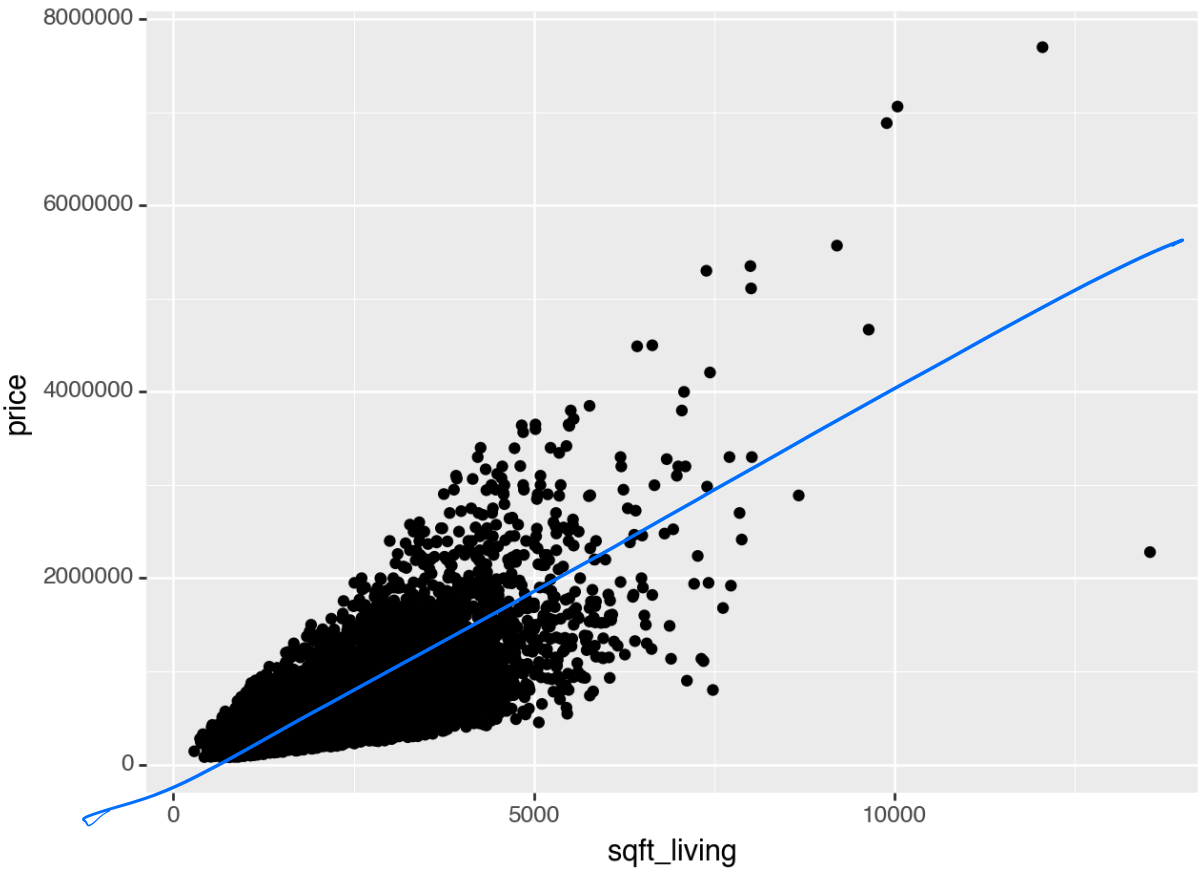
In [1]:
```
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import statsmodels.api as sm
import statsmodels.formula.api as smf
from plotnine import *
```

Examples of Heteroskedasticity

In [3]:
```
df = pd.read_csv('../Data/kc_house_data.csv')
```

In [4]:
```
(ggplot(df, aes(x = 'sqft_living', y = 'price')) +
       geom_point())
```

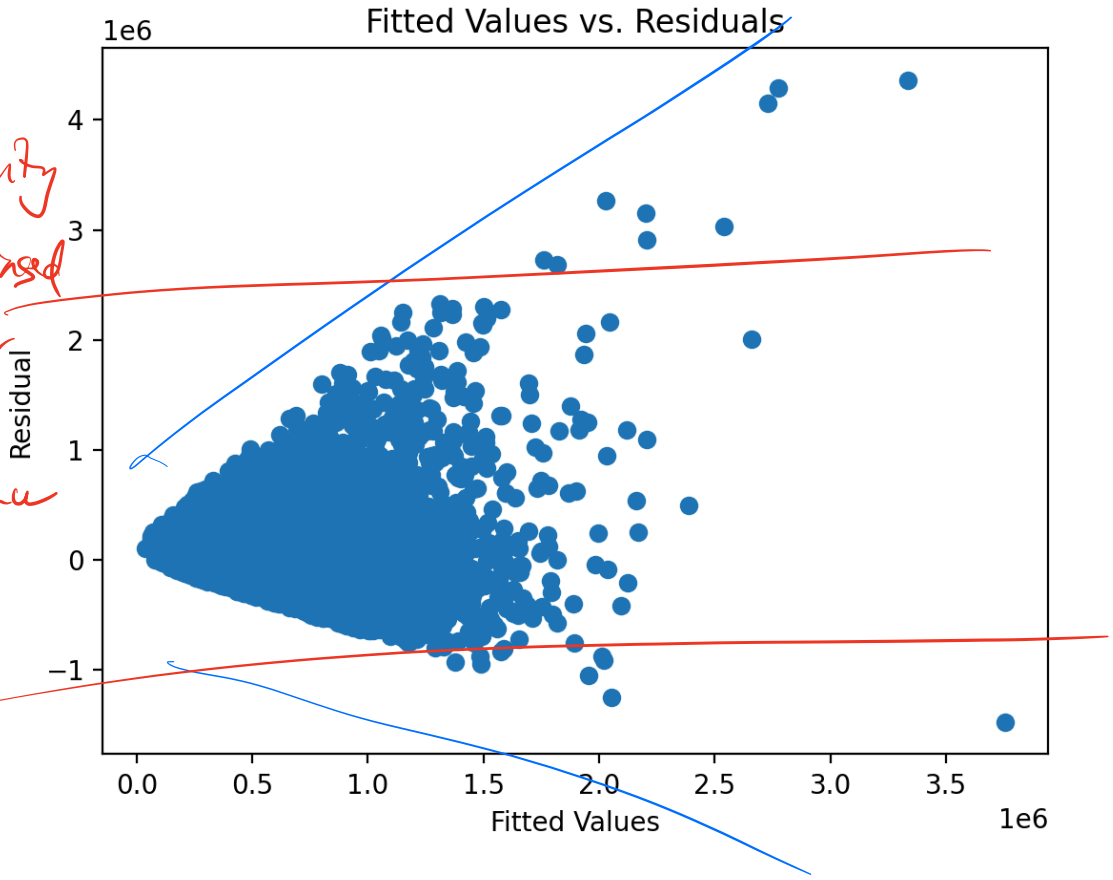


Out[4]: <Figure Size: (640 x 480)>

In [5]:
```
f = 'price~sqft_living'
model = smf.ols(formula=f, data=df).fit()
p = model.fittedvalues
res = model.resid
plt.scatter(p,res)
plt.xlabel("Fitted Values")
plt.ylabel("Residual")
plt.title("Fitted Values vs. Residuals")
```

```
plt.show()
```

Problems caused by heteroskedasticity
- $\hat{\beta}$ is still unbiased but it's no longer BLUE
- not min variance
- $\hat{SE}(\hat{\beta}_k)$ are wrong
- CI & PI widths are wrong



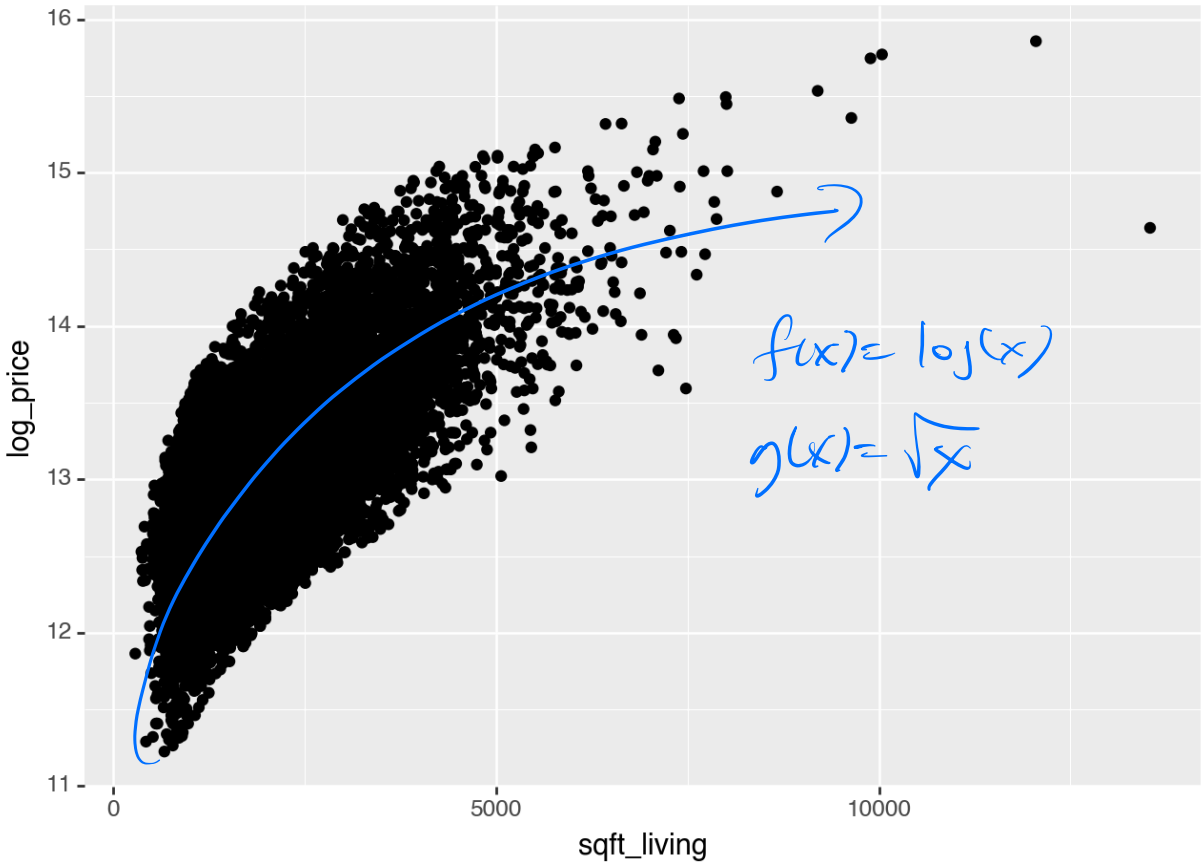
The classic fan shape emblematic of heteroskedasticity

One option: Transforming the Response

Normally when we see the "fan" shape in the residual plot, a log transformation on y helps mitigate heteroskedasticity.

In [8]:
```
df['log_price'] = np.log(df['price'])
```

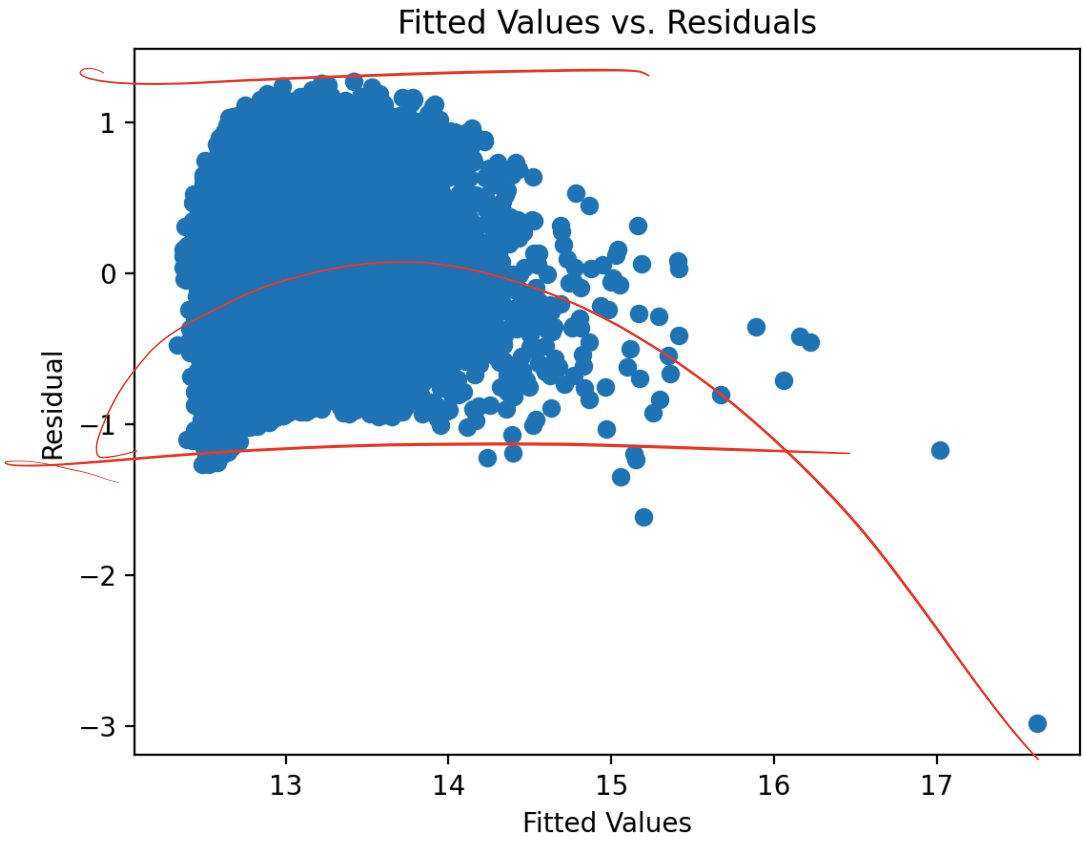
In [9]:
```
(ggplot(df, aes(x = 'sqft_living', y = 'log_price')) +
       geom_point())
```



Out[9]: <Figure Size: (640 x 480)>

In [10]:
```
f2 = 'log_price~sqft_living'
model2 = smf.ols(formula=f2, data=df).fit()
p2 = model2.fittedvalues
res2 = model2.resid
plt.scatter(p2,res2)
plt.xlabel("Fitted Values")
plt.ylabel("Residual")
plt.title("Fitted Values vs. Residuals")

plt.show()
```



The heteroskedasticity is improved, but now we see some non-linearity between x and y.

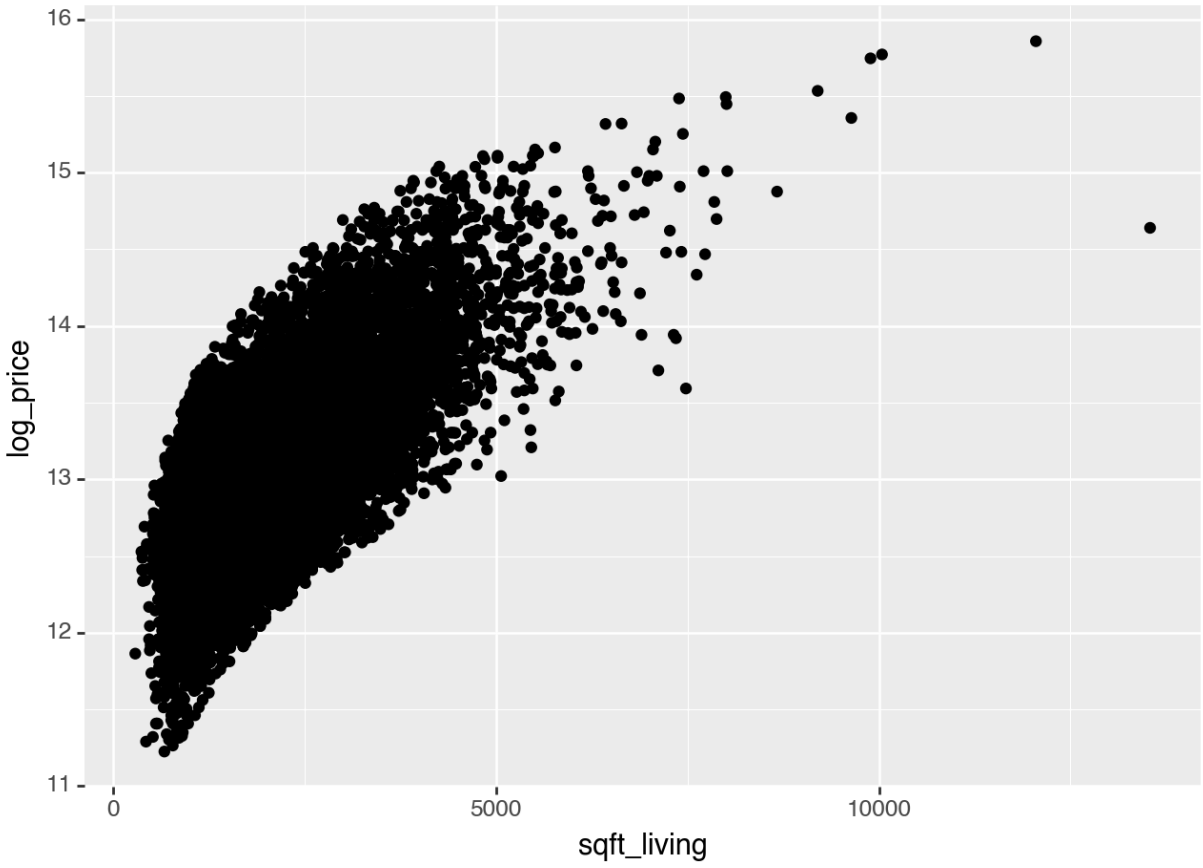
This suggests we may need to transform the predictor too, in order to find a linear relationship between the response and predictor.

Transforming the Predictor

A heuristic for transforming x is just to look at the scatterplot of x vs. the response in question and see what function the scatterplot approximates.

If we are using `sqft_living` to predict `log_price`, the scatterplot looks like $y = \log(x)$ or maybe $y = \sqrt{x}$.

In [14]:
```
(ggplot(df, aes(x = 'sqft_living', y = 'log_price')) +
       geom_point())
```



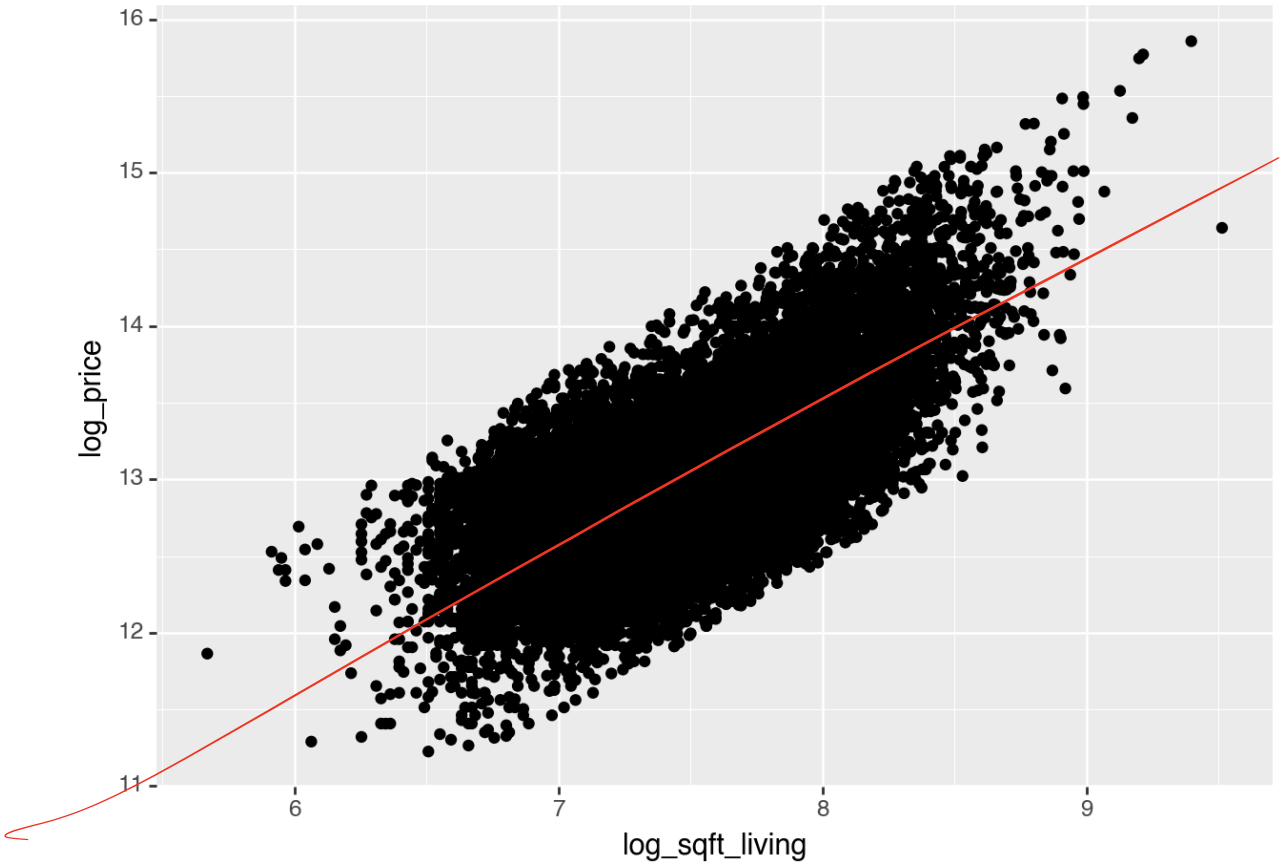
Out[14]: <Figure Size: (640 x 480)>

We can try both...

If we use log:

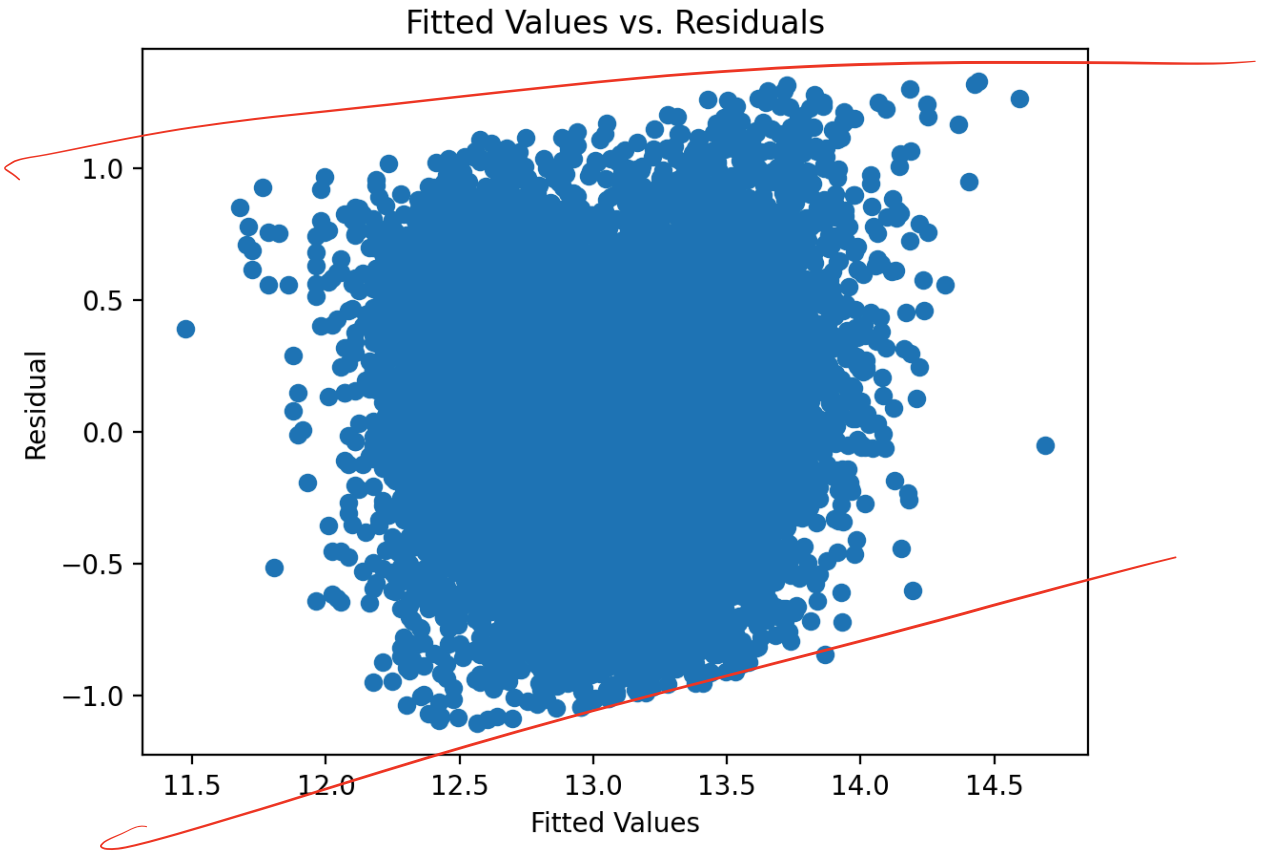
In [17]:
```
df['log_sqft_living'] = np.log(df['sqft_living'])

(ggplot(df, aes(x = 'log_sqft_living', y = 'log_price')) +
       geom_point())
```



Out[17]: <Figure Size: (640 x 480)>

In [18]:
```
f3 = 'log_price~log_sqft_living'
model3 = smf.ols(formula=f3, data=df).fit()
p3 = model3.fittedvalues
res3 = model3.resid
plt.scatter(p3,res3)
plt.xlabel("Fitted Values")
plt.ylabel("Residual")
plt.title("Fitted Values vs. Residuals")
plt.show()
```



In [19]:
```
model3.summary()
```

Out[19]:

OLS Regression Results

Dep. Variable:	log_price	**R-squared:**	0.456
Model:	OLS	**Adj. R-squared:**	0.455
Method:	Least Squares	**F-statistic:**	1.808e+04
Date:	Tue, 24 Sep 2024	**Prob (F-statistic):**	0.00
Time:	12:37:07	**Log-Likelihood:**	-10240.
No. Observations:	21613	**AIC:**	2.048e+04
Df Residuals:	21611	**BIC:**	2.050e+04
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	6.7299	0.047	143.001	0.000	6.638	6.822
log_sqft_living	0.8368	0.006	134.459	0.000	0.825	0.849

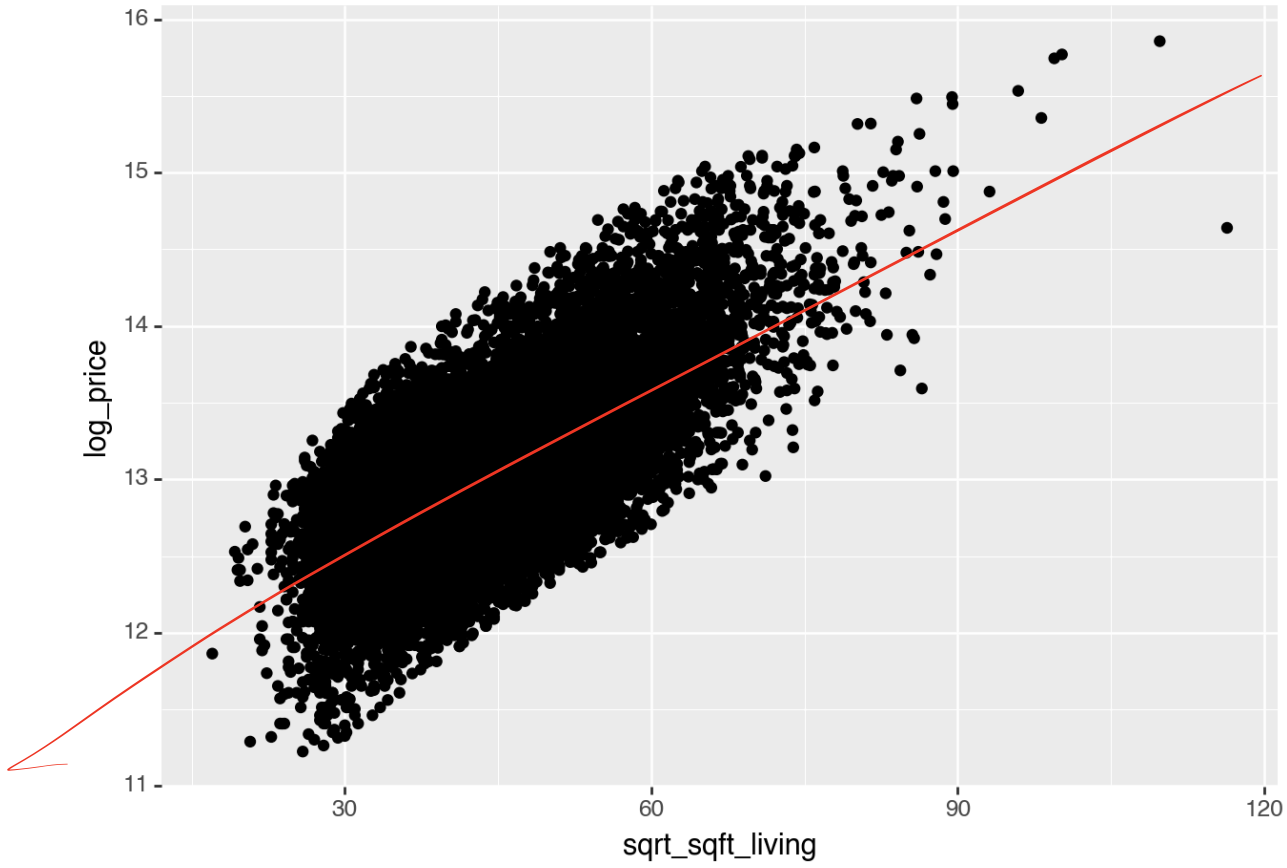
Omnibus:	123.344	**Durbin-Watson:**	1.978
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	113.759
Skew:	0.142	**Prob(JB):**	1.98e-25
Kurtosis:	2.787	**Cond. No.**	137.

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

If we use sqrt:

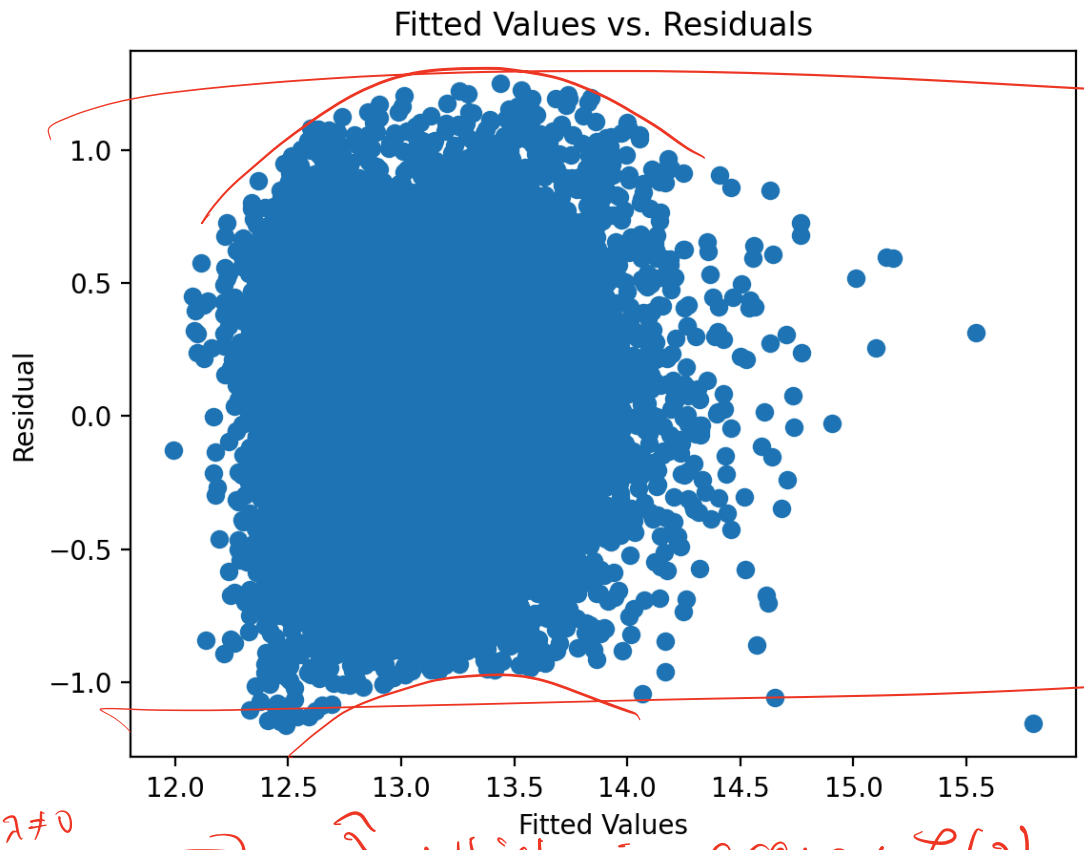
In [21]:
```
df['sqrt_sqft_living'] = np.sqrt(df['sqft_living'])


(ggplot(df, aes(x = 'sqrt_sqft_living', y = 'log_price')) +
       geom_point())
```



Out[21]: <Figure Size: (640 x 480)>

In [22]:
```
f4 = 'log_price~sqrt_sqft_living'
model4 = smf.ols(formula=f4, data=df).fit()
p4 = model4.fittedvalues
res4 = model4.resid
plt.scatter(p4,res4)
plt.xlabel("Fitted Values")
plt.ylabel("Residual")
plt.title("Fitted Values vs. Residuals")
plt.show()
```



$y \to \begin{cases} y^\lambda & \lambda \neq 0 \\ \log(y), & \lambda = 0 \end{cases}$ $\Rightarrow$ $\hat{\lambda}$ which is $\arg\max \mathcal{L}(\lambda)$

Either works – I think log price ~ log sqft looks a little bit nicer in terms of the residual plot.

Note: Transformation can be applicable even when we don't have heteroskedasticity -- for example if there's a nonlinear relationship between x and y.

A general strategy to identify which transformations are most appropriate is to use the Box-Cox transformation method. This is helpful, for example, if I couldn't guess that log or sqrt were reasonable functions to use based off of a visual inspection. See, e.g., https://www.css.cornell.edu/faculty/dgr2/_static/files/R_html/Transformations

Another solution to address heteroskedasticity: Robust standard errors

It is common to use "robust standard errors" in econometric applications. We can access the robust standard errors, or Heteroskedasticity-Consistent Standard Errors (HCE) using `statsmodels` methods:

The robust estimated variance matrix is calculated as:

$$Var_{HC}(\hat{\beta}) = (X^T X)^{-1} X^T diag(e_i^2) X (X^T X)^{-1}$$

$\sigma^2 (X^T X)^{-1}$

in line with White (1980).

This adjustment tries to downweigh the impact of increasing residuals in the variance estimation step.

Notice that if $e_i^2 \approx \sigma^2$ for all i, the formula reduces to the usual variance matrix for beta hat:

$Var(\hat{\beta}) = \sigma^2(X^T X)^{-1}.$

In [27]:
```
credit = pd.read_csv('../Data/Credit.csv')

#orginal model that we know having heteroskedasticity problem
model =smf.ols('Balance ~ Income + Rating + Cards +Age + Education \
                + Gender + Student + Married + Ethnicity',data=credit).fit()

#extact HCE variance-coviance matrix
var_HC =model.cov_HC0
var_HC
```

Out[27]:
```
array([[ 1.28979518e+03, -2.85353537e+01, -1.93737996e+01,
        -1.91773618e+01, -1.21102507e+02, -1.31108552e+02,
         2.15189155e+00, -9.87186508e-01, -2.83436775e+01,
        -4.19545031e+00, -4.02601264e+01],
       [-2.85353537e+01,  1.04540615e+02, -1.76173734e+01,
        -9.13676854e+00,  3.60555120e+00, -2.99505322e+00,
         1.95705491e-01, -3.60836602e-02,  1.37253190e+00,
        -1.93788690e-01, -5.71573694e-01],
       [-1.93737996e+01, -1.76173734e+01,  2.60173417e+02,
         1.19331093e+01, -1.82455379e+01, -6.48625401e+00,
        -7.32623436e-01,  1.69110939e-01, -1.18638848e+00,
        -2.39572478e-02, -1.69790633e+00],
       [-1.91773618e+01, -9.13676854e+00,  1.19331093e+01,
         1.09805461e+02, -2.66797583e+01, -1.78231341e+01,
         2.22313056e-01, -5.20147833e-02,  1.69160764e+00,
        -1.79079061e-01, -9.93094775e-01],
       [-1.21102507e+02,  3.60555120e+00, -1.82455379e+01,
        -2.66797583e+01,  2.11832833e+02,  1.03655948e+02,
         1.16961731e-01,  1.30085214e-03, -5.00674435e+00,
         6.48215640e-01,  4.77855921e-01],
       [-1.31108552e+02, -2.99505322e+00, -6.48625401e+00,
        -1.78231341e+01,  1.03655948e+02,  1.54647620e+02,
         3.56773320e-01, -4.99088021e-02, -2.83152775e-01,
         5.56792020e-01,  1.05395794e+00],
       [ 2.15189155e+00,  1.95705491e-01, -7.32623436e-01,
         2.22313056e-01,  1.16961731e-01,  3.56773320e-01,
         5.46879998e-02, -1.16917379e-02,  1.03503126e-01,
        -5.85418584e-03, -2.66999700e-02],
       [-9.87186508e-01, -3.60836602e-02,  1.69110939e-01,
        -5.20147833e-02,  1.30085214e-03, -4.99088021e-02,
        -1.16917379e-02,  4.09522670e-03, -2.38449033e-02,
        -1.79848498e-03,  9.53202836e-03],
       [-2.83436775e+01,  1.37253190e+00, -1.18638848e+00,
         1.69160764e+00, -5.00674435e+00, -2.83152775e-01,
         1.03503126e-01, -2.38449033e-02,  1.14035045e+01,
        -1.15145126e-01,  4.51416457e-01],
       [-4.19545031e+00, -1.93788690e-01, -2.39572478e-02,
        -1.79079061e-01,  6.48215640e-01,  5.56792020e-01,
        -5.85418584e-03, -1.79848498e-03, -1.15145126e-01,
         9.20241639e-02,  1.93567461e-02],
       [-4.02601264e+01, -5.71573694e-01, -1.69790633e+00,
        -9.93094775e-01,  4.77855921e-01,  1.05395794e+00,
        -2.66999700e-02,  9.53202836e-03,  4.51416457e-01,
         1.93567461e-02,  2.64627072e+00]])
```

In [28]:
```
#sometimes we may not care to see the corrected SE values themselves,
#we can extract the corrected t test and p values directly if we just care a
print(np.round(model.get_robustcov_results(cov_type = "HC0").summary2().tabl
```

```
                            Coef.  Std.Err.         t    P>|t|     [0.025  \
Intercept                -549.31402  35.91372 -15.29538  0.00000 -619.92330
Gender[T.Female]          -10.71056  10.22451  -1.04754  0.29550  -30.81278
Student[T.Yes]            416.43756  16.12989  25.81775  0.00000  384.72489
Married[T.Yes]            -15.10961  10.47881  -1.44192  0.15013  -35.71180
Ethnicity[T.Asian]         21.76158  14.55448   1.49518  0.13568   -6.85370
Ethnicity[T.Caucasian]     10.64919  12.43574   0.85634  0.39234  -13.80048
Income                     -7.77460   0.23385 -33.24542  0.00000   -8.23437
Rating                      3.97896   0.06399  62.17709  0.00000    3.85314
Cards                       3.96537   3.37691   1.17426  0.24101   -2.67390
Age                        -0.64159   0.30335  -2.11500  0.03507   -1.23801
Education                  -0.37986   1.62674  -0.23351  0.81549   -3.57815

                             0.975]
Intercept                -478.70473
Gender[T.Female]            9.39166
Student[T.Yes]            448.15024
Married[T.Yes]              5.49258
Ethnicity[T.Asian]         50.37687
Ethnicity[T.Caucasian]     35.09886
Income                     -7.31482
Rating                      4.10478
Cards                      10.60465
Age                        -0.04517
Education                   2.81844
```

In [29]:
```
#Compare to the orginal model summary, numbers are slightly changed.
model.summary()
```

Out[29]:

OLS Regression Results

Dep. Variable:	Balance	**R-squared:**	0.951
Model:	OLS	**Adj. R-squared:**	0.950
Method:	Least Squares	**F-statistic:**	757.8
Date:	Tue, 24 Sep 2024	**Prob (F-statistic):**	4.46e-248
Time:	12:37:08	**Log-Likelihood:**	-2415.4
No. Observations:	400	**AIC:**	4853.
Df Residuals:	389	**BIC:**	4897.
Df Model:	10		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	-549.3140	35.085	-15.657	0.000	-618.293	-480.335
Gender[T.Female]	-10.7106	10.325	-1.037	0.300	-31.010	9.589
Student[T.Yes]	416.4376	17.336	24.021	0.000	382.353	450.522
Married[T.Yes]	-15.1096	10.728	-1.408	0.160	-36.202	5.983
Ethnicity[T.Asian]	21.7616	14.678	1.483	0.139	-7.096	50.619
Ethnicity[T.Caucasian]	10.6492	12.716	0.837	0.403	-14.351	35.649
Income	-7.7746	0.244	-31.878	0.000	-8.254	-7.295
Rating	3.9790	0.055	72.332	0.000	3.871	4.087
Cards	3.9654	3.793	1.045	0.296	-3.492	11.422
Age	-0.6416	0.306	-2.096	0.037	-1.243	-0.040
Education	-0.3799	1.659	-0.229	0.819	-3.642	2.882

Omnibus:	15.651	**Durbin-Watson:**	1.987
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	16.769
Skew:	0.490	**Prob(JB):**	0.000228
Kurtosis:	2.789	**Cond. No.**	2.73e+03

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 2.73e+03. This might indicate that there are strong multicollinearity or other numerical problems.

Another solution: Weighted least squares

In [ ]:

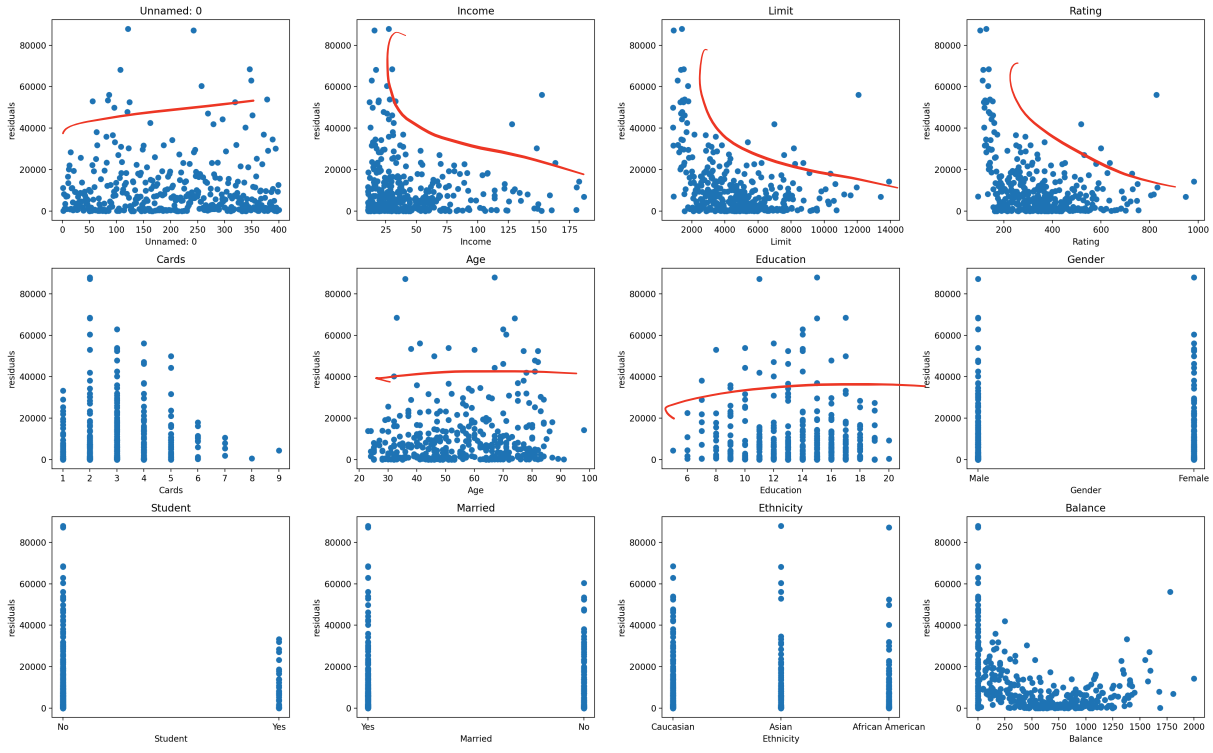
In [31]:
```
plt.figure(figsize=(20, 16))

# i: index
for i, col in enumerate(credit.columns):
    plt.subplot(4, 4, i+1)
    x = credit[col]
    y = model.resid**2
    plt.plot(x, y, 'o')

    plt.tight_layout()
    plt.title(col)
    plt.xlabel(col)
    plt.ylabel('residuals')

    # looks like residuals are proportional to Income, Limit, and Rating

plt.show()
```



One method of performing weighted least squares is to regress each observation's squared residual onto the predictors causing

heteroskedasticity, and then use the fitted value's absolute value to weigh each observation:

In [33]:
```
credit["res_sq"] = model.resid**2
model_res = smf.ols("res_sq ~ Rating", data=credit).fit()  #Here we don't al
weight = model_res.fittedvalues
weight = abs(weight)
weight = weight**(-1/2)
credit['weight'] = weight
```

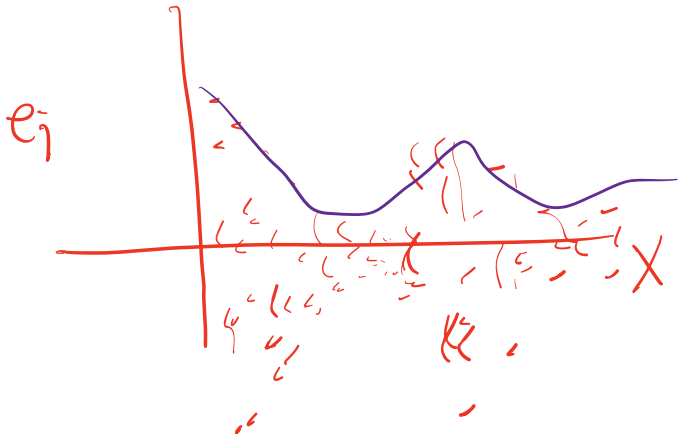
$\hat{\Sigma} = \text{diag}(\hat{\sigma}_i^2)$

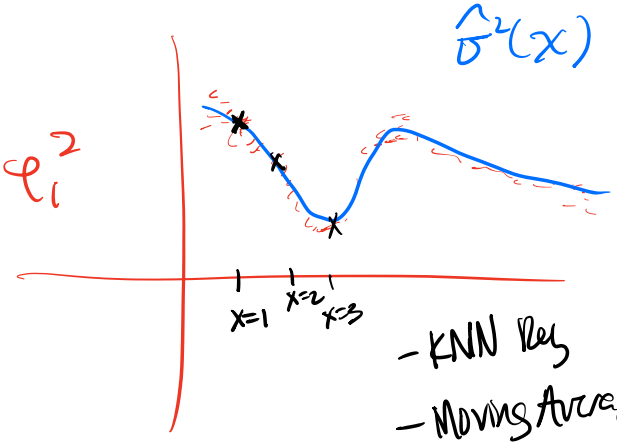
In [34]:
```
y=credit.Balance
x=credit[['Income', 'Rating','Cards','Age','Education']] #or some other comb
```

In [35]:
```
mod_wls = sm.WLS(y,x.astype(float), credit['weight'])
mod_res = mod_wls.fit()
print(mod_res.summary())
```

$e_i^2 \sim X_1 \ (+X_2 + X_3)$

What if e_i is a nonlinear f'n of predictors?





Nonparametric Regression

$y_i = g(x_i) + e_i$

```
                            WLS Regression Results
==============================================================================
==========
Dep. Variable:                Balance   R-squared (uncentered):
0.952
Model:                            WLS   Adj. R-squared (uncentered):
0.952
Method:                 Least Squares   F-statistic:
1581.
Date:                Tue, 24 Sep 2024   Prob (F-statistic):
1.27e-258
Time:                        12:37:09   Log-Likelihood:
-2673.0
No. Observations:                 400   AIC:
5356.
Df Residuals:                     395   BIC:
5376.
Df Model:                           5
Covariance Type:            nonrobust
==============================================================================
==
                 coef    std err          t      P>|t|      [0.025      0.97
5]
------------------------------------------------------------------------------
--
Income        -6.6570      0.400    -16.636      0.000      -7.444      -5.8
70
Rating         3.7287      0.095     39.271      0.000       3.542       3.9
15
Cards        -18.7933      6.417     -2.929      0.004     -31.408      -6.1
79
Age           -3.8213      0.467     -8.176      0.000      -4.740      -2.9
02
Education    -15.2975      2.172     -7.042      0.000     -19.568     -11.0
27
==============================================================================
==
Omnibus:                      120.287   Durbin-Watson:                   1.8
05
Prob(Omnibus):                  0.000   Jarque-Bera (JB):              406.6
06
Skew:                           1.341   Prob(JB):                     5.09e-
89
Kurtosis:                       7.148   Cond. No.                        32
4.
==============================================================================
==

Notes:
[1] R² is computed without centering (uncentered) since the model does not c
ontain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is corre
ctly specified.
```

In [ ]:

Usual OLS setup:

$$Y \sim N(X\beta, \sigma^2 I)$$

↑ Homoskedastic error

But in heteroskedastic situations:

$$Y \sim N(X\beta, \Sigma)$$

↓ general cov matrix

e.g. $e_i \perp e_j \ \forall i \neq j$ but $Var(e_i) \neq \sigma^2 \ \forall i$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ & & \sigma_3^2 & \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n^2 \end{pmatrix}$$

$$Y = X\beta + \varepsilon, \quad Var(\varepsilon) = \Sigma$$

Consider the transformation:

$$\tilde{Y} = \Sigma^{-1/2} Y$$

$$\tilde{X} = \Sigma^{-1/2} X$$

$$\tilde{\varepsilon} = \Sigma^{-1/2} \varepsilon$$

$$\mathrm{Var}(\tilde{\varepsilon}) = \mathrm{Var}(\Sigma^{-1/2}\varepsilon)$$

$$= \Sigma^{-1/2} \mathrm{Var}(\varepsilon) (\Sigma^{-1/2})^T$$

$$= \Sigma^{-1/2} \mathrm{Var}(\varepsilon) \Sigma^{-1/2}$$

$$= \Sigma^{-1/2} \Sigma \Sigma^{-1/2}$$

$$= \Sigma^{-1/2} \Sigma^{1/2} \cdot \Sigma^{1/2} \Sigma^{-1/2}$$

$$= I_n$$

$$\tilde{Y} = \tilde{X}\beta + \tilde{\varepsilon}, \quad \mathrm{Var}(\tilde{\varepsilon}) = I_n$$

This is just OLS!

$$\Rightarrow \hat{\beta} = (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T\tilde{Y}$$

$$= \left((\Sigma^{-1/2}X)^T(\Sigma^{-1/2}X) \right)^{-1} (\Sigma^{-1/2}X)^T(\Sigma^{-1/2}Y)$$

$$= (X^T\Sigma^{-1/2}\Sigma^{-1/2}X)^{-1}(X^T\Sigma^{-1/2}\Sigma^{-1/2}Y)$$

$$= (X^T\Sigma^{-1}X)^{-1}(X^T\Sigma^{-1}Y)$$

↑ WLS solution