

On the linearity of the periods of subtraction games

Shenxing Zhang

School of Mathematics, Hefei University of Technology, Hefei, 230009, Anhui, China

Article Info

Keywords:

periodic sequence
subtraction games
combinatorial games
nim-sequence

MSC: 91A46, 91A05

Abstract

A subtraction game is an impartial combinatorial game involving a finite set S of positive integers. The nim-sequence \mathcal{G}_S associated with this game is ultimately periodic. In this paper, we study the nim-sequence $\mathcal{G}_{S \cup \{c\}}$ where S is fixed and c varies. We conjecture that there is a multiple q of the period of \mathcal{G}_S , such that for sufficiently large c , the pre-period and period of $\mathcal{G}_{S \cup \{c\}}$ are linear in c if c modulo q is fixed. We prove it in several cases.

We also give new examples with period 2 inspired by this conjecture.

1. Introduction

Let S be a finite set of positive integers. The (finite) subtraction game $\text{SUB}(S)$ is a two-player game involving a heap of $n \geq 0$ counters. The two players move alternately, subtracting some $s \in S$ counters. The player who cannot make a move loses.

We always write the subtraction set as $S = \{s_1, \dots, s_k\}$ with an order $s_1 < s_2 < \dots < s_k$. Denote by $\mathcal{G}(n) = \mathcal{G}_S(n)$ the nim-value (or Grundy-value), i.e.,

$$\mathcal{G}(n) = \text{mex} \{ \mathcal{G}(n - s) : s \in S, s \leq n \}, \quad \forall n \geq 0,$$

where mex means the minimal non-negative integer not in the set. The sequence $\mathcal{G} = \mathcal{G}_S = \{\mathcal{G}(n)\}_{n \geq 0}$ is called the nim-sequence (or Sprague-Grundy sequence).

If $d = \gcd(S) = \gcd\{s : s \in S\} > 1$ and $S' = \{s/d : s \in S\}$, then $\mathcal{G}_S(n) = \mathcal{G}_{S'}(m)$, where $md \leq n < (m+1)d$. Hence we may assume that $\gcd(S) = 1$ if necessary.

Definition 1. A subtraction game $\text{SUB}(S)$ (or its nim-sequence \mathcal{G}) is called *ultimately periodic*, if there exist integers $p \geq 1$ and $\ell \geq 0$ such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq \ell$. The minimal p is called the *period* and the minimal ℓ is called the *pre-period*.

Since $\mathcal{G}(n) \leq k$, one can show that \mathcal{G} is ultimately periodic with $\ell, p \leq (k+1)^{s_k}$ by the pigeonhole principle, see [1, Theorem 7.33].

Since $\mathcal{G}(n+s_k)$ only depends on $\mathcal{G}(n), \mathcal{G}(n+1), \dots, \mathcal{G}(n+s_k-1)$, we have the following lemma to determine the period and pre-period.

Lemma 1.1 ([1, Corollary 7.34]). *The minimal integers $\ell \geq 0, p \geq 1$ such that $\mathcal{G}(n) = \mathcal{G}(n+p)$ for $\ell \leq n < \ell + s_k$ are the pre-period and period of $\text{SUB}(S)$ respectively.*

The nim-sequence \mathcal{G} is known when $k \leq 2$. For $k \geq 3$, even the pre-period and the period are not known in general. In §§2-3, we will recall some known results with $k \leq 3$, and give several new results with $k = 3$. We also give the nim sequence when $k \geq 4$ and S have a special form in §4. Based on these results and some computer-assistant calculations, we propose a conjecture on the inductive behavior of ℓ and p as follows:

✉ zhangshenxing@hfut.edu.cn (S. Zhang)

ORCID(s): 0000-0001-8126-1015 (S. Zhang)

Conjecture 1.2 (Asymptotic linearity). *Fix a subtraction set S . Then the pre-period and the period of the $\text{SUB}(S \cup \{c\})$ grow at most linearly in c .*

Moreover, the pre-period and the period should increase piecewise linearly on c :

Conjecture 1.3 (Piecewise linearity). *Fix a subtraction set S . There are*

- *positive integers q, N ;*
- *integers $\alpha_r, \beta_r, \lambda_r, \mu_r$ for each $0 \leq r < q$,*

such that if $c \geq N$ and $c \equiv r \pmod{q}$,

- *the pre-period of $\text{SUB}(S \cup \{c\})$ is $\alpha_r c + \beta_r$;*
- *the period of $\text{SUB}(S \cup \{c\})$ is $\lambda_r c + \mu_r$.*

In many cases, q is the period of $\text{SUB}(S)$.

Theorem 1.4. *Conjecture 1.3 holds in the following cases:*

1. $1 \in S$ and the elements of S are all odd;
2. $S = \{1, b\}$;
3. $S = \{a, 2a\}$;
4. $S = \{a, a+1, \dots, b-1, b\}$.

We will also give new ultimately bipartite nim-sequences inspired by this conjecture. See Theorem 6.3.

Remark 1. Once Conjecture 1.3 holds with effective q, N , then one can get the pre-period and period of $\text{SUB}(S \cup \{c\})$ for all c effectively. That is because we only need to calculate the pre-periods and periods of $\text{SUB}(S \cup \{c\})$ for $c \leq N + 2q$.

Remark 2. Denote by $\mathcal{P}(n) \in \{0, 1\}$ the sign of $\mathcal{G}(n)$. Then $\mathcal{P}(n) = 1$ if and only if the starting position with heap size n is a win for the player to move. One can easily see that \mathcal{P} is ultimately periodic with pre-period $\leq \ell$, period $\leq p$ and both of them $\leq 2^{s_k}$. We can propose a similar conjecture on the \mathcal{P} -sequence of $\text{SUB}(S \cup \{c\})$, which is a consequence of Conjecture 1.3.

Remark 3. In [2], Althöfer and Bültmann studied the pre-period and period of the \mathcal{P} -sequence of $\text{SUB}(S)$, where all elements of S are linear in a variable s . For example, they conjectured that $\text{SUB}(s, 4s, 12s+1, 16s+1)$ has no pre-period and period $56s^3 + 52s^2 + 9s + 1$. Our conjecture is in a different direction since we do not require the subtraction set $S \cup \{c\}$ to have a special form.

Let's introduce some notations we will use. Let t, a be non-negative integers and $\mathcal{H} = (h_1 \dots h_k)$ a sequence of integers with finite length. As usual, we denote by a^t the sequence $a \dots a$ (t copies of a) and \mathcal{H}^t the sequence $\mathcal{H} \dots \mathcal{H}$ (t copies of \mathcal{H}). Denote by $\underline{\mathcal{H}}$ the infinite-length sequence with periodic sequence \mathcal{H} , i.e., $\underline{\mathcal{H}} = \mathcal{H}\mathcal{H} \dots$. For example, if a nim-sequence \mathcal{G} has pre-period ℓ and period p , then we can write

$$\mathcal{G} = \mathcal{G}(0)\mathcal{G}(1)\mathcal{G}(2) \dots = \mathcal{G}(0) \dots \mathcal{G}(\ell-1) \underline{\mathcal{G}(\ell) \dots \mathcal{G}(\ell+p-1)}.$$

We will not give detailed proofs of all nim-sequences, since these proofs tend to involve lengthy and tedious inductions.

2. The case $S = \{1, b, c\}$

In this section, we will consider the nim-sequences of $S = \{1, b, c\}$, where $1 < b < c$. Let's recall some classical cases first.

Lemma 2.1. *Let p be the period of $\text{SUB}(S)$. Let $S' = S \cup \{x + pt\}$ for some $x \in S$ and $t \geq 1$. If the pre-period of $\text{SUB}(S)$ is zero, then $\mathcal{G}_{S'} = \mathcal{G}_S$.*

PROOF. Certainly $\mathcal{G}_{S'}(0) = \mathcal{G}_S(0) = 0$. Suppose that $\mathcal{G}_{S'}(i) = \mathcal{G}_S(i)$ for $0 \leq i \leq n-1$. If $n < x + pt$, then

$$\mathcal{G}_{S'}(n) = \text{mex} \{ \mathcal{G}_S(n-s) : s \in S, s \leq n \} = \mathcal{G}_S(n).$$

If $n \geq x + pt$, then

$$\begin{aligned} \mathcal{G}_{S'}(n) &= \text{mex} \{ \mathcal{G}_S(n-x-pt), \mathcal{G}_S(n-s) : s \in S, s \leq n \} \\ &= \text{mex} \{ \mathcal{G}_S(n-x), \mathcal{G}_S(n-s) : s \in S, s \leq n \} = \mathcal{G}_S(n). \end{aligned}$$

The lemma then follows by induction.

Example 2.2. Certainly, $\mathcal{G}_{\{1\}} = \underline{01}$. If $1 \in S$ and all elements of S are odd, then $\mathcal{G}_S = \underline{01}$ by applying Lemma 2.1 several times. This condition is also necessary for $\mathcal{G}_S = \underline{01}$, see [4].

Example 2.3. Let $S = \{a, c\}$ with $1 \leq a < c$. Write $c = at + r, 0 \leq r < a$. Then

$$\mathcal{G}_S = \begin{cases} (0^a 1^a)^{t/2} 0^r 2^{a-r} 1^r, & \text{if } t \text{ is even;} \\ (0^a 1^a)^{(t+1)/2} 2^r, & \text{if } t \text{ is odd,} \end{cases}$$

$\ell = 0$ and $p = c + a$ or $2a$. See [3] and [2, Theorem 2].

Example 2.4. 1. Let $S = \{1, b, c\}$ with odd b and $1 < b < c$. Note that $\mathcal{G}_{\{1,b\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 01$. We have

c	\mathcal{G}_S	ℓ	p
odd	$\underline{\mathcal{H}}$	0	2
even	$\underline{\mathcal{H}^{c/2} (23)^{(b-1)/2} 2}$	0	$c + b$

See [5, Theorem 4].

2. Let $S = \{1, 2, c\}$ with $c > 2$. Note that $\mathcal{G}_{\{1,2\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 012$. Write $c = 3t + r, 0 \leq r < 3$. Then

r	\mathcal{G}_S	ℓ	p
0	$\underline{(012)^t 3}$	0	$c + 1$
1, 2	$\underline{012}$	0	3

3. Let $S = \{1, 4, c\}$ with $c > 4$. Note that $\mathcal{G}_{\{1,4\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 01012$. Write $c = 5t + r, 0 \leq r < 5$. Then

r, c	\mathcal{G}_S	ℓ	p
$r = 0, c = 5$	$\underline{\mathcal{H} 323}$	0	8
$r = 0, c > 5$	$\underline{\mathcal{H}^t 323013 \mathcal{H}^{t-1} 012012}$	$c + 6$	$c + 1$
$r = 1, 4$	$\underline{\mathcal{H}}$	0	5
$r = 2$	$\underline{\mathcal{H}^t 012}$	0	$c + 1$
$r = 3$	$\underline{\mathcal{H}^{t+1} 32}$	0	$c + 4$

Proposition 2.5. Let $S = \{1, b, c\}$ with even $b = 2k \geq 6$. Write $c = t(b + 1) + r$ with $0 \leq r \leq b, t \geq 1$.

1. If $r = 1, b$, then $\ell = 0$ and $p = b + 1$.
2. If $3 \leq r \leq b - 1$ is odd, then $\ell = 0$ and $p = c + b$.
3. If $r = b - 2$, then $\ell = 0$ and $p = c + 1$.
4. If $c = b + 1$, then $\ell = 0, p = 2b$.
5. If $c > b + 1, 0 \leq r \leq b - 4$ is even and $t + r/2 \geq k$, then $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b$ and $p = c + 1$.
6. If $c > b + 1, 0 \leq r \leq b - 4$ is even and $t + r/2 \leq k - 1$, then $\ell = t(c + b + 2) - b$. If $t + r/2 < k - 1$, then $p = c + b$; if $t + r/2 = k - 1$, then $p = b - 1$.

PROOF. Note that $\mathcal{G}_{\{1,b\}} = \underline{H}$ where $H = (01)^k 2$.

1. In this case, $\mathcal{G} = \underline{H}$, $\ell = 0$ and $p = b + 1$ by Lemma 2.1.
2. In this case, $\mathcal{G} = \underline{H^{t+1} (32)^{(r-1)/2}}$, $\ell = 0$ and $p = c + b$.
3. In this case, $\mathcal{G} = \underline{H^t (01)^{k-1} 2}$, $\ell = 0$ and $p = c + 1$.
4. In this case, $\mathcal{G} = \underline{(01)^k (23)^k} = \underline{H 3(23)^{k-1}}$, $\ell = 0$ and $p = 2b$.
5. Write $r = 2v$. If $1 \leq v \leq k - 2$, the leading $(c + 1)(k - v + 1)$ terms of \mathcal{G} are (the waved part is the first periodic nim-sequence)

i	$\mathcal{G}((c + 1)i + j), 0 \leq j \leq c$
0	$H^t, (01)^v 2$
1	$(32)^{k-v-1} (01)^{v+1} 2, H^{t-1}, (01)^v 0$
2	$1(01)^{k-v-2} 2(01)^{v+1} 2, (32)^{k-v-2} (01)^{v+2} 2, H^{t-2}, (01)^v 0$
i	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^{k-v-i+1} 2(01)^{v+i-2} 0,$ $1(01)^{k-v-i} 2(01)^{v+i-1} 2, (32)^{k-v-i} (01)^{v+i} 2, H^{t-i}, (01)^v 0$
$k - v - 1$	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^2 2(01)^{k-3} 0, 1(01) 2(01)^{k-2} 2,$ <u>$(32)(01)^{k-1} 2, H^{t-k+v+1}, (01)^v 0$</u>
$k - v$	<u>$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01) 2(01)^{k-2} 0, 12(01)^{k-1} 2,$</u> $H^{t-k+v-1}, (01)^v 0.$

If $v = 0$, the leading $(c + 1)(k + 1)$ terms of \mathcal{G} are

i	$\mathcal{G}((c + 1)i + j), 0 \leq j \leq c$
0	$H^t 3$
1	$(23)^{k-1} 013, H^{t-1} 0$
2	$1(01)^{k-2} 2(01) 2, (32)^{k-2} (01)^2 2, H^{t-2} 0$
i	$1(01)^{k-2} 2(01) 0, \dots, 1(01)^{k-i+1} 2(01)^{i-2} 0, 1(01)^{k-i} 2(01)^{i-1} 2,$ $(32)^{k-i} (01)^i 2, H^{t-i} 0$
$k - 1$	$1(01)^{k-2} 2(01) 0, \dots, 1(01)^2 2(01)^{k-3} 0, 1(01)^1 2(01)^{k-2} 2,$ <u>$(32)(01)^{k-1} 2, H^{t-k+1} 0$</u>
k	<u>$1(01)^{k-2} 2(01) 0, \dots, 1(01)^1 2(01)^{k-2} 0, 12(01)^{k-1} 2, H^{t-k+1} 0.$</u>

In both cases, we have $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b, p = c + 1$ and

$$\mathcal{G} = \dots 2(01)^{k-1} \left(2(01)^k\right)^{t-k+v+1} \left(2(01)^{k-1}\right)^{k-v-1}.$$

6. If $1 \leq v \leq k - 2$, the leading $(c + 1)(t + 2)$ terms of \mathcal{G} are

i	$\mathcal{G}((c+1)i+j), 0 \leq j \leq c$
0	$\mathcal{H}^t (01)^v 2$
1	$(32)^{k-v-1} (01)^{v+1} 2, \mathcal{H}^{t-1} (01)^v 0$
2	$1(01)^{k-v-2} 2(01)^{v+1} 2, (32)^{k-v-2} (01)^{v+2} 2, \mathcal{H}^{t-2} (01)^v 0$
i	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^{k-v-i+1} 2(01)^{v+i-2} 0,$ $1(01)^{k-v-i} 2(01)^{v+i-1} 2, (32)^{k-v-i} (01)^{v+i} 2, \mathcal{H}^{t-i} (01)^v 0$
$t-1$	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^{k-v-t+2} 2(01)^{v+t-3} 0,$ $1(01)^{k-v-t+1} 2(01)^{v+t-2} 2, (32)^{k-v-t+1} (01)^{v+t-1} 2, \mathcal{H}^1 (01)^v 0$
t	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^{k-v-t+1} 2(01)^{v+t-2} 0,$ $1(01)^{k-v-t} 2(01)^{v+t-1} 2, (32)^{k-v-t} (01)^{v+t} 2, (01)^v 0$
$t+1$	$1(01)^{k-v-2} 2(01)^{v+1} 0, \dots, 1(01)^{k-v-t+1} 2(01)^{v+t-2} 0,$ $1(01)^{k-v-t} 2(01)^{v+t-1} 0, 1(01)^{k-v-t-1} 2(01)^{v+t} 2, (32)^{k-v-t-1} 01 \dots$

Therefore, $\ell = t(c+b+2) - b$. If $t+v < k-1$, then $p = c+b$ and

$$\mathcal{G} = \dots 2(32)^{k-v-t-1} (01)^{v+t} 2((01)^{k-1} 2)^t (01)^{v+t}.$$

If $t+v = k-1$, then $p = b-1$ and $\mathcal{G} = \dots 2(01)^{k-1}$.

If $v = 0$, the leading $(c+1)(t+2)$ terms of \mathcal{G} are

i	$\mathcal{G}((c+1)i+j), 0 \leq j \leq c$
0	$\mathcal{H}^t 3$
1	$(23)^{k-1} 013, \mathcal{H}^{t-1} 0$
2	$1(01)^{k-2} 2(01)2, (32)^{k-2} (01)^2 2, \mathcal{H}^{t-2} 0$
i	$1(01)^{k-2} 2(01)0, \dots, 1(01)^{k-i+1} 2(01)^{i-2} 0, 1(01)^{k-i} 2(01)^{i-1} 2,$ $(32)^{k-i} (01)^i 2, \mathcal{H}^{t-i} 0$
$t-1$	$1(01)^{k-2} 2(01)0, \dots, 1(01)^{k-t+2} 2(01)^{t-3} 0, 1(01)^{k-t+1} 2(01)^{t-2} 2,$ $(32)^{k-t+1} (01)^{t-1} 2, \mathcal{H}^1 0$
t	$1(01)^{k-2} 2(01)0, \dots, 1(01)^{k-t+1} 2(01)^{t-2} 0, 1(01)^{k-t} 2(01)^{t-1} 2,$ $(32)^{k-t} (01)^t 20$
$t+1$	$1(01)^{k-2} 2(01)0, \dots, 1(01)^{k-t} 2(01)^{t-1} 0, 1(01)^{k-t-1} 2(01)^t 0,$ $1(01)^{k-t-1} 2(01)^t 2, (32)^{k-t-1} 01 \dots$

Therefore, $\ell = t(c+b+2) - b$. If $t < k-1$, then $p = c+b$ and

$$\mathcal{G} = \dots 2(32)^{k-t-1} (01)^t 2((01)^{k-1} 2)^t (01)^t.$$

If $t = k-1$, then $p = b-1$ and $\mathcal{G} = \dots 2(01)^{k-1}$.

Remark 4. The case $c < 4b$ is studied in [5], but there are some incorrect data. In Table 1, $p = a-1$ if $r = a-3 \geq 3$. In Table B.11, $n_0 = a+2b+4$ if $2 \leq r \leq a-4$. In Table B.12, $n_0 = 2a+3b+6$ if $3 \leq r \leq a-5$. The corresponding pre-period nim-values also need to be modified.

3. The case $S = \{a, 2a, c\}$

Proposition 3.1. *Let $S = \{a, 2a, c\}$ with $2a < c$. Write $c = 3at + r$ with $0 \leq r < 3a$. Then*

$$\ell = \begin{cases} c + a - r, & 0 < r < a; \\ 0, & \text{otherwise,} \end{cases} \quad p = \begin{cases} 3a/2, & r = a/2; \\ 3a, & a/2 < r \leq 2a; \\ c + a, & \text{otherwise.} \end{cases}$$

PROOF. Denote by $\mathcal{H} = 0^a 1^a 2^a$. Then $\mathcal{G}_{\{a, 2a\}} = \underline{\mathcal{H}}$ with period $q = 3a$. Write $a = 2k - 1$ if a is odd; $a = 2k$ if a is even.

1. If $a \leq r \leq 2a$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = 3a$.
2. If $r = 0$, then $\mathcal{G} = \underline{\mathcal{H}^t 3^a}$, $\ell = 0$ and $p = c + a$.
3. If $0 < r < k$, then

$$\mathcal{G} = \underline{\mathcal{H}^t 0^r 3^{a-r} (1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r})^t 1^r 0^r 3^{a-2r} 2^r},$$

$$\ell = c + a - r \text{ and } p = c + a.$$

4. If $k \leq r < a$, then

$$\mathcal{G} = \underline{\mathcal{H}^t 0^r 3^{a-r} 1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r}},$$

$$\ell = c + a - r \text{ and } p = 3a \text{ or } 3a/2.$$

5. If $r > 2a$, then $\mathcal{G} = \underline{\mathcal{H}^{t+1} 3^{r-2a}}$, $\ell = 0$ and $p = c + a$.

Remark 5. The pre-period and period of $\text{SUB}(S)$ are not easy to determine, even if $S = \{s_1, s_2, s_3\}$ is a 3-element set. In [2, §4, Conjecture (i)], Althofer and Bultermann conjectured that the period of $\text{SUB}(S)$ is bounded by a quadratic polynomial in s_3 . Ho also studied $\text{SUB}(S)$ for 3-element set S in [5].

4. The case S contains successive numbers

Proposition 4.1. *Let $S = \{a, a+1, \dots, b-1, b, c\}$ with $a < b < c$. Write $c = t(a+b) + r$ with $0 \leq r < a+b$. Then*

$$\ell = 0, \quad p = \begin{cases} a + b, & a \leq r \leq b; \\ c + a, & r = 0 \text{ or } r > b; \\ c + b, & 0 < r < a. \end{cases}$$

PROOF. Write $b = ak + s$, $0 \leq s \leq a - 1$ and denote by $\mathcal{H} = 0^a 1^a \dots k^a (k+1)^s$, then $\mathcal{G}_{\{a, a+1, \dots, b\}} = \underline{\mathcal{H}}$ with period $q = a + b = a(k+1) + s$.

1. If $a \leq r \leq b$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = a + b$ by Lemma 2.1.
2. If $r = 0$, then

$$\mathcal{G} = \underline{\mathcal{H}^t (k+1)^{a-s} (k+2)^s}.$$

If $r > b$ and $r + s > q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a-s} (k+2)^{r+s-q}}.$$

If $r > b$ and $r + s \leq q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a+r-q}}.$$

In all cases, we have $\ell = 0$ and $p = c + a$.

3. If $0 < r < a - 2s$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+1)^{a-s-r}(k+2)^s, 1^r(k+2)^{a-s-r}(k+3)^s, \dots, (k-1)^r(2k)^{a-s-r}(2k+1)^s, k^r(2k+1)^s}.$$

If $a - 2s \leq r < a - s$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+1)^{a-s-r}(k+2)^s, 1^r(k+2)^{a-s-r}(k+3)^s, \dots, (k-1)^r(2k)^{a-s-r}(2k+1)^s, k^r(2k+1)^{a-s-r}(2k+2)^{2s+r-a}}.$$

If $a - s \leq r < a$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+2)^{a-r}, 1^r(k+3)^{a-r}, \dots, (k-1)^r(2k+1)^{a-r}, k^r(k+1)^s},$$

In all cases, we have $\ell = 0$ and $p = c + b$.

5. Piecewise linearity of pre-periods and periods

Let S be a fixed subtraction set. Denote by \mathcal{G}_S the nim-sequence of S with pre-period ℓ and period p . Denote by $\mathcal{G}_{S \cup \{c\}}$ the nim-sequence of $S \cup \{c\}$ with pre-period ℓ_c and period p_c . The following examples are due to computer-assistant calculations.

Example 5.1. Let $S = \{6, 17\}$. Then $\mathcal{G}_S = \underline{0^6 1^6 0^5 2^1 5}$ with period 23. For $116 \leq c \leq 500$, we have

$$\ell_c = \begin{cases} (9 - 2\lambda)c + (147 - 35\lambda), & c \equiv \lambda \text{ or } \lambda + 12 \pmod{23}, \lambda \in [0, 4]; \\ 0, & \text{otherwise,} \end{cases}$$

$$p_c = \begin{cases} c + 6, & c \equiv 0, 1, 2, 3, 4, 5, 12, 13, 14, 15, 16 \pmod{23}; \\ c + 17, & c \equiv 7, 8, 9, 10, 11, 18, 19, 20, 21, 22 \pmod{23}; \\ 23, & r = 6 \text{ or } 17. \end{cases}$$

See <https://zhangshenxing.github.io/nim/example5.1.html>.

Example 5.2. Let $S = \{3, 5, 8\}$. Then $\mathcal{G}_S = \underline{0^3 1^3 2^3 3^2}$ with period 11. For $13 \leq c \leq 500$, we have

$$\ell_c = \begin{cases} c + 18, & c \equiv 1, 2 \pmod{11}; \\ 0, & \text{otherwise,} \end{cases} \quad p_c = \begin{cases} c + 3, & c \equiv 0, 1, 9, 10 \pmod{11}; \\ c + 25, & c \equiv 2 \pmod{11}; \\ 11, & \text{otherwise.} \end{cases}$$

See <https://zhangshenxing.github.io/nim/example5.2.html>.

Example 5.3. Let $S = \{2, 3, 5, 7\}$. Then $\mathcal{G}_S = \underline{0^2 1^2 2^2 3^2 4}$ with period 9. For $11 \leq c \leq 500$, we have

$$\ell_c = \begin{cases} 2c - 4, & c \equiv 1 \pmod{18}; \\ c + 5, & c \equiv 10 \pmod{18}; \\ 0, & \text{otherwise,} \end{cases} \quad p_c = \begin{cases} c + 2, & c \equiv 0, 8, 9, 10, 17 \pmod{18}; \\ 4, & c \equiv 1 \pmod{18}; \\ 9, & \text{otherwise.} \end{cases}$$

See <https://zhangshenxing.github.io/nim/example5.3.html>.

Example 5.4. Let $S = \{4, 11, 12, 14\}$. Then $\mathcal{G}_S = \dots \underline{20^4 1^4 0^3 31^3 2^3 03^3 12}$ with pre-period 24 and period 25. Write $r \equiv c \pmod{25}, 0 \leq r < 25$. For $101 \leq c \leq 500$, we have

$$\ell_c = \begin{cases} 4c + 91, & r = 0; & 2c + 8, & r = 1; & 2c + 34, & r = 2; \\ c - 6, & r = 3; & 2c + 16, & r = 4; & 2c + 36, & r = 5; \\ 3c + 4, & r = 6; & c + 26, & r = 9; & c + 12, & r = 12; \\ 0, & r = 13; & 2c + 37, & r = 18; & c + 14, & r = 19; \\ c + 2, & r = 20; & 12, & r = 21; & 3c + 5, & r = 22; \\ c + 52, & r = 23; & 2c + 33, & r = 24; & 24, & \text{otherwise,} \end{cases}$$

$$p_c = \begin{cases} c + 37, & r = 0, 1, 9, 18; & c + 14, & r = 2, 10; \\ c + 11, & r = 6, 7, 8, 15, 16, 17; & c + 12, & r = 13; \\ 2c + 41, & r = 19; & c + 4, & r = 21; \\ c + 28, & r = 22; & 25, & \text{otherwise.} \end{cases}$$

See <https://zhangshenxing.github.io/nim/example5.4.html>.

Based on these observations, we propose the Conjecture 1.3. By the results in §§2-4, Conjecture 1.3 is valid in the cases mentioned in Theorem 1.4.

PROOF (PROOF OF THEOREM 1.4). 1. The period of $\text{SUB}(S)$ is $q = 2$. If c is odd, then $\mathcal{G}_{S \cup \{c\}} = \underline{01}$. If c is even, denote by s the maximal number in S . Then

$$\mathcal{G}_{S \cup \{c\}} = \underline{(01)^{c/2} (23)^{(s-1)/2} 2},$$

$$\ell_c = 0 \text{ and } p_c = c + s.$$

2. follows from Example 2.4 and Proposition 2.5.
3. follows from Proposition 3.1.
4. follows from Proposition 4.1.

6. Ultimately bipartite nim-sequences

A subtraction game (or its nim-sequence) is said to be *ultimately bipartite* if the period is 2. It is known that \mathcal{G}_S is ultimately bipartite with pre-period 0 if and only if $1 \in S$ and all elements in S are odd, see [4].

Example 6.1. Let $a \geq 3$ be an odd integer. If S is one of the following:

- $S = \{3, 5, 9, \dots, 2^a + 1\};$
- $S = \{3, 5, 2^a + 1\};$
- $S = \{a, a + 2, 2a + 3\};$
- $S = \{a, 2a + 1, 3a\},$

then $\text{SUB}(S)$ is ultimately bipartite. See [4, Theorem 2] and [5, Theorem 5].

Lemma 6.2. If $\mathcal{G} = \mathcal{G}_S$ is ultimately bipartite, then all elements in S are odd.

PROOF. As shown in [4, Theorem 3], there exists an integer n_0 such that for $n \geq n_0$, $\mathcal{G}(n) = 0$ if n is even; $\mathcal{G}(n) = 1$ if n is odd. Take an even number $n \geq n_0 + s_k$, where s_k is the maximal element in S . Then

$$0 = \mathcal{G}(n) = \text{mex} \{ \mathcal{G}(n - s) : s \in S \},$$

which implies that $\mathcal{G}(n - s) = 1$ for all $s \in S$. Hence all $s \in S$ are odd.

We have the following new ultimately bipartite subtraction sets inspired by our conjecture.

Theorem 6.3. *Let $a \geq 3$ be an odd integer and $t \geq 1$. The subtraction game $\text{SUB}(S)$ is ultimately bipartite in the following cases:*

1. $S = \{a, a + 2, (2a + 2)t + 1\}$;
2. $S = \{a, 2a + 1, (3a + 1)t - 1\}$;
3. $S = \{a, 2a - 1, (3a - 1)t + a - 2\}$.

PROOF. Let c be the maximal element in S . Write $a = 2k + 1$.

1. If $k \geq 2$, then the leading $(k + 1)(a + 1)(2t + 1)$ terms of \mathcal{G} are

i	$\mathcal{G}((a + 1)(2t + 1)i + j), 0 \leq j < (a + 1)(2t + 1) = c + a$		
0	$0^a 1$	$[\begin{array}{cc} 1^{a-1} 22 & 0^a 1 \\ 1^{a-1} 22 & 02^{a-3} 331 \end{array}]^{t-1},$	
1	$030^{a-2} 1$	$[\begin{array}{cc} 01^{a-2} 21 & 020^{a-2} 1 \\ 01^{a-2} 21 & 0202^{a-5} 321 \end{array}]^{t-1},$	
i	$(01)^{i-1} 030^{a-2i} 1$	$[\begin{array}{cc} (01)^{i-1} 01^{a-2i} 21 & (01)^{i-1} 020^{a-2i} 1 \\ (01)^{i-1} 01^{a-2i} 21 & (01)^{i-1} 0202^{a-2i-3} 321 \end{array}]^{t-1},$	
$k - 1$	$(01)^{k-2} 030^3 1$	$[\begin{array}{cc} (01)^{k-2} 01^3 21 & (01)^{k-2} 020^3 1 \\ (01)^{k-2} 01^3 21 & (01)^{k-2} 020321 \end{array}]^{t-1},$	
k	$[(01)^{k-1} 0301$	$(01)^{k-1} 0121]^{t-1}$	$(01)^{k-1} 0301, \\ (01)^{k-1} 0101 \quad (01)^{k-1} 0101.$

Hence the pre-period is

$$\ell = (k + 1)(c + a) - 2a - 4 = (k + 1)c + 2k^2 - k - 5$$

and the period is $p = 2$. The case $a = 3$ will be shown in Case 3.

2. The leading $(k + 1)((3a + 1)t + a - 1)$ terms of \mathcal{G} are

i	$\mathcal{G}(((3a + 1)t + a - 1)i + j), 0 \leq j < (3a + 1)t + a - 1 = c + a$			
0	$[\begin{array}{cc} 0^a & 1^a \end{array}]^t,$	$02^{a-1} 1$	3^{a-1}	
1	$[\begin{array}{cc} 020^{a-2} & 101^{a-2} \\ 020^{a-2} & 101^{a-2} \end{array}]^{t-1},$	$(01)^1 32^{a-3} 1$	$(01)02^{a-3} 1$	$(01)3^{a-3}$
i	$[\begin{array}{cc} (01)^{i-1} 020^{a-2i} & 1(01)^{i-1} 01^{a-2i} \\ (01)^{i-1} 020^{a-2i} & 1(01)^{i-1} 01^{a-2i} \end{array}]^{t-1},$	$(01)^i 32^{a-2i-1} 1$	$(01)^i 02^{a-2i-1} 1$	$(01)^i 3^{a-2i-1}$
$k - 1$	$[\begin{array}{cc} (01)^{k-2} 020^3 & 1(01)^{k-2} 01^3 \\ (01)^{k-2} 020^3 & 1(01)^{k-2} 01^3 \end{array}]^{t-1},$	$(01)^{k-1} 32^2 1$	$(01)^{k-1} 02^2 1$	$(01)^{k-1} 3^2$
k	$[\begin{array}{cc} (01)^{k-1} 020 & 1(01)^{k-1} 01 \\ (01)^{k-1} 020 & 1(01)^{k-1} 01 \end{array}]^{t-1},$	$(01)^k 31$	$(01)^k 01$	$(01)^k.$

Hence the pre-period is

$$\ell = (k+1)(c+a) - 3a - 1 = (k+1)c + 2k^2 - 3k - 3$$

and the period is $p = 2$.

(3) The leading $(k+1)(3a-1)(t+1)$ terms of \mathcal{G} are

i	$\mathcal{G}((3a-1)(t+1)i+j), 0 \leq j < (3a-1)(t+1) = c+2a+1$					
0	$[0^{a-1}$	01^{a-1}	12^{a-1}	$]^t, 0^{a-2}3$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1$
1	$[0^{a-3}(01)^1$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1$	$]^t, 0^{a-4}3(01)^1$	$31^{a-5}(10)^2$	$2^{a-4}(01)^2$
i	$[0^{a-2i-1}(01)^i$	$31^{a-2i-1}(10)^i$	$2^{a-2i}(01)^i$	$]^t, 0^{a-2i-2}3(01)^i$	$31^{a-2i-3}(10)^{i+1}$	$2^{a-2i-2}(01)^{i+1}$
$k-1$	$[0^2(01)^{k-1}$	$31^2(10)^{k-1}$	$2^3(01)^{k-1}$	$]^t, 0^13(01)^{k-1}$	$3(10)^k$	$2^1(01)^k$
k	$[(01)^k$	$3(10)^k$	$2(01)^k$	$]^{t-1}, (01)^{6k+2}.$		

Hence the pre-period is

$$\ell = (k+1)(c+2a+1) - 2(7k+2) = (k+1)c + 4k^2 - 7k - 1$$

and the period is $p = 2$.

Remark 6. One may expect that if $\text{SUB}(a, b, c)$ is ultimately bipartite, then so is $\text{SUB}(a, b, d)$ for sufficient large d with $d \equiv c \pmod{a+b}$. This is not true in general. For example, $\text{SUB}(3, 11, 13)$ is ultimately bipartite but $\text{SUB}(3, 11, 14t+13)$ has period $14t+16$, $t \geq 1$.

Remark 7. Write $a = 2k+1$. Consider the four-element subtraction set $S = \{a, 2a+1, 3a, c\}$ with odd $c > 3a$. For $3 \leq a \leq 25$, $c < 500$, we find the following phenomenon.

- If $c = 4a+1$, then $\ell = 0$ and $p = 5a+1$.
- If $c = (4i+2)a-1$ with $1 \leq i < k$, then $\ell = (8i-1)a+2i-1$ and $p = 4a$.
- Otherwise, $\text{SUB}(S)$ is ultimately bipartite.

Acknowledgments. The author would like to express appreciation for the reviewers' meticulous review and valuable suggestions. This work is partially supported by NSFC (Grant No. 12001510) and the Fundamental Research Funds for the Central Universities (No. JZ2023HG TB0217).

References

- [1] Albert, M., Nowakowski, R.J., Wolfe, D., 2007. Lessons in Play An Introduction to Combinatorial Game Theory. Second ed., A K Peters Ltd., Wellesley, MA. URL: <https://doi.org/10.1201/9781482243048>, doi:10.1201/9781482243048.
- [2] Althöfer, I., Bühlmann, J., 1995. Superlinear period lengths in some subtraction games. Theoret. Comput. Sci. 148, 111–119. URL: [https://doi.org/10.1016/0304-3975\(95\)00019-S](https://doi.org/10.1016/0304-3975(95)00019-S), doi:10.1016/0304-3975(95)00019-S.
- [3] Berlekamp, E.R., Conway, J.H., Guy, R.K., 2003. Winning ways for your mathematical plays. Vol. 3. Second ed., A K Peters, Ltd., Natick, MA.
- [4] Cairns, G., Ho, N.B., 2010. Ultimately bipartite subtraction games. Australas. J. Combin. 48, 213–220.
- [5] Ho, N.B., 2015. On the expansion of three-element subtraction sets. Theoret. Comput. Sci. 582, 35–47. URL: <https://doi.org/10.1016/j.tcs.2015.03.025>, doi:10.1016/j.tcs.2015.03.025.