RESEARCH STATEMENT

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I work on ***algebraic combinatorics***, which is an area of research that often involves using abstract algebra to solve combinatorial problems, and conversely, using techniques from discrete mathematics (combinatorics) to answer algebraic questions.

As the Fields medalist Timothy Gowers [8] noted, there are two cultures of mathematics, one focuses on *problem-solving* while the other emphasizes *theory-building*. On the first side we have ***combinatorics***, which studies discrete objects (e.g. permutations, graphs, integers, networks, algorithms, etc.) with a primary focus on enumeration (counting). It often emphasizes problem-solving techniques (tricks) over abstractions. In fact, many elegant combinatorial proofs, despite being extremely technical, are elementary enough to be understood by high school students. Combinatorics has found many applications since the prosperity of computer sciences. For example, the navigation systems we use everyday, finding the best route between two points, can be naturally translated to a graph theory problem. And how efficient the navigation algorithm performs heavily depends on the combinatorial theory behind. Moreover, many other subjects like social networks, sorting and ranking algorithms, are all combinatorial problems in their essences. However, despite being appreciated for having fruitful real-world applications in computer science and optimization theory, combinatorics is often criticized by many main stream mathematicians for "having very little structure and consists of nothing but a large number of problems." ([8], p. 9) On the other side, ***abstract algebra*** is the study of symmetries and algebraic structures (such as vector fields, groups, algebras, etc.), which are often highly abstract and theorized. Although being regarded as part of the "central core" of mathematics, abstract algebra, along with many other so-called "pure mathematics", have always been accused to be useless in real world scenarios and consist of noting but abstract theories.

Algebraic Combinatorics is a young field of mathematics, born roughly in the 1970's, which is the magical combination of these two seemingly completely unrelated fields. Almost miraculously, many algebraic structures can be interpreted in terms of simple (and often pretty) diagrams or graphs, which then allows us to tackle those abstract algebraic questions using concrete techniques from discrete mathematics. Just as Sara Billey said, "Combinatorics is the nanotechnology of mathematics". On the other hand, abstract algebra has also proved to be very helpful in various kinds of problems in discrete mathematics which otherwise could be very technical and hard to solve. For example, in one of my undergraduate research projects on finding spanning trees in certain networks [4], we struggled and failed to come up with an elementry proof, but the problem got resolved immediately once we incorporated certain ideas from abstract group theory (specifically, Galois theory).

I found myself fascinated by these kinds of connections, and am always excited about the discovery of new connections during my research. In particular, my research interests often lie in the crossroad of algebra, combinatorics, and statistical physics. Specifically, I am interested the following topics: cluster algebras, affine combinatorics, and integrable models in statistical physics, of which I will elaborate on the first two in this statement. The plan of this statement is as follows. In the first section, I will talk about my previous major contributions, which is in the area of ***cluster algebras***. And in the second section, I will discuss my current main focuses, which is my thesis research project on ***affine combinatorics*** including several future directions.

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1. Cluster Algebras

Cluster algebras are special kinds of commutative algebras. Loosely speaking, a *cluster algebra* is a collection of Laurent polynomials (i.e. a polynomial divided by a monomial, e.g. $\frac{ab+cd+ef}{ace}$), which is defined recursively and has vert nice combinatorial properties. They were first introduced by Fomin and Zelevinsky in 2002 as an algebraic framework for canonical basis of quantum groups, but have been found to be connected to many other areas in mathematics (e.g. hyperbolic geometry, homological algebra, plane curve singularity, etc.) and physics (thermodynamic Bethe Ansatz, Feynman integrals, $\mathcal{N} = 4$ Yang-Mills, string theory, etc.)

Among my several research interests, cluster algebra is the topic where I spent most of my time on. My main contribution in this are is a generalization of cluster algebras called ***supersymmetric cluster algebras***.

Background on Supersymmetric Cluster Algebras. In particle physics, supersymmetry (SUSY) is a simple principle proposing a relationship between two kinds of basic particles, bosons and fermions. It appeared last century as an effort to unifying all physics theories. The mathematical framework for SUSY is a new kind of numbers called *Grassmann numbers* or simply "super numbers". These numbers are defined in a similar spirit as complex numbers $a+b\mathbf{i}$ where a, b are usual real numbers and $\mathbf{i}$ satisfy the relation $\mathbf{i}^2 = -1$. Grassmann numbers looks like $a+b\theta_1+c\theta_2+d\theta_3+\cdots+e\theta_1\theta_2+f\theta_1\theta_3\cdots$, where $a, b, c, \cdots$ are usual numbers and $\theta_1, \theta_2, \cdots$ are new variables[1] which satisfy the relation $\theta_i\theta_j = -\theta_j\theta_i$ (anti-commutativity). Under this framework, the usual numbers $(a, b, c, \cdots)$ correspond to bosons and the anti-commutative variables θ_i's correspond to fermions.

In recent years, a lot of progress have been made on extending the current theory of cluster algebras to a supersymmetric version (naively speaking, adding anti-commutative variables to cluster algebras), such as [13, 16] to name a few. This line of research is not only a natural mathematical question to pursue, but also has found application in theoretical physics [7].

My contribution. In a serious of joint works with G. Musiker and N. Ovenhouse [10, 11, 12], we took the first step of understanding supersymmetric cluster algebras through a geometric lens. In particular, we introduced the notion of ***super cluster algebras of geometric type***, motivated by Penner-Zeitlin's work on super-geometry [14]. Our works generalize the notion of (classical) cluster algebras of geometric type, which is a canonical and motivating example in the theory of cluster algebras.

In [10, 11], we proved that, elements of a super cluster algebra (of geometric type) can be interpreted using ***double dimer configurations***, a statistical-physics model first used to describe crystal structures of certain molecules. In the recent paper [12], we took a more algebraic approach and relate the combinatorics of double dimer models from super cluster algebras to the ortho-symplectic super-group $\mathrm{OSp}(1|2)$[2].

Future Directions. There are several future directions that my collaborators and I are planning to pursue.

(1) We plan to investigate the mathematical meanings of the statistical physics models (the double dimer models) which show up in our work. It is known that single dimer models are related to continued fractions. In a current joint work-in-progress with G. Musiker, N. Ovenhouse, and R. Schiffler, we are working on a generalization of this correspondence which suggests a higher-dimensional analogue of continued fractions.
(2) Our works on super cluster algebras of geometric type only provide an special class of super cluster algebras, despite being an important one. A more general, axiomatic theory of super cluster algebras is still out of reach. However, our works shed an light on this darkness since we provided the very first explicit formulae. We hope to continue in this direction and make further progress on a more general theory.

[1] Think of the $\theta_1, \theta_2, \cdots$ as the $\mathbf{i}$ in complex numbers, but this time we have many of them instead of just one.

[2] $\mathrm{OSp}(1|2)$ is an abstract group of 3×3 matrices whose entries are "super numbers", which is of fundamental importance in the algebraic theory of supersymmetry.

Another Related Project. I've also worked on a closely related project involving ***Laurent Phenomenon (LP) Algebras***, which is another generalization of cluster algebras. In cluster theory, one of the most important achievements is the proof of the positivity conjecture [9], which states that all elements of a cluster algebras only have positive coefficients[3].

The same positivity property was conjectured for LP algebras as well, which has been wildly open. In the collaboration with E. Banaian, S. Chepuri, and E. Kelley [1], we made the first progress on proving this conjecture in a special case. In a subsequent work (in-progress) [2], we took the dimer-model approach and strengthened our previous result to all acyclic graph LP algebras, with only small restrictions on the chosen element.

2. AFFINE COMBINATORICS.

Permutations are one of the most fundamental mathematical objects, which are ways to arrange finitely many numbers $\{1, 2, \cdots, n\}$ into a sequence. For example, the permutations of $\{1, 2, 3\}$ are 123, 132, 231, 312, 213, and 321. A rather new and less-studied variation of permutations are ***affine permutations***. An *affine permutation of order* n is an arrangement the entire line of integers $\{\cdots, -3, -2, -1, 0, 1, 2, 3 \cdots\}$ under the requirement of being periodic modulo the number n. For example,

$$\cdots, -3, -8, -1, 0, -5, 2, 3, -2, 5, 6, 1, 8, 9, \cdots$$

is an affine permutation of order 3. It can be seen that, when colored periodically in three colors, numbers of the same color are all equivalent modulo 3 and form an increasing sequence. In fact, affine permutations are really ***infinite periodic permutations***, where the buzzword "affine" actually came from Lie theory.

Many other combinatorial objects have very similar infinite-periodic analogues. I will refer to the study of these objects roughly as ***affine combinatorics***. Under the guidance of my advisor P. Pylyavskyy, my thesis research is centered around affine combinatorics. There is one project on *affine posets* (see the following section) that I am working on right now with my advisor, and several others that I plan to think about in the future.

2.1. Current Project: Affine partially ordered sets.

We say that a set is *ordered* if there is an ordering on the elements, such that one can compare certain pairs of elements of the set. For example, the set of all integers (or real numbers) is a ordered set, where the order is the obvious one: $\cdots -1 < 0 < 1 < 2 < 3 \cdots$. The ordering on numbers is linear, and is called a *total order*, since any two numbers are comparable: for two different numbers x and y, either $x < y$ or $x > y$. However, one may not require an ordering on a set to be total, that is to say, there might exist two elements a and b that are not comparable, i.e. $a \not< b$, $a \not> b$, and $a \neq b$. We call a set with such an ordering a *partially ordered set*, or *poset* for short. An important example of posets is human's preferences, because in real life people are often indifference between several options, which is captured by the "incomparability" in a poset. The combinatorics of posets has played an important rule in some of the recent developments of socioeconomical analysis.

$$\begin{tikzcd} & e \arrow[dl] \arrow[dr] & & d \arrow[dl] \\ b \arrow[dr] & & c \arrow[dl] & \\ & a & & \end{tikzcd}$$

A poset can be abstractly represented by a directed graph, which consists of nodes which are connected by arrows. For example, the above graph represents a poset, in which nodes correspond to elements of the poset and arrows tell us how to compare the elements. In particular, an element x is greater than y when there is a directed sequence of arrows from x to y. For example, $c < e$ because of the arrow $e \to c$, and $a < d$ because of $d \to c \to a$. At the same time, b and c are incomparable because there's no arrow between then, as well as b and d are incomparable, etc.

[3]In other words, you will only see '+' signs in them.

The object of interest to us is called ***affine posets***. They are, loosely speaking, periodic infinite posets — a poset with infinitely many elements such that the patterns of the arrows are periodic.[4]

My Result. In 1950, Dilworth [6] proved a famous decomposition theorem for posets, now known as the *Dilworth's theorem*. Dilworth's theorem studies the minimal way to decompose a poset in to totally ordered sets (called *chains*). In the above example, the poset can be decomposed into two chains: $e \to b$ and $d \to c \to a$, using the smallest number of chains. More specifically, Dilworth's theorem says that the minimal number of chains that we need to decompose a poset equals to the maximal cardinality of a pair-wise incomparable subset of the poset. For example, in the above poset, a maximal pairwise incomparable subset is $\{b, c\}$, having two elements. Dilworth's theorem is directly related the *min-cut max-flow* problem in network theory, and has many other implications in lattice theory and network flows.

In a work-in-progress with my advisor P. Pylyavskyy [15], we proved an analogue of Dilworth's theorem for affine posets. Our ***affine Dilworth's theorem*** is also related to certain problems in network theory. In particular, our theorem implies, as a corollary, a 2007 theorem of Bessy and Thomassé [3] about the minimal way to cover a network by circuits, which was first conjectured by Gallai in 1963 and remained open for nearly 50 years.

2.2. **Future Plans.** I plan to pursue in my thesis research several directions under this line of research, which I will summarize in below.

Affine Greene-Kleitman Correspondence. An important generalization of Dilworth's theorem is called the *Greene–Kleitman correspondence*, which associates to each poset a "Young diagram"[5] Right now, we have a conjectural ***affine Greene-Kleitman correspondence*** that we are working towards a proof.

Affine Permutations and Flag Manifolds. A *flag manifold* is, loosely speaking, a sequence of nested vector spaces. They are important objects in algebraic geometry and are related to Hilbert's 15th problem. Despite being sophisticated geometric objects, certain aspects of flag manifolds can be understood by combinatorial means. For a pair of two flag manifolds, their "relative position" is given by a permutation. Moreover, given two flag manifolds, one can use the above-mentioned Greene-Kleitman correspondence to find their relative position. And of course, there should be an affine version of the story.

A flag manifold can also be represented using a matrix, then *affine flag manifolds* can be defined in an analogous way where the representing matrix is an infinite-periodic matrix. Affine flag manifolds are also important in algebraic geometry and Lie theory, but are much more complicated and less understood. There is also a notion of relative position, which is given by an affine permutation. And it remains an open question how to find the relative position of given two affine flag manifolds. Since GK correspondence is proved to be useful in the non-affine setting, we hope that, after proving our conjectural affine GK correspondence, it can be used to give an answer to this question.

Jordan Forms of Infinite-Periodic Matrices. In linear algebra, it is very important to find the *Jordan canonical form* of nilpotent (i.e. upper-triangular) matrices[6]. As another application, this can be done using the GK correspondence. The affine version of matrices are infinite-periodic matrices, which has a fancier name called *loop groups*. To our knowledge, not much is know about nilpotent elements in loop groups, and we hope to make some progress along this direction.

Possible Connection to Cluster Algebras. Matrices and Flag manifolds are known to have cluster algebra structures. However, the cluster algebra attached to loop groups and affine flag manifold is unknown, though is believed by the experts to exist. In the future, as a more far-reaching goal, I hope to be able make some progress on this question by combining my knowledge in these two different topics.

[4]It's worth noting here that, there are many ways to present a permutation, and one of them is using a poset. When the permutation is affine, its poset presentation is indeed an affine poset. This is actually one of the motivations behind the definition of affine posets.

[5]Young diagrams are important "pictures" which carries a lot of information about the abstract algebras and groups.

[6]A matrix M is nilpotent if multiplying by itself a certain times gives the zero matrix, i.e. $M^k = 0$ for some k.

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