{m-1}{m}, 1\right]$$

Let $h = 1/m$, v_j be the number of observations in bin B_j, and define $\hat{p}_j = v_j/n$ and $p_j = \int_{B_j} f(u)du$. Then the histogram estimator of the density f is

$$\hat{f}_n(x) = \sum_{j=1}^{m} \frac{\hat{p}_j}{h} I\{x \in B_j\}$$

1. Find the (exact) mean and variance of $\hat{f}_n(x)$.
2. Explain why increasing the number of bins decreases the bias of $\hat{f}_n(x)$.
3. If our goal is to minimize the mean-squared error

$$MSE = E\left[\int (f(x) - \hat{f}_n(x))^2 dx\right],$$

please give some advice on how to choose m.

Problem 5. Let $X_i \sim N(\theta_i, 1)$ independently for $i = 1, \ldots, k$. We are interested in estimating $\tau = \theta_1^2 + \cdots + \theta_k^2$ given observations $X_1, \ldots, X_k$.

1. A possible estimator of τ is $\tilde{\tau} = \sum_{i=1}^{k} X_i^2 - k$. Show that it is unbiased and compute its sampling variance.
2. Now assume the proper prior $\theta_i \sim N(0, A)$, independently for $i = 1, \ldots, k$ and a given $A > 0$. Since A is unknown, please provide an estimator $\hat{A}$ of A and also derive the empirical Bayes estimator of τ, denoted as $\hat{\tau}_B$. (Hint: $\hat{\tau}_B = E(\tau \mid X_1, \ldots, X_k, \hat{A})$).
3. How do you compare the two estimators, $\tilde{\tau}$ and $\hat{\tau}_B$?

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Algebra and Number Theory

Individual (5 problems)

1) Let G be a finite group. Assume that for any representation V of G over a field of characteristic zero, the character χ_V takes value in $\mathbb{Q}$. Assume g is an element in G such that $g^{2019} = 1$.

   Prove that g and g^{19} are conjugate in G.

2) Let p be a prime number, and let $\mathbb{F}_p$ be the finite field with p elements. Let $F = \mathbb{F}_p(t)$ be the field of rational functions over $\mathbb{F}_p$. Consider all subfields C of F such that F/C is a finite Galois extension.

   1. Show that among such subfields, there is a smallest one C_0, i.e., C_0 is contained in any other C.
   2. What is the degree of F/C_0 ?

3) Let $R \subset R'$ be an integral extension of commutative rings. Let $\mathfrak{p}'$ be a prime ideal of R'. Prove that $\mathfrak{p}'$ is a maximal ideal of R' if and only if $\mathfrak{p}' \cap R$ is a maximal ideal of R.

4)
   1. Prove that $\mathrm{GL}_n(\mathbb{C})$ is path-connected.
   2. Let
   $$X = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^m = \mathrm{Id}\},$$
   Describe the path-connected component of X and prove your answer.

5) The Fibonacci sequence is defined by

   $$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

   Let p be a prime number.

   1. Show that if $p \equiv 1, 4 \pmod 5$, then p divides F_{p-1}.
   2. Let $\mathbb{F}_{p^2}$ be the finite field of p^2 elements. Show that the norm map $N : \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$ is surjective, and deduce the cardinality of the kernel of N.
   3. Show that if $p \equiv 2, 3 \pmod 5$, then p divides F_{p+1}.

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Algebra and Number Theory

Team (5 problems)

1) Let S_n be the group of permutations of $\{1, 2, ..., n\}$. Let $\sigma \in S_n$ be the permutation

$$(1, n)(2, n-1) \cdots (k, n-k+1) \cdots .$$

Prove that the centraliser $Z_{S_n}(\sigma)$ is isomorphic to $S_{[\frac{n}{2}]} \ltimes (\mathbb{Z}/2\mathbb{Z})^{[\frac{n}{2}]}$.

2) Recall that the algebra of regular functions on a vector space W is the symmetric algebra of linear forms on W.

Let $V = \mathbb{C}^2$. Let $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]$ be the algebra of regular functions on $\mathrm{End}_{\mathbb{C}}(V)$. The natural action of the group $G = SL_2(\mathbb{C})$ on V induces an action of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$ by left and right multiplication. Thus we get an action of $G \times G$ on $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]$.

Compute the algebra of fixed points $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]^{G \times G}$.

3) Let R be a Noetherian ring and $I \subset R$ be an ideal. Define the Rees algebra as

$$Rees(I, R) := \bigoplus_{n \geqslant 1} I^n t^n \subset R[t].$$

Prove that $Rees(I, R)$ is Noetherian.

4) Let p be a odd prime. Let Φ_p be the p-th cyclotomic field, *i.e.*, $\Phi_p = \mathbb{Q}(\zeta_p)$ where ζ_p is a primitive p-th root of unity.

   1. Show that $\Phi_p/\mathbb{Q}$ is a Galois extension with Galois group $(\mathbb{Z}/p\mathbb{Z})^\times$.
   2. Deduce that Φ_p contains a unique quadratic extension of $\mathbb{Q}$.
   3. Write g_p for the Gauss sum $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a$. Show that
      - $\overline{g_p} = \left(\frac{-1}{p}\right) g_p$,
      - $|g_p|^2 = p$.
   4. Determine the unique quadratic extension of $\mathbb{Q}$ contained in Φ_p.

5)
   1. Let E/F be a finite Galois extension. Assume that the Galois group $Gal(E/F)$ is generated by a single element σ. Let x be an element of E such that $tr_{E/F}(x) = 0$. Show that there exists $y \in E$ such that $x = \sigma(y) - y$.
   2. Let F be a field of characteristic p, and let E/F be a Galois extension of degree p. Show that there exists $x \in F$ such that $E \cong F[T]/(T^p - T - x)$.

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Analysis and Differential Equations

Individual (5 problems)

1) Let $F: \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let $u: [0,1] \to \mathbb{R}$ be a continuous function, with

$$\int_0^1 u(x)\,dx = 0.$$

Show that

$$\int_0^1 F(u(x))\,dx \leqslant \frac{F(\|u\|_\infty) + F(-\|u\|_\infty)}{2}$$

where $\|u\|_\infty := \sup_{x\in[0,1]} |u(x)|$. Also determine when equality occurs.

2) Prove that there exists a universal constant K, for all C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$, if $f \in L^1\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right)$ and $|\nabla f| \in L^2(\mathbb{R}^2)$, we have the following inequality:

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leqslant K\, \|f\|_{L^1(\mathbb{R}^2)}\, \|\nabla f\|_{L^2(\mathbb{R}^2)}\,.$$

Can you provide constant K so that $K < 10$? In the problem, all the L^p-spaces are defined with respect to the Lebesgue measure.

3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. Suppose

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\ln|x|} = 0.$$

Prove or disprove that f is a constant.

4) (a) Show that there does not exist a holomorphic function f on $\mathbb{C} \setminus \{1, -1\}$ so that

$$f'(z) = \frac{1}{(z^2-1)^{2019}} \quad \text{for all } z \in \mathbb{C} \setminus \{1, -1\}.$$

(b) Show that there exist a set $L \subset \mathbb{C}$ and a holomorphic function F on $\mathbb{C} \setminus L$ so that L has Hausdorff dimension 1, and

$$F'(z) = \frac{1}{(z^2-1)^{2019}} \quad \text{for all } z \in \mathbb{C} \setminus L.$$

5) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Prove that, for all $p > 1$ and $1 \leqslant q < \infty$, for all $f \in L^p(\Omega)$, there exists a unique $u \in H^1_0(\Omega)$, such that

$$\triangle u = |u|^{q-1}u + f \text{ in } \Omega.$$

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Analysis and Differential Equations

Team (5 problems)

1) Show that there is no non-zero $f \in C_0^\infty(\mathbb{R}^2)$ (compactly supported smooth function) so that its Fourier transform $\widehat{f}(\xi)$ is also compactly supported.

2) Prove the following classical interior Schauder estimates:

   There exists a universal constant C, for all smooth compactly supported functions $u, f \in C_0^\infty(\mathbb{R}^3)$ with $\triangle u = f$, we have
$$\|u\|_{C^{2,\alpha}} \leqslant C\|f\|_{C^{0,\alpha}},$$
   where $0 < \alpha < 1$ and $\|\cdot\|_{C^{k,\alpha}}$ are Hölder norms.

3) Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \to X$ be a measurable and measure preserving map, i.e., for all $A \in \mathcal{A}$, we have $\mu\big(T^{-1}(A)\big) = \mu(A)$. For $A, B \in \mathcal{A}$, if $\mu(A - B) = \mu(B - A) = 0$, we say that $A = B$ *a.e.*

   Assume $A \in \mathcal{A}$ such that $T^{-1}(A) = A$ *a.e.*.Prove that there exists a set $B \in \mathcal{A}$ so that $T^{-1}B = B$ and $A = B$ *a.e.*

4) Is there an entire function f with infinitely many zeroes, so that for every $r \in (0, 1)$, there exist constants $A_r, B_r < \infty$ such that
$$|f(z)| \leqslant A_r e^{B_r|z|^r}$$
   for every $z \in \mathbb{C}$?

5) Let $u(t, x, y)$ be a smooth real function defined on $\mathbb{R} \times \mathbb{R}^2$ where $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. We assume that it solves the following semilinear wave equation:
$$-\frac{\partial^2}{\partial t^2}u + \triangle u = u^3.$$
   If the supports of the initial data $u(0, x)$ and $\dfrac{\partial u}{\partial t}(0, x)$ are compact, prove that, for all $t_0 \in \mathbb{R}$, the supports of $u(t_0, x)$ and $\dfrac{\partial u}{\partial t}(t_0, x)$ are compact.

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Applied and Computational Math

Individual (4 problems)

1) (20 points)

Given a set $\mathcal{X}$, $m \in \mathbb{N}$ and a hypothesis space $\mathcal{H}$, define

$$\Pi_{\mathcal{H}}(m) = \max_{\{x_1,x_2,\ldots,x_m\}\subseteq\mathcal{X}} |\{(h(x_1), h(x_2), \ldots, h(x_m))|h \in \mathcal{H}\}|$$

where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. The VC dimension of $\mathcal{H}$ is

$$\mathrm{VC}(\mathcal{H}) = \max\{m : \Pi_{\mathcal{H}}(m) = 2^m\}.$$

(i) Let $\mathcal{X} = \mathbf{R}$. If $a \leqslant b$, define $h(x; a, b) = 1$ if $x \in [a, b]$ and $h(x) = -1$ if $x \notin [a, b]$. Find the VC dimension of the hypothesis space $\mathcal{H} = \{h(x; a, b)|a, b \in \mathbf{R}, a \leqslant b\}$.
(ii) Let $\mathcal{X} = \mathbf{R}^d$, $\mathcal{H}$ to be the set of linear classifiers, i.e. $\mathcal{H} = \{f(x)|f(x) = \mathrm{sign}(w^\top x + b), w \in \mathbf{R}^d, b \in \mathbf{R}\}$ where $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(x) = 0$ if $x = 0$. Show that the VC dimension of $\mathcal{H}$ is $d + 1$.

2) (25 points)

Consider Richardson's difference scheme for the heat equation $u_t = u_{xx}$:

$$\frac{1}{2k}\left(u(x, t + k) - u(x, t - k)\right) = \frac{1}{h^2}\left(u(x - h, t) - 2u(x, t) + u(x + h, t)\right).$$

(i) Show that this scheme has second-order truncation error.
(ii) Use either ODE principles or von Neumann analysis to show that this scheme is unconditionally unstable.
(iii) Demonstrate a minor modification of the left-side of Richardson's scheme that yields a familiar unconditionally stable scheme and prove it.

3) (25 points)

Let $\emptyset \neq K$ be a closed convex set in $\mathbf{R}^n$, i.e., K is a closed set and for any $x, y \in K$ and $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in K$. For any $z \in \mathbf{R}^n$, let $\Pi_K(z)$ denote the metric projection of z onto K, which is the unique optimal solution of following problem:

$$\min \frac{1}{2}\|y - z\|_2^2, \quad \text{s.t.} \quad y \in K. \tag{1}$$

Show that
(i) the point $y \in K$ solves (1) if and only if

$$(z - y)^T(d - y) \leqslant 0, \quad \forall\, d \in K;$$

(ii) for any $y, z \in \mathbf{R}^n$,

$$\|\Pi_K(y) - \Pi_K(z)\|_2 \leq \|y - z\|_2;$$

(iii) $\Theta(\cdot)$ is continuously differentiable with its gradient given by

$$\nabla\Theta(z) = z - \Pi_K(z),$$

where for any $z \in \mathbf{R}^n$, $\Theta(z) := \frac{1}{2}\|z - \Pi_K(z)\|_2^2$.

4) (25 points) The scientists FitzHugh (1961) and Nagumo, Arimoto, Yoshizawa (1962) derived a mathematical model to characterize the behavior of a neuron under the externally injected current I:

$$\begin{cases} \frac{dV}{dt} & = V - \frac{1}{3}V^3 - W + I, \\ \frac{dW}{dt} & = \frac{1}{\tau}\big(V + a - bW\big), \end{cases}$$

where the variable V describes the membrane potential of the neuron, the variable W describes the current arising from opening and closing of ion channels on the neurons membrane. The variables τ, a and b are parameters with typical values: $a = 0.7$,$b = 0.8$ and $\tau = 13$.

(i) For a small positive constant current I, how the neuron behaves.

(ii) For a large positive constant current I, how the neuron behaves.

(iii) Suppose one injects a pulse current with different magnitude at some time t_0, i.e., $I = I_0\delta(t - t_0)$, where I_0 describes the magnitude of the pulse, analyze the dynamical behavior of the neuron when I_0 is small or large.

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Applied and Computational Math

Team (5 problems)

1) (10 points)

Show that the quadrature formula $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n} \sum_{k=0}^{n-1} f\left(\cos\pi \frac{2k+1}{2n}\right)$ is exact for all polynomials of degree up to and including $2n-1$.

2) (15 pointes) Let $x = (x_0, \ldots, x_{N-1}) \in \mathbf{R}^N$, $x \neq 0$ and $\hat{x}$ be its discrete Fourier transform, i.e.

$$\hat{x}_w = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x_t \exp(-2\pi i w t/N), \; w = 0, \ldots, N-1.$$

Prove that $\|x\|_0 \|\hat{x}\|_0 \geqslant N$ where $\|x\|_0$ denotes the number of nonzero entries in x. (Hint: show that $\hat{x}$ can not have $\|x\|_0$ consecutive zeros.)

3) (20 pointes)

Let $m \leqslant n$. Consider the $(n+m) \times (n+m)$ real matrix defined by

$$A = \begin{bmatrix} I & X \\ X^\top & O \end{bmatrix},$$

where I is the $n \times n$ identity matrix, X is a full-rank $n \times m$ matrix, O is the $m \times m$ zero matrix.

(i) Show that A is nonsingular.

(ii) Find the eigenvalues of A, some of which are in terms of the singular values of X.

(iii) Under what conditions on X would the iteration

$$x_{n+1} = x_n - (Ax_n - b)$$

converge to the solution of $Ax = b$ for any $(n+m) \times (n+m)$ real vector b?

4) (25 pointes)

Let f be a continuously differentiable convex function defined on $\mathbf{R}^n$, i.e., $f : \mathbf{R}^n \to \mathbf{R}$ is continuously differentiable and for any $x, y \in \mathbf{R}^n$ and any $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$. Suppose that the gradient of f is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L \|x - y\|_2.$$

Prove the following inequalities:

(i). $f(y) \leq f(x) + (\nabla f(x))^T (y-x) + \frac{L}{2} \|y-x\|_2^2, \quad \forall x, y \in \mathbf{R}^n;$

(ii). $f(y) \geq f(x) + (\nabla f(x))^T (y-x) + \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|_2^2, \quad \forall x, y \in \mathbf{R}^n;$

(iii). $\frac{1}{L} \|\nabla f(y) - \nabla f(x)\|_2^2 \leq (\nabla f(y) - \nabla f(x))^T (y-x), \quad \forall x, y \in \mathbf{R}^n.$

5) (30 pointes) Consider the following problems.

(i) Determine the order of Störmer's method,

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(t_{n+1}, y_{n+1}), \qquad n \geqslant 0,$$

for solving the second order system of ODE's

$$y'' = f(t, y), \qquad t \geqslant 0,$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$.

(ii) Using the second order central differences in space and Störmer's method in time, construct a scheme to solve the wave equation,

$$u_{tt} = u_{xx}.$$

(iii) Determine the condition for its stability.

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Geometry and Topology

Individual (5 problems)

1) Let Conf_n be the following submanifold of $\mathbb{C}^n$:

$$\mathrm{Conf}_n = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for any } i \neq j\}.$$

For every pair (i, j) with $i \neq j$, we define the complex valued 1-form

$$\omega_{ij} := \frac{dz_i - dz_j}{z_i - z_j}.$$

(a) Show that for any $i \neq j$, ω_{ij} represents a non-zero de Rham cohomology class in $H^1(\mathrm{Conf}_n, \mathbb{C})$.

(b) Show that for any pair-wise distinct indices i, j, k,

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0.$$

2) Let M be a compact oriented manifold of (real) dimension 4. Consider the following symmetric bilinear form on $H^2(M)$

$$H^2(M) \times H^2(M) \to \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

Let $\tau(M)$ be the signature of this bilinear form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues. Compute $\tau(M)$ for $M = S^4, \mathbb{CP}^2$ and $S^2 \times S^2$.

3) Let $X = \mathbb{R}^4 / \sim$, where

$$\begin{aligned}
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2 + 1, x_3, x_4) \\
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2, x_3, x_4 + 1) \\
(x_1, x_2, x_3, x_4) &\sim (x_1 + 1, x_2, x_3, x_4) \\
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2 + x_4, x_3 + 1, x_4)
\end{aligned}$$

Compute $H_1(X, \mathbb{Z})$.

4) Let E be a vector bundle over a smooth manifold M. Let ∇^E be a connection E and $R^E \in \Omega^2(M, \mathrm{End}(E))$ be its curvature tensor. For any polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, we denote

$$f(R^E) = a_0 + a_1 R^E + a_2 (R^E)^2 \cdots + a_n (R^E)^n \in \Omega^*(M, \mathrm{End}(E)).$$

Here $(R^E)^k \in \Omega^{2k}(M, \mathrm{End}(E))$ is the k-th wedge product on forms combined with matrix multiplications on $\mathrm{End}(E)$.

(a) Show that the differential form $\mathrm{tr}\left[f(R^E)\right] \in \Omega^*(M)$ is closed

$$d\mathrm{tr}\left[f(R^E)\right] = 0.$$

Here tr is the trace on $\mathrm{End}(E)$.

(b) Let ∇^E, $\widetilde{\nabla}^E$ be two connections on E and R^E, $\widetilde{R}^E$ be the corresponding curvature tensors. Show that there exists a differential form $\omega \in \Omega^*(M)$ such that

$$\mathrm{tr}\left[f(R^E)\right] - \mathrm{tr}\left[f(\tilde{R}^E)\right] = d\omega.$$

5) (a) Let u be a smooth function over a Riemannian manifold (M, g). Prove the following Bochner's formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla\nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla\Delta u, \nabla u)$$

where Δ is the Laplacian and $|\bullet|^2 = g(\bullet, \bullet)$.

(b) Let (S^2, g) be the standard unit sphere and E be a constant. Show that the only smooth positive solution to

$$\Delta \ln f + Ef^2 = 1$$

is $f = \frac{1}{A+\phi}$ where A is a constant and ϕ is some first eigenfunction of S^2.

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Geometry and Topology

Team (5 problems)

1) Is TS^2 diffeomorphic to $S^2 \times \mathbb{R}^2$? Verify your answer. Here TS^2 is the total space of the tangent bundle of S^2.

2) Solve the problem which Russell Crowe assigns to his students in the movie "A beautiful mind" (2001):

$$V = \{F : \mathbb{R}^3 \backslash X \to \mathbb{R}^3 \ s.t. \ \nabla \times F = 0\}$$
$$W = \{F = \nabla g\}$$
$$\dim (V/W) = ?$$

First give the general answer for any closed $X \subset \mathbb{R}^3$, and then specialize it to (a) $X = \{x = y = z = 0\}$, (b) $X = \{x = y = 0\}$ and (c) $X = \{x = 0\}$.

3) Let $T^2 = S^1 \times S^1$ be the 2-torus with the standard orientation, and let $F : T^2 \to T^2$ be a smooth map of degree 1 such that $F \circ F = \mathrm{Id}$ and F has no fixed points. Prove that the induced map $F^* : H^1(T^2) \to H^1(T^2)$ is the identity.

4) Let $U(n)$ be the group of $n \times n$ unitary matrices, and $O(n)$ be the group of $n \times n$ orthogonal matrices. Let $SU(n) = \{A \in U(n) | \det A = 1\}$ be the special unitary group and $SO(n) = \{A \in O(n) | \det A = 1\}$ be the special orthogonal group. All $U(n), SU(n), O(n), SO(n)$ are Lie groups with natural manifold structures.

   (a) Compute the dimensions of $SU(n)$ and $SO(n)$.

   (b) Compute the fundamental groups of $SU(n)$ and $SO(n)$ ($n \geqslant 2$).

5) Let (M, g) be a compact Riemannian manifold and R be its Riemannian curvature tensor. (M, g) will be called *weakly negative* if for any point $p \in M$ and for any nonzero vector field $X \in T_pM$, there exists a nonzero vector field $Y \in T_pM$ such that $R(X, Y, Y, X) < 0$.

   (a) Let X be a Killing vector field and $f = \frac{1}{2}g(X, X) = \frac{1}{2}|X|^2$. Show that for any vector field V

$$(\mathrm{Hess} f)(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V).$$

   Here the Hessian of f is $(\mathrm{Hess} f)(Y, Z) := g(\nabla_Y \mathrm{grad}(f), Z)$ for any vector fields Y, Z, where $\mathrm{grad}(f)$ is the gradient vector of f.

   (b) Prove that if (M, g) is weakly negative, then there are no nontrivial Killing vector fields.

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Probability and Statistics

Individual (4 problems)

1) Suppose $(X_n)_{n\geq 1}$ is a sequence of positive random variables. There exists a constant $C > 0$ such that,

$$\mathbb{E}[X_n] \leq C, \quad \mathbb{E}[\max\{0, -\log X_n\}] \leq C, \quad \forall n.$$

Show that

$$\limsup_{n\to\infty} X_n^{1/n} = 1.$$

2) Suppose γ is a probability measure on $\{0, 1, 2\}$ such that $\gamma(0) > \gamma(2) > 0$. Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law γ. Define the sequence

$$Y_0 = 0, \quad Y_{n+1} = \max\{0, Y_n + \xi_{n+1} - 1\}, \quad \forall n \geq 0.$$

Show that $(Y_n)_{n\geq 0}$ is an irreducible Markov chain on the state space $\mathbb{N} = \{0, 1, 2, \ldots\}$ and it is positive recurrent.

3) Suppose $(\epsilon_n)_{n\geq 1}$ is a sequence of i.i.d. random variables and the common law is Bernoulli:

$$\mathbb{P}[\epsilon_1 = 1] = \mathbb{P}[\epsilon_1 = -1] = 1/2.$$

Consider the random series $f(x) = \sum_{n=1}^{\infty} \epsilon_n x^n$. Show that the random series attains zero infinitely many times on $x \in [0, 1)$ almost surely.

4) Consider a randomized experiment with $2N$ units, half to be randomly assigned an active treatment, and the other half to be assigned the control treatment; the objective is to measure the effect of the active versus control treatments on an outcome, called Y. For example, the units could be people with high blood pressure, where Y is blood pressure one week after receiving the active drug or an inactive drug, a placebo, where the patient is blinded to which drug is being given.

The estimand, the goal of the experiment, is the average value of Y if all $2N$ units received active minus the average value of Y if all $2N$ units received control. Assume that these $2 \times 2N$ numbers are fixed quantities (in the statistical literature this assumption is known as SUTVA the stable-unit-treatment-value assumption), which means that the outcome Y for the i-th unit receiving a particular treatment is a proper function of that unit and the treatment that unit i received.

Derive the following results in this simple situation.

a) Find the expectation of the estimator, the difference in the observed sample means (between those assigned treatment and those assigned control), in terms of the estimand (defined above), where expectation in this context refers to averaging over all possible random allocations.

b) The variance of the estimator described in part a) (again, with variance defined as averaging over all possible random allocations).

c) Find an unbiased estimator of the variance in part b), assuming additive treatment effects, that is, the treatment minus control values of Y are constant across the $2N$ units, so that the treatment versus control condition adds a constant value for all $2N$ units.

d) Find the bias of the estimator in part c) when the treatment effects are non-additive.

e) Generalize the results in parts a),b), c) and d) to the situation where $2N$ is replaced by $N_t + N_c$, with N_t units getting active treatment and N_c units getting control, where these sample sizes are unequal.

f) Argue that the estimator in part c), when the sample sizes are large, will look gaussian, and conduct a small simulation to indicate that this often happens with relatively small sample sizes.

g) Modify the first four parts to consider a different randomized experiment, but still with $2N$ units, half to be allocated to active and half to be allocated to control, but now we have a covariate, X, a background variable that is suspected to be related to Y. For example, X could be blood pressure today, pre-treatment. In this experiment, many randomized allocations are considered, but all allocations are rejected if the sample X means of the treated and controls are too different for example more than a standard deviation apart. Be careful here to note which results generalize and which do not.

h) Finally, consider the randomized experiment in part g) when N_t units are treated and Nc are control, where N_t and N_c are not equal: Which results in parts a),b),c) and d) generalize without modification? In particular, describe how the conclusion in part f) changes. Note that this is an interesting situation where the asymptotic distributions of sample means are not gaussian. What are these distributions?

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Probability and Statistics

Team (4 problems)

1) Suppose $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables and the common law is exponential with parameter one. Show that
$$\mathbb{P}\left[\limsup_{n\to\infty}\frac{X_n}{\log n}=1\right]=1.$$

2) Let $(X_n)_{n\geq 1}$ be i.i.d. real random variables and set $S_n=\sum_{i=1}^n X_i$ for $n\geq 1$. Suppose that for some constant $c\in\mathbb{R}$ we have $S_n/n\to c$ as $n\to\infty$ almost surely. Show that X_1 has a finite first moment and $\mathbb{E}[X_1]=c$.

3) Consider uniform permutation of $\{1,2,\ldots,n\}$ and denote by X_n the number of cycles in the permutation. Find a sequence of reals $(a_n)_{n\geq 1}$ such that
$$\lim_{n\to\infty}\frac{\mathbb{E}[X_n]}{a_n}=1,$$
and justify your answer.

4) The Erdös-Rényi random graph $G(n,p)$ with parameters $n\geq 1$ and $p\in[0,1]$ is the random graph whose vertex set is $V=\{1,2,\ldots,n\}$ and where for each pair $i\neq j\in V$ the edge $i\leftrightarrow j$ is present with probability p independently of all the other pairs.

   (a) For $\epsilon>0$, if $p_n\geq(1+\epsilon)\frac{\log n}{n}$, then
   $$\mathbb{P}[G(n,p_n)\text{ has an isolated vertex}]\to 0,\quad \text{as } n\to\infty.$$

   (b) For $\epsilon>0$, if $p_n\leq(1-\epsilon)\frac{\log n}{n}$, then
   $$\mathbb{P}[G(n,p_n)\text{ has an isolated vertex}]\to 1,\quad \text{as } n\to\infty.$$

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Algebra and Number Theory

Solve every problem.

Problem 1. Let F be a field of characteristic zero. Consider the polynomial ring $F[x_1, \ldots, x_n]$.

(a) Prove Newton's identity over the field F

$$p_k - p_{k-1}e_1 + \cdots + (-1)^{k-1}p_1 e_{k-1} + (-1)^k k e_k = 0,$$

where

$$e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < \text{ß}_k \le n} x_{i_1} \cdots x_{i_k}$$

for $1 \le k \le n$, $e_0 = 1$, $e_k = 0$ when $k > n$, and

$$p_k(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k.$$

(b) Prove that over the field of F of characteristic zero, an $n \times n$ matrix A is nilpotent if and only if the trace of A^k is equal to zero for all $k = 1, 2 \ldots$

Hint: use Part (a).

(c) Prove that over the field of F of characteristic zero, two $n \times n$ matrix A and B have the same characteristic polynomial if and only if the trace of A^k and trace of B^k are equal for all $k = 1, 2 \ldots$

Hint: use Part (a).

Problem 2.

(a) Let M be a finitely generated R-module and $\mathfrak{a} \subset R$ an ideal. Suppose $\phi : M \longrightarrow M$ is an R-module map such that $\phi(M) \subseteq \mathfrak{a}M$. Prove that there is a monic polynomial $p(t) \subset R[t]$ with coefficients from $\mathfrak{a}$ such that $p(\phi) = 0$.

Hint: $p(t)$ is basically just the characteristic polynomial.

(b) If M is a finitely generated R-module such that $\mathfrak{a}M = M$ for some ideal $\mathfrak{a} \subset R$, then there exits $x \in R$ such that $1 - x \in \mathfrak{a}$ and $xM = 0$.

Problem 3. Let $R = F[x, y]/(y^2 - x^2 - x^3)$ for some field F.

(a) Prove that R is an integral domain.

(b) Compute the normalization of R (*i.e.*, the integral closure of R in its field of fraction).

Problem 4. Let p and ℓ be two prime numbers and $[\ell_x]$ denote the ℓ-th cyclotomic polynomial $1 + x + \cdots + x^{\ell-1}$.

(a) Prove that $[\ell_x]$ is an irreducible element of $\mathbb{Q}[x]$.

(b) Show that $[\ell_x]$ is divisible by $x - 1$ in $\mathbb{F}_p[x]$ if $p = \ell$. Here $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$.

(c) Suppose $p \neq \ell$. let a be the order of p in $\mathbb{F}_\ell$. Show that a is the first value of m for which the group $\mathrm{GL}_m(\mathbb{F}_p)$ of invertible $m \times m$ matrices with entries from $\mathbb{F}_p$ contains an element of order ℓ.

Hint: Derive and use the formula for the number of elements in $\mathrm{GL}_m(\mathbb{F}_p)$.

Problem 5. Let $p \geq 3$ be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers.

(a) Show that an element in $1 + p\mathbb{Z}_p$ is a p-th power in $\mathbb{Z}_p$ if and only if it lives in $1 + p^2\mathbb{Z}_p$.

(b) Let $\mathbb{Z}_p^\times$ denote the group of units in $\mathbb{Z}_p$. Show that there exist $a, b, c \in \mathbb{Z}_p^\times$ such that $a^p + b^p = c^p$ if and only if

$$\sum_{i=1}^{p-1} i^{p-2} t^i \equiv 0 \pmod{p}$$

for some integer $t \in \{2, 3, \ldots, p-1\}$. (In particular, this condition holds for $p = 7$ by taking $t = 3$. Therefore, Fermat's Last Theorem does not hold for $\mathbb{Z}_7$.)

Problem 6. Recall that a metric space is called *spherically complete* if any decreasing sequence of closed balls has nonempty intersection.

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p-adic numbers. For every integer $n \geq 1$, consider the finite extension $\mathbb{Q}_p(\mu_{p^n})$ of $\mathbb{Q}_p$ generated by all p^n-th roots of unity. Let $\mathbb{Q}_p(\mu_{p^\infty}) = \cup_{n\geq 1}\mathbb{Q}_p(\mu_{p^n})$. All of these algebraic extensions of $\mathbb{Q}_p$ are equipped with the unique norm $|\cdot|$ extending the usual p-adic norm on $\mathbb{Q}_p$.

Question: Which of the following are spherically complete? Explain why.

(a) $\mathbb{Q}_p$;

(b) $\mathbb{Q}_p(\mu_{p^n})$;

(c) $\mathbb{Q}_p(\mu_{p^\infty})$;

(d) $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, the completion of $\mathbb{Q}_p(\mu_{p^\infty})$.

Hint: Show that there exists a sequence $a_1, a_2, \ldots \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ such that $|a_1| > |a_2| > \cdots$ and $\lim |a_i| > 0$, and such that the closed balls

$$B_i := \left\{ x \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})} \ : \ |x - a_1 - a_2 - \cdots - a_i| \leq |a_i| \right\}$$

have empty intersection.

S.-T. Yau College Student Mathematics Contests 2020

Algebra and Number Theory

Solve every problem.

Problem 1. Let F be a field of characteristic zero. Consider the polynomial ring $F[x_1, \dots, x_n]$.

(a) Prove Newton's identity over the field F

$$p_k - p_{k-1}e_1 + \cdots + (-1)^{k-1}p_1 e_{k-1} + (-1)^k k e_k = 0,$$

where

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < ß_k \le n} x_{i_1} \cdots x_{i_k}$$

for $1 \le k \le n$, $e_0 = 1$, $e_k = 0$ when $k > n$, and

$$p_k(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k.$$

(b) Prove that over the field of F of characteristic zero, an $n \times n$ matrix A is nilpotent if and only if the trace of A^k is equal to zero for all $k = 1, 2 \dots$

Hint: use Part (a).

(c) Prove that over the field of F of characteristic zero, two $n \times n$ matrix A and B have the same characteristic polynomial if and only if the trace of A^k and trace of B^k are equal for all $k = 1, 2 \dots$

Hint: use Part (a).

Solution: Part (a): Consider

$$E(y) = \prod_{i=1}^{n}(1 - x_i y) = 1 - e_1 y + e_2 y^2 + \cdots (-1)^n e_n y^n.$$

Using $-\ln(1-t) = \sum_{j=1}^{\infty} t^j/j$, we obtain

$$\ln(E(y)) = \sum_{i=1}^{n} \ln(1 - x_i y) = -\sum_{i=1}^{n}\sum_{j=1}^{\infty}(x_i y)^j/j = -\sum_{j=1}^{\infty} p_j y^j/j.$$

Apply d/dy to the above, we have

$$E'(y)/E(y) = -\sum_{j=1}^{\infty} p_j y^{j-1}, \quad \text{or} \quad -E'(y) = E(y)\sum_{j=1}^{\infty} p_j y^{j-1}.$$

By comparing the coefficients of y^{k-1} of both sides, we obtain

$$-(-1)^k k e_k = \sum_{j=0}^{k-1}(-1)^j e_j p_{k-j}.$$

Part (b): Suppose A is nilpotent. Then, the minimal polynomial of A is x^r for some integer r. It follows that the characteristic of A is $f(x) = x^n$. The trace of A is equal to a_{n-1} where $-a_{n-1}$ is the coefficient of x^{n-1} of $f(x)$, hence is equal to 0. Similarly, A^k is nilpotent, hence its trace is zero.

Conversely, assume trace of A^k equals 0 for all $k \geq 1$. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k. Since the trace is the sum of eigenvalues, we see that (the sums of powers) $p_k(\ldots, \lambda, \ldots) = 0$. By Part (a), we see that $e_k(\ldots, \lambda, \ldots) = 0$. Since the coefficients of the characteristic polynomial $f(t)$ of A are precisely $e_k(\ldots, \lambda, \cdots)$ for $0 \leq k \leq n$ (possibly up to $\pm$ signs), we obtain $f(t) = t^n$, hence $A^n = 0$.

Part (c): Suppose that A and B have the same characteristic polynomials. Let λ_A (resp. λ_B) be an eigenvalue of A (resp. B). Then, $e_k(\ldots, \lambda_A, \ldots) = e_k(\ldots, \lambda_B, \ldots)$ for all $k \geq 0$. By (a), $p_k(\ldots, \lambda_A, \ldots) = p_k(\ldots, \lambda_B, \ldots)$. Since the trace is the sum of eigenvalues, we obtain the trace of A^k and trace of B^k are equal. Conversely, if the trace of A^k and trace of B^k are equal for all k, then $p_k(\ldots, \lambda_A, \ldots) = p_k(\ldots, \lambda_B, \ldots)$. Hence, $e_k(\ldots, \lambda_A, \ldots) = e_k(\ldots, \lambda_B, \ldots)$ for all $k \geq 0$. Thus, A and B have the same characteristic polynomials.

Problem 2.

(a) Let M be a finitely generated R-module and $\mathfrak{a} \subset R$ an ideal. Suppose $\phi : M \longrightarrow M$ is an R-module map such that $\phi(M) \subseteq \mathfrak{a}M$. Prove that there is a monic polynomial $p(t) \subset R[t]$ with coefficients from $\mathfrak{a}$ such that $p(\phi) = 0$.

Hint: $p(t)$ is basically just the characteristic polynomial.

(b) If M is a finitely generated R-module such that $\mathfrak{a}M = M$ for some ideal $\mathfrak{a} \subset R$, then there exits $x \in R$ such that $1 - x \in \mathfrak{a}$ and $xM = 0$.

Solution: Part (a): Let $x_1, \ldots, x_m$ be a generating set for M as an R-module. We have

$$\phi(x_i) = \sum a_{ij} x_j$$

for some $a_{ij} \in \mathfrak{a}$. Let A_{ij} be the operator $\delta_{ij}\phi - a_{ij}\mathrm{Id_M}$ where $\mathrm{Id_M} : M \longrightarrow M$ is the identity hom and δ_{ij} is the Kronecker's symbol. Then we have $\sum_j A_{ij} x_j = 0$ for all j. The matrix $A = (A_{ij})$ annihilates the column vector $v = (x_j)_{j=1}^m$, Consider M as an $R[\phi]$-module, then $A_{ij} \in R[\phi]$. Thus, A is a matrix with entries in $R[\phi]$. Its adjugate is well-defined. Multiplying $Av = 0$ on the left by the adjugate gives rise to $\det A\, x_j = 0$ for all j. Let $p(\phi) = \det A(\phi)$ (recall $A = (\delta_{ij}\phi - a_{ij}\mathrm{Id_M})$). Then, $p(t)$ is a monic polynomial and $p(\phi) = 0$ on M.

Part (b): By Part (a), $\mathrm{Id_M} : M \longrightarrow M$ satisfies

$$\mathrm{Id_M}^r + a_1 \mathrm{Id_M}^{r-1} + \cdots + a_r \mathrm{Id_M} = 0$$

for some $a_j \in \mathfrak{a}$. Let $x = 1 + a_1 + \cdots + a_r$, then $x - 1 \in \mathfrak{a}$ and $xM = 0$.

Problem 3. Let $R = F[x, y]/(y^2 - x^2 - x^3)$ for some field F.

(a) Prove that R is an integral domain.

(b) Compute the normalization of R (*i.e.*, the integral closure of R in its field of fraction).

Solution: Part (a): It suffices to prove that $y^2 - x^2 - x^3$ is irreducible in $F(x)[y]$. It is reducible if it has a root $f(x)/g(x) \in F(x)$, where $f(x)$ and $g(x)$ are co-prime. But $(f(x)/g(x))^2 - x^2 - x^3 = 0$ implies $f(x)^2 = g(x)^2(x^2 + x^3) = (g(x)x)^2(x + 1)$. Thus, $(x + 1)$ divides $f(x)$. Hence, $(x + 1)^2$ divides $f(x)^2$. It follows that $(x + 1)$ divides $g(x)$, a contradiction. This implies that R is an integral domain.

Part (b): We have $0 = y^2 - x^3 - x^2 = x^2(y^2/x^2 - x - 1) = x^2(t^2 - x - 1)$. As K is an integral domain, $t^2 - x - 1 = 0$, that is, $x = t^2 - 1$. Then $y = xy/x = (t^2 - 1)t$. It follows that any element of R is a polynomial in t, hence $R \subset F[t]$. Therefore $K \subset F(t)$. Thus, $K = F(t)$.

Now let S be the integral closure of R in K. We claim $S = F[t]$. Let $f(t) \in F[t]$. Let $s = 2k$ be an even integer. Then

$$t^s = (t^2)^k = ((t^2 - 1) + 1)^k = \sum_{i=0}^{k} \binom{k}{i} (t^2 - 1)^i = \sum_{i=0}^{k} \binom{k}{i} x^i.$$

Let $s = 2k+1$ be an odd integer with $s > 3$, using the above, we obtain

$$t^s = t^s - t^{s-2} + t^{s-2} = t^{s-3}(t^2-1)t + t^{s-2} = \left(\sum_{i=0}^{k-1}\binom{k-1}{i}x^i\right)y + t^{s-2}.$$

Repeat the above for the odd integer $s-2$, by induction, we see that t^s is of the form $g(x,y)+at$. Combing all the above, we see that $f(t)$ is of the form $h(x,y)+bt$ for some $b \in \mathbb{Z}$ and $h(x,y) \in R$. Then, $f(t)$ is a root of

$$(X - h(x,y))^2 - b^2 - b^2x \in R[X].$$

it follows that $f(t) \in S$. Hence, $F[t] \subset S$. But, $R \subset F[t]$ and $F[t]$ is integrally closed in $F(t)$, hence $S \subset F[t]$. Therefore $S = F[t]$.

Problem 4. Let p and ℓ be two prime numbers and $[\ell_x]$ denote the ℓ-th cyclotomic polynomial $1 + x + \cdots + x^{\ell-1}$.

- **(a)** Prove that $[\ell_x]$ is an irreducible element of $\mathbb{Q}[x]$.
- **(b)** Show that $[\ell_x]$ is divisible by $x-1$ in $\mathbb{F}_p[x]$ if $p = \ell$. Here $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$.
- **(c)** Suppose $p \neq \ell$. let a be the order of p in $\mathbb{F}_\ell$. Show that a is the first value of m for which the group $\mathrm{GL}_m(\mathbb{F}_p)$ of invertible $m \times m$ matrices with entries from $\mathbb{F}_p$ contains an element of order ℓ.

  Hint: Derive and use the formula for the number of elements in $\mathrm{GL}_m(\mathbb{F}_p)$.

Solution: Part (a): $[\ell_x]$ is irreducible over $\mathbb{Q}$ if and only if $[\ell_{x+1}]$ is irreducible$\mathbb{Q}$.

$$[\ell_{x+1}] = ((x+1)^\ell - 1)/((x+1)-1) = x^{\ell-1} + \ell x^{\ell-2} + \cdots + \ell(\ell-1)/2x + \ell.$$

This is irreducible by Eisenstein's criterion.

Part (b): $p = \ell$. If $p = 2$, then $[2]_x = 1 + x = x - 1$ If $p > 2$, then

$$[p]_x = (x^p - 1)/(x-1) = (x-1)^{p-1}.$$

Part (c): Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{F}_q^n$, where q is a prime power. If $A \in \mathrm{GL}_m(\mathbb{F}_q)$, then the columns of A, $\{Ae_1, \ldots, Ae_n\}$, form a basis for $\mathbb{F}_q^n$. Conversely, any basis form columns of an element $A \in \mathrm{GL}_m(\mathbb{F}_q)$. Thus, it is equivalent to count the number of bases $\mathcal{B} = (f_1, \ldots, f_n)$ for $\mathbb{F}_q^n$. The first vector has $q^m - 1$ choices. The second, not a multiple of the first, has $q^m - q$ choices. The third vector $f_3 \in \mathbb{F}_q^n \setminus \{af_1 + bf_2 \mid a, b \in \mathbb{F}_q\}$ has $q^m - q^2$ choices. Inductively, f_i has $q^m - q^i$ choices. Therefore

$$\left|\mathrm{GL}_m(\mathbb{F}_q)\right| = (q^m - 1)(q^m - q)\cdots(q^m - q^{m-1}).$$

If $\mathrm{GL}_m(\mathbb{F}_p)$ contains an element of order ℓ, then ℓ divides

$$|\mathrm{GL}_m(\mathbb{F}_p)| = p^{\binom{m}{2}}\prod_{i=1}^{m}(p^i - 1).$$

Sicne $\ell \neq p$, the first value of m such that ℓ divides the above is when ℓ divides the highest term $p^m - 1$ for the first time. This happens when $p^a - 1 = 0 \bmod \ell$.

Problem 5. Let $p \geq 3$ be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers.

- **(a)** Show that an element in $1 + p\mathbb{Z}_p$ is a p-th power in $\mathbb{Z}_p$ if and only if it lives in $1 + p^2\mathbb{Z}_p$.
- **(b)** Let $\mathbb{Z}_p^\times$ denote the group of units in $\mathbb{Z}_p$. Show that there exist $a, b, c \in \mathbb{Z}_p^\times$ such that $a^p + b^p = c^p$ if and only if

$$\sum_{i=1}^{p-1} i^{p-2}t^i \equiv 0 \pmod p$$

for some integer $t \in \{2, 3, \ldots, p-1\}$. (In particular, this condition holds for $p = 7$ by taking $t = 3$. Therefore, Fermat's Last Theorem does not hold for $\mathbb{Z}_7$.)

Solution: Part (a): If an element in $1 + p\mathbb{Z}_p$ is a p-th power, it must have form $(1 + p\alpha)^p$ for some $\alpha \in \mathbb{Z}_p$. A simple calculation yields

$$(1 + p\alpha)^p = 1 + \binom{p}{1} p\alpha + \binom{p}{2} (p\alpha)^2 + \cdots \in 1 + p^2\mathbb{Z}_p.$$

To prove sufficiency, recall the two functions

$$\exp : p\mathbb{Z}_p \longrightarrow 1 + p\mathbb{Z}_p, \qquad \log : 1 + p\mathbb{Z}_p \longrightarrow p\mathbb{Z}_p$$

which are inverses to each other. For any $a = 1 + p^2 x \in 1 + p^2\mathbb{Z}_p$, consider

$$a^{\frac{1}{p}} := \exp\left(\frac{1}{p}\log(a)\right).$$

Notice that

$$\frac{1}{p}\log(a) = \frac{1}{p}\log\left(1 + p^2 x\right) = \frac{1}{p}\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}\left(p^2 x\right)^i \in p\mathbb{Z}_p$$

and hence $a^{\frac{1}{p}}$ is well-defined. It is clear that $\left(a^{\frac{1}{p}}\right)^p = a$.

Part (b): As an immediate corollary from Part (a), if we write an element $a \in \mathbb{Z}_p^\times$ in terms of Witt coordinates $a = (a_0, a_1, \ldots)$, then a is a p-th power in $\mathbb{Z}_p$ if and only if $a_1 = 0$. In particular, whether an element in $\mathbb{Z}_p^\times$ is a p-th power can be detected by its image under the projection $\mathbb{Z}_p = W(\mathbb{F}_p) \longrightarrow W_2(\mathbb{F}_p)$.

Hence, there exist $a, b, c \in \mathbb{Z}_p^\times$ such that $a^p + b^p = c^p$ if and only if there exist $a_0, b_0, c_0 \in \mathbb{F}_p^\times$ such that $(a_0, 0) + (b_0, 0) = (c_0, 0)$ in $W_2(\mathbb{F}_p)$. Using the addition formula of Witt coordinates, the later equation translates to $a_0 + b_0 = c_0$ and

$$\frac{1}{p}\left(a_0^p + b_0^p - (a_0 + b_0)^p\right) = 0.$$

Direct calculation gives

$$\begin{aligned}
\frac{1}{p}(a_0^p + b_0^p - (a_0 + b_0)^p) &= -\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i} \\
&= -\sum_{i=1}^{p-1} \frac{1}{i} \frac{(p-1)(p-2)\cdots(p-i+1)}{(i-1)\cdots 1} a_0^i b_0^{p-i} \\
&\equiv \sum_{i=1}^{p-1} \frac{1}{i}(-1)^i a_0^i b_0^{p-i} \equiv \sum_{i=1}^{p-1} i^{p-2}\left(-\frac{a_0}{b_0}\right)^i \pmod{p}
\end{aligned}$$

Since $a_0 + b_0 = c_0 \neq 0$, we have $-\frac{a_0}{b_0} \neq 1$. Namely, there exists $t \in \{2, 3, \ldots, p-1\}$ such that

$$\sum_{i=1}^{p-1} i^{p-2} t^i \equiv 0 \pmod{p}.$$

All steps above are clearly reversible and hence cover both the "if" and "only if" parts. This completes the proof.

Problem 6. Recall that a metric space is called *spherically complete* if any decreasing sequence of closed balls has nonempty intersection.

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p-adic numbers. For every integer $n \geq 1$, consider the finite extension $\mathbb{Q}_p(\mu_{p^n})$ of $\mathbb{Q}_p$ generated by all p^n-th roots of unity. Let $\mathbb{Q}_p(\mu_{p^\infty}) =$

$\cup_{n\geq 1}\mathbb{Q}_p(\mu_{p^n})$. All of these algebraic extensions of $\mathbb{Q}_p$ are equipped with the unique norm $|\cdot|$ extending the usual p-adic norm on $\mathbb{Q}_p$.

Question: Which of the following are spherically complete? Explain why.

(a) $\mathbb{Q}_p$;

(b) $\mathbb{Q}_p(\mu_{p^n})$;

(c) $\mathbb{Q}_p(\mu_{p^\infty})$;

(d) $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, the completion of $\mathbb{Q}_p(\mu_{p^\infty})$.

Hint: Show that there exists a sequence $a_1, a_2, \ldots \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ such that $|a_1| > |a_2| > \cdots$ and $\lim |a_i| > 0$, and such that the closed balls

$$B_i := \left\{x \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})} \;:\; |x - a_1 - a_2 - \cdots - a_i| \leq |a_i|\right\}$$

have empty intersection.

Solution: (a) and (b) are spherically complete. In fact, every finite extension of $\mathbb{Q}_p$ is spherically complete. Such a field is discretely valued and complete. In this case, a decreasing sequence of closed balls either eventually stabilizes, or has radius converging to 0. In both cases, the intersection is nonempty.

(c) is not spherically complete. Notice that spherical completeness implies completeness. (Why? From any Cauchy sequence, one can construct a decreasing sequence of closed balls whose intersection gives the limit of the Cauchy sequence.) However, it is well-known that $\mathbb{Q}_p(\mu_{p^\infty})$ is not complete, hence not spherically complete.

(d) is not spherically complete. Assume that $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ is spherically complete. Notice that

$$\left|\widehat{\mathbb{Q}_p(\mu_{p^\infty})}\right| = 0 \cup \left\{p^{\frac{m}{p^n(p-1)}} \;:\; m \in \mathbb{Z},\, n \geq 0\right\}.$$

In particular, $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ is not discretely valued. Choose and fix a sequence of negative rational numbers $r_1 > r_2 > \cdots$ such that

$$r_i \in \left\{-\frac{m}{p^n(p-1)} \;:\; m \in \mathbb{Z}_{>0},\, n \geq 0\right\}$$

and $r := \lim_i r_i$ exists. We can find a sequence of elements $a_1, a_2, \ldots \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ such that $|a_i| = p^{r_i}$ for all i. In particular, we have $|a_1| > |a_2| > \cdots$ and $\lim |a_i| = p^r > 0$. Consider closed balls

$$B_i := \left\{x \in \widehat{\mathbb{Q}_p(\mu_{p^\infty})} \;:\; |x - a_1 - a_2 - \cdots - a_i| \leq |a_i|\right\}.$$

If $|x - a_1 - a_2 - \cdots - a_{i+1}| \leq |a_{i+1}|$, then

$$|x - a_1 - a_2 - \cdots - a_i| \leq |a_{i+1}| < |a_i|.$$

This means $B_1 \supsetneq B_2 \supsetneq \cdots$ is a strictly decreasing sequence of closed balls. By assumption, $B := \cap_{i=1}^\infty B_i$ is nonempty. It is necessarily an open subset of $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, and hence contains at least an element $q \in \mathbb{Q}_p(\mu_{p^\infty})$.

Now, we vary $a = (a_1, a_2, \ldots)$ and write "B_a," "q_a" instead of "B," "q." Running through all possible a's, we obtain uncountably many disjoint B_a's. (Why? If two a's have the same $a_1, \ldots, a_{i-1}$ but $|a_i - a_i'| > |a_{i+1}|$, then the two B_{i+1}'s are disjoint.) On the other hand, from each of these B_a, we have an element

$$q_a \in B_a \cap \mathbb{Q}_p(\mu_{p^\infty}).$$

These q_a's map to distinct elements in $\mathbb{Q}_p(\mu_{p^\infty})/(s)$ where $s \in \mathbb{Q}_p(\mu_{p^\infty})$ has $0 < |s| \leq p^r$. However, $\mathbb{Q}_p(\mu_{p^\infty})/(s)$ is a countable set, a contradiction.

S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Solve every problem.

Problem 1. Let χ be a real valued smooth function with compact support on $\mathbb{R}$. We assume that

$$\int_{\mathbb{R}^1} \chi(x)dx = 1.$$

For all $\varepsilon > 0$, we define

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon}\chi\left(\frac{x}{\varepsilon}\right).$$

Prove that for any given $f \in L^1(\mathbb{R})$, for almost every $x \in \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0}(\chi_\varepsilon * f)(x) = f(x).$$

Problem 2. In last year's Yau College Student Mathematics Contests, four students Tintin, Haddock, Dupont and Dupond made it to the last round of the oral exam in analysis. Professor Yau asked them to compute the Fourier coefficients of the 2π-periodic function F (defined on $\mathbb{R}$):

$$\begin{aligned} &F : (0, 2\pi) \longrightarrow R, \\ &\qquad x \mapsto F(x) = \arctan\left(\frac{x}{2\pi}e^{\sin(x)} + x^{2019}(x - 2\pi) + 2019\sin(x)\right). \end{aligned}$$

Here were their solutions: for $k \neq 0$,

$$\begin{aligned} \text{Tintin} : \widehat{F}(k) &= \frac{\cos(k\pi)}{|k|^{\frac{1}{2}}} + \frac{a}{|k|} + \frac{b}{|k|^3}, \\ \text{Haddock} : \widehat{F}(k) &= \frac{c}{k^2} + \frac{d}{k^4} + \frac{e}{k^6}, \\ \text{Dupont} : \widehat{F}(k) &= \frac{1}{k} + \frac{1}{k^2} + \frac{f}{k^3} + \frac{g}{k^5}, \\ \text{Dupond} : \widehat{F}(k) &= \frac{2019\sqrt{-1}}{k} + \frac{h_k}{k}, \end{aligned}$$

where $a, b, c, d, e, f, g, h_k (k \in \mathbb{Z})$ were constants and $\sum_{k\in\mathbb{Z}} |h_k|^2 < \infty$.

Whose solutions were correct?

Problem 3. Let B_1 be the unit ball centered at the origin in $\mathbb{R}^4$ and $u \in W^{1,2}(B_1) \cap C^\infty(\mathbb{R}^4)$ be a nonnegative real valued function so that

$$-\triangle u \leq u^2,$$

where $\triangle = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$. Prove that, there exists a constant $\varepsilon > 0$, so that if $\|u\|_{L^2(B_1)} \leq \varepsilon$, we have

$$\|\nabla u\|_{L^2(B_{\frac{1}{2}})} \leq 10000\|u\|_{L^2(B_1)},$$

where $B_{\frac{1}{2}}$ is the ball of radius $\frac{1}{2}$ centered at the origin.

Problem 4. Let f and g be two holomorphic functions defined on the entire complex plane $\mathbb{C}$ so that for all $z \in \mathbb{C}$, we have

$$f(z)^{2020} + g(z)^{2020} = 1.$$

Prove that f and g are constants.

Problem 5. We consider the following ordinary differential equation:

$$\begin{cases} x''(t) + x(t) + x(t)^3 = 0, \\ (x(0), x'(0)) = (x_0, 0), \end{cases}$$

where $x(t)$ takes values in $\mathbb{R}$. Prove that for all $x_0 \in \mathbb{R}$, the solution of the above system is periodic.

Problem 6. Let $\alpha \in \mathbb{R}$ and $a_k \in \mathbb{C}$ with $|a_k| < 1$, where $k = 1, 2, \ldots, n$. We consider the following holomorphic map

$$\begin{aligned} f :& \mathbb{C} - \left\{(\overline{a_1})^{-1}, \ldots, (\overline{a_n})^{-1}\right\} \to \mathbb{C}, \\ z \mapsto & f(z) = e^{2\sqrt{-1}\pi\alpha} z \cdot \frac{z - a_1}{1 - \overline{a_1} z} \cdots \frac{z - a_n}{1 - \overline{a_n} z}. \end{aligned}$$

Let $\mathbf{S}^1$ be the unit circle in $\mathbb{C}$, *i.e.*,

$$\mathbf{S}^1 = \left\{z \in \mathbb{C} \;\middle|\; |z| = 1\right\}.$$

Prove that f maps $\mathbf{S}^1$ to itself and f preserves the surface measure (*i.e.*, $d\theta$ in terms of standard polar coordiantes of $\mathbb{R}^2$) of $\mathbf{S}^1$.

S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Solve every problem.

Problem 1. Let χ be a real valued smooth function with compact support on $\mathbb{R}$. We assume that

$$\int_{\mathbb{R}^1} \chi(x)dx = 1.$$

For all $\varepsilon > 0$, we define

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon}\chi\left(\frac{x}{\varepsilon}\right).$$

Prove that for any given $f \in L^1(\mathbb{R})$, for almost every $x \in \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0}(\chi_\varepsilon * f)(x) = f(x).$$

Solution: Let $x_0 \in \mathbb{R}$ be a Lebesgue point of f, *i.e.*, for $t \to 0$, we have

$$\int_{|y|\le t} |f(x_0+y) - f(x_0)|\, dy = o(t),$$

it suffices to show that

$$\lim_{K\to\infty}(\chi_\varepsilon * f)(x_0) = f(x_0).$$

We assume that $\operatorname{supp}(\chi) \subset [-M, M]$. We have

$$\begin{aligned}(\chi_\varepsilon * f)(x_0) - f(x_0) &= \int_{\mathbb{R}} (f(x_0-y) - f(x_0))\, \chi_\varepsilon(y)dy \\ &= \varepsilon^{-1}\int_{|y|\le M} (f(x_0-y) - f(x_0))\, \chi(y)dy.\end{aligned}$$

Thus,

$$\begin{aligned}|\chi_\varepsilon * f)(x_0) - f(x_0)| &\le \varepsilon^{-1}\|\chi\|_{L^\infty}\int_{|y|\le\varepsilon M} |f(x_0-y) - f(x_0)|\, dy \\ &= \|\chi\|_{L^\infty}\varepsilon^{-1}o(\varepsilon M) \\ &= o(1).\end{aligned}$$

This proves the statement.

Problem 2. In last year's Yau College Student Mathematics Contests, four students Tintin, Haddock, Dupont and Dupond made it to the last round of the oral exam in analysis. Professor Yau asked them to compute the Fourier coefficients of the 2π-periodic function F (defined on $\mathbb{R}$):

$$\begin{aligned}&F : (0, 2\pi) \to R, \\ &\qquad x \mapsto F(x) = \arctan\left(\frac{x}{2\pi}e^{\sin(x)} + x^{2019}(x-2\pi) + 2019\sin(x)\right).\end{aligned}$$

Here were their solutions: for $k \neq 0$,

$$\text{Tintin}: \widehat{F}(k) = \frac{\cos(k\pi)}{|k|^{\frac{1}{2}}} + \frac{a}{|k|} + \frac{b}{|k|^3},$$
$$\text{Haddock}: \widehat{F}(k) = \frac{c}{k^2} + \frac{d}{k^4} + \frac{e}{k^6},$$
$$\text{Dupont}: \widehat{F}(k) = \frac{1}{k} + \frac{1}{k^2} + \frac{f}{k^3} + \frac{g}{k^5},$$
$$\text{Dupond}: \widehat{F}(k) = \frac{2019\sqrt{-1}}{k} + \frac{h_k}{k},$$

where $a, b, c, d, e, f, g, h_k (k \in \mathbb{Z})$ were constants and $\sum_{k\in\mathbb{Z}} |h_k|^2 < \infty$.

Whose solutions were correct?

Solution: We remark that F is smooth on $(0, 2\pi)$. At 0, its right limit is 0; at 2π, its left limit is $\frac{\pi}{2}$. In particular, F is not continous on $2\pi\mathbb{Z}$.

Tintin was wrong: the function F is bounded on $(0, 2\pi)$ hence in L^2. Parsevel's identity implies $\widehat{F}(k) \in \ell^2(\mathbb{Z})$. But according to Tintin, $|\widehat{F}(k)| \sim \frac{1}{|k|^{\frac{1}{2}}}$. This is not in ℓ^2.

Haddock was wrong: otherwise, since his coefficients were absolutely summable, the function F should be a continous function (since its Fourier series should be absolutely convergent).

Dupont was wrong: since F is a real valued function, we must have $\widehat{F}(k) = \widehat{F}(-k)$. Dupont's coefficients did not satisfy this condition.

To show that Dupond was wrong, we do the following computations:

$$\begin{aligned}
\widehat{F}(k) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} F(x) dx \\
&= \frac{e^{-k\pi i x}\Phi(x)}{2\pi i k}\Big|_0^{2\pi} + \frac{1}{2\pi i k} \underbrace{\int_0^{2\pi} e^{-ikx} F'(x) dx}_{\in \ell^2, \text{ since } F' \in L^2((0,2\pi))} \\
&= \frac{\sqrt{-1}}{8k} + \frac{h'_k}{k}.
\end{aligned}$$

Now since $h'_k \in \ell^2$, if Dupond was correct, it would imply that $2019\sqrt{-1} - \frac{\sqrt{-1}}{8}$ is square summable. This is absurd.

Problem 3. Let B_1 be the unit ball centered at the origin in $\mathbb{R}^4$ and $u \in W^{1,2}(B_1) \cap C^\infty(\mathbb{R}^4)$ be a nonnegative real valued function so that

$$-\triangle u \leq u^2,$$

where $\triangle = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$. Prove that, there exists a constant $\varepsilon > 0$, so that if $\|u\|_{L^2(B_1)} \leq \varepsilon$, we have

$$\|\nabla u\|_{L^2(B_{\frac{1}{2}})} \leq 10000 \|u\|_{L^2(B_1)},$$

where $B_{\frac{1}{2}}$ is the ball of radius $\frac{1}{2}$ centered at the origin.

Solution: We can choose a smooth cut-off function η so that $\mathrm{supp}(\eta) \subset B_1$, $\eta\big|_{B_{\frac{1}{2}}} \equiv 1$, $\eta \geq 0$ and $|\nabla \eta| \leq 4$. We multiply the $-\triangle u \leq u^2$ by $\eta^2 u$ and integrate by parts. This leads to

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq \int_{B_1} \eta^2 u^3 + 2\int_{B_1} \eta u |\nabla \eta||\nabla u|.$$

According to the Cauchy-Schwarz inequality, we have

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq \int_{B_1} \eta^2 u^3 + 2\int_{B_1} |\nabla \eta|^2 |u|^2 + \frac{1}{2}\int_{B_1} \eta^2 |\nabla u|^2.$$

Hence,

$$\begin{aligned}
\int_{B_1} \eta^2 |\nabla u|^2 &\leq 2\int_{B_1} \eta^2 u^3 + 4\int_{B_1} |\nabla \eta|^2 |u|^2 \\
&\leq 2\int_{B_1} \eta^2 u^3 + 64\int_{B_1} |u|^2.
\end{aligned}$$

On the other hand, we can use four dimension Sobolev inequality to derive

$$\begin{aligned}
\int_{B_1} \eta^2 u^3 &\leq \|u\|_{L^2(B_1)} \left(\int_{B_1} (\eta u)^4\right)^{\frac{1}{2}} \\
&\leq \|u\|_{L^2(B_1)} \times C_4 \int_{B_1} |\nabla(\eta u)|^2,
\end{aligned}$$

where C_4 is the constant coming from the Sobolev inequality. Since $\|u\|_{L^2(B_1)} \leq \varepsilon$, we can proceed as follows

$$\begin{aligned}
\int_{B_1} \eta^2 u^3 &\leq C_4 \varepsilon \int_{B_1} |\nabla(\eta u)|^2 \\
&\leq 2C_4\varepsilon \left(\int_{B_1} |\nabla \eta|^2 |u|^2 + \int_{B_1} |\nabla \eta|^2 |u|^2\right) \\
&\leq 32 C_4 \varepsilon \int_{B_1} |u|^2 + 2C_4\varepsilon \int_{B_1} |u|^2.
\end{aligned}$$

Putting all the inequalities together, we obtain

$$\begin{aligned}
\int_{B_1} \eta^2 |\nabla u|^2 &\leq 2\int_{B_1} \eta^2 u^3 + 4\int_{B_1} |\nabla \eta|^2 |u|^2 \\
&\leq 64 C_4 \varepsilon \int_{B_1} |u|^2 + 4C_4\varepsilon \int_{B_1} |u|^2 + 64\int_{B_1} |u|^2.
\end{aligned}$$

If we take $\varepsilon < \frac{1}{128 C_4}$, we have

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq \frac{1}{2}\int_{B_1} |u|^2 + (64 + \frac{1}{32})\int_{B_1} |u|^2.$$

This leads to the final estimate:

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq (128 + \frac{1}{16})\int_{B_1} |u|^2.$$

Problem 4. Let f and g be two holomorphic functions defined on the entire complex plane $\mathbb{C}$ so that for all $z \in \mathbb{C}$, we have

$$f(z)^{2020} + g(z)^{2020} = 1.$$

Prove that f and g are constants.

Solution: We consider the following holomorphic map

$$\varphi : \mathbb{C} \to \mathbf{P}^3(\mathbb{C}), \ \ z \mapsto (f(z) : g(z) : 1),$$

where $(z_1 : z_2 : z_3)$ is the homogenous coordinates on $\mathbf{P}^3(\mathbb{C})$. Let $\mathscr{C}$ be the curve defined by the homogenous equation

$$\mathscr{C} = \left\{(z_1 : z_2 : z_3) \mid z_1^{2020} + z_2^{2020} = z_3^{2020}\right\} \subset \mathbf{P}^3(\mathbb{C}).$$

It is obviously a smooth curve (by the Jacobian criterion). The maps φ factor through $\mathscr{C}$, *i.e.*, we have a holomorphic map

$$\varphi : \mathbb{C} \to \mathscr{C}.$$

On the other hand, the genus $g(\mathscr{C})$ of the plance curve $\mathscr{C}$ can be computed through the genus formula

$$g(\mathscr{C}) = \frac{1}{2}(2020-1)(2020-2) \geq 2.$$

Therefore, according to the uniformization theorem, the universal covering of $\mathscr{C}$ must be the Poincaré disk $\mathbf{D}$. Hence, we can lift φ to a holomorphic map

$$\widehat{\varphi} : \mathbb{C} \to \mathbf{D},$$

i.e., we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathbf{D} \arrow[d, "\pi"] \\
\mathbb{C} \arrow[ur, dashed, "\widehat{\varphi}"] \arrow[r, "\varphi"] & \mathscr{C}
\end{tikzcd}$$

In particular, $\widehat{\varphi}$ is a bounded entire function, hence a constant map by Liouville's theorem. Therefore, φ is also a constant map.

Problem 5. We consider the following ordinary differential equation:

$$\begin{cases} x''(t) + x(t) + x(t)^3 = 0, \\ (x(0), x'(0)) = (x_0, 0), \end{cases}$$

where $x(t)$ takes values in $\mathbb{R}$. Prove that for all $x_0 \in \mathbb{R}$, the solution of the above system is periodic.

Solution: The key is to observe that

$$E(t) = \frac{1}{2}\xi(t)^2 + \frac{1}{2}x(t)^2 + \frac{1}{4}x(t)^4$$

is conserved, where $\xi(t) = x'(t)$. Therefore, on the phase plane (x, ξ), the solution lies on

$$\frac{1}{2}\xi^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = \frac{1}{2}x_0^2 + \frac{1}{4}x_0^4.$$

The curve is bounded. Hence, $x(t)$ is bounded and the solution to the equation is defined on the whole of $t \in \mathbb{R}$. To show that $x(t)$ is periodic: when $x'(t_0) = 0$, $x(t_0)$ is either the maximum or the minimum of $x(t)$ for all $t \in \mathbb{R}$ (using the fact that $E(t)$ is conserved). Therefore, it suffices to show that $x(t)$ will reach its local maximum and local minimum for some later time. If not, then $x(t)$ must be monotone which is impossible since this would pose a sign condition on $\xi'(t)$.

Problem 6. Let $\alpha \in \mathbb{R}$ and $a_k \in \mathbb{C}$ with $|a_k| < 1$, where $k = 1, 2, \ldots, n$. We consider the following holomorphic map

$$\begin{aligned} f : & \mathbb{C} - \left\{(\overline{a_1})^{-1}, \ldots, (\overline{a_n})^{-1}\right\} \to \mathbb{C}, \\ & z \mapsto f(z) = e^{2\sqrt{-1}\pi\alpha} z \cdot \frac{z - a_1}{1 - \overline{a_1}z} \cdots \frac{z - a_n}{1 - \overline{a_n}z}. \end{aligned}$$

Let $\mathbf{S}^1$ be the unit circle in $\mathbb{C}$, *i.e.*,

$$\mathbf{S}^1 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}.$$

Prove that f maps $\mathbf{S}^1$ to itself and f preserves the surface measure (*i.e.*, $d\theta$ in terms of standard polar coordiantes of $\mathbb{R}^2$) of $\mathbf{S}^1$.

Solution: It suffices to show that

$$\int_{\mathbf{S}^1} \varphi d\theta = \int_{\mathbf{S}^1} \varphi \circ f d\theta, \;\; \forall \varphi \in C^0(\mathbf{S}^1).$$

Let $\overline{\varphi}$ be the harmonic extension of φ to the unique disk

$$\mathbf{D} = \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq 1 \right\},$$

i.e., $\overline{\varphi}$ is the unique solution of the following Dirichlet problem:

$$\begin{cases} -\triangle \overline{\varphi} & = 0, \;\; \text{in } \mathbf{D}, \\ \overline{\varphi}\big|_{\mathbf{S}^1} & = \varphi, \;\; \text{on } \mathbf{S}^1. \end{cases}$$

Since f is holomophic and f preserves $\mathbb{D}$, $\overline{\varphi} \circ f$ is still a harmonic function in D and it is the harmonic extension of $\varphi \circ f$.

We observe that f maps 0 to 0. Therefore, by the mean value property, we have

$$\int_{\mathbf{S}^1} \varphi d\theta = \overline{\varphi}(0) = \overline{\varphi} \circ f(0) = \int_{\mathbf{S}^1} \varphi \circ f d\theta.$$

This completes the proof.

S.-T. Yau College Student Mathematics Contests 2020

Computational and Applied Mathematics

Solve every problem.

Problem 1. Let $f \in C^{k+1}[-1,1]$ and $[-1,1]$ be partitioned into subintervals $I_j = [(j-1)h, jh]$ of width h. Assume p is a polynomial of degree k which approximates f in I_j with

$$\max_{x \in I_j} |p_j(x) - f(x)| \le C_0 h^{k+1},$$

where C_0 is a constant independent of j. Show that there exists an another constant C, independent of j, such that

$$\max_{x \in I_{j\pm1}} |p_j(x) - f(x)| \le C h^{k+1}.$$

(as long as $I_{j\pm1} \subset [-1,1]$, of course).

Problem 2. Consider the iteration

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)} \right) f(x_n)$$

for finding the roots of a two times continuous differentiable function $f(x)$. Assuming the method converges to a simple root x^*, what is the rate of convergence? Justify your answer.

Problem 3. Suppose $\mathbf{A}$ is an $m \times m$ matrix with a complete set of orthonormal eigenvectors $\mathbf{q_1}, \ldots, \mathbf{q_m}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. Assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$ and $\lambda_j \ge \lambda_{j+1}$ for $j = 3, \ldots, m$. Consider the power method $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}/\lambda_1$, with $\mathbf{v}^{(0)} = \alpha_1 \mathbf{q}_1 + \cdots + \alpha_m \mathbf{q}_m$ where α_1 and α_2 are both nonzero. Show that the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{\infty}$ converges linearly to $\alpha_1 \mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

Problem 4. For the initial value problem $y' = f(t,y),\ \ y(0) = y_0$ on the interval $[0,\ T]$, consider the implicit two-step method

$$\begin{aligned} y_{n+1} &= \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(t_{n+1}, y_{n+1}), \\ y_1 &= y_0 + h f(t_1, y_0), \end{aligned}$$

where h is the step size and $t_n = nh$.

(a) What is the order of the accuracy of the scheme?

(b) Check the stability of the scheme by analyzing the stability polynomial?

(c) Find the stability region of the scheme.

Problem 5. Suppose the difference scheme $u^{n+1} = Bu^n$ is stable, and $C(\Delta t)$ is a bounded family of operators. Show that the scheme

$$u^{n+1} = (B + \Delta t C(\Delta t))u^n$$

is stable.

Problem 6. Let A be an $m \times m$ nonsingular matrix. Suppose $\inf_{p_n \in P^n} ||p_n(A)|| = ||p^*(A)|| > 0$ where P^n denotes the set of all degree-n monic polynomials:

$$P^n = \{p : p \text{ is a polynomial of degree } n,\, p(z) = z^n + \cdots\}\,.$$

Prove that p^* is unique.

S.-T. Yau College Student Mathematics Contests 2020

Computational and Applied Mathematics

Solve every problem.

Problem 1. Let $f \in C^{k+1}[-1,1]$ and $[-1,1]$ be partitioned into subintervals $I_j = [(j-1)h, jh]$ of width h. Assume p is a polynomial of degree k which approximates f in I_j with

$$\max_{x\in I_j} \left|p_j(x) - f(x)\right| \le C_0 h^{k+1},$$

where C_0 is a constant independent of j. Show that there exists an another constant C, independent of j, such that

$$\max_{x\in I_{j\pm 1}} \left|p_j(x) - f(x)\right| \le C h^{k+1}.$$

(as long as $I_{j\pm 1} \subset [-1,1]$, of course).

Solution: Pick points $0 \le x_0 < x_1 < \cdots < x_k \le 1$, and let

$$L_i(x) = \prod_{l\neq i} \frac{x - x_l}{x_i - x_l}$$

be the l-th Lagrange polynomial. Let

$$\Lambda = \sum_{i=0}^{k} \max_{x\in[-1,2]} |L_i(x)|\,.$$

On I_j we use rescaled versions with $x_{ji} = (j-1)h + hx_i$, and

$$L_{ji}(x) = \prod_{l\neq i} \frac{x - x_{jl}}{x_{ji} - x_{jl}}.$$

Note that Λ is unchanged with

$$\Lambda = \sum_i \max_{x\in L_{j\pm 1}} \left|L_{ji}(x)\right|.$$

Let f_j be the interpolating polynomial on I_j

$$f_j(x) = \sum_{i=0}^{k} f(x_{ji}) L_{ji}(x),$$

and note that also

$$p_j(x) = \sum_i p(x_{ji}) L_{ji}(x).$$

Then for $x \in I_{j\pm 1}$,

$$\begin{aligned}
\left|p_j(x) - f(x)\right| &\le \left|p_j(x) - f_j(x)\right| + \left|f_j(x) - f(x)\right| \\
&= \left|\sum_{i=0}^{k} (f(x_{ji}) - p_j(x_{ji}) L_{ji}(x)\right| + |R_k f(x)| \\
&\le \max_{x\in I_j} \left|f(x) - p_j(x)\right| \Lambda + \frac{\|f^{(k+1)}\|}{(k+1)!} \max_{x\in I_{j\pm 1}} \left|\prod_{i=0}^{k} (x - x_{ji})\right| \\
&\le C_0 \Lambda h^{k+1} + \frac{\|f^{(k+1)}\|}{(k+1)!} \max_x \left|(2h)^{k+1}\right| \\
&= C h^{k+1},
\end{aligned}$$

$$C = C_0\Lambda + \frac{2^{k+1}\|f^{(k+1)}\|}{(k+1)!}.$$

Problem 2. Consider the iteration

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f(x_n)$$

for finding the roots of a two times continuous differentiable function $f(x)$. Assuming the method converges to a simple root x^*, what is the rate of convergence? Justify your answer.

Solution: The iteration may be rewritten as

$$x_{n+1} = \frac{[x_n f(x_n) - x_n f(x_0)] - [x_n f(x_n) - x_0 f(x_n)]}{f(x_n) - f(x_0)} = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)}.$$

Thus

$$x_{n+1} - x^* = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)} - x^* = \frac{(x_0 - x^*) f(x_n) - (x_n - x^*) f(x_0)}{f(x_n) - f(x_0)}.$$

Taylor's Theorem asserts that there is ξ_n between x_n and x^* such that

$$0 = f(x^*) = f(x_n) + f'(\xi_n)(x^* - x_n) \quad \Rightarrow \quad f(x_n) = f'(\xi_n)(x_n - x^*).$$

This implies

$$x_{n+1} - x^* = \frac{(x_0 - x^*) f'(\xi_n) - f(x_0)}{f(x_n) - f(x_0)}(x_n - x^*).$$

Evaluating the limit as $n \to \infty$, $\xi_n \to x^*$ and

$$\lim_{n\to\infty}\left|\frac{x_{n+1} - x^*}{x_n - x^*}\right| = \lim_{n\to\infty}\left|\frac{(x_0 - x^*) f'(\xi_n) - f(x_0)}{f(x^*) - f(x_0)}\right| = \left|\frac{(x_0 - x^*)\lim_{n\to\infty} f'(\xi_n) - f(x_0)}{0 - f(x_0)}\right|.$$

Applying Taylor's expression one more time, we know there is η between x^* and x_0 such that

$$f(x_0) = f(x^*) + f'(x^*)(x_0 - x^*) + \frac{f''(\eta)}{2}(x_0 - x^*)^2,$$

So

$$f'(x^*)(x_0 - x^*) - f(x_0) = -\frac{f''(\eta)}{2}(x_0 - x^*)^2.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{x_{n+1} - x^*}{x_n - x^*}\right| = \left|\frac{f''(\eta)}{2f(x_0)}\right|(x_0 - x^*)^2.$$

Note the right hand side is dependent only upon x^* and x_0. Since we know $x_n \to x^*$, this shows the rate of convergence is linear.

Problem 3. Suppose $\mathbf{A}$ is an $m \times m$ matrix with a complete set of orthonormal eigenvectors $\mathbf{q_1}, \ldots, \mathbf{q_m}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. Assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$ and $\lambda_j \ge \lambda_{j+1}$ for $j = 3, \ldots, m$. Consider the power method $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}/\lambda_1$, with $\mathbf{v}^{(0)} = \alpha_1\mathbf{q}_1 + \cdots + \alpha_m\mathbf{q}_m$ where α_1 and α_2 are both nonzero. Show that the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{\infty}$ converges linearly to $\alpha_1\mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

Solution: Matrix $\mathbf{A}$ has following eigen-decomposition

$$\mathbf{A} = \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right]\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m\end{bmatrix}\left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right]^{-1},$$

thus

$$\mathbf{A}^k = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m] \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_m^k \end{bmatrix} [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m]^{-1}.$$

The power method reduces to

$$\begin{aligned}
\mathbf{v}^{(k)} &= \mathbf{A}^k \frac{\mathbf{v}^{(0)}}{\lambda_1^k} \\
&= [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m] \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_m^k \end{bmatrix} [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m]^{-1} [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m] \begin{bmatrix} \frac{\alpha_1}{\lambda_1^k} \\ \frac{\alpha_2}{\lambda_1^k} \\ \vdots \\ \frac{\alpha_m}{\lambda_1^k}, \end{bmatrix} \\
&= \alpha_1 \mathbf{q}_1 + \sum_{j=2}^{m} \left(\frac{\lambda_j}{\lambda_1}\right)^k \alpha_j \mathbf{q}_j,
\end{aligned}$$

from this we deduce $\mathbf{v}^{(k)} \to \alpha_1 \mathbf{q}_1$ as $k \to \infty$, since $|\lambda_j/\lambda_1| < 1$ for $j = 2, \ldots, m$.

To show the convergencee is linear with asymptotic constant $C = |\lambda_2/\lambda_1|$ we need to verify the limit

$$\lim_{k\to\infty} \frac{||e^{(k+1)}||}{||e^{(k)}||} = \lim_{k\to\infty} \frac{||\mathbf{v}^{(k+1)} - \alpha_1 \mathbf{q}_1||}{||\mathbf{v}^{(k)} - \alpha_1 \mathbf{q}_1||} = \left|\frac{\lambda_2}{\lambda_1}\right| \qquad \text{(here } ||\cdot|| \text{ denotes the } L_2\text{-norm).}$$

Note that $e^{(k)} = \sum_{j=2}^{m} \left(\frac{\lambda_j}{\lambda_1}\right)^k \alpha_j \mathbf{q}_j$, using the orthonormality of the eigenvectors we have

$$||e^{(k)}||^2 = \sum_{j=2}^{m} \left|\frac{\lambda_j}{\lambda_1}\right|^{2k} |\alpha_j|^2 = \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2k} |\alpha_j|^2\right),$$

similarly

$$||e^{(k+1)}||^2 = \left|\frac{\lambda_2}{\lambda_1}\right|^{2(k+1)} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2(k+1)} |\alpha_j|^2\right).$$

Thus

$$\begin{aligned}
\lim_{k\to\infty} \frac{||e^{(k+1)}||}{||e^{(k)}||} &= \lim_{k\to\infty} \left(\frac{\left|\frac{\lambda_2}{\lambda_1}\right|^{2(k+1)} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2(k+1)} |\alpha_j|^2\right)}{\left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2k} |\alpha_j|^2\right)} \right)^{\frac{1}{2}} \\
&= \left|\frac{\lambda_2}{\lambda_1}\right| \frac{|\alpha_2|}{|\alpha_2|} \qquad \text{(we have used } |\lambda_2| > |\lambda_3| \geq |\lambda_j| \text{ for } j > 3 \text{)} \\
&= \left|\frac{\lambda_2}{\lambda_1}\right| \qquad \text{(since } \alpha_2 \neq 0 \text{).}
\end{aligned}$$

Problem 4. For the initial value problem $y' = f(t, y)$, $y(0) = y_0$ on the interval $[0, T]$, consider the implicit two-step method

$$\begin{aligned}
y_{n+1} &= \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(t_{n+1}, y_{n+1}), \\
y_1 &= y_0 + h f(t_1, y_0),
\end{aligned}$$

where h is the step size and $t_n = nh$.

(a) What is the order of the accuracy of the scheme?

(b) Check the stability of the scheme by analyzing the stability polynomial?

(c) Find the stability region of the scheme.

Solution: **(a)** Let $y(t)$ be the exact solution, then the truncation error of form

$$h\tau_{n+1} := y(t_{n+1}) - \left(\frac{4}{3}y(t_n) - \frac{1}{3}y(t_{n-1}) + \frac{2h}{3}f(t_{n+1}, y(t_{n+1}))\right)$$

can be estimated by using Taylor expansion to every term involved:

$$\begin{aligned}
y(t_{n+1}) &= y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + O(h^4),\\
-\frac{1}{3}y_{n-1} &= -\frac{1}{3}y_n + \frac{1}{3}hy_n' - \frac{1}{6}h^2y_n'' + \frac{1}{18}h^3y_n''' + O(h^4),\\
\frac{2h}{3}f(t_{n+1}, y_{n+1}) &= \frac{2h}{3}y_{n+1}' = \frac{2}{3}hy_n' + \frac{2}{3}h^2y_n'' + \frac{1}{3}h^3y_n''' + O(h^4).
\end{aligned}$$

Hence

$$\begin{aligned}
h\tau_{n+1} &= \left[y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + O(h^4)\right] - \left[y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{7}{18}h^3y_n''' + O(h^4)\right]\\
&= -\frac{2}{9}h^3y_n''' + O(h^4).
\end{aligned}$$

The method has order of accuracy 2.

(b) Apply the method to the case $f = 0$, then

$$y_{n+1} - \frac{4}{3}y_n + \frac{1}{3}y_{n-1} = 0,$$

when for ansatz of form $y_n = \gamma^n$ gives the stability polynomial

$$\gamma^2 - \frac{4}{3}\gamma + \frac{1}{3} = 0,$$

which has nonzero roots $\gamma = 1, \frac{1}{3}$. Since $|\gamma| \le 1$ and $\gamma = 1$ as a single root, the method is stable.

(c) Consider the stiff problem $y' = \lambda y$. The method becomes

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}h\lambda y_{n+1},$$

which has stability polynomial

$$(3 - 2h\lambda)\gamma^2 - 4\gamma + 1 = 0.$$

So the stability region is given by

$$\left|\frac{4 \pm \sqrt{16 - 4(3 - 2h\lambda)}}{2(3 - 2h\lambda)}\right| < 1,$$

i.e.,

$$R = \left\{h\lambda \in C : \quad \left|\frac{2 \pm \sqrt{1 - 2h\lambda}}{3 - 2h\lambda}\right| < 1\right\}.$$

Problem 5. Suppose the difference scheme $u^{n+1} = Bu^n$ is stable, and $C(\Delta t)$ is a bounded family of operators. Show that the scheme

$$u^{n+1} = (B + \Delta t C(\Delta t))u^n$$

is stable.

Solution: Suppose $\|B^k\| \le K_1$ for $0 \le k \le n$ and $\|C(\Delta t)\| \le K_2$. It suffices to show $(B + \Delta t C(\Delta t))^n$ is bounded for $n\Delta t \le T$. This will consist of 2^n terms, of which $\left(\begin{array}{c} n \\ j \end{array}\right)$ terms involve j factors $\Delta t C$ interspersed in $n - j$ factors B; the latter can occur in at most $j + 1$ sequences of consecutive factors, the norm of each sequence being bounded by K_1, and hence the norm of each such term by $K_2^j K_1^{j+1}$. Thus overall we obtain the bound

$$\begin{aligned} \|(B + \Delta t C(\Delta t))^n\| &\le \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) K_1^{j+1} (\Delta t K_2)^j \\ &= K_1 (1 + \Delta t K_1 K_2)^n \\ &\le K_1 e^{n\Delta t K_1 K_2} \end{aligned}$$

which is bounded for $n\Delta t \le T$.

Problem 6. Let A be an $m \times m$ nonsingular matrix. Suppose $\inf_{p_n \in P^n} ||p_n(A)|| = ||p^*(A)|| > 0$ where P^n denotes the set of all degree-n monic polynomials:

$$P^n = \{p : p \text{ is a polynomial of degree } n,\ p(z) = z^n + \cdots\}.$$

Prove that p^* is unique.

Solution: We prove by contradiction. Assuming there are two p_i for $i = 1, 2$ as minimizers, then $p = (p_1+p_2)/2$ shares the same 2-norm,

$$\|p_1\| = \|p_2\| = \|p\| = \sigma_1,$$

where σ_1 is the largest singular value. Let the SVD of p be

$$p(A) = U\mathrm{diag}(\sigma_1, \ldots, \sigma_n)V^*.$$

Suppose σ_1 is $J-$fold, with left and right singular vectors $u_1, \ldots, u_J$ and $v_1, \ldots, v_J$, respectively.

By convexity of the 2-norm, we have

$$\sigma_1 = \|p(A)v_j\| \le \frac{1}{2}\left(\|p_1(A)v_j\| + \|p_2(A)v_j\|\right) \le \sigma_1,$$

which implies that

$$\|p_1(A)v_j\| = \|p_2(A)v_j\| = \sigma_1$$

and

$$(p_1 - p_2)v_j = 0, 1 \le j \le J.$$

Similarly we can show that $(p_1^* - p_2^*)u_j = 0$.

Construct $q \in P^n$ using $p_1 - p_2$ so that $qv_j = 0$ and $q^* u_j = 0$. For a fixed $\epsilon \in (0, 1)$, define

$$p_\epsilon = (1 - \epsilon)p + \epsilon q \in P^n.$$

Hence

$$p_\epsilon^* p_\epsilon v_j = (1 - \epsilon)p_\epsilon^* p(A)v_j = (1 - \epsilon)\sigma_1 p_\epsilon^* u_j = (1 - \epsilon)^2\sigma_1^2 v_j.$$

This says that p_ϵ has right singular vector $v_1, \ldots, v_J$ corresponding to the singular value $(1-\epsilon)\sigma_1$.

There are two cases to consider:

(1) $\|p_\epsilon\| = (1-\epsilon)\sigma_1 < \sigma_1$ is not the largest singular value, we see a contradiction.

(2) None of $v_1, \ldots, v_J$ correspond to the largest singular value of p_ϵ. Using this fact and

$$p(A) = U\Sigma V^*,$$

we have

$$\begin{aligned}
\|p_\epsilon(A)\| &= \|p_\epsilon(A)V\| = \|p_\epsilon(A)[v_{J+1}, \ldots, v_n]\| \\
&= \|(1-\epsilon)p(A)[v_{J+1}, \ldots v_n] + \epsilon q(A)[v_{J+1}, \ldots, v_n]\| \\
&\leq (1-\epsilon)\|[u_{J+1}, \ldots u_n]\mathrm{diag}(\sigma_{J+1}, \ldots, \sigma_n)\| + \epsilon\|q(A)[v_{J+1}, \ldots, v_n]\| \\
&\leq (1-\epsilon)\sigma_{J+1} + \epsilon\|q(A)[v_{J+1}, \ldots, v_n]\| \to \sigma_{J+1} < \sigma_J = \sigma_1
\end{aligned}$$

for ϵ small. This again leads to a contradiction. The uniqueness proof is complete.

S.-T. Yau College Student Mathematics Contests 2020

Geometry and Topology

Solve every problem.

Problem 1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$.

(a) Find a 6-form α on $\mathbb{R}^7 \setminus \{0\}$ such that

$$d\alpha = 0, \quad \text{and} \quad \int_{S^6} \alpha = 1.$$

(b) For any smooth map $f : S^{11} \to S^6$, show that there exists a 5-form φ on S^{11} such that

$$f^*\alpha = d\varphi. \tag{1}$$

(c) Let

$$H(f) = \int_{S^{11}} \varphi \wedge d\varphi.$$

Show that $H(f)$ is independent of the choice of φ satisfying (1).

(d) Show that $H(f)$ is an even integer, for any smooth map $f : S^{11} \to S^6$.

Problem 2. For any $h \in C^\infty(\mathbb{R}^2)$ and $h > 0$ on $\mathbb{R}^2$, define the Ricci curvature $\mathrm{Ric}(h)$ associated with h by

$$\mathrm{Ric}(h) = \frac{1}{h}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\log h,$$

where (x, y) are the standard Cartesian coordinates in $\mathbb{R}^2$. Either construct a positive smooth function h_1 such that $\mathrm{Ric}(h_1) = 1$, or show that no such function h_1 exists.

Problem 3. Let M be an n-dimensional Riemannian manifold, and $p \in M$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, and let $\{x^1, \ldots, x^n\}$ be a coordinate system of M centered at p such that

$$\exp_p^{-1}(q) = \sum_{j=1}^{n} x^j(q)e_j,$$

where $\exp_p$ denotes the exponential map. Let $\gamma(t) = \exp_p(te_1)$, $0 \le t \le \delta$, where δ is a positive constant less than 1.

(a) For $2 \le \alpha \le n$, which one of the following,

$$t\frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)} \quad \text{or} \quad \frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)},$$

is a Jacobi field along $\gamma(t)$? Prove your assertion.

(b) Denote

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \le i, j \le n.$$

Compute

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \quad \text{at the point } p.$$

(c) Show that

$$\max_{0 \le t \le \delta} \left| \frac{\partial g_{22}}{\partial x^1}(\gamma(t)) \right| \le C\delta A,$$

where $C > 0$ is a constant depending only on n, and A is the C^0-bound of the curvature tensor of M along $\gamma(t)$, for $0 \le t \le \delta$.

Problem 4. Let $\mathrm{SO}(n)$ be the set of $n \times n$ orthogonal real matrices with determinant equal to 1. Endow $\mathrm{SO}(n)$ with the relative topology as a subspace of Euclidean space $\mathbb{R}^{n^2}$.

(a) Show that $\mathrm{SO}(n)$ is compact.

(b) Is $\mathrm{SO}(3)$ homeomorphic to the real projective space $\mathbb{RP}^3$? Prove your assertion.

(c) Compute the fundamental group of $\mathrm{SO}(2020)$.

Problem 5. Let X be a topological space and $\pi : \mathbb{R}^2 \to X$ a covering map. Let K be a compact subset of X and B the closed unit ball centered at the origin in $\mathbb{R}^2$.

(a) Suppose $\pi : \mathbb{R}^2 \setminus B \to X \setminus K$ is a homeomorphism. Show that $\pi : \mathbb{R}^2 \to X$ is a homeomorphism.

(b) Suppose $\mathbb{R}^2 \setminus B$ is homeomorphic to $X \setminus K$, where the homeomorphism may not be π. Is X necessarily homeomorphic to $\mathbb{R}^2$? Prove your assertion.

Problem 6. Let F_n be the free group of rank n,

(a) Give an example of a finite connected graph such that its fundamental group is F_2.

(b) Does F_2 contain a proper subgroup isomorphic to F_2?

(c) Does F_2 contain a proper finite index subgroup isomorphic to F_2?

S.-T. Yau College Student Mathematics Contests 2020

Geometry and Topology

Solve every problem.

Problem 1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$.

(a) Find a 6-form α on $\mathbb{R}^7 \setminus \{0\}$ such that

$$d\alpha = 0, \quad \text{and} \quad \int_{S^6} \alpha = 1.$$

(b) For any smooth map $f : S^{11} \to S^6$, show that there exists a 5-form φ on S^{11} such that

$$f^*\alpha = d\varphi. \tag{1}$$

(c) Let

$$H(f) = \int_{S^{11}} \varphi \wedge d\varphi.$$

Show that $H(f)$ is independent of the choice of φ satisfying (1).

(d) Show that $H(f)$ is an even integer, for any smooth map $f : S^{11} \to S^6$.

Solution: Problem 1 is essentially taken from [BT82, p. 227ff] (originally due to Hopf and J. H. C. Whitehead) and [Hat02, §4.B] (originally due to Steenrod). It covers differential forms, exterior product, de Rham cohomology, behavior under pull-back, integration on manifolds in geometry; one approach to Part (d) uses singular cohomology and cohomology operations in topology.

For Part (a), the 6-form can be given by

$$\alpha = \frac{\sum_{j=1}^{7}(-1)^{j-1}x^j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^7}{\mathrm{Vol}(S^6)\cdot[(x^1)^2 + \cdots + (x^7)^2]^{7/2}} \quad \text{in } \mathbb{R}^7 \setminus \{0\},$$

Part (b) follows from the fact that $H^6(S^{11}) = 0$.

Part (c) follows from Stokes' theorem. Note that $H(f)$ is also independent of the choice of generator α of $H^6_{\mathrm{deR}}(S^6)$.

To see Part (d), one way is to consider the CW complex $K = B \cup_f S^6$ obtained by attaching a closed 12-cell B to S^6 via $f : S^{11} = \partial B \to S^6$. Then, the cohomology $H^q(K;\mathbb{Z})$ is equal to $\mathbb{Z}$ for $q = 0, 6, 12$ and 0 else. Let α and β be the generators of $H^q(K;\mathbb{Z})$ for $q = 6$ and 12, respectively. We claim the cup-product $\alpha^2 = H(f)\beta$ in $H^{12}(K;\mathbb{Z})$. Indeed, in our previous formulation $H(f) = \int_{\partial B} \varphi \wedge f^*(\alpha) = \int_K (f^*\alpha) \wedge (f^*\alpha)$ which is the same as taking the cap product $[K] \frown \alpha^2 = H(f)[K] \frown \beta$. This shows $H(f) \in \mathbb{Z}$. Suppose $H(f)$ is an odd number for some $f : S^{11} \to S^6$. Then $\mathrm{Sq}^6(\sigma) = \sigma^2 = \tau$ in $H^{12}(K;\mathbb{Z}_2)$, where $\mathrm{Sq}^i : H^q(K;\mathbb{Z}_2) \to H^{q+i}(K;\mathbb{Z}_2)$ is the Steenrod square and σ and τ are generators of $H^q(K;\mathbb{Z}_2)$ for $q = 6, 12$. But $\mathrm{Sq}^6 = \mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^5\mathrm{Sq}^1$ by the Adem relation [Hat02, §4.L]. Note that $\mathrm{Sq}^i(\sigma) = 0$ for $1 \le i \le 5$ for dimension reasons. Hence, $\tau = \mathrm{Sq}^6(\sigma) = 0$, a contradiction. An alternate way to see $H(f) \in \mathbb{Z}$ is to use the intersection-theoretic linking number as in [BT82] or [Whi57, §33].

Problem 1 counts 30 points in total. Part (a)–(c) counts 5 points each, and Part (d) counts 15 points, for instance, $H(f) \in \mathbb{Z}$ (7pts), and $H(f)$ is even (8pts).

Problem 2. For any $h \in C^\infty(\mathbb{R}^2)$ and $h > 0$ on $\mathbb{R}^2$, define the Ricci curvature $\mathrm{Ric}(h)$ associated with h by

$$\mathrm{Ric}(h) = \frac{1}{h}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) \log h,$$

where (x, y) are the standard Cartesian coordinates in $\mathbb{R}^2$. Either construct a positive smooth function h_1 such that $\mathrm{Ric}(h_1) = 1$, or show that there is no such function h_1 exists.

Solution: Problem 2 covers manifolds of nonpositive curvature, Riemannian metrics, curvatures. It can be viewed as a Schwarz type lemma, or a baby version of the generalized maximum principle of Yau [Yau75], and Cheng-Yau [CY75].

The problem counts 20 points. The first part has negative answer (10 points, which rewards those students with good intuition or right sense). The second part, proof of the assertion, counts 10 points.

One way is to show f has an upper bound and apply the generalized maximum principle. Alternatively one can also apply integration method. Since we have equality here, there will be more interesting solutions.

Problem 3. Let M be an n-dimensional Riemannian manifold, and $p \in M$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, and let $\{x^1, \ldots, x^n\}$ be a coordinate system of M centered at p such that

$$\exp_p^{-1}(q) = \sum_{j=1}^n x^j(q) e_j,$$

where $\exp_p$ denotes the exponential map. Let $\gamma(t) = \exp_p(te_1)$, $0 \le t \le \delta$, where δ is a positive constant less than 1.

(a) For $2 \le \alpha \le n$, which one of the following,

$$t\frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)} \quad \text{or} \quad \frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)},$$

is a Jacobi field along $\gamma(t)$? Prove your assertion.

(b) Denote

$$g_{ij} = \Big\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \Big\rangle, \quad 1 \le i, j \le n.$$

Compute

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \quad \text{at the point } p.$$

(c) Show that

$$\max_{0 \le t \le \delta} \Big| \frac{\partial g_{22}}{\partial x^1}(\gamma(t)) \Big| \le C\delta A,$$

where $C > 0$ is a constant depending only on n, and A is the C^0-bound of the curvature tensor of M along $\gamma(t)$, for $0 \le t \le \delta$.

Solution: Problem 3 covers Jacobi fields, Riemannian metrics, curvature, the exponential map, and geodesic.

For Part (a), only $t\partial/\partial x^\alpha$ is a Jacobi field along $\gamma(t)$. For Part (b), one way is to let

$$g_{22} = \frac{1}{t^2}\langle J, J\rangle, \quad J \equiv t\frac{\partial}{\partial x^2},$$

and use the Jacobi equation to obtain $|J|^2 = 1 + t^2 R_{1212}/3 + O(t^3)$; hence,

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}(0) = \frac{2}{3}R_{1212}.$$

For Part (c), one way is to write

$$\frac{\partial g_{22}}{\partial x^1} = \frac{1}{t^3}(t\langle J, J\rangle' - 2\langle J, J\rangle)$$

and apply the calculus identity

$$th'(t) - 2h(t) = -2h(0) - h'(0)t + t^2\int_0^1 (2\tau - 1)h''(\tau t)d\tau$$

to $h(t) = |J|^2(t)$. Note that $h'' = 2R(\gamma', J, \gamma', J) + 2\langle J', J'\rangle$. One still needs to express $\langle J', J'\rangle$ in terms of integration of curvature; thus,

$$\begin{aligned}\frac{\partial g_{22}}{\partial x^1} &= \frac{2}{t}\int_0^1 (2\tau - 1)R(\gamma', J, \gamma', J)(\tau t)d\tau \\ &+ \int_0^1 (2\tau - 1)\tau\int_0^1 [R(\gamma', J, \gamma', J) + R(\gamma', J, \gamma', J')](\tau t\rho)d\rho d\tau.\end{aligned}$$

The estimate follows from the comparison theorem $|R(\gamma', J, \gamma', J)|(\tau t) \le CAt^2\tau^2$ and $|R(\gamma', J, \gamma', J')|(\tau\rho t) \le CA\tau\rho t$.

Part (b) is taken from [LP87, p. 60] and [SY94, p. 210, Lemma 3.4].

The point of Part (c) is that the constant A does not depend on the bounds of derivatives of curvature; a version of (c) is used in [WY20, (2.24)].

This problem counts 30 points, with Part (a) 5 points, Part (b) 10 points, and Part (c) 15 points.

Problem 4. Let $\mathrm{SO}(n)$ be the set of $n \times n$ orthogonal real matrices with determinant equal to 1. Endow $\mathrm{SO}(n)$ the relative topology as a subspace of Euclidean space $\mathbb{R}^{n^2}$.

- **(a)** Show that $\mathrm{SO}(n)$ is compact.
- **(b)** Is $\mathrm{SO}(3)$ homeomorphic to the real projective space $\mathbb{RP}^3$? Prove your assertion.
- **(c)** Compute the fundamental group of $\mathrm{SO}(2020)$.

Solution: Problem 4 covers basics of matrix Lie group $\mathrm{SO}(n)$ in geometry, and fundamental groups, covering spaces, fibrations and the long exact sequence of fibration in topology.

Part (a) follows from the closedness and boundedness of $\mathrm{SO}(n)$.

For Part (b), note that $SO(3)$ consists of all rotations in $\mathbb{R}^3$ about the origin. Each such rotation, except the identity, is fixed by its axis and an angle $-\pi \le \theta \le \pi$. Defines a map ψ from the closed unit ball $B \subset \mathbb{R}^3$ to $\mathrm{SO}(3)$, by sending $x \in B \setminus \{0\}$ to the rotation of angle $|x|\pi$ around axis $x/|x|$, and sending $x = 0$ to the identity. Then, ψ is continuous. Note that a rotation of angle π is the same as a rotation of angle $-\pi$. Thus, $\psi(x) = \psi(-x)$ for $x \in \partial B = S^2$. This means ψ induces a map $B/\sim_{\partial B} \to \mathrm{SO}(3)$, where $x_1 \sim x_2$, $x_1, x_2 \in \partial B$ if and only if $x_1 = -x_2$. Observe that $B/\sim_{\partial B}$ is homeomorphic to the quotient of the upper hemi-sphere $S^3_+/\sim_{S^2}$ by identifying the antipodal points on its equator S^2; the latter is precisely $\mathbb{RP}^3$. One can show the induced maps from $\mathbb{RP}^3$ to $\mathrm{SO}(3)$ is continuous and bijective, and hence, it is an homeomorphism.

For Part (c), one way is to apply the exact sequence of homotopy groups to fibration

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r] & \mathrm{SO}(n+1) \arrow[d] \\ & S^n \end{tikzcd} \tag{2}$$

to obtain $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n+1))$ for $n \geq 3$. On the other hand, by Part (b) we obtain $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, by the fact that S^3 is a double cover of $\mathbb{RP}^3$. Combining these yields $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for all $n \geq 3$; this in particular holds for $n = 2020$. Part (b) is taken from [Hat02, §3D], the fibration 2 or the fact of homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ can be found in [Hat02, 4D.3] and [War83, p. 126], respectively.

This problem counts 30 points, with Part (a) 5 points, Part (b) 10 points, and Part (c) 15 points.

Problem 5. Let X be a topological space and $\pi : \mathbb{R}^2 \to X$ a covering map. Let K be a compact subset of X and B the closed unit ball centered at the origin in $\mathbb{R}^2$.

(a) Suppose $\pi : \mathbb{R}^2 \setminus B \to X \setminus K$ is a homeomorphism. Show that $\pi : \mathbb{R}^2 \to X$ is a homeomorphism.

(b) Suppose $\mathbb{R}^2 \setminus B$ is homeomorphic to $X \setminus K$, where the homeomorphism may not be π. Is X necessarily homeomorphic to $\mathbb{R}^2$? Prove your assertion.

Solution: Problem 5 is taken from a problem that I made for the PhD Preliminary Topology Exam at the University of Connecticut, August 2018. It covers the covering spaces, which was motivated by a conversation with L. H. Huang on general relativity.

For Part (a), it suffices to show $\#\pi^{-1}(y) = 1$, $y \in X$. Pick a sequence $\{x_k\}$ in $\mathbb{R}^2 \setminus B$ that tends to infinity; say x_k has coordinates $(2k, 0)$ for $k \geq 1$. Then, $\pi(x_k)$ has no limit in X, in view of the homeomorphism $\pi : \mathbb{R}^2 \setminus B \to X \setminus K$. Suppose $\pi^{-1}(\pi(x_k))$ contains more than one point; say $x_k' \neq x_k$. Then, x_k' has to lie in B. By compactness of B, a subsequence $\{x_{k_l}'\}$ converges to a point $x_* \in B$. This implies $\pi(x_{k_l})$ converges to a point $p(x_*)$ in X, a contradiction.

The answer to (b) is "no." For example, let X be the torus $S^1 \times S^1$ and $K = (\{a\} \times S^1) \cup (S^1 \times \{b\}) \cup D$ where D is a small disk away from the two circles. Then, $X \setminus K$ is homeomorphic to $\mathbb{R}^2 \setminus B$, but $\mathbb{R}^2$ is not homeomorphic to X.

This problem counts 20 points, with 10 points each part.

Problem 6. Let F_n be the free group of rank n,

(a) Give an example of a finite connected graph such that its fundamental group is F_2.

(b) Does F_2 contain a proper subgroup isomorphic to F_2?

(c) Does F_2 contain a proper finite index subgroup isomorphic to F_2?

Solution: Let $\pi_1(X)$ and $\chi(X)$ be the fundamental group and Euler characteristic of a finite connected graph X respectively. Recall several facts:

(1) $\pi_1(X)$ is a free group of rank $1 - \chi(X)$.

(2) The correspondence between the subgroups of $\pi_(X)$ and the covering spaces of X, and coverings of graphs are still graphs.

(3) $\chi(\tilde{X}) = d\chi(X)$ for a finite covering $\tilde{X}$ of degree d of X.

Now we have the answer as below:

Part (b), Yes: pick any graph X of $\chi(X) = -1$, say the figure eight, let $\tilde{X}$ be the double covering of X. Then we have $\chi(\tilde{X}) = -2$ by (3), $\chi(X) = F_2$, $\chi(\tilde{X}) = F_3$ by (1), so F_2 has a subgroup F_3 by (2). Clearly F_3 has a proper subgroup isomorphic to F_2. So F_2 contains a proper subgroup isomorphic to F_2.

Part (c), No: Pick X as in Part (a). If F_2 contains a subgroup G of index d isomorphic to F_2, $d > 1$, then there is a degree d covering $\tilde{X}$ of X with $\pi_(\tilde{X}) = G$ by (2), and $\chi(\tilde{X}) = -d$, so the rank of $\pi_1(\tilde{X}) = 1 + d > 2$ by (1), which contradicts that $G = F_2$.

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Probability and Statistics

Solve every problem.

Part I: Probability

Problem 1. Let X be an essentially bounded random variable with mean zero. Show that

$$\mathbb{E}e^X \leq \cosh \|X\|_\infty,$$

where $\cosh x = \frac{e^x+e^{-x}}{2}$ is the hyperbolic cosine function.

Problem 2. Let λ be a positive number. Suppose that X is a random variable with $\mathbb{E}|X| < \infty$. Suppose that

$$\lambda \mathbb{E} f(X+1) = \mathbb{E}\{X f(X)\}$$

for all bounded smooth functions. Show that X has the Poisson distribution Poisson(λ).

Problem 3. Consider the random walk

$$S_n = a + X_1 + X_2 + \cdots + X_n,$$

where a is a positive integer and $\{X_i\}$ are independent and identically distributed random variables with a common distribution

$$\mathbb{P}\{X_i = 1\} = p, \quad \mathbb{P}\{X_i = -1\} = 1 - p.$$

Let $\tau_0 = \inf\{n : S_n = 0\}$ be the first time the random walk reaches the state $x = 0$. For all $p \in [0, 1]$ find the probability $\mathbb{P}_a\{\tau_0 < \infty\}$ that the random walk will eventually hit the state $x = 0$.

Problem 4. Let $Z = (X, Y)$ be an $\mathbb{R}^2$-valued random variable such that (1) X and Y are independent; (2) both X and Y have mean zero and finite (nonvanishing) second moments; (3) the distribution of Z is invariant under the rotation counter-clockwise around the origin by an angle θ not a multiple of 90 degrees. Show that X and Y must be normal random variables with the same variance.

Part II: Statistics

The following collection of questions concerns the design of a randomized experiment where the N units to be randomized to drug A or drug B are people, for whom we have a large number of background covariates, collectively labelled X (e.g., age, sex, blood pressure, height, weight, occupational status, history of heart disease, family history of heart disease). The objective is to assign approximately half to drug A and half to drug B where the means of each of the X variables (and means of non-linear functions of them, such as squares or products) are close to equal in the two groups. Instead of using classical methods of design, such as blocking or stratification, the plan is to use modern computers to try many random allocations and discard those allocations that are considered unacceptable according to a pre-determined criterion for balanced X means, in particular an affinely invariant measure such as the Mahalanobis distance between the means of X in the two groups. After an acceptable allocation is found, outcome variables will be measured, and their means will be compared in group A and group B to estimate a treatment effect.

Problem 5. Prove that if the two groups are of the same size (*i.e.*, $N/2$ for even N), this plan will result in unbiased estimates of the A versus B casual effect based on the sample means of Y in groups A and B, where Y is any linear function of X.

Problem 6. Provide a counter-example to the assertion that **Problem 5** is true in small samples with odd N.

S.-T. Yau College Student Mathematics Contests 2020

Probability and Statistics

Solve every problem.

Part I: Probability

Problem 1. Let X be an essentially bounded random variable with mean zero. Show that

$$\mathbb{E}e^X \leq \cosh \|X\|_\infty,$$

where $\cosh x = \frac{e^x+e^{-x}}{2}$ is the hyperbolic cosine function.

Solution: The equation of the straight line connecting the two points $(-M, e^{-M})$ and (M, e^M) on the curve $y = e^x$ is

$$y = \frac{e^M + e^{-M}}{x} + \left(\frac{e^M - e^{-M}}{2M}\right) x = \cosh M + \left(\frac{\sinh M}{M}\right) x$$

where $\sinh M = \frac{e^M - e^{-M}}{2}$ is the hyperbolic sine function. Since the function $y = e^x$ is a convex function, it lies under the the straight line on the interval $[-M, M]$, hence

$$e^x \leq \cosh M \left(\frac{\sinh M}{M}\right) x. \tag{1}$$

The above inequality can also be proved as follows. We write $x \in [-M, M]$ as a convex combination of M and $-M$ as follows:

$$x = \frac{1}{2}\left(1 + \frac{x}{M}\right) \cdot M + \frac{1}{2}\left(1 - \frac{x}{M}\right) \cdot (-M).$$

From the convexity of the function e^x we have

$$e^x \leq \frac{1}{2}\left(1 - \frac{x}{M}\right) e^M + \frac{1}{2}\left(1 - \frac{x}{M}\right) e^{-M} = \cosh M + \left(\frac{\sinh M}{M}\right) x.$$

Now let $M = \|X\|_\infty$. Because $|X| \leq M$ almost surely from (1) we have

$$e^X \leq \cosh M + \left(\frac{\sinh M}{M}\right) X$$

almost surely. Taking the expectation and using the assumption that $\mathbb{E}X = 0$ we obtain $\mathbb{E}e^X \leq \cosh M$.

Problem 2. Let λ be a positive number. Suppose that X is a random variable with $\mathbb{E}|X| < \infty$. Suppose that

$$\lambda \mathbb{E} f(X+1) = \mathbb{E}\{X f(X)\}$$

for all bounded smooth functions. Show that X has the Poisson distribution Poisson(λ).

Solution: We will apply the identity in the problem to the bounded smooth function $f(x) = e^{itx}$ (for a fixed $t \in \mathbb{R}$). The left-hand side is equal to $\lambda e^{it}\phi(t)$, where $\phi(t) = \mathbb{E}e^{itX}$ is the characteristic function of X. The right-hand side is $\mathbb{E}[Xe^{itX}] = -i\phi'(t)$. Note that under the assumption $\mathbb{E}|X| < \infty$ the function $\phi(t)$ is continuously differentiable and the differentiation under the expectation is justified by the dominated convergence theorem. Therefore $\phi'(t) = i\lambda e^{it}\phi(t)$.

Let $g(t) = \phi(t)e^{-\lambda(e^{it}-1)}$. We have

$$g'(t) = \left\{\phi'(t) - i\lambda e^{it}\phi(t)\right\} e^{-\lambda(e^{it}-1)} = 0.$$

Hence $g(t)$ is a constant, *i.e.*, $g(t) = g(0) = 1$. It follows that the characteristic function of the random variable X is $\phi(t) = e^{\lambda(e^{it}-1)}$. This is the characteristic function of the distribution Poisson(λ). By the uniqueness theorem for characteristic functions the distribution of X must be Poisson(λ).

Problem 3. Consider the random walk

$$S_n = a + X_1 + X_2 + \cdots + X_n,$$

where a is a positive integer and $\{X_i\}$ are independent and identically distributed random variables with a common distribution

$$\mathbb{P}\{X_i = 1\} = p, \quad \mathbb{P}\{X_i = -1\} = 1 - p.$$

Let $\tau_0 = \inf\{n : S_n = 0\}$ be the first time the random walk reaches the state $x = 0$. For all $p \in [0, 1]$ find the probability $\mathbb{P}_a\{\tau_0 < \infty\}$ that the random walk will eventually hit the state $x = 0$.

Solution: It is clear that $\mathbb{P}_a\{\tau_0 < \infty\} = 1$ and 0 for $p = 0$ and $p = 1$, respectively. We assume that $0 < p < 1$. Let $q = 1 - p$ for simplicity. Consider the function

$$f(x) = \left(\frac{q}{p}\right)^x.$$

It is easy to verify that $f(S_n)$ is a martingale. For each integer $b > a$ let $\tau_b = \inf\{n : S_n = b\}$ and $\tau = \tau_0 \wedge \tau_b$. Then the stopped martingale $f(S_{n\wedge\tau})$ is bounded. By the martingale convergence theorem, it must converge almost surely. Since $\mathbb{P}_a\{\tau < \infty\} = 1$, the walk $S_{n\wedge\tau}$ stops at either a or b and we have

$$f(a) = \mathbb{P}\{\tau_0 < \tau_b\} + f(b)\mathbb{P}\{\tau_0 > \tau_b\}.$$

Note that $f(0) = 1$. If $0 \le p > 1/2$, then letting $b \to \infty$ we have $\tau_b \to \infty$ and $f(b) \to 0$, hence $\mathbb{P}\{\tau_0 < \infty\} = f(a)$. If $p < 1/2$, then $f(b) \to \infty$ as $b \to \infty$ and we must have $\mathbb{P}\{\tau_0 > \tau_b\} \to 0$. This means that $\mathbb{P}\{\tau_0 < \infty\} = 1$.

For the case $p = 1/2$, we can use the same argument with the function $f(x) = x$. We have $a = b\mathbb{P}\{\tau_0 > \tau_b\}$ or $\mathbb{P}\{\tau_0 > \tau_b\} = a/b$. Letting $b \to \infty$ we have again $\mathbb{P}\{\tau_0 < \infty\} = 1$. We have

$$\mathbb{P}\{\tau_0 < \infty\} = \begin{cases} 1 & 0 \le p \le 1/2, \\ \left(\frac{1-p}{p}\right)^a & 1/2 < p \le 1. \end{cases}$$

Problem 4. Let $Z = (X, Y)$ be an $\mathbb{R}^2$-valued random variable such that (1) X and Y are independent; (2) both X and Y have mean zero and finite (nonvanishing) second moments; (3) the distribution of Z is invariant under the rotation counter-clockwise around the origin by an angle θ not a multiple of 90 degrees. Show that X and Y must be normal random variables with the same variance.

Solution: We first show that X and Y have the same variance $\sigma^2 = \mathbb{E}|X|^2 = \mathbb{E}|Y|^2$. By hypothesis, we have

$$\begin{aligned} \mathbb{E}|X|^2 &= \cos^2\theta\mathbb{E}|X|^2 + \sin^2\theta\mathbb{E}|Y|^2, \\ \mathbb{E}|Y|^2 &= \sin^2\theta\mathbb{E}|X|^2 + \cos^2\theta\mathbb{E}|Y|^2. \end{aligned} \tag{2}$$

Subtracting the two equations we have

$$(1 - \cos 2\theta)\left(\mathbb{E}|X|^2 - \mathbb{E}|Y|^2\right) = 0.$$

If $\cos 2\theta \neq 1$ we have $\mathbb{E}|X|^2 = \mathbb{E}|Y|^2$; otherwise $\cos^\theta = \sin^2\theta = 1/2$ and the equality obviously holds.

From (2) by induction we can show that

$$X \sim \sum i = 1^{2^n} a_{ni} Z_i$$

where $\{Z_i\}$ are independent and have the same distribution as either X or Y, say $X_i \sim X$ for $i \in I$ and $Z_i \sim Y$ for $i \in J$. Each coefficient has the form $a_{ni} = \pm \cos^k \theta \sin^{n-k} \theta$ for some k. Since θ is not a multiple of 90 degrees, there is a constant $0 < \lambda < 1$ such that $|a_{ni}| \leq \lambda^n$. Furthermore, since X and Y have the same variance we have

$$\sum_{i=1}^{2^n} |a_{ni}|^2 = 1.$$

We show that Lindeberg's condition is satisfied for the sum $\sum_{i=1}^{2^n} a_{ni} Z_i$. Indeed, by Chebyshev's inequality

$$\begin{aligned}
\sum_{i=1}^{2^n} \mathbb{E}\left\{|a_{ni} z_i|^2 \,;\, |a_{ni} Z_i| \geq \epsilon\right\} &\leq \left(\sum_{i \in I} |a_{ni}|^2\right) \mathbb{E}\left\{|X|^2 \,;\, |X| \geq \frac{\epsilon}{\lambda^n}\right\} + \left(\sum_{i \in J} |a_{ni}|^2\right) \mathbb{E}\left\{|Y|^2 \,;\, |Y| \geq \frac{\epsilon}{\lambda^n}\right\} \\
&\leq \frac{\lambda^n}{\epsilon} \left\{\left(\sum_{i \in I} |a_{ni}|^2\right) \mathbb{E}|X|^2 + \left(\sum_{i \in J} |a_{ni}|^2\right) \mathbb{E}|Y|^2\right\} \\
&= \frac{\lambda^n}{\epsilon} \mathbb{E}|X|^2 \to 0.
\end{aligned}$$

Now by the Lindeberg central limit theorem, the above sum converges in distribution to a normal distribution, hence X (and also Y by the same argument) are normal random variables.

We can also work with characteristic functions without using the Lindeberg central limit theorem. We have

$$\phi_X(t) = \prod_{i \in I} \phi_X(a_{ni} t) \prod_{i \in J} \phi_Y(a_{ni} t).$$

Since X and Y have mean zero and the same variance σ^2

$$\phi_X(t) = 1 - \frac{\sigma^2}{2} t^2 + o(t^2) = e^{-\sigma^2 t^2/2 + o(t^2)}.$$

The same equality holds for $\phi_Y(t)$. It follows from the fact that a_{ni}'s tend to zero uniformly as $n \to \infty$ and $\sum_{i=1}^{2^n} |a_{ni}|^2 = 1$ that

$$\phi_X(t) = \prod_{i=1}^{2^n} e^{-|a_{ni}|^2 \sigma^2/2 + o\left(|a_{ni}|^2 t^2\right)} = e^{-\sigma^2 t^2 + o(t^2)} \to e^{-\sigma^2 t^2/2}.$$

Part II: Statistics

The following collection of questions concerns the design of a randomized experiment where the N units to be randomized to drug A or drug B are people, for whom we have a large number of background covariates, collectively labelled X (e.g., age, sex, blood pressure, height, weight, occupational status, history of heart disease, family history of heart disease). The objective is to assign approximately half to drug A and half to drug B where the means of each of the X variables (and means of non-linear functions of them, such as squares or products) are close to equal in the two groups. Instead of using classical methods of design, such as blocking or stratification, the plan is to use modern computers to try many random allocations and discard those allocations that are considered unacceptable according to a pre-determined criterion for balanced X means, in particular an affinely invariant measure such as the Mahalanobis distance between the means of X in the two groups. After an acceptable allocation is found, outcome variables will be measured, and their means will be compared in group A and group B to estimate a treatment effect.

Problem 5. Prove that if the two groups are of the same size (*i.e.*, $N/2$ for even N), this plan will result in unbiased estimates of the A versus B casual effect based on the sample means of Y in groups A and B, where Y is any linear function of X.

Solution: Let X_i denote the covariate vector of unit i, and $\boldsymbol{X} = (X_1, \ldots, X_N)^\top$ denote the covariate matrix for all N units. Let z_i denote the treatment allocation for unit i, which equals 1 if the unit is assinged to drug A and 0 otherwise, and $z \in \{0,1\}^N$ denote the treatment allocation for all N units.

Let $\phi(\boldsymbol{X}, z)$ denote the pre-determined criterion for balanced covariate means, which equals 1 if the allocation z is acceptable and 0 otherwise. By construction, the criterion is invariant when we switch treatment and control groups, *i.e.*, $\phi(\boldsymbol{X}, z) = \phi(\boldsymbol{X}, \boldsymbol{1} - z)$. Note that under a completely randomized experiment (CRE) with half of the units assinged to each treatment group, $\boldsymbol{Z}$ and $\boldsymbol{1-Z}$ follows the same distribution. This implies that, under the randomization (*i.e*, the CRE with balance criterion ϕ),

$$\boldsymbol{Z} \mid \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1 \sim \boldsymbol{1} - \boldsymbol{Z} \mid \phi(\boldsymbol{X}, 1 - \boldsymbol{Z}) = 1 \sim \boldsymbol{1} - \boldsymbol{Z} \mid \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1.$$

Therefore, under re-randomization, $\boldsymbol{Z}$ and $\boldsymbol{1} - \boldsymbol{Z}$ must have the same distribution. Consequently, for any $1 \le i \le N$,

$$\mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = \mathbb{E}\left(1 - Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = 1 - \mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right),$$

which immediately implies that $\mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = 0.5$.

We consider potential outcomes $Y_i(1) = Y_i(0) = \alpha + \beta^\top X_i$ that are some linear function of the covariate. Obviously, the treatment effect of drug A versus B on outcome Y is zero. Let $Y_i = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$ be the observed outcome for unit i. The estimator of this causal effect based on the samples means of Y in groups A and B is

$$\hat{\tau} = \frac{1}{N/2}\sum_{i=1}^N Z_i Y_i - \frac{1}{N/2}\sum_{i=1}^N (1 - Z_i) Y_i = \frac{1}{N/2}\sum_{i=1}^N Z_i Y_i(1) - \frac{1}{N/2}\sum_{i=1}^N (1 - Z_i) Y_i(0).$$

Under re-randomization, $\hat{\tau}$ has expectation

$$\begin{aligned}
\mathbb{E}(\hat{\tau} \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1) &= \frac{2}{N}\sum_{i=1}^N \mathbb{E}(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1) \cdot Y_i(1) - \frac{2}{N}\sum_{i=1}^N \{1 - \mathbb{E}(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1)\} \cdot Y_i(0) \\
&= \frac{2}{N}\sum_{i=1}^N \frac{1}{2} \cdot Y_i(1) - \frac{2}{N}\sum_{i=1}^N \left(1 - \frac{1}{2}\right) \cdot Y_i(0) \\
&= \frac{1}{N}\sum_{i=1}^N Y_i(1) - \frac{1}{N}\sum_{i=1}^N Y_i(0) \\
&= 0.
\end{aligned}$$

Therefore, $\hat{\tau}$ is an unbiased estimator for the A versus B causal effect on the outcome Y.

Problem 6. Provide a counter-example to the assertion that **Problem 5** is true in small samples with odd N.

Solution: We consider an experiment with three units, among which two will be assigned to drug A and the remaining one to drub B. Suppose each unit has a scalar covariate with covariate values $(X_1, X_2, X_3) = (0, 0, 10)$ and the potential outcomes $Y_i(1) = Y_i(0) = X_i$ for $1 \le i \le 3$.

We consider re-randomization with the following balance criterion:

$$\phi(\boldsymbol{X}, z) = 1\left\{\left|\frac{1}{2}\sum_{i=1}^N z_i X_i - \sum_{i=1}^N (1 - z_i) X_i\right| \le 8\right\},$$

i.e., a treatment allocation is acceptable if and only if the absolute value of the difference between covariate means in groups A and B are less than or equal to 8. Consequently, for all $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ treatment allocations, only 2 are acceptable:

$\boldsymbol{z} = (z_1, z_2, z_3)^\top$	$\hat{\tau}_X \equiv \frac{1}{2}\sum_{i=1}^N z_i X_i - \sum_{i=1}^N (1 - z_i) X_i$	$\phi(\boldsymbol{X}, \boldsymbol{z})$
$(1, 1, 0)$	-10	0; Not acceptable
$(1, 0, 1)$	5	1; Acceptable
$(0, 1, 1)$	5	1; Acceptable

By the definition of potential outcomes, $\hat{\tau}$ is the same as $\hat{\tau}_X$ in the above table. From the above table, under re-randomization with the balance criterion ϕ, $\hat{\tau}$ has probability 1 to be 5. Note that the true casual effect of A versus B on outcome Y is zero. Therefore, in this example, $\hat{\tau}$ is not unbiased for the casual effect.

Algebra and Number Theory

Solve every problem.

Problem 1. For any prime p and a nonzero element $a \in \mathbf{F}_p$, prove that the polynomial $A(x) = x^p - x - a$ is irreducible and separable over $\mathbf{F}_p$.

Problem 2. Determine the automorphism group of the splitting field of $f(x) = x^3 - 3x + 1$ over $\mathbf{Q}$.

Problem 3. Let $R = F[x, y]/(x^2 - y^3)$ for some field F.

(a) Prove that R is an integral domain.

(b) If t denotes the element x/y in the fraction field K of R, prove that K is equal to $F(t)$.

(c) Prove that $F[t]$ is the integral closure of R in $K = F[t]$.

Problem 4. Let $p_1, \ldots, p_n$ be n distinct prime numbers. Show: $\sqrt{p_1} + \cdots + \sqrt{p_n}$ is not rational.

Problem 5. Find all integral solutions (x, y) for the equation $x^2 + 13 = y^3$. (**Hint:** You can use the fact that $\mathbf{Q}(\sqrt{-13})$ has class number 2).

Problem 6. Let p be a prime number and $\mathbf{Q}_p$ be the field of p-adic numbers. Fix an algebraic closure $\overline{\mathbf{Q}_p}$ of $\mathbf{Q}_p$. Let $g : \mathbf{Z}_{\geq 0} \longrightarrow \mathbf{N}$ be a strictly increasing function. For each $i \in \mathbf{Z}_{\geq 0}$, pick a primitive $(p^{g(i)} - 1)$-th root of unity ζ_i in $\overline{\mathbf{Q}_p}$.

(a) Show that for each $i \geq 0$, $K_i := \mathbf{Q}_p(\zeta_i)$ is an unramified Galois extension of $\mathbf{Q}_p$ of degree $g(i)$.

(b) Give an explicit function g as above such that $K_{i-1} \subset K_i$ for all $i > 0$. Let $0 = N_0 < N_1 < N_2 \cdots$ be an increasing sequence of nonnegative integers. Let $\alpha_i := \sum_{j=0}^{i} \zeta_j p^{N_j}$. Show that for each $i \geq 0$, $K_i = \mathbf{Q}_p(\alpha_i)$ and that (α_i) is a Cauchy sequence in $\overline{\mathbf{Q}_p}$.

(c) Let $\eta \in \overline{\mathbf{Q}_p}$ be of degree g over $\mathbf{Q}_p$, prove that there exists $M \in \mathbf{N}$ such that ζ_i does not satisfy any congruence

$$s_{g-1}\eta^{g-1} + s_{g-2}\eta^{g-2} + \cdots + s_1\eta + s_0 \equiv 0 \pmod{p^M}$$

in which the s_i's are p-adic integers not all of which are divisible by p.

(d) Take a suitable sequence (N_i) as above such that (a_i) does not converge in $\overline{\mathbf{Q}_p}$. Conclude that $\mathbf{Q}_p$ is not complete with respect to the p-adic topology.

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Algebra and Number Theory

Solve every problem.

Problem 1. For any prime p and a nonzero element $a \in \mathbf{F}_p$, prove that the polynomial $A(x) = x^p - x - a$ is irreducible and separable over $\mathbf{F}_p$.

Solution: First of all, $A(x)$ is separable because $A'(x) = -1$ is non-zero. Second, $A(x)$ has no root in $\mathbf{F}_p$, since $A(b) = b^p - b - a = -a$ for any $b \in \mathbf{F}_p$. Notice that $A(x+a) = (x+a)^p - (x+a) - a = x^p + a^p - x - a - a = x^p - x - a = A(x)$. Therefore, by iteration, we see that $A(x + ba) = A(x)$ for all $b = 1, 2, \ldots$. As b runs over all positive integers, ba runs over $\mathbf{F}_p$. This means that in the splitting field K of $A(x)$ over $\mathbf{F}_p$, if a is a root, then, so is $a + k$, $k \in \mathbf{F}_p$. It follows that

$$A(x) = \prod_{k=0}^{p-1} (x - a - k).$$

If $A(x)$ is not irreducible over $\mathbf{F}_p$, then there exist $r(x), s(x) \in \mathbf{F}_p[x]$ such that $A(x) = r(x)s(x)$ and $0 < d = \deg r(x) < p$. Being a divisor of $A(x)$, $r(x)$ has the form

$$r(x) = \prod_{k \in J} (x - a - k),$$

for some subset $J \subset \mathbf{F}_p$. The coefficient of x^{d-1} of $r(x)$ is the negative of the sum of all roots of $r(x)$, hence it is equal to

$$\sum_{i \in J} (a + i) = da + \sum_{i \in J} i \in \mathbf{F}_p.$$

Then, $a \in \mathbf{F}_p$, a contradiction.

Problem 2. Determine the automorphism group of the splitting field of $f(x) = x^3 - 3x + 1$ over $\mathbf{Q}$.

Solution: $f(x)$ is irreducible over $\mathbf{Z}$ (hence over $\mathbf{Q}$ as well). To see this we reduce mod 2 and evaluate: $f(0) = 1, f(1) = 1$.

Next, $f'(x) = 3x^2 - 3$ is positive outside the interval $(-1, 1)$ and negative in the interval $(-1, 1)$. Since $f(-1) = 3$ and $f(1) = 1$, we see that there are three real roots, denoted a_1, a_2 and a_3. Since $f(x)$ is irreducible of degree 3, $\dim_{\mathbf{Q}} \mathbf{Q}(a_i) = 3$. Let K denote the splitting field of $f(x)$. Thus, K is either degree 6 over $\mathbf{Q}$, or it is degree 3 over $\mathbf{Q}$. In the latter case the automorphism group is cyclic group of order 3. Let us prove that the former case is not possible. In fact, in the former case the automorphism group $G = \mathrm{Aut}_{\mathbf{Q}} K = S_3$, the symmetric group on 3 letters. We let σ be the automorphism that maps a_1 to a_3, a_3 to a_1 and a_2 to a_2. Write $f(x) = (x - a_1)(x - a_2)(x - a_3)$. By taking the derivative of $f(x)$ and substituting a_i into this equation gives

$$(a_1 - a_2)(a_1 - a_3) = 3a_1^2 - 3,$$

$$(a_2 - a_1)(a_2 - a_3) = 3a_2^2 - 3,$$

$$(a_3 - a_1)(a_3 - a_2) = 3a_3^2 - 3.$$

Take the product of the above three equations and then take the square roots, we have

$$\prod_{1 \le i < j \le 3} \left(a_i - a_j\right) = 9.$$

Then apply the permutation σ to the above, the left is sent to its negative, but 9 is obviously fixed by σ. Therefore, σ cannot be in G. It follows that the automorphism group G must be the cyclic group of order 3. Moreover, $\mathbf{Q}(a_1) = \mathbf{Q}(a_1, a_2, a_3)$.

Problem 3. Let $R = F[x, y]/(x^2 - y^3)$ for some field F.

(a) Prove that R is an integral domain.

(b) If t denotes the element x/y in the fraction field K of R, prove that K is equal to $F(t)$.

(c) Prove that $F[t]$ is the integral closure of R in $K = F[t]$.

Solution:

(a) To prove $x^2 - y^3$ is irreducible, it suffices to show that it is irreducible in $F(y)[x]$. Since it is a quadratic polynomial in x over $F(y)$, it is reducible if and only if it has a root in $F(y)$. Suppose $f(y)/g(y)$ is a root, where $f(y)$ and $g(y)$ are co-prime. But, $(f(y)/g(y))^2 - y^3 = 0$ implies $f(y)^2 = y^3 g(y)^2$. Thus, an irreducible factor of $g(y)$ divides $f(y)^2$, hence divides $f(y)$, a contradiction. This implies the ideal generated by $x^2 - y^3$ is prime since $F[x, y]$ is a UFD. Hence, R is an integral domain.

(b) Since $x^2 = y^3$ in R, we have $y = (x/y)^2 = t^2$ in K. Also, $x = yx/y = yt = t^3$. Thus, any element $f(x, y) \in R$ is a polynomial in t in K. Therefore, any element $f(x, y)/g(x, y) \in K$ belongs to $F(t)$. On the other hand, $F(t) \subset K$, hence $K = F(t)$.

(c) Let $h(t) \in F[t] \subset F(t)$. Replacing t^2 by y and t^3 by x (from **(b)**), we have $h(t) = at + g(x, y)$ for some $g(x, y) \in R$. Then, we have $(h(t) - g(x, y))^2 = (at)^2 = a^2 y$. Thus, $h(t)$ is a root of $X^2 - 2g(x, y)X + g(x, y)^2 - a^2 y^2 \in R[X]$. This implies that $F[t]$ is integral over R. Suppose now that $h(t) \in F(t)$ is integral over R. Then $h(t)$ is also integral over $F[t]$. But, $F(t)$ is the fraction field of $F[t]$ and $F[t]$ is a UFD, hence $F[t]$ is integrally closed. Therefore, $h(t) \in F[t]$.

Problem 4. Let $p_1, \ldots, p_n$ be n distinct prime numbers. Show: $\sqrt{p_1} + \cdots + \sqrt{p_n}$ is not rational.

Solution: It suffices to show $[\mathbf{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) : \mathbf{Q}] = 2^n$ (this implies that the 2^n numbers $1, \sqrt{p_{\alpha_1} \cdots p_{\alpha_k}}$ $(1 \le \alpha_1 < \cdots < \alpha_k \le n)$ are linear independent over $\mathbf{Q}$). We show the following stronger result:

Lemma: *Suppose K is a field with characteristic 0, and $\{x_1, \ldots, x_n\} \subset K$ is a subset such that the product of elements of any non-empty subset of $\{x_1, \ldots, x_n\}$ is not a square in K. Then $[K(\sqrt{x_1}, \ldots, \sqrt{x_n}) : K] = 2^n$.*

We prove the lemma by induction on n. The case $n = 1$ is trivial. Suppose $n = 2$ and we aim to show $[K(\sqrt{x_1}, \sqrt{x_2}) : K] = 4$. Since we have $[K(\sqrt{x_1}) : K] = 2$, it suffices to prove $[K(\sqrt{x_1}, \sqrt{x_2}) : K(\sqrt{x_1})] = 2$. We only need to prove $\sqrt{x_2} \notin K(\sqrt{x_1})$. If not, then $\sqrt{x_2} = a + b\sqrt{x_1}$ with $a, b \in K$. Since $\sqrt{x_1} \notin K$, we have $b \neq 0$. Then $a^2 = (\sqrt{x_2} - b\sqrt{x_1})^2$ implies that $\sqrt{x_1 x_2} \in K$, which is a contradiction. We conclude the lemma for $n = 2$.

Now suppose $n \ge 3$ and suppose for smaller n the lemma holds. By the induction assumption, we have $[K(\sqrt{x_3}, \ldots, \sqrt{x_n}) : K] = 2^{n-2}$. Denote $L = K(\sqrt{x_3}, \ldots, \sqrt{x_n})$. Then we only need to show $[L(\sqrt{x_1}, \sqrt{x_2}) : L] = 4$. It suffices to prove $\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_1 x_2} \notin L$. Suppose one of $\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_1 x_2}$, say y, belong to L. Then $L(y) = L$, which implies that $[K(y, \sqrt{x_3}, \ldots, \sqrt{x_n}) : K] = 2^{n-2}$. But on the other hand, by the induction-assumption, we must have $[K(y, \sqrt{x_3}, \ldots, \sqrt{x_n}) : K] = 2^{n-1}$, a contradiction!

Problem 5. Find all integral solutions (x, y) for the equation $x^2 + 13 = y^3$. (**Hint:** You can use the fact that $\mathbf{Q}(\sqrt{-13})$ has class number 2).

Solution: The only solutions are $(x, y) = (\pm 70, 17)$.

Suppose $x^2 + 13 = y^3$ has integral solution (x, y). We may first assume x, y are positive. It is easy to see that $\gcd(x, y) = 1$. If y is even, then x is odd and $x^2 + 13 \equiv 6 \pmod 8$. But $y^3 \equiv 0 \pmod 8$. This is impossible. Therefore, x is even and y is odd.

In $\mathbf{Z}[\sqrt{-13}]$ we have $(x + \sqrt{-13})(x - \sqrt{-13}) = y^3$. Consider the principal ideals $(x + \sqrt{-13}), (x - \sqrt{-13}), (y) \subset \mathbf{Z}[\sqrt{-13}]$. Suppose $(x + \sqrt{-13})$ and $(x - \sqrt{-13})$ are not coprime to each other, then there exists a prime ideal $P \subset \mathbf{Z}[\sqrt{-13}]$ such that $P \big| (x + \sqrt{-13})$ and $P \big| (x - \sqrt{-13})$. Then $x \pm \sqrt{-13} \in P$, which implies that $2\sqrt{-13} \in P$ and $2x \in P$. We have also $P \big| (y)^3$ which implies that $P \big| (y)$. Thus $y \in P$. But $\gcd(2x, y) = 1$, a contradiction.

We conclude that $(x+\sqrt{-13})$ and $(x-\sqrt{-13})$ are two ideals coprime to each other. Since $\mathbf{Z}[\sqrt{-13}]$ is a Dedekind ring, there exists ideals P_1, P_2 such that $(x+\sqrt{-13}) = P_1^3$, $(x-\sqrt{-13}) = P_2^3$ and $(y) = P_1P_2$.

Since the class number of $\mathbf{Z}[\sqrt{-13}]$ is 2, the square of any fractional ideal is principal. Since P_i^3 is principal, we know that P_i is principal. Note that the units of $\mathbf{Z}[\sqrt{-13}]$ are ± 1. Thus there exists $a, b \in \mathbf{Z}$, such that $x+\sqrt{-13} = (a+b\sqrt{-13})^3$. Comparing the coefficient of $\sqrt{-13}$, we conclude that $1 = 3a^2b - 13b^3$. This implies $b = -1$ and $a = \pm 2$. Thus $x = \pm 70$, $y = 17$.

Problem 6. Let p be a prime number and $\mathbf{Q}_p$ be the field of p-adic numbers. Fix an algebraic closure $\overline{\mathbf{Q}_p}$ of $\mathbf{Q}_p$. Let $g\colon \mathbf{Z}_{\geq 0} \to \mathbf{N}$ be a strictly increasing function. For each $i \in \mathbf{Z}_{\geq 0}$, pick a primitive $(p^{g(i)}-1)$-th root of unity ζ_i in $\overline{\mathbf{Q}_p}$.

- **(a)** Show that for each $i \geq 0$, $K_i := \mathbf{Q}_p(\zeta_i)$ is an unramified Galois extension of $\mathbf{Q}_p$ of degree $g(i)$.
- **(b)** Give an explicit function g as above such that $K_{i-1} \subset K_i$ for all $i > 0$. Let $0 = N_0 < N_1 < N_2 \cdots$ be an increasing sequence of nonnegative integers. Let $\alpha_i := \sum_{j=0}^{i} \zeta_j p^{N_j}$. Show that for each $i \geq 0$, $K_i = \mathbf{Q}_p(\alpha_i)$ and that (α_i) is a Cauchy sequence in $\overline{\mathbf{Q}_p}$.
- **(c)** Let $\eta \in \overline{\mathbf{Q}_p}$ be of degree g over $\mathbf{Q}_p$, prove that there exists $M \in \mathbf{N}$ such that ζ_i does not satisfy any congruence
$$s_{g-1}\eta^{g-1} + s_{g-2}\eta^{g-2} + \cdots + s_1\eta + s_0 \equiv 0 \pmod{p^M}$$
in which the s_i's are p-adic integers not all of which are divisible by p.
- **(d)** Take a suitable sequence (N_i) as above such that (a_i) does not converge in $\overline{\mathbf{Q}_p}$. Conclude that $\mathbf{Q}_p$ is not complete with respect to the p-adic topology.

Solution:

- **(a)** Let v_p be the usual p-adic valuation of $\mathbf{Q}_p$, then v_p has a unique extension to K_i, which is $v_{K_i}(x)\colon = [K_i\colon \mathbf{Q}_p]^{-1}v_p(\mathcal{N}_{K_i/\mathbf{Q}_p}(x))$. The relation $\zeta_i^{p^{g(i)}-1} = 1$ implies that $v_{K_i}(\zeta_i) = 0$, so $\zeta_i \in \mathcal{O}_{K_i}$. Let $P(X) \in \mathbf{Z}_p[X]$ be the minimal polynomial of ζ_i over $\mathbf{Q}_p$, then $P(X)$ is a factor of $X^{p^{g(i)}-1} - 1$. Since ζ_i is primitive, we see that $\zeta_i^l \in K_i, 0 \leq l \leq p^{g(i)} - 1$ are all roots of $X^{p^{g(i)}-1} - 1$. Thus each root of $P(X)$ has shape $\zeta_i^l \in K_i$, which implies that $K_i/\mathbf{Q}_p$ is normal. Since $\mathrm{char}(\mathbf{Q}_p) = 0$, all its extensions are separable. So $K/\mathbf{Q}_p$ is a Galois extension.

  The reduction $\overline{P}(X) \in \mathbf{F}_p[X]$ of $P(X)$ is also a factor of $X^{p^{g(i)}-1} - 1 \in \mathbf{F}_p[X]$. Since p and $p^{g(i)} - 1$ are coprime, $X^{p^{g(i)}-1} - 1 \in \mathbf{F}_p[X]$ has no multiple roots. Using Hensel's lemma and the fact that $P(X) \in \mathbf{Z}_p[X]$ is irreducible, we see that $\overline{P}(X)$ is also irreducible.
  Let $\overline{K_i}$ be the residue field of K_i, then $\overline{\zeta_i} \in \overline{K_i}$ is a root of $\overline{P}(X)$. We see that $[\overline{K_i}\colon \mathbf{F}_p] \geq [\mathbf{F}_p(\overline{\zeta_i})\colon \mathbf{F}_p] = \deg(\overline{P}) = \deg(P) = [K_i\colon \mathbf{Q}_p]$. It follows that the ramification index e satisfies
$$1 \leq e = [K_i\colon \mathbf{Q}_p]/[\overline{K_i}\colon \mathbf{F}_p] \leq 1.$$
  Thus $e = 1$, i.e. $K/\mathbf{Q}_p$ is an unramified extension.

  The argument above also implies the equality $[\overline{K_i}\colon \mathbf{F}_p] = [K_i\colon \mathbf{Q}_p]$, the relation $\overline{K_i} = \mathbf{F}_p[\overline{\zeta_i}]$ and the property that $\overline{K_i}$ contains all $(p^{g(i)}-1)$-th roots of unity. Thus $\overline{K_i}/\mathbf{F}_p$ is cyclotomic and $\overline{\zeta_i}$ is primitive. Now, since $\overline{K_i}^\times$ is a cyclic group of order $p^{[K_i\colon \mathbf{Q}_p]} - 1$, we get that $[K_i\colon \mathbf{Q}_p] = g(i)$. Thus the degree of ζ_i over $\mathbf{Q}_p$ is $g(i)$.
- **(b)** Take any sequence $g(i)$ such that $g(i)/g(i-1)$ is an integer at least 2, *e.g.*, $g(i) = 2^i$. Then $(p^{g(i-1)}-1) \mid (p^{g(i)}-1)$, so ζ_{i-1} is a power of ζ_i. Thus $K_{i-1} \subset K_i$ holds for each $i > 0$. For any $1 < i < i'$, we have
$$v_{K_{i'}}(\alpha_{i'} - \alpha_i) = v_{K_{i'}}(p^{i+1}) + v_{K_{i'}}(\text{an element in } \mathcal{O}_{K_{i'}}) \geq i+1,$$
  which means that (α_i) is a Cauchy sequence.
  Take any $\sigma \in \mathrm{Gal}(K_i/\mathbf{Q}_p)\backslash\{\mathrm{id}\}$. Notice that $\sigma(\alpha_i) = \sum_{j=0}^{i} \sigma(\zeta_i)p^{N_i}$. Since $\zeta_i \neq \sigma(\zeta_i)$, we get $v_{K_i}(\alpha - \sigma(\alpha_i)) = v_{K_i}((\zeta_i - \sigma(\zeta_i))p^{N_i} + \cdots) = N_i < \infty$. So $\sigma(\alpha_i) \neq \alpha_i$, *i.e.*, σ does not fix α_i. Thus $\mathbf{Q}_p(\alpha_i) = K_i$.

(c) Suppose the contrary, then for any $M \in \mathbf{N}$, there exist $s_{g-1}^M, \cdots, s_0^M \in \mathbf{Z}_p$, not all divisible by p, such that

$$s_{g-1}^M \eta^{g-1} + s_{g-2}^M \eta^{g-2} + \cdots + s_1^M \eta + s_0^M \equiv 0 \pmod{p^M}.$$

By the pigenhole principle, there are an $0 \le j \le g-1$ and an infinite subset $R \subset \mathbf{Z}_{\ge 0}$ such that $p \nmid s_i^M$ for all $M \in R$. Since $\mathbf{Z}_p$ is sequentially compact, so is $\mathbf{Z}_p^g$. Thus the sequence $(s_{g-1}^M, \cdots, s_0^M)_{M \in R} \in \mathbf{Z}_p^g$ has a convergent subsequence. Let $(s_{g-1}, \cdots, s_0) \in \mathbf{Z}_p^g$ be the limit of this subsequence. Then $p \nmid s_j$ and

$$s_{g-1}\eta^{g-1} + s_{g-2}\eta^{g-2} + \cdots + s_\eta + s_0 \equiv 0 \pmod{p^r}, \text{ for any } r \in \mathbf{N}.$$

In other words, $v_{\mathbf{Q}_p(\eta)}(s_{g-1}\eta^{g-1} + s_{g-2}\eta^{g-2} + \cdots + s_\eta + s_0) \ge r$ for all $r \in \mathbf{N}$, thus equals to ∞. So $s_{g-1}\eta^{g-1} + s_{g-2}\eta^{g-2} + \cdots + s_\eta + s_0 = 0$, contradicting the assumption that η has degree g over $\mathbf{Q}_p$.

(d) We will take an increasing sequence (N_i) by induction. We have $N_0 = 0$ already given. Suppose we have defined N_j for all $j \le i$, so that we have $\alpha_i = \sum_{j=0}^i \zeta_j p^{N_j}$ determined. Since α_i has degree $g(i)$ over $\mathbf{Q}_p$, by **(c)**, there exists $N_{i+1} > N_i$ such that ζ_i does not satisfy any congruence

$$t_n\alpha_i^n + t_{n-1}\alpha_i^{n-1} + \cdots + t_1\alpha_i + t_0 \equiv 0 \pmod{p^{N_{i+1}}},$$

for any $n < g(i)$ and $t_j \in \mathbf{Z}_p$ not all divisible by p. Then the sequence (N_i) is completely defined.

Suppose $\overline{\mathbf{Q}_p}$ is complete, then (α_i) converges to certain $\alpha \in \overline{\mathbf{Q}_p}$. Then there exist $t_n, t_{n-1}, \ldots, t_0 \in \mathbf{Z}_p$, not all divisible by p, such that

$$t_n\alpha^n + t_{n-1}\alpha^{n-1} + \cdots + t_1\alpha + t_0 = 0.$$

Choose i with $g(i) > n$. Since $\alpha \equiv \alpha_i \pmod{p^{N_i}}$, we have

$$t_n\alpha_i^n + t_{n-1}\alpha_i^{n-1} + \cdots + t_1\alpha_i + t_0 \equiv 0 \pmod{p^{N_{i+1}}},$$

a contradiction. This proves the assertion.

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Analysis and Differential Equations

Solve every problem.

Problem 1. Prove that $f(x) \equiv 0$ is the only solution in $L^2(\mathbf{R^n})$ such that

$$\triangle f = 0.$$

Problem 2. Let $X \subset C([0,1])$ be a finite dimensional linear subspace of the space of real-valued continuous functions on $[0,1]$. Show that, for a sequence of functions $\{f_k\}_{k\geqslant 1} \subset X$, if it converges pointwise, it converges uniformly.

Problem 3.

(a) For $f \in L^1(\mathbf{R^n})$, $g \in L^\infty(\mathbf{R^n})$, show that their convolution $f * g$ is a well-defined continuous function.

(b) Let $E \subset \mathbf{R^n}$ be a Lebesgue measurable set with Lebesgue measure $m(E) > 0$. Prove that

$$E - E := \left\{x - y \,\middle|\, x \in E, y \in E\right\}$$

contains an open neighborhood of $0 \in \mathbf{R^n}$.

Problem 4. Assume that P is a polynomial with complex coefficients. Prove that there exists infinitely many solutions of the following equations on $\mathbf{C}$:

$$e^z = P(z).$$

Problem 5. Let f be a bounded holomorphic function defined on $B = \left\{z \,\middle|\, 0 < \mathrm{Re}(z) < 1\right\}$ that can be extended as a continuous function on $\overline{B}$. Let

$$A_0 = \sup_{\mathrm{Re}(z)=0} |f(z)| > 0, \quad A_0 = \sup_{\mathrm{Re}(z)=1} |f(z)| > 0.$$

Prove that for all $z \in B$, we have

$$|f(z)| \leqslant (A_0)^{1-\mathrm{Re}(z)}(A_1)^{\mathrm{Re}(z)}.$$

Problem 6. Assume that $\Omega \subset \mathbf{R^n}$ is a bounded domain with smooth boundary. Prove that there exists a positive constant ε_0 so that for all real numbers $\varepsilon < \varepsilon_0$, for all $f \in L^2(\Omega)$, there exist a unique $u \in H^1_0(\Omega)$ so that

$$-\triangle u + \varepsilon \sin(u) = f$$

in the sense of distributions.

S.-T. Yau College Student Mathematics Contests 2021

Analysis and Differential Equations

Solve every problem.

Problem 1. Prove that $f(x) \equiv 0$ is the only solution in $L^2(\mathbf{R}^\mathbf{n})$ such that

$$\triangle f = 0.$$

Solution: Consider the Fourier transform of the equation $\triangle f = 0$. This yields

$$-|\xi|^2 \widehat{f}(\xi) = 0.$$

By the Plancherel Theorem, $f \in L^2(\mathbf{R}^\mathbf{n})$ implies that $\widehat{f} \in L^2(\mathbf{R}^\mathbf{n})$. Therefore, the above equation shows $\operatorname{supp}\left(\widehat{f}\right) \subset \{0\}$. Hence $\widehat{f} = 0$ in L^2. Hence, $f = 0$ in L^2. Since harmonic functions are smooth, $f \equiv 0$.

Problem 2. Let $X \subset C([0,1])$ be a finite dimensional linear subspace of the space of real-valued continuous functions on $[0,1]$. Show that, for a sequence of functions $\{f_k\}_{k\geqslant 1} \subset X$, if it converges pointwise, it converges uniformly.

Solution: Let $\varphi_1, \ldots, \varphi_n$ be a basis of X. We first show that there exists $t_1, t_2, \ldots, t_n \in [0,1]$ so that $\det\left(\varphi_i(t_j)\right) \neq 0$, where $1 \leqslant i, j \leqslant n$. Consider the linear functionals

$$\ell_t : X \longrightarrow \mathbb{R}, \quad f \longmapsto f(t).$$

We have $\cap_{t\in[0,1]}\ker(\ell_t) = \{0\}$. Therefore, there exists $t_1, t_2, \ldots, t_n \in [0,1]$ so that $\cap_{i\leqslant n}\ker(\ell_{t_i}) = \{0\}$. This means that $\det\left(\varphi_i(t_j)\right) \neq 0$.

We write $\{f_k\}_{k\geqslant 1}$ in terms of our basis:

$$f_k(x) = \sum_{j=1}^{n} \alpha_j^{(k)} \varphi_j(x).$$

Therefore,

$$\begin{pmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{pmatrix} = \left(\varphi_i(x_j)\right) \begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix} = A \begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix}.$$

We obtain that

$$\begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix} = A^{-1} \begin{pmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{pmatrix}.$$

Since $\{f_k\}_{k\geqslant 1}$ converges pointwise, it converges on $x_1, \ldots, x_n$. Therefore,$\{\alpha_i^{(k)}\}_{k\geqslant 1}$ converges to some α_i. This implies that $\{f_k\}_{k\geqslant 1}$ converges uniformly to $\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n$.

Problem 3.

(a) For $f \in L^1(\mathbf{R}^\mathbf{n})$, $g \in L^\infty(\mathbf{R}^\mathbf{n})$, show that their convolution $f * g$ is a well-defined continuous function.

(b) Let $E \subset \mathbf{R^n}$ be a Lebesgue measurable set with Lebesgue measure $m(E) > 0$. Prove that

$$E - E := \left\{x - y \,\middle|\, x \in E, y \in E\right\}$$

contains an open neighborhood of $0 \in \mathbf{R^n}$.

Solution:

(a) This is standard: In fact, we have $\|f * g\|_{L^\infty} \leqslant \|f\|_{L^1}\|g\|_{L^\infty}$. Therefore, by the continuity argument, it suffices to prove the theorem for $f \in C_0^\infty(\mathbf{R^n})$. In this case, we have

$$\begin{aligned}|f * g(x_0 + x) - f * g(x_0)| &= \left|\int_{\mathbf{R^n}} \big(f(x_0 + x - y) - f(x_0 - y)\big)g(y)\, dy\right| \\ &\leqslant \|g\|_{L^\infty} \int_{\mathbf{R^n}} \left|f(x_0 + x - y) - f(x_0 - y)\right| dy\end{aligned}$$

Now let $x \to 0$, the integrand converges to 0 uniformly. This yields **(a)**.

(b) It suffices to consider the case where $m(E) < \infty$. We take $f = \mathbf{1}_E$, $g = \mathbf{1}_{-E}$, thus $h(x) = f * g$ is a continuous function. In particular, $h(0) = m(E) > 0$. Therefore, there exists an open set U such that $0 \in U$ and $h\big|_u > \delta > 0$ for some $\delta > 0$. For $x \in U$, by definition,

$$h(z) = \int_{\mathbf{R^n}} \mathbf{1}_E(x - y)\mathbf{1}_{-E}(y)dy > 0.$$

Therefore, there must be some $y \in -E$, such that $x - y = x + (-y) \in E$. This implies $x \in E - (-y) \subset E - E$. Hence $U \subset E - E$.

Problem 4. Assume that P is a polynomial with complex coefficients. Prove that there exists infinitely many solutions of the following equations on $\mathbf{C}$:

$$e^z = P(z).$$

Solution: This is an application of big Picard's theorem at 0.

Problem 5. Let f be a bounded holomorphic function defined on $B = \left\{z \,\middle|\, 0 < \mathrm{Re}(z) < 1\right\}$ that can be extended as a continuous function on $\overline{B}$. Let

$$A_0 = \sup_{\mathrm{Re}(z)=0} |f(z)| > 0, \ \ A_0 = \sup_{\mathrm{Re}(z)=1} |f(z)| > 0.$$

Prove that for all $z \in B$, we have

$$|f(z)| \leqslant (A_0)^{1-\mathrm{Re}(z)}(A_1)^{\mathrm{Re}(z)}.$$

Solution: We consider the function $g(z) = f(z)(A_0)^{z-1}(A_1)^{-z}$. This is a holomorphic function defined on B and bounded by 1. We consider the function $h(z) = g(z)e^{\varepsilon z^2}$. This function is bounded for $z \to \pm i\infty$, therefore, it is bounded by its maximal value on the boundary. Letting $\varepsilon \to 0$ proves the statement.

Problem 6. Assume that $\Omega \subset \mathbf{R^n}$ is a bounded domain with smooth boundary. Prove that there exists a positive constant ε_0 so that for all real numbers $\varepsilon < \varepsilon_0$, for all $f \in L^2(\Omega)$, there exist a unique $u \in H_0^1(\Omega)$ so that

$$-\triangle u + \varepsilon \sin(u) = f$$

in the sense of distributions.

Solution: We consider the functional

$$E(u) = \int_\Omega \tfrac{1}{2} |\nabla u|^2 - \cos(u) - fu$$

defined on $H_0^1(\Omega)$. $E(u)$ is bounded below by Poincaré's inequality. Therefore, a minimizing sequence gives a solution. To show that the solution is unique, we assume that $u_1, u_2 \in H_0^1(\Omega)$ so that

$$-\triangle u_i + \varepsilon \sin(u_i) = f \implies -\triangle(u_1 - u_2) + \varepsilon\left(\sin(u_1) - \sin(u_2)\right) = 0.$$

We multiply the equation by $u_1 - u_2$ and the integrate by parts, this leads to

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 = \varepsilon \left|\int_\Omega (u_1 - u_2)(\sin(u_1) - \sin(u_2))\right| \leqslant \varepsilon \left|\int_\Omega |u_1 - u_2||u_1 - u_2|\right|.$$

Therefore,

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 \leqslant \varepsilon \left\|u_1 - u_2\right\|_{L^2}^2.$$

If $\varepsilon_0 < \lambda_1(\Omega)$, the Poincaré inequality implies that

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 \leqslant \frac{\varepsilon}{\lambda_1(\Omega)} \left\|\nabla(u_1 - u_2)\right\|_{L^2}^2.$$

Hence, $u_1 = u_2$.

S.-T. Yau College Student Mathematics Contests 2021

Computational and Applied Mathematics

Solve every problem.

Problem 1.

(a) Show that

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1],$$

is a polynomial of degree n with extrema at

$$x_k = \cos\left(k\frac{\pi}{n}\right), \quad k = 0, 1, \dots, n$$

and leading coefficient 2^{n-1}.

(b) Show that if $f \in C^{n+1}[-1, 1]$ and if $P(x)$ is the polynomial with degree at most n that interpolates f at x_k, $k = 0, 1, \dots, n$ then

$$\|f(x) - P(x)\|_\infty \le \frac{1}{2^{n-1}(n+1)!} \left\|f^{n+1}\right\|_\infty .$$

Problem 2. Let $S(x)$ be a cubic spline with knots $\{t_i\}_{i=0}^n$. If it is determined that $S(x)$ is linear over $[t_1, t_2]$ and $[t_3, t_4]$. Prove that $S(x)$ is also linear over $[t_2, t_3]$.

Problem 3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - \cos x$.

(a) Prove that the equation $f(x) = 0$ has a unique solution $x^* \in \mathbb{R}$ that lies in the interval $(\frac{1}{4}, \frac{1}{2})$.

(b) Prove that the sequence defined by the fixed point iteration

$$\begin{aligned} &x_0, \\ &x_n = \tfrac{1}{2}\cos x_{n-1}, \quad n = 1, 2, \dots \end{aligned}$$

converges to x^* with any initial guess x_0.

(c) For the fixed point iteration in **(b)** with $x_0 = \frac{\pi}{6}$, determine an n that guarantees $|x_n - x^*| < \frac{1}{2} \times 10^{-8}$. For the fixed point iteration in **(b)** with $x_0 = 20$, determine an n that guarantees $|x_n - x^*| < \frac{1}{4}$.

Problem 4. Let matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ with $m \ge n$ and $r = \text{rank}(\mathbf{A}) < n$, and assume A has the following SVD decomposition

$$\mathbf{A} = [\mathbf{U_1}, \mathbf{U_2}] \begin{bmatrix} \mathbf{\Sigma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} [\mathbf{V}_1, \mathbf{V}_2]^T = \mathbf{U}_1 \Sigma_1 \mathbf{V}_1^T,$$

where Σ_1 is $r \times r$ nonsingular and $\mathbf{U}_1$ and $\mathbf{V}_1$ have r columns. Let $\sigma = \sigma_{\min}(\Sigma_1)$, the smallest nonzero singular value of $\mathbf{A}$. Consider the following least square problem, for some $\mathbf{b} \in \mathbf{R}^{\mathbf{m}}$,

$$\min_{\mathbf{x} \in \mathbf{R}^{\mathbf{n}}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 .$$

(a) Show that all solutions $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{V_1}\Sigma_\mathbf{1}^{-1}\mathbf{U_1^T}\mathbf{b} + \mathbf{V_2}\mathbf{z_2},$$

with $\mathbf{z_2}$ an arbitrary vector.

(b) Show that the solution $\mathbf{x}$ has minimal norm $\|\mathbf{x}\|_2$ precisely when $\mathbf{z_2} = \mathbf{0}$, and in which case,

$$\|\mathbf{x}\|_2 \le \frac{\|\mathbf{b}\|_2}{\sigma}.$$

Problem 5. Consider the family of semi-implicit Runge-Kutta methods

$$k_1 = f(y_n + \beta h k_1), \quad k_2 = f(y_n + hk_1 + \beta h k_2),$$
$$y_{n+1} = y_n + h\Big((\tfrac{1}{2} + \beta)k_1 + (\tfrac{1}{2} - \beta)k_2\Big).$$

(a) Determine the order and the principal part of the local truncation error.

(b) Show that if $\beta > \frac{1}{2}$, then the negative real axis $\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) = 0\}$ is contained in the region of absolute stability of the method.

Problem 6. Consider the Beam equation from mechanics with boundary conditions that model a cantilever beam:

$$\begin{aligned} &u^{(4)} = f(x), \quad x \in (0,\ 1), \\ &u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{aligned} \tag{1}$$

(a) Recast this equation into a variational problem, stating the trial and test function spaces.

(b) Interpret the variational problem as an energy minimization problem, clearly stating the energy functional. Prove that the variational problem and the energy minimization problems are equivalent.

(c) Develop a CG(3) (cubic continuous Galerkin method) finite element method for this problem.

(d) Prove an *a priori* error estimate for this method in the energy norm:

$$\|e\|_E = \left(\int_0^1 (e'')^2 dx\right)^{\frac{1}{2}},$$

Where $e = u(x) - U(x)$, in which, $u(x)$ is the exact solution to VP (variational problem), $U(x)$ is the FEM (finite element method) solution.

(e) Prove an *a priori* error estimate for this method in the L_2 norm:

$$\|e\|_{L_2} =: \|e\| = \left(\int_0^1 e^2 dx\right)^{\frac{1}{2}}.$$

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(b) Show that if $f \in C^{n+1}[-1, 1]$ and if $P(x)$ is the polynomial with degree at most n that interpolates f at x_k, $k = 0, 1, \ldots, n$ then

$$\|f(x) - P(x)\|_\infty \leq \frac{1}{2^{n-1}(n+1)!} \left\|f^{n+1}\right\|_\infty.$$

Solution:

(a)

$$\cos(n \arccos(\cos(k\pi/n))) = \cos(k\pi) = (-1)^k \quad \text{for } k = 0, 1, \ldots, n.$$

To show the degree of T_n, we use induction. $T_0(x) = 1$ and $T_1(x) = x$.
Induction hypothesis: T_k is a polynomial of degree k and leading coefficient 2^{k-1} for $k \leq n$.
Note that

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= \cos\left((n+1)\arccos x\right) + \left((n-1)\arccos x\right) \\ &= 2x\cos(n \arccos x) \\ &= 2xT_n(x). \end{aligned}$$

Now suppose T_n is of degree n and with leading coefficient 2^{n-1}. From the above calculation,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

which shows that T_{n+1} is of degree $n+1$ and has leading coefficient 2^n. This completes the proof to part **(a)**.

(b) Essentially, this boils down to proving the following,

$$\max_{[-1,1]} |(x - x_0)(x - x_1)\cdots(x - x_n)| \leq \frac{1}{2^{n-1}}.$$

To show this, we define a new Chebyshev-like polynomial. Define

$$Q_n(x) := \sin(n \arccos x)\sqrt{1 - x^2}, \quad n = 1, 2, \ldots$$

Claim: Q_n is of degree $n+1$ with leading coefficient -2^{n-1}.
We prove this claim by induction.
$Q_1(x) = 1 - x^2$ and $Q_2(x) = 2x\sin(\arccos x)\sqrt{1 - x^2} = 2x(1 - x^2)$. This satisfies the claim and serves as the

base case.
Induction hypothesis: Q_k is a polynomial of degree $k+1$ and leading coefficient -2^{k-1} for $k \leq n$.
Note,

$$\begin{aligned}
Q_{n+1}(x) - Q_{n-1}(x) &= [\sin((n+1)\arccos x) - \sin((n-1)\arccos x)]\sqrt{1-x^2} \\
&= 2\cos(n\arccos x)\sin(\arccos x)\sqrt{1-x^2} \\
&= 2(1-x^2)T_n(x).
\end{aligned}$$

Therefore,

$$Q_{n+1}(x) = 2(1-x^2)T_n(x) + Q_{n-1}(x).$$

Using part **(a)** and the induction hypothesis, Q_{n+1} is a polynomial of degree $n+2$ and leading coefficient -2^n. This completes the proof to claim.
Also note that $x_0, x_1, \ldots, x_n$ are the roots of Q_n. As a result,

$$Q_n = -2^{n-1}(x-x_0)(x-x_1)\ldots(x-x_n).$$

Since $\max_{[-1,1]} Q_n = 1$, the result follows.

Problem 2. Let $S(x)$ be a cubic spline with knots $\{t_i\}_{i=0}^n$. If it is determined that $S(x)$ is linear over $[t_1, t_2]$ and $[t_3, t_4]$. Prove that $S(x)$ is also linear over $[t_2, t_3]$.

Solution: First define $p : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
p(x) = &\frac{S''(t_3)(x-t_2)^3}{6(t_3-t_2)} + \frac{S''(t_2)(t_3-x)^3}{6(t_3-t_2)} + \left[\frac{S(t_3)}{t_3-t_2} - \frac{S''(t_3)(t_3-t_2)}{6}\right](x-t_2) \\
&+ \left[\frac{S(t_2)}{t_3-t_2} - \frac{S''(t_2)(t_3-t_2)}{6}\right](t_3-x).
\end{aligned}$$

We claim that $p = S$ in $[t_2, t_3]$. Since $\deg(p) = \deg(S) = 3$, we will be done if we show p and S match at four distinct constraints. Observe

$$p(t_2) = 0 + \frac{S''(t_2)(t_3-t_2)^2}{6} + 0 + \left[\frac{S(t_2)}{t_3-t_2} - \frac{S''(t_2)(t_3-t_2)}{6}\right](t_3-t_2) = S(t_2).$$

In a similar fashion, we also have

$$p(t_3) = S(t_3).$$

Moreover,

$$p''(x) = S''(t_3)\frac{x-t_2}{t_3-t_2} + S''(t_2)\frac{x-t_3}{t_2-t_3},$$

which is the Lagrange interpolating polynomial between $S''(t_2)$ and $S''(t_3)$, *i.e.*,

$$p''(t_i) = S''(t_i),$$

for $i = 2, 3$. This shows four degrees of freedom for which p matches S, and so we conclude $p = S$ in $[t_2, t_3]$.

We use the fact $p = S$ to show S is linear over $[t_2, t_3]$. Because S is linear over $[t_1, t_2]$ and $[t_3, t_4]$, we have $S''(t_2) = S''(t_3) = 0$. This implies for $[t_2, t_3]$,

$$S(x) = p(x) = 0 + 0 + \left[\frac{S(t_3)}{t_3-t_2} - 0\right](x-t_2) + \left[\frac{S(t_2)}{t_3-t_2} - 0\right](t_3-x) = S(t_3)\frac{x-t_2}{x_3-x_2} + S(t_2)\frac{x-t_3}{x_2-x_3},$$

i.e., $\deg(S) = 1$. Thus S is linear over $[t_2, t_3]$.

Problem 3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - \cos x$.

(a) Prove that the equation $f(x) = 0$ has a unique solution $x^* \in \mathbb{R}$ that lies in the interval $(\frac{1}{4}, \frac{1}{2})$.

(b) Prove that the sequence defined by the fixed point iteration

$$
\begin{aligned}
&x_0,\\
&x_n = \tfrac{1}{2}\cos x_{n-1}, \quad n = 1, 2, \ldots
\end{aligned}
$$

converges to x^* with any initial guess x_0.

(c) For the fixed point iteration in **(b)** with $x_0 = \frac{\pi}{6}$, determine an n that guarantees $|x_n - x^*| < \frac{1}{2} \times 10^{-8}$. For the fixed point iteration in **(b)** with $x_0 = 20$, determine an n that guarantees $|x_n - x^*| < \frac{1}{4}$.

Solution:

(a) $f(\frac{1}{4}) = \frac{1}{2} - \cos\frac{1}{4} < \frac{1}{2} - \cos\frac{\pi}{4} < 0$. Also, $f(\frac{1}{2}) = 1 - \cos\frac{1}{2} > 0$. Therefore, by the Intermediate Value Theorem, there exists a root in the said interval. However, since $f' = 2 + \sin x$, the function is strictly increasing and the root is unique.

(b) Set $\phi(x) := \frac{1}{2}\cos x$. The iteration scheme is $x_{n+1} = \phi(x_n)$.

$$
\begin{aligned}
|x_{n+1} - x^*| &= |\phi(x_n) - x^*| \\
&= |\phi(x_n) - \phi(x^*)| \\
&= |\phi'(\xi)| \cdot |x_n - x^*| \\
&\leq \tfrac{1}{2}\,|x_n - x^*|\,.
\end{aligned}
$$

Therefore,

$$
|x_n - x^*| \leq \frac{1}{2^n}|x_0 - x^*|,
$$

which converges.

(c) For $x_0 = \frac{\pi}{6}$, using part **(b)**, a necessary condition to ensure the required bound is,

$$
\frac{\left|\frac{\pi}{6} - \frac{1}{4}\right|}{2^n} < \frac{10^{-8}}{2},
$$

which is,

$$
n > 1 + \log_2\left[10^8\left(\frac{\pi}{6} - \frac{1}{4}\right)\right] \approx 25.71.
$$

Hence, $n = 26$ would suffice. For $x_0 = 20$,

$$
\frac{20 - \frac{1}{4}}{2^n} < \frac{1}{4}.
$$

So,

$$
n > 2 + \log_2 19.75 > 6.
$$

Hence, $n = 7$ would suffice.

Problem 4. Let matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ with $m \geq n$ and $r = \text{rank}(\mathbf{A}) < n$, and assume A has the following SVD decomposition

$$
\mathbf{A} = [\mathbf{U_1}, \mathbf{U_2}] \begin{bmatrix} \mathbf{\Sigma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} [\mathbf{V}_1, \mathbf{V}_2]^T = \mathbf{U}_1\Sigma_1\mathbf{V}_1^T,
$$

where Σ_1 is $r \times r$ nonsingular and $\mathbf{U}_1$ and $\mathbf{V}_1$ have r columns. Let $\sigma = \sigma_{\min}(\Sigma_1)$, the smallest nonzero singular value of $\mathbf{A}$. Consider the following least square problem, for some $\mathbf{b} \in \mathbf{R}^{\mathbf{m}}$,

$$
\min_{\mathbf{x}\in\mathbf{R}^{\mathbf{n}}} \|\mathbf{Ax} - \mathbf{b}\|_2\,.
$$

(a) Show that all solutions $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b} + \mathbf{V_2 z_2},$$

with $\mathbf{z_2}$ an arbitrary vector.

(b) Show that the solution $\mathbf{x}$ has minimal norm $\|\mathbf{x}\|_2$ precisely when $\mathbf{z_2} = \mathbf{0}$, and in which case,

$$\|\mathbf{x}\|_2 \le \frac{\|\mathbf{b}\|_2}{\sigma}.$$

Solution:

(a) Set $\mathbf{U} = [\mathbf{U_1}, \mathbf{U_2}], \Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, and $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]^T$, then the SVD decomposition of $\mathbf{A}$ is $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V^T}$, where $\mathbf{U_{m\times m}}, \mathbf{V_{n\times n}}$ are orthogonal matrices such that

- $\mathbf{U^T} = \mathbf{U^{-1}}$ and $\mathbf{V^T} = \mathbf{V^{-1}}$
- $\mathbf{U}$ and $\mathbf{V}$ are l_2-norm preserving.

As a consequence

$$\|\mathbf{Ax} - \mathbf{b}\|_2 = \left\|\mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} - \mathbf{UU^T b}\right\|_2 = \left\|\Sigma\mathbf{V}^T\mathbf{x} - \mathbf{U^T b}\right\|_2.$$

Let $\mathbf{z} = \mathbf{V^T x} = (\mathbf{z_1}, \mathbf{z_2})^\mathbf{T}$ and $\mathbf{c} = \mathbf{U^T b} = (\mathbf{c_1}, \mathbf{c_2})^\mathbf{T}$. Then

$$\|\mathbf{Ax} - \mathbf{b}\|_2 = \left\|\begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix} - \begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} \Sigma_1\mathbf{z_1} - \mathbf{c_1} \\ \mathbf{c_2} \end{bmatrix}\right\|_2.$$

The l_2-norm is minimized when the vector $\mathbf{z}$ is chosen with $\mathbf{z_1} = \Sigma_1^{-1}\mathbf{c_1}$, $\mathbf{z_2}$ arbitrary. Then

$$\mathbf{x} = \mathbf{Vz} = \mathbf{V}\begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{z_2} \end{bmatrix} = (\mathbf{V_1}, \mathbf{V_2})\begin{bmatrix} \Sigma_1^{-1}\mathbf{U_1^T b} \\ \mathbf{z_2} \end{bmatrix} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b} + \mathbf{V_2 z_2}.$$

(b) Let $\tilde{\mathbf{x}} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b}$ (*i.e.*, $\mathbf{z_2} = \mathbf{0}$), so $\tilde{\mathbf{z}} = \mathbf{V^T}\tilde{\mathbf{x}}$, implies $\tilde{\mathbf{z}} = \begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{0} \end{bmatrix}$, then $\|\tilde{\mathbf{x}}\|_2 = \|\mathbf{V}\tilde{\mathbf{z}}\|_2 = \left\|\Sigma_1^{-1}\mathbf{c_1}\right\|_2$.

For any solution $\mathbf{x}$, we have

$$\|\mathbf{x}\|_2 = \|\mathbf{Vz}\|_2 = \left\|\begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{z_2} \end{bmatrix}\right\|_2 \ge \left\|\Sigma_1^{-1}\mathbf{c_1}\right\|_2 = \|\tilde{\mathbf{x}}\|_2.$$

Finally,

$$\|\tilde{\mathbf{x}}\|_2 = \left\|\mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b}\right\|_2 = \left\|\Sigma_1^{-1}\mathbf{U_1^T b}\right\|_2 \le \left\|\Sigma_1^{-1}\mathbf{U_1^T}\right\|_2 \|\mathbf{b}\|_2 = \left\|\Sigma_1^{-1}\right\|_2 \|\mathbf{b}\|_2 = \frac{\|\mathbf{b}\|_2}{\sigma}.$$

Problem 5. Consider the family of semi-implicit Runge-Kutta methods

$$k_1 = f(y_n + \beta h k_1), \quad k_2 = f(y_n + hk_1 + \beta h k_2),$$
$$y_{n+1} = y_n + h\Big((\tfrac{1}{2} + \beta)k_1 + (\tfrac{1}{2} - \beta)k_2\Big).$$

(a) Determine the order and the principal part of the local truncation error.

(b) Show that if $\beta > \frac{1}{2}$, then the negative real axis $\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) = 0\}$ is contained in the region of absolute stability of the method.

Solution:

(a) Apply this method to the problem $f(y) = \lambda y$, we get

$$\begin{aligned}
k_1 = \lambda y_n + \beta\lambda h k_1 &\implies (1-\beta\lambda h)k_1 = \lambda y_n \\
&\implies k_1 = (1-\beta\lambda h)^{-1}\lambda y_n \\
k_2 = \lambda y_n + \lambda h k_1 + \beta\lambda h k_2 &\implies (1-\beta\lambda h)k_2 = \lambda y_n + (1-\beta\lambda h)^{-1}\lambda^2 h y_n \\
&\implies k_2 = (1-\beta\lambda h)^{-1}\lambda y_n + (1-\beta\lambda h)^{-2}\lambda^2 h y_n.
\end{aligned}$$

Then the method can be written as

$$\begin{aligned}
y_{n+1} &= y_n + (1-\beta\lambda h)^{-1}\lambda h y_n + (\tfrac{1}{2}-\beta)(1-\beta\lambda h)^{-2}\lambda^2 h^2 y_n \\
&= \Big(1 + (1-\beta z)^{-1} z + (\tfrac{1}{2}-\beta)(1-\beta z)^{-2} z^2\Big) y_n \quad (z := \lambda h) \\
&= \Big(1 + z\sum_{i=0}^{\infty}\beta^i z^i + (\tfrac{1}{2}-\beta) z^2 \Big(\textstyle\sum_{i=0}^{\infty}\beta^i z^i\Big)^2\Big) y_n \quad (|z\beta| < 1) \\
&= \Big(1 + z\Big(1 + \beta z + \beta^2 z^2 + \beta^3 z^3 + O(z^4)\Big) + (\tfrac{1}{2}-\beta) z^2 \Big(1 + \beta z + \beta^2 z^2 + O(z^3)\Big)^2\Big) y_n \\
&= \Big(1 + z + \tfrac{1}{2} z^2 + \left(\beta - \beta^2\right) z^3 + \left(\tfrac{3}{2}\beta^2 - 2\beta^3\right) z^4 + O(h^5)\Big) y_n.
\end{aligned}$$

Assume $y_n = y(x_n)$, the exact solution $y(x_{n+1}) = e^z y_n$ can be written as

$$y(x_{n+1}) = \Big(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{1}{24} + O(h^5)\Big) y_n.$$

Comparing the coefficients of z^3, we conclude that

- If $\beta - \beta^2 \neq \frac{1}{6}$, *i.e.* $\beta \neq \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, then the method is second order, and $\tau_n \sim (\beta - \beta^2 - \frac{1}{6})h^3$.
- If $\beta - \beta^2 = \frac{1}{6}$, *i.e.* $\beta = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, then $\frac{3}{2}\beta^2 - 2\beta^3 \neq \frac{1}{24}$, the method is third order, and $\tau_n \sim (\frac{3}{2}\beta^2 - 2\beta^3 - \frac{1}{24})h^4$.

(b) It suffices to show if $\beta > \frac{1}{2}$ and $z < 0$, then

$$-1 < 1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2 < 1.$$

First note $\beta > \frac{1}{2}$ and $z < 0$ imply $\frac{z}{1-\beta z} < 0$, and $(\frac{1}{2}-\beta)(\frac{z}{1-\beta z})^2 < 0$, hence

$$1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2 < 1.$$

Now it remains to show

$$-1 < 1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2.$$

Observing that $-\frac{1}{\beta} \leq \frac{z}{1-\beta z}$ (since $\beta z - 1 \leq \beta z$), we only need to verify

$$-1 < 1 - \frac{1}{\beta} + (\tfrac{1}{2}-\beta)\left(-\tfrac{1}{\beta}\right)^2.$$

For $\beta \neq 0$, this is equivalent to

$$0 < (2\beta - 1)^2,$$

which certainly holds for $\beta > \frac{1}{2}$.

Problem 6. Consider the Beam equation from mechanics with boundary conditions that model a cantilever beam:

$$\begin{aligned}
&u^{(4)} = f(x), \quad x \in (0,\ 1), \\
&u(0) = u'(0) = u''(1) = u'''(1) = 0.
\end{aligned} \tag{1}$$

(a) Recast this equation into a variational problem, stating the trial and test function spaces.

(b) Interpret the variational problem as an energy minimization problem, clearly stating the energy functional. Prove that the variational problem and the energy minimization problems are equivalent.

(c) Develop a CG(3) (cubic continuous Galerkin method) finite element method for this problem.

(d) Prove an *a priori* error estimate for this method in the energy norm:

$$\|e\|_E = \left(\int_0^1 (e'')^2 dx\right)^{\frac{1}{2}},$$

Where $e = u(x) - U(x)$, in which, $u(x)$ is the exact solution to VP (variational problem), $U(x)$ is the FEM (finite element method) solution.

(e) Prove an *a priori* error estimate for this method in the L_2 norm:

$$\|e\|_{L_2} =: \|e\| = \left(\int_0^1 e^2 dx\right)^{\frac{1}{2}}.$$

Solution:

(a) Multiply both sides of $u^{(4)} = f(x)$ with test function v and integrate on $[0,\ 1]$ to get

$$\int_0^1 u^{(4)} v dx = \int_0^1 f(x) v dx,$$

integration by parts twice yields

$$u''' v\Big|_0^1 - u'' v'\Big|_0^1 + \int_0^1 u'' v'' dx = \int_0^1 f(x) v dx.$$

Assume $v(0) = 0,\ v'(0) = 0$ so that $u''' v\Big|_0^1 - u'' v'\Big|_0^1 = 0$, then

$$\int_0^1 u'' v'' dx = \int_0^1 f(x) v dx.$$

Define

$$V = \{w : \int_0^1 w^2 + (w')^2 + (w'')^2 dx < \infty, \quad w(0) = w'(0) = 0\},$$

then the Variational Problem(VP) is:

Find $u \in V$, such that

$$\int_0^1 u'' v'' dx = \int_0^1 f(x) v dx, \quad \forall v \in V. \tag{2}$$

(b) Define the total energy $F : V \to \mathbb{R}$ as

$$F(w) = \tfrac{1}{2} \int_0^1 (w'')^2 dx - \int_0^1 f(x) w dx,$$

then the energy minimization problem(MP) is:

Find $u \in V$ such that

$$F(u) \le F(w), \quad \forall w \in V. \tag{3}$$

We can prove the equivalence of VP and MP:

(VP$\Rightarrow$MP) Assume $u \in V$ such that $\int_0^1 u''v''dx = \int_0^1 f(x)vdx$ for all $v \in V$. Let $w = u + v \in V$, then

$$
\begin{aligned}
F(w) =& \tfrac{1}{2}\int_0^1 (u''+v'')^2 dx - \int_0^1 f(x)(u+v)dx \\
=& \tfrac{1}{2}\int_0^1 (u'')^2 dx + \tfrac{1}{2}\int_0^1 (v'')^2 dx + \int_0^1 u''v''dx - \int_0^1 f(x)udx - \int_0^1 f(x)vdx \\
=& \left(\tfrac{1}{2}\int_0^1 (u'')^2 dx - \int_0^1 f(x)udx\right) + \left(\int_0^1 u''v''dx - \int_0^1 f(x)udx\right) + \tfrac{1}{2}\int_0^1 (v'')^2 dx \\
=& F(u) + 0 + \tfrac{1}{2}\int_0^1 (v'')^2 dx \\
\geq& F(u),
\end{aligned}
$$

where the last equality is obtained by the definition of total energy and the fact that u is solution to the VP. Which implies solution to VP is also solution to MP.

(VP$\Leftarrow$ MP) Assume $u \in V$ such that $F(u) \leq F(w)$ for all $w \in V$. Let $g(\epsilon) = F(u + \epsilon v)$, here $v \in V$ is arbitrary but fixed, then $g'(0) = 0$.

Note that

$$
g'(\epsilon) = \int_0^1 (u'' + \epsilon v'')v''dx - \int_0^1 fvdx,
$$

substitute $\epsilon = 0$ and use the fact that $g'(0) = 0$, we have

$$
\int_0^1 u''v''dx = \int_0^1 fvdx,\ \forall v \in V.
$$

Which implies solution to MP is solution to VP.

(c) (i) Partition: Let τ_h: $0 = x_0 < \cdots < x_M < x_{M+1} = 1$ be a partition of $[0,\ 1]$, let $h_j = x_j - x_{j-1}$ for $j = 1, \ldots, M+1$ be the size of j-th mesh $I_j = [x_{j-1},\ x_j]$, define $h := \max_{1\leq j\leq M+1} h_j$.

(ii) Finite element space: Let $V_h^3 \subseteq V$ be our finite element space defined as

$$
V_h^3 := \{u \in C^1(0,\ 1) | \quad u|_{I_j} \text{ is cubic polynomial for all } j = 1, \ldots, M+1, \text{ and } u(0) = u'(0) = 0\}. \tag{4}
$$

(iii) CG(3) Finite Element Method: Find $U(x)$ in V_h^3 such that

$$
\int_0^1 U''v''dx = \int_0^1 fvdx,\quad \forall v \in V_h^3.
$$

(d) Let $u(x) \in V$ be the exact solution to VP (variational problem), $U(x) \in V_h^3$ be the FEM solution, we estiamte the error $e = u - U$ as follows:

$$
\begin{aligned}
\|u\|_E^2 &= \int_0^1 (e'')^2\, dx \\
&= \int_0^1 e''(u - v + v - U)''dx \quad (v \in V_h^3) \\
&= \int_0^1 e''(u - v)''dx + \int_0^1 e''(v - U)''dx \quad \text{(by Galerkin orthogonality)} \\
&= \int_0^1 e''(u - v)''dx \quad \text{(by Cauchy inequality)} \\
&\leq \|u - U\|_E \cdot \|u - v\|_E\ .
\end{aligned}
$$

Hence $\|e\|_E \leq \|u - v\|$ for all $v \in V_h^3$, take $v = \pi_h u$ as interpolation of u, then

$$
\|e\|_E \leq Ch^2\|u^{(4)}\|,
$$

where C comes from interpolation error.

(e) Consider the following dual problem

$$\phi^{(4)} = e, \quad \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0.$$

We have

$$\begin{aligned}
\|e\|^2 &= \int_0^1 e\phi^{(4)} dx \\
&= \int_0^1 e''\phi'' dx + e\phi'''\Big|_0^1 - e'\phi''\Big|_0^1 \quad \text{(by integration by parts twice)} \\
&= \int_0^1 e''\phi'' dx \quad \left(\text{subtract } \int_0^1 e''(\pi_h\phi)'' dx = 0\right) \\
&= \int_0^1 e''(\phi - \pi_h\phi)'' dx \\
&\leq \|e\|_E \cdot \|\phi - \pi_h\phi\|_E \quad \text{(by the interpolation error and energy norm estimate)} \\
&\leq C^2 h^4 \|u^{(4)}\| \cdot \|\phi^{(4)}\| \quad (\text{ since } \|\phi^{(4)}\| = \|e\|) \\
&= C^2 h^4 \|u^{(4)}\| \cdot \|e\|,
\end{aligned}$$

that is, $\|e\| \leq C^2 h^4 \|u^{(4)}\|$.

S.-T. Yau College Student Mathematics Contests 2021

Geometry and Topology

Solve every problem.

Problem 1.

(a) Show that $\mathbf{P^{2n}}$ can not be the boundary of a compact manifold.

(b) Show that $\mathbf{P^3}$ is the boundary of some compact manifold.

Problem 2. Suppose M is a noncompact, complete n-dimensional manifold, and suppose there is an open subset $U \subset M$ and an open set $V \subset \mathbf{R^n}$ such that $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash V$. If $\mathrm{Ric} M \geq 0$, show that M is isometric to $\mathbf{R^n}$.

Problem 3. Compute all the homotopy groups of the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, $n \geq 2$.

Problem 4. Consider the upper half space $\mathbf{H^3} = \{(x, y, z) \mid z > 0\}$ equipped with hyperbolic metric $g = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$. Let P be the surface defined by $\{z = x \tan\alpha,\ z > 0\}$ for some $\alpha \in (0, \frac{\pi}{2})$. Compute the mean curvature of P.

Problem 5. Suppose M is a compact 2-dimensional Riemannian manifold without boundary, with positive sectional curvature. Show that any two compact closed geodesics on M must intersect with each other.

Problem 6. Suppose Σ is a smooth compact embedded hypersurface (*i.e.* a codimension 1 submanifold) without boundary in $\mathbf{R^n}$ for $n \geq 3$. Show that Σ is orientable.

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Geometry and Topology

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Problem 1.

(a) Show that $\mathbf{P^{2n}}$ can not be the boundary of a compact manifold.

(b) Show that $\mathbf{P^3}$ is the boundary of some compact manifold.

Solution:

(a) Suppose M is a compact manifold and ∂M is its boundary. We can glue together two copies of M, say M_1, M_2, to get a closed manifold $\widetilde{M}$. From the Mayer-Vietoris long exact sequence for the triad $(\widetilde{M}; M_1, M_2)$, we have the identity

$$\chi(\widetilde{M}) = 2\chi(M) - \chi(\partial M),$$

where χ is the Euler characteristic. If the dimension of ∂M is even, then the dimension of M is odd, and so is the dimension of $\widetilde{M}$. By Poincate duality, $\chi(\widetilde{M}) = 0$. So $\chi(\partial M)$ has to be an even number. However, $\mathbf{RP^{2n}}$ has odd Euler characteristic. Thus $\mathbf{RP^{2n}}$ can not be the boundary of a compact manifold.

(b) Since $\mathbf{RP^3}$ is diffeomorphic to SO(3), it is actually the circle bundle on S^2 in the tangent bundle of S^2. Thus $\mathbf{RP^3}$ is the boundary of the disk bundle of S^2 in the tangent bundle of S^2.

Problem 2. Suppose M is a noncompact, complete n-dimensional manifold, and suppose there is an open subset $U \subset M$ and an open set $V \subset \mathbf{R^n}$ such that $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash V$. If Ric$M \geq 0$, show that M is isometric to $\mathbf{R^n}$.

Solution: Without loss of generality, we may assume $V = B_R(0)$ for some $R > 0$. Let $p \in M$ be a point. If Ric$M \geq 0$, by the Bishop-Gromov inequality, we know $\dfrac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^n)}$ is non-increasing. Here, B_r^n is the standard Euclidean ball in $\mathbf{R^n}$ with radius r. On one hand, as $r \to 0$, because M is a smooth manifold, we have $\lim\limits_{r\to 0} \dfrac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^n)} = 1$; one the other hand, when $r \to \infty$, because $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash B_R(0)$, we also have $\lim\limits_{r\to\infty} \dfrac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^n)} = 1$. As a consequence, $\dfrac{\text{Vol}(B_r(p))}{\text{Vol}(B_r^n)}$ is a constant 1 for all $r > 0$. Then by the rigidity case of Bishop-Gromov theorem, M is isometric to $\mathbf{R^n}$.

Remark: *Some students may want to use the Cheeger-Gromoll splitting theorem to show $M \cong N \times \mathbf{R}$, then conclude that $M \cong \mathbf{R^n}$. To my knowledge, it is actually hard to show that one can find a straight line in M. In fact, for a straight line in $\mathbf{R^n} \backslash V$, its corresponding line in M may not be straight, because there could be a shorter path going through U which connects two points on the line.*

Problem 3. Compute all the homotopy groups of the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, $n \geq 2$.

Solution: In the following homotopy groups we always assume that we have fixed a base point.

Because T^n is connected, $\pi_0(T^n)$ is a trivial group.

$\pi_1(T^n)$ is the fundamental group of T^n. Because the fundamental group of a product space is just the product of each fundamental group, and the fundamental group of S^1 is $\mathbf{Z}$, so $\pi_1(T^n) = \mathbf{Z^n}$.

The universal cover of T^n is $\mathbf{R^n}$, which is contractible. So for all $k \geq 2$, $\pi_k(T^n) \cong \pi_k(\mathbf{R^n}) = 0$, which is the trivial group.

Problem 4. Consider the upper half space $\mathbf{H^3} = \{(x, y, z) \mid z > 0\}$ equipped with hyperbolic metric $g = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$. Let P be the surface defined by $\{z = x \tan\alpha,\ z > 0\}$ for some $\alpha \in (0, \frac{\pi}{2})$. Compute the mean curvature of P.

Solution: We use ∂_x, ∂_y and ∂_z to denote the vector fields on $\mathbf{H^3}$ induced from $\mathbf{R^3}$. Then we can compute the Christoffel symbols

$$\begin{cases} \Gamma^i_{zi} = -z^{-1} \\ \Gamma^z_{jj} = z^{-1}, & j \neq z \\ \Gamma^k_{ij} = 0, & \text{other cases.} \end{cases}$$

So the covariant derivatives are

$$\begin{cases} \nabla_{\partial_x}\partial_x = z^{-1}\partial_z \\ \nabla_{\partial_y}\partial_y = z^{-1}\partial_z \\ \nabla_{\partial_z}\partial_z = -z^{-1}\partial_z \\ \nabla_{\partial_z}\partial_x = \nabla_{\partial_x}\partial_z = -z^{-1}\partial_x \\ \nabla_{\partial_z}\partial_y = \nabla_{\partial_y}\partial_z = -z^{-1}\partial_y \\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0 \end{cases}$$

Now consider the surface P parametrized by $F : (u, v) \to (u, v, u\tan\alpha)$. Then at any fixed point the tangent space is spanned by

$$F_u = \partial_x + \tan\alpha\partial_z,\ F_v = \partial_y,$$

with the metric

$$g_{uv} = z^{-2}\begin{pmatrix} 1 + \tan^2\alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

We can also find a unit normal vector

$$\mathbf{n} = \frac{z}{\sqrt{1 + \tan^2\alpha}}(\tan\alpha\partial_x - \partial_z).$$

Next, we can compute that

$$\begin{aligned} \nabla_{F_u}F_u &= (1 - \tan^2\alpha)z^{-1}\partial_z - 2\tan\alpha z^{-1}\partial_x, \\ \nabla_{F_v}F_u &= -\tan\alpha z^{-1}\partial_y, \\ \nabla_{F_v}F_v &= z^{-1}\partial_z. \end{aligned}$$

Moreover,

$$\begin{aligned} \langle\nabla_{F_u}F_u, \mathbf{n}\rangle &= -\tfrac{1}{z^2\sqrt{1+\tan^2\alpha}}(1 + \tan^2\alpha), \\ \langle\nabla_{F_v}F_u, \mathbf{n}\rangle &= 0, \\ \langle\nabla_{F_v}F_v, \mathbf{n}\rangle &= -\tfrac{1}{z^2\sqrt{1+\tan^2\alpha}}. \end{aligned}$$

So the mean curvature is

$$H = -g^{ij}\langle\nabla_{F_i}F_j, \mathbf{n}\rangle = \frac{2}{\sqrt{1 + \tan^2\alpha}} = 2\cos\alpha.$$

Remark: *There are different conventions for the definition of mean curvature, so the final answer could be* $\cos\alpha$*,* $-2\cos\alpha$*, or* $-\cos\alpha$*, depending on the choice of definitions.*

Problem 5. Suppose M is a compact 2-dimensional Riemannian manifold without boundary, with positive sectional curvature. Show that any two compact closed geodesics on M must intersect with each other.

Solution: We prove by contradiction. Suppose there exist two compact closed geodesics γ_1 and γ_2 that do not intersect with each other. Then we can find $p \in \gamma_1$ and $q \in \gamma_2$ such that the distance between p, q is the shortest distance among all pairs of points on γ_1 and γ_2. Let $\tilde{\gamma} : [a, b] \to M$ be a length parametrized geodesic connecting $\tilde{\gamma}(a) = p$ and $\tilde{\gamma}(b) = q$, whose length realizes this shortest distance. Let ℓ be the length functional of curves. By the first variational formula,

$$\delta\ell(\tilde{\gamma}) = 0.$$

Namely, if V is a normal variational vector field along $\tilde{\gamma}$, suppose $\tilde{\gamma}_s$ is a family of curves generating this variational vector field, then

$$0 = \left.\frac{d\ell(\tilde{\gamma}_s)}{ds}\right|_{s=0} = -\left\langle V(a), \dot{\tilde{\gamma}}(a)\right\rangle + \left\langle V(b), \dot{\tilde{\gamma}}(b)\right\rangle.$$

As a consequence, we know that $\dot{\tilde{\gamma}}(a)$ is perpendicular to γ_1 at p and $\dot{\tilde{\gamma}}(b)$ is perpendicular to γ_2 at q.

Next we consider the second variational formula. Suppose X is a vector field along $\tilde{\gamma}$, where $|X(a)| = 1$ and $X(a)$ is perpendicular to $\tilde{\gamma}(a)$, and $X(t)$ is defined by parallel transport along $\tilde{\gamma}$ for $a < t \leq b$. Suppose $\tilde{\gamma}_s$ is a family of curves that generate X, then

$$0 \leq \left.\frac{d^2\ell(\tilde{\gamma}_s)}{ds^2}\right|_{s=0} = \int_a^b -R(\dot{\tilde{\gamma}}, X, X, \dot{\tilde{\gamma}})dt + \langle \nabla_{X(a)}X(a), \dot{\tilde{\gamma}}(a)\rangle - \langle \nabla_{X(b)}X(b), \dot{\tilde{\gamma}}(b)\rangle.$$

Notice that γ_1 and γ_2 are geodesics and $X(a)$, $X(b)$ are both unit vectors in the direction of γ_1, γ_2 at p, q respectively, so $\nabla_{X(a)}X(a) = 0$ and $\nabla_{X(b)}X(b) = 0$, and as a consequence

$$0 \leq \left.\frac{d^2\ell(\tilde{\gamma}_s)}{ds^2}\right|_{s=0} = \int_a^b -R(\dot{\tilde{\gamma}}, X, X, \dot{\tilde{\gamma}})dt = \int_a^b -\sec(\dot{\tilde{\gamma}}, X)dt < 0.$$

This is a contradiction.

Problem 6. Suppose Σ is a smooth compact embedded hypersurface (*i.e.* a codimension 1 submanifold) without boundary in $\mathbf{R^n}$ for $n \geq 3$. Show that Σ is orientable.

Solution: We first claim that it suffices to show Σ has a trivial normal bundle in $\mathbf{R^n}$. In fact, the trivial bundle has the splitting $\mathbf{R^n} \times \Sigma = T\Sigma \oplus N\Sigma$, so the first Stiefel-Whitney class of the bundles satisfies

$$0 = w_1(T\Sigma) + w_1(N\Sigma).$$

If the line bundle of Σ is trivial, we must have $w_1(N\Sigma) = 0$, therefore $w_1(T\Sigma) = 0$. This is equivalent to Σ being orientable.

Thus it remains to show that Σ has a trivial normal bundle. We prove by contradiction. We can view the tubular neighbourhood $\mathcal{T}$ of Σ as a part of $N\Sigma$. If Σ has a non-trivial normal bundle, then there exists a closed curve γ in $\mathcal{T}$ that only intersects Σ at a single point transversely. Consider a smoothly embedded disk D bounded by γ that intersects Σ transversely. Then the intersection of D and Σ consists of finitely many smooth curves whose endpoints lie on the boundary $\partial D = \gamma$. This implies that γ intersects Σ at an even number of points, which is a contradiction.

Probability and Statistics

Solve every problem.

Part I: Probability

Problem 1. Suppose that a sequence $\{X_n\}$ of real-valued random variables converges to X in distribution and there are positive constants r and C such that $\mathbb{E}|X_n|^r \le C$ for all n. Show that

$$\lim_{n\to\infty} \mathbb{E}|X_n|^s = \mathbb{E}|X|^s$$

for all $0 < s < r$.

Problem 2. Let $p(x, y)$ be the (one-step) transition function of a Markov chain on a discrete state space S and $p_n(x, y)$ be the n-step transition function. Show that for any positive integers L and N and any two states x and y we have

$$\sum_{n=L}^{N+L} p_n(x, y) \le \sum_{n=0}^{N} p_n(y, y).$$

Problem 3. Let $\{X_n\}$ be an independent, identically distributed sequence of random variables with the symmetric Bernoulli distribution

$$\mathbb{P}\{X = 1\} = \mathbb{P}\{X = -1\} = \tfrac{1}{2}.$$

Let $S_n = \sum_{i=1}^{N} X_i$ be the partial sum. Show that for all $\alpha > \frac{1}{2}$,

$$\mathbb{P}\left\{\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0\right\} = 1.$$

Problem 4. Let $X^n = \left\{X_{ij}\right\}$ be an $n \times n$ random matrix whose entries are independent and identically distributed random variables with the symmetric Bernoulli distribution

$$\mathbb{P}\{X = 0\} = \mathbb{P}\{X = 1\} = \tfrac{1}{2}.$$

Let $p_n = \mathbb{P}\{\det X_N \text{ is odd}\}$. Show that $\lim_{n\to\infty} p_n > 0$.

Part II: Statistics

Problem 5. You have been asked to help design a randomized trial of a new drug, call it drug A, to be used in place of the current drug, call it drug B, for a particular medical condition. The budget is fixed to have 1000 patients treated with A and 1000 treated with drug B. The issue is how to do the allocation of patients, because we have many pre-randomization measurements on each patient, roughly 200, such as blood pressure recordings, age, sex, and a large collection of genetics measurements. Obviously it is desirable to have the A group similar to the B group with respect to all pre-treatment covariates and non-linear functions of them that are expected to influence the effectiveness of the drugs with respect to the outcome variables.

Complete (or simple) randomization does this in expectation, but with many covariates, some covariates will not be balanced between the A and B groups in any one single randomized allocation. Standard blocking used in traditional experimental design can force balance on a few covariates, but the designer of drug A wants to have an experimental design that creates balance on many covariates, and feels that you, as a modern applied mathematician/statistician, should be able to do this.

Describe a class of methods that achieves this goal where each patient has a positive probability of receiving drug A and a positive probability of receiving drug B. Provide enough detail that you are describing an explicit algorithm.

Problem 6. You are given the results of a randomized experiment of two drugs, A and B. The experiment was not conducted in the usual way, however, but rather by allocating patients by a machine-learning algorithm under which each patient has a positive probability of receiving A and of receiving B; moreover the algorithm is completely specified and is built to create better than random balance on the covariates.

- **(a)** Can unbiased estimates of the causal effect of drug A versus B be found, and if so, show why.
- **(b)** Can exact small sample, non-parametric inferences for the causal effect in part **(a)** be derived, based solely on the randomization distribution of some statistic? For example, can we find exact significance levels under a sharp null hypothesis? If so, outline how to accomplish this goal.

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$$\lim_{n\to\infty} \mathbb{E}|X_n|^s = \mathbb{E}|X|^s$$

for all $0 < s < r$.

Solution: $X_n \to X$ in distribution if and only if $\lim_{n\to\infty} \mathbb{E}_x f(X_n) = \mathbb{E} f(X)$ for all bounded continuous functions f. Let M be a fixed large positive number and define

$$f_M(x) = \begin{cases} |x|^s, & |x| < M; \\ M^s, & |x| \ge M. \end{cases}$$

Then f_M is a bounded continuous function and $0 \le f_M(x) \uparrow |x|^s$ as $M \uparrow \infty$. We have

$$\lim_{n\to\infty} \mathbb{E} f_M(X_n) = \mathbb{E} f_M(X).$$

On the other hand,

$$0 \le \mathbb{E}|X_n|^s - \mathbb{E} f_M(X_n) \le \mathbb{E}\left[|X_n|^s;\ |X_n| \ge M\right] \le \frac{1}{M^{r-s}} \mathbb{E}\left[|X_n|^r;\ |X_n| \ge M\right] \le \frac{C}{M^{r-s}}.$$

Hence

$$\mathbb{E} f_M(X_n) \le \mathbb{E}|X_n|^s \le \mathbb{E} f_M(X_n) + \frac{C}{M^{r-s}}.$$

Letting $n \to \infty$ and then $M \uparrow \infty$ we have $\lim_{n\to\infty} \mathbb{E}|X_n|^s = \mathbb{E}|X|^s$. In the second limit we use the monotone convergence theorem.

Problem 2. Let $p(x, y)$ be the (one-step) transition function of a Markov chain on a discrete state space S and $p_n(x, y)$ be the n-step transition function. Show that for any positive integers L and N and any two states x and y we have

$$\sum_{n=L}^{N+L} p_n(x, y) \le \sum_{n=0}^{N} p_n(y, y).$$

Solution: The intuition of this problem is that for counting how many steps the chain starting from x to be at y the chain needs to spend time to reach y first.

By the Markov property at L we have for $n \ge L$,

$$p_n(x, y) = \mathbb{P}_x \{X_n = y\} = \mathbb{E}_x p_{n-L}(X_L, y).$$

If the assertion holds for $L = 0$, then $(l = n - L)$

$$\sum_{n=L}^{N+L} p_n(x, y) = \mathbb{E}_x \left[\sum_{l=0}^{N} p_l(X_L, y)\right] \le \sum_{l=0}^{N} p_l(y, y).$$

Thus it is enough to show for the case $L = 0$. By the strong Markov property at T_y, we have

$$p_n(x, y) = \mathbb{P}_x \left\{X_n = y,\ T_y \le n\right\} = \mathbb{E}_x \left[p_{n-T_y}(y, y); T_y \le n\right].$$

It follows that ($l = n - T_y$)

$$\sum_{n=0}^{N} p_n(x, y) = \sum_{n=0}^{N} \mathbb{E}_x \left[p_{n-T_y}(y, y); T_y \le n\right] = \mathbb{E}_x \left[\sum_{l=0}^{N-T_y} p_l(y, y); T_y \le N\right] \le \sum_{l=0}^{N} p_n(y, y).$$

Problem 3. Let $\{X_n\}$ be an independent, identically distributed sequence of random variables with the symmetric Bernoulli distribution

$$\mathbb{P}\{X = 1\} = \mathbb{P}\{X = -1\} = \tfrac{1}{2}.$$

Let $S_n = \sum_{i=1}^{N} X_i$ be the partial sum. Show that for all $\alpha > \frac{1}{2}$,

$$\mathbb{P}\left\{\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0\right\} = 1.$$

Solution: Let k be a positive integer. By Markov's inequality we have

$$\mathbb{P}\left\{\left|\frac{S_n}{n^\alpha}\right|^{2k} \ge \frac{1}{n}\right\} \le n^{-(2k\alpha-1)}\mathbb{E}|S_n|^{2k}.$$

We have

$$\mathbb{E}|S_n|^{2k} = \sum_{1\le i_1,\ldots,i_{2k}\le n} \mathbb{E}\left[X_{i_1}X_{i_2}\cdots X_{i_{2k}}\right].$$

Since the sequences are independent and each random variable has mean zero, each expectation is zero unless the indices $i_1, i_2, \ldots, i_{2k}$ can be grouped into k pairs of identical indices. The number of such groups is at most $(2k)! \cdot n^k$. Hence

$$\mathbb{P}\left\{\left|\frac{S_n}{n^\alpha}\right|^{2k} \ge \frac{1}{n}\right\} \le C_k n^{-(2k\alpha-k-1)}.$$

Choosing $k > 2/(2\alpha - 1)$ we have $2k\alpha - k - 1 > 1$. By the Borel-Cantelli lemma, the probability that $|S_n|/n^\alpha \ge n^{-1}$ happens infinitely often is zero, hence $\lim_{n\to\infty} S_n/n^\alpha = 0$ with probability one.

Problem 4. Let $X^n = \left\{X_{ij}\right\}$ be an $n \times n$ random matrix whose entries are independent and identically distributed random variables with the symmetric Bernoulli distribution

$$\mathbb{P}\{X = 0\} = \mathbb{P}\{X = 1\} = \tfrac{1}{2}.$$

Let $p_n = \mathbb{P}\{\det X_N \text{ is odd}\}$. Show that $\lim_{n\to\infty} p_n > 0$.

Solution: Consider the (algebraic) field $\mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ with two elements, so we need $\det X_n \ne 0$ in $\mathbf{Z}/2\mathbf{Z}$, which means that p_n is the probability that the n rows of the random matrix are independent.

Each row has 2^n choices. There are $2^n - 1$ choices for the first row X_1 (the zero column is not allowed). The linear span for the first row has two vectors (the zero vector and X_1), and we cannot choose from this linear span, hence the number of choices for X_2 is $2^n - 2$. In general after the first l rows are fixed, the linear span of these rows has 2^l vectors, and we cannot choose X_{l+1} from this linear span, hence the number of choices for X_{l+1} is $2^n - 2^l$. It follows that the total number of choices for the matrix det X^n with nonvanishing determinant is

$$(2^n - 1) \cdot (2^n - 2) \cdot (2^n - 2^2) \cdots (2^n - 2^{n-1}).$$

The total number of choices for the random matrix X^n is $(2^n)^n = 2^{n^2}$ and the probability distribution is uniform, hence

$$p_n = \prod_{l=1}^{n} \left(1 - \frac{1}{2^l}\right) \to \prod_{l=1}^{\infty} \left(1 - \frac{1}{2^l}\right) > 0.$$

Part II: Statistics

Problem 5. You have been asked to help design a randomized trial of a new drug, call it drug A, to be used in place of the current drug, call it drug B, for a particular medical condition. The budget is fixed to have 1000 patients treated with A and 1000 treated with drug B. The issue is how to do the allocation of patients, because we have many pre-randomization measurements on each patient, roughly 200, such as blood pressure recordings, age, sex, and a large collection of genetics measurements. Obviously it is desirable to have the A group similar to the B group with respect to all pre-treatment covariates and non-linear functions of them that are expected to influence the effectiveness of the drugs with respect to the outcome variables.

Complete (or simple) randomization does this in expectation, but with many covariates, some covariates will not be balanced between the A and B groups in any one single randomized allocation. Standard blocking used in traditional experimental design can force balance on a few covariates, but the designer of drug A wants to have an experimental design that creates balance on many covariates, and feels that you, as a modern applied mathematician/statistician, should be able to do this.

Describe a class of methods that achieves this goal where each patient has a positive probability of receiving drug A and a positive probability of receiving drug B. Provide enough detail that you are describing an explicit algorithm.

Solution: Let X_i denote the covariate vector of unit i, which can include the pre-treatment measurements as well as their non-linear functions, and let $\boldsymbol{X} = (X_1, \ldots, X_n)^T$ denote the convariate matrix for all $n = 2000$ units. Let z_i denote the treatment allocation for unit i, which equals 1 if the unit is assigned to drug A and 0 otherwise, and $z \in \{0, 1\}^n$ denote the treatment allocation for all n units.

Let $\phi(\boldsymbol{X}, z)$ denote a pre-determined covariate balance criterion, which equals 1 if the allocation z is acceptable and 0 otherwise. Moreover, we construct the criterion ϕ such that it is invariant when we switch treatment and control groups, *i.e.*, $\phi(\boldsymbol{X}, z) = \phi(\boldsymbol{X}, 1 - z)$. We then consider the following re-randomization design to randomly allocate the patients to groups A and B:

(i) Completely randomize the patients to groups A and B, with 1000 patients within each group.

(ii) If the balance criterion is satisfied (*i.e.*, $\phi(\boldsymbol{X}, \boldsymbol{Z}) = 1$), proceed to step **(iii)**; otherwise, return to step **(i)**

(iii) Conduct the experiment using the final randomization obtained in step **(ii)**.

Note that under the completely randomized experiment (CRE) with the same number of units assigned to each treatment group, $\boldsymbol{Z}$ and $1 - \boldsymbol{Z}$ follows the same distribution. This implies that, under re-randomization described above (*i.e.*, the CRE with balance criterion ϕ,

$$\boldsymbol{Z} \mid \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1 \sim \boldsymbol{1} - \boldsymbol{Z} \mid \phi(\boldsymbol{X}, 1 - \boldsymbol{Z}) = 1 \sim \boldsymbol{1} - \boldsymbol{Z} \mid \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1.$$

Thus, under re-randomization, $\boldsymbol{Z}$ and $\boldsymbol{1} - \boldsymbol{Z}$ must have the same distribution. Consequently, for any $1 \leq i \leq N$,

$$\mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = \mathbb{E}\left(1 - Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = 1 - \mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right),$$

which immediately implies that $\mathbb{E}\left(Z_i \mid \boldsymbol{X}, \phi(\boldsymbol{X}, \boldsymbol{Z}) = 1\right) = 0.5$. Therefore, under our proposed design, each patient has a positive probability (*i.e.*, $\frac{1}{2}$) to receive either drug A or drug B.

Problem 6. You are given the results of a randomized experiment of two drugs, A and B. The experiment was not conducted in the usual way, however, but rather by allocating patients by a machine-learning algorithm under which each patient has a positive probability of receiving A and of receiving B; moreover the algorithm is completely specified and is built to create better than random balance on the covariates.

(a) Can unbiased estimates of the causal effect of drug A versus B be found, and if so, show why.

(b) Can exact small sample, non-parametric inferences for the causal effect in part **(a)** be derived, based solely on the randomization distribution of some statistic? For example, can we find exact significance levels under a sharp null hypothesis? If so, outline how to accomplish this goal.

Solution:

(a) Yes. For each patient i, let $Y_i(1)$ and $Y_i(0)$ denote the potential outcomes under drug A and drug B, Z_i be the treatment assignment indicator, where $Z_i = 1$ if the patient receives drug A and 0 otherwise, and $Y_i = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$ be the observed outcome.

For each patient i, let $e_i = \Pr(Z_i = 1)$ denote the probability of receiving drug A. Then $1 - e_i$ is the probability of receiving drug B for patient i. Because the algorithm for allocating the patients are completely specified and each patient has a positive probability of receiving A and of receiving B, e_i is known for all i and $e_i \in (0, 1)$. We can construct the following unbiased estimator for the average causal effect of drug A versus B for patients in the experiment:

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} \frac{Z_i}{e_i} Y_i - \frac{1}{n}\sum_{i=1}^{n} \frac{1 - Z_i}{1 - e_i} Y_i.$$

Below we show that $\hat{\tau}$ is unbiased for the average causal effect $\tau \equiv n^{-1}\sum_{i=1}^{n}(Y_i(1) - Y_i(0))$. By definition we have

$$\begin{aligned}
\mathbb{E}\hat{\tau} =& \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} \frac{Z_i}{e_i} Y_i - \frac{1}{n}\sum_{i=1}^{n} \frac{1 - Z_i}{1 - e_i} Y_i\right) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} \frac{Z_i}{e_i} Y_i(1) - \frac{1}{n}\sum_{i=1}^{n} \frac{1 - Z_i}{1 - e_i} Y_i(0)\right) \\
=& \frac{1}{n}\sum_{i=1}^{n} \frac{\mathbb{E}(Z_i)}{e_i} Y_i(1) - \frac{1}{n}\sum_{i=1}^{n} \frac{1 - \mathbb{E}(Z_i)}{1 - e_i} Y_i(0) = \frac{1}{n}\sum_{i=1}^{n} \frac{e_i}{e_i} Y_i(1) - \frac{1}{n}\sum_{i=1}^{n} \frac{1 - e_i}{1 - e_i} Y_i(0) \\
=& \frac{1}{n}\sum_{i=1}^{n} Y_i(1) - \frac{1}{n}\sum_{i=1}^{n} Y_i(0) = \tau.
\end{aligned}$$

Therefore, $\hat{\tau}$ is unbiased for τ.

(b) Yes. For example we consider the sharp null hypothesis of no treatment effect:

$$H_0 : Y_i(1) = Y_i(0), \quad i = 1, 2, \ldots, n.$$

Let $t(\boldsymbol{Z}, \boldsymbol{Y})$ denote a general test statistic, based on which we test the null hypothesis H_0.

First, we can impute all the unknown potential outcomes based on the observed data and the null hypothesis H_0. Specifically, under H_0, $Y_i(1) = Y_i(0) = Y_i$, *i.e.*, both potential outcomes are the same as the observed one.

Second, we can know the distribution of the test statistic under the null hypothesis H_0. Because the algorithm for allocating the patients are completely specified, we know the distribution of the treatment assignment (*i.e.*, treatment assignment mechanism). Let $\boldsymbol{W}$ denote a random vector following this treatment assignment mechanism. Under H_0, the test statistic follows the same distribution as

$$t(\boldsymbol{W}, \boldsymbol{W} \circ \boldsymbol{Y}(1) + (\boldsymbol{1} - \boldsymbol{W}) \circ \boldsymbol{Y}(0)),$$

where $\boldsymbol{Y}(1)$ and $\boldsymbol{Y}(0)$ denote the potential outcome vectors for all units, and $\circ$ denotes element-wise multiplication.

Third, we can then evaluate the tail probability of the null distribution of the test statistic at its observed value to get the p-value. Specifically, let

$$G(c) = \Pr\left\{t(\boldsymbol{W}, \boldsymbol{W} \circ \boldsymbol{Y}(1) + (\boldsymbol{1} - \boldsymbol{W}) \circ \boldsymbol{Y}(0)) \geq c\right\}$$

denote the tail probability of the null distribution. Then the p-value is

$$p = G(t(\boldsymbol{Z}, \boldsymbol{Y})),$$

which is valid for testing H_0.

Note that, for a general sharp null hypothesis, we can obtain a valid p-value in a similar way.