ON NON-CONGRUENT NUMBERS AS MULTIPLES OF NON-CONGRUENT NUMBERS

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Abstract. Let $n = PQ$ be a square-free positive integer, where P is a product of primes congruent to 1 mod 8, and Q is a non-congruent number with a trivial 2-primary Shafarevich-Tate group. Under certain conditions on the Legendre symbols $\left(\frac{q}{p}\right)$ for primes $p \mid P, q \mid Q$, we establish a criteria characterizing when n is non-congruent with a minimal or a second minimal 2-primary Shafarevich-Tate group. We also provide a sufficient condition for n to be non-congruent with a larger 2-primary Shafarevich-Tate group. These results involve the 4-rank and 8-rank of certain class groups.

1. Introduction

1.1. **Background.** A square-free positive integer n is called *congruent* if it is the area of a right triangle with rational lengths. This is equivalent to say, the Mordell-Weil rank of E_n over $\mathbb{Q}$ is positive, where

$$E_n : y^2 = x^3 - n^2 x$$

is the associated congruent elliptic curve. Denote by $\mathrm{Sel}_2(E_n)$ the 2-*Selmer group* of E_n over $\mathbb{Q}$ and

$$s_2(n) := \dim_{\mathbb{F}_2} \left(\frac{\mathrm{Sel}_2(E_n)}{E_n(\mathbb{Q})[2]} \right) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E_n) - 2$$

the *pure* 2-*Selmer rank*. Then

$$s_2(n) = \mathrm{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) + \dim_{\mathbb{F}_2} Ш(E_n)[2]$$

by the exact sequence

$$0 \to E_n(\mathbb{Q})/2E_n(\mathbb{Q}) \to \mathrm{Sel}_2(E_n) \to Ш(E_n)[2] \to 0,$$

where $Ш(E_n)$ is the Shafarevich-Tate group of $E_n/\mathbb{Q}$.

Certainly, $s_2(n) = 0$ implies that n is non-congruent with $Ш(E_n)[2^\infty] = 0$. The examples of $s_2(n) = 0$ can be found in [Fen97], [Isk96] and [OZ15], which are corollaries of Monsky's formula (2.8) for $s_2(n)$. This case is fully characterized in terms of the 2-primary class groups of imaginary quadratic fields, and the full Birch-Swinnerton-Dyer conjecture holds, see [TYZ17, Theorem 1.1, Corollary 1.3] and [Smi16, Theorem 1.2].

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The examples of non-congruent n with $Ш(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ can be found in [LT00], [OZ14], [OZ15] and [Zha23]. Denote by

$$h_{2^a}(m) = \dim_{\mathbb{F}_2}\left(\frac{2^{a-1}\mathcal{A}_m}{2^a\mathcal{A}_m}\right) \tag{1.1}$$

the 2^a-rank of the narrow class group $\mathcal{A}_m$ of the quadratic field $\mathbb{Q}(\sqrt{m})$. Denote by $(a,b)_v$ the Hilbert symbol.

Theorem 1.1 ([Wan16, Theorem 1.1])**.** *Let $n = p_1 \cdots p_k \equiv 1 \bmod 8$ be a square-free positive integer with prime factors p_i such that $p_i \equiv 1 \bmod 4$ for all i. The following are equivalent:*

- *n is non-congruent with $Ш(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;*
- *$h_4(-n) = 1$ and $h_8(-n) \equiv (d-1)/4 \bmod 2$,*

where d is a positive divisor of n such that either $(d,-n)_v = 1, \forall v, d \neq 1, n$, or $(2d,-n)_v = 1, \forall v$.

Theorem 1.2 ([WZ22, Theorem 1.1])**.** *Let $n = p_1 \cdots p_k \equiv 1 \bmod 8$ be a square-free positive integer with prime factors p_i such that $p_i \equiv \pm 1 \bmod 8$ for all i. The following are equivalent:*

- *n is non-congruent with $Ш(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;*
- *$h_4(-n) = 1, h_8(-n) = 0$.*

Theorem 1.3 ([Zha23, Theorem 5.3])**.** *Let $n = p_1 \cdots p_k \equiv 1 \bmod 8$ be a square-free positive integer with prime factors p_i such that $p_i \equiv \pm 1 \bmod 8$ for all i. The following are equivalent:*

- *$2n$ is non-congruent with $Ш(E_{2n})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;*
- *$h_4(-n) = 1$ and $d \equiv 9 \bmod 16$,*

where d is the unique divisor of n such that $(d,n)_v = 1, \forall v$ and $d \neq 1, d \equiv 1 \bmod 4$.

The condition that $d \equiv 9 \bmod 16$ is equivalent to $h_8(-n) + h_8(-2n) = 1$, see Theorem 4.2. This recovers [LQ23, Theorem 1.6].

1.2. **Main results.** In this paper, we want to construct non-congruent numbers n with the form $n = PQ$, where

- P is a product of different primes $\equiv 1 \bmod 8$,
- Q is a non-congruent number prime to P, such that $Ш(E_Q)[2^\infty] = 0$.

Denote the prime decomposition of n by

$$n = \gcd(2,Q)p_1 \cdots p_k q_1 \cdots q_\ell,$$

where $P = p_1 \cdots p_k, Q = \gcd(2,Q)q_1 \cdots q_\ell$. Assume that there exists two vectors

$$\mathbf{u} = (u_1, \dots, u_k)^{\mathrm{T}} \in \mathbb{F}_2^k \quad \text{and} \quad \mathbf{v} = (v_1, \dots, v_\ell)^{\mathrm{T}} \in \mathbb{F}_2^\ell$$

such that the Legendre symbol $\left(\frac{p_i}{q_j}\right) = (-1)^{u_i v_j}$. Denote by

$$\mathbf{U}_P = \mathrm{diag}\{u_1, \dots, u_k\} \quad \text{and} \quad \mathbf{A}_P = (a_{ij})_{k\times k}$$

such that the Hilbert symbol $(p_j, -d)_{p_i} = (-1)^{a_{ij}}$.

1.2.1. $s_2(n) = 0$.

Theorem 1.4. *Assume that* $\sum_{i=1}^k u_i = 0, \sum_{j=1}^\ell v_j = 1$, $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ *and* Q *is non-congruent with* $\text{Ш}(E_Q)[2^\infty] = 0$. *The following are equivalent:*

- n *is non-congruent with* $\text{Ш}(E_n) = 0$;
- $\mathbf{A}_P + \mathbf{U}_P$ *is invertible.*

1.2.2. $s_2(n) = 2$.

Theorem 1.5. *Assume that* $\sum_{i=1}^k u_i = 0, \sum_{j=1}^\ell v_j = 1$, $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ *and* Q *is non-congruent with* $\text{Ш}(E_Q)[2^\infty] = 0$. *The following are equivalent:*

- n *is non-congruent with* $\text{Ш}(E_n) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $\operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ *and* $\left(\frac{\gamma}{d}\right) = -\left(\frac{\sqrt{2}+1}{d}\right)$,

where $(\delta_1, \ldots, \delta_k)$ *is the non-zero vector in the kernel of* $\mathbf{A}_P + \mathbf{U}_P$, $d = p_1^{\delta_1} \cdots p_k^{\delta_k}$ *and* (α, β, γ) *is a primitive positive solution of* $d\alpha^2 - \frac{n}{d}\beta^2 = 4\gamma^2$.

Here, a *primitive positive solution* of $d\alpha^2 - \frac{n}{d}\beta^2 = 4\gamma^2$ is an integer solution such that $\alpha, \beta, \gamma > 0$ and $\gcd(\alpha, \beta, \gamma) = 1$.

When $\mathbf{u} = \mathbf{0}$, we obtain the following result:

Corollary 1.6. *Assume that* $\left(\frac{p_i}{q_j}\right) = 1, \forall i, j$, $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ *and* Q *is non-congruent with* $\text{Ш}(E_Q)[2^\infty] = 0$. *The following are equivalent:*

- n *is non-congruent with* $\text{Ш}(E_n) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $h_4(-P) = 1$ *and* $\left(\frac{\gamma}{P}\right) = (-1)^{h_8(-P)}$,

where (α, β, γ) *is a primitive positive solution of* $P\alpha^2 - Q\beta^2 = 4\gamma^2$.

When $\ell = 0$, we obtain the following results, which are special cases of Theorems 1.1,1.2 and 1.3.

Corollary 1.7. *Let* $n = p_1 \cdots p_k$ *be a square-free integer where* $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$.

(1) *The following are equivalent:*
   - n *is non-congruent with* $\text{Ш}(E_n) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
   - $h_4(-n) = 1$ *and* $h_8(-n) = 0$.

(2) *The following are equivalent:*
   - $2n$ *is non-congruent with* $\text{Ш}(E_{2n}/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
   - $h_4(-n) = 1$ *and* $h_8(-n) + h_8(-2n) = 1$.

In fact, $h_8(-2n) = h_8(2n)$ in this case, see Corollary 4.5.

1.2.3. *General case.*

Theorem 1.8. *Assume that* $\left(\frac{p_i}{q_j}\right) = 1, \forall i, j$, $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ *and* Q *is non-congruent with* $\text{Ш}(E_Q)[2^\infty] = 0$. *If there is a decomposition* $P = f_1 \cdots f_r$ *such that*

- $h_4(f_i) = 0, \forall i$;
- $\left(\frac{p}{p'}\right) = 1$ *for any* $i \neq j$ *and prime factors* $p \mid f_i, p' \mid f_j$;
- $\left(\frac{\gamma_i}{f_j}\right) = 1$ *if* $i \neq j$; $\left(\frac{\gamma_i}{f_i}\right) = (-1)^{h_8(-f_i)}$,

then n *is non-congruent with* $\text{Ш}(E_n) \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$, *where* $(\alpha_i, \beta_i, \gamma_i)$ *is a primitive positive solution of* $f_i\alpha_i^2 - \frac{n}{f_i}\beta_i^2 = 4\gamma_i^2$.

When $\ell = 0$, we obtain the following results, where (1) is just [Wan16, Theorem 1.2].

Corollary 1.9. *Let $n = p_1 \cdots p_k$ be a square-free integer where $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$.*

(1) If there is a decomposition $n = f_1 \cdots f_r$ such that
- $h_4(f_i) = h_8(-f_i) = 0, \forall i$ *and* $h_8(n) = r - 1$;
- $\left(\frac{p}{p'}\right) = 1$ *for any* $i \neq j$ *and prime factors* $p \mid f_i, p' \mid f_j$,

then n is non-congruent with $Ш(E_n) \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$.

(2) If there is a decomposition $n = f_1 \cdots f_r$ such that
- $h_4(f_i) = h_8(-f_i) = 0, \forall i$ *and* $h_8(2n) = r$;
- $\left(\frac{p}{p'}\right) = 1$ *for any* $i \neq j$ *and prime factors* $p \mid f_i, p' \mid f_j$,

then $2n$ is non-congruent with $Ш(E_{2n}) \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$.

1.3. **Notations.** Denote by
- $\gcd(m, n)$ the greatest common divisor of integers m, n, where $m \neq 0$ or $n \neq 0$;
- $(a, b)_v$ the Hilbert symbol;
- $[a, b]_v$ the additive Hilbert symbol, i.e., the image of $(a, b)_v$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$;
- $\left(\frac{a}{b}\right) = \prod_{p \mid b} (a, b)_p$ the Jacobi symbol, where $\gcd(a, b) = 1$ and $b > 0$;
- $\left[\frac{a}{b}\right]$ the additive Jacobi symbol, i.e., the image of $\left(\frac{a}{b}\right)$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$;
- v_p the normalized valuation on $\mathbb{Q}_p$;
- $\mathbf{0} = (0, \ldots, 0)^{\mathrm{T}}$ and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$.

If n be a square-free positive integer, then we denote by
- $E_n : y^2 = x^3 - n^2 x$ the congruent elliptic curve associated to n;
- $\mathrm{Sel}_2(E_n)$ the 2-Selmer group of $E_n/\mathbb{Q}$;
- $Ш(E_n)$ the Shafarevich-Tate group of $E_n/\mathbb{Q}$;
- $\mathrm{Sel}_2'(E_n) := \mathrm{Sel}_2(E_n)/E_n(\mathbb{Q})[2]$ the pure 2-Selmer group of $E_n/\mathbb{Q}$;
- $s_2(n) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2'(E_n)$ the pure 2-Selmer rank of E_n.

If n is odd with a fixed ordered prime decomposition $n = p_1 \cdots p_k$, then we denote by
- $\mathbf{A}_n = \big([p_j, -n]_{p_i}\big)_{k \times k}$ a matrix associated to n, see (2.2);
- $\mathbf{D}_{n,\varepsilon} = \mathrm{diag}\left\{\left[\frac{\varepsilon}{p_1}\right], \ldots, \left[\frac{\varepsilon}{p_k}\right]\right\}$ a matrix associated to n and ε, see (2.3);
- $\mathbf{b}_{n,\varepsilon} = \mathbf{D}_{n,\varepsilon}\mathbf{1} = \left(\left[\frac{\varepsilon}{p_1}\right], \ldots, \left[\frac{\varepsilon}{p_k}\right]\right)^{\mathrm{T}}$;
- $\mathbf{M}_n$ (resp. $\mathbf{M}_{2n}$) the Monsky matrix of E_n (resp. E_{2n}), see (2.4) and (2.6);
- $\psi_n(d) = \big(v_{p_1}(d), \ldots, v_{p_1}(d)\big)^{\mathrm{T}}$ a vector over $\mathbb{F}_2$ associated to $0 < d \mid n$.

If $m \neq 0, 1$ is a square-free integer, then we denote by
- $F_m = \mathbb{Q}(\sqrt{m})$ a quadratic field;
- $\mathbf{R}_m$ the Rédei matrix of F_m, with a submatrix $\mathbf{R}_m'$, see (2.9) and (2.12);
- $\mathcal{A}_m$ the narrow class group of F_m;
- $D_m, \omega_m, \mathcal{O}_m, \mathscr{D}_m$ objects associated to F_m, see §2.3;
- $h_{2^a}(m)$ the 2^a-rank of $\mathcal{A}_m$, see (1.1);
- $\mathscr{D}_m$ the set of all square-free positive integers of the discriminant of F_m;
- $\theta_m : \mathscr{D}_m \to \mathcal{A}_m[2]$ a two-to-one onto homomorphism, see Proposition 2.2.

2. Preliminaries

2.1. **The Monsky matrix.** By the 2-descent method, Monsky in [HB94, Appendix] represented the pure 2-Selmer group

$$\mathrm{Sel}_2'(E_n) := \frac{\mathrm{Sel}_2(E_n)}{E_n(\mathbb{Q})[2]}$$

as the kernel of a matrix $\mathbf{M}_n$ over $\mathbb{F}_2$. Let's recall it roughly. One can identify $\mathrm{Sel}_2(E_n)$ with

$$\{\Lambda = (d_1, d_2, d_3) \in (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^3 : D_\Lambda(\mathbb{A}_\mathbb{Q}) \neq \emptyset, d_1 d_2 d_3 \equiv 1 \bmod \mathbb{Q}^{\times 2}\},$$

where D_Λ is a genus one curve defined by

$$\begin{cases} H_1: & -nt^2 + d_2 u_2^2 - d_3 u_3^2 = 0, \\ H_2: & -nt^2 + d_3 u_3^2 - d_1 u_1^2 = 0, \\ H_3: & 2nt^2 + d_1 u_1^2 - d_2 u_2^2 = 0. \end{cases} \tag{2.1}$$

Under this identification, $O, (n, 0), (-n, 0), (0, 0)$ and non-torsion $(x, y) \in E_n(\mathbb{Q})$ correspond to $(1, 1, 1), (2, 2n, n), (-2n, 2, -n), (-n, n, -1)$ and $(x - n, x + n, x)$ respectively.

Let n be an odd positive square-free integer with an ordered prime decomposition $n = p_1 \cdots p_k$. Denote by

$$\mathbf{A}_n := (a_{ij})_{k\times k} \quad \text{where} \quad a_{ij} = [p_j, -n]_{p_i} = \begin{cases} \left[\frac{p_j}{p_i}\right], & i \neq j; \\ \left[\frac{n/p_i}{p_i}\right], & i = j, \end{cases} \tag{2.2}$$

and

$$\mathbf{D}_{n,\varepsilon} := \mathrm{diag}\Big\{\Big[\frac{\varepsilon}{p_1}\Big], \dots, \Big[\frac{\varepsilon}{p_k}\Big]\Big\}. \tag{2.3}$$

Then $\mathbf{A}_n \mathbf{1} = \mathbf{0}$ and $\operatorname{corank} \mathbf{A}_n \geqslant 1$.

Monsky showed that each element in $\mathrm{Sel}_2'(E_n)$ can be represented as (d_1, d_2, d_3), where d_1, d_2, d_3 are all positive divisors of n. The system D_Λ is locally solvable everywhere if and only if certain conditions on the Hilbert symbols hold. Then we can express $\mathrm{Sel}_2'(E_n)$ as the kernel of the *Monsky matrix*

$$\mathbf{M}_n := \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{D}_{n,2} \\ \mathbf{D}_{n,2} & \mathbf{A}_n + \mathbf{D}_{n,-2} \end{pmatrix} \tag{2.4}$$

via the isomorphism

$$\begin{aligned} \mathrm{Sel}_2'(E_n) &\to \operatorname{Ker} \mathbf{M}_n \\ (d_1, d_2, d_3) &\mapsto \begin{pmatrix} \psi_n(d_2) \\ \psi_n(d_1) \end{pmatrix}, \end{aligned} \tag{2.5}$$

where $\psi_n(d) := \big(v_{p_1}(d), \dots, v_{p_k}(d)\big)^{\mathrm{T}} \in \mathbb{F}_2^k$ for any positive divisor d of n.

Similarly, each element in $\mathrm{Sel}_2'(E_{2n})$ can be represented as (d_1, d_2, d_3), where d_1, d_2, d_3 are all divisors of n and $d_2 > 0, d_3 \equiv 1 \bmod 4$. Then we can express $\mathrm{Sel}_2'(E_{2n})$ as the kernel of the *Monsky matrix*

$$\mathbf{M}_{2n} := \begin{pmatrix} \mathbf{A}_n^{\mathrm{T}} + \mathbf{D}_2 & \mathbf{D}_{n,-1} \\ \mathbf{D}_{n,2} & \mathbf{A}_n + \mathbf{D}_{n,2} \end{pmatrix} \tag{2.6}$$

via the isomorphism

$$\begin{aligned} \mathrm{Sel}_2'(E_{2n}) &\to \mathrm{Ker}\,\mathbf{M}_{2n} \\ (d_1,d_2,d_3) &\mapsto \begin{pmatrix} \psi_n(|d_3|) \\ \psi_n(d_2) \end{pmatrix}. \end{aligned} \tag{2.7}$$

In both cases, we have

$$s_2(n) := \dim_{\mathbb{F}_2} \mathrm{Sel}_2'(E_n) = \mathrm{corank}\,\mathbf{M}_n. \tag{2.8}$$

2.2. **The Cassels pairing.** Cassels in [Cas98] defined a (skew-)symmetric bilinear pairing $\langle -,- \rangle$ on the $\mathbb{F}_2$-vector space $\mathrm{Sel}_2'(E_n)$. For any $\Lambda \in \mathrm{Sel}_2(E_n)$, the equation H_i in (2.1) is locally solvable everywhere. Thus H_i is solvable over $\mathbb{Q}$ by the Hasse-Minkowski principal. Choose $Q_i \in H_i(\mathbb{Q})$ and let L_i be a linear form such that $L_i = 0$ defines the tangent plane of H_i at Q_i. For any $\Lambda' = (d_1', d_2', d_3') \in \mathrm{Sel}_2(E_n)$, define the *Cassels pairing*

$$\langle \Lambda, \Lambda' \rangle = \sum_v \langle \Lambda, \Lambda' \rangle_v \in \mathbb{F}_2 \quad \text{where} \quad \langle \Lambda, \Lambda' \rangle_v = \sum_{i=1}^{3} \big[L_i(P_v), d_i' \big]_v,$$

where $P_v \in D_\Lambda(\mathbb{Q}_v)$ for each place v of $\mathbb{Q}$. This pairing is independent of the choice of P_v, Q_i and the representative Λ. It is (skew-)symmetric and satisfies $\langle \Lambda, \Lambda \rangle = 0$.

Lemma 2.1 ([Cas98, Lemma 7.2])**.** *The local Cassels pairing $\langle -,- \rangle_v = 0$ if*

- $v \nmid 2\infty$,
- *the coefficients of H_i and L_i are all integral at v for $i = 1,2,3$, and*
- *modulo D_Λ and L_i by v, they define a curve of genus 1 over $\mathbb{F}_v$ together with tangents to it.*

2.3. **The Rédei matrix.** Denote by

- $F_m = \mathbb{Q}(\sqrt{m})$ a quadratic field, where $m \neq 0, 1$ is a square-free integer;
- D_m the discriminant of F_m;
- $\omega_m = (D_m + \sqrt{D_m})/2$;
- $\mathcal{O}_m = \mathbb{Z} + \mathbb{Z}\varepsilon$ the ring of integers of F_m;
- $\mathcal{A}_m$ the narrow class group of F_m;
- $\mathbf{N} = \mathbf{N}_{F_m/\mathbb{Q}}$ the norm map;
- $\mathscr{D}_m$ the set of all square-free positive integers $d \mid D_m$.

Proposition 2.2 ([Hec81, Chapter 7])**.** *(1) The map $\theta_m : \mathscr{D}_m \to \mathcal{A}_m[2]$ defined as*

$$\theta_m(d) = [(d, \omega_m)]$$

is a two-to-one onto homomorphism. In particular, $h_2(m) = \dim_{\mathbb{F}_2} \mathcal{A}_m[2] = t - 1$.

(2) Let $\mathfrak{a}$ be a non-zero fractional ideal of F_m. Then the ideal class $[\mathfrak{a}] \in \mathcal{A}_m^2$ if and only if $\mathbf{N}\mathfrak{a} \in \mathbf{N}F_m$.

Fix an ordered decomposition

$$D_m = p_1^* \cdots p_t^*, \quad \text{where} \quad p^* = \begin{cases} (-1)^{\frac{p-1}{2}} p, & \text{if } p \text{ is an odd prime;} \\ -4, 8, -8, & \text{if } p = 2. \end{cases}$$

To calculate $h_4(m)$, we need the Rédei matrix, which is defined as

$$\mathbf{R}_m = ([p_j, m]_{p_i})_{t \times t}. \tag{2.9}$$

Example 2.3. Let $n = p_1 \cdots p_k$ be an odd positive square-free integer. Denote by

$$\mathbf{b}_{n,\varepsilon} := \left(\left[\frac{\varepsilon}{p_1}\right], \dots, \left[\frac{\varepsilon}{p_k}\right]\right)^{\mathrm{T}} = \mathbf{D}_{n,\varepsilon}\mathbf{1}.$$

When $n \equiv 1 \bmod 4$, we have

$$\mathbf{R}_n = \mathbf{A}_n + \mathbf{D}_{n,-1}, \qquad \mathbf{R}_{-n} = \begin{pmatrix} \mathbf{A}_n & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-1}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix},$$
$$\mathbf{R}_{2n} = \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,2}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix}, \qquad \mathbf{R}_{-2n} = \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-2}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix}.$$

When $n \equiv -1 \bmod 4$, we have

$$\mathbf{R}_n = \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-1} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-1}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix}, \qquad \mathbf{R}_{-n} = \mathbf{A}_n,$$
$$\mathbf{R}_{2n} = \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-2}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix}, \qquad \mathbf{R}_{-2n} = \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,2}^{\mathrm{T}} & \left[\frac{2}{n}\right] \end{pmatrix}.$$

The following are equivalent:

- $d \in \mathscr{D}_m \cap \mathbf{N}F_m$;
- $X^2 - mY^2 = dZ^2$ is solvable over $\mathbb{Q}$;
- the Hilbert symbols $(d, m)_v = 1, \forall v$;
- $\mathbf{R}_m\mathbf{d} = \mathbf{0}$, where $\mathbf{d} = \left(v_{p_1}(d), \dots, v_{p_t}(d)\right)^{\mathrm{T}}$.

Rédei showed that θ_m induces a two-to-one onto homomorphism

$$\theta_m : \mathscr{D}_m \cap \mathbf{N}F_m \to \mathcal{A}_m[2] \cap \mathcal{A}_m^2, \tag{2.10}$$

which induces that

$$h_4(m) = \operatorname{corank} \mathbf{R}_m - 1. \tag{2.11}$$

Denote by

$$\mathbf{R}'_m = \begin{cases} \text{the submatrix of } \mathbf{R}_m \text{ by removing the last row}, & \text{if } 2 \mid D_m; \\ \mathbf{R}_m, & \text{otherwise}. \end{cases} \tag{2.12}$$

Since $\mathbf{1}^{\mathrm{T}}\mathbf{R}_m = \mathbf{0}^{\mathrm{T}}$, we have

$$\operatorname{rank} \mathbf{R}'_m = \operatorname{rank} \mathbf{R}_m. \tag{2.13}$$

When $m < 0$ or m has no prime factor congruent to -1 modulo 4, the kernel of θ_m is $\{1, |m|\}$. See [Rè34] and [LY20, Example 2.6].

Proposition 2.4. *Let $n = p_1 \cdots p_k$ be an odd positive square-free integer. If all $p_i \equiv \pm 1 \bmod 8$ and $n \equiv 1 \bmod 8$, then*

$$h_4(n) + 1 = h_4(2n) = h_4(-n) = h_4(-2n) = \operatorname{corank} \mathbf{A}_n.$$

Proof. By the quadratic reciprocity law, we have

$$\mathbf{A}_n^{\mathrm{T}} = \mathbf{A}_n + \mathbf{D}_{n,-1} + \mathbf{b}_{n,-1}\mathbf{b}_{n,-1}^{\mathrm{T}}. \tag{2.14}$$

Since $\mathbf{b}_{n,-1}^{\mathrm{T}}\mathbf{b}_{n,-1} = \mathbf{b}_{n,-1}^{\mathrm{T}}\mathbf{1} = \left[\frac{-1}{n}\right] = 0$, one can show that

$$\mathbf{A}_n^{\mathrm{T}}(\mathbf{I} + \mathbf{1}\mathbf{b}_{n,-1}^{\mathrm{T}}) = \mathbf{A}_n + \mathbf{D}_{n,-1},$$

where $\mathbf{I} + \mathbf{1}\mathbf{b}_{n,-1}^{\mathrm{T}}$ is invertible by $(\mathbf{I} + \mathbf{1}\mathbf{b}_{n,-1}^{\mathrm{T}})^2 = \mathbf{I}$. Thus

$$\operatorname{rank} \mathbf{R}_n = \operatorname{rank} \mathbf{R}'_{-n} = \operatorname{rank} \mathbf{R}'_{\pm 2n},$$

which concludes the result by (2.11) and (2.13). $\square$

The 8-rank $h_8(m)$ can be obtained by the following proposition, which is similar to [Wan16, Proposition 3.6]. See also [JY11, Lu15].

Proposition 2.5. *For any $d \in \mathscr{D}_m \cap \mathbf{N}F_m$, let (α,β,γ) be a primitive positive solution of*

$$d\alpha^2 - \frac{m}{d}\beta^2 = 4\gamma^2.$$

Then

(1) $\theta_m(d) \in \mathcal{A}_m^4$ if and only if $\big([\gamma,m]_{p_1},\dots,[\gamma,m]_{p_t}\big)^{\mathrm{T}} \in \operatorname{Im}\mathbf{R}_m$;

(2) $\sum_{i=1}^t [\gamma,m]_{p_i} = 0$.

In particular, $\theta_m(d) \in \mathcal{A}_m^4$ if and only if $\mathbf{b}_{n,\gamma} \in \operatorname{Im}\mathbf{R}'_m$, where n is the positive odd part of m.

Proof. Denote by σ the non-trivial automorphism of $\mathbb{Q}(\sqrt{m})$. If p is an odd prime factor of γ, then $p \nmid m$ and $\left(\frac{m}{p}\right) = 1$. Thus $(p) = \mathfrak{p}\mathfrak{p}^\sigma$ is split in F_m and $[\gamma,m]_p = 0$. We will show that $x = (d\alpha + \beta\sqrt{m})/2 \in \mathcal{O}_m$.

- If d is odd and m is even, then both of α and β are even and $x \in \mathcal{O}_m$.
- If d, m are odd, then α and β have same parities. If moreover both of α and β are odd, then $4 \mid (d - m/d)$, $m \equiv 1 \bmod 4$ and $x \in \mathcal{O}_m$.
- If d is even, then β is even and $x \in \mathcal{O}_m$.

Certainly, $p \mid d\gamma^2 = \mathbf{N}(x)$. If both $\mathfrak{p},\mathfrak{p}^\sigma$ divide $x\mathcal{O}_m$, then $p\mathcal{O}_m \mid x\mathcal{O}_m$ and $p \mid \alpha,\beta,\gamma$, which contradicts to $\gcd(\alpha,\beta,\gamma) = 1$. Hence only one of $\mathfrak{p}$ and $\mathfrak{p}^\sigma$ divides $x\mathcal{O}_m$. We may assume that $\mathfrak{p}^\sigma \mid x\mathcal{O}_m$ for each odd $p \mid \gamma$.

Assume that d is odd. If γ is odd, we have

$$x\mathcal{O}_m = \mathfrak{d}\prod_{p\mid\gamma}(\mathfrak{p}^\sigma)^{2v_p(\gamma)} = \gamma^2\mathfrak{d}\mathfrak{c}^{-2}, \quad \text{where } \mathfrak{c} := \prod_{p\mid\gamma}\mathfrak{p}^{v_p(\gamma)} \text{ with } \mathbf{N}\mathfrak{c} = \gamma \tag{2.15}$$

and $\mathfrak{d} = (d,\omega_m)$. If γ is even, one can show that m is odd. Then both of α and β are odd, $8 \mid (d - m/d)$ and $m \equiv 1 \bmod 8$. Thus $2\mathcal{O}_m = \mathfrak{q}\mathfrak{q}^\sigma$ is split in F. Similarly, only one of $\mathfrak{q}$ and $\mathfrak{q}^\sigma$ divides $x\mathcal{O}_m$. We may assume that $\mathfrak{q}^\sigma \mid x\mathcal{O}_m$. Hence we also have (2.15), where $\mathfrak{p}$ is $\mathfrak{q}$ for $p = 2$.

Assume that d is even. Then D_m is even, $m \not\equiv 1 \bmod 4$ and $2\mathcal{O}_m = \mathfrak{q}^2$ is ramified in F. Similarly, we have (2.15), where $\mathfrak{p} = \mathfrak{p}^\sigma = \mathfrak{q}$ for $p = 2$.

(1) By (2.15), we have $[\mathfrak{d}] = [\mathfrak{c}]^2$. Clearly, $[\mathfrak{d}] \in \mathcal{A}_m^4$ if and only if $[\mathfrak{c}] + [(a,\omega_m)] \in \mathcal{A}_m^2$ for some $a \in \mathscr{D}_m$. This is equivalent to $a\mathbf{N}\mathfrak{c} = a\gamma \in \mathbf{N}F_m$ by Proposition 2.2. Note that

   - $[a\gamma,m]_p = 1$ for any odd prime $p \mid \gamma$;
   - $[a\gamma,m]_\infty = 1$ because $a\gamma > 0$;
   - if $2 \nmid D_m$ and γ is odd, then a is odd and $m \equiv 1 \bmod 4$; if $2 \nmid D_m$ and γ is even, then $m \equiv 1 \bmod 8$.

   In other words, $[a\gamma,m]_v = 1$ for all $v \nmid D_m$. Thus $a\gamma \in \mathbf{N}F_m$ if and only if $[a,m]_{p_i} = [\gamma,m]_{p_i}$ for all $p_i \mid D_m$, if and only if

$$\mathbf{R}_m\big(v_{p_1}(a),\dots,v_{p_t}(a)\big)^{\mathrm{T}} = \big([\gamma,m]_{p_1},\dots,[\gamma,m]_{p_t}\big)^{\mathrm{T}}.$$

(2) Denote by γ_0 the odd part of γ. If $m \not\equiv 1 \bmod 4$, then D_m is even and

$$\sum_{i=1}^t [\gamma,m]_{p_i} = \sum_{p\mid\gamma_0}[\gamma,m]_p = 0.$$

Here, $[\gamma,m]_\infty = 0$ because $\gamma > 0$. If $m \equiv 1 \bmod 4$ and γ is odd, then $[\gamma,m]_2 = 0$; if $m \equiv 1 \bmod 4$ and γ is even, then $m \equiv 1 \bmod 8$ and $[\gamma,m]_2 = 0$, as shown in the proof of (1). Therefore

$$\sum_{i=1}^{t}[\gamma,m]_{p_i} = \sum_{p\mid\gamma_0}[\gamma_0,m]_p + [\gamma,m]_2 = 0. \qquad \square$$

3. The Selmer groups and the Cassles pairings

Let $n = PQ$ be a square-free positive integer with an ordered prime decomposition

$$n = \gcd(2,n)p_1\cdots p_k q_1\cdots q_\ell,$$

where $P = p_1\cdots p_k, Q = \gcd(2,n)q_1\cdots q_\ell$. Assume that $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ and there exists

$$\mathbf{u} = (u_1,\ldots,u_k)^{\mathrm{T}} \in \mathbb{F}_2^k, \qquad \mathbf{v} = (v_1,\ldots,v_\ell)^{\mathrm{T}} \in \mathbb{F}_2^\ell$$

such that the Legendre symbol $\left[\frac{p_i}{q_j}\right] = u_i v_j$. Clearly,

$$\mathbf{1}^{\mathrm{T}}\mathbf{u} = \sum_{i=1}^{k} u_i \quad \text{and} \quad \mathbf{1}^{\mathrm{T}}\mathbf{v} = \sum_{j=1}^{\ell} v_j.$$

Lemma 3.1. *Assume that $\mathbf{1}^{\mathrm{T}}\mathbf{u} = 0, \mathbf{1}^{\mathrm{T}}\mathbf{v} = 1$, $p_1 \equiv \cdots \equiv p_k \equiv 1 \bmod 8$ and Q is non-congruent with $\text{Ш}(E_Q)[2^\infty] = 0$. Then*

$$\operatorname{Ker}\mathbf{M}_n = \left\{ \begin{pmatrix}\mathbf{x}\\ \mathbf{0}\\ \mathbf{z}\\ \mathbf{0}\end{pmatrix} \;\middle|\; \mathbf{x},\mathbf{z} \in \operatorname{Ker}(\mathbf{A}_P + \mathbf{U}_P) \right\}$$

In particular, $s_2(n) = 2\operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P)$.

Proof. Note that $\mathbf{A}_n\mathbf{1} = \mathbf{0}$ and $\mathbf{A}_P^{\mathrm{T}} = \mathbf{A}_P$. By our assumptions,

$$\mathbf{A}_n = \begin{pmatrix}\mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}}\\ \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q\end{pmatrix} \quad \text{and} \quad \mathbf{A}_n^{\mathrm{T}} = \begin{pmatrix}\mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}}\\ \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q^{\mathrm{T}}\end{pmatrix}.$$

Note that $\mathbf{D}_{P,\pm 2} = \mathbf{O}_k$. If Q is odd, we have

$$\mathbf{M}_n = \begin{pmatrix}\mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}} & \mathbf{O}_k & \\ \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q + \mathbf{D}_{Q,2} & & \mathbf{D}_{Q,2}\\ \mathbf{O}_k & & \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}}\\ & \mathbf{D}_{Q,2} & \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q + \mathbf{D}_{Q,-2}\end{pmatrix}.$$

If Q is even, we have

$$\mathbf{M}_n = \begin{pmatrix}\mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}} & \mathbf{O}_k & \\ \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q^{\mathrm{T}} + \mathbf{D}_{Q,2} & & \mathbf{D}_{Q,-1}\\ \mathbf{O}_k & & \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^{\mathrm{T}}\\ & \mathbf{D}_{Q,2} & \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{A}_Q + \mathbf{D}_{Q,2}\end{pmatrix}.$$

If

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\\ \mathbf{w}\end{pmatrix} \in \operatorname{Ker}\mathbf{M}_n,$$

then

$$(\mathbf{A}_P + \mathbf{U}_P)\mathbf{x} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{y}, \qquad (\mathbf{A}_P + \mathbf{U}_P)\mathbf{z} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{w}$$

and

$$\mathbf{M}_Q \begin{pmatrix} \mathbf{y} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{v}\mathbf{u}^{\mathrm{T}}\mathbf{x} \\ \mathbf{v}\mathbf{u}^{\mathrm{T}}\mathbf{z} \end{pmatrix}.$$

Since $\mathbf{A}_P = \mathbf{A}_P^{\mathrm{T}}$, we have $\mathbf{1}^{\mathrm{T}}\mathbf{A}_P = \mathbf{0}^{\mathrm{T}}$ and

$$0 = \mathbf{1}^{\mathrm{T}}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{y} = \mathbf{1}^{\mathrm{T}}(\mathbf{A}_P + \mathbf{U}_P)\mathbf{x} = \mathbf{1}^{\mathrm{T}}\mathbf{U}_P\mathbf{x} = \mathbf{u}^{\mathrm{T}}\mathbf{x}. \tag{3.1}$$

Similarly, $\mathbf{u}^{\mathrm{T}}\mathbf{z} = 0$. Thus

$$\mathbf{M}_Q \begin{pmatrix} \mathbf{y} \\ \mathbf{w} \end{pmatrix} = \mathbf{0}.$$

Since $s_2(Q) = 0$, $\mathbf{M}_Q$ is invertible and we have $\mathbf{y} = \mathbf{w} = \mathbf{0}$. Thus $\mathbf{x}, \mathbf{z} \in \mathrm{Ker}(\mathbf{A}_P + \mathbf{U}_P)$,

$$\mathrm{Ker}\,\mathbf{M}_n = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \\ \mathbf{z} \\ \mathbf{0} \end{pmatrix} \middle| \mathbf{x}, \mathbf{z} \in \mathrm{Ker}(\mathbf{A}_P + \mathbf{U}_P) \right\}$$

and $s_2(n) = 2\,\mathrm{corank}(\mathbf{A}_P + \mathbf{U}_P)$. □

Proposition 3.2. *Let f_i, f_j be two positive divisors of P such that* $\gcd(f_i, f_j) = 1$ *and $\psi_P(f_i), \psi_P(f_j) \in \mathrm{Ker}(\mathbf{A}_P + \mathbf{U}_P)$. Denote by*

$$\Lambda_t = (f_t, 1, f_t) \quad \text{and} \quad \Lambda_t' = (f_t, f_t, 1)$$

for $t = i, j$. Then

$$\begin{aligned}
\langle \Lambda_i', \Lambda_i \rangle &= \left[\frac{\sqrt{2}+1}{f_i}\right] + \left[\frac{\gamma_i}{f_i}\right] = \left[\frac{\sqrt{2}+1}{f_i}\right] + \left[\frac{\gamma_i'}{f_i}\right], \\
\langle \Lambda_i', \Lambda_j \rangle &= \left[\frac{\gamma_j}{f_i}\right] = \left[\frac{\gamma_i'}{f_j}\right], \\
\langle \Lambda_i', \Lambda_i' \rangle &= \left[\frac{\gamma_i\gamma_i'}{f_i}\right], \qquad \langle \Lambda_i', \Lambda_j' \rangle = \left[\frac{\gamma_i\gamma_i'}{f_j}\right],
\end{aligned}$$

where $(\alpha_i, \beta_i, \gamma_i)$ $\big($resp. $(\alpha_i', \beta_i', \gamma_i'), (\alpha_i'', \beta_i'', \gamma_i'')\big)$ is a primitive positive solution of

$$f_i\alpha_i^2 - \frac{n}{f_i}\beta_i^2 = 4\gamma_i^2 \quad \left(\text{resp. } f_i\alpha_i'^2 + \frac{n}{f_i}\beta_i'^2 = 4\gamma_i'^2, f_i\alpha_i''^2 - \frac{2n}{f_i}\beta_i''^2 = 4\gamma_i''^2\right).$$

Proof. It's easy to see that $\alpha_i, \beta_i, \gamma_i, \alpha_i', \beta_i', \gamma_i', \alpha_i'', \beta_i'', \gamma_i''$ are coprime to $n/\gcd(2, n)$.

(1) Recall that D_{Λ_i} is defined by

$$\begin{cases} H_1: & -nt^2 + u_2^2 - f_iu_3^2 = 0, \\ H_2: & -\frac{n}{f_i}t^2 + u_3^2 - u_1^2 = 0, \\ H_3: & 2nt^2 + f_iu_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned}
Q_1 &= (\beta_i, f_i\alpha_i, 2\gamma_i) \in H_1(\mathbb{Q}), & L_1 &= \frac{n}{f_i}\beta_i t - \alpha_i u_2 + 2\gamma_i u_3, \\
Q_2 &= (0, 1, -1) \in H_2(\mathbb{Q}), & L_2 &= u_3 + u_1, \\
Q_3 &= (\beta_i'', 2\gamma_i'', f_i\alpha_i'') \in H_3(\mathbb{Q}), & L_3 &= \frac{2n}{f_i}\beta_i'' t + 2\gamma_i'' u_1 - \alpha_i'' u_2.
\end{aligned}$$

Since $\mathbf{u}^{\mathrm{T}}\psi_P(f_t) = 0$ by (3.1), we have

$$\left[\frac{f_t}{q_s}\right] = \sum_{p_r \mid f_t} u_r v_s = v_s \mathbf{u}^{\mathrm{T}} \psi_P(f_t) = 0. \tag{3.2}$$

If $v = p_s \mid P$, then $\left[\frac{q_t}{p_s}\right] = \left[\frac{p_s}{q_t}\right] = u_s v_t$ and $p_s \equiv 1 \bmod 8$. Thus we have

$$\left[\frac{Q}{p_s}\right] = u_s \mathbf{v}^{\mathrm{T}} \mathbf{1} = u_s.$$

One can see that the s-th entry of the vector $(\mathbf{A}_P + \mathbf{U}_P)\psi_P(f_i)$ is

$$0 = u_s + \sum_{p \mid f_i} [p, -P]_{p_s} = \left[\frac{Q}{p_s}\right] + [f_i, -P]_{p_s} = \left[\frac{Q}{p_s}\right] + \left[\frac{P/f_i}{p_s}\right] = \left[\frac{n/f_i}{p_s}\right]$$

if $p_s \mid f_i$;

$$0 = \sum_{p \mid f_i} [p, -P]_{p_s} = [f_i, -P]_{p_s} = \left[\frac{f_i}{p_s}\right].$$

if $p_s \mid \frac{P}{f_i}$. In particular, $[f_i, f_i]_v = [f_i, f_j]_v = 0$ for any $v \mid P$.

(i) The case $v = p_s \mid f_i$. Take

$$P_v = (t, u_1, u_2, u_3) = \Big(1, \sqrt{-\frac{2n}{f_i}}, 0, \sqrt{-\frac{n}{f_i}}\Big).$$

Note that

$$\Big(\beta_i \sqrt{-\frac{n}{f_i}} + 2\gamma_i\Big)\Big(-\beta_i \sqrt{-\frac{n}{f_i}} + 2\gamma_i\Big) = f_i \alpha_i^2$$

and one of $\pm\beta_i\sqrt{-n/f_i} + 2\gamma_i$ is congruent to $4\gamma_i$ modulo v. Thus

$$\big[L_1(P_v), f_t\big]_v = \Big[4\gamma_i \sqrt{-\frac{n}{f_i}}, f_t\Big]_v = \Big[\gamma_i \sqrt{-\frac{n}{f_i}}, f_t\Big]_v.$$

Similarly,

$$\big[L_2(P_v), f_t\big]_v = \Big[(\sqrt{2}+1)\sqrt{-\frac{n}{f_i}}, f_t\Big]_v,$$
$$\big[L_3(P_v), f_t\big]_v = \Big[4\sqrt{2}\gamma_i'' \sqrt{-\frac{n}{f_i}}, f_t\Big]_v = \Big[\sqrt{2}\gamma_i'' \sqrt{-\frac{n}{f_i}}, f_t\Big]_v.$$

Then

$$\big[L_1 L_2(P_v), f_t\big]_v = \big[(\sqrt{2}+1)\gamma_i, f_t\big]_v,$$
$$\big[L_1 L_3(P_v), f_t\big]_v = \big[\sqrt{2}\gamma_i \gamma_i'', f_t\big]_v.$$

(ii) The case $v = p_s \mid \frac{P}{f_i}$. Take

$$P_v = (t, u_1, u_2, u_3) = \big(0, 1, \sqrt{f_i}, 1\big).$$

Similarly, we have

$$\big[L_1(P_v), f_t\big]_v = [4\gamma_i, f_t]_v = [\gamma_i, f_t]_v,$$
$$\big[L_2(P_v), f_t\big]_v = [2, f_t]_v = 0,$$
$$\big[L_3(P_v), f_t\big]_v = [4\gamma_i'', f_t]_v = [\gamma_i'', f_t]_v,$$

and then

$$\begin{aligned}\big[L_1L_2(P_v), f_t\big]_v &= [\gamma_i, f_t]_v,\\ \big[L_1L_3(P_v), f_t\big]_v &= [\gamma_i\gamma_i'', f_t]_v.\end{aligned}$$

By Lemma 2.1 and (3.2), we have

$$\begin{aligned}\langle\Lambda_i,\Lambda_i\rangle &= \sum_{v|f_i}\big[\sqrt{2}\gamma_i\gamma_i'', f_i\big]_v + \sum_{v|\frac{P}{f_i}}[\gamma_i\gamma_i'', f_i]_v = \Big[\frac{\sqrt{2}\gamma_i\gamma_i''}{f_i}\Big],\\ \langle\Lambda_i,\Lambda_j\rangle &= \sum_{v|f_i}\big[\sqrt{2}\gamma_i\gamma_i'', f_j\big]_v + \sum_{v|\frac{P}{f_i}}[\gamma_i\gamma_i'', f_j]_v = \Big[\frac{\gamma_i\gamma_i''}{f_j}\Big],\\ \langle\Lambda_i,\Lambda_i'\rangle &= \sum_{v|f_i}\big[(\sqrt{2}+1)\gamma_i, f_i\big]_v + \sum_{v|\frac{P}{f_i}}[\gamma_i, f_i]_v = \Big[\frac{(\sqrt{2}+1)\gamma_i}{f_i}\Big],\\ \langle\Lambda_i,\Lambda_j'\rangle &= \sum_{v|f_i}\big[(\sqrt{2}+1)\gamma_i, f_j\big]_v + \sum_{v|\frac{P}{f_i}}[\gamma_i, f_j]_v = \Big[\frac{\gamma_i}{f_j}\Big],\end{aligned} \tag{3.3}$$

(2) Recall that $D_{\Lambda_i'}$ is defined by

$$\begin{cases} H_1: & -nt^2 + f_iu_2^2 - u_3^2 = 0,\\ H_2: & -nt^2 + u_3^2 - f_iu_1^2 = 0,\\ H_3: & \frac{2n}{f_i}t^2 + u_1^2 - u_2^2 = 0.\end{cases}$$

Choose

$$\begin{aligned} Q_1 &= (\beta_i', 2\gamma_i', f_i\alpha_i') \in H_1(\mathbb{Q}), & L_1 &= -\frac{n}{f_i}\beta_i't + 2\gamma_i'u_2 - \alpha_iu_3,\\ Q_2 &= (\beta_i, f_i\alpha_i, 2\gamma_i) \in H_2(\mathbb{Q}), & L_2 &= \frac{n}{f_i}\beta_it - \alpha_iu_3 + 2\gamma_iu_1,\\ Q_3 &= (0,1,-1) \in H_3(\mathbb{Q}), & L_3 &= u_1 + u_2.\end{aligned}$$

Similar to (1), we have

$$\langle\Lambda_i,\Lambda_i'\rangle = \sum_{v|P}\big[L_1L_2(P_v), f_i\big]_v$$

for any $P_v \in D_\Lambda(\mathbb{Q}_v)$.

(i) The case $v \mid f_i$. Take

$$P_v = (t, u_1, u_2, u_3) = \Big(1, \sqrt{-\frac{n}{f_i}}, \sqrt{\frac{n}{f_i}}, 0\Big).$$

Similarly, we have

$$\begin{aligned}\big[L_1(P_v), f_t\big]_v &= \Big[4\gamma_i'\sqrt{\frac{n}{f_i}}, f_t\Big]_v = \Big[\gamma_i'\sqrt{\frac{n}{f_i}}, f_t\Big]_v,\\ \big[L_2(P_v), f_t\big]_v &= \Big[4\gamma_i\sqrt{-\frac{n}{f_i}}, f_t\Big]_v = \Big[\gamma_i\sqrt{-\frac{n}{f_i}}, f_t\Big]_v,\\ \big[L_3(P_v), f_t\big]_v &= \Big[(\sqrt{-1}+1)\sqrt{\frac{n}{f_i}}, f_t\Big]_v,\end{aligned}$$

and then

$$\begin{aligned}\big[L_1L_2(P_v), f_t\big]_v &= \big[\sqrt{-1}\gamma_i\gamma_i', f_t\big]_v,\\ \big[L_1L_3(P_v), f_t\big]_v &= \big[(\sqrt{-1}+1)\gamma_i', f_t\big]_v.\end{aligned}$$

(ii) The case $v \mid \frac{P}{f_i}$. Take

$$P_v = (t, u_1, u_2, u_3) = \big(0, 1, 1, \sqrt{f_i}\big).$$

Similarly, we have

$$\begin{aligned}\big[L_1(P_v), f_t\big]_v &= [4\gamma_i', f_t]_v = [\gamma_i', f_t]_v,\\ \big[L_2(P_v), f_t\big]_v &= [4\gamma_i, f_t]_v = [\gamma_i, f_t]_v,\\ \big[L_3(P_v), f_t\big]_v &= [2, f_t]_v = 0,\end{aligned}$$

and then

$$\begin{aligned}\big[L_1L_2(P_v), f_t\big]_v &= [\gamma_i\gamma_i', f_t]_v,\\ \big[L_1L_3(P_v), f_t\big]_v &= [\gamma_i', f_t]_v.\end{aligned}$$

By Lemma 2.1 and (3.2), we have

$$\begin{aligned}\langle\Lambda_i', \Lambda_i'\rangle &= \sum_{v\mid f_i}[\sqrt{-1}\gamma_i\gamma_i', f_i]_v + \sum_{v\mid\frac{P}{f_i}}[\gamma_i\gamma_i', f_i]_v = \Big[\frac{\gamma_i\gamma_i'}{f_i}\Big],\\ \langle\Lambda_i', \Lambda_j'\rangle &= \sum_{v\mid f_i}[\sqrt{-1}\gamma_i\gamma_i', f_j]_v + \sum_{v\mid\frac{P}{f_i}}[\gamma_i\gamma_i', f_j]_v = \Big[\frac{\gamma_i\gamma_i'}{f_j}\Big],\\ \langle\Lambda_i', \Lambda_i\rangle &= \sum_{v\mid f_i}[(\sqrt{-1}+1)\gamma_i', f_i]_v + \sum_{v\mid\frac{P}{f_i}}[\gamma_i', f_i]_v = \Big[\frac{(\sqrt{2}+1)\gamma_i'}{f_i}\Big],\\ \langle\Lambda_i', \Lambda_j\rangle &= \sum_{v\mid f_i}[(\sqrt{-1}+1)\gamma_i', f_j]_v + \sum_{v\mid\frac{P}{f_i}}[\gamma_i', f_j]_v = \Big[\frac{\gamma_i'}{f_j}\Big],\end{aligned} \tag{3.4}$$

Here, we use the fact that

$$\begin{aligned}4\sqrt{-1} &= (\sqrt{2}+\sqrt{-2})^2,\\ (\sqrt{2}+1)(\sqrt{-1}+1) &= \frac{1}{2}(\sqrt{2}+\sqrt{-1}+1)^2\end{aligned}$$

are squares in $\mathbb{Q}_v$. Finally, we conclude the results by (3.3) and (3.4). $\square$

4. Proof of main theorems

Lemma 4.1. *The following are equivalent:*

- *n is non-congruent with $\text{Ш}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{s_2(n)}$;*
- *the Cassels pairing on $\mathrm{Sel}_2'(E_n)$ is non-degenerate.*

Proof. The proof is due to [Wan16, pp 2146, 2157]. Since

$$0 \to E_n[2] \to E_n[4] \xrightarrow{\times 2} E_n[2] \to 0$$

is exact, we have the long exact sequence

$$0 \to \frac{E_n(\mathbb{Q})[2]}{2E_n(\mathbb{Q})[4]} \to \mathrm{Sel}_2(E_n) \to \mathrm{Sel}_4(E_n) \to \mathrm{Im}\,\mathrm{Sel}_4(E_n) \to 0,$$

where $\operatorname{Im}\mathrm{Sel}_4(E_n)$ is the image of $\mathrm{Sel}_4(E_n) \xrightarrow{\times 2} \mathrm{Sel}_2(E_n)$. It's known that the kernel of the Cassels pairing on $\mathrm{Sel}_2(E_n)$ is $\operatorname{Im}\mathrm{Sel}_4(E_n)$. Thus

$$\operatorname{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) = 0, \quad Ш(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{s_2(n)}$$

if and only if $\#\mathrm{Sel}_2(E_n) = \#\mathrm{Sel}_4(E_n)$, if and only if $\operatorname{Im}\mathrm{Sel}_4(E_n) = E_n[2]$ in $\mathrm{Sel}_2(E_n)$, if and only if the Cassels pairing on $\mathrm{Sel}_2'(E_n)$ is non-degenerate. □

Theorem 4.2. *Let $n = p_1 \cdots p_k \equiv 1 \bmod 8$ be a square-free positive integer with odd prime factors p_i such that $p_i \equiv \pm 1 \bmod 8$ for all i. The following are equivalent:*

- *$2n$ is non-congruent with $Ш(E_{2n})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;*
- *$h_4(-n) = 1$ and $h_8(-n) + h_8(-2n) = 1$.*

Proof. Assume that $h_4(-n) = 1$. By Theorem 1.3, we only need to show that

$$h_8(-n) + h_8(-2n) \equiv \frac{d-1}{8} \bmod 2,$$

where d is a divisor of n such that $d \neq 1, d \equiv 1 \bmod 4$ and $(d, n)_v = 1, \forall v$. Notice that $d = \left(\frac{-1}{|d|}\right)|d|$ and

$$\begin{aligned} 0 = [d, n]_{p_i} &= [d, -1]_{p_i} + [d, -n]_{p_i} \\ &= [d, -1]_{p_i} + [|d|, -n]_{p_i} + \left[\frac{-1}{|d|}\right][-1, -n]_{p_i} \\ &= [d, -1]_{p_i} + [|d|, -n]_{p_i} + \left[\frac{-1}{|d|}\right]\left[\frac{-1}{p_i}\right], \end{aligned}$$

we have

$$\begin{aligned} \mathbf{0} &= \mathbf{D}_{n,-1}\psi_P(|d|) + \mathbf{A}_n\psi_P(|d|) + \left[\frac{-1}{|d|}\right]\mathbf{b}_{n,-1} \\ &= (\mathbf{A}_n + \mathbf{D}_{n,-1})\psi_P(|d|) + \mathbf{b}_{n,-1}\mathbf{b}_{n,-1}^{\mathrm{T}}\psi_P(|d|) = \mathbf{A}_n^{\mathrm{T}}\psi_P(|d|) \end{aligned}$$

by (2.14). Since $h_4(-n) = 1$, by Proposition 2.4, we have $\operatorname{corank} \mathbf{A}_n = 1$ and

$$\mathbf{R}'_{-n} = \mathbf{R}'_{-2n} = \begin{pmatrix} \mathbf{A}_n & \mathbf{0} \end{pmatrix}.$$

Since $\psi_P(|d|)^{\mathrm{T}}\mathbf{A}_n = \mathbf{0}^{\mathrm{T}}$, we have

$$\operatorname{Im}\mathbf{R}'_{-n} = \operatorname{Im}\mathbf{R}'_{-2n} = \left\{\mathbf{x} \;\middle|\; \psi_P(|d|)^{\mathrm{T}}\mathbf{x} = 0\right\}.$$

Since $h_4(-n) = 1$, we have $\mathcal{A}_{-n}[2] \cap \mathcal{A}_{-n}^2 = \{[(1)], [\mathfrak{d}]\}$ by (2.10), where $\mathfrak{d} = (2, \sqrt{-n})$. By Proposition 2.5, $h_8(-n) = 1$ is equivalent to $\psi_P(|d|)^{\mathrm{T}}\mathbf{b}_{n,\gamma} = 0$, where (α, β, γ) is a primitive positive solution of $2\alpha^2 + \frac{n}{2}\beta^2 = 4\gamma^2$. Write $n = 2a^2 - b^2$. Then we may take $(\alpha, \beta, \gamma) = (b, 2, a)$. Thus

$$1 - h_8(-n) = \psi_P(d)^{\mathrm{T}}\mathbf{b}_{n,a} = \left[\frac{a}{d}\right].$$

Similarly, $(a + b, 2, 2a + b)$ is a primitive positive solution of $2\alpha''^2 + n\beta''^2 = 4\gamma''^2$ and then

$$1 - h_8(-2n) = \psi_P(d)^{\mathrm{T}}\mathbf{b}_{n,2a+b} = \left[\frac{2a+b}{|d|}\right].$$

Therefore,

$$h_8(-n) + h_8(-2n) = \left[\frac{a(2a+b)}{|d|}\right] = \left[\frac{2 + b/a}{|d|}\right] = \left[\frac{2+\sqrt{2}}{|d|}\right],$$

which is congruent to $(d-1)/8$ modulo 2 by [Zha23, Lemma 5.4]. □

Proof of Theorem 1.4. It follows from Lemma 3.1 that $s_2(n) = 0$ if and only if $\mathbf{A}_P + \mathbf{U}_P$ is invertible. This concludes the result. $\square$

Proof of Theorem 1.5. By Lemma 3.1, $s_2(n) = 2$ if and only if $\operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$. Assume that $\operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ from now on. Then

$$\operatorname{Ker} \mathbf{M}_n = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{d} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{0} \\ \mathbf{d} \\ \mathbf{0} \end{pmatrix} \right\},$$

where $\mathbf{d} = (\delta_1, \ldots, \delta_k) = \psi_P(d)$. Thus

$$\operatorname{Sel}_2'(E_n) = \{(1,1,1), (d,1,d), (1,d,d), (d,d,1)\}$$

by (2.5) and (2.7).

Denote by $\Lambda = (d,1,d)$ and $\Lambda' = (d,d,1)$. Then

$$\langle \Lambda, \Lambda' \rangle = \left[\frac{\sqrt{2}+1}{d}\right] + \left[\frac{\gamma}{d}\right]$$

by Proposition 3.2. Hence the Cassels pairing on $\operatorname{Sel}_2'(E_n)$ is non-degenerate if and only if $\left(\frac{\sqrt{2}+1}{d}\right)\left(\frac{\gamma}{d}\right) = -1$. Conclude the results by Lemma 4.1. $\square$

Lemma 4.3 ([Wan16, Theorem 4.2])**.** *Let P be a square-free odd positive integer whose prime factors are all congruent to 1 modulo 8. If $h_4(-P) = 1$, then $h_8(-P) = 1 - \left[\frac{\sqrt{2}+1}{P}\right]$.*

Proof of Corollary 1.6. Take $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = (1, 0, \ldots, 0)^{\mathrm{T}}$ in Theorem 1.5, we obtain that $\mathbf{U}_P = \mathbf{O}$. Thus $\operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ if and only if $\operatorname{corank} \mathbf{A}_P = 1$, if and only if $h_4(-n) = 1$ by (2.11).

Since $\mathbf{A}_P \mathbf{1} = \mathbf{0}$, the non-zero vector in $\operatorname{Ker} \mathbf{A}_P$ is $\psi_P(d) = \mathbf{1}$. Thus $d = P$ and we only need to show that $h_8(-P) = 1 - \left[\frac{\sqrt{2}+1}{P}\right]$, which follows from Lemma 4.3. $\square$

Proof of Corollary 1.7.

(1) By Corollary 1.6, we only need to show that $\left[\frac{\gamma}{n}\right] = 0$. Since the ideal $(n, \omega_n) = (\sqrt{n})$ is principal in $\mathcal{O}_n$, its class is trivial and lies in $\mathcal{A}_n^4$. By Proposition 2.5, we have $\mathbf{b}_{n,\gamma} \in \operatorname{Im} \mathbf{R}_n = \operatorname{Im} \mathbf{A}_n$. Since $\mathbf{1}^{\mathrm{T}} \mathbf{A}_n = \mathbf{0}^{\mathrm{T}}$, we have $\left[\frac{\gamma}{n}\right] = \mathbf{1}^{\mathrm{T}} \mathbf{b}_{n,\gamma} = 0$.

(2) Since $h_4(-n) = 1$, we have $\operatorname{corank} \mathbf{A}_n = 1$ and $h_4(2n) = 1$ by Proposition 2.4. Since $\theta_{2n}(2n) = [(\sqrt{2n})] = [(1)]$ is the trivial class, we have $\mathcal{A}_{2n}[2] \cap \mathcal{A}_{2n}^2 = \{[(1)], \theta_{2n}(n)\}$, where $\theta_{2n}(n) = \theta_{2n}(2)$. Since $\mathbf{A}_n^{\mathrm{T}} = \mathbf{A}_n$ and $\mathbf{A}_n \mathbf{1} = \mathbf{0}$, the image of $\mathbf{R}_{2n}' = \begin{pmatrix} \mathbf{A}_n & \mathbf{0} \end{pmatrix}$ is

$$\operatorname{Im} \mathbf{R}_{2n}' = \left\{ \mathbf{x} \;\middle|\; \mathbf{1}^{\mathrm{T}} \mathbf{x} = 0 \right\}.$$

Now $h_8(2n) = 1$ if and only if $\theta_{2n}(n) \in \mathcal{A}_{2n}^4$. Let (α, β, γ) be a primitive positive solution of $n\alpha^2 - 2\beta^2 = 4\gamma^2$. By Proposition 2.5, $h_8(2n) = 1$ is equivalent to $\mathbf{b}_{n,\gamma} \in \operatorname{Im} \mathbf{R}_{2n}'$, i.e., $0 = \mathbf{1}^{\mathrm{T}} \mathbf{b}_{n,\gamma} = \left[\frac{\gamma}{n}\right]$. In other words, $h_8(2n) = 1 - \left[\frac{\gamma}{n}\right]$.

Similarly, we have $h_8(-2n) = 1 - \left[\frac{\gamma'}{n}\right]$, where $(\alpha', \beta', \gamma')$ is a primitive positive solution of $n\alpha'^2 + 2\beta'^2 = 4\gamma'^2$. Take $f_i = P, n = 2P$ in Proposition 3.2, we obtain that $\left[\frac{\gamma}{n}\right] = \left[\frac{\gamma'}{n}\right]$, which implies that $h_8(2n) = h_8(-2n) = 1 - \left[\frac{\gamma}{n}\right]$. This concludes the result by Corollary 1.6. $\square$

Remark 4.4. Corollary 1.7 (1) can be shown directly. Write $n = a^2 + b^2$ where b is odd and positive, then we may take $(\alpha, \beta, \gamma) = (2, 2a, b)$. Thus

$$\Big[\frac{b}{n}\Big] = \Big[\frac{n}{b}\Big] = \Big[\frac{a^2+b^2}{b}\Big] = \Big[\frac{a^2}{b}\Big] = 0.$$

As shown in the proof of Corollary 1.7 (2), we obtain the following corollary.

Corollary 4.5. *If n is a square-free odd positive integer whose prime factors are all congruent to 1 modulo 8 and $h_4(-n) = 1$, then $h_8(2n) = h_8(-2n)$.*

Proof of Theorem 1.8. By our assumptions (we rearrange the order of prime factors of P),

$$\mathbf{A}_P + \mathbf{U}_P = \mathbf{A}_P = \operatorname{diag}\big\{\mathbf{A}_{f_1}, \cdots \mathbf{A}_{f_r}\big\}.$$

Since $h_4(-f_i) = 1$, we have $\operatorname{corank} \mathbf{A}_{f_i} = 1$ by Proposition 2.4. Since $\mathbf{A}_{f_i}\mathbf{1} = \mathbf{0}$, we have $s_2(n) = 2r$ and the kernel of $\mathbf{M}_n$ is consists of vectors

$$\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_r \\ \mathbf{0} \\ \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_r \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{c}_i, \mathbf{d}_i = \mathbf{0}$ or $\mathbf{1}$ are vectors in $\operatorname{Ker} \mathbf{A}_{f_i}$. Thus $\mathrm{Sel}_2'(E_n)$ is generated by $\Lambda_1, \dots, \Lambda_s, \Lambda_1', \dots, \Lambda_s'$, where

$$\Lambda_i = (f_i, 1, f_i), \quad \Lambda_i' = (f_i, f_i, 1)$$

by (2.5) and (2.7). By Proposition 3.2, we have $\big[\frac{\gamma_i'}{f_j}\big] = \big[\frac{\gamma_j}{f_i}\big]$ and the Cassles pairing with respect to this basis is

$$\mathbf{X} = \begin{pmatrix} * & \mathbf{B} + \mathbf{C} \\ \mathbf{B}^{\mathrm{T}} + \mathbf{C} & \mathbf{B} + \mathbf{B}^{\mathrm{T}} \end{pmatrix},$$

where

$$\mathbf{B} = \Big(\Big[\frac{\gamma_i}{f_j}\Big]\Big)_{r\times r} \quad \text{and} \quad \mathbf{C} = \operatorname{diag}\Big\{\Big[\frac{\sqrt{2}+1}{f_1}\Big], \cdots, \Big[\frac{\sqrt{2}+1}{f_r}\Big]\Big\}.$$

Since $h_4(-f_i) = 1$, we have

$$\mathbf{C} = \operatorname{diag}\big\{1 - h_8(-f_1), \cdots, 1 - h_8(-f_r)\big\}$$

by Lemma 4.3. By our assumptions,

$$\mathbf{B} = \operatorname{diag}\Big\{h_8(-f_1), \cdots, h_8(-f_r)\Big\}.$$

Therefore, $\mathbf{X} = \begin{pmatrix} * & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}$ is invertible, i.e., the Cassles pairing on $\mathrm{Sel}_2'(E_n)$ is non-degenerate. Conclude the results by Lemma 4.1. $\square$

Proof of Corollary 1.9. (1) Since
$$\mathbf{R}_n = \mathbf{A}_n = \mathrm{diag}\{\mathbf{A}_{f_1}, \cdots \mathbf{A}_{f_r}\},$$
we have $h_4(n) = r-1$ and $\mathcal{A}_n[2] \cap \mathcal{A}_n^2$ is generated by $\theta_n(f_1), \ldots, \theta_n(f_{r-1})$ by (2.10) and (2.11). Here, one notice that
$$\theta_n(f_1)\cdots\theta_n(f_r) = \theta_n(n) = [(\sqrt{n})]$$
is the trivial class. If $h_8(n) = r-1$, then all $\theta_n(f_i) \in \mathcal{A}_n[2] \cap \mathcal{A}_n^4$. By Proposition 2.5, this implies that $\mathbf{b}_{n,\gamma_i} \in \mathrm{Im}\,\mathbf{A}_n$, where $(\alpha_i, \beta_i, \gamma_i)$ is a primitive positive solution of $f_i\alpha_i^2 - \frac{n}{f_i}\beta_i^2 = 4\gamma_i^2$. Thus $\mathbf{b}_{f_j,\gamma_i} \in \mathrm{Im}\,\mathbf{A}_{f_j}$ for all j. Since $\mathbf{1}^{\mathrm{T}}\mathbf{A}_{f_j} = \mathbf{0}^{\mathrm{T}}$, we have
$$0 = \mathbf{1}^{\mathrm{T}}\mathbf{b}_{f_j,\gamma_i} = \left[\frac{\gamma_i}{f_j}\right].$$
Conclude the results by Theorem 1.8.

(2) Since
$$\mathbf{R}_{2n} = \mathrm{diag}\{\mathbf{A}_n, 0\} = \mathrm{diag}\{\mathbf{A}_{f_1}, \cdots \mathbf{A}_{f_r}, 0\},$$
we have $h_4(2n) = r$ and $\mathcal{A}_{2n}[2] \cap \mathcal{A}_{2n}^2$ is generated by $\theta_{2n}(f_1), \ldots, \theta_{2n}(f_r)$ by (2.10) and (2.11). Here, one notice that $\theta_{2n}(2n) = [(\sqrt{2n})]$ is the trivial class. If $h_8(2n) = r$, then all $\theta_{2n}(f_i) \in \mathcal{A}_{2n}[2] \cap \mathcal{A}_{2n}^4$. By Proposition 2.5, this implies that $\mathbf{b}_{n,\gamma_i} \in \mathrm{Im}\,\mathbf{A}_n$, where $(\alpha_i, \beta_i, \gamma_i)$ is a primitive positive solution of $f_i\alpha_i^2 - \frac{2n}{f_i}\beta_i^2 = 4\gamma_i^2$. Thus $\mathbf{b}_{f_j,\gamma_i} \in \mathrm{Im}\,\mathbf{A}_{f_j}$ for all j. Since $\mathbf{1}^{\mathrm{T}}\mathbf{A}_{f_j} = \mathbf{0}^{\mathrm{T}}$, we have
$$0 = \mathbf{1}^{\mathrm{T}}\mathbf{b}_{f_j,\gamma_i} = \left[\frac{\gamma_i}{f_j}\right].$$
Conclude the results by Theorem 1.8. □

Example 4.6. Clearly, $\mathbf{M}_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $q = 3$ is a non-congruent number with $\text{Ш}(E_3)[2^\infty] = 0$. If $p = 193$, then $\left(\frac{p}{q}\right) = 1$, $\mathbf{A}_p = 0$ and $h_4(-p) = 1$. Since $52^2 \equiv 2 \bmod p$, we have
$$h_8(-p) = 1 - \left[\frac{\sqrt{2}+1}{p}\right] = 1 - \left[\frac{53}{193}\right] = 0.$$
Since $193 \times 2^2 - 3 \times 16^2 = 4 \times 1^2$ and $\left(\frac{1}{p}\right) = 1$, we obtain that $n = pq = 3 \times 193$ is non-congruent with $\text{Ш}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ by Corollary 1.6.

Example 4.7. Clearly, $\mathbf{M}_{10} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus $Q = 2q = 10$ is a non-congruent number with $\text{Ш}(E_{10})[2^\infty] = 0$. If $p = 241 = 23^2 - 2 \times 12^2$, then $\left(\frac{p}{q}\right) = 1$, $\mathbf{A}_p = 0$ and $h_4(-p) = 1$. Since $22^2 \equiv 2 \bmod p$, we have
$$h_8(-p) = 1 - \left[\frac{\sqrt{2}+1}{p}\right] = 1 - \left[\frac{23}{241}\right] = 0.$$
Since $241 \times 2^2 - 10 \times 8^2 = 4 \times 9^2$ and $\left(\frac{9}{p}\right) = 1$, we obtain that $n = 2pq = 10 \times 241$ is non-congruent with $\text{Ш}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ by Corollary 1.6.

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