ALGEBRA ORAL

(1) Let k be a field and $B \subset \mathrm{GL}_2(k)$ the subgroup of upper triangles. Describe all $B \times B$-orbits in $\mathrm{GL}_2(k)$, where $B \times B$ acts on $\mathrm{GL}_2(k)$ by

$$(b_1, b_2) \cdot g = b_1 g b_2^{-1},\ \forall (b_1, b_2) \in B \times B, g \in \mathrm{GL}_2(k).$$

(2) Let k be a field and $A, B \in M_{n\times n}(k)$. Prove that $\det(I - AB) = \det(I - BA)$.

(3) Find $\mathbb{Q}$-algebras R such that in $R[X, Y]$, $X^2 + Y^2 = (aX + bY)^2$ for some $a, b \in R$.

ALL ROUND TEST: ALGEBRA

Let C be a double cover over $\mathbb{P}^1$ ramified at 6 points.

(1) Classify all unramified 7-cyclic covers D over C.

(2) Prove that all such D are Galois over $\mathbb{P}^1$ with Galois group $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

(3) Classify all intermediate curves between D and $\mathbb{P}^1$.

GROUP TEST: ALGEBRA

(1) (a) The group SO(2) acts on $\mathbb{C}[X,Y]$ as follows:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot P(X,Y) = P(\cos\theta X + \sin\theta Y, -\sin\theta X + \cos\theta Y).$$

Prove that its invariant subspace $\mathbb{C}[X,Y]^{\mathrm{SO}(2)} = \mathbb{C}[X^2+Y^2]$.

(b) Let $\mathcal{S} = \mathbb{C}[X,Y]e^{-\pi(X^2+Y^2)}$ and let F be the Fourier transformation:

$$F\phi(x,y) = \int_{\mathbb{R}^2} \phi(u,v)e^{2\pi i(ux+vy)}dudv.$$

Prove that for any $P \in \mathbb{C}[X,Y], \phi \in S$,

$$F(P\cdot\phi) = P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) F(\phi).$$

(c) Describe the space $\mathbb{C}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]^{\mathrm{SO}(2)}$.

(2) (a) What is the Galois group of $\mathbb{Q}(\zeta_7)$ over $\mathbb{Q}$, where $\zeta_7 = e^{2\pi i/7}$?

(b) Find a Galois extension field F of $\mathbb{Q}$ such that $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$? Are there infinitely many such Galois extensions?

(c) Find the quadratic subfield of $\mathbb{Q}(\zeta_7)$?

1. Consider thefollowing initial boundary-value problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \qquad on \quad \Re^+(0,1) \\ u(0,x) = u(1,x) = 0 \\ u|_{t=0} = u_0(x) \in C_c^\infty(0,1), \qquad u_t|_{t=0} = 0 \end{cases} \tag{a}$$

1) Solving the ODE problem:

$$\begin{cases} \dfrac{d^2}{dx^2} f_n(x) = -\lambda_n f_n(x) \\ f_n(0) = f_n(1) = 0 \end{cases} \tag{b}$$

2) Prove that all the solutions $\{f_n(x)\}$ of (b)j forms on orthornal basis of $L^2(0,1)$. Moreover, it is complete orthornal basis.

3) Try to (a) by the method of separable variables.

4) Prove the uniqueness of thus obtained solution.

2. Let f be an entire function with $f(0)=0$ and $f'(0)=1$. Let $f^n = f \circ \cdots \circ f$ denote the n-th iteration on f. Suppose that these exist $r>0$ such that $|f^n(z)|<1$ for all $|z|<r$ and all $n=1,2,3,\cdots$. Show that $f(z)=z$ for all z.

3. 1) If the restriction of $f(x,y)$ to each line in the plane is continuous, is f necessarily continuous?

2) Same question for differentiability(smoothness).

If yes, prove; no, give example.

3) Same question for $f:\ C^2 \to C$

INDIVIDUAL TEST / ORAL EXAM
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied and Computational Mathematics

1. Let $f(x)$ defined on [0, 1] be a smooth function with sufficiently many derivatives. $x_i = ih$, where $h = \frac{1}{N}$ and $i = 0, 1, \cdots, N$ are uniformly distributed points in [0, 1]. What is the highest integer k such that the numerical integration formula
(1)
$$I_N = \frac{1}{N}\left(a_0(f(x_0) + f(x_N)) + a_1(f(x_1) + f(x_{N-1})) + \sum_{i=2}^{N-2} f(x_i)\right)$$
is k-th order accurate, namely
$$\left| I_N - \int_0^1 f(x)dx \right| \le Ch^k \tag{2}$$
for a constant C independent of h? Please describe the procedure to obtain the two constants a_0 and a_1 for this k.

2. The classical Euclidean Algorithm to find the greatest common divisor $\gcd(m, n)$ of two positive integers $m < n$ requires only $O(\log n)$ arithmetic operation. However, it uses division with a reminder, which is a rather slow operation. Your task is to design and analyze a **division-free** algorithm.

More precisely, using that for non-zero integers k and l we have
$$\gcd(2k, 2l) = 2\gcd(k, l),$$
$$\gcd(2k+1, 2l) = \gcd(2k+1, l),$$
$$\gcd(2k+1, l) = \gcd(2k+1-l, l)$$

- design an efficient algorithm to compute $\gcd(m, n)$ that uses only subtraction and division by 2 (the latter is very fast as it is equivalent to a shift of the bit representation of the operand);
- give a motivated estimate on the complexity of your algorithm.

ALL-AROUND TEST / ORAL EXAM
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied and Computational Mathematics

1. Let

$$V_h = \{v : v|_{I_j} \in P^k(I_j) \qquad 1 \leq j \leq N\}$$

where

$$I_j = (x_{j-1}, x_j), \qquad\qquad 1 \leq j \leq N$$

with

$$x_j = jh, \qquad h = \frac{1}{N}.$$

Here $P^k(I_j)$ denotes the set of polynomials of degree at most k in the interval I_j.

Recall the L^2 projection of a function $u(x)$ into the space V_h is defined by the unique function $u_h \in V_h$ which satisfies

$$||u - u_h|| \leq ||u - v|| \qquad \forall v \in V_h$$

where the norm is the usual L^2 norm. We assume $u(x)$ has at least $k+2$ continuous derivatives.

(1) Prove the error estimate

$$||u - u_h|| \leq Ch^{k+1}$$

Explain how the constant C depends on the derivatives of $u(x)$.

(2) If another function $\varphi(x)$ also has at least $k+2$ continuous derivatives, prove

$$\left|\int_0^1 (u(x) - u_h(x))\varphi(x)dx\right| \leq Ch^{2k+2}$$

Explain how the constant C depends on the derivatives of $u(x)$ and $\varphi(x)$.

GROUP TEST / ORAL EXAM
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied and Computational Mathematics

1. We would like to solve the following PDE

$$u_t + u_x = u \tag{1}$$

by a finite difference scheme

$$u_j^{n+1} = \sum_{k=-p}^{q} a_k u_{j+k}^n \tag{2}$$

where a_k are constants depending on the mesh sizes Δx and Δt, and u_j^n are approximations to the exact solution $u(x_j, t^n)$ with $x_j = j\Delta x$ and $t^n = n\Delta t$. A student defines stability of the scheme by

$$\|u^{n+1}\| \leq \|u^n\| \tag{3}$$

where $\|\cdot\|$ is the usual discrete L^2-norm. Do you believe this definition is reasonable, namely do you believe there are consistent and accurate schemes (2) approximating the PDE (1) which will be stable under the definition (3)? If yes, give your reasons. If not, modify the definition (3).

2. Let T be a rooted tree with the root r. Show that there is an injection f from the set of vertices of degree at least 3 to the set of leaves (not the root) such that for each vertex v of degree at least 3, v lies in the path from r to $f(v)$.

S-T Yau College Math Contests 2012 Oral Exam

Geometry and Topology

1. (a) Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : S^1 \times S^1 \to S^2$ of degree k.

(b) Let Σ_2 be the closed surface of genus 2. Show that any continuous map $f : S^1 \times S^1 \to \Sigma_2$ has degree 0.

2. Let Σ_2 be the closed orientable surface of genus 2.

(a) What is $G = \pi_1(\Sigma_2)$?

(b) Why is G non-abelian?

(c) Why does G contain a subgroup of index 7?

(d) Show that Σ_2 is not a non-trivial cover of any orientable surface.

(e) Show that Σ_2 is a non-trivial cover of a space.

3. State and prove the Crofton formula.

4. State and sketch the proof of Bonnet-Meyer's theorem.

5. What is a Jacobi field? Compute it in the space form M_k^n. What is the relationship between Jacobi fields and the exponential map.

6. What is the Cartan-Hadamard theorem for non-positively curved spaces? Why is the space a $K(\pi, 1)$ space?

7. Let S be a compact surface in $\mathbb{R}^3$ with smooth boundary and S has negative curvature. Show that S is contained in convex hull of ∂S.

2012 S. T. Yau College Math Contests Oral Exam on Statistics

Saturday, August 4, morning

Problem 1. Let $X_1, \cdots, X_n$ be n independent and identically distributed observations from the exponential distribution with density function $f(x) = \frac{1}{\beta} e^{-x/\beta}$, $x \geq 0$.

a) Let T be an unbiased estimator of the scale parameter β. Prove that

$$\mathrm{Var}(T) \geq \frac{\beta^2}{n}.$$

b) Can you find an unbiased estimator T that attains the lower bound in part a)? If yes, please construct one. If no, please show why such an estimator does not exist.

Solutions to 2012 S. T. Yau College Math Contests Oral Exam on Statistics

Saturday, August 4, morning

Problem 1. Let $X_1, \cdots, X_n$ be n independent and identically distributed observations from the exponential distribution with density function $f(x) = \frac{1}{\beta} e^{-x/\beta}$, $x \geq 0$.

a) Let T be an unbiased estimator of the scale parameter β. Prove that

$$\mathrm{Var}(T) \geq \frac{\beta^2}{n}.$$

Solution: The above lower bound on the variance of an unbiased estimator T of the scale parameter β is given by the Cramér-Rao bound $1/I(\beta)$. The log-likelihood function is

$$\ell(\beta) = \sum_{i=1}^{n} \left\{ -\log \beta - \frac{X_i}{\beta} \right\},$$

which leads to

$$\ell'(\beta) = \sum_{i=1}^{n} \left\{ -\frac{1}{\beta} + \frac{X_i}{\beta^2} \right\} \quad \text{and} \quad \ell''(\beta) = \sum_{i=1}^{n} \left\{ \frac{1}{\beta^2} - \frac{2X_i}{\beta^3} \right\}.$$

Thus the Fisher information is

$$I(\beta) = -E\ell''(\beta) = \frac{n}{\beta^2}.$$

b) Can you find an unbiased estimator T that attains the lower bound in part a)? If yes, please construct one. If no, please show why such an estimator does not exist.

Solution: The answer is yes. The maximum likelihood estimator $\widehat{\beta}$, which solves the score equation $\ell'(\beta) = 0$, is identical to the sample mean $\frac{1}{n} \sum_{i=1}^{n} X_i$. It is easy to show that such an estimator is unbiased and attains the lowest variance.

2012 S. T. Yau College Math Contests Oral Exam on Statistics

Saturday, August 4, afternoon

Problem 1. Let $X_1, \cdots, X_n$ be n independent and identically distributed observations from the Cauchy distribution with density function $f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}$, $x \in \mathbb{R}$.

a) Let T be an unbiased estimator of the location parameter θ. Prove that

$$\mathrm{Var}(T) \geq \frac{2}{n}.$$

b) Can you find an unbiased estimator T that attains the lower bound in part a)? If yes, please construct one. If no, please show why such an estimator does not exist.

c) Can you provide an estimator T that can attain the lower bound on $\mathrm{Var}(T)$ in part a) asymptotically, by removing the constraint of unbiasedness?

Solutions to 2012 S. T. Yau College Math Contests Oral Exam on Statistics

Saturday, August 4, afternoon

Problem 1. Let $X_1, \cdots, X_n$ be n independent and identically distributed observations from the Cauchy distribution with density function $f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}$, $x \in \mathbb{R}$.

a) Let T be an unbiased estimator of the location parameter θ. Prove that

$$\mathrm{Var}(T) \geq \frac{2}{n}.$$

Solution: The above lower bound on the variance of an unbiased estimator T of the location parameter θ is given by the Cramér-Rao bound $1/I(\theta)$. The log-likelihood function is

$$\ell(\theta) = \sum_{i=1}^{n} \left\{-\log\pi - \log\left[1 + (X_i - \theta)^2\right]\right\},$$

which leads to

$$\ell'(\theta) = \sum_{i=1}^{n} \frac{2(X_i - \theta)}{1 + (X_i - \theta)^2} \quad \text{and} \quad \ell''(\theta) = \sum_{i=1}^{n} \frac{-2 + 2(X_i - \theta)^2}{[1 + (X_i - \theta)^2]^2}.$$

Thus the Fisher information is

$$I(\theta) = -E\ell''(\theta) = \frac{n}{2}.$$

b) Can you find an unbiased estimator T that attains the lower bound in part a)? If yes, please construct one. If no, please show why such an estimator does not exist.

Solution: The answer is no. From the proof of the Cramér-Rao theorem, we see that the above lower bound on variance can be attained only if the following Cauchy-Schwarz inequality becomes an equation

$$\left(E\left\{\ell'(\theta)(T-\theta)\right\}\right)^2 \leq E\left\{\ell'(\theta)\right\}^2 E(T-\theta)^2.$$

It is well known that the equation holds only when

$$T - \theta = (\text{some constant}) \cdot \ell'(\theta) = (\text{some constant}) \cdot \sum_{i=1}^{n} \frac{2(X_i - \theta)}{1 + (X_i - \theta)^2},$$

which entails that

$$T = \theta + (\text{some constant}) \cdot \sum_{i=1}^{n} \frac{2(X_i - \theta)}{1 + (X_i - \theta)^2}.$$

The above representation shows that such an "optimal" estimator T should always depend on the location parameter θ, which cannot be an estimator in the first place.

c) Can you provide an estimator T that can attain the lower bound on $\mathrm{Var}(T)$ in part a) asymptotically, by removing the constraint of unbiasedness?

Solution: The answer is yes by the classical asymptotic theory of the maximum likelihood estimator (MLE). The MLE $\widehat{\theta}$, which solves the score equation $\ell'(\theta) = 0$, is known to be asymptotically normal with mean θ and variance $1/I(\theta) = \frac{2}{n}$.

2012 S. T. Yau College Math Contests Oral Exam on Statistics

Sunday, August 5, morning

Problem 1. Consider the linear regression model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}_0 + \boldsymbol{\varepsilon},$$

where $\mathbf{y} = (y_1, \cdots, y_n)^T$ is an n-dimensional vector of response, $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_p)$ is an $n \times p$ design matrix, $\boldsymbol{\beta}_0 = (\beta_{0,1}, \cdots, \beta_{0,p})^T$ is a p-dimensional vector of regression coefficients, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)^T$ is an n-dimensional vector of independent and identically distributed noise with mean 0 and variance σ^2. It is well known that the ordinary least-squares estimator becomes unstable or even inapplicable when p is large compared to n. One idea for remedying this issue is the ridge regression which gives the ridge estimator

$$\widehat{\boldsymbol{\beta}}_{\text{ridge}} = (\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\mathbf{X}^T\mathbf{y},$$

where $\lambda > 0$ is called the ridge parameter.

a) Calculate the mean of $\widehat{\boldsymbol{\beta}}_{\text{ridge}}$.

b) Calculate the covariance matrix of $\widehat{\boldsymbol{\beta}}_{\text{ridge}}$.

Solutions to 2012 S. T. Yau College Math Contests Oral Exam on Statistics

Sunday, August 5, morning

Problem 1. Consider the linear regression model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}_0 + \boldsymbol{\varepsilon},$$

where $\mathbf{y} = (y_1, \cdots, y_n)^T$ is an n-dimensional vector of response, $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_p)$ is an $n \times p$ design matrix, $\boldsymbol{\beta}_0 = (\beta_{0,1}, \cdots, \beta_{0,p})^T$ is a p-dimensional vector of regression coefficients, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)^T$ is an n-dimensional vector of independent and identically distributed noise with mean 0 and variance σ^2. It is well known that the ordinary least-squares estimator becomes unstable or even inapplicable when p is large compared to n. One idea for remedying this issue is the ridge regression which gives the ridge estimator

$$\widehat{\boldsymbol{\beta}}_{\text{ridge}} = (\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\mathbf{X}^T\mathbf{y},$$

where $\lambda > 0$ is called the ridge parameter.

a) Calculate the mean of $\widehat{\boldsymbol{\beta}}_{\text{ridge}}$.

Solution:

$$E\widehat{\boldsymbol{\beta}}_{\text{ridge}} = (\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}_0 = \boldsymbol{\beta}_0 - \lambda(\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\boldsymbol{\beta}_0.$$

b) Calculate the covariance matrix of $\widehat{\boldsymbol{\beta}}_{\text{ridge}}$.

Solution:

$$\begin{aligned}\mathrm{Cov}(\widehat{\boldsymbol{\beta}}_{\text{ridge}}) &= (\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\mathbf{X}^T\mathrm{Cov}(\mathbf{y})\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1} \\ &= \sigma^2(\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda I_p)^{-1}.\end{aligned}$$

2012 S. T. Yau College Math Contests Oral Exam on Statistics

Sunday, August 5, afternoon

Problem 1. Let $X_i \sim N(\theta_i, \frac{1}{n})$, $i = 1, \cdots, n$, be independent. Find an estimator $\widehat{T}$ of $T = \sum_{i=1}^n \theta_i^2$ and calculate $E(\widehat{T} - T)^2$.

2012 S. T. Yau College Math Contests Oral Exam on Probability

August 4, morning

Problem. Take two points ξ and η randomly and independently with respect to the uniform distribution from the unit interval $[0, 1]$. Then in general these two points divide the interval $[0, 1]$ into three subintervals with lengths X, Y and Z, respectively.

(1) What is the probability that X, Y and Z constitute the lengths of three sides of a triangle in the plane?

(2) What are the probability distributions of X, Y and Z?

2012 S. T. Yau College Math Contests Oral Exam on Probability

August 4, afternoon

Problem. Suppose that $\{\xi_k\}$ are independent and identically distributed random variables with uniform distribution on the interval $[0, 1]$. Let

$$Y = \max_{1\le k\le n} \xi_k.$$

(1) What is the joint distribution of (ξ_1, Y)?
(2) Evaluate the probability $\mathrm{P}(\xi_1 = Y)$.
(3) Evaluate the conditional expectation $\mathrm{E}(\xi_1|Y)$.

2012 S. T. Yau College Math Contests Oral Exam on Probability

August 5, morning

Problem. Discuss the following issue by constructing an appropriate probability model. You may make some further reasonable assumptions.

Suppose that there are 1000 persons, and only one of them is your ideal friend. Suppose that when you meet a person which is your ideal friend, you can identify whether he/she is your ideal friend with a success probability 99/100, and when you meet a person who is not your ideal friend, you may wrongly identify him/her as your ideal friend with a probability 1/100. Now if you have already met a person that you regard as an ideal friend, what is the probability that this person REALLY is your ideal friend?

2012 S. T. Yau College Math Contests Oral Exam on Probability

August 5, afternoon

Problem. Let $\{X_n\}$ be independent and identically distributed random variables with expectation $\mathrm{E}X$, variance $\mathrm{D}X < \infty$ and characteristic function $\phi_X(t)$, respectively. Let N be a non-negative integer valued random variable with expectation $\mathrm{E}N$, variance $\mathrm{D}N < \infty$ and characteristic function $\phi_N(t)$, respectively. Furthermore, $\{X_n\}$ and N are independent. Let $Y = \sum_{k=1}^{N} X_k$.

(1) What is the characteristic function of Y?

(2) Evaluate the variance of Y.

Solutions to 2012 S. T. Yau College Math Contests Oral Exam on Probability

August 4, morning

Problem. Take two points ξ and η randomly and independently with respect to the uniform distribution from the unit interval $[0, 1]$. Then in general these two points divide the interval $[0, 1]$ into three subintervals with lengths X, Y and Z, respectively.

(1) What is the probability that X, Y and Z constitute the lengths of three sides of a triangle in the plane?

(2) What are the probability distributions of X, Y and Z?

Solution

(1) If $0 \leq \xi < \eta$, then the lengths of the three subintervals are ξ, $\eta - \xi$ and $1 - \eta$, respectively. These constitute the lengths of three sides of a triangle is equivalent to

$$\begin{aligned} \xi + (\eta - \xi) &> 1 - \eta, \\ (\eta - \xi) + (1 - \eta) &> \xi, \\ \xi + (1 - \eta) &> \eta - \xi, \end{aligned}$$

which are further equivalent to

$$\xi < \frac{1}{2}, \quad \eta - \xi < \frac{1}{2}, \quad \eta > \frac{1}{2}.$$

These constraints form a region with area $1/8$.

By symmetry, the probability that X, Y and Z constitute the lengths of three sides of a triangle in the plane is $1/8 + 1/8 = 1/4$

(2) We have

$$X = \min\{\xi, \eta\}, \quad Y = |\xi - \eta|, \quad Z = 1 - \max\{\xi, \eta\}.$$

The distribution of X is

$$\begin{aligned} F_X(x) &:= \mathrm{P}(\min\{\xi, \eta\} \leq x) \\ &= 1 - \mathrm{P}(\min\{\xi, \eta\} > x) \\ &= 1 - \mathrm{P}(\xi > x)\mathrm{P}(\eta > x) \\ &= 1 - (1 - x)^2, \quad x \in [0, 1]. \end{aligned}$$

The distribution of Y is

$$\begin{aligned} F_Y(y) &:= \mathrm{P}(|\xi - \eta| \leq y) \\ &= \int\int_{-y \leq t - s \leq y} dtds \\ &= 1 - (1 - y)^2, \quad y \in [0, 1]. \end{aligned}$$

The distribution of Z is

$$\begin{aligned} F_Z(z) &:= \mathrm{P}(1 - \max\{\xi, \eta\} \leq z) \\ &= \mathrm{P}(\max\{\xi, \eta\} \geq 1 - z) \\ &= 1 - \mathrm{P}(\max\{\xi, \eta\} < 1 - z) \\ &= 1 - \mathrm{P}(\xi < 1 - z)\mathrm{P}(\eta < 1 - z) \\ &= 1 - (1 - z)^2, \quad z \in [0, 1]. \end{aligned}$$

August 4, afternoon

Problem. Suppose that $\{\xi_k\}$ are independent and identically distributed random variable with uniform distribution on the interval $[0, 1]$. Let

$$Y = \max_{1\le k\le n} \xi_k.$$

(1) What is the joint distribution of (ξ_1, Y)?
(2) Evaluate the probability $\mathrm{P}(\xi_1 = Y)$.
(3) Evaluate the conditional expectation $\mathrm{E}(\xi_1|Y)$.

Solution
(1) The joint distribution of (ξ_1, Y) is

$$\begin{aligned} F_{\xi_1,Y}(x, y) &:= \mathrm{P}(\xi_1 \le x, \max_{1\le k\le n} \xi_k \le y) \\ &= \mathrm{P}(\xi_1 \le x, \xi_2 \le y, \cdots, \xi_n \le y) \\ &= xy^{n-1}, \qquad 0 \le x \le y \le 1. \end{aligned}$$

(2) By symmetry, we have $\mathrm{P}(\xi_1 = Y) = \mathrm{P}(\xi_2 = Y) = \cdots = \mathrm{P}(\xi_n = Y)$. But

$$\sum_{k=1}^{n} \mathrm{P}(\xi_k = Y) = 1.$$

Therefore $\mathrm{P}(\xi_1 = Y) = \frac{1}{n}$.

(3) The distribution of Y is $F_Y(y) := \mathrm{P}(\max_{1\le k\le n} \xi_k \le y) = y^n, \quad y \in [0, 1]$. The conditional distribution of ξ_1 under Y is $F(x|y) = \frac{n-1}{n} \cdot \frac{x}{y}$ for $0 < x < y$, and $F(x|y) = 1$ for $1 \ge x \ge y \ge 0$. Therefore

$$\begin{aligned} E(\xi_1|Y = y) &= \frac{1}{n}y + \int_0^y x \cdot \frac{n-1}{n} \cdot \frac{1}{y}dx \\ &= \frac{1}{n}y + \frac{n-1}{2n}y \\ &= \frac{n+1}{2n}y, \end{aligned}$$

and $\mathrm{E}(\xi_1|Y) = \frac{n+1}{2n}Y$.

August 5, morning

Problem. Discuss the following issue by constructing an appropriate probability model. You may make some further reasonable assumptions.

Suppose that there are 1000 persons, and only one of them is your ideal friend. Suppose that when you meet a person which is your ideal friend, you can identify whether he/she is your ideal friend with a success probability 99/100, and when you meet a person who is not your ideal friend, you may wrongly identify him/her as your ideal friend with a probability 1/100. Now if you have already met a person that you regard as an ideal friend, what is the probability that this person REALLY is your ideal friend?

Solution

Let P(+) denote the probability that you meet an ideal friend, P(−) the probability that the person you meet is not your ideal friend, then

$$P(+) = 1/1000, \qquad P(-) = 999/1000.$$

Let P(“ + ”) denote the probability that you meet a person and identify him/her as your ideal friend and P(“ − ”) denote the probability that you meet a person and do not regard him/her as an ideal friend. Let P(“ + ”|−) denote the conditional probability that you regard a person as your ideal friend while in fact he/she is not, and other conditional probabilities are defined similarly. Then

$$P(\text{“}+\text{”}|+) = 99/100, \quad P(\text{“}-\text{”}|+) = 1/100.$$

$$P(\text{“}+\text{”}|-) = 1/100, \quad P(\text{“}-\text{”}|-) = 99/100.$$

What we need to calculate is in fact the conditional probability P(+|“+”), that is, the probability that the person you identify as an ideal friend is really your ideal friend. This can be evaluated by the Bayesian formula as follows:

$$\begin{aligned}
P(+|\text{“}+\text{”}) &= \frac{P(+, \text{“}+\text{”})}{P(\text{“}+\text{”})} \\
&= \frac{P(+)P(\text{“}+\text{”}|+)}{P(+)P(\text{“}+\text{”}|+) + P(-)P(\text{“}+\text{”}|-)} \\
&= \frac{\frac{1}{1000} \times \frac{99}{100}}{\frac{1}{1000} \times \frac{99}{100} + \frac{999}{1000} \times \frac{1}{100}} \\
&= 11/122 \approx 0.090.
\end{aligned}$$

August 5, afternoon

Problem. Let $\{X_n\}$ be independent and identically distributed random variables with expectation EX, variance D$X < \infty$ and characteristic function $\phi_X(t)$, respectively. Let N be a non-negative integer valued random variable with expectation EN, variance D$N < \infty$ and characteristic function $\phi_N(t)$, respectively. Furthermore, $\{X_n\}$ and N are independent. Let $Y = \sum_{k=1}^{N} X_k$.

(1) What is the characteristic function of Y?

(2) Evaluate the variance of Y.

Solution

(1) The characteristic function of Y is

$$\begin{aligned}
\phi_Y(t) &:= \mathrm{E}e^{it\sum_{k=1}^{N} X_k} \\
&= \mathrm{E}(\mathrm{E}(e^{it\sum_{k=1}^{N} X_k}|N)) \\
&= \mathrm{E}((\phi_X(t))^N).
\end{aligned}$$

(2) We have

$$\phi_Y'(t) = \mathrm{E}\{N(\phi_X(t))^{N-1}\phi_X'(t)\},$$

$$\phi_Y''(t) = \mathrm{E}\{N(N-1)(\phi_X(t))^{N-2}(\phi_X'(t))^2 + N(\phi_X(t))^{N-1}\phi_X''(t)\}.$$

Therefore, buy putting $t = 0$, we have

$$\mathrm{E}Y = \mathrm{E}N \cdot \mathrm{E}X,$$

and

$$\mathrm{E}Y^2 = \mathrm{E}\{N(N-1)(\mathrm{E}X)^2 + N\mathrm{E}X^2\},$$

from which we obtain

$$\mathrm{D}Y = \mathrm{E}Y^2 - (\mathrm{E}Y)^2 = \mathrm{D}N \cdot (\mathrm{E}X)^2 + \mathrm{E}N \cdot \mathrm{D}X.$$

Team Analysis

Problem 1

1) Let $f \in C([0,2\pi])$ with

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n cosn\theta + b_n sin\theta)$$

a) $u_n = \frac{a_0}{2} + \sum_{k=1}^{n} r^k (a_R cos\theta + b_R sin\theta)$ converges to a harmonic function $u(x, y)$ on every compact subset of $D = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$, where $z = re^{i\theta} = x + iy$.

b) $\iint_D (u_x^2 + u_y^2) dxdy = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2)$.

2) Let $f \in C([0,2\pi])$ which satisfies for some $\alpha \in \left(\frac{1}{2}, 1\right)$

$$\|f\|_{\dot{C}^\alpha} = \sup_{x,h \in [0,2\pi]} \frac{|f(x^{-1}h) - f(x)|}{h^\alpha} < +\infty$$

Then we have

$$\frac{|a_0|}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \leq C\|f\|_{\dot{C}^\alpha}$$

Problem 2

Let ζ_j, $j \geq 1$ be points with $|\zeta_j| = 1$ which is dense in the unit circle. For $j \geq 1$, let $\nu_j > 0$ with $\sum_j \nu_j = 1$. We let $\ell^1(\nu) = \{\alpha = (a_j) : \sum_j |a_j| \nu_j < \infty\}$ be the space of sequences. For a polynomial p, we define the sequence or restrictions $R(p) = (p(\zeta_j))_{j \geq 1}$, so it is evident that $R(p) \in \ell^1(\nu)$.

(a) Show that there is no $C < \infty$ such that

$$|p(0)| \leq C||R(p)||_{\ell^1(\nu)}$$

for every polynomial p.
Hint. It may be useful to use properties of the Poisson kernel function

$$Re\left(\frac{1 + re^{i\theta}}{1 - e^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

(b) Let $\alpha \in \ell^1(\nu)$ be given. Show that for any polynomial g, there is a sequence of polynomials g_j such that g_j converges to g uniformly on compact subsets of $\{|z| < 1\}$ and $R(g_j)$ converges to α in $\ell^1(\nu)$.

(c) Show that in (b) we may have, in addition, that $g_j(z)$ converges pointwise to $g(z)$ for every z such that $|z| \leq 1$ and $z \neq \zeta_j$ for any j.

Problem 3

Consider the following initial-boundary value problem

$$
(*)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = u^2 & on\ B_1(0) \equiv \mathbb{B}_1 \\ \dfrac{\partial u}{\partial n}\Big|_{\partial \mathbb{B}_1} = 0 & n\ is\ the\ outnormal\ of\ \partial \mathbb{B}_1 \\ u\big|_{t=0} = u_{0(x)} \end{cases}
$$

(a) Let $u_1(t,x)$ and $u_2(t,x)$ be two smooth solutions to $(*)$ on $\mathbb{B}_1 \times [0,T]$.

If $u_1\big|_{t=0}(x) \geq u_2\big|_{t=0}(x)$, $\forall\ x \in \mathbb{B}_1$, then

$$u_1(t,x) \geq u_2(t,x) \qquad on\ \mathbb{B}_1 \times [0,T]$$

(b) Let a be a positive constant. Solve $(*)$ with $u_0(x) \equiv a$ for all $x \in \mathbb{B}_1$.

(c) Assume that $u_0(x) > 0$ on $\mathbb{B}_1$ and $u(x,t)$ be the solution to $(*)$. Then $\exists\ \mathrm{T} > 0$ such that

$$\lim_{t \to T^-} \sup_{x \in \mathbb{B}_1} u(x_1, t) = +\infty$$

Problem 4

Let $f \geq 0, f \in L^1(0,1)$, monotone decrease

Suppose $\forall\ a, x \in (0,1)$ such that $x - \mathrm{a}, x + \mathrm{a} \in (0,1)$,

$$\int_{x-a}^{x} f(t)dt < \frac{5}{4}\int_{x}^{x+a} f(t)dt$$

Prove $f \in L^p(0,1), \forall\ 1 \leq P \leq \frac{\log 2}{\log 3 - \log 2}$

Algebra and Number Theory
Individual Oral Test

1. Consider $f \in \mathbb{Z}_{>0}$ and nonzero vector spaces V_i indexed by $i \in \mathbb{Z}/f\mathbb{Z}$. Suppose that there are linear maps $\phi_i : V_i \to V_{i+1}$ and $\psi_i : V_i \to V_{i-1}$ such that

$$\phi_{i-1} \circ \psi_i = 0, \quad \psi_{i+1} \circ \phi_i = 0.$$

(We may think of a circular graph with oriented edges such that the "Orpheus condition" holds: *Whenever you turn back while traveling through the graph you are killed.*)

Prove that there exists lines $\ell_i \subset V_i$ for every $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\phi_i(\ell_i) \subset \ell_{i+1}, \quad \psi_i(\ell_i) \subset \ell_{i-1}$$

under one of the following two conditions:

1. all $\psi_i = 0$, or
2. $\dim V_i$ are equal to each other.

Hint: use induction

2. For k non-negative integer, let $V_k := \mathbb{R}[x]_{\leq k}$ be the vector space of real polynomials of degree at most k with an action by $\mathrm{SL}_2(\mathbb{R})$ by

$$\gamma \cdot P(x) = (cx+d)^k P\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

1. Show that V_k is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$;

2. For non-negative integers m, n, consider $V_{m,n} := V_m \otimes V_n$ as a subspace of $\mathbb{C}[x, y]$ of polynomials with both x, y-degrees at most k. Assume $m \geq n \geq 1$. Show that following exact sequence is exact and split as representations of $\mathrm{SL}_2(\mathbb{R})$.

$$0 \longrightarrow V_{m-1,n-1} \xrightarrow{\cdot(y-x)} V_{m,n} \xrightarrow{y=x} V_{m+n} \longrightarrow 0.$$

   This implies the following decomposition of representations:

$$V_m \otimes V_n = \oplus_{i=0}^{n} V_{m+n-2i}.$$

3. For non-negative integers $\ell \geq m \geq n$ consider the space of invariants $(V_\ell \otimes V_m \otimes V_n)^{\mathrm{SL}_2(\mathbb{R})}$. Show that this space is either trivial or one-dimensional; it is non-trivial if and only if

$$\ell + m + n \equiv 0 \mod 2, \qquad \ell + m \geq n.$$

3. Is $(x^2 + 3)(x^3 + 2)$ solvable mod p for every prime p?

Algebra and Number Theory
Team Oral Test

1. Let E/F be a field extension. Let A be an $m \times m$ matrix with entries in E such that $tr(A^n)$ belongs to F for every $n \geq 2$. Show that $tr(A)$ belongs to F by following steps.

a Show that there is a polynomial $P(x) = \sum_i a_i x^i \in \bar{E}[x]$ with $a_0 = 1$ such that
$$\sum_i a_i tr(A^{i+k}) = 0, \qquad \forall k \geq 1.$$

b Show that we have a polynomial $Q = \sum_i b_i x^i \in F[x]$ with $b_0 = 1$ such that
$$\sum_i b_i tr(A^{i+k}) = 0, \qquad \forall k \geq 2.$$

c Let $t \in \bar{E}$ be an eigenvalue of A with multiplicity m invertible in F. Show that $Q(t) = 0$.

d Show that $tr(A)$ belongs to F.

Hint: Let $t_i \in \bar{E}$ be all distinct non-zero eigen values of A with multiplicity m_i invertible in F. Then
$$tr(A^n) = \sum_i m_i t_i^n.$$

2. a Prove that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not a UFD.

b Prove that $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is a PID.

Individual Oral Test

Analysis and PDE

1. Suppose $f(\mathrm{x},\mathrm{y})$ is a nonsmooth bounded convex function defined on $R^1 \times [0,1]$. Prove that f is independent of x.
2. Consider the property

$$f\left(\frac{1}{n}\right) = \frac{1}{n^3} + e^{-2n} \qquad for\ \ n = 2,3,4\ldots \qquad\qquad (*)$$

   (a) Show that there is no analytic function f on $\{z \in \mathbb{C}:\ |z| < 1\}$ with the property $(*)$.
   (b) Show that there is an analytic function g on $\{0 < |z| < 1\}$ with property $(*)$.
   (c) Is there an analytic function g on $\{0 < |z| < 1\}$ with property such that $f(\mathrm{z})$ is never an integer, i.e. $f(\mathrm{z}) \notin \mathbb{Z}$ for all $0 < |z| < 1$?
3. Real Analysis.

   Let E be a measurable subset of $\mathbb{R}^d$ with $\mathrm{m(E)} < +\infty$, g is a measureable function on E.
   (a) For all $f \in L^1(E)$, there holds $f(\mathrm{x})\mathrm{g}(\mathrm{x}) \in L^1(E)$. Then there exists a subset F of E with $\mathrm{m(F)} = 0$, so that $\mathrm{g} \in L^\infty(E\backslash F)$.
   (b) If $\mathrm{g} \in L^\infty(E)$, then we have

$$\|g\|_{L^\infty} = \sup_{\|f\|_{L^1}=1}\left\{\left|\int_E f(x)g(x)dx\right|\right\}$$

4. (a) Suppose $f \in C^1[-1,1]$, prove that there exists a absolute constant C, such that

$$\int_{-1}^{1}\left(f(x) - \frac{1}{2}\int_{-1}^{1} f(y)dy\right)^2 dx \le C\int_{-1}^{1} f'^2(x)\ \ dx$$

   (b) Suppose $f \in C^1(B_1)$, B_1 is the unit ball in $\mathbb{R}^2$. Prove that there exists a absolute constant C, such that

$$\int_{B_1}\left(f(x) - \frac{1}{|B_1|}\int_{B_1} f(y)dy\right)^2 dx \le C\int_{B_1} |\nabla f|^2\, dx$$

Oral exam, applied and computational mathematics, individual, 2013

Problem 1. Suppose $A \in R^{n\times n}$ is nonsingular, u and v are two vectors.

(a) Find condition such that $A + uv^\top$ is invertible, in that case find $(A + uv^\top)^{-1}$.

(b) Change the first column of A: $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ by $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ resulting in a new matrix $\bar{A}$. Find $(\bar{A})^{-1}$.

Problem 2. A discrete surface is represented as a simplicial complex, such that each face is a Euclidean triangle, which is also called a triangle mesh. Suppose $M = (V, E, F)$ is a triangle mesh, where V, E, F represents the set of vertices, edges and faces respectively. The Euler number of M is defined as

$$\chi(M) := |V| + |F| - |E|.$$

An edge is called an interior edge if it is adjacent to two faces; an edge is called a boundary edge if it is adjacent to only one face. A vertex is called an interior vertex, if all the edges adjacent to it are interior; a vertex is called a boundary vertex, if it attaches to at least one boundary edge. A triangle mesh is called a closed mesh, if it has no boundary edges.

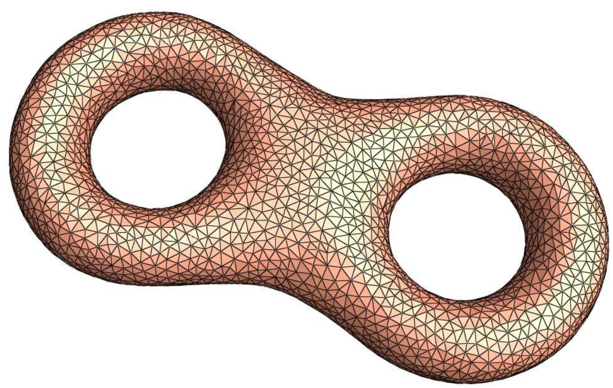


Figure 1: A discrete surface is represented as a triangle mesh.

Suppose $v_i \in V$ is an interior vertex on M, $[v_i, v_j, v_k] \in F$ is a face on M. θ_i^{jk} is the corner angle on the face $[v_i, v_j, v_k]$ with apex v_i. Then the discrete Gaussian curvature at v_i is defined as

$$K(v_i) := 2\pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}.$$

If v_i is an boundary vertex, then the discrete Gaussian curvature at v_i is defined as

$$K(v_i) := \pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}.$$

1. Suppose M is a closed triangle mesh, prove the Discrete Gauss-Bonnet theorem:

$$\sum_{v_i\in V} K(v_i) = 2\pi\chi(M).$$

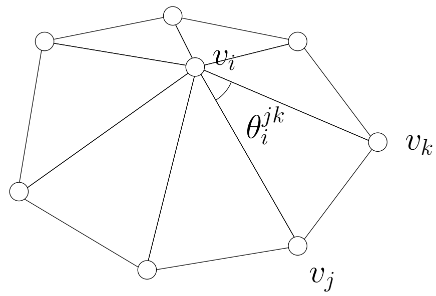


Figure 2: Discrete Gaussian curvature for an interior vertex.

2. (optional) Suppose the faces of M are not only triangles, but also general planar polygons, prove the discrete Gauss-Bonnet theorem.
3. (Optional) Suppose M is a triangle mesh with boundaries, different boundary connect components have no intersection, prove the discrete Gauss-Bonnet theorem.

Oral exam, applied and computational mathematics, all-around, 2013

Let $D_n = \{x \in R^n : |x| < 1\}$, f be a smooth map from $\overline{D_n}$ to $\bar{D}_n$. For homotopy $H : \overline{D_n} \times [0,1] \to R^n$

$$H(x,t) = (1-t)(x-a) + t(x - f(x))$$

we can find $x = f(x)$ from $H(x,t) = 0$. Here $a \in D_n$.

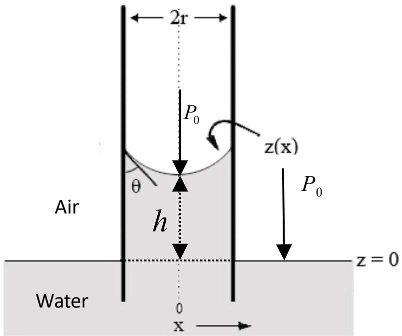
1. Prove $H^{-1}(0)$ is a smooth curve for almost all $a \in D_n$. (Hint: Sard's theorem and implicit function theorem)

2. Prove this curve starts from $x = a$ will stay inside D_n for $0 \le t < 1$ and when $t = 1$ yields x_0 s.t. $f(x_0) = x_0$.

3. How to follow this curve? (Hint: parametrize the curve by arc length)

Oral Exam for Teams: Applied and Computational Math, 2013

Modeling the capillary rise: why is the meniscus curved?



In a capillary tube, gravity and surface tension both act on water and a chemical equilibrium is attained (see the above Figure. We consider only the two dimensional case). The existence of surface tension caused a pressure difference on the surface, which leads to the capillary rise h and the curved surface z(x) with a contact angle $\theta < 90^o$. The contact angle is determined by the surface property of the tube and is given. The Young-Laplace equation is an equation that relates the capillary pressure difference sustained across the interface between two static fluids to the shape of the surface:

$$p_0 - p_1 = 2\gamma H$$

$p_0 - p_1$ is the pressure difference across the air-water interface, γ is the surface tension, H is the curvature of the meniscus surface z(x). We also assume that ρ is the density of the water, p_0 is the atmospheric pressure, g is the gravitational constant and r is the radius of the tube.

In this question, γ, ρ, r, θ, g, p_0 are given parameters. Our goal is to determine the shape of the surface z(x) and the capillary rise h.

(a). Calculate the pressure difference and curvature in terms of z(x) and then derive a differential equation for the meniscus surface $z = z(x)$.

(b). Design a shooting method to solve z(x) and determine h numerically. Explain analytically why the shooting method will work. Give the details of the numerical scheme.

Geometry and Topology

Oral Exam (Individual)

1. How many S^2 bundles on S^2 are there?

2. Describe normal coordinate what is derivative?

   When is the derivative nonsingular?

   State Catan Hademand Theorem.

   And find Taylor expansion of g_{ij} in normal coordinate.

3. Define Hopf Algebra.

   Which space has Hopf Algebra structure.

   Why dose Lie group structure => Hopf Algebra structure.

   Show π_i (closed loop space) = π_{i+1} .

All-around Geometry

(1) Prove $S^{2n}, n \geq 1$ is not Lie group.

(2) Torus T^2 immersed into $\mathbb{R}^3$, $\varphi\colon T^2 \to \mathbb{R}^3$ with nontrivial self-intersection, then

$$\int_{\varphi(T^2)} |H|^2 dv \geq 8\pi$$

Geometry and Topology

Oral Exam (Team)

1. State and prove Synge Theorem.

2. Why does Euler number of odd dimensional manifold equal to zero?

3. State Hopf-Poincare Theorem.

   Idea of the proof.

   Any interesting application.

4. State clear Euler Formula.

   Prove there are 5 regular Spheres.

2013 概率统计个人竞赛复试题

1. 甲乙二人各出资 100 元反复掷一枚均匀硬币玩公平游戏，正面甲赢，负面乙赢. 双方约定谁先赢三次将获得全部赌资 200 元. 当硬币掷到第三次时因故停止，结果是甲方赢两次，乙方赢一次. 在这种情况下，应该如何在甲乙之间公平分配 200 元赌资.

2. 设 X 与 Y 为独立并且二维随机变量 $Z=(X,Y)$ 的分布旋转不变（即对于任何正交矩阵 O, 二维随机变量 OZ 与 Z 同分布. 证明 X 和 Y 是均值为零且方差相同的正态随机变量.

3. 考虑一元线性回归模型

$$Y_k = aX_k + \epsilon_k, k = 1,2,\ldots,n,$$

其中 $\{\epsilon_k : k=1,2,\ldots,n,\}$ 是独立同分布正态 $N(0,\sigma^2)$ 随机变量序列，$\{X_k : k=1,2,\ldots,n,\}$ 是非随机、人为设计的自变元序列. 请给出一种估计方法，并找出估计量的均值及方差。

1. If X, Y are independent and X has a density, does $X+Y$ also have a density?
2. Assume that $0 \leq X \leq 1$. For what distributions of X does $\operatorname{Var} X$ have the largest value?
3. Assume that X, Y haNve Independent Identically Distribution $N(0,1)$. What is the distribution of $\left(\frac{X_1}{\sqrt{X_1^2+X_2^2}}, \frac{X_2}{\sqrt{X_1^2+X_2^2}}\right)$?
4. Assume that $P(N=i)=\frac{1}{3}$, X_1, X_2, X_3 have Independent Identically Distribution $\mathrm{N}(0,1)$ and are independent of N. Is $X=\sum_{i=1}^{N} X_i$ also normal?

2013 概率统计团体竞赛复试题

1. 设 ξ_i $(i=1,2,\cdots)$ 为一列独立同分布的随机变量且它们的期望与方差分别是 μ 与 σ, 又设 N 为取值正整数的随机变量，它的期望与方差分别是 m 与 θ. 假定 N 与 $(\xi_i, i=1,2,\cdots)$ 独立，试求 $X=\sum_{i=1}^{N}\xi_i$ 的期望与方差.

2. 若 X,Y,Z 为独立的 $[0\ 1]$ 上均匀分布的随机变量, 则 $W=(XY)^Z$ 也服从 $[0\ 1]$ 上的均匀分布.

3. 用投掷方法，考察一枚硬币的对称性，记出现正面的概率为 p.

1) 当独立投掷 n 次时，给出 p 的最大似然估计.

2) 当投掷次数 n 很大时，（可用渐近正态性）对假设

$$H_0: p=0.5, H_1: p\neq 0.5,$$

在给定水平 $\alpha=0.10$, 给出假设检验的具体做法.

S.T. YAU COLLEGE MATH COMPETITION 2014 ORAL EXAM

Algebra

Problem 1. Solve the equation $x^2 = x$ in End k^n where k is a field.
Problem 2. Let $n \geq 1$ be an integer. Construct a Galois extension over $\mathbb{Q}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Problem 3. Let $p > 3$ be a prime. Consider the equation

$$x^3 + y^3 = 1 \ (*)$$

in $\mathbb{Z}/p\mathbb{Z}$.
(1) When $p \equiv 2$ (3), find the number of solutions.
(2) When $p \equiv 1$ (3), prove that there exists a pair (a, b) of integers such that
   (a) $4p = a^2 + 27b^2$
   (b) $a \equiv 1 \mod 3$. (Note: a is unique.)
(3) (Continuation of (2)) When $p \equiv 1$ (3). Prove that (*) has $p-2+a$ solution in $\mathbb{Z}/p\mathbb{Z}$.

S.-T Yau College Student Mathematics Contests

Algebra and Number Theory, Team, 2014

1. Let α, β be linear transformations of $\mathbb{C}[x,y]$ defined by

$$\alpha(x^m y^n) = (-x)^m (x+y)^n$$
$$\beta(x^m y^n) = (x+y)^m (-y)^n$$

Please find all polynomials in $\mathbb{C}[x,y]$ which are invariant under α and β.

2. (1) Show that the following map

$$\varphi: u \to 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!}$$

defines a bijection between the sets

$\Lambda = \{n \times n$ complex matrices A whose eigenvalues are all $0\}$

and

$\Sigma = \{n \times n$ complex matrices B whose eigenvalues are all $1\}$

(2) Assume $n = 2m$. If $\mathrm{A} \in \Lambda$

Satisfies $\mathrm{A}\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}\mathrm{A}^{\mathrm{t}} = 0$

Show that $\varphi(\mathrm{A})\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}\varphi(\mathrm{A}^{\mathrm{t}}) = I_{2m}$

3. Solve the equation $\begin{cases} x^2 - 2y^2 = 7 \\ x, y \in Z \end{cases}$

For which integer n, the equation $\begin{cases} x^2 - 2y^2 = n \\ x, y \in Z \end{cases}$ is solvable

4. Let G be a finite group such that $[\mathrm{G},\mathrm{G}] \subset \mathrm{G}$ is a $p-$group $(\mathrm{i.e.}\ |[\mathrm{G},\mathrm{G}]| = p^{\mathrm{m}})$ .If V is an irreducible G-module over an algebraically closed field R of $\mathrm{Char(R)} = p > 0$.

Prove that $dim(V) = 1$

Analysis and Differential Equations

Oral Exam (Individual)

1. Let S be the complex plane C, the open unit disk Δ, or the Riemann sphere $\hat{C}$. A map $f: S \to S$ is called a conformal automorphism if it is one-to-one, onto, and analytic. Let $Aut(S)$ denote the group of all conformal automorphisms of S.

   1) What are the $Aut(\hat{C})$, $Aut(C)$, and $Aut(\Delta)$?

   2) Give a proof for $Aut(C)$.

2. Suppose Δ is the open unit disk. Suppose

   $$f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

   is a convergent power series on Δ. Prove the Schwarz Lemma:

   1) $|f(z)| \le |z|$ for all $z \in \Delta$ and

   2) $|a_1| \le 1$.

3. 设 $p_1 = 2$， $p_2 = 3, \cdots$ 为素数全体，证明 $\sum_{n=1}^{\infty} \frac{1}{p_n}$ 发散。

4. 设 f 为区间 $[0,1]$ 上定义的连续实值函数，证明 f 不能是二到一的映射。

5. a) 假设 $f \in L^1(\Re)$， $\hat{f} \in L^1(\Re)$ 且对某一 $\delta > 0$， $|f(x)| + |\hat{f}(x)| \le \frac{C}{(1+|x|)^{1+\delta}}$，

   则 $\sum_{n \in Z} \hat{f}(n) = \sum_{n \in Z} f(n)$

   b) $$\sum_{n \in Z} e^{-2\pi n^2} = \frac{1}{\sqrt{\pi}} \sum_{n \in Z} e^{-\frac{\pi n^2}{2}}$$

提示，$\hat{f}(\xi) = \int_{\Re} e^{-2\pi ix\xi} f(x)\,dx$

6. 对如下 Burgers 方程

$$(*) \qquad \begin{cases} \partial_t u + u\partial_x u = 0 & \Re^+ \times \Re \\ u|_{t=0} = u_0 \in C^1(\Re) \end{cases}$$

证明：a) If u_0 满足 $u_0'(x) \geq 0$，则（*）存在一整体 C^1 解

b) If u_0 满足 $u_0'(x) < 0$，对某一 $x_0 \in \Re$，则证明存在 t_0 使得

$$\lim_{t \to t_0} u_x(t, \cdot) = -\infty$$

1. Consider the map $f(z) = z^2$ on the complex plane $\mathbb{C}$. Let f^n denote n times compositions of f, that is, $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$. It is easy to see that $f^n(z) = z^{2^n}$, so $|z| < 1$ are only points such that $|f^n(z)| \to 0$ as $n \to \infty$ and $|z| > 1$ are only points such that $|f^n(z)| \to \infty$ as $n \to \infty$. Now find out what are all points such that $|g^n(z)| \to \infty$ as $n \to \infty$ for $g(z) = z^2 - 2$.

2. Find a conformal map from the region

$$A = \left\{z: 0 < \arg z < \frac{\pi}{2}, 0 < |z| < 1\right\}$$

to the open unit disk Δ.

3. 设 $f(\theta)$ 是 $\mathbb{R}$ 上的周期为 2π 的连续函数，且

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

试证：**a):**

$$u_n = \frac{a_0}{2} + \sum_{k=1}^{n} r^k \left(a_k \cos k\theta + b_k \sin k\theta\right)$$

在单位圆盘 $D = \{z \in \mathbb{C} \mid |z| < 1\}$ 内的紧子集上一致收敛于一调和函数 $u(x, y)$，其中 $z = re^{i\theta} = x + iy$.

b) :

$$\iint_D \left(u_x^2 + u_y^2\right) dx\, dy = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2).$$

4. 设 $\omega(R)$ 是定义于 $(0,R_0)$ 上的非减非负函数，如果其满足

$$\omega(\gamma R) \leq \eta\omega(R) + KR^{\alpha}, \qquad \forall R \in (0,R_0],$$

其中 $0 < \gamma,\eta < 1, 0 < \alpha \leq 1, K \geq 0$ 是常数，则 $\exists 0 < \beta \leq \alpha, C \geq 1$ 使得

$$\omega(R) \leq \left(\frac{R}{R_0}\right)^{\beta} [\omega(R_0) + KR_0^{\alpha}], \qquad \forall R \in (0,R_0]$$

其中 β, C 只依赖于 γ,η 与 α.

5. 设 $l^{\infty}(\mathbb{N})$ 为自然数 $\mathbb{N}$ 上的一致有界函数全体，在上确界范数下构成巴那赫（Banach）空间，在通常的函数乘积下还构成代数. 再设 θ 为无理数，定义 $f(n) = e^{2\pi i n\theta}$.

(1) 证明 f 的值域在复平面上的单位圆周里稠密.

(2) 令 $\mathcal{A}$ 为 f 和常值函数 1 生成的对复共轭运算封闭的 $l^{\infty}(\mathbb{N})$ 的闭子代数，证明 $\mathcal{A}$ 的极大理想空间和复平面上的单位圆周同胚.

5. 证明

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

S.-T. Yau College Student Mathematics Contest

Applied Mathematics, Individual, 2014

Find the eigenvalues and eigenvectors of the following $N \times N$ tridiagonal matrix

$$A = \begin{pmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & b & c \\ & & & & a & b \end{pmatrix}$$

where a, b and c are 3 constants, and $a\,c > 0$.

Definition:

* A *propre k-edge-coloring* of a graph $G(V, E)$ is a mapping $f\colon E \to \{1, 2, 3, \cdots, k\}$ such that $f(e) \neq f(e')$ for any pair of edges e,e' that have a common end vertex.

** Suppose that f is a propre k-edge-coloring of a graph $G(V, E)$. f is called a *uniform propre k-edge-coloring* of $G(V, E)$ if for any $i, j \in \{1, 2, 3, \cdots, k\}$, $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$.

Problem:

Prove that if a graph $G(V, E)$ has a propre k-edge-coloring, then $G(V, E)$ has a uniform propre k-edge-coloring.

Consider the following equation over an one-dimensional (1-D) domain $\Omega = (0, 1)$:

$$\partial_t \phi = -\phi^3 + \phi + \epsilon^2 \phi_{xx}, \quad \text{in } \Omega, \tag{1}$$

$$\phi_x = 0, \quad \text{at } x = 0,\, x = 1, \tag{2}$$

with $\epsilon > 0$ a given constant.

The following semi-implicit, semi-discrete numerical scheme is formulated:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -\left(\phi^{n+1}\right)^3 + \phi^n + \epsilon^2 \phi_{xx}^{n+1}, \quad \text{in } \Omega, \tag{3}$$

$$\phi_x^{n+1} = 0, \quad \text{at } x = 0,\, x = 1. \tag{4}$$

in which ϕ^k denotes the numerical solution at t^k, with $t^k = k\Delta t$, Δt being the time step size.

Prove the following energy stability for the numerical solution (3)-(4):

$$E(\phi^{n+1}) \le E(\phi^n), \quad \text{for any } \Delta t > 0, \tag{5}$$

with the energy functional given by

$$E(\phi) = \int_\Omega \left(\frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 + \frac{\epsilon^2}{2}|\phi_x|^2 \right) dx = \frac{1}{4}\|\phi\|_{L^4}^4 - \frac{1}{2}\|\phi\|_{L^2}^2 + \frac{\epsilon^2}{2}\|\phi_x\|_{L^2}^2 . \tag{6}$$

Hint. Take an L^2 inner product with (3) by $\tilde{\mu}^{n+1} = \left(\phi^{n+1}\right)^3 - \phi^n - \epsilon^2 \phi_{xx}^{n+1}$.

Oral exam, applied and computational mathematics, individual, 2014

1 Problem 1. Discrete Optimal Mass Transportation

Suppose $\Omega \subset \mathbb{R}^2$ is a convex planar domain, $P = \{p_1, p_2, \cdots, p_n\}$ are discrete points on $\mathbb{R}^2$, each point has a Dirac measure $\{A_i \delta(p - p_i)\}$, such that

$$Area(\Omega) = \sum_{i=1}^{n} A_i.$$

A mapping $f : \Omega \to P$ is called *measure preserving*, if the area of the pre-image of p_i equals to A_i,

$$Area(f^{-1}(p_i)) = A_i.$$

The transportation cost of a mapping is given by

$$E(f) := \int_\Omega |p - f(p)|^2 dp.$$

Among all the measure preserving mappings, the one which minimizes the transportation cost is called *the optimal mass transportation map*.

We want to show that: if f is the optimal mass transportation map, then there exists a convex function $u : \Omega \to \mathbb{R}$, such that f is the gradient map of u, $f = \nabla u$.

1. Let $H = \{h_1, h_2, \cdots, h_n\}$ be weights. The *power voronoi diagram* induced by (P, H) is a cell decomposition of $\mathbb{R}^2$,

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} W_i,$$

where

$$W_i = \{q \in \mathbb{R}^2 | \ |q - p_i|^2 + h_i \leq |q - p_j|^2 + h_j, \forall 1 \leq j \leq n\}.$$

Define a map: $\varphi : W_i \to p_i$. Show that there is a piecewise linear convex function $u : \Omega \to \mathbb{R}$, such that

$$W_i = \{q \in \mathbb{R}^2 \ |\nabla u(q) = p_i\},$$

namely φ is the gradient map of u.

2. Suppose there is another cell decomposition

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} \overline{W}_i,$$

such that

$$Area(W_i \cap \Omega) = Area(\overline{W}_i \cap \Omega),$$

The map induced by this cell decomposition is $\overline{\varphi} : \overline{W}_i \to p_i$. prove the transportation cost of φ is no greater than that of $\overline{\varphi}$,

$$\int_\Omega |q - \varphi(q)|^2 dq \leq \int_\Omega |q - \overline{\varphi}(q)|^2 dq.$$

Namely, the discrete optimal mass transportation map must be induced by a power voronoi diagram.

3. Suppose given any $A = \{A_1, A_2, \cdots, A_n\}$, such that $A_i > 0$ and $\sum_{i=1}^{n} A_i = Area(\Omega)$, we can always find H, such that the power voronoi diagram induced by H satisfies the condition $Area(W_i) = A_i$, then show that the discrete optimal mass transportation is given by the gradient map of a convex function.

2 Problem 2. Circle Packing

A discrete surface is represented as a simplicial complex, such that each face is a Euclidean triangle, which is also called a triangle mesh. Suppose $M = (V, E, F)$ is a triangle mesh, where V, E, F represents the set of vertices, edges and faces respectively. The Euler number of the mesh is $\chi(M) = |V| + |F| - |E|$. Furthermore, a circle packing defined on the mesh. Each vertex v_i is associated with a circle (v_i, r_i), two circles on an edge are tangent to each other.

Figure 1: A discrete surface is represented as a triangle mesh.

Suppose $v_i \in V$ is an interior vertex on M, $[v_i, v_j, v_k] \in F$ is a face on M. θ_i^{jk} is the corner angle on the face $[v_i, v_j, v_k]$ with apex v_i. The discrete curvature at v_i is defined as

$$K_i = 2\pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}$$

The the total curvature satisfies the Gauss-Bonnet theorem $\sum_i K_i = 2\pi\chi(M)$. Let $u_i = \log r_i$, which is called the discrete conformal factor. We want to show the mapping from the discrete conformal factor to the discrete curvature

$$\varphi : (u_1, u_2, \cdots, u_n) \mapsto (K_1, K_2, \cdots, K_n),$$

where $\sum_i u_i = 0$, is deffeomorphic.

1. Derivative Cosine Law

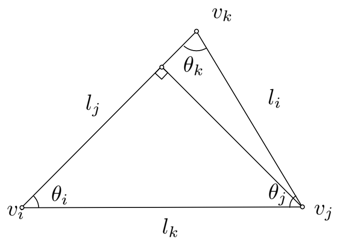


Figure 2: A Euclidean triangle.

Consider one triangle $[v_i, v_j, v_k]$, the corner angles are the functions of edge lengths, $\theta_i(l_i, l_j, l_k)$, prove

$$\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{2A}, \frac{\partial \theta_i}{\partial l_j} = -\frac{l_i}{2A}\cos\theta_k,$$

where A is the triangle area.

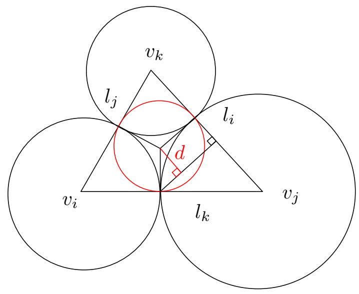


Figure 3: A Euclidean triangle with a circle packing.

2. Circle Packing on one Triangle. Suppose we associate each vertex v_i with a circle $c_i(v_i, r_i)$ centered at v_i with radius r_i. All three circles are tangent to each other, the inner circle has radius r, let $u_i = \log r_i$, prove

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{r}{l_k}$$

and

$$\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_i}{\partial u_k}.$$

Prove that the mapping

$$\varphi : \{(u_i, u_j, u_k) | u_i + u_j + u_k = 0\} \rightarrow \{(\theta_i, \theta_j, \theta_k) | \theta_i + \theta_j + \theta_k = \pi\}$$

is a diffeomorphism.

3. Consider the whole triangle mesh, prove the mapping

$$\varphi : (u_1, u_2, \cdots, u_n) \mapsto (K_1, K_2, \cdots, K_n),$$

where $\sum_i u_i = 0$, is deffeomorphic.

S.-T. Yau College Student Mathematics Contest

Applied Mathematics, Team, 2014

For the interval $[0, \pi]$, we divide it into $N + 1$ equally spaced subintervals by using the nodal points:

$$0 = x_0 < x_1 < \cdots < x_{N+1} = \pi,$$

with

$$x_i = i\,h, \quad h = \pi/(N+1).$$

For any continuous function w on $[0, \pi]$, we define $\Pi_h w$ to be the piecewise linear interpolation of w, namely $\Pi_h w$ is linear on each subinterval (x_i, x_{i+1}) for $i = 0, 1, \cdots, N$, and it takes the same values as w at all nodal points x_i, $i = 0, 1, \cdots, N+1$. For any function w, we define

$$\|w\| = \left(\int_0^\pi w^2(x) dx \right)^{1/2}.$$

Prove the following estimates for any function $u \in C^2[0, \pi]$:

$$\|u - \Pi_h u\| \leq \frac{1}{\pi^2} h^2 \|u''\|, \quad \|u' - (\Pi_h u)'\| \leq \frac{1}{\pi} h \|u''\|.$$

claw-free graphs

A graph $G(V, E)$ is claw-free if it has no induced subgraph isomophic to the bipartite complete graph $K_{1,3}$,(i.e, $V = \{w, u_1, u_2, u_3\}$, $E = \{wu_1, wu_2, wu_3\}$).

Let G be a claw-free graph of order n. Let δ be the minimum degree of G and α the size of a maximum independent set. Prove that

$$\alpha \leq \frac{2n}{\delta + 2}.$$

Over $\Omega = (0,1)$, consider the heat equation with a homogeneous Dirichelt boundary condition

$$\partial_t u = u_{xx} + f, \quad \text{in } \Omega, \tag{1}$$
$$u(0,t) = u(1,t) = 0, \tag{2}$$

in which $f(x,t)$ is a given force term, with $\|f(\cdot,t)\|_{L^2} \le M$, for any $t \ge 0$. The following semi-discrete implicit scheme is given

$$\frac{u^{n+1} - u^n}{\Delta t} = u_{xx}^{n+1} + f^{n+1}, \quad \text{in } \Omega, \tag{3}$$
$$u^{n+1}(0) = u^{n+1}(1) = 0, \tag{4}$$

in which u^k denotes the numerical solution at t^k, with $t^k = k\Delta t$, Δt being the time step size.

The final time is set as $T > 0$ and the initial data is given by $u^0(x)$. Prove the following uniform in time L^2 bound for the numerical scheme (3)-(4):

$$\left\| u^k \right\|_{L^2}^2 \le \tilde{C} := \left\| u^0 \right\|_{L^2}^2 + C_2^4 M^2, \quad \text{for any } k \ge 0, \tag{5}$$

in which $\tilde{C}$ is independent on the time step t^k, and C_2 is given by the following Pincaré inequality

$$\|v\|_{L^2} \le C_2 \|v_x\|_{L^2}, \quad \text{if } v(0) = v(1) = 0. \tag{6}$$

Hint. Take an L^2 inner product with $2u^{n+1}$, use Poincaré inequality, and apply an induction in time to derive a uniform in time L^2 bound.

Geometry and Topology

Oral Exam (Individual)

1. How many S^1 bundles on S^2 are there?

   How many S^2 bundles on S^2 are there?

2. State and prove the maximal diameter theorem.

3. There is no non-constant holomorphic map from a compact Riemann surface of genus g to a compact Riemann surface of genus h, where $g<h$.

4. There is at least one point p in a closed surface, s.t., $K(p)>0$.

Geometry and Topology

Oral Exam (Team)

1. Brawer fixed point theorem

   $$B^n \xrightarrow{f} B^n$$

   has a fixed point.

2. There is no geodesic rectangle on the sphere S^2 .

3. State Hopf-Poincare Theorem idea of proof.

   Interesting application.

4. No closed minimal surface in $\mathbf{R}^3$.

S.-T Yau College Student Mathematics Contests

Geometry and Topology, Team, 2014

1. Let $\phi: S^2 \to T^2$ smooth map, show any top de Rham cohomology class $[v] \in H^2(T^2)$,we have $\phi^*[v] = 0$
2. $\mathrm{f}: \mathrm{D}^2 \to \mathrm{D}^2$ homomorphism, Show f must map boundary point to boundary point. (D^2 closed unit disc)
3. Let $\omega_1, \omega_2, \cdots \omega_k$ be one-forms. Show $\{\omega_i\}$ linearly independent if and only if $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$
4. Show every map

$$\mathrm{F}: S^n \to S^{n_1} \times \cdots \times S^{n_k}$$

$$(k \geq 2, n_1 + \cdots + n_k = n, n_j > 0)$$

has degree 0

5. Proof any smooth map

$$\mathrm{f}: \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$$

has a fixed point

6. Let M be the closed surface generated by carrying a small circle with radius r around a closed and knotted curve C embedded in $\mathbb{R}^3$ such that the center moves along C and the circle is in the normal plane to C at each point. Show that

$$\int_{\mathrm{M}} \mathrm{H}^2 \, \mathrm{d}\sigma \geq 4\pi^2$$

Where H is the mean curature of M and $\mathrm{d}\sigma$ is the area element of M

2014 S.-T Yau College Math Contests 概率口试题（个人）

1. 设 Z 为一标准指数随机变量，即 $P(Z \in dt) = e^{-t}dt$. 定义 $\{Z\}$ 和 $[Z]$ 分别为 Z 的分数部分和整数部分．证明 $\{Z\}$ 和 $[Z]$ 独立，并分别求出它们的分布.

2. 设 X 和 Y 为具有均值为零、方差分别为 σ^2 和 τ^2, 相关系数为 ρ 的二元正态分布．试求条件期望 $E[X|X+Y]$.

3. 假定有甲、乙两个乒乓球运动员参加比赛，已知甲的实力强于乙。现有两个备选的竞赛规则：“3 局 2 胜制”，或“5 局 3 胜制”。试问：哪一种竞赛规则对甲有利？

2014 年 概率统计面试题

1. 设 X 和 Y 是独立可积随机变量，且 $E[X]=0$. 证明 $E[X+Y]\geq E[Y]$.
2. 设 $X_1, X_2, \ldots$ 为 $(0,\infty)$ 上一列相互独立随机变量，每个服从参数为 1 的指数分布. 令 $N(x)=\inf\{n:\sum_{i=1}^{n} X_i > x\}$，求 $N(x)$ 的数学期望.

2014 S.-T Yau College Math Contests 概率口试题（团体）

1. 设 X 和 Y 为两个相互独立的随机变量，其中 Y 服从 $(0,1)$ 上的均匀分布，X 服从二项分布，且 $P(X=0)=P(X=1/2)=1/2$. 令 $W=Y/2+X$,

(1) 证明 W 也服从 $(0,1)$ 上的均匀分布；

(2) 求 $E[Y|W]$.

2. 设 X_n 是 n 维标准高斯随机向量．证明下述概率当 n 趋于无穷大时趋于 1：

$$P\big(\sqrt{n}(1-\epsilon)\leq\|X_n\|\leq\sqrt{n}(1+\epsilon)\big).$$

3. 设 (X,Y) 和 (Y',Z) 为两个离散值随机向量，且 Y 和 Y' 有相同分布。证明存在随机向量 $(\hat{X},\hat{Y},\hat{Z})$，使得 $(\hat{X},\hat{Y})$ 和 $(\hat{Y},\hat{Z})$ 分别与 (X,Y) 和 (Y',Z), 有相同分布。

4. 设 (ξ_n) 为一列相互独立随机变量，满足 $P(\xi_n=1)=p_n,P(\xi_n=0)=1-p_n$. 假定 $\sum_{k=1}^{\infty}p_kp_{k+1}<\infty$, 证明级数 $\sum_{k=1}^{\infty}\xi_k\xi_{k+1}$ 几乎处处收敛。

1. 考虑欧氏空间$\mathbb{R}^n$，其元素是列向量。对$n\times n$实矩阵空间$M_n(\mathbb{R})$中的任一元素M，定义$\|M\| = \sup\limits_{\|x\|=1}\|Mx\|$，$x\in\mathbb{R}^n$。命$GL_n(\mathbb{R})$ 为$n\times n$可逆实矩阵群，G是$GL_n(\mathbb{R})$的子群，包含在单位开球中：

$$B(I,1)=\{M\in M_n(\mathbb{R})\mid \|M-I\|<1\}.$$

(a) 设$g\in G$，$\lambda\in\mathbb{C}$是g的一个特征根，证明$|\lambda-1|<1$.

(b) 证明g有唯一的复特征根，其值等于1.

(c) 证明$g=I$.

2. 设G 是$GL_n(\mathbb{Z})$的有限子群，$p\geq 3$是素数，$\mathbb{F}_p$ 是p元域。

(a) 证明：自然映射$GL_n(\mathbb{Z})\to GL_n(\mathbb{F}_p)$ 在G上的限制是单射。

(b) 设$M\in GL_2(\mathbb{Z})$ 是阶为$m<\infty$ 的元素，证明: $m\in\{1,2,3,4,5,6\}$.

3. 设$f\in\mathbb{R}(x)$ 是有理函数，在整数上取整数值，证明f是多项式。

1. 设 $P \in \mathbb{Z}[x]$, $d(P)$ 表示集合 $\{P(a) : a \in \mathbb{Z}\}$ 中元的最大公因子. 如果 P 是 k 次本原多项式, 证明 $d(P)$ 整除 $k!$.

2. 对任意 $A, B \in M_n(\mathbb{C})$, 定义 $\varphi(A, B) = Tr(AB)$. 如果 $\{X_i\}_{1 \leq i \leq n^2}$ 是线性空间 $M_n(\mathbb{C})$ 的一组基。证明存在另一组基 $\{X_i\}'_{1 \leq i \leq n^2}$ 满足

(a) $\varphi(X_i, X'_j) = \delta_{i,j}$

(b) $\forall A \in M_n(\mathbb{C}), \quad \sum_{i=1}^{n^2} X_i A X'_i = Tr(A) I_n$

1. 设p是素数，w是p次本元单位根。令$\mathbb{F}_p$为p元域。

   (a) 证明$\mathbb{Z}[w]/(1-w)$ 同构于$\mathbb{F}_p$。

   (b) 设$\varphi \in \mathbb{F}_p[x]$是次数小于$p$的非零多项式。证明$\varphi$在$\mathbb{F}_p$中的根的重数一定小于$\varphi$的非零系数的个数。

   (c) 设$I = \{a_1, \cdots, a_n\}$, $J = \{b_1, \cdots, b_n\}$为$\mathbb{F}_p$的两个n元子集。证明: $n \times n$ 矩阵$(w^{a_i b_j})_{i,j}$ 可逆。

2. (a) 设$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ 满足$a_n \geq 1$, $a_{n-1} \geq 0$，对$i \in \{0, 1, \cdots, n-2\}$,　$|a_i| \leq H$，其中H为正常数。令α为$P(x)$的一个复根。证明：或者α 的实部非正，或者$|\alpha| < \frac{1+\sqrt{1+4H}}{2}$ 。

   (b) 设$b > 2$ 为整数，p是素数。考虑p的b –进制展开

   $$p = a_n b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0$$

   证明: 多项式$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ 在$\mathbb{Q}$ 上不可约。

3. 用$< \cdot, \cdot >$表示$\mathbb{C}^n$的标准内积。对任意$A \in M_n(\mathbb{C})$，令

   $$W(A) = \{< x, Ax >;\ \|x\| = 1\}.$$

   设$z_1, z_2, z_3 \in \mathbb{C}$且$|z_i| = 1$, $\triangle(z_1, z_2, z_3)$ 表示以z_1, z_2, z_3为顶点的三角形。试证明：$W(A) \subseteq \triangle(z_1, z_2, z_3)$ 当且仅当存在半正定的矩阵$A_1, A_2, A_3 \in M_n(\mathbb{C})$ 满足

   $$A = z_1 A_1 + z_2 A_2 + z_3 A_3, \quad \sum_{i=1}^{3} A_i = I.$$

Problems for 24/7

1. Show that the series

$$2\sin\frac{1}{3x} + 4\sin\frac{1}{9x} + \cdots + 2^n \sin\frac{1}{3^n x} + \cdots$$

converges absolutely for $x \neq 0$ but does not converge uniformly on any interval $(0,\epsilon)$ with $\epsilon > 0$.

2. Suppose that $f(t) \in C^{(n)}[0,1]$. If

$$\int_0^1 f(t)t^j\,dt = 0, \quad j = 1,2,\ldots,n,$$

prove that $\exists \xi \in (0,1)$ such that $f^{(n)}(\xi) = 0$.

3. (a) Suppose that $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Suppose $f : \Delta \to \Delta$ is an analytic function and can be extended continuously to an arc $I \subset \partial\Delta$. Prove that if $f|_I$ is constant, then f is constant on Δ.
   (b) Can you prove a similar result if D is a Jordan domain?

4. Let $\mathcal{L}^*$ denote Lebesgue outer measure. What are the (not necessarily measurable) sets E such that $\mathcal{L}^*(E \cap [a,b]) \leq (b-a)/2$ for all intervals $[a,b]$?

Analysis questions 24/7 Evening

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.
(a) Show that if f is continuous, then it is linear.
(b) Show that if f is Borel measurable, then it is continuous and thus linear.

2. Suppose that μ is a Borel measure of total mass 1, and let $\hat{\mu}(\xi) = \int e^{-ix\xi}\mu(dx)$ be its Fourier transform. Show that if the $\hat{\mu}(\xi_0) = 1$ for some ξ_0, then the support of μ is contained in a translation and dilation of $\mathbf{Z}$, i.e., it is contained in $\{ak + b : k \in \mathbf{Z}\}$ for some real a and b.

Analysis questions for teams

1. Suppose that $f \in C_c^\infty(\mathbf{R}^2)$ has the property that $\int_L f = 0$ for every line $L \subset \mathbf{R}^2$. Show that $f = 0$.

2. Let $D \subset \mathbf{C}$ be an open set. Let $Aut(D)$ denote the set of invertible conformal self-maps $\psi : D \to D$.

(a) Show that there is a domain D such that $Aut(D)$ is countably infinite.

(b) Show that there is a domain D such that $Aut(D)$ is isomorphic to the integers $\mathbf{Z}$.

(c) Is it possible for $Aut(D)$ to be isomorphic to the real numbers $\mathbf{R}$?

3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is of class C^1 and satisfies

$$\int_{-\infty}^{\infty} \frac{1}{|f(x)|} dx = +\infty$$

Prove that the maximal interval of existence for the solution of

$$\frac{dx}{dt} = f(x)$$

is $(-\infty, \infty)$.

S.-T. Yau College Student Mathematics Contest

Applied and Computational Mathematics 2015 (Individual)

1. Suppose an n by n matrix A is given by

$$A = \begin{pmatrix} 1 & r & & & & \\ & 1 & r & & & \\ & & 1 & r & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & r \\ r & & & & & 1 \end{pmatrix}_{n\times n}$$

$A\mathbf{x} = \mathbf{b}$, prove that

$$\|\mathbf{x}\| \le C\|\mathbf{b}\|,$$

where the constant C is independent of the dimension n.

2. For an interval $[a, b]$, we divide it into $N + 1$ equally spaced subintervals by using the nodal points:

$$a = x_0 < x_1 < \cdots < x_{N+1} = b,$$

with

$$x_i = a + i\,h, \quad h = (b-a)/(N+1).$$

For any continuous function w on $[0, \pi]$, we define $\Pi_h w$ to be the piecewise linear interpolation of w, namely $\Pi_h w$ is linear on each subinterval (x_i, x_{i+1}) for $i = 0, 1, \cdots, N$, and it takes the same values as w at all nodal points x_i, $i = 0, 1, \cdots, N+1$. For any function w, we define

$$\|w\| = \left(\int_0^\pi w^2(x)dx\right)^{1/2}.$$

Prove the following estimates for any function $u \in C^2[0, \pi]$:

$$\|u - \Pi_h u\| \le \frac{1}{\pi^2}h^2\,\|u''\|\,, \quad \|u' - (\Pi_h u)'\| \le \frac{1}{\pi}h\,\|u''\|\,.$$

3. Newton iteration for computing the kth root $(k \ge 2)$ of $C > 0$ is

$$x_{n+1} = x_n - \frac{x_n^k - C}{kx_n^{k-1}}.$$

Show that the iteration converges for any initial value $x_0 > 0$.

S.-T. Yau College Student Mathematics Contest

Applied Mathematics (All Round) 2015

1. There are three roots of the cubic equation

$$\epsilon x^3 - x - 2 = 0.$$

Find the asymptotic expansion of these roots in terms of small ϵ.

S.-T. Yau College Student Mathematics Contest

Applied Mathematics (Group)2015

1. Find the following generalized eigen value λ and the eigen vector $(x_1, x_2, \cdots, x_n)^T$, such that

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}_{n\times n} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

2. Let A be a positive definite matrix. Consider the quadratic function $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Suppose the eigenvalues of A are given by

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, k = \frac{\lambda_n}{\lambda_1}.$$

Let the sequence $\{\mathbf{x}_k\}$ be generated by the steepest descent method:

$$x_{k+1} = x_k - \alpha_k \nabla g(x_k), \quad k = 0, 1, 2, \cdots,$$

where α_k is selected such that

$$g(x_k - \alpha_k \nabla g(x_k)) = \min_{\alpha \ge 0} g(x_k - \alpha \nabla g(x_k)).$$

Prove that

$$g(\mathbf{x}_{k+1}) \le \left(\frac{k-1}{k+1}\right)^2 g(\mathbf{x}_k) \le \left(\frac{k-1}{k+1}\right)^{2(k+1)} g(\mathbf{x}_0).$$

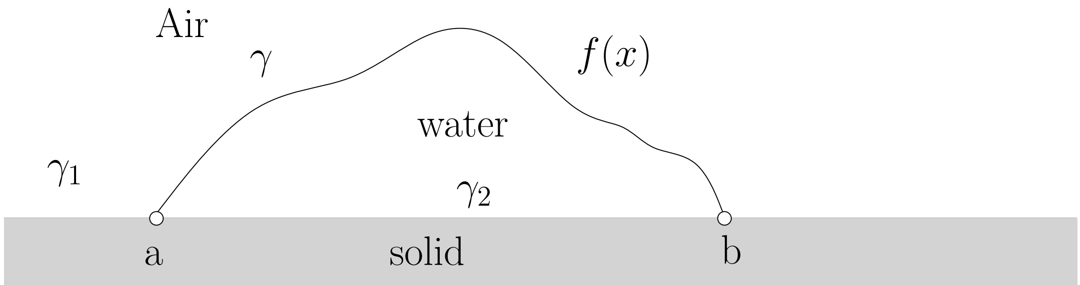


Figure 1: Equilibrium shape of water droplet.

3. Consider a water droplet on a solid surface as shown in Figure 1. The air-water surface tension is γ, the air-solid surface tension is γ_1, the water-solid surface tension is γ_2. The interface energy = surface tension $\times$ length of the interface. The equilibrium state should minimizes the total interface energy. (neglecting gravity)

   Formulate a variational problem to show that the equilibrium droplet should have circular shape (i.e. the curvature of the curve $f(x)$ is a constant).

Oral-Example-Geometry-Topology

1 Personal

1. Let $\pi : S^3 \to S^2$ be the Hopf fibration. Recall that if we identify $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$ and $S^2 = \mathbb{CP}^1$ with homogenous coordinate $[z_1, z_2]$, then

$$\pi : (z_1, z_2) \to [z_1, z_2].$$

   Show that there doesn't exist a section for π, i.e., a smooth map $s : S^2 \to S^3$ such that $\pi \circ s : S^3 \to S^2$ is the identity.

2. Show that there exists no degree one map from $S^2 \times S^2$ to $\mathbb{CP}^2$.

3. Show that there exists no metric on $\mathbb{RP}^2 \times \mathbb{RP}^2$ with positive sectional curvature.

Oral-Example-Geometry-Topology

All-round

1. (a) Let $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ be a continuous map of degree d with $|d| \geq 2$. Show that there are at least 3 fixed points.

   (b) Show that $\mathbb{CP}^{2n}$ does not cover any manifold except itself.

2. Let D be a bounded and simply connected domain in $\mathbb{R}^2$, and Γ be the boundary of D. Set
$$A := Area(D), \quad L := Length(\Gamma).$$

   (a) Prove that
$$4\pi A \leq L^2,$$
   and the equality holds iff Γ is a circle.

   (b) Generalize the theorem above to the case where D is a compact and simply connected minimal surface with boundary in $\mathbb{R}^3$.

Oral-Example-Geometry-Topology

1 Group

1. Show that $S^2 \times S^2$ and the connected sum $\mathbb{CP}^2 \sharp \mathbb{CP}^2$ are not homotopic equivalent.
2. Show that any closed surface has a closed geodesic.
3. Compute the Euler number of a smooth degree d hypersurface in $\mathbb{CP}^3$.

概率统计面试题（个人）

1. 设ξ_i $(i=1,2,\cdots)$为一列独立同分布的随机变量且它们的期望与方差分别是μ与σ，又设N为取值正整数的随机变量，它的期望与方差分别是m与θ. 假定N与$(\xi_i, i=1,2,\cdots)$ 独立，试求$X=\sum_{i=1}^{N}\xi_i$的期望与方差.

2. 若X,Y,Z为独立的$[0\ 1]$上均匀分布的随机变量，则$W=(XY)^Z$ 也服从$[0\ 1]$上的均匀分布.

3. 若$E[Y|X]=X$,而且$E[X|Y]=Y$，则$Y=X(a.s.)$.

概率统计面试题（全能）

1. 设(ξ_n)为一列相互独立随机变量，满足$P(\xi_n = 1) = p_n, P(\xi_n = 0) = 1 - p_n$. 假定$\sum_{k=1}^{\infty} p_k p_{k+1} < \infty$，证明级数$\sum_{k=1}^{\infty} \xi_k \xi_{k+1}$几乎处处收敛。

2. 设Z 为一标准指数随机变量，即$P(Z \in dt) = e^{-t}dt$. 定义$\{Z\}$ 和$[Z]$ 分别为Z的分数部分和整数部分. 证明$\{Z\}$ 和$[Z]$ 独立, 并分别求出它们的分布.

概率统计面试题（团体）

1. 设$X_1, X_2, \cdots$为一列独立同分布的随机变量, 每个X_i 服从参数为1的指数分布. 设$x > 0$. 令$N(x) = \inf\{n : \sum_{i=1}^{n} X_i > x\}$. 试求$N(x)$的数学期望.

2. 设X 与Y 是独立可积随机变量，且$E[X] = 0$. 证明：$E[|X + Y|] \geqslant E[|Y|]$.

3. 用投掷方法,考察一枚硬币的对称性,记出现正面的概率为p.

1)当独立投掷n次时,给出p的最大似然估计.

2)当投掷次数n很大时,（可用渐近正态性）对假设

$$H_0 : p = 0.5, H_1 : p \neq 0.5,$$

在给定水平$\alpha = 0.10$, 给出假设检验的具体做法.

PROBLEMS FOR ALGEBRA SECTION

1. Individual test

Problem 1 (Individual 1)**.** Let K be the splitting field of $f(x) = x^4 - 4x^2 - 1$ over $\mathbb{Q}$.

(1) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
(2) Describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

Problem 2 (Individual 2)**.** Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ (p is a prime). Describe the abelian group $\overline{\mathbb{F}}_p^\times$ in more elementary terms. What is the action of the Frobenius in terms of your description?

Problem 3 (Individual 3)**.** Let A and B be two $n \times n$ matrices with coefficients in $\mathbb{Q}$. For any field extension K of $\mathbb{Q}$, we say that A and B are similar over K if $A = PBP^{-1}$ for some $n \times n$ invertible matrix P with coefficients in K. Prove that A and B are similar over $\mathbb{Q}$ if and only if they are similar over $\mathbb{C}$.

PROBLEMS FOR ALGEBRA SECTION

1. Around test

Problem 1. For p any prime and $n \in \mathbb{N}$, we let $\zeta_{p^n} = \exp(2\pi i/p^n)$.

(1) When p is an odd prime, show that there is a unique subfield $K \subset \mathbb{Q}(\zeta_{p^n})$ that is quadratic over $\mathbb{Q}$.
(2) When p is an odd prime, determine which quadratic field K is in $\mathbb{Q}(\zeta_{p^n})$? (Hint: you may consider the sum $g = \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{a^2}$.)
(3) When $p = 2$ and $n \geq 3$, how many quadratic subfields does $\mathbb{Q}(\zeta_{p^n})$ have? What are they?

PROBLEMS FOR ALGEBRA SECTION

1. Team test

Problem 1. Let A be a finitely generated $\mathbb{Z}$-algebra and let $\mathfrak{m}$ be a maximal ideal of A. Show that $A/\mathfrak{m}$ is a finite field.

Problem 2. Let C be a category. Denote by $1_C : C \to C$ the identity functor on C. A natural transform from 1_C to 1_C consists of a collection $\{\eta_X\}_{X \in Ob(C)}$ such that

- for any $X \in Ob(C)$, η_X is a morphism from X to X;
- for any morphism $f : X \to Y$ in C, the following square is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\eta_Y} & Y. \end{array}$$

We call the set of all natural transforms from 1_C to 1_C the center of C.

(1) Determine the center of the category of abelian groups.
(2) Determine the center of the category of groups.

Problem 3. Let W be the Weyl algebra over a field k, which is the associative algebra generated by $x_1, ..., x_n, y_1, ..., y_n$ such that $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j] = \delta^i_j$ for all $1 \le i, j \le n$. ($\delta^i_j = 0$ or 1 is the Kronecker symbol.)

(1) In case $\mathrm{char}(k) = 0$, show that W does not have finite dimensional representation.
(2) In case $\mathrm{char}(k) > 0$, find all finite dimensional representations of W.

ANALYSIS 2016.

1. Individual

Problem 1.1. Suppose that f is analytic on $\Delta = \{|z| < 1\}$ and $|f(z)| \leq 1$ for $z \in \Delta$. If $f(\frac{1}{2}) = f(-\frac{1}{2}) = 0$, then

$$|f(0)| \leq \frac{1}{4}.$$

Problem 1.2. Let $a_1 = \sin(x)$ where $x \in (0, \frac{\pi}{2})$. For $n \geq 2$, $a_n = \sin(a_{n-1})$. Show that the series

$$\sum_{n=1}^{\infty} a_n^2$$

is divergent.

Problem 1.3. Let $L^p[0,1]$ be the space of L^p-integrable functions on $[0,1]$ where

$$\|f\|_p = \left(\int_0^1 |f|^p dx\right)^{1/p}, \quad p > 0.$$

Show the $\| \bullet \|_p$ satisfies the parallelogram law if and only if $p = 2$.

Problem 1.4. Find all the bounded solutions of

$$\begin{cases} \Delta u(x,y) = 0, \quad \text{for} \quad (x,y) \in \mathbb{R}^2_+, \\ u(x,0) = \begin{cases} 1, x > 0, \\ 0, x < 0 \end{cases} \end{cases}$$

Problem 1.5. Suppose $f(x) \in C[0,+\infty)$, and for any non-negative real number a, we have

$$\lim_{x \to +\infty} (f(x+a) - f(x)) = 0. \tag{1}$$

Show that there exist $g(x) \in C[0,+\infty)$ and $h(x) \in C^1[0,+\infty)$ such that $f(x) = g(x) + h(x)$ and

$$\lim_{x \to +\infty} g(x) = 0, \quad \lim_{x \to +\infty} h'(x) = 0. \tag{2}$$

Problem 1.6. Let f be a Riemann integrable function on $[-\pi, \pi]$ with Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}. \tag{3}$$

Suppose

$$|a_n| \leq \frac{K}{|n|}$$

for some positive constant K and all $n \neq 0$. Show that

$$\left| \sum_{n=-N}^{N} a_n e^{inx} \right| \leq \sup_{y \in [-\pi,\pi]} |f(y)| + 2K \tag{4}$$

for all $x \in [-\pi, \pi]$ and all $N \in \mathbb{N}$.

3. All-round

Problem 3.1. Let A be an $N \times N$ positive definite symmetric matrix with $N \geq 2$. Assume that there exists $\varepsilon \in (0,1)$ with

$$tr\, A \leq N + \varepsilon, \quad \det A \geq 1 - \varepsilon.$$

Then there exists a constant C_N depending only on N such that

$$\|A - I\| \leq C_N \sqrt{\varepsilon},$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm, and I is the $N \times N$ identity matrix.

Problem 3.2. Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers. Let $\{e_n\}$ be the standard orthonormal basis of $\ell^2(\mathbb{N})$. Define $T_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T_n(e_m) = e_{nm}. \tag{10}$$

Prove that

(1) The elements in $\{T_n, T_n^*\}$ can commute with the elements in $\{T_m, T_m^*\}$ if and only if $(n, m) = 1$.
(2) Let T be a bounded linear operator which can commute with all T_n, T_n^*, then $T = c \cdot I$ for some constant c.

2. Team

Problem 2.1. Let $f(z) : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n. For any $r > 0$, let

$$M(r) := \max_{|z| \le r} |f(z)|. \tag{5}$$

Show that if $R > r > 0$, then

$$\frac{M(R)}{R^n} \le \frac{M(r)}{r^n}. \tag{6}$$

Moreover, "=" holds in (6) if and only if $f(z) = cz^n$ for some constant c.

Problem 2.2. If $b_1 = 1$, $b_2 = 2$ and

$$b_{n+1} = b_n + b_{n-1}$$

for $n \ge 2$. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{b_n}$$

converge? Show all your work.

Problem 2.3. Find all solutions of

$$\begin{cases} \Delta u = 0, & \text{in} \quad B_1 \setminus \{0\} \subset \mathbb{R}^2, \\ u(x) = 0 & \text{on} \quad \partial B_1, \\ u \ge 0, \end{cases}$$

where $B_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

Problem 2.4. Assume the function $f : [0, 1] \to \mathbb{R}$ is of $C^1[0, 1]$. Prove that the set of critical values of f has measure zero.

Problem 2.5. Let $w(r)$ and $\sigma(r)$ be non-decreasing functions in an interval $(0, R]$. Suppose there holds for all $r < R$

$$w(\tau r) \le \gamma w(r) + \sigma(r) \tag{7}$$

for some $\gamma, \tau \in (0, 1)$. Then for any $\mu \in (0, 1)$ and $r < R$ we have

$$w(r) \le C \left\{ \left(\frac{r}{R}\right)^{\alpha} w(R) + \sigma(r^{\mu} R^{1-\mu}) \right\} \tag{8}$$

where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants.

Problem 2.6. Let $P_k(x)$ denote the k-th Chebyshev polynomial, i.e. $P_k(\cos\theta) = \cos(k\theta)$ for $k \in \mathbb{N}$. Suppose

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

is a real monic polynomial (i.e. $a_n = 1$) with all roots in $(-1, 1)$. Prove that all the roots of

$$g(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{9}$$

are all real numbers and in $(-1, 1)$.

Problem 1. Given a vector $\mathbf{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times m}$, the Arnoldi process is a systematic way of constructing an orthonormal bases for the sucessive Krylov subspaces

$$\mathcal{K}_n = \langle \mathbf{b}, A\mathbf{b}, \ldots, A^{n-1}\mathbf{b} \rangle, \quad n = 1, 2, \ldots.$$

It gives

$$AQ_n = Q_{n+1}\tilde{H}_n,$$

where $Q_n \in \mathbb{R}^{m\times n}$, $Q_{n+1} \in \mathbb{R}^{m\times(n+1)}$ are with orthonormal columns and $\tilde{H}_n \in \mathbb{R}^{(n+1)\times n}$ is upper-Hessenberg. Let $H_n \in \mathbb{R}^{n\times n}$ be obtained by deleting the last row of $\tilde{H}_n$.

(a) Write out the Arnoldi algorithm.

(b) Assume that at step n, the $(n+1, n)$-th entry of $\tilde{H}_n$ is zero.

   i. Show that $\mathcal{K}_n$ is an invariant subspace of A and that $\mathcal{K}_n = \mathcal{K}_{n+1} = \mathcal{K}_{n+2} = \ldots$.

   ii. Show that each eigenvalue of H_n is an eigenvalue of A for $n > 1$.

(c) Let P_n be the set of monic polynomials of degree n. Show that the minimizer of

$$\min_{p_n \in P_n} \|p_n(A)\mathbf{b}\|_2$$

is given by the characteristic polynomial of H_n.

Problem 2. The following FitzHugh-Nagumo model is a simplified version of the Hodgkin-Huxley model (1963 Nobel Prize in Physiology or Medicine) which models in a detailed manner activation and deactivation dynamics of a spiking neuron.

$$\epsilon \dot{v} = v - \tfrac{1}{3}v^3 - w + I_{\text{ext}}$$

$$\dot{w} = v + a - bw$$

where v is the membrane voltage, w is a linear recovery variable, I_{ext} is the external stimulus. It contains the van der Pol oscillator as a special case for $a = b = I_{\text{ext}} = 0$. Fig 1 gives a qualtitative description of the four-stage structure of FitzHugh-Nagumo limit cycle solution and Fig 2 sketchs the time-profile illustrating the four stages in the limit cycle.

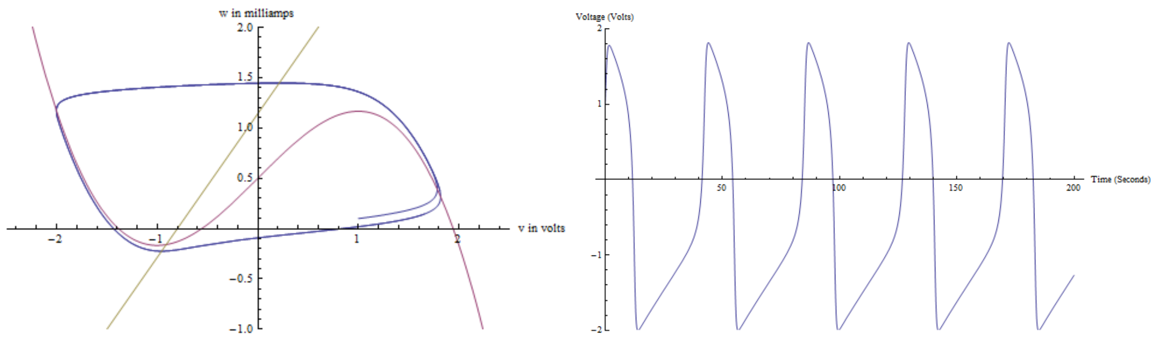


Figure 1: (a) The blue line is the trajectory of the FHN model in phase space. The pink line is the cubic nullcline $w = v - \frac{1}{3}v^3 + I_{\text{ext}}$ and the yellow line is the linear nullcline $w = a/b + v/b$. (b) Graph of v with parameters $I_{\text{ext}} = 0.5, a = 0.7, b = 0.8$, and $\epsilon = 1/12.5$.

(a) (10 pts) Use the expansions $v(t) = v_0(t) + \epsilon v_0(t) + O(\epsilon^2)$, $w(t) = w_0(t) + \epsilon w_0(t) + O(\epsilon^2)$ to determine the equations for the leading order slow solution. Point out the slow manifold (w_0 as a function of v_0) in Fig 1(a) and indicate the direction of the motion on each part, and identify the two attracting points on the curve.

(b) (5 pts) Use the expansion $v(t) = V_0(T) + \epsilon V_1(T) + O(\epsilon^2)$, $w(t) = W_0(T) + \epsilon W_1(T) + O(\epsilon^2)$ with $T = t/\epsilon$ to obtain the equations for the leading order fast solution.

(c) (5 pts) Use the phase plane to determine the maximum and the minimum values of $v(t)$ during an oscillation. Point out in Fig 1(a) and Fig 1 (b) the part of slow dynamics and fast dynamics, estimate the period for $v(t)$ as a function of time.

S.-T. Yau College Student Mathematics Contests 2016

Oral Exam of Computational and Applied Math

1. Consider n scalars: $a(1), a(2), \ldots, a(n)$. Our goal is to find the max partial sum, that is, find two indices p and q, where $p \leq q$, such that the partial sum, $a(p:q) = a(p) + a(p+1) + \cdots + a(q)$, is maximized. Develop an algorithm that finds p, q, and $a(p:q)$ by accessing each a(i) only once. For solution existence, assume that *not* all scalars are negative. Please provide a convincing oral proof to the correctness of your algorithm

S.-T. Yau College Student Mathematics Contests 2016

Oral Exam of Geometry and Topology

Individual

1. Let M be an orientable closed regular surface in the 3-dimensional Euclidean space with positive Gaussian curvature. Prove that the intersection of any two simple (i.e. no self- intersection) closed geodesics on M is non-empty.

2. Let M be a embedded compact surface with positive genus in $\mathbb{R}^3$, show that the Gaussian curvature of M must vanish somewhere on M.

3. Prove the Cartan formulas: $L_X = di_X + i_X d$ and $i_{[X,Y]} = [L_X, i_Y]$.

4.

(1) State Künneth formula for product manifold $M \times N$. Apply it to $S^2 \times S^2$.

(2) Let $f : S^2 \to S^2$ be a degree 2 map. Determine the cohomology defined by the graph of f in $H^*(S^2 \times S^2, \mathbb{Q})$.

(3) Compute the intersection of the graph with the diagonal in $S^2 \times S^2$.

S.-T. Yau College Student Mathematics Contests 2016

All Around of Geometry and Topology

1. Let $P_1 = (a_1, b_1), \cdots, P_n = (a_n, b_n)$ be n distinct points in $\mathbf{R}^2$. Define a 1-form ω on $\mathbf{R}^2 - \{P_1, \cdots, P_n\}$ by

$$\omega = \sum_{i=1}^{n} \frac{(x-a_i)dy - (y-b_i)dx}{(x-a_i)^2 + (y-b_i)^2}.$$

Let C be a simple closed curve containing $P_1, \cdots, P_n$ inside and rotate in the positive direction. Compute the line integral

$$\int_C \omega.$$

2.

(a) Define complex projective space CP^n.

(b) Compute the homology and cohomology of CP^n.

S.-T. Yau College Student Mathematics Contests 2016

Oral Exam of Geometry and Topology

1.

(1) Let $F : S^n \to S^n$ be a continuous map.
Define the degree of F and show that when F is smooth,

$$degF \int_{S^n} \omega = \int_{S^n} F^* \omega$$

for all $\omega \in \Omega^n(S^n)$.

(2) Show that if F has no fixed point then $degF = (-1)^{n+1}$.

2. Let $n \geq 0$ be an integer, M be a compact smooth manifold of dimension $4n+2$, show that $dimH^{2n+1}(M, \mathbb{R})$ is even.

3. Let M be a compact, simply connected smooth manifold of dimension n, prove that there is no smooth immersion

$$f : M \to T^n,$$

where T^n is n-torus.

4. Let $a > 0$ be a real number.
Let $S'(a)$ denote the circle obtained by identifying the end points of the interral $[0, a]$.
Consider the Riemannian metric defined by

$$r^2 \left(1 - \frac{1}{r^2}\right) dx^2 + \left(1 - \frac{1}{r^2}\right)^{-1} r^{-2} dr^2,$$

where $x \in S'(a)$ and $r \in (1, \infty)$.
Find the value of a such that the metric can be smoothly extended to $r = 1$.

2016 Oral Exam, Probability and Statistics

1. (Individual) Let X_λ be a Poisson random variable with parameter $\lambda > 0$:

$$\mathbb{P}[X_\lambda = k] = \frac{\lambda^k}{k!} e^{-\lambda}, \quad \forall k \in \{0, 1, 2, 3, \ldots\}.$$

   What is the limiting distribution of $\sqrt{X_\lambda} - \sqrt{\lambda}$ as $\lambda \to \infty$?

2. (Individual) Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables in L^1. Define, for $1 \le k \le n$,

$$S_k = \sum_{j=1}^{k} X_j.$$

   What is the conditional expectation of S_{n-1} given S_n?

3. (Individual) Three players A, B, C play chess in turn. In each game, the winning rate is half-half. The one who wins twice in row (i.e, wins twice consecutively) will be the final winner. The game starts with A *v.s.* B.

   What is the probability for A to be the final winner?

   How about 5 players, i.e., there are players A, B, C, D, E.?

4. (Overall) Suppose that X, Y, Z are i.i.d. uniform in $[0,1]$. Show that $W = (XY)^Z$ is uniform.

5. (Overall) A boy tries to collect some special tennis cards. There are 100 different types. Each time he put 1\$ into the card machine, he will randomly get a tennis card. The type of the card is uniformly distributed. Let T be the total money he will spend to collect all different types of cards.

   What is the expectation and variance of T?

6. (Team) Let X,Y be two real valued random variables such that $X-Y$ and X are independent, and that $X-Y$ are Y are independent. Show that $X-Y$ is almost surely constant.

7. (Team) For each n, let X_n be an exponential random variable with parameter $q_n > 0$:

$$\mathbb{P}[0 \le X_n \le t] = 1 - \exp(-tq_n).$$

Suppose that $X_1, X_2, \ldots$ are independent.

   (a) What is $\mathbb{E}[\exp(-X_n)]$?
   (b) Suppose $\sum 1/q_n < \infty$. Show that $\sum X_n < \infty$ almost surely.
   (c) Suppose $\sum 1/q_n = \infty$. Show that $\sum X_n = \infty$ almost surely.

8. (Team) Consider the numbers 1, 2, . . . 12 written around a ring as they usually are on a clock. A random walker starts at 12 and at each step moves at random to one of its two nearest neighbors (with probability half-half)

   What is the probability that she will visit all the other numbers before her first returning back to 12.

ALGEBRA (INDIVIDUAL)

Problem 1.

(1) Classify the groups of order 8.
(2) For each finite group of order 8, classify the irreducible finite dimensional representations over $\mathbb{C}$.

Problem 2. For each integer $m > 1$, let $\mathcal{K}_m = \mathbb{Q}(e^{\frac{2\pi i}{m}})$.

(1) Prove that the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.
(2) Prove that $\mathcal{K}_5$ is a Galois extension of $\mathbb{Q}$ whose Galois group is cyclic of order 4.
(3) Prove that $\mathcal{K}_5 \supseteq \mathbb{Q}(\cos \frac{2\pi}{5})$.
(4) Prove that $\mathcal{K}_{20}$ is a Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to $(\mathbb{Z}/20\mathbb{Z})^\times$.

(Recall that $(\mathbb{Z}/20\mathbb{Z})^\times$ is the group of units of the ring $\mathbb{Z}/20\mathbb{Z}$.)

Problem 3. Let f be a nonconstant polynomial in $\mathbb{C}[x]$ and $R = \mathbb{C}[x]/(f)$. Show that the following are equivalent:

(1) R has no nonzero nilpotent element.
(2) any finitely generated indecomposable R-module is projective.

ALGEBRA (OVERALL)

Problem 1.

(1) Prove that any finite field has order p^n for some prime p and integer n.
(2) State the law of quadratic reciprocity. For which odd prime p is -1 a square modulo p? For which odd prime p is 2 a square modulo p?
(3) Assuming that 691 is a prime, prove that 439 is not a square modulo 691.
(4) For which odd prime p does the polynomial $x^2 + 6x + 1$ have two roots in $\mathbb{Z}/p\mathbb{Z}$?

Problem 2. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$

(1) Prove that the subgroup of upper triangular matrices with 1's on the diagonal is a Sylow p-subgroup of G.
(2) Compute the number of Sylow p-subgroups of G.

ALGEBRA (TEAM)

Problem 1. Let $G = \mathrm{GL}_2(\mathbb{C})$.

(1) Prove all finite dimensional representations of G over $\mathbb{C}$ are completely reducible.
(2) Find all irreducible finite dimensional representations of G over $\mathbb{C}$.

Problem 2. Let K and L over $\mathbb{Q}$ be field extensions of prime degrees. Show that if $[KL : \mathbb{Q}] < [K : \mathbb{Q}][L : \mathbb{Q}]$, then the Galois closure of $K/\mathbb{Q}$ equals to the Galois closure of $L/\mathbb{Q}$.

Problem 3. Let G be a finite group. Let N be a minimal nontrivial normal subgroup of G (i.e., N does not properly contain any other nontrivial normal subgroup of G). Show that N is isomorphic to a direct product $L \times ... \times L$ of copies of a single simple group L.

S.-T. Yau College Student Mathematics Contests 2017

Analysis and Differential Equations

Individual

Please solve the following problems.

1. Suppose $f(x)$ is a positive and continuous function over $[a,b]$ and $g(x)$ is a positive and decreasing function over $[a,b]$. Prove that

$$\frac{\int_a^b xf(x)g(x)dx}{\int_a^b f(x)g(x)dx} \le \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}.$$

2. Let E be a closed subset of $\mathbb{R}^n$ and

$$E_r = \{x \in \mathbb{R}^n : \ d(x,E) = r\}, \quad \text{for } r > 0.$$

Prove that E_r is measurable and is of Lebesgue measure zero.

3. (1). Prove that

$$w \longrightarrow z = \int_0^w (1-x^n)^{-\frac{2}{n}} dx$$

is a conformal mapping from the unit disk onto the interior of a regular n-gon Γ.

(2). Find an explicit formula for the boundary value problem of the Laplace equation over an n-gon Γ, where ϕ is a continuous function on $\partial\Gamma$:

$$\begin{cases} \Delta u = 0, \\ u|_{\partial\Gamma} = \phi. \end{cases}$$

4. Prove the uniqueness of the following problem for $u(x,t)$ of

$$\begin{cases} u_t = u_{xx}, & \text{in} \quad (-1,1)\times(0,+\infty), \\ u(\pm 1,t) = 1, & \text{for} \quad t>0, \\ u(x,0) = 0, & \text{for} \quad x \in (-1,1), \\ u \text{ is uniformly bounded.} & \qquad\qquad (*) \end{cases}$$

Prove that the last condition $(*)$ is necessary for the result.

5. Let $(v,u)(x,t)$ $(0 \le x \le 1, t \ge 0)$ with $v>0$ for $0 \le x \le 1$, $t \ge 0$ be the C^∞ smooth solution of the following initial boundary value problem

$$\begin{cases} v_t - u_x = 0, \quad u_t + \left(\frac{1}{v^\gamma}\right)_x = \left(\frac{u_x}{v}\right)_x, \quad 0<x<1, \ t>0 \\ \left(\frac{u_x}{v} - \frac{1}{v^\gamma}\right)(0,t) = \left(\frac{u_x}{v} - \frac{1}{v^\gamma}\right)(1,t) = 0, \quad t>0 \\ v(x,0) = v_0(x), u(x,0) = u_0(x), 0 \le x \le 1. \end{cases}$$

where $\gamma > 1$ is a constant.

Prove that

(1)
$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}v^{1-\gamma}\right)(x,t)dx \le \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}v_0^{1-\gamma}\right)(x)dx, \qquad \text{for} \qquad t > 0.$$

(2) there exist positive constants c_1, c_2, c_3, c_4 independent of x and t such that
$$c_1 v_0(x)\left(1 + c_2 v_0^{-\gamma}(x)t\right)^{\frac{1}{\gamma}} \le v(x,t) \le c_3 v_0(x)\left(1 + c_4 v_0^{-\gamma}(x)t\right)^{\frac{1}{\gamma}}$$
for $0 < x < 1$, $t > 0$.

ANALYSIS 2017.

1. All-around

Problem 1.1. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, where $\mathbb{D}$ is the unit disk. If $f(0) = 0$, then

$$|z|\frac{|f'(0)| - |z|}{1 - |f'(0)||z|} \leq |f(z)| \leq |z|\frac{|f'(0)| + |z|}{1 + |f'(0)||z|}$$

Problem 1.2. Suppose $f(x)$ is a convex function defined on the whole real axis with the condition that

$$|f(x)| \leq C(1 + |x|)$$

for some constant C. Prove

(1) $\lim_{x\to+\infty} \frac{f(x)}{x} = \alpha$ exists for some real number α.
(2) There is a β such that $f(x) \leq \alpha x + \beta$ for x large.
(3) Is there a constant γ such that

$$f(x) \geq \alpha x + \gamma$$

for all x?

Problem 1.3. Can you solve the equation

$$\begin{cases} \Delta u = \frac{x}{x^2+y^2}, & \text{in } \{(x,y) : x^2 + y^2 < 1\}, \\ u = 0, & \text{on } \{(x,y) : x^2 + y^2 = 1\} \end{cases}$$

in the weak sense? What is the optimal continuity property of u?

ANALYSIS 2017.

1. Team

Problem 1.1. Let μ and ν be two finite Borel measures on $\mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n$ and $\forall R > 1$, we have

$$\mu(B(x,R)) = \nu(B(x,R))$$

where $B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$.

(1) Prove that if μ and ν are absolutely continuous with respect to the Lebesgue measure, then $\mu = \nu$.
(2) What if μ and ν are not absolutely continuous with respect to the Lebesgue measure?

Problem 1.2. Let Ω be a bounded simply connected domain. If $f : \Omega \to \Omega$ is analytic and there exists a point $z_0 \in \Omega$ such that

$$f(z_0) = z_0, \quad \text{and} \quad f'(z_0) = 1$$

then $f(z) = z$ for all $z \in \Omega$.

Problem 1.3. Find all the solutions of

$$\begin{cases} \Delta u = 0, \quad u > 0 \quad \text{in} \quad \{(x,y) : x > 0, y > 0\}, \\ u(x,0) = 0, \ u(0,y) = 0, \quad \text{for} \quad x > 0, y > 0. \end{cases}$$

Here we assume all the conditions hold in the classical sense.

Problem 1.4. For $0 \le t \le T (T > 0)$, Ω_t is an open bounded region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega_t$. Set

$$D_\Gamma = \{(x,t) : x \in \Omega_t, 0 \le t \le T\}, \qquad \partial D_t = \{(x,t) : x \in \partial\Omega_t, 0 \le t \le T\}.$$

Let $\eta(x,t) = (\eta_0, \eta_1, \eta_2, \eta_3)(x,t)$ be the unit outward normal of ∂D_Γ at (x,t). Suppose that $v(x,t) = (v_1, v_2, v_3)(x,t)$ is a smooth vector field and $P(x,t)$ is a smooth scalar function defined on D_Γ satisfying

$$\begin{cases} \frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = 0, \quad i = 1,2,3 \quad \text{in} \quad D_\Gamma, \\ \sum_{j=1}^3 \frac{\partial v_j}{\partial x_j} = 0, \quad \text{in} \quad D_\Gamma, \end{cases}$$

$$\begin{cases} \eta_0 + \sum_{j=1}^3 v_j \eta_j = 0, \qquad \text{on} \quad \partial D_\Gamma \\ P = 0, \qquad\qquad \text{on} \quad \partial D_\Gamma. \end{cases}$$

(1) Prove that $\int_{\Omega_t} |v|^2(x,t)dx$ is a constant for $0 < t < T$.
(2) Suppose that

$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = 0$$

for all $1 \le i, j \le 3$ in D_Γ, and

$$\frac{\partial v_{i_0}(x, t_0)}{\partial x_{j_0}} \ne 0$$

for $x \in \Omega_{t_0}$ and $t_0 \in (0,T)$ and some $i_0, j_0 \in \{1,2,3\}$. Prove that

$$\frac{\partial P}{\partial n} < 0$$

on $\partial\Omega_{t_0}$, where $n = (n_1.n_2,n_3)$ is the unit outer normal to $\partial\Omega_{t_0}$.

Oral Exam for Individuals: Applied and Computational Mathematics 2017

1. Consider the following Burgers' equation:

$$\begin{cases} u_t + uu_x = \nu u_{xx}, \quad x \in R, \quad t > 0 \\ u(x,0) = u_0(x), \quad x \in R\,, \end{cases} \tag{1}$$

which can be used to model the motion of a viscous compressible gas, where $u(x,t)$ is the speed of the gas, $\nu > 0$ is the kinematic viscosity, x is the spatial coordinate, and t is the time. As it is shown below, we can apply the Hopf-Cole transformation to solve the strongly nonlinear Burgers equation (1).

   (a) Let $U_x = u$ and introduce the Hopf-Cole transformation $U(x,t) = -2\nu \log(\phi(x,t))$. Derive the equation for $\phi(x,t)$.

   (b) Solve the equation for $\phi(x,t)$, and then obtain the solution to the Burgers equation (1).

   (c) When $\nu = 0$ the Burgers' equation (1) becomes the inviscid Burgers' equation. Show that if we solve the inviscid Burgers' equation with smooth initial data $u_0(x)$, for which $u_0'(x)$ is negative and bounded from below, then the solution will break at time $T_b = \frac{-1}{\min u_0'(x)}$. The term "break" here means the solution $u(x,t)$ has an infinite slope and a shock form.

2. (a) Consider the following Rudin-Osher-Fatemi model for image regularization

$$\min_u \int_\Omega \|\nabla u\| + \lambda\,(F-u)^2 \tag{2}$$

   for some given function F in a boundary rectangular domain Ω, where $\|\cdot\|$ is the usual 2-norm. Show that the Euler-Lagrange equation corresponding to the variational problem (2) is given by

$$u - \frac{1}{2\lambda}\nabla\cdot\left(\frac{\nabla u}{\|\nabla u\|}\right) = F\,.$$

   (b) Consider the following variational problem for the cartoon-texture decomposition in image processing

$$\min_{u,g,h} \int_\Omega \|\nabla u\| + \lambda\left(f - \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} - u\right)^2 + \mu\sqrt{g^2+h^2}\,, \tag{3}$$

   where f is the given image, λ and μ are two positive constants, and $f, u, g, h : \Omega \to \mathbb{R}$. With appropriate boundary conditions, derive the system of Euler-Lagrangian equations for (3) that the minimizers u, g and h should satisfy.

   (c) Let $\kappa(u) = \nabla\cdot\left(\frac{\nabla u}{\|\nabla u\|}\right)$ be the curvature operator. If u solves the variational problem (3), show that $\|\nabla\kappa(u)\| = \mu$.

Oral Exam for All Round: Applied and Computational Mathematics 2017

1. Consider a vector-valued ODE:

$$m_t = -a \times m\,,$$

where $a = (a_1, a_2, a_3)^T$ is a non-zero real constant vector, and $m(t) = (m_1(t), m_2(t), m_3(t))^T$ is the unknown vector-valued function.

Consider a uniform time sequence

$$0 = t_0 < t_1 < \cdots < t_k < \cdots\,,$$

with $t_k = k\Delta t$, and the explicit time marching scheme:

$$\frac{m^{n+1} - m^n}{\Delta t} = -a \times m^n\,,$$

(a) Analyse the stability of the scheme.

(b) Propose a numerical strategy to improve the scheme so that the new scheme has better stability.

(Applied Math. of TEAM)

Given a region $\Omega \subset \mathbb{R}^3$, please deduce the partial di erential equations of the fluid movement according to the conservation law of the mass and momentum.

Assume $\rho = \rho(t, x)$ is the density, $u = (u_1, u_2, u_3)(u_i = u_i(t, x))$ is the velocity(speed) of the fluid, and $p = p(t, x)$ is the pressure, where $x = (x_1, x_2, x_3)$ is the space variables.

Question 1: Given two di erent expression of the change of the mass in Ω of the fluid with the density $\rho(t, x)$ within the time interval $[t_1, t_2]$.

Question 2: Please deduce the partial di erential equations of ρ, u according to the conservation of the mass in Ω.

Question 3: Assume there is no outside force acting on the fluid in Ω, please give two di erent expression of the change of the momentum of the fluid with the density $\rho(t, x)$ and the velocity $u(t, x)$ in Ω within the time interval $[t_1, t_2]$.

Question 4: According to the conservation of the momentum of the fluid in Ω, deduce the partial di erential equation of ρ, u, p.

Oral Exam for Teams: Applied and Computational Mathematics 2017

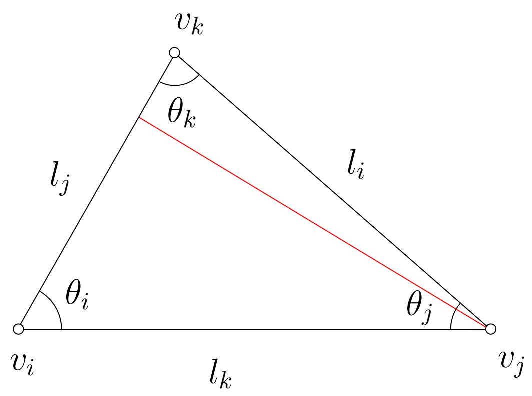


Figure 1: reference triangle.

1. Given a Euclidean triangle $[v_i, v_j, v_k]$ with edge lengths l_i, l_j, l_k and corner angles $\theta_i, \theta_j, \theta_k$ (see Figure 1), we treat the angles as the functions of edge lengths, namely, $\theta_i = \theta_i(l_i, l_j, l_k)$.

   (a) Show that
   $$\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{2A}, \quad \frac{\partial \theta_i}{\partial l_j} = -\frac{l_i}{2A}\cos\theta_k,$$
   where A is the area of the triangle.

   (b) Suppose the initial edge lengths are (l_i^0, l_j^0, l_k^0), the conformal factor (u_i, u_j, u_k) are three real numbers associated with the vertices, the vertex scaling operator changes each edge length by multiplying the exponential of conformal factors at its two end vertces, namely:
   $$l_i = e^{u_j} l_i^0 e^{u_k}, \quad l_j = e^{u_k} l_j^0 e^{u_i}, \quad l_k = e^{u_i} l_k^0 e^{u_j},$$
   Show that
   $$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \cot\theta_k, \quad \frac{\partial \theta_i}{\partial u_i} = -\cot\theta_j - \cot\theta_k$$

   (c) If the initial triangle is an acute triangle, then in a neighborhood of $(u_i, u_j, u_k) = (0, 0, 0)$, the mapping $\varphi : \{(u_i, u_j, u_k) | u_i + u_j + u_k = 0\} \to \{(\theta_i, \theta_j, \theta_k) | \theta_i + \theta_j + \theta_k = \pi\}$ is diffeomorphic.

2. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let $\|q_1\|_2 = 1$. Consider the following Lanczos iteration:

$$
\begin{array}{l}
r_0 = q_1, \quad \beta_0 = 1, \quad q_0 = 0, \quad k := 0 \\
\textbf{while } \ \beta_k \neq 0 \\
\quad q_{k+1} := r_k / \beta_k \\
\quad k := k + 1 \\
\quad \alpha_k := q_k^T A q_k \\
\quad r_k := (A - \alpha_k I) q_k - \beta_{k-1} q_{k-1} \\
\quad \beta_k := \|r_k\|_2 \\
\textbf{end}
\end{array}
$$

Let $K_n = \text{span}\{q_1, Aq_1, \cdots, A^{n-1} q_1\}$.

(a) Show that

$$AQ_k = Q_k T_k + r_k e_k^T$$

where e_k is the k-th unit vector, $Q_k = [q_1 \cdots q_k]$ and

$$
T_k = \begin{bmatrix}
\alpha_1 & \beta_1 & & \cdots & 0 \\
\beta_1 & \alpha_2 & \ddots & & \vdots \\
 & \ddots & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & \beta_{k-1} \\
0 & \cdots & & \beta_{k-1} & \alpha_k
\end{bmatrix}
$$

(b) Assume that the iteration does not terminate. Show that Q_k has orthonormal columns, and that they span K_k.

(c) Show that the Lanczos iteration will stop when $k = m$, where $m = \text{rank}(K_n)$.

(d) What is the purpose of this algorithm? Briefly justify your answer.

1 (Optimal Mass Transport)**.** Suppose $\mathbb{D}$ is the unit disk in the plane, $P = \{p_1, p_2, \cdots, p_n\}$ is a discrete planar point set. Each point p_i is associated with a weight r_i, the power distance between any point $p \in \mathbb{R}^2$ to p_i is defined as

$$Pow(p, p_i) = |p - p_i|^2 + r_i.$$

The power Voronoi diagram is a partition of the whole plane

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} W_i,\ W_i = \{p \in \mathbb{R}^2 | Pow(p, p_i) \leq Pow(p, p_j), \forall 1 \leq j \leq n\}.$$

The power Vornoi diagram induces a cell decomposition of $\mathbb{D}$,

$$\mathbb{D} = \bigcup_{i=1}^{n} W_i \cap \mathbb{D},$$

suppose the area of each cell $\mathbb{D} \cap W_i$ is A_i. Construct a mapping $\varphi : \mathbb{D} \to P$, such that each cell $W_i \cap \mathbb{D}$ is mapped to the point p_i,

$$\varphi : W_i \cap \mathbb{D} \mapsto p_i,\ \forall 1 \leq i \leq n.$$

(1) Suppose given another cell decomposition

$$\mathbb{D} = \bigcup_{i=1}^{n} \tilde{W}_i \cap \mathbb{D},$$

and construct a mapping $\tilde{\varphi}$, such that

$$\tilde{\varphi} : \tilde{W}_i \cap \mathbb{D} \mapsto p_i,$$

and the area of each cell $\tilde{W}_i \cap \mathbb{D}$ equals to A_i as well. The L^2 transportation cost of φ is defined as

$$E(\varphi) := \int_{\mathbb{D}} |p - \varphi(p)|^2 dA,$$

show that the mapping φ is optimal, i.e.

$$E(\varphi) \leq E(\tilde{\varphi}).$$

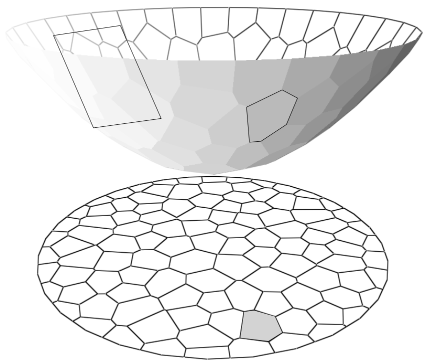
(2) Show that there exists real numbers $h_1, h_2, \cdots, h_n$, which determine n planes

$$\pi_i(p) := \langle p, p_i \rangle + h_i,$$

the upper envelope of the planes $\{\pi_i\}$ is the graph of the convex PL function

$$f(p) = \max_{1 \leq i \leq n} \pi_i(p).$$

The power Voronoi diagram is induced by the projection of the upper envelope of these planes $\{\pi_i, i = 1, 2, \cdots, n\}$



S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Individual Problems

1. Show that $S^2 \times S^2$ and $\mathbb{CP}^2 \vee S^2$ are not homotopically equivalent.

2 (Bonnet-Myers theorem)**.** Prove that a complete Riemannian manifold M whose sectional curvature is everywhere bounded below by a constant k has diameter at most $\pi/\sqrt{k}$. In particular, M is compact.

3 (Isoperimetric inequality)**.** For the length L of a closed curve and the area A of the planar region that it encloses, show that $4\pi A \leq L^2$ and that equality holds if and only if the curve is a circle.

4. (a) Compute the cohomology of the unitary group $U(n)$.
(b) Compute $\pi_1(U(n))$ and $\pi_2(U(n))$.

S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Team Problems

1. (a) Describe the loop space ΩS^2 and path space PS^2 of the sphere S^2 in the following fibration:

$$\begin{array}{ccc} \Omega S^2 & \longrightarrow & PS^2 \\ & & \downarrow \\ & & S^2. \end{array}$$

(b) Compute the cohomology of the loop space ΩS^2. What is the ring structure of $H^*(\Omega S^2)$?

2 (Synge theorem)**.** Let M be an even-dimensional compact Riemannian manifold with positive sectional curvature.

(a) When M is orientable, show that M is simply connected.

(b) When M is unorientable, what is $\pi_1(M)$?

3. (a) Let C be a smooth curve on the sphere. The Crofton formula expresses the arc length $L(C)$ of the curve C as

$$L(C) = \frac{1}{4} \int_{S^2} n(C \cap W^\perp) dW.$$

Here $W^\perp$ is the plane with normal W going through the origin and $n(C \cap W^\perp)$ is the number of points in the intersection of C and $W^\perp$.

(b) Sketch a proof of Crofton formula.

4. Let $\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $\alpha > 0$. Suppose D is the surface in $\mathbb{R}^3$ defined by $\{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1, z \geq \alpha\sqrt{x^2 + y^2}\}$.

(a) Show that $\Omega|_D$ is an orientation form and makes D an oriented manifold with boundary.

(b) Evaluate $\int_D \Omega$. Your answer should be in terms of α.

S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Overall Problems

1. Let Σ_g be a compact Riemann surface of genus $g > 1$, $Aut(\Sigma_g)$ be the automorphism group of biholomorphic maps of Σ_g. Let $V = H^0(\Sigma_g, K)$ be the space of holomorphic 1-forms on Σ_g.

(a) Show that the natural group homomorphism

$$\rho : Aut(\Sigma_g) \to GL(V)$$

is injective.

(b) V carries a natural hermitian structure

$$< \omega_1, \omega_2 >= i \int_{\Sigma_g} \omega_1 \wedge \overline{\omega_2}, \quad \omega_i \in V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside the unitary subgroup.

(c) V carries a natural integral structure from the lattice

$$H^1(\Sigma, \mathbb{Z})(\simeq \mathbb{Z}^{2g}) \subset V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside $GL(\mathbb{Z}^{2g})$.

(d) Conclude that $Aut(\Sigma_g)$ is a finite group.

2. (a) What is a Killing field on a Riemannian manifold?

(b) Explain why a Killing field on a connected Riemannian manifold is determined by its value and the value of its first derivative at a given point.

(c) Show that the maximal dimension of the space of Killing fields on a three dimensional connected Riemannian manifold is six.

3. (a) Let X be an n-dimensional compact Riemannian manifold. Show that

$$\dim(\mathrm{Isom}(X)) \leq \frac{n(n+1)}{2}.$$

(b) List all possible M when the equality in the above is achieved.

S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Overall Problems

1. Let Σ_g be a compact Riemann surface of genus $g > 1$, $Aut(\Sigma_g)$ be the automorphism group of biholomorphic maps of Σ_g. Let $V = H^0(\Sigma_g, K)$ be the space of holomorphic 1-forms on Σ_g.

(a) Show that the natural group homomorphism

$$\rho : Aut(\Sigma_g) \to GL(V)$$

is injective.

(b) V carries a natural hermitian structure

$$< \omega_1, \omega_2 >= i \int_{\Sigma_g} \omega_1 \wedge \overline{\omega_2}, \quad \omega_i \in V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside the unitary subgroup.

(c) V carries a natural integral structure from the lattice

$$H^1(\Sigma, \mathbb{Z})(\simeq \mathbb{Z}^{2g}) \subset V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside $GL(\mathbb{Z}^{2g})$.

(d) Conclude that $Aut(\Sigma_g)$ is a finite group.

2. (a) What is a Killing field on a Riemannian manifold?

(b) Explain why a Killing field on a connected Riemannian manifold is determined by its value and the value of its first derivative at a given point.

(c) Show that the maximal dimension of the space of Killing fields on a three dimensional connected Riemannian manifold is six.

3. (a) Let X be an n-dimensional compact Riemannian manifold. Show that

$$\dim(\mathrm{Isom}(X)) \leq \frac{n(n+1)}{2}.$$

(b) List all possible M when the equality in the above is achieved.

S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Team Problems

1. (a) Describe the loop space ΩS^2 and path space PS^2 of the sphere S^2 in the following fibration:

$$\begin{tikzcd} \Omega S^2 \arrow[r] & PS^2 \arrow[d] \\ & S^2. \end{tikzcd}$$

(b) Compute the cohomology of the loop space ΩS^2. What is the ring structure of $H^*(\Omega S^2)$?

2 (Synge theorem)**.** Let M be an even-dimensional compact Riemannian manifold with positive sectional curvature.

(a) When M is orientable, show that M is simply connected.

(b) When M is unorientable, what is $\pi_1(M)$?

3. (a) Let C be a smooth curve on the sphere. The Crofton formula expresses the arc length $L(C)$ of the curve C as

$$L(C) = \frac{1}{4}\int_{S^2} n(C \cap W^{\perp})dW.$$

Here $W^{\perp}$ is the plane with normal W going through the origin and $n(C \cap W^{\perp})$ is the number of points in the intersection of C and $W^{\perp}$.

(b) Sketch a proof of Crofton formula.

4. Let $\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $\alpha > 0$. Suppose D is the surface in $\mathbb{R}^3$ defined by $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq \alpha\sqrt{x^2 + y^2}\}$.

(a) Show that $\Omega|_D$ is an orientation form and makes D an oriented manifold with boundary.

(b) Evaluate $\int_D \Omega$. Your answer should be in terms of α.

2017 Oral Exam: Probability and Statistics
Individual

Problem 1. *Let X be a random variable with finite variance. Denote by m, μ, σ the median, mean and standard deviation of X:*

$$m := \inf\{c : \mathbb{P}[X \le c] \ge 1/2\}, \quad \mu = \mathbb{E}[X], \quad \sigma^2 = \mathbb{E}[(X-\mu)^2].$$

Show that $|m - \mu| \le \sigma$.

Problem 2. *Let $(X_n)_{n\ge 1}$ be a sequence of non-negative random variables. Let $(\mathcal{F}_n)_{n\ge 1}$ be a filtration (i.e. a sequence of increasing σ-algebras). Assume that*

$$\mathbb{E}[X_n \,|\, \mathcal{F}_n] \to 0, \quad \textit{in probability.}$$

Show that

$$X_n \to 0, \quad \textit{in probability.}$$

Is it true reversely? If yes, prove it; if not, give a counterexample.

Problem 3. *Let $X_1, ..., X_n$ be independent random variables following common Poisson distribution with mean λ. Let $\eta = e^{-\lambda}$. Does there exist a uniformly unbiased minimum variance estimator UMVUE of η? (Recall that an estimator is UMVUE if it is unbaised estimator and has smallest variance among all unbiased estimators.) If yes, find it; if no, prove it.*

2017 Oral Exam: Probability and Statistics
Overall

Problem 1. *Let* $\epsilon_1, ..., \epsilon_n, ...$ *be a sequence of i.i.d. random variables with mean 0 and finite variance* σ^2*. Let* x_1 *be a constant, and*

$$y_n = m(x_n) + \epsilon_n; \quad x_{n+1} = x_n - cy_n/n, \quad \forall n \geq 1,$$

where c *is a positive constant. Suppose that* $m(x)$ *is smooth and strictly increasing and* $m(x^*) = 0$*. Show that* x_n *converges to* x^* *in probability.*

2017 Oral Exam: Probability and Statistics
Team

Problem 1. *An ant randomly travels along edges on a unit cube. Each step, it travels from one vertex to an adjacent vertex. Starting from a vertex, what is the expected number of steps the ant is to travel to first reach the diagonal vertex?*

Problem 2. *An algorithm called MM to maximize a function $g(x)$ is as follows. We find another easy-to-compute function $Q(x, z)$ such that*

$$g(z) \geq Q(x, z), \quad g(x) = Q(x, x).$$

Starting from an initial x_0, let $x_{k+1} = argmax_z Q(x_k, z)$. Argue that $g(x_k)$ is a nondecreasing sequence.

Let Y and X be two random vectors. Denote by $f_Y(\cdot;\theta), f_{X\,|\,Y}(\cdot,\cdot;\theta), f_{XY}(\cdot,\cdot;\theta)$ the marginal, conditional and joint densities that depend on parameter θ. We wish to maximize $l(\theta) := \log f_Y(y;\theta)$. Define

$$Q(\theta, \tilde{\theta}) = \int \log f_{XY}(x, y; \tilde{\theta}) f_{X|Y}(x, y; \theta) dx$$

Consider

$$\theta_{k+1} = argmax_{\tilde{\theta}}\ Q(\theta_k, \tilde{\theta}).$$

Argue that this algorithm, called EM, can be viewed as a special case of MM. As a result $l(\theta_{k+1}) \geq l(\theta_k)$.

YAU COLLEGE MATH CONTESTS INDIVIDUAL ALGEBRA 2018

Problem 1

Factorize the polynomial

$$f(x) = 6x^5 + 3x^4 - 9x^3 + 15x^2 - 13x - 2$$

into a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.

Problem 2

Prove that any group of order 588 is solvable, given that any group of order 12 is solvable.

Problem 3

Decide which field F has the following property: for each integer $n > 0$, and for every $n \times n$ matrix A with entries in F, we can conjugate A to an upper triangular matrix under $GL_n(F)$.

Problem 4

Let n be a positive integer.

(1) Find the image of the map

$$M_{n\times n}(\mathbb{C}) \longrightarrow M_{n\times n}(\mathbb{C}), \quad A \to A^t A.$$

Here $M_{n\times n}(\mathbb{C})$ denotes the space of all $(n \times n)$ matrices with complex entries.

(2) Find the image of the map

$$M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}), \quad A \to A^t A.$$

Here $M_{n\times n}(\mathbb{R})$ denotes the space of all $(n \times n)$ matrices with real entries.

Problem 5

Let k be a field of characteristic $p > 0$, and let x, y be algebraically independent over k. Prove the following

(a). $k(x, y)$ has field extension degree p^2 over $k(x^p, y^p)$.

(b). There are infinitely many intermediate field extensions between $k(x, y)$ and $k(x^p, y^p)$.

YAU COLLEGE MATH CONTESTS ALL AROUND ALGEBRA 2018

Problem 1

Let ζ_n be a primitive n-th root of unity, where $n > 1$ is a positive integer. Show that the number

$$N := \prod_{1 \le i \le n, (i,n)=1} (1 - \zeta_n^i)$$

is p if n is a power of a prime p, and is 1 if n is not a power of a prime (i.e., n is divisible by at least two distinct primes).

Problem 2

Let F be a field. Let $G = GL_2(F)$ and B be the subgroup of G consisting of all upper triangular matrices. Then $B \times B$ acts on G by left and right multiplication, i.e.

$$B \times B \times G \longrightarrow G, \quad (b_1, b_2, a) = b_1 a b_2^{-1}.$$

Prove that there are exactly two orbits, and they can be represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Generalize this to $GL_n(F)$.

Problem 3

Let $M_n(\mathbb{Q})$ be the ring of all $n \times n$ matrices with coefficients in $\mathbb{Q}$ for a positive integer n. Describe all field extensions K of $\mathbb{Q}$ such that there is an injective ring homomorphism $K \to M_n(\mathbb{Q})$. (Note: we take the convention that a ring homomorphism maps the additive identity and the multiplicative identity respectively to the additive identity and the multiplicative identity.)

Remark. This is a special case of classification of commutative sub-algebras in central simple algebras.

Problem 4

Fix positive integers $n > m > k$, and fix a $\mathbb{C}$-linear subspace $E \subset \mathbb{C}^n$ of dimension k. Let

$$X = \{\mathbb{C}\text{-linear subspace } V \subset \mathbb{C}^n | V \supset E,\ \dim V = m\}.$$

Does X naturally have the structure of a compact manifold? If so, what is $\dim_{\mathbb{R}} X$? Does X naturally have the structure of the coset space of a group?

YAU COLLEGE MATH CONTESTS TEAM ALGEBRA 2018

Problem 1

Let d_i $(1 \le i \le n)$ be positive integers such that $\sum_{i=1}^{n} \frac{1}{d_i} > 1$. For a prime number p, let $\mathbf{F}_p$ be the finite field of p elements. For

$$f(x_1, \cdots, x_n) = x_1^{d_1} + x_2^{d_2} + \cdots + x_n^{d_n},$$

prove that the number

$$N := \#\{(x_1, \cdots, x_n) \in \mathbf{F}_p^n | f(x_1, \cdots, x_n) = 0\}$$

is divisible by p. Hint: consider the sum $\sum_{(x_1,\cdots,x_n)\in \mathbf{F}_p^n} f(x_1, \cdots, x_n)^{p-1}$.

Problem 2

Let G be a finite group acting on the polynomial ring $R = \mathbf{k}[x_1, \cdots, x_n]$ with n variables $x_1, \cdots, x_n$. Let $S := \{f \in R \mid g \cdot f = f, \ \forall g \in G\}$ be the subring of invariants. Prove that S is a finitely generated $\mathbf{k}$-algebra.

Problem 3

Let k be any field. Let $R = k[[t]]$ be the ring of formal power series over k. Let M be a finitely generated free R-module. Let $v_1, \cdots, v_n \in M$, and denote their images in M/tM by $\bar{v}_1, \cdots, \bar{v}_n$. Assume that $\{\bar{v}_1, \cdots, \bar{v}_n\}$ is a basis of the vector space M/tM over k. Prove that $\{v_1, \cdots, v_n\}$ is an R-basis of the module M.

Remark. This is a special case of Nakayama's lemma.

Problem 4

Let $\mu(t) \in k[t]$ be the minimal polynomial of $A \in M_n(k)$ and

$$W_A = \{ \ AX - XA \mid \forall \ X \in M_n(k) \} \subset M_n(k).$$

Prove that

$$\dim(W_A) \leq n^2 - \deg(\mu(t))$$

and the equality holds if and only if $\deg(\mu(t)) = n$.

S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Individual

Please solve the following problems.

1. A map $f : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$ is a $(L-)$Lipschitz map if there is a constant $L < \infty$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in A.$$

The smallest such constant L is called the Lipschitz constant of f and denoted by $\mathrm{Lip}(f)$.

1) Let $f : A \to \mathbb{R}, A \subset \mathbb{R}^m$. Let $G = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^m \times \mathbb{R}$ denote its graph. Suppose that there exists $\alpha > 0$, such that for any $z \in G$, the cone

$$C_{z,\alpha} := z + \{(y_1, y_2) \in \mathbb{R}^m \times \mathbb{R} : |y_2| > \alpha|y_1|\} \subset \mathbb{R}^m \times \mathbb{R}$$

does not meet G. Prove that f is Lipschitz, with $\mathrm{Lip}(f) \leq \alpha$.

2) Suppose that $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^m$ is an L-Lipschitz map. Prove that f can be extended to a L-Lipschitz map on $\mathbb{R}^m$, that is, there exists an L-Lipschitz map $g : \mathbb{R}^m \to \mathbb{R}$, such that $g\lfloor_A = f$.

2. Let D be a region in the complex plane, z_0 a point in D. Let U be the open unit disk. If D is simply connected and $D \neq C$, then there exists at least one univalent function from $f : D \to U$ such that $f^{'}(z_0) > 0$.

You can not use Riemann mapping theorem directly.

3. Let $P(x)$ be polynomial of degree n. Show that

$$|P(0)| \leq C \int_{-1}^{1} |P(x)| dx$$

4. Prove uniqueness of solutions to the following problem

$$\Delta u + \sqrt{u} = 0 \text{ in } \Omega$$

$$u > 0 \text{ in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

5. Let ℓ_2 be the Hilbert space of all square summable complex sequences $x = (x_k)_{k\geq 1}$, equipped with the following inner product and norm

$$\langle x,\, y\rangle = \sum_{k\geq 1} x_k\, \overline{y_k} \quad \text{and} \quad \|x\| = \Big(\sum_{k\geq 1} |x_k|^2\Big)^{1/2}.$$

Let $u : \ell_2 \to \ell_2$ be the linear operator defined by

$$\forall\, x = (x_k)_{k\geq 1} \in \ell_2, \quad u(x) = \Big(\sum_{k=1}^{\infty} \frac{x_k}{j+k}\Big)_{j\geq 1}.$$

The aim of this exercise is to calculate the norm $\|u\|$.

a) Let φ be the 2π-periodic function defined by $\varphi(t) = \mathrm{i}(\pi - t)$ for $0 \leq t < 2\pi$, where $\mathrm{i} = \sqrt{-1}$. Show that

$$\sum_{j,k\geq 1} \frac{x_j y_k}{j+k} = \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{j\geq 1} x_j e^{-\mathrm{i}jt}\Big)\Big(\sum_{k\geq 1} y_k e^{-\mathrm{i}kt}\Big)\varphi(t)dt\,, \quad \forall\, x, y \in \ell_2.$$

b) Deduce that u is bounded and $\|u\| \leq \pi$.

c) For any given $n \geq 1$ let $a_n = (1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots)$. Show that

$$\langle u(a_n),\, a_n\rangle \geq \pi \ln n + \mathrm{O}(1).$$

Deduce $\|u\| \geq \pi$.

d) For any given $n \geq 1$ let $a_n = (1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots)$. Show that

$$\langle u(a_n),\, a_n\rangle \geq \pi \ln n + \mathrm{O}(1).$$

Deduce $\|u\| \geq \pi$.

S.-T. Yau College Student Mathematics Contests 2018

Analysis and Di erential Equations

Individual Overall

Please solve the following problems.

1. Let f be an analytic function in a neighborhood of $\bar{U}$ where $U = \{|z| < 1\}$. Show that if f is real on ∂U, then f must be constant.

2. Find the Green's function for

$$-u'' = f$$

$$u(0) = u(1), u'(0) = u'(1)$$

$$\int_0^1 u = 0, \int_0^1 f = 0$$

3. Let $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$ for $n > 2$ and $R > 0$. Prove that there exists a constant C independent of R such that

$$\int_{B_R(0)} \frac{|v(x)|^2}{|x|^2} dx \leq C \int_{B_R(0)} (|v_r|^2 + R^{-2} v^2) dx$$

for any function $v \in C^\infty(B_R(0))$, where $v_r(x) = \frac{x \cdot \nabla v}{|x|}$.

S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Team

Please solve the following problems.

1. Isoperimetric inequality. and Steiner symmetrization.

It is well known that in $\mathbb{R}^2$, any region Ω with continuous piecewise C^1 boundary $\partial\Omega$ satisfies that

$$4\pi|\Omega| \leq \text{Length}(\partial\Omega)^2.$$

When the equality holds for a region Ω with continuous piecewise C^1 boundary, it is then called an isoperimetric set. For example, all disks are isoperimetric sets.

1) Prove that any isoperimetric set is convex;

2) Given a line V in $\mathbb{R}^2$ passing through the origin, the Steiner symmetrization with respect to V is the operation which associates with each bounded convex subset C of $\mathbb{R}^2$ the subset $S(C)$ of $\mathbb{R}^2$ such that, for every line L (not necessarily passing through the origin) perpendicular to V,

either $L \cap C = \emptyset$ and $L \cap S(C) = \emptyset$,

or $L \cap C \neq \emptyset$ and $L \cap S(C)$ is a closed segment with center in V, and Length($L \cap S(C)$)=Length($L \cap C$). In other words, $L \cap S(C)$ is a translation of the segment $L \cap C$ along L, such that the segment $L \cap S(C)$ is symmetric with respect to V.

Prove that $|S(C)| = |C|$, and Length($\partial S(C)$) $\geq$Length(C), equality holds if and only if C is symmetric with respect to V.

3) Deduce that the only isoperimetric sets are the disks.

2. Define $\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}, s = \sigma + it$. Show that

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}dz$$

where $\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx, (-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < Im \log(-z) < \pi.C$ is the contor consisting of upper real axis in the direction from $+\infty$ to 0, a small circle around 0 counter-clockwise, and lower real axis in the direction from 0 to $+\infty$.

Show that $\zeta(s)$ is a meromorphic function of the complex plane.

3. Consider the following linear parabolic equation

$$\begin{cases} \partial_t u(t;r,z) = (\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u(t;r,z), \text{ on } \mathbb{R}^+ \times \Omega \\ u(0;r,z) = u_0(r,z), u(t;0,z) = 0 \end{cases}$$

where $\Omega = \{(r,z); r > 0, z \in \mathbb{R}\}$. Given initial data u_0 satisfying

$$\int_\Omega |u_0| drdz < \infty.$$

Prove that $u(t;r,z)$ is given by

$$u(t;r,z) = \frac{1}{4\pi t}\int_\Omega |\frac{r'}{r}|^{\frac{1}{2}} H(\frac{t}{rr'}) \exp(-\frac{(r-r')^2 + (z-z')^2}{4t}) u_0(r',z') dr' dz'$$

where

$$H(y) = \frac{1}{\sqrt{\pi y}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\frac{\sin^2\phi}{y}} \cos(2\phi) d\phi$$

4. Let Ω be a bounded and connected smooth domain in $\mathbb{R}^2$, and $\tilde{N} = (\tilde{N}_1, \tilde{N}_2)$ be a smooth vector field defined on $\bar{\Omega}$ that equals to the outer unit normal N on the boundary $\partial\Omega$ satisfying $\sup_{x\in\bar{\Omega}} |\tilde{N}|(x) \le 1$ and $\sup_{x\in\bar{\Omega}} |\partial_{x_j}\tilde{N}|(x) \le K$ $(j = 1,2)$ for some constant K. Prove that there exists a constant C independent of K such that

$$\int_\Omega \sum_{i,j=1}^2 |\partial_i v_j|^2 dx$$

$$\le C \int_\Omega \sum_{1\le i,k,l\le 2} (\tilde{N}^k \tilde{N}^l \partial_i v_k \partial_i v_l + |curl v|^2 + |\mathrm{div} v|^2 + K|v|^2)(x) dx,$$

for any smooth vector field $v = (v_1, v_2)$ on $\bar{\Omega}$, where $|curl v|^2 = \sum_{i,j=1}^2 (\partial_i v_j - \partial_j v_i)^2$, $|\mathrm{div} v|^2 = \sum_{i=1}^2 (\partial_i v_i)^2$.

S.-T. Yau College Student Mathematics Contest, 2018
Applied Mathematics, Individual

1. Consider the definite integral

$$I = \int_a^b f(x)\, dx.$$

   a. Construct an approximation to I by using the composite trapezoidal rule with a uniform partition of the interval $[a, b]$.

   b. Suppose $f \in C^2[a, b]$, show that the above approximation is second-order accurate.

   c. Suppose f is periodic and smooth on the interval $[a, b]$, show that the above approximation is spectral order accurate.

2. Let $(\mathbf{u}, p)$ be the solution of the Stokes equation on a domain Ω

$$\begin{cases} -\nabla p + \Delta \mathbf{u} = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}|_{\partial\Omega} = g \end{cases} \tag{1}$$

where $\mathbf{u}(x, y) = (u(x, y), v(x, y))$, $\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and g is the given boundary value. Show that $\mathbf{u}$ is a minimizer of the dissipation functional

$$E(\mathbf{v}) = \int_\Omega |\nabla \mathbf{v}|^2 dxdy$$

among all $\mathbf{v}$ such that $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v}|_{\partial\Omega} = g$.

S.-T. Yau College Student Mathematics Contest, 2018
Applied Mathematics, All-around

Let $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ be a solution of the ODE

$$\frac{d\mathbf{u}}{dt} = \mathbf{a} \times \mathbf{u}.$$

where $\times$ denotes the cross product and $\mathbf{a} = (a_1, a_2, a_3) \neq 0$.

Consider a forward difference scheme in time

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\, (\mathbf{a} \times \mathbf{u}^n).$$

1. Show that the scheme is always unstable.

2. Discuss ways to improve the stability of the scheme.

S.-T. Yau College Student Mathematics Contest, 2018
Applied Mathematics, team

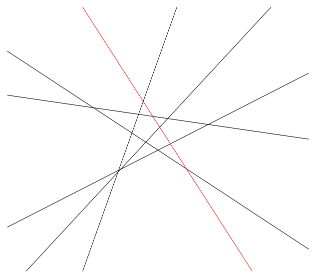
1. Given a set of column vectors $y_1, \cdots, y_n \in \mathcal{R}^m$, we set $\mathcal{V}$ =span$\{y_1, \cdots, y_n\} \subset \mathcal{R}^m$. How to find $\ell \leq$dim$\mathcal{V}$ orthonormal vectors $\{\psi_i\}_{i=1}^{\ell}$ in $\mathcal{R}^m$ that minimize

$$J(\psi_1, \cdots, \psi_\ell) = \sum_{j=1}^{n} ||y_j - \sum_{i=1}^{\ell} (y_j^T \psi_i)\psi_i||^2$$

with the Euclidean norm $||y|| = \sqrt{y^T y}$.

2. Suppose there are n hyper-planes in the d dimensional Euclidean space $\mathbb{R}^d$, the planes partition the space into convex cells, the maximal number of cells is denoted as $f_d(n)$,

   1. Find the formula for $d = 2$ case, namely, the plane is partitioned by n lines, prove it.
   2. Find the formula for general d and n and prove.



S.-T. Yau College Student Mathematics Contests 2018

Oral Exams in Geometry and Topology

Individual (4 problems)

1. Is the surface of genus 3 a covering space of the surface of genus 2. Is the surface of genus 2 a covering space of the surface of genus 3. If so, then exhibit an example of such a covering.

2. Consider a smooth map $f : S^{2n-1} \to S^n$ with $n \geq 2$. Let ν be a volume form on S^n with volume 1.

a). Show that $f^*\nu$ is exact.

b). Write $f^*\nu = d\alpha$. Show that the integral

$$\int_{S^{2n-1}} \alpha \wedge f^*\nu$$

is independent of the choice of α.

c). Show that the integral above is actually an invariant of the homotopy class of f.

d). Show that the integral is 0 if n is odd.

e). Calculate this integral if f is the Hopf map $(z_0, z_1) \to [z_0, z_1]$.

3. (D. Hilbert) There does not exist a complete regular surface in $\mathbb{R}^3$ whose Gaussian curvature is a negative constant K_0.

4. Let $\overline{CP^2}$ denote the complex projective plane with the opposite orientation.

a). Show that $S^2 \times S^2$ and $CP^2 \# \overline{CP^2}$ have the same cohomology group but different cohomology ring.

b). Show that $(S^2 \times S^2) \# \overline{CP^2}$ and $CP^2 \# 2\overline{CP^2}$ have the same cohomology.

All-round (3 problems)

1. Suppose $f(z_0, z_1, z_2, z_3)$ is a degree 2 homogeneous polynomial and its zero set S is a smooth submanifold in CP^3. Then S is called a smooth degree 2 complex hypersurface in CP^3.

a). Compute the Euler number of S.

b). Suppose that $f(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_2$. Identify the zero set as a familiar 4 dimensional manifold.

2. Show that a positively curved noncompact surface in $\mathbb{R}^3$ has infinite area.

3. If M is a compact manifold with negative sectional curvature, then the fundamental group of M is of exponential growth.

Team (5 problems)

1. Show that a closed simply connected 3-manifold is homotopically equivalent to the 3-sphere.

2. Recall that a symplectic form ω on a smooth manifold M is a degree 2 differential form which is closed and non-degenerate. Here non-degeneracy means that if there is a point $x \in M$ and a tangent vector $u \in T_x M$ such that $\omega(u, v) = 0$ for any $v \in T_x M$, then $u = 0$. Let ω be a symplectic form on M.

a). Show that M is orientable.

b). A vector field V on M is called a Liouville vector field with respect to ω if $L_V \omega = \omega$. Here L_V denotes the Lie derivative wit respect to V. Show that there isn't any Liouville vector field on M if M is a closed manifold.

3. Prove that there is no nonconstant continuous function on R which is periodic with respect to two periods 1 and π.

4. If M is a compact manifold with negative sectional curvature, then the isometry group of M is finite.

5. (E. Cartan) Let K be a compact Lie group acting by isometries on a simply connected, complete Riemannian manifold M of negative curvature. Then there is a common fixed point of all $k \in K$.

Yau Mathematical Competition 2018
Probability and Statistics Individual

Problem 1 (Probabability) Suppose that for each n, the $(n \times n)$ random matrix X^n has the uniform distribution on the orthogonal group $O(n)$.

(1) What is the distribution of the first row vector

$$X_1^n = (X_{11}^n, X_{12}^n, \dots X_{1n}^n)?$$

(2) Show that in distribution

$$X_{11}^n \sim \frac{Z_1}{\sqrt{\sum_{i=1}^n Z_i^2}},$$

where Z_i are independent, identically distributed random variables with the standard normal distribution.

(3) Find the limit in distribution of the random variables $\sqrt{n}X_{11}^n$ as $n \to \infty$.

Problem 2 (Statistics) Suppose we toss an unbiased coin and record K_1, the number of tosses needed to obtain the first head. Then, we draw X_1 from a normal distribution with mean $K_1\mu$ and variance $K_1\sigma^2$, and record the pair (K_1, X_1). By repeating the experiment $n-1$ times, we obtain the pairs $(K_2, X_2), \dots, (K_n, X_n)$. Using all the n data pairs $(K_1, X_1), \dots, (K_n, X_n)$:

(1) How would you best estimate μ and σ^2?

(2) Can you give a 95% confidence interval for μ?

Yau Mathematical Competition 2018
Probability and Statistics Overall

Problem 1 (Probability) Let X be a real valued random variable such that

$$\mathbb{E}f(X+1) = \mathbb{E}(Xf(X))$$

for all smooth function f with compact support.

(1) Show that X has a rapidly vanishing positive tail probability, i.e.,

$$\mathbb{P}\{X > N\} \le \frac{1}{N!}.$$

(2) Use (1) or otherwise to show that X has the standard Poisson distribution $P(1)$.

Problem 2 (Statistics) Let $X_1, \cdots, X_m$ be a random sample from a $N(\mu_1, \sigma_1^2)$ distribution and let $Y_1, \cdots, Y_n$ be another random sample from $N(\mu_2, \sigma_2^2)$. Further, assume that

(a) $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are independent;

(b) $m \ge 2$ and $n \ge 2$;

(c) μ_1 and μ_2 are real, and σ_1 and σ_2 are positive.

Finally, the parameters are all unknown.

(1) How to test $H_0 : \sigma_1^2 \le \sigma_2^2$ versus $H_1 : \sigma_1^2 > \sigma_2^2$?

(2) If we cannot assume that the distributions for the X_i's and Y_i's are normal, how would you test whether the two samples come from the same distribution?

Yau Mathematical Competition 2018
Probability and Statistics Team

Problem 1 (Probability) Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with the distribution $\mathbb{P}\{X_n = 1\} = \mathbb{P}\{X_n = -1\} = 1/2$. Define

$$Z = \sqrt{\frac{1}{2} + \frac{X_1}{2}\sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2} + \cdots}}}.$$

(1) Let

$$Z_N = \sqrt{\frac{1}{2} + \frac{X_1}{2}\sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2} + \cdots X_N\sqrt{\frac{1}{2}}}}}$$

be the random variable Z truncated at the nth step. Show that

$$Z_N = \sin\left(\frac{\pi}{4}\sum_{n=0}^{N}\frac{X_1X_2\cdots X_n}{2^n}\right).$$

(2) Let

$$Y_n = X_1X_2\cdots X_n, \quad n = 1, 2, \ldots.$$

What is the joint distribution of the random variables $\{Y_n\}$?

(3) Find the distribution function F_Z of the random variable Z.

Problem 2 (Statistics) For $n \geq 2$, let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be independent, identically distributed random vectors, with a common distribution which is bivariate normal with two component means μ_1 and μ_2 and the variance-covariance elements

$$\mathrm{var}(X_1) = \sigma_1^2, \quad \mathrm{var}(X_2) = \sigma_2^2, \quad \mathrm{cov}(X_1, X_2) = \rho\sigma_1\sigma_2.$$

We assume that σ_1 and σ_2 are both positive. Let $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)^T$.

(1) Assuming that the parameter θ is known, show that if one desires to predict Y_1 by using a function $g(X_1, \cdots, X_n)$ that minimizes $\mathbb{E}_\theta(Y_1 - g(X_1, \cdots, X_n))^2$, then the solution is given by

$$g(X_1, \cdots, X_n) = \beta_0 + \beta_1 X_1.$$

Provide expressions for β_0 and β_1 in terms of θ.

(2) Assuming that the parameter θ is unknown, how do you predict Y_1 and how do you measure the uncertainty of your prediction?

S.T. Yau College Student Mathematics Contests 2019
Algebra and Number Theory Individual

1. Consider the group $\mathrm{GL}(m,\mathbb{R})$. Given $g \in \mathrm{GL}(m,\mathbb{R})$, prove there is a decomposition of g as

$$g = k_1\, d\, k_2$$

where k_1 and k_2 are orthogonal matrices, and d is a diagonal matrix whose diagonal entries are positive.

2. (a) Let $P \in \mathbb{Q}[X]$ be a monic, irreducible polynomial of degree n.

- (i) Prove that its roots in $\mathbb{C}$ are simple.
- (ii) Prove there exists a matrix $M \in M_{n\times n}(\mathbb{Q})$ whose characteristic polynomial is P.

(b) Let $P \in \mathbb{Z}[X]$ be a monic polynomial of degree n. Prove there exists a matrix $M \in M_{n\times n}(\mathbb{Z})$ whose characteristic polynomial is P and which is diagonalizable over the field $\mathbb{C}$.

3. Prove that a group G of order 48 cannot be a simple group.

4. (a) Describe, in as simple terms as possible, the splitting field of the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

regarded as a matrix with entries in $\mathbb{Q}$. Is A semisimple?

(b) Same problem as (a), but with A now regarded as a matrix with entries in the field $\mathbb{F}_2 =^{\mathrm{def}} \mathbb{Z}/(2\mathbb{Z})$. How many elements does the splitting field have?

(c) Once more the same problem as (a), but over the field $\mathbb{F}_5 =^{\mathrm{def}} \mathbb{Z}/(5\mathbb{Z})$. Is A semisimple now?

(d) Recall the inductive definition of the sequence of the Fibonacci numbers $\{f_n\}$: $f_0 = 1$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. Argue that the sequence $\{\sqrt[n]{f_n}\}$ must have a limit as $n \to \infty$, and compute that limit.

(e) What does (c) tell you about the behavior of the Fibonacci numbers modulo 5, i.e., as a sequence of values in $\mathbb{F}_5$?

5. Let k an infinite field and take K to be an extension field of k. Let $A, B \in M_{n\times n}(k)$. Prove that if A and B are similar in $M_{n\times n}(K)$, then they are similar in $M_n(k)$.

S.T. Yau College Student Mathematics Contests 2019 Ma
Algebra and Number Theory Overall

1. This problem is meant to show for any prime number p, that there exists an irreducible polynomial of degree p with $(p-2)$ real roots and 2 (nonreal) complex conjugate roots. Let

$$P_0(X) = \prod_{k=1}^{p-2} (X+k)\,(X^2+1)$$

(a) Prove that for any integer k, that $P_k(X) = kp^2P_0(X) + X^p - p$ is an irreducible polynomial in $\mathbb{Z}[X]$

(b) Deduce from P_k a sequence of polynomials in $\mathbb{Q}[X]$ converging uniformly to P_0 on any compact subset of $\mathbb{C}$.

(c) Prove that for k large enough, P_k has two complex conjugates roots and $(p-2)$ real roots.

2. Let k be a field and take $L = k(x,y)$, where x is transcendental over k and $x^2+y^2=1$. Find the Galois group of L over k.

3. Let G be a finite group and $\rho : G \longrightarrow GL(V)$ a representation on a complex finite dimensional vector space V. Let $\rho^* : G \longrightarrow GL(V^*)$ be the dual representation.

Consider the symmetric algebra $S(V)$, viewed as polynomial functions on V^* For $l \in V^*$ and $p \in S(V)$, denote by p_l the fonction on G given by: $p_l(g) = p(g(l))$; this defines a function

$$\begin{aligned} S(V) &\xrightarrow{\phi_l} \mathbb{C}[G] \\ p &\longrightarrow p_l \end{aligned}$$

(a) Prove $\exists\, u \in V^*$ whose stabilizer is the neutral element of G.

(b) Prove that ϕ_u is surjective.

(c) Prove that for any faithful irreducible representation ρ of G, there exists an integer n such that ρ is equivalent to a subrepresentation de $S^n(V)$.

S.T. Yau College Student Mathematics Contests 2019
Algebra and Number Theory Team

1. (a) Let p be a (positive) prime integer, and $P \in \mathbb{Q}[X]$ an irreducible polynomial of degree p having two complex conjugate roots and $(p-2)$ real roots. Let K the subfield of $\mathbb{C}$ generated by the roots of P. Prove that K is a Galois extension of $\mathbb{Q}$, whose Galois group is the symmetric group on p elements.

(b) What is the Galois group of the polynomial $P(X) = X^5 - 6X + 3$?

2. Let K be a **nonalgebraically** closed field. Let $f_1, f_2, \ldots, f_m \in K[x_1, \ldots, x_n]$ and let $S \subset K^n$ be the set of solutions of the system of equations $f_1 = \cdots = f_m = 0$. Show that there exists a polynomial P such that S is the set of solutions of the equation $P = 0$.

3. (a) Let K be a field and $K[X]$ the ring of polynomials with coefficients in K. Define $v_0 : K[X] - \{0\} \to \mathbb{N}$ by the rule

$$v_0\Big(\sum_{0 \le k \le d} a_k X^k \Big) \;=\; \min\{\, k \mid a_k \neq 0 \,\} \; .$$

Fix a real number $C > 1$ (the particular choice will not matter). For $p \in K[X]$, define $\|p\|_0 = c^{-v_0(p)}$ if $p \neq 0$, and $\|0\|_0 = 0$. Show that

$$d_0(p,q) \;=\; \|p - q\|_0$$

defines an ultrametric on $K[X]$. Recall the notion of an ultrametric: a metric $d(\ ,\ )$ such that $d(p,r) \le \max\big(d(p,q)\,,\, d(q,r) \big)$ for all $p,\ q,\ r \in K[X]$.

(b) A **formal power series** with coefficients in K is a formal sum

$$\sum_{k=0}^{\infty} a_k\, X^k \quad , \quad a_k \in K \; .$$

The set $K[[X]]$ of all formal power series is a commutative ring with 1 under formal addition and multiplication of series. Note that every polynomial can be regarded as a formal power series, with only finitely many non-zero coefficients; thus $K[X] \subset K[[X]]$.

Show that v_0, $\|\ \|_0$, and d_0 extend naturally from $K[X]$ to $K[[X]]$ and identify $K[[X]]$ with the completion of $K[X]$: a metric space containing $K[X]$ whose metric agreeds with d_0 on $K[X]$, such that $K[X]$ is dense in the completion.

(c) Prove that $K[[X]]$, equipped with the ultrametric $d_0(\ ,\ )$, is a compact metric space, provided the field K is finite.

4. A integer n is said to be a **Congruent Number** if it is the area of a right triangle with each of the three sides rational numbers. For example, 6 is a congruent number since it is the area of the right triangle of sides length $(3, 4, 5)$.

Prove the following:

(a) $n \in \mathbb{N}$ is a congruent number if and only if there exist $m, a, b \in \mathbb{N}$ such that

$$nm^2 = ab(a+b)(a-b)$$

(b) For each of $r \in \{1, 2, 3, 5, 6, 7\}$, there exists infinitely many square free congruent numbers $n \equiv r \pmod 8$.

5. Let R be a commutative ring, and suppose

$$0 \to K \longrightarrow P \xrightarrow{f} M \to 0 \qquad \text{and} \qquad 0 \longrightarrow K' \longrightarrow P' \xrightarrow{f'} M \to 0$$

are short exact sequences with P and P' projective. Prove that $K \oplus P'$ is isomorphic to $K' \oplus P$.

Hint. With the indicated f and f', consider $Z = \{(p, p') \in P \oplus P'; f(p) = f'(p')\}$ and the natural maps from Z to P and P'.)

Question 1.

Let $M_n(\mathbf{C})$ be the algebra of all $n \times n$ matrices of complex entries. Suppose $E, F \in M_n(\mathbf{C})$ are projections, i.e., E, F are selfadjoint matrices so that $E = E^2$ and $F = F^2$. Assume that $||E - F|| \leq 1/2$, where $|| \cdot ||$ is the operator norm. Show that there is a unitary matrix U such that $UEU^* = F$.

Question 2. Let E be a measurable subset of R^d with $mes(E) < +\infty$, g is a measureable function on E.

(a) Suppse that for all $f \in L^1(E)$, there holds $f(x)g(x) \in L^1(E)$. Then $g \in L^\infty(E)$.

(b) Given $g \in L^\infty(E)$, we have

$$\|g\|_{L^\infty} = \sup_{\|f\|_{L^1}=1} \{| \int_E f(x)g(x)dx|\}$$

Question 3.

Let u be a harmonic function over $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$, prove there exist constants α, β such that for all r,

$$\frac{1}{2\pi} \int_{|z|=r} u d\theta = \alpha \log r + \beta.$$

Question 4.

Let $D = (0,1) \times (0, \infty)$. Find all the solutions:

$$\begin{cases} \Delta u &= 0 \text{ in } D \\ u &> 0 \text{ in } D \\ u &= 0 \text{ on } \partial D \end{cases}$$

Question 5. Consider Legendre polynomials (up to a multiple constant), i.e.,

$$P_0(x) = 1,$$
$$P_n(x) = \frac{d^n}{dx^n}\big((x^2 - 1)^n\big), \quad n \geq 1,$$

1) Show that P_n is orthogonal to all polynomials with degree less than n in the sense

$$\int_{-1}^{1} P_n(x)P(x)dx = 0 \quad \text{if deg} P < n.$$

2) Show that all roots of Legendre polynomial $P_n(x)$ are real, simple and lie in $(-1, 1)$.

3) Consider approximation of integral $\int_{-1}^1 f(x)dx \approx \sum_{i=1}^n w_i f(x_i)$. How to choose n points $x_1, x_2, \cdots, x_n$ on $(-1, 1)$ and the weights $w_1, w_2, \cdots, w_n$ such that the approximation of the above integral is exact for all polynomials with degree $\leq 2n - 1$?

4) Given a function, say $f(x) = \sin x, e^x$ or e^{-x^2}, how to find a degree 2 polynomial $P(x)$ such that $\int_{-1}^1 |f(x) - P(x)|^2 dx$ is minimized?

Question 1. If the restriction of $f(x, y)$, where $(x, y) \in \mathbb{R}^2$, to each line in the plane is continuous, is f necessarily continuous?

Question 2.

Prove: $\frac{\pi^2}{\sin^2 \pi z} = \Sigma_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

Question 3.

Let $D = \{(x, y) : x^2 + y^2 < 1\}$. Solve the equation

$$\begin{cases} \Delta u &= \dfrac{\cos(xy)}{\sqrt{x^2 + y^2}} \text{ in } D \\ u &= 0 \text{ on } \partial D, \end{cases}$$

Explain in what sense the equation holds and indicate where the equation holds in the classical sense.

Question 1. Suppose $f(x, y)$ is a non-smooth bounded convex function defined on $\mathbb{R}^1 \times [0, 1]$. Prove that f is independent of x.

Question 2.

Let $f : [0, 1] \to [0, 1]$ be the tent map given by

$$f(x) = \begin{cases} 2x, & \text{when } 0 \leq x \leq 1/2, \quad \textit{and} \\ 2(1-x), & \text{when } 1/2 < x \leq 1. \end{cases}$$

Prove that there is a point $x_0 \in (0, 1)$ such that the orbit of x_0 under f is dense in $[0, 1]$, i.e., the set $\{f^n(x_0)|n = 1, 2, 3, ...\}$ is a dense subset of $[0, 1]$.

Question 3.

Consider the Cauchy problem for the Burger's equation (B):

$$\begin{cases} \partial_t u + u\partial_x u = \epsilon \partial_x^2 u, x \in \mathbf{R}, t > 0 \\ u(x, t = 0) = u_0(x) \end{cases}$$

where $\epsilon \in (0, 1)$ is a constant, and $u_0(x)$ is a smooth periodic function of period 1.

(1) Prove that if $u^\epsilon(x, t)$ is a solution to (B), then $u^\epsilon(x, t)$ is uniformly bounded independent of ϵ.

(2) Prove that the solution $u^\epsilon(x, t)$ is periodic in x with period 1.

(3) Show that if $u^\epsilon(x, t)$ is a solution to (B), then $\partial_x u^\epsilon(x, t) \leq \frac{1}{t}$ for all $x \in \mathbf{R}, t > 0$.

(4) Show that total variation of $u^\epsilon(x, t)$ in x is uniformly bounded independent of ϵ for $t > 0$, i.e.

$$TV_{[0,1]} u^\epsilon(x, t) \leq C(t).$$

(5)(*) Discuss the convergence property of $u^\epsilon(x, t)$ as $\epsilon \to 0^+$.

Question 4.

Let $D^+ = \{(x, y) : x^2 + y^2 < 1, y > 0\}$. Find all the solutions:

$$\begin{cases} \Delta u &= 0 \text{ in } D^+ \\ u &> 0 \text{ in } D^+ \\ u &= 0 \text{ on } \{(x, y) : x^2 + y^2 = 1, y > 0\} \cup \{(x, 0) : |x| < 1\}. \end{cases}$$

PROBLEMS FOR PERSONAL CONTEST

CHOOSE ANY 3 OUT OF 5

Problem 1. Show that if

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

is a companion matrix with distinct eigenvalues $\lambda_1, ..., \lambda_n$, then

$$VCV^{-1} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$$

where

$$V = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

Problem 2. Prove the following error estimate for the Simpson's rule:

$$\int_a^b f(x)\,dx - \frac{b-a}{6}\Big(f(a) + 4f(\frac{a+b}{2}) + f(b)\Big) = -\Big(\frac{b-a}{2}\Big)^5 \frac{f^{(4)}(\xi)}{90}$$

where $a < \xi < b$.

Problem 3. Consider the fixed point iteration method to solve non-linear equation $f(x) = 0$

$$x_{n+1} = g(x_n). \tag{1}$$

a. State the necessary conditions for existence and uniqueness of a fixed-point $x = \alpha$ in (1), and derive the criteria that determines the order of convergence.

b. Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n}.$$

Show that if α is a fixed-point of $g(x)$, then it is also a fixed point of $G(x)$.

Problem 4. Consider the following parabolic equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

where $a > 0$ is a constant. Consider the following finite difference scheme

$$\frac{1}{12}\frac{v_{m+1}^{n+1} - v_{m+1}^{n}}{k} + \frac{5}{6}\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + \frac{1}{12}\frac{v_{m-1}^{n+1} - v_{m-1}^{n}}{k}$$
$$= a\frac{(\delta^2 v)_m^{n+1} + (\delta^2 v)_m^n}{2}$$

where δ^2 is the standard central difference operator in space, namely,

$$(\delta^2 v)_m^n = \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

and $k, h > 0$ are the time step size and the mesh size respectively. It is known that the scheme is second order in time and fourth order in space.
How to modify the above scheme for the equation

$$\frac{\partial u}{\partial t} = \sigma(x)\frac{\partial^2 u}{\partial x^2}$$

where $\sigma(x) \geq \sigma_0 > 0$, so that the resulting scheme is still second order in time and fourth order in space?

Problem 5. Let A and B be $n \times n$ matrices, A non-singular. Consider solving the linear system of equations

$$\begin{aligned} A\,\mathbf{x}_1 + B\,\mathbf{x}_2 &= \mathbf{b}_1 \\ B\,\mathbf{x}_1 + A\,\mathbf{x}_2 &= \mathbf{b}_2 \end{aligned}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^n$. Find the necessary and sufficient condition(s) for the convergence of the iterative method

$$\begin{aligned} A\,\mathbf{x}_1^{k+1} &= \mathbf{b}_1 - B\,\mathbf{x}_2^k \\ A\,\mathbf{x}_2^{k+1} &= \mathbf{b}_2 - B\,\mathbf{x}_1^k \end{aligned}$$

for some initial guess $\mathbf{x}_1^0$ and $\mathbf{x}_2^0$.

PROBLEMS FOR OVEALL CONTEST

Problem 1. Show that for any integer $n \geq 3$ there are infinitely many irreducible polynomials of the form

$$x^n + (6a-1)x^2 + (7b-3)x + 25c \in \mathbb{Z}[x]$$

for some $a, b, c \in \mathbb{Z}[x]$.

Problem 2. The matrix $\mathbf{A}$ is defined by $a_{ij} = 1$, when $i + j$ is even and $a_{ij} = 0$, when $i + j$ is odd. The order of the matrix is $2n$. Show that

$$\|\mathbf{A}\|_F = \|\mathbf{A}\|_\infty = n,$$

where $\|\mathbf{A}\|_F$ is the Frobenius norm, and that

$$\sum_{k=1}^{\infty} \left(\frac{1}{2n}\right)^k \mathbf{A}^k = \frac{1}{n}\mathbf{A}.$$

Problem 3. Let X and Y be two Hilbert spaces, with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, and the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Consider a bounded operator T mapping from X to Y, with its adjoint operator given by T^*. For any $\beta > 0$ and $z \in Y$, consider the minimization

$$\min_{f\in X} J(f) := \frac{1}{2}\|Tf - z\|_Y^2 + \frac{\beta}{2}\|f\|_X^2\,,$$

and write its minimizer f as $f(\beta)$, and its minimal value function as $F(\beta)$, i.e., $F(\beta) = J(f(\beta))$.

(1) Prove $f(\beta) \in X$ satisfies

$$(Tf,\ Tg)_Y + \beta\,(f,\ g)_X = (z,\ Tg)_Y \quad \text{for all} \quad g \in X.$$

(2) Prove the n-th derivative $w = f^{(n)}(\beta) \in X$ satisfies

$$(Tw, Tg)_Y + \beta\,(w,\ g)_X = -n\,(f^{(n-1)}(\beta),\ g)_X \quad \text{for all} \quad g \in X.$$

(3) Prove the first and second derivatives of $F(\beta)$ are given by

$$F'(\beta) = \frac{1}{2}\|f(\beta)\|_X^2, \quad F''(\beta) = (f(\beta), f'(\beta))_X\,.$$

(4) If $z \notin \ker T^*$, prove $F(\beta)$ is strictly monotonically increasing and strictly concave.

PROBLEMS FOR TEAM CONTEST

ANSWER ALL QUESTIONS

Problem 1. The following statement *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \le n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \le H, \quad 1 \le i \le m,\ 1 \le j \le n.$$

(i) Show that for $K \ge 0$ there are at most $(2K+1)^n$ vectors $\mathbf{x} \in \mathbb{Z}^n$ with

$$\|\mathbf{x}\|_\infty \le K,$$

where $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$.
(ii) Apply (i) and Dirichlet's pigeon hole principle to prove that there is an integer **non-zero** vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \le (2nH)^{n-1}.$$

Problem 2. Let $u(x), a(x)$ and $f(x)$ be smooth functions on $[0,1]$.
(i) Determine the order of accuracy of the following approximation

$$\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]\Bigg|_{x=x_i} \simeq \frac{(a_{i+1}+a_i)(u_{i+1}-u_i) - (a_i + a_{i-1})(u_i - u_{i-1})}{2h^2}$$

where $h = \frac{1}{m+1}$ is the mesh size, $x_i = ih$, $a_i = a(x_i)$, and $u_i = u(x_i)$ for $i = 1, \cdots, m$ such that $x_{m+1} = 1$.
(ii) For given functions $a(x) > 0$ and $f(x)$, one determines the function u that solves the following second order ordinary differential equation

$$u - \frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x)$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$. Apply the discretization given in (i) and let $f_i = f(x_i)$. Denote the linear system that one has to solve by $A\mathbf{u} = \mathbf{f}$ where $A \in \mathbb{R}^{m\times m}$ and $\mathbf{u}, \mathbf{f} \in \mathbb{R}^m$. If Gauss-Seidel method is used to solve this linear system, show that the iterative method converges for any initial guess.

Problem 3. *Maximal entropy principle.* Consider probability distributions on a discrete random variable X taking on possible values of $x_1, x_2, \ldots, x_n$. Denote the probability $\Pr(X = x_i) = p_i$, $i = 1, \ldots, n$, and recall that its Shannon entropy S is

$$S = -\sum_{i=1}^{n} p_i \log p_i$$

Now suppose we have some knowledge of $p_1, \ldots, p_n$, specified in terms of its expectation values E_j with respects to k known functions $f_j(\cdot)$ of the random variable X

$$\sum_{i=1}^{n} p_i f_j(x_i) = E_j, \qquad j = 1, 2, \ldots, k < n.$$

(i) Show that the probability distribution $p = [p_1, \ldots, p_n]$ that maximized the entropy S has the form of an exponential family:

$$p_i = \frac{e^{\sum_{j=1}^{k} \lambda_j f_j(x_i)}}{Z}$$

where λ_j are all constants, and Z is the normalization constant given by

$$Z = \sum_{i=1}^{n} e^{\sum_{j=1}^{k} \lambda_j f_j(x_i)}$$

(ii) Show that the constants λ_j are related to E_j by

$$E_j = \frac{\partial \log Z}{\partial \lambda_j}.$$

Geometry and Topology: Individual

(1) Let $M = \{(x^1, y^1, \cdots, x^n, y^n) \mid \sum_{i=1}^n (x^i)^2 = 1, \sum_{i=1}^n x^i y^i = 0\} \subset \mathbb{R}^{2n}$. Show that

(a) M is a smooth manifold and a vector bundle over the $(n-1)$ dimensional sphere. Compute the Euler class of this vector bundle.

(b) M is a symplectic manifold, i.e. there exists a non-degenerate closed 2-form on M.

(2) Let f be a smooth function on $\mathbb{R}^n$ that satisfies $|\nabla f| < 1$ and f vanishes at the origin, and let M be the graph of f in $\mathbb{R}^{n+1}$ with standard coordinates $x^1, \cdots, x^{n+1}$. Show that the function

$$g = -(x^{n+1})^2 + \sum_{i=1}^n (x^i)^2$$

restricts to a proper function on M, i.e for any $c > 0$, the intersection of $g^{-1}((-\infty, c])$ with M is always compact.

(3) Let X and Y be two compact Riemann surfaces with Euler characteristics $\chi(X)$ and $\chi(Y)$, respectively. Suppose $\chi(X) > \chi(Y)$, prove that there exists no non-trivial holomorphic map from X to Y.

(4) Show that a complete surface in $\mathbb{R}^3$ with finite area and negative curvature has at least four ends.

Geometry and Topology: Overall

(1) Show that a compact torus $T^n = (S^1)^n$ cannot admit a Riemannian metric of negative sectional curvature.

(2) Let M be an n-dimensional closed submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Prove the following inequality
$$\int_M H^n dV \geq vol(S^n),$$
where H and dV are the mean curvature and the volume element of M, respectively, and S^n is the standard unit sphere of dimension n.

(3) Compute the index of a closed geodesic of length 4π on the standard unit 2-sphere.

Geometry and Topology: Team

(1) Let M be a surface in $\mathbb{R}^3$. Suppose for each point of M, there exist two families of geodesics which intersect at a constant angle. Prove that M has constant zero Gaussian curvature.

(2) Prove that two closed minimal hypersurfaces in S^n must intersect each other.

(3) Let M be a compact closed hypersurface in $\mathbb{R}^{n+1}$. Prove that M is an n-sphere if M has constant mean curvature and nonnegative Ricci curvature.

(4) Let $X \in \mathfrak{k} := Lie(K)$ be a real vector field on a compact connected smooth manifold M with an effective action of a compact real Lie group K. By choosing a K-invariant real symplectic form ω on M, assume $f \in C^\infty(M)_\mathbb{R}$ is such that

$$df = i_X \omega$$

Show that the value $\max_M f - \min_M f$ is independent of the choice of ω as far as ω defines the same de Rham cohomology class $[\omega]$.

Statistics problems: individual contest

May 2019

1. Suppose we have n pairs of observations $(X_i, Y_i),\ i = 1, \ldots, n$. Suppose we fit a simple linear regression with Y as the response variable and the value of the regression coefficient estimator is 1. What happens if the role of X and Y are switched, i.e., we fit a simple regression with X as the response variable?

2. A math test consists of 10 questions. For each question, one either answers it correctly ($Y = 1$) or incorrectly ($Y = 0$). Thus for a test taker, his/her answers consist of $Y_1, \ldots, Y_{10}$, where Y_i is the answer to the ith question and takes value 1 or 0. Suppose a reasonable statistical model is that for each student, his/her responses to the 10 questions are independent Bernoulli variables with the following specification:

$$P(Y_i = 1) = 1 - P(Y_i = 0) = \frac{e^{\theta - b_i}}{1 + e^{\theta - b_i}}, \quad i = 1, \ldots, 10,$$

where θ is his/her math ability (different students have different θ values) and b_i is the difficulty level for the ith question. The test is designed, of course, to find out the test taker's θ value. This model implies that a person with higher θ value has a larger probability to answer a question correctly, while a more difficult question (larger b value) make the probability of a correct answer smaller. The teacher allocates 10 points equally to each of the 10 questions for the total of 100 points for the test.

Suppose that student A answered two easiest questions (2 smallest b_i values) incorrectly thus scoring 80 out of 100 and that student B answered two most difficult questions (2 largest b_i values) incorrectly thus also scoring 80 out of 100. Student A claims that it is unfair to him (in comparison to student B) because his 8 correct answers are on the more difficult questions. And more difficult questions should worth more points. Do you think student A has a valid point? Do you think the teacher's scoring system is fair? Explain your thinking from the statistical perspective.

3. Two research centers, A and B, collected two separate data sets to study relationship between two variables X and Y. Center A looked at its data, denoted by $(X_1, Y_1), \ldots, (X_m, Y_m)$, and found a positive correlation. Center B also looked at its own data, denoted by $(X_{m+1}, Y_{m+1}), \ldots, (X_{m+n}, Y_{m+n})$, and also found a positive correlation. Now a new researcher pooled the two data set together into a larger one, $(X_1, Y_1), \ldots, (X_{m+n}, Y_{m+n})$. He claims that for the pooled data set, X and Y are negatively correlated. Do you think this is possible? Explain your answer.

Probability problems: individual contest

May 2019

1. Let $p \geq 1$ and $f, g \in L^p[0,1]$ such that $\int_0^1 g(y)dy = 0$. Show that

$$\int_0^1 \int_0^1 |f(x) + g(y)|^p dxdy \geqslant \int_0^1 |f(x)|^p dx.$$

2. Consider an urn with p plus balls and m minus balls in it, where m and p are given nonnegative numbers. You are allowed to pick a random ball from the urn or quit the game. If you decide to pick and get a plus ball you gain a dollar; if you get a minus ball you lose a dollar. You can continue the game but picked balls are not replaced. Denote by $V(m,p)$ the expected value of playing the game. Find a recurrence for $V(m,p)$

3.

3-1 Let X_n be increasing, integrable random variables and converges a.s. to $X \in L^1$, show that, for any sigma algebra $\mathcal{G}$, $\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G})$.

3-2 Let $X_n \geq 0$, Show that

$$\liminf_n \mathrm{E}(X_n|\mathcal{G}) \geq \mathrm{E}(\liminf_n X_n|\mathcal{G}).$$

3-3 Let X_n be random variables in L^1 and $X_n \to X$ a.s. with $|X_n| \leq Z$ in L^1. Show that

$$\mathrm{E}(X|\mathcal{G}) = \lim \mathrm{E}(X_n|\mathcal{G}) \text{ a.s. and in } L^1.$$

3-4 Show that, if f convex and $X, f(X)$ integrable, then

$$f(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(f(X|\mathcal{G})).$$

3-5 Show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent i.f.f. for all X $\mathcal{G}_2$-mesurable, $\mathrm{E}(X|\mathcal{G}_1) = \mathrm{E}(X)$.

Statistics problems: personal-overall

May 2019

1. Consider the multiple linear regression model

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \beta + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix},$$

where $Y_1 \in \mathbb{R}^{n_1}$, $Y_2 \in \mathbb{R}^{n_2}$, X_1 is a $(n_1 \times p)$ matrix, X_2 is a $(n_2 \times p)$ matrix, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ has rank p, and error terms ϵ_1 and ϵ_2 are independent of each other with

$$\epsilon_1 \sim N_{n_1}(0, \sigma^2 I_{n_1}) \text{ and } \epsilon_2 \sim N_{n_2}(0, \rho\sigma^2 I_{n_2}) \qquad (\rho > 0).$$

The unknown parameters are $\beta \in \mathbb{R}^p$ and σ^2.

1. Treat ρ as a constant and driver the maximum likelihood estimates of β and σ^2, denoted by $\hat{\beta}_\rho$ and $\hat{\sigma}^2_\rho$.
2. Suppose that X_1 has full rank p and X_2 has full rank $n_2 < p$. Prove that as ρ goes to zero, $\hat{\beta}_\rho$ converges to
$$\hat{\beta} + \left(X_1'X_1\right)^{-1} X_2' \left[X_2 \left(X_1'X_1\right)^{-1} X_2'\right]^{-1} \left(Y_2 - X_2\hat{\beta}\right)$$
where $\hat{\beta} = (X_1'X_1)^{-1}X_1'Y_1$.
3. Interpret the above limit in some context of multiple linear regression with constraints on β.

2. Consider the simple linear regression

$$Y_i = \alpha + \beta X_i + \epsilon_i, \qquad i = 1, \ldots, n.$$

Define quadratic function

$$Q(\alpha, \beta) = \sum_{i=1}^{n} (Y_i - \alpha - \beta X_i)^2.$$

Let $\hat{\alpha}$ and $\hat{\beta}$ be the estimators of α and β, which minimizes $Q(\alpha, \beta)$. Let $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$.
1. Find the gradient vector of $\hat{Y}_i$ with respect to the vector $Y = (Y_1 \ldots, Y_n)'$.
2. The degree of freedom, d_{LM} of the fitted model is defined to be the trace of matrix $\frac{\partial \hat{Y}}{\partial Y}$, where $\hat{Y} = (\hat{Y}_1 \ldots, \hat{Y}_n)'$. Find d_{LM}. How is it related to the model?

Probability problems: personal-overall

May 2019

1. Let $\{S_n\}$ be a simple random walk in one dimension, with $S_0 = 0$, and let

$$\tau = \tau_{[0,5]^c} = \inf\{n : S_n \notin [0, 5]\}$$

be the first time the random walk exits the set $\{0, 1, 2, 3, 4, 5\}$. Evaluate $\mathbb{E}[S_{\tau-1}]$ (Hint: use Wald's identity)

2. Prove that for any two events,

$$\Big|\mathbf{P}\{A \cap B\} - \mathbf{P}\{A\}\mathbf{P}\{B\}\Big| \leq \frac{1}{4}$$

and

$$\mathbf{P}\{A \cup B\}\mathbf{P}\{A \cap B\} \leq \mathbf{P}\{A\}\mathbf{P}\{B\}.$$

Statistics problems: team contest

May 2019

1. Consider the simple linear regression

$$Y_i = \alpha + \beta X_i + \epsilon_i, \qquad i = 1, \ldots, 2n.$$

(i) Suppose that only the first half of Y's are observed, i.e. only Y_i, $i = 1, \ldots, n$ are observed and Y_i, $i = n+1, \ldots, 2n$ are missing, while all X's are observed. What would you suggest for the estimation of β? Are there assumptions you need to make for your estimator to be valid?

(ii) Suppose for the second half of Y's, their absolute values are observed, i.e. we observe $|Y_i|$, $i = n+1, \ldots, 2n$. What would you suggest for the estimation of β? Are there assumptions you need to make for your estimator to be valid? Any good properties for your estimator?

2. Suppose that $X = (X_1, \ldots, X_n)'$ is an observation from the n-dimensional multivariate normal distribution $N_n(\theta, I)$ with unknown parameter $\theta \in R^n$, that is, X_i's are independent of each other with $X_i \sim N(\theta_i, 1)$ for $i = 1, \ldots, n$. (i). Derive the maximum likelihood estimator (MLE) of $\|\theta\|^2 = \sum_{i=1}^n \theta_i^2$.

(ii). Show that the MLE is a biased estimator.

(iii). Find the distribution of the MLE and describe how to use this distribution to construct exact confidence intervals.

3. Suppose that $X_1, \cdots, X_n$ is a sample of size n from the Student-t distribution $t_\nu(\mu, 1)$ with known degrees of freedom $\nu \geq 1$, unit scale, and known center μ. The Student-t distribution $t_\nu(\mu, 1)$ has density function of the form

$$f_X(x; \mu, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2})} \left(1 + \frac{(x-\mu)^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad (x \in \mathcal{R}^1).$$

(i). Write $X_i = \mu + U_i$, where U_i are independently and identically distributed (iid) with $t_\nu(0, 1)$ for $i = 1, \cdots, n$. Find the conditional distribution of U_1 given $U_i - U_1 = X_i - X_1$ for $i = 2, \cdots, n$.

(ii). Describe a method to construct confidence intervals by making use of the above result, and argue for its efficiency and coverage probability.

(iii). Use the limiting case of $\nu \to \infty$ to verify your answers.

Probability problems: team contest

May 2019

1. Let Z be a real random variable, the log Laplace transform of Z is the function taking values in $\mathbb{R} \cup \{\infty\}$ defined for $\lambda \in \mathbb{R}$ by:

$$\Psi_Z(\lambda) = \ln \mathrm{E}(\exp(\lambda Z)).$$

1-1 For Z a random variable taking value in a bounded interval I, derive twice Ψ_Z and identify, for all λ, this second derivative to the variance of a random variable Z_λ, which we precise its distribution.

1-2 If Z is a random variable taking value in a bounded interval I, show that the variance of Z is upper bounded by $|I|^2/4$.

1-3 If Z is a random variable taking value in a bounded interval I. Show that for all $\lambda \in \mathbb{R}^+$,

$$\Psi_Z(\lambda) \leq \frac{|I|^2\lambda^2}{8}.$$

1-4 Let $Z_1, \ldots, Z_n$ be intendant random variables, such that Z_i take value in $[a_i, b_i]$; denote $Z = \sum Z_i$ and $\tilde{Z} = Z - \mathrm{E}(Z)$. Show that

$$\Psi_{\tilde{Z}}(\lambda) \leq \frac{\lambda^2}{8}\sum(b_i - a_i)^2.$$

1-5 With same notations as in 1-4, show that for all $\epsilon > 0$,

$$\mathbf{P}(|\tilde{Z}| \geq \epsilon) \leq 2\exp(-\frac{2\epsilon^2}{\sum(b_i - a_i)^2}).$$

2. Let $\{X_n\}$ be independent indentically distributed random variables with finite mean, and $S_n = \sum_{i=1}^n X_i$ be the partial sum. Show that the sequence $\{S_n/n\}$ is a reverse martingale. This means that

$$\mathbb{E}\left[\frac{S_n}{n}\middle|\mathscr{F}_{n+1}\right] = \frac{S_{n+1}}{n+1},$$

where

$$\mathscr{F}_n = \sigma\{S_n, S_{n+1}, \cdots\}$$

is the smallest σ-algebra generated by the random variables S_k for $k \geq n$.

3. Let $n \geq 2$ be an integer, show that, a random vector $X \in \mathbb{R}^n$ having independent components and its distribution is $O(n)$ invariant i.f.f. $X \sim \mathcal{N}(0, \sigma^2)$.
(Distribution is $O(n)$ invariant means that for any orthogonal matrix, that is a rotation, $P \in O(n)$, X and PX have the same distribution.)

Problem 1. Let R be the subring of $\mathbb{C}[x]$ consisting of all polynomials $f(x) \in \mathbb{C}[x]$ such that $f'(0) = 0$; i.e, the derivative of f at 0 is 0. Is R a finitely generated ring over $\mathbb{C}$? If yes, find an isomorphism from R to some quotient of a polynomial ring over $\mathbb{C}$ (with finitely many indeterminates). If no, justify your answer.

Problem 2. In this problem K is either the field of real numbers or the field of complex numbers. Let $\mathcal{N}$ denote the set of nilpotents in $M_{n\times n}(K)$, the set of $n \times n$ matrices with entries in K, and $\mathcal{U}$ the set of unipotents – i.e., matrices A such that $A - 1_{n\times n}$ is nilpotent. Show that the exponential map

$$\exp : M_{n\times n}(K) \longrightarrow M_{n\times n}(K) , \quad \exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

maps $\mathcal{N}$ one-to-one onto $\mathcal{U}$. Can you describe the inverse of this map?

2b) What can you say about the exponential map from $M_{n\times n}(K)$ to itself? What is its image? Is it invertible in any sense?

Problem 3. Let p be a prime, $\mathbb{F}_p$ the prime field of p elements, and ζ_p a primitive p-th root of unity in $\mathbb{C}$. For a positive integer d, define the algebraic integer

$$G_d = \sum_{x\in\mathbb{F}_p} \zeta_p^{x^d}.$$

Prove that the degree over $\mathbb{Q}$ of the algebraic integer G_d is equal to $(d, p-1)$.

Problem 1. Let $d \in \mathbb{N}$ and $\zeta := e^{\frac{2\pi\sqrt{-1}}{d}}$ be a d-th primitive root of unity. Let A be the $(d-1) \times (d-1)$ matrix whose (i, j) entry is $A_{ij} = \zeta^{ij} - \zeta^{(i-1)j}$. Show that $\det(A)^2 \in \mathbb{Z}$. For $d \equiv 0, 3 \pmod 4$ show that $\det(A) \notin \mathbb{Z}$.

Problem 2. Let k be any field of characteristic not equal to 2. Let M be an $n \times n$ orthogonal matrix over k; that is, the coefficients of M are in k, and $MM^t = I_n$. Assume that n is odd. Prove that M has an eigenvalue equal to $\det(M)$.

Problem 3. Let $G \subset GL_n(\mathbb{Z})$ a finite subgroup. Prove that there is a constant c_n depending only on n such that the order of G satisfies: $|G| \leq c_n$.

Overall Exam Problems for Algebra, Number Theory and Combinatorics

1. Show that any finite abelian group is the Galois group of some field extension of $\mathbb{Q}$. Give an example of a finite extension of $\mathbb{Q}$ with non-abelian Galois group.

2. Let $p > 2$ be a prime number and F_p the finite field with p elements.
i) Determine the order of the group $\mathrm{SL}_2(F_p)$.
ii) How many Sylow p-subgroups are there in $\mathrm{SL}_2(F_p)$?
iii) Show that there exists an element $\mu \in F_p$ which is not a square. For $a, b \in F_p$ with $a^2 - b^2\mu = 1$, consider the following element in $\mathrm{SL}_2(F_p)$:

$$M_{a,b} := \begin{pmatrix} a & b\mu \\ b & a \end{pmatrix}.$$

Show that $M_{a,b}$ is conjugate to $M_{a',b'}$ in $\mathrm{SL}_2(F_p)$ if and only if $a = a'$ and $b = \pm b'$.
iv) How many conjugacy classes of the form $M_{a,b}$ are there in $\mathrm{SL}_2(F_p)$?

S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Individual-II

Please solve the following problems.

1. (a) Assume that the function $f : [a, b] \to \mathcal{R}$ with $a < b$ is differentiable and satisfies $|f'(t)| \geq \beta$ for all $t \in [a, b]$ for some $\beta > 0$. Prove the following estimate

$$m\{t \in [a, b] : |f(t)| \leq \varepsilon\} \leq \frac{2\varepsilon}{\beta} \quad \text{for } \varepsilon > 0,$$

where $m\{B\}$ denotes the Lebesgue measure of set B.

(b) Assume that the function $f : [a, b] \to \mathcal{R}$ with $a < b$ is q-times continuously differentiable and satisfies $|f^{(q)}(t)| \geq \beta$ for all $t \in [a, b]$ for some positive integer q and $\beta > 0$. Prove the following estimate

$$m\{t \in [a, b] : |f(t)| \leq \varepsilon\} \leq 4\left(q!\frac{\varepsilon}{2\beta}\right)^{\frac{1}{q}} \quad \text{for } \varepsilon > 0,$$

where $m\{B\}$ denotes the Lebesgue measure of set B.

2. Suppose that $E_p(z) = (1 - z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$, $p \in \mathbb{N}$. Then prove that:

$$|1 - E_p(z)| \leq |z|^{p+1}, \ \ |z| \leq 1.$$

3. Solve the following ordinary differential equation by elementary integration

$$x\frac{dy}{dx} = \sqrt{x^6 - y^2} + 3y.$$

S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Individual-I

Please solve the following problems.

1. (a) Assume that f is a non-decreasing function in $[0,1]$. Show that f' is integrable and that $\int_0^1 f'(x)dx \leq f(1) - f(0)$. (b) Let $f_n(x)$ be a sequence of non-decreasing functions in $[0,1]$, and $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [0,1]$. Show that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere.

2. Suppose that $\Omega \subseteq \mathbb{C}$ is a domain and the unit disk $\overline{\mathbb{D}} \subseteq \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f(0) = 0$, and $|f(e^{i\theta})| \geq 3$ for any $\theta \in \mathbb{R}$.

If $\lambda_1, \cdots, \lambda_N$ are all zeroes (counting multiplicities) of $1 - f(z)$ in $\mathbb{D}$, then we have

$$|\lambda_1 \cdots \lambda_N| < \frac{1}{2}.$$

3. Let u satisfy

$$\sum_{i,j=1}^{n} a_{ij} u_{ij} - u \geq 0 \text{ in } \mathbb{R}^n$$

where $(a_{ij}) > 0$ is a bounded and positive definite matrix. Furthermore assume that

$$u(x) \leq 2020 + |x|^{2020}.$$

Show that

$$u(x) \leq 0 \text{ in } \mathbb{R}^n.$$

S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Individual Overall

Please solve the following problems.

1. Construct real numbers through decimal numbers. Prove that a bounded increasing sequence of decimal numbers has a limit of decimal number.

2. Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$ centered at the origin. Assume that the function $f \in C^2(B_1(0))$. Prove that
1) If f satisfies

$$\sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial_j x_j} = 0$$

on $B_1(0)$, and $\nabla f(0) = 0$, then f is constant in $B_1(0)$.

2) If f satisfies

$$x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} = 0, i, j = 1, \cdots, n$$

on $B_1(0)$, then f is constant on the sphere $\{x : x \in B_1(0), |x| = 1/2\}$.

Question I

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $q_1 \in \mathbb{R}^n$ satisfy $\|q_1\|_2 = 1$. Consider the following iteration:

$$
\begin{aligned}
& r_0 = q_1, \quad \beta_0 = 1, \quad q_0 = 0, \quad k := 0 \\
& \textbf{while } \beta_k \neq 0 \\
& \quad q_{k+1} := r_k/\beta_k \\
& \quad k := k+1 \\
& \quad \alpha_k := q_k^T A q_k \\
& \quad r_k := (A - \alpha_k I)q_k - \beta_{k-1}q_{k-1} \\
& \quad \beta_k := \|r_k\|_2 \\
& \textbf{end}
\end{aligned}
$$

Let $K_k = \text{span}\{q_1, Aq_1, \cdots, A^{k-1}q_1\}$ and $Q_k = [q_1 \cdots q_k]$ be a matrix whose columns are $\{q_i\}_{i=1}^k$ obtained from the above iteration.

1. Assume that the iteration does not terminate. Show that Q_k has orthonormal columns, and that they span K_k.
2. What is the purpose of this algorithm? Justify your answer.

Question II

Consider the oscillatory second order ordinary differential equation (ODE)

$$y''(t) + \lambda^2 y(t) + g(y(t)) = 0, \qquad 0 < t \leq T,$$

with the initial data

$$y(0) = \alpha, \qquad y'(0) = \beta,$$

where $\lambda >> 1$, α and β are given constants, and $g(y)$ is a given Lipschitz continuous function.

Choose a time step $\tau > 0$ and denote $t_n = n\tau$ for $n \geq 0$. Let y^n be the numerical approximation of $y(t_n)$ for $n \geq 0$.

1. Re-write the above ODE into its equivalent integral formulation near $t = t_n$ with $t = t_n + s$ for $s \in \mathbb{R}$ via the variation-of-constant formula.
2. Based on the integral formulation, design the following time integrator via proper numerical quadratures

$$y^{n+1} = 2\cos(\lambda\tau)y^n - y^{n-1} - \frac{\sin(\lambda\tau)}{\lambda}g(y^n), \qquad n \geq 1,$$

with

$$y^0 = \alpha, \qquad y^1 = \alpha\cos(\lambda\tau) + \frac{\beta}{\lambda}\sin(\lambda\tau) - \frac{\sin(\lambda\tau)}{2\lambda}g(\alpha).$$

Under proper stability assumption, prove the following error bound

$$|y(t_n) - y^n| \le C\tau^2, \qquad 0 \le n \le \frac{T}{\tau},$$

where $C > 0$ is a constant independent of τ.

Question III

Consider an energy functional for $\rho \in \mathcal{P}(\mathbb{R})$ (probability distribution on the real line), given by

$$F[\rho] = \int_{\mathbb{R}\times\mathbb{R}} \rho(x)K(x-y)\rho(y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} \rho(x)(1-\rho(x))\,\mathrm{d}x + \int_{\mathbb{R}} \rho(x)\ln\rho(x)\,\mathrm{d}x,$$

where $K : \mathbb{R} \to \mathbb{R}_+$ is a given kernel.

1. Write down explicitly the Euler-Lagrange equation corresponding to F.
2. Consider the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x,t) = \nabla \cdot \Big(\rho(x,t)\nabla\frac{\delta F}{\delta\rho}\left[\rho(x,t)\right]\Big).$$

   Show that $F[\rho(x,t)]$ is decreasing in t.

Question I

Let A be an $m \times n$ matrix and its singular value decomposition (SVD) be

$$V^T A U = F = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_r & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

where the matrices U and V are orthogonal, and the singular values μ_i satisfy

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0.$$

Define the $n \times m$ matrix

$$F^+ = \begin{bmatrix} \mu_1^{-1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mu_2^{-1} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_r^{-1} & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A^+ = U F^+ V^T.$$

The matrix A^+ is called as the *generalized inverse* of A.

1. Show that $x^* = A^+ b$ is the least square solution of the linear system $Ax = b$.
2. Show that
$$\lim_{\alpha \to 0^+} (\alpha I + A^T A)^{-1} A^T = A^+,$$
where $\alpha > 0$.

Question II

Consider the following finite difference scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2\tau} + a\Big(1 + \frac{h^2}{6}\delta^2\Big)^{-1} \delta_0 v_m^n = f_m^n$$

for the transport equation $u_t + a u_x = f$, where a is a constant, $\tau > 0$ is the time step, δ_0 denotes the standard second order central difference operator for u_x, and δ^2

denotes the standard second order central difference operator for u_{xx}. Assume that $\lambda = \tau/h$ is a constant with h being the spatial mesh size. Show that the scheme is stable if and only if

$$|a\lambda| < \frac{1}{\sqrt{3}}.$$

How do you modify the scheme when a is not a constant without affecting the accuracy of the scheme? Justify your answer.

Question III

Show that the following system of ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= 0.5x + 2.5y - x(x^2 + y^2),\\ \frac{dy}{dt} &= -0.5x + 1.5y - y(x^2 + y^2).\end{aligned}$$

has at least one periodic solution.

Question I
Consider the following gradient flow with discrete normalization (GFDN)

$$
\begin{aligned}
&\partial_t u(x,t) = \left[\partial_{xx} - V(x) - \beta|u|^2\right] u, \quad 0 < x < 1,\ t_n \le t < t_{n+1},\ n \ge 0, \\
&u(x, t_{n+1}) = u(x, t_{n+1}^+) := \frac{u(x, t_{n+1}^-)}{\|u(\cdot, t_{n+1}^-)\|_{L^2}}, \qquad 0 \le x \le 1, \quad n \ge 0, \\
&u(x,0) = g_0(x), \qquad 0 \le x \le 1, \\
&u(0,t) = u(1,t) = 0, \qquad t \ge 0,
\end{aligned}
$$

where $u = u(x,t)$ is a real-valued function, $V(x) \ge 0$ for $0 \le x \le 1$ is a given function, $\beta \ge 0$ is a given constant, $t_n = n\tau$ for $n = 0, 1, 2, \ldots$ with $\tau > 0$ being the time step, $u(x, t_n^{\pm}) := \lim_{t \to t_n^{\pm}} u(x,t)$, $\|u\|_{L^2}^2 = \int_0^1 |u(x)|^2\, dx$ and $g_0(x)$ is a given function satisfying $\|g_0\|_{L^2} = 1$. Define the mass and energy as

$$
\begin{aligned}
M(t) &:= M(u(\cdot,t)) = \int_0^1 |u(x,t)|^2\, dx, \qquad t \ge 0, \\
E(t) &:= E(u(\cdot,t)) = \int_0^1 \left[|\partial_x u(x,t)|^2 + V(x)|u(x,t)|^2 + \frac{\beta}{2}|u(x,t)|^4\right] dx.
\end{aligned}
$$

1. Show that the mass and energy are diminishing in each time interval $[t_n, t_{n+1})$, i.e.

$$
M(t_2) \le M(t_1), \qquad E(t_2) \le E(t_1), \qquad t_n \le t_1 \le t_2 < t_{n+1}, \quad n \ge 0.
$$

2. When $\beta = 0$, show that

$$
E(g_0) = E(u(x,t_0)) \ge E(u(x,t_1)) \ge \cdots \ge E(u(x,t_n)) \ge \cdots, \qquad n \ge 0,
$$

for any given time step $\tau > 0$ and initial data $g_0(x)$.

3. Let $\tau \to 0^+$ in the problem GFDN, what partial differential equations can you get? Show that the limiting equation is mass conservative and energy diminishing.

Question II
Define a dynamical system

$$
\frac{dy_t}{dt} = -A^T(Ax_t - b), \tag{1a}
$$

$$
y_t \in \partial\psi(x_t), \tag{1b}
$$

where $A \in \mathbb{R}^{n \times k}$ satisfies that A^TA has smallest eigenvalue $\gamma > 0$, $b = Ax^*$, and $\psi(x) = \|x\|_1 + \frac{\|x\|_2^2}{2\alpha}$ has its subgradient set $\partial\psi(x)$ at $x \in \mathbb{R}^k$, where $\alpha > 0$ is a given constant. For a convex function $\psi : \mathbb{R}^k \to \mathbb{R}$, define the *Bregman divergence* function associated with ψ, $D_\psi : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, by

$$
D_\psi(x', x) := \psi(x') - \psi(x) - \langle \partial\psi(x), x' - x \rangle. \tag{2}
$$

1. Show that (1) leads to the following ODE

$$\frac{dD_\psi(x^*, x_t)}{dt} = -\|A(x_t - x^*)\|_2^2. \tag{3}$$

2. For x_t in (1), define

$$\tau := \inf\{t > 0 : \mathrm{sign}(x_t) = \mathrm{sign}(x^*)\}. \tag{4}$$

   Denote by $\underline{x}^*$ the smallest nonzero magnitude of $|x_i|$ ($i = 1, \ldots, k$). Find an upper bound of τ, as tight as possible.

Geometry and Topology
For morning of October 24th

Problem 1 *Let Ω be a domain in $\mathbb{R}^n$, containing the ball D_r of radius r with center at the origin. Consider $u : \Omega \to \mathbb{R}$ satisfying the minimal graph equation:*

$$div\left(\frac{\bigtriangledown u}{\sqrt{1+\left|\bigtriangledown u\right|^2}}\right) = 0.$$

Namely, $\Gamma_u = \{(x, u(x))\,|x \in \Omega\}$ is a minimal graph in $\mathbb{R}^{n+1}$. Let $B_r \subset \mathbb{R}^{n+1}$ be the ball of radius r centered at the origin. Show that

$$Vol\left(B_r \cap \Gamma_u\right) \leq \frac{Vol\left(S^n\right)}{2} r^n$$

where S^n is the unit hypersphere of $\mathbb{R}^{n+1}$.

Problem 2 *Show that $\int_\Sigma H^2 \geq 16\pi$ for any closed embedded surface Σ in $\mathbb{R}^3$. When does equality hold? (Here H is the mean curvature of the surface Σ, namely H is the sum of principal curvatures).*

Problem 3 *Let I be the interval $[0,1]$. For a topological space B, say homeomorphisms $g_0, g_1 : B \to B$ are isotopic if they are homotopic via a homotopy $G : B \times I \to B \times I$ with each $G_t : B \to B$ defined by $G_t(b) = G(b,t)$ also a homeomorphism.*

(a) Show that any order-preserving homeomorphism $f : I \to I$ is isotopic to the identity.

(b) Show that any homeomorphism $f : S^1 \to S^1$ of the unit circle is isotopic to the identity or the reflection along the x-axis.

Geometry and Topology
For morning of October 24th

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(b) Show that any homeomorphism $f : S^1 \to S^1$ of the unit circle is isotopic to the identity or the reflection along the x-axis.

Geometry and Topology
For morning of October 25th

Problem 1 *Consider the mean curvatur flow (MCF for short) $f : M \times I \to \mathbb{R}^n$ of a hypersurface M in $\mathbb{R}^n$ where $f_t = f(\cdot, t) : M \to \mathbb{R}^n$. This means*

$$\left(\frac{\partial f}{\partial t}\right)^{\perp} = H(t)$$

where $^{\perp}$ denotes the normal component, and $H(t)$ is the mean curvature vector of $f_t(M)$. In particular when $\frac{\partial f}{\partial t} = T$, a constant vector, M_t is called a translating soliton, which is just a parallel transport of M in the direction T.

(1) When $n = 2$, show that the MCF of the grim reaper γ in $\mathbb{R}^2$

$$y = -\log \cos x$$

is the translating soliton with $T = (0, 1)$.

(2) Show that $M = \gamma \times \mathbb{R}^k$ is a translating soliton in $\mathbb{R}^{k+2}$.

Problem 2 *Let (M^n, g) be a closed, orientable n-dimensional Riemannian manifold with positive Ricci curvature.*

(a) Prove that the first Betti number b_1 of M vanishes.

(b) Suppose $Ric_M \geq (n-1)\kappa > 0$, show that $\lambda_1 \geq n\kappa$, where λ_1 is the first eigenvalue of the Laplace-Beltrami operator Δ on (M, g).

Problem 3 *Let I be the interval $[0, 1]$. For a topological space B, say homeomorphisms $g_0, g_1 : B \to B$ are isotopic if they are homotopic via a homotopy $G : B \times I \to B$ with each $G_t : B \to B$ defined by $G_t(b) = G(b, t)$ also a homeomorphism.*

(1) Show that any orientation-preserving homeomorphism $f : D^2 \to D^2$ of the closed unit disc is isotopic to a homeomorphism which is the identity on the boundary S^1.

(You can use the fact that any orientation-preserving homeomorphim $\phi : S^1 \to S^1$ is isotopic to the identity).

(2) Show that a homeomorphism $f : D^2 \to D^2$ of the unit disc is isotopic to the identity or the reflection along the x-axis.

Yau College Math Competition 2020 Final Probability and Statistics

Individual Exam Problem Set 1 (Saturday, October 24, 2020)

PROBLEM 1. Suppose that $\{X_n\}$ is a sequence of independent, identically distributed random variables with the uniform distribution on the unit interval $[0,1]$. For each $x \in [0,1]$, define

$$X_n^x = \begin{cases} 1, & X_n \leq x; \\ 0, & X_n > x. \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be an nondecreasing continuous function on $[0,1]$ and

$$B_n(x; f) = \mathbb{E}\left[f\left(\frac{X_1^x + \cdots + X_n^x}{n} \right) \right].$$

Show that

(1) $B_n(x; f)$ is a polynomial in x of degree n; (2) $B_n(x; f)$ is nondecreasing in x;

(3) $B_n(x; f) \to f(x)$ uniformly on $[0,1]$.

PROBLEM 2. An urn contains N balls marked $1, 2, ..., N$. A ball is drawn from the urn repeatedly and independently with replacement. Let T_N be the first time every ball in the turn has been drawn at least once. Show that $T_N / N \log N$ converges to 1 in probability.

PROBLEM 3. Suppose $\{X_1, \ldots, X_n\}$ is a random sample from an unknown probability distribution with finite mean, variance, and third central moment, denoted by μ, σ^2, and $\mu_3 = \mathbb{E}(X_1 - \mu)^3$, respectively. It is of interest to study the relationship between

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

(1) Show that they are independent when the underlying distribution is Gaussian.
(2) For a general distribution, what is $\text{cov}(\bar{X}, S^2)$? Find an expression.
(3) Suppose the random sample is from Bernoulli(1/2). Show that $\bar{X}$ and S^2 are uncorrelated, but are not independent by showing that

$$\mathbb{P}(S^2 = 0 \mid \bar{X} = 1) \neq \mathbb{P}(S^2 = 0)$$

Yau College Math Competition 2020 Final Probability and Statistics

Individual Exam Problem Set 2 (Sunday, October 25, 2020)

PROBLEM 1. Suppose that $\{X_n\}$ is a sequence of real valued, independent, identically distributed random variables and B is a Borel set in $\mathbb{R}$. Assume that $\mathbb{P}(X_1 \in B) > 0$. Let $T = \inf\{n : X_n \in B\}$ be the first time the sequence is in the set B.

(1) Show that $\mathbb{P}(T < \infty) = 1$.

(2) Suppose $\mathbb{E}|X_1| < \infty$. Show that $\mathbb{E}X_T = \mathbb{E}[X_1 I_B(X_1)]\mathbb{E}T$.

PROBLEM 2. We flip a fair coin repeatedly and independently. Let N_n be the number of consecutive heads beginning from the n^{th} flip. (For example, $N_n = 0$ if the n^{th} flip is a tail, and $N_n = 2$ if the n^{th} and $(n+1)^{\text{th}}$ flips are heads but the $(n+2)^{\text{th}}$ flip is a tail. Show that

$$\limsup_{n\to\infty} \frac{N_n}{\log n} = \frac{1}{\log 2}.$$

PROBLEM 3. Let $\{X_1, \ldots, X_n\}$ be independent and identically distributed from the uniform distribution on the interval $(-\theta, \theta)$ with $\theta > 0$.

(1) Find a minimal sufficient statistic T for θ.

(2) Define $V = \bar{X}/|X|_{(n)}$, where $|X|_{(n)} = \max(|X_1|, \ldots, |X_n|)$. Show that V is independent of T.

Yau College Math Competition 2020 Final Probability and Statistics

Overal Exam Problem Set (Saturday, October 24, 2020)

PROBLEM 1. Let $\{X_n\}$ be a sequence of independent, identically distributed random variables taking values in $\mathbb{N}$, the set of positive integers. Define

$$R_n = \text{Card}\{X_1, \cdots, X_n\},$$

i.e., R_n is the number of distinct integers in the set $\{X_1, \cdots, X_n\}$. Suppose that $\mathbb{E}[X_1] < \infty$. Prove that $R_n/\sqrt{n} \to 0$ in probability.

PROBLEM 2. Consider the mixture experiment whose components are E_1 and E_2, taken with equal probabilities and each with the sample space $\Omega = \{0, 1, 2, ...\}$. It is postulated that

(1) the outcome of E_1 follows the Binomial model Binomial(n, θ) with the total number of trials $n > 0$ and the unknown probability of success parameter $\theta \in [0, 1]$, and
(2) the outcome of E_2 follows the Negative-Binomial model Binomial(r, θ) with the target for number of successful trials $r > 0$ and the unknown probability of success parameter $\theta \in [0, 1]$.

Let H be the observed index to the experiment that is actually conducted, and let X denote the observed outcome of the conducted experiment.

Find a minimal sufficient statistic for θ and prove your claim.

Algebra, Number Theory and Combinatorics (2021)

Problem 1. (Individual round.) Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers. Let $n \geq 1$ be an integer and $L = \mathbb{Q}_p(\zeta_{p^n})$, where ζ_{p^n} denotes a primitive p^n-th roots of unity. Determine the image of the norm map $N_{L/\mathbb{Q}_p} : L^\times \to \mathbb{Q}_p^\times$. You may use the inequality $[L : \mathbb{Q}_p] \leq (\mathbb{Q}_p^\times : N_{L/\mathbb{Q}_p}(L^\times))$ without proof in the case $n \geq 2$.

Problem 2. (Individual round.) Let k be a field and V a k-vector space of dimension n. Consider the group homomorphism:

$$\phi : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^2 V), f \mapsto \wedge^2 f.$$

(1) Determine the kernel of ϕ.

(2) Show that ϕ induces a group homomorphism $\psi : \mathrm{SL}(V) \to \mathrm{SL}(\wedge^2 V)$. Express $det(\wedge^2 f)$ in terms of $det(f)$.

Problem 3. (Individual round.) Let A be a rank 2 integer matrix of size 5×3. Classify all possible groups of the form $\mathbb{Z}^5 / A\mathbb{Z}^3$.

Solution to Problem 1. We will show that $N_{L/\mathbb{Q}_p}(L^\times) = p^{\mathbb{Z}}(1+p^n\mathbb{Z}_p)$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Let $\Phi(X) = (X^{p^n}-1)/(X^{p^{n-1}}-1)$ be the p^n-th cyclotomic polynomial. Then $\Phi(X+1)$ is an Eisenstein polynomial. Thus $\Phi(X)$ is the minimal polynomial of ζ_{p^n}, so that

$$N_{L/\mathbb{Q}_p}(1-\zeta_{p^n}) = \Phi(1) = p.$$

We have $[L:\mathbb{Q}_p] = \phi(p^n) = p^n - p^{n-1}$. For p odd, the $\phi(p^n)$-th power map on $1+p\mathbb{Z}_p$ is the composition

$$1+p\mathbb{Z}_p \xrightarrow[\sim]{\log} p\mathbb{Z}_p \xrightarrow[\sim]{\phi(p^n)} p^n\mathbb{Z}_p \xrightarrow[\sim]{\exp} 1+p^n\mathbb{Z}_p.$$

Thus

$$N_{L/\mathbb{Q}_p}(L^\times) \supseteq N_{L/\mathbb{Q}_p}(1+p\mathbb{Z}_p) = 1+p^n\mathbb{Z}_p.$$

For $p=2$, we may assume $n \geq 2$. The $\phi(2^n)$-th power map on $1+4\mathbb{Z}_2$ is the composition

$$1+4\mathbb{Z}_2 \xrightarrow[\sim]{\log} 4\mathbb{Z}_2 \xrightarrow[\sim]{\phi(2^n)} 2^{n+1}\mathbb{Z}_2 \xrightarrow[\sim]{\exp} 1+2^{n+1}\mathbb{Z}_2.$$

Thus

$$N_{L/\mathbb{Q}_2}(L^\times) \supseteq N_{L/\mathbb{Q}_2}(1+4\mathbb{Z}_2) = 1+2^{n+1}\mathbb{Z}_2.$$

It is easy to see that

$$1+2^n\mathbb{Z}_2 = (1+2^{n+1}\mathbb{Z}_2)\coprod 5^{2^{n-2}}(1+2^{n+1}\mathbb{Z}_2)$$

and

$$5^{2^{n-2}} = N_{L/\mathbb{Q}_2}(2+\zeta_4).$$

This finishes the proof of $N_{L/\mathbb{Q}_p}(L^\times) \supseteq p^{\mathbb{Z}}(1+p^n\mathbb{Z}_p)$ in all cases.

Let $\mathcal{O}_L$ denote the integral closure of $\mathbb{Z}_p$ in L. The residue field of L is $\mathbb{F}_p$, so that $N_{L/\mathbb{Q}_p}|_{\mathcal{O}_L^\times}$ is compatible with the $\phi(p^n)$-th power map on $\mathbb{F}_p^\times$, which carries every element to 1. In other words, $N_{L/\mathbb{Q}_p}(L^\times)\cap\mathbb{Z}_p^\times = N_{L/\mathbb{Q}_p}(\mathcal{O}_L^\times) \subseteq 1+p\mathbb{Z}_p$. This finishes the proof in the case $n=1$. For $n\geq 2$, it suffices to apply the given inequality $(\mathbb{Q}_p^\times : N_{L/\mathbb{Q}_p}(L^\times)) \geq \phi(p^n) = (\mathbb{Q}_p^\times : p^{\mathbb{Z}}(1+p^n\mathbb{Z}_p))$. □

Solution to Problem 2. (1) If $n=2$, then $\wedge^2 V \simeq k$ and $\phi(f)\in \mathrm{GL}(k)$ is just the multiplication by $det(f)$, hence the kernel is just $\mathrm{SL}(V)=\mathrm{SL}_2$. Now assume $n\geq 3$. By definition, $f\in\mathrm{Ker}(\phi)$ if and only if $f(x)\wedge f(y) = x\wedge y$ for all $x,y\in V$. We claim x and $f(x)$ are proportional: otherwise expand to a basis $e_1 = x, e_2 = f(x), e_3, ..., e_n$, then we have $e_2\wedge f(e_3) = e_1\wedge e_3$ which is not possible as $e_i\wedge e_j$ is a basis of $\wedge^2 V$. Hence x and $f(x)$ are proportional for all $x\in V$. This implies that $f(x) = ax$ for some $a\in k$. Then $f(x)\wedge f(y) = a^2 x\wedge y = x\wedge y$, thus $a=\pm 1$. So the kernel is just $\pm Id$.

(2) First we show the case for elementary matrices: Take a basis $e_1,\cdots,e_n$ of V and consider the endomorphism $f\in\mathrm{GL}(V)$ defined by $f(e_i) = e_i + b\delta_{1,i}e_2$ for all i, where b is a constant. We have $(\wedge^2 f)(e_1\wedge e_j) = e_1\wedge e_j + be_2\wedge e_j$ for all $j\geq 2$, and for $2\leq i<j$, $(\wedge^2 f)(e_i\wedge e_j) = e_i\wedge e_j$. Thus $\wedge^2 f$ is lower triangular in the basis of $e_i\wedge e_j$ with 1 on the diagonal, which gives $det(\wedge^2 f) = 1$.

Recall that any matrix of determinant 1 is a product of elementary matrices, hence by (ii) ϕ sends SL(V) to SL($\wedge^2 V$). Take t in an extension of k, such that $t^n det(f) = 1$. Then $det(tf) = 1$ and

$$1 = det(\wedge^2(tf)) = det(t^2 \wedge^2 f) = (t^2)^{\binom{n}{2}} det(\wedge^2 f),$$

which gives $det(\wedge^2 f) = t^{-n(n-1)} = det(f)^{n-1}$. $\square$

Solution to Problem 3. We can change Z bases of Z^5 and Z^3 to turn A in to a "canonical form". Equivalently, we can do the usual row-column reduction on A. Since A rank 2 means that AZ^3 is a rank 2 subgroup of Z^5, which means the free part of G is Z^3. So, in the reduced form A has 3 zero rows, and 2 positive diagonal entries of all possibilities. We can arrange so that

$$G \simeq Z/a \oplus Z/b \oplus Z^3 \text{with } a \geq b > 0.$$

Finally list all possible non-isomorphic torsion part $Z/a \oplus Z/b$, by factorizing a, b.

Possible follow up: Generalize this as follows: A is rank k of size $m \times n$ with $m > n > k$. Classify all possible abelian groups of the form $G = \mathbb{Z}^m / A\mathbb{Z}^n$. The same method would likewise yield $G \simeq Z/a_1 \oplus \cdots \oplus Z/a_{n-k} \oplus Z^{m-k}$ with $a_1 \geq a_2 \geq \cdots \geq a_{n-k} > 0$. $\square$

Algebra, Number Theory and Combinatorics (Overall individual round, 2021)

Problem 1. (Overall individual round). For any $n \geq 1$, let A denote the $\mathbb{C}$-algebra consisting of $n \times n$ upper triangular complex matrices $\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}_{n\times n}\}$. We shall consider the left A-modules (that is, $\mathbb{C}$-vector spaces V with $\mathbb{C}$-algebra homomorphisms $\rho : A \to \mathrm{End}(V)$).

(1) Show that all simple modules of A are finite dimensional.
(2) Determine all simple modules of A.

Problem 2. (Overall individual round). Let p be a prime number. Prove the following theorem of Euler: the equation $p = x^2 + 3y^2$ has a solution with $x, y \in \mathbb{Z}$ if and only if $p = 3$ or $p \equiv 1 \pmod 3$. (You may use the fact that the ring of integers of $\mathbb{Q}(\sqrt{-3})$ is a principal ideal domain.)

Solution to Problem 1 (Overall individual round). (1) Let E_{ij} denote the matrix whose (i,j)-entry is 1 and other entries vanish. Then E_{ij}, $i \le j$, form a basis of A. In particular, A is finite dimensional.

Let V denote a non-zero module. Take any $0 \neq v \in V$. Then Av is a non-zero submodule of V. And Av is finite dimensional because A is. Since V is simple, $V = Av$ is finite dimensional.

(2a) Let S_i, $1 \le i \le n$, denote the 1-dimensional modules such that E_{ii} acts by 1 and E_{ij}, E_{jj} acts by 0 for $j \neq i$. They are simple modules.

(2b) It remains to show that the S_i we have constructed are the only simple modules. Let S denote any finite dimensional simple module.

We claim that E_{ij}, $i < j$, form a nilpotent 2-sided ideal N (because the product of an upper triangular matrix with a strictly upper one is strictly upper).

Then N acts on S by 0 (To see this, NS is a submodule of S. It is proper because N is nilpotent. Since S is simple, we deduce that $NS = 0$.)

Note that the action of E_{ii} commute with each other (and with the 0-action by E_{ij}), thus they are module endomorphisms. By Schur's Lemma, E_{ii} acts on S as a scalar. Since $E_{ii}E_{jj} = 0$ for $i \neq j$, at most one E_{ii} acts as a non-zero scalar. Recall that $1 = \sum_i E_{ii}$ acts by the identity. The claim follows. $\square$

Solution to Problem 2 (Overall individual round). The "only if" part is clear. We prove the "if" part. For $p = 3$ one can take $(x, y) = (0, 1)$. Assume $p \equiv 1 \pmod 3$. By quadratic reciprocity, $(\frac{-3}{p}) = (\frac{p}{3}) = 1$. Thus p splits in $\mathbb{Q}(\sqrt{-3})$. The ring of integers of $\mathbb{Q}(\sqrt{-3})$ is $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$. Since $\mathbb{Z}[\omega]$ is a PID, there exists $\pi \in \mathbb{Z}[\omega]$ such that $N_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(\pi) = p$. We claim that at least one of π, $\pi\omega$, and $\pi\omega^2$ belongs to $\mathbb{Z}[\sqrt{-3}]$ and thus is of the form $x + y\sqrt{-3}$ with $x, y \in \mathbb{Z}$. Taking norms, we then get $p = x^2 + 3y^2$.

To prove the claim, we may assume $\pi = \frac{a+b\sqrt{-3}}{2}$, where a and b are odd integers. Then either $4 \mid a - b$ (which is equivalent to $\pi\omega \in \mathbb{Z}[\sqrt{-3}]$) or $4 \mid a + b$ (which is equivalent to $\pi\omega^2 \in \mathbb{Z}[\sqrt{-3}]$). $\square$

12th Oral Exam of S.-T. Yau College Student Mathematics Contests 2021

Analysis and differential equations
Individual Contest

(∗ The forth and fifth problems are optional)

1. Let $\mathbb{Z}$ be the set of integers. Recall that $\vec{a} := \{a_j\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$ if and only if $(\sum_{j\in\mathbb{Z}} |a_j|^q)^{\frac{1}{q}} < \infty$. Answer the following questions and justify your answers:

   (i) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ continuous ?

   (ii) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ compact?

   (iii) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ compact modulo translation? More precisely, let $\{\vec{a}_n\}_n$ be a sequence in $l^2(\mathbb{Z})$ with $\vec{a_n} = \{a_{j,n}\}_{j\in\mathbb{Z}}$. Can one find $k_n \in \mathbb{Z}$ for $n \geq 1$, such that $\{\vec{b}_n\}_n$, with $\vec{b}_n := \{b_{j,n}\}_{j\in\mathbb{Z}}$ and $b_{j,n} = a_{j-k_n,n}$ has a convergent subsequence in $l^4(\mathbb{Z})$?

2. Let S^1 be the unit circle in $\mathbb{R}^2$. For any $v \in S^1$, let $\pi_v : \mathbb{R}^2 \to \mathbb{R} : \pi_v(x) = \langle v, x\rangle$. Let A, B be two bounded open convex sets in $\mathbb{R}^2$ and λ_1 be the Lebesgue measure on $\mathbb{R}^1$.

   (i) If for all $v \in S^1$, $\lambda_1(\pi_v(A)) = \lambda_1(\pi_v(B))$, can one conclude that $A = B$? Can one conclude that $A = B$ modulo isometries?

   (ii) If for all $v \in S^1$, $\pi_v(A) = \pi_v(B)$, can one say that $A = B$?

   Justify your answers.

3. Assume that $\rho \in C_0^1(\mathbb{R}^3)$ satisfies $\rho(x) \geq 0$ for $x \in \mathbb{R}^3$ and

   $$\nabla(\rho^{4/3}) + \rho\nabla\phi = 0, \ \ on\ \mathbb{R}^3$$

   where

   $$\phi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}dy.$$

   Evaluate the integral

   $$\int_{\mathbb{R}^3} (3\rho^{4/3} + \frac{1}{2}\rho\phi)dx,$$

   and justify your answer.

4*. Let $D = \{z, |z| < 1\}$. Determine $Aut(D)$, the group of holomorphic automorphisms of the unit disk.

5*. Assume that H is space with inner product, and $x_k \xrightarrow{w} x$ $(k \to \infty)$. Prove that there exists a subsequence $\{x_{k_n}\} \subset \{x_k\}$ such that $\frac{x_{k_1}+x_{k_2}+\cdots+x_{k_n}}{n} \to x(n \to \infty)$.

12th Oral Exam of S.-T. Yau College Student Mathematics Contests 2021

Analysis and differential equations
Group Contest

1. Explain that the dual space of $L^\infty(\mathbb{R})$ is not $L^1(\mathbb{R})$.

2. Let $\varphi : [0,1] \to \mathbb{R}$ be integrable for the Lebesgue measure. Define $G : \mathbb{R} \to \mathbb{R}_+$ by
$$G(t) = \int_{[0,1]} |\varphi(x) - t| dx.$$
   (i) Show that G is continuous on $\mathbb{R}$;
   (ii) Show that G is derivable at $t \in \mathbb{R}$ if and only if
$$\lambda_1(\{x : \varphi(x) = t\}) = 0,$$
   here λ_1 denotes the Lebesgue measure on $\mathbb{R}^1$.

3. Let $B_1(0)$ be the unit ball in $\mathbb{R}^3$ centered at the origin. Assume that the function v is a smooth function defined on $\mathbb{R}^3$ with $v_r = \frac{x\cdot\nabla v}{|x|} \in L^2(B_1(0)$. Prove that
$$\begin{aligned}\int_{B_1(0)} \frac{|v(x)|^2}{|x|^2} dx &\le C(\int_{B_1(0)} |v_r|^2 dx + \int_{\partial B_1(0)} |v|^2 d\sigma) \\ &\le C_1 \int_{B_1(0)} (|v_r|^2 + |v|^2) dx,\end{aligned}$$
   where C and C_1 are some constants independent of v.

4. Prove that the life span of any solution to the following differential equation
$$\frac{dy}{dx} = x^2 + y^2$$
   is finite.

Question I

Let A be an $n \times n$ matrix with real and positive eigenvalues and b be a given vector. Consider the solution of $Ax = b$ by the following Richardson's iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

where ω is a damping coefficient. Let λ_1 and λ_n be the smallest and the largest eigenvalues of A. Let $G_\omega = I - \omega A$.

1. Prove that the Richardson's iteration converges if and only if

$$0 < \omega < \frac{2}{\lambda_n}.$$

2. Prove that the optimal choice of ω is given by

$$\omega_{\text{opt}} = \frac{2}{\lambda_1 + \lambda_n}.$$

Prove also that

$$\rho(G_\omega) = \begin{cases} 1 - \omega\lambda_1, & \omega \le \omega_{\text{opt}} \\ (\lambda_n - \lambda_1)/(\lambda_n + \lambda_1), & \omega = \omega_{\text{opt}} \\ \omega\lambda_n - 1, & \omega \ge \omega_{\text{opt}} \end{cases}$$

where $\rho(G_\omega)$ is the spectral radius of G_ω.

3 Prove that, if A is symmetric and positive definite, then

$$\rho(G_{\omega_{\text{opt}}}) = \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}$$

where $\kappa_2(A)$ is the spectral condition number of A.

Question II

Let the energy functional for $u(x) \in \mathbb{R}$ ($x \in [0, 1]$) be given as

$$E[u] = \int_0^1 \left[\frac{1}{2} |\partial_x u(x)|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right] dx, \quad 0 < \varepsilon \ll 1.$$

Consider the dynamical equation (natural boundary condtions)

$$\frac{\mathrm{d}}{\mathrm{dt}} u(x, t) = -\frac{\delta E}{\delta u}[u(x, t)], \quad u(x, 0) = u_0,$$

where u_0 is a sufficiently smooth function.

1. Show that $E[u(x,t)]$ is decreasing in t.
2. If initially $u_0(x) \in [-1,1]$, show that $u(x,t) \in [-1,1]$ for all $t > 0$.
3. Design a semi-discrete-in-time scheme such that the energy functional is decreasing for the discrete scheme.

Question III

Let $a_k(t), b_k(t) \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$ satisfy the differential equations:

$$\frac{\mathrm{d}}{\mathrm{dt}} a_k(t) = 2(b_k^2 - b_{k-1}^2), \quad \frac{\mathrm{d}}{\mathrm{dt}} b_k(t) = b_k(a_{k+1} - a_k), \quad k = 1, 2, \cdots, n,$$

where $b_0(t) = b_n(t) = 0$. Consider the $n \times n$ tri-diagonal matrix $L(a,b)$

$$L(a,b) = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & & & 0 \\ & & \ddots & & \\ & & & & b_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix},$$

show that:

1. The eigenvalues of $L(t) = L(a(t), b(t))$ are independent of t.
2. $\lim_{t \to \infty} b_k(t) = 0$, $k = 1, 2, \cdots, n-1$.

Question I

Suppose $A \in M_{n\times n}(\mathbb{C})$ is normal, i.e., $A = QDQ^*$ for some unitary Q. Also, we suppose there are k dominant eigenvalues of A, i.e.,

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|$$

Consider the power iteration:

$$\mathbf{x}^{m+1} = A\mathbf{x}^m.$$

Define the angle between a non-zero vector $\mathbf{x}$ and a non-trivial subspace W.

$$\begin{aligned}
\cos\angle(\mathbf{x}, W) &:= \max\left\{\cos\angle(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in W \setminus \{0\}\right\}, \\
\sin\angle(\mathbf{x}, W) &:= \min\left\{\sin\angle(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in W \setminus \{0\}\right\}, \\
\tan\angle(\mathbf{x}, W) &:= \frac{\sin\angle(\mathbf{x}, W)}{\cos\angle(\mathbf{x}, W)}.
\end{aligned}$$

Using the given definitions, prove that:

1. Let W be a nontrivial subspace of $\mathbb{C}^n$ and P be the orthogonal projection onto W. Then, for $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \notin W$,

$$\cos\angle(\mathbf{x}, W) = \frac{\|P\mathbf{x}\|}{\|\mathbf{x}\|},\ \sin\angle(\mathbf{x}, W) = \frac{\|\mathbf{x} - P\mathbf{x}\|}{\|\mathbf{x}\|},\ \tan\angle(\mathbf{x}, W) = \frac{\|\mathbf{x} - P\mathbf{x}\|}{\|P\mathbf{x}\|}$$

   Here, $||\cdot||$ is the usual Euclidean norm in $\mathbb{C}^n$.

2. Let $\{\delta_i : i = 1, \cdots, n\}$ be the standard ordered basis for $\mathbb{C}^n$. Let $W_k := \operatorname{span}\{Q\delta_j : j = 1, \cdots, k\}$ and let $\cos\angle(\mathbf{x}^0, W_k) \neq 0$. Then,

$$\tan\angle(\mathbf{x}^{m+1}, W_k) \leq \frac{|\lambda_{k+1}|}{|\lambda_k|}\tan\angle(\mathbf{x}^m, W_k).$$

Question II

Consider the linear system $Ax = b$. The (Generalized minimal residual method) GMRES method is a projection method which obtains a solution in the m-th Krylov subspace K_m so that the residual is orthogonal to AK_m. Let r_0 be the initial residual and let $v_0 = r_0$. The Arnoldi process (see the hint below) is applied to build an orthonormal system $v_1, v_2, \ldots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|_2$ ($\|\cdot\|_2$- the l^2 norm). The approximate solution is obtained from the following space

$$K_m = \operatorname{span}\{v_0, v_1, \ldots, v_{m-1}\}.$$

1. Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

2. Prove that the residual r_k is orthogonal to $\{v_1, v_2, \ldots, v_{k-1}\}$.

3. Find a formula for the residual norm.

4. Derive the complete GMRES algorithm.

Hint: The Arnoldi process uses the stabilized Gram-Schmidt process to produce a sequence of orthonormal vectors $v_1, v_2, v_3, \ldots$. Explicitly, the algorithm is as follows:

(1.) Start with an arbitrary vector v_1 with norm 1.

(2.) Repeat for $k = 2, 3, \ldots$

   i. $v_k \leftarrow Av_{k-1}$

   ii. for j from 1 to $k-1$

      A. $h_{j,k-1} \leftarrow v_j^* v_k$

      B. $v_k \leftarrow v_k - h_{j,k-1} v_j$

   iii. $h_{k,k-1} \leftarrow \|v_k\|_2$

   iv. $v_k \leftarrow \frac{v_k}{h_{k,k-1}}$

S.-T. Yau College Student Mathematics Contests 2021

Oral Exams in Geometry and Topology

Individual (4 problems)

1. Consider the manifold

$$M = \left\{ \big((x_1, \cdots, x_n), [y_1 : \cdots : y_n]\big) \in \mathbb{R}^n \times \mathbb{RP}^{n-1} : x_i y_j = x_j y_i \ \forall i, j \right\},$$

and the projection map $\pi : M \to \mathbb{R}^n$ onto its $\mathbb{R}^n$-factor. Determine whether or not π is a submersion.

2. The suspension SX of a topological space X is defined as the quotient space of $X \times [0, 1]$ modulo the equivalence relation generated by

$$(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X.$$

Show that for all n there are isomorphisms $\widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$.

3. Let M be a compact orientable Riemannian manifold with nonnegative Ricci curvature. Then prove the following:

(a) The first Betti number $b_1(M) \leq \dim M$.
(b) The above equality holds if and only if M is isometric to a flat torus.
(c) If we further assume M has positive Ricci curvature, then $b_1(M) = 0$.

4. Let M be a Riemannian manifold, let $p \in M$, and let Π be a plane in T_pM (i.e., a 2-dimensional linear subspace of T_pM). Let $D_r \subset T_pM$ be the open disc of radius r in the plane Π, centered at 0. For r sufficiently small, we know that $\exp_p(D_r)$ is an embedded 2-dimensional submanifold of M; call its area A_r. Prove that the sectional curvature

$$K(\Pi) = \lim_{r \to 0+} 12 \frac{\pi r^2 - A_r}{\pi r^4}.$$

If you could not give a general proof, maybe try when M is surface, i.e., $\dim M = 2$.

S.-T. Yau College Student Mathematics Contests 2021

Oral Exams in Geometry and Topology

All-round (2 problems)

1. Let Σ be a regular surface in $\mathbb{R}^3$, and ν be its Gauss map. Let $f : \Sigma \to (0, \infty)$ be a smooth function on Σ, and consider the set:

$$\hat{\Sigma} = \{p + f(p)\nu(p) : p \in \Sigma\}.$$

Show that if $\mathrm{id} + fh : T_p\Sigma \to T_p\Sigma$ is invertible for all $p \in \Sigma$, then $\hat{\Sigma}$ is also a regular surface. Here h is the second fundamental form.

2. Suppose M is a compact manifold with boundary. Glue together a pair of M's along their boundaries to form a new manifold $\widetilde{M}$. Explain how to use a Mayer-Vietoris sequence to find a relation between $\chi(\widetilde{M})$, $\chi(\partial M)$ and $\chi(M)$.

Yau College Math Competition 2021

Final Probability and Statistics

Individual Exam Problems (May 29, 2021)

Problem 1. Let $\{X_n\}_{n\geq 1}$ be a sequence of real valued, nonnegative random variables. Assume that there are constants $C > 0$ and $\lambda > 0$ such that $\mathbb{E}X_n \leq Ce^{-\lambda n}$, $\forall n \geq 1$. Prove that

$$P\Big(\limsup_{n\to\infty} \frac{1}{n} \ln X_n \leq -\lambda\Big) = 1.$$

Solution

For any $\lambda_0 \in (0, \lambda)$, define the events

$$A_n = \{\omega \in \Omega : X_n(\omega) > e^{-\lambda_0 n}\}, \quad n \geq 1.$$

By Chebyshev's inequality,

$$\mathbb{P}(A_n) \leq e^{\lambda_0 n}\mathbb{E}X_n \leq Ce^{(\lambda_0-\lambda)n}, \quad \forall n \geq 1.$$

Since $\lambda_0 < \lambda$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} Ce^{(\lambda_0-\lambda)n} < +\infty.$$

Borel-Cantelli's lemma implies that for $\mathbb{P}$-a.s. $\omega \in \Omega$, there exists $n(\omega) \in \mathbb{N}$ such that for all $n \geq n(\omega)$, we have $\omega \in A_n^c$, that is $X_n(\omega) \leq e^{-\lambda_0 n}$. Therefore,

$$\frac{1}{n} \ln X_n(\omega) \leq -\lambda_0, \quad \forall n \geq n(\omega).$$

This implies the desired result since λ_0 is an arbitrary number less than λ.

Problem 2. Assume that $X_1, \ldots, X_n \sim U[0, 1]$ (uniform distribution) are i.i.d. Denote $X_{(1)} = \min_{1\leq k\leq n} X_k$ and $X_{(n)} = \max_{1\leq k\leq n} X_k$. Let $R = X_{(n)} - X_{(1)}$ be the sample range and $V = (X_{(1)} + X_{(n)})/2$ be the sample midvalue.

(1). Find the joint density of $(X_{(1)}, X_{(n)})$.

(2). Find the joint density of (R, V).

(3). Find the density of R and the density of V.

Solution

(1). Denote $F(x_1, x_n) = P(X_{(1)} \le x_1,\ X_{(n)} \le x_n)$, then $F(x_1, x_n) = 0$ for $x_1 \notin [0,1]$ or $x_n \notin [0,1]$. If $x_1 \ge x_n$, then $\{X_{(n)} \le x_n\} \subset \{X_{(1)} \le x_1\}$, and therefore

$$F(x_1, x_n) = P(X_{(n)} \le x_n).$$

If $0 \le x_1 \le x_n \le 1$, then

$$\begin{aligned} P(X_{(1)} \ge x_1,\ X_{(n)} \le x_n) &= P(\cup_{k=1}^{n}\{x_1 \le X_k \le x_n) \\ &= \prod_{k=1}^{n} P(x_1 \le X_k \le x_n) \\ &= (x_n - x_1)^n, \end{aligned}$$

which implies that

$$\begin{aligned} F(x_1, x_n) &= P(X_{(n)} \le x_n) - P(X_{(1)} \ge x_1,\ X_{(n)} \le x_n) \\ &= P(X_{(n)} \le x_n) - (x_n - x_1)^n. \end{aligned}$$

Thus,

$$\begin{aligned} f(x_1, x_n) &= \frac{\partial^2 F(x_1, x_n)}{\partial x_1 \partial x_n} \\ &= \begin{cases} n(n-1)(x_n - x_1)^{n-2}, & \text{if } 0 \le x_1 \le x_n \le 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

(2). Note that

$$\begin{pmatrix} X_{(1)} \\ X_{(n)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} R \\ V \end{pmatrix} \equiv A \begin{pmatrix} R \\ V \end{pmatrix},$$

thus the joint density of (R, V) is

$$\begin{aligned} f_{R,V}(r, v) &= f(x_1, x_n) \times |\det A| \\ &= f\Big(v - \frac{r}{2}, v + \frac{r}{2}\Big) \\ &= n(n-1)r^{n-2}, \end{aligned}$$

where $(r, v) \in D \equiv \{(r, v) : 0 \le v - \frac{r}{2} \le v + \frac{r}{2} \le 1\}$ and

$$f_{R,V}(r, v) = 0,$$

if $(r, v) \notin D$.

(3) The density of R is

$$
\begin{aligned}
f_R(r) &= \int_{-\infty}^{+\infty} f_{R,V}(r,v)dv \\
&= \int_{r/2}^{1-r/2} f_{R,V}(r,v)dv = n(n-1)r^{n-2}(1-r), \quad 0 \le r \le 1.
\end{aligned}
$$

For the density of V, if $v \in [0,1/2]$, then

$$
f_V(v) = \int_{-\infty}^{+\infty} f_{R,V}(r,v)dr = \int_0^{2v} n(n-1)r^{n-2}dr = n(2v)^{n-1},
$$

if $v \in [1/2,1]$, then

$$
f_V(v) = \int_{-\infty}^{+\infty} f_{R,V}(r,v)dr = \int_0^{2(1-v)} n(n-1)r^{n-2}dr = n(2(1-v))^{n-1}.
$$

Problem 3. A binary tree is a tree in which each node has exactly two descendants. Suppose that each node of the tree is coloured black with probability p, and white otherwise, independently of all other nodes. For any path π containing n nodes beginning at the root of the tree, let $B(\pi)$ be the number of black nodes in π, and let $X_n(k)$ be the number of such paths π for which $B(\pi) \ge k$.

(1) Show that there exists β_c such that

$$
\lim_{n\to\infty} \mathbb{E}(X_n(\beta n)) = \begin{cases} 0, & \text{if } \beta > \beta_c, \\ \infty, & \text{if } \beta < \beta_c. \end{cases}
$$

How to determine the value of β_c?

(2) For $\beta \ne \beta_c$, find the limit $\lim_{n\to\infty} \mathbb{P}(X_n(\beta n) \ge 1)$.

Solution

The number of paths π containing exactly n nodes is 2^{n-1}, and each such π satisfies $\mathbb{P}(B(\pi) \ge k) = \mathbb{P}(S_n \ge k)$ where $S_n = Y_1 + Y_2 + \cdots + Y_n$ is the sum of n independent Bernoulli variables having parameter p. Therefore $\mathbb{E}(X_n(k)) = 2^{n-1}\mathbb{P}(S_n \ge k)$. We set $k = n\beta$, and need to estimate $\mathbb{P}(S_n \ge n\beta)$. It is a consequence of the large deviation theorem that, if $p \le \beta < 1$,

$$
\mathbb{P}(S_n \ge n\beta)^{1/n} \overset{n\to\infty}{\longrightarrow} \inf_{t>0} \left\{ e^{-t\beta} M(t) \right\}
$$

where $M(t) = \mathbb{E}\left(e^{tY_1}\right) = q + pe^t, q = 1-p$. With some calculus, we find that

$$
\mathbb{P}(S_n \ge n\beta)^{1/n} \overset{n\to\infty}{\longrightarrow} \left(\frac{p}{\beta}\right)^{\beta} \left(\frac{1-p}{1-\beta}\right)^{1-\beta}, \qquad p \le \beta < 1
$$

Hence

$$\mathbb{E}(X_n(\beta n)) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0, & \text{if } \gamma(\beta) < 1 \\ \infty, & \text{if } \gamma(\beta) > 1 \end{cases}$$

where

$$\gamma(\beta) = 2\left(\frac{p}{\beta}\right)^{\beta}\left(\frac{1-p}{1-\beta}\right)^{1-\beta}$$

is a decreasing function of β. If $p < \frac{1}{2}$, there is a unique $\beta_c \in [p, 1)$ such that $\gamma(\beta_c) = 1$; if $p \geq \frac{1}{2}$ then $\gamma(\beta) > 1$ for all $\beta \in [p, 1)$ so that we may take $\beta_c = 1$.

Turning to the final part,

$$\mathbb{P}(X_n(\beta n) \geq 1) \leq \mathbb{E}(X_n(\beta n)) \overset{n\to\infty}{\longrightarrow} 0, \quad \text{if } \beta > \beta_c.$$

As for the other case, we will use the Payley-Zygmund inequality

$$\mathbb{P}(N \neq 0) \geq \frac{\mathbb{E}(N)^2}{\mathbb{E}(N^2)}$$

for nonnegative random variable N.

We have that $\mathbb{E}(X_n(\beta n)^2) = \sum_{\pi,\rho} \mathbb{E}(I_\pi I_\rho)$, where the sum is over all such paths π, ρ, and I_π is the indicator function of the event $\{B(\pi) \geq \beta n\}$. Hence

$$\mathbb{E}(X_n(\beta n)^2) = \sum_{\pi} \mathbb{E}(I_\pi) + \sum_{\pi \neq \rho} \mathbb{E}(I_\pi I_\rho) = \mathbb{E}(X_n(\beta n)) + 2^{n-1} \sum_{\rho \neq L} \mathbb{E}(I_L I_\rho)$$

where L is the path which always takes the left fork (there are 2^{n-1} choices for π, and by symmetry each provides the same contribution to the sum). We divide up the last sum according to the number of nodes in common to ρ and L, obtaining $\sum_{m=1}^{n-1} 2^{n-m-1}\mathbb{E}(I_L I_M)$ where M is a path having exactly m nodes in common with L. Now

$$\mathbb{E}(I_L I_M) = \mathbb{E}(I_M \mid I_L = 1)\,\mathbb{E}(I_L) \leq \mathbb{P}(T_{n-m} \geq \beta n - m)\,\mathbb{E}(I_L),$$

where T_{n-m} has the Binomial$(n-m, p)$ distribution (the 'most value' to I_M of the event $\{I_L = 1\}$ is obtained when all m nodes in $L \cap M$ are black). However

$$\mathbb{E}(I_M) = \mathbb{P}(T_n \geq \beta n) \geq p^m \mathbb{P}(T_{n-m} \geq \beta n - m),$$

so that $\mathbb{E}(I_L I_M) \leq p^{-m}\mathbb{E}(I_L)\,\mathbb{E}(I_M)$. It follows that $N = X_n(\beta n)$ satisfies

$$\mathbb{E}(N^2) \leq \mathbb{E}(N) + 2^{n-1} \sum_{m=1}^{n-1} 2^{n-m-1} \cdot \frac{1}{p^m}\mathbb{E}(I_L)\,\mathbb{E}(I_M) = \mathbb{E}(N) + \frac{1}{2}(\mathbb{E}(N))^2 \sum_{m=1}^{n-1} \left(\frac{1}{2p}\right)^m$$

whence, by the Payley-Zygmund inequality,

$$\mathbb{P}(N \neq 0) \geq \frac{1}{\mathbb{E}(N)^{-1} + \frac{1}{2}\sum_{m=1}^{n-1}(2p)^{-m}}.$$

If $\beta < \beta_c$ then $\mathbb{E}(N) \to \infty$ as $n \to \infty$. It is immediately evident that $\mathbb{P}(N \neq 0) \to 1$ if $p \leq \frac{1}{2}$. Suppose finally that $p > \frac{1}{2}$ and $\beta < \beta_c$. By the above inequality,

$$\mathbb{P}\left(X_n(\beta n) > 0\right) \geq c(\beta), \qquad \forall\ n \tag{0.1}$$

where $c(\beta)$ is some positive constant. Take $\epsilon > 0$ such that $\beta + \epsilon < \beta_c$. Fix a positive integer m, and let $\mathcal{P}_m$ be a collection of 2^m disjoint paths each of length $n - m$ starting from depth m in the tree. Now

$$\mathbb{P}\left(X_n(\beta n) = 0\right) \leq \mathbb{P}\left(B(v) < \beta n \text{ for all } v \in \mathcal{P}_m\right) = \mathbb{P}(B(v) < \beta n)^{2^m},$$

where $v \in \mathcal{P}_m$. However

$$\mathbb{P}(B(v) < \beta n) \leq \mathbb{P}(B(\nu) < (\beta + \epsilon)(n - m))$$

if $\beta n < (\beta + \epsilon)(n - m)$, which is to say that $n \geq (\beta + \epsilon)m/\epsilon$. Hence, for all large n,

$$\mathbb{P}\left(X_n(\beta n) = 0\right) \leq \left(1 - c(\beta + \epsilon)\right)^{2^m}$$

by (0.1). We let $n \to \infty$ and $m \to \infty$ in that order, to obtain $\mathbb{P}\left(X_n(\beta n) = 0\right) \to 0$ as $n \to \infty$. In summary,

$$\mathbb{P}(X_n(\beta n) \geq 1) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0, & \text{if } \beta > \beta_c, \\ 1, & \text{if } \beta < \beta_c. \end{cases}$$

Yau College Math Competition 2021

Final Probability and Statistics

Individual Overall Exam Problems (May 30, 2021)

Problem 1. Let $X_1, X_2, \cdots, X_n$ be independent exponential random variables with parameter 1, and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be their order statistics. Let $X_{(0)} = 0$.

(1) Find the joint density function of

$$Y_k = (n+1-k)(X_{(k)} - X_{(k-1)}), \qquad k = 1, 2, \cdots, n.$$

(2) Find the limit

$$\lim_{n\to\infty} \mathbb{P}\left(X_{(n)} - \ln n \le x\right).$$

(3) Find the limit

$$\lim_{n\to\infty} \int_0^\infty \mathbb{P}\left(X_{(n)} - \ln n > x\right) \mathrm{d}x.$$

Solution

(1) Notice that the joint density function of $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ is

$$h(x_1, \cdots, x_n) = \begin{cases} n! e^{-\sum_{i=1}^n x_i}, & \text{if} \quad x_1 \le x_2 \le \cdots \le x_n, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

Let $x_0 = 0$ and de ne

$$y_k = (n+1-k)(x_k - x_{k-1}), \qquad k = 1, 2, \cdots, n,$$

then

$$x_k = \sum_{i=1}^{k} \frac{y_i}{n-i+1}, \qquad k = 1, 2, \cdots, n,$$

and the Jacobian is $1/n!$. So the density function of $Y_1, \cdots, Y_n$ is $e^{-\sum_{i=1}^n y_i}$.

(2) Since

$$\mathbb{P}(X_{(n)} \le x) = (1 - e^{-x})^n,$$

we have

$$\mathbb{P}(X_{(n)} \le x + \ln n) = \left(1 - \frac{e^{-x}}{n}\right)^n \xrightarrow{n\to\infty} e^{-e^{-x}}.$$

(3) According to the above two steps and the lack-of-memory property, we have

$$\mathbb{E}(X_{(n)}) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Consequently,

$$\lim_{n\to\infty}\int_0^\infty \mathbb{P}\left(X_{(n)}-\ln n>x\right)\mathrm{d}x=\lim_{n\to\infty}\mathbb{E}\left(X_{(n)}-\ln n\right)=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=\gamma.$$

Problem 2. Let $\{X_n\}_{n\geq 1}$ be i.i.d. random variables such that $\mathbb{P}(X_1=1)=1-\mathbb{P}(X_1=-1)=p>\frac{1}{2}$. Let $S_0=0$, $S_n=\sum_{i=1}^n X_i$. De ne the range of $\{S_n\}_{n\geq 0}$ by $R_n=\#\{S_0,S_1,S_2,\cdots,S_n\}$, which is the number of distinct points visited by the random walk $\{S_n\}_{n\geq 0}$ up to time n.

(1) Prove $\mathbb{E}(R_n)=\mathbb{E}(R_{n-1})+P(S_1S_2\cdots S_n\neq 0),\quad n=1,2,\cdots.$

(2) Find $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}(R_n)$.

Solutions

(1)

$$\begin{aligned}P(R_n=R_{n-1}+1)&=P(S_n\notin\{S_0,S_1,\cdots S_{n-1}\})\\&=P(S_n\neq S_0,S_n\neq S_1,\cdots,S_n\neq S_{n-1})\\&=P(X_1+X_2+\cdots+X_n\neq 0,X_2+X_3+\cdots+X_n\neq 0,\cdots,X_n\neq 0)\\&=P(X_1\neq 0,X_1+X_2\neq 0,\cdots,X_1+X_2+\cdots+X_n\neq 0)\quad\text{(by i.i.d)}\\&=P(S_1S_2\cdots S_n\neq 0).\end{aligned}$$

Thus

$$\mathbb{E}(R_n)=\mathbb{E}(R_{n-1})+P(S_1S_2\cdots S_n\neq 0).$$

(2) Using the above relation recursively, one has

$$\frac{1}{n}\mathbb{E}(R_n)=\frac{1}{n}+\frac{1}{n}\sum_{k=1}^n P(S_1S_2\cdots S_k\neq 0)\overset{n\to\infty}{\longrightarrow}P(S_k\neq 0,\forall\ k\geq 1).$$

On the other hand, according to law of large numbers,

$$\lim_{n\to\infty}\frac{S_n}{n}=2p-1>0,\qquad\text{a.s.}$$

Thus

$$\begin{aligned}P(S_k\neq 0,\forall\ k\geq 1)&=P(S_k>0,\forall\ k\geq 1)\\&=\lim_{n\to\infty}P(S_k>0,\ k=1,2,\cdots,n)\end{aligned}$$

By the re ection principle,

$$P(S_k > 0,\ k = 1, 2, \cdots, n) = \frac{1}{n}\mathbb{E}(S_n \vee 0) \stackrel{n\to\infty}{\longrightarrow} 2p - 1.$$

Thus $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(R_n) = 2p - 1$.