MATH 4242 Applied Linear Algebra

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Contents

1. Systems of Linear Equations

A $m \times n$ system of linear equation is of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{n1}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{n2}x_n &= b_n \\ \cdots \quad \cdots & \quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

Such equation can be represented using product of matrices.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or by an augmented matrix.

$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{m1} & b_1 \\ a_{21} & a_{22} & \cdots & a_{m2} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Definition 1.1. We have three types of elementary row operations.

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(1) Multiply the i-th equation (or the i-th row of the augmented matrix), then add it to the j-th equation (or the j-th row of the augmented matrix).
(2) Permute the equations (or the rows of the augmented matrix)
(3) Multiply one equation (or one row of the augmented matrix) by a non-zero number.

1.1. **Systems of $n \times n$ Equations.** Matrices considered in this sections are all $n \times n$.

Definition 1.2. A matrix is *regular* if it can be turned into a upper triangular matrix such that every entry on the diagonal is non-zero.

Proposition 1.3. *Let E be the matrix with 1's on the diagonal and $E_{ij} = k \neq 0$ is the only other non-zero entry in the lower triangular part. Then for any matrix M, EM is the matrix obtained by multiplying the j-th row of M then adding to the i-th row of M.*

Proposition 1.4. *A matrix A is regular if and only if it has an LU factorization, i.e.*

$$A = LU$$

where L is a lower uni-triangular matrix, and U is a upper triangular matrix with non-zero diagonal entries.

Definition 1.5. Let $w \in S_n$ be a permutation, then define $P_w = \{a_{ij}\}$ to be the matrix such that

$$a_{i,j} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.6. *For any matrix M, $P_w M$ is the matrix obtained by permuting the rows of M according to the permutation w.*

Definition 1.7. A matrix A is called *non-singular* if it can be turned into a upper triangular matrix without non-zero diagonal entry via row operations of the first two types.

Proposition 1.8. *A matrix A is non-singular if and only if it has a permuted LU factorization: $PA = LU$ where P is some permutation matrix.*

Proposition 1.9. *Denote A^T the transpose of A. We have that $AB = (BA)^T$.*

Proposition 1.10. *A matrix A is regular if it admits an LDV factorization, $A = LDU$ where L is lower-unitriangular matrix, D is a diagonal matrix, and U is a uni-upper triangular matrix.*

1.2. **Systems of $m \times n$ Equations.**

Definition 1.11. A matrix is in row echelon form if it looks like,

$$\begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\bullet$'s are non-zero entries (called *pivots*) and $*$ represent generic entries. The pivots are the first non-zero entries in each rows. We require the pivots occupy the first several rows consecutively.

Proposition 1.12. *Every matrix can be turned into a row echelon form using elementary row operations of type I and II. In other words, every matrix A has a factorization $PA = LU$ where P is a permutation matrix, L is a lower uni-triangular matrix, and U a matrix in row-echelon form.*

Definition 1.13. Since every matrix can be turned in to row-echelon form using elementary row operations, we define its *rank* to be the number of pivots.

Proposition 1.14. *A square $n \times n$ matrix is non-singular if its rank is n (full-rank).*

2. Vector Spaces

2.1. **Some Basic Setup.**

Definition 2.1. [1] A field is a set $\mathbb{F}$ with two binary operations $\times$ (multiplication) and $+$ (addition), satisfying the following axioms.

- $a + b = b + a$ and $a \times b = b \times a$ for all $a, b \in \mathbb{F}$.
- There exists an additive identity 0 such that $0 + a = a + 0 = a$ for all $a \in \mathbb{F}$.
- There exists a multiplication identity 1 such that $1 \times a = a \times 1 = a$ for all $a \in \mathbb{F}$.
- For every $a \in \mathbb{F}$, there exists an element denoted $-a$, such that $a + (-a) = 0$.
- $0 \neq 1$.
- For every $a \in \mathbb{F}$ and $a \neq 0$, there exists an element denoted a^{-1}, such that $a \times (a^{-1}) = 1$.
- For every $a, b, c \in \mathbb{F}$, $a \times (b + c) = ab + ac$.

For most part of this class, we will take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}$.

Definition 2.2. For a field $\mathbb{F}$, denote $\mathbb{F}[x]$ the ring[2] of polynomials over $\mathbb{F}$.

$$\mathbb{F}[x] = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n | a_0, \cdots, a_n \in \mathbb{F}, n \geqslant 0, x^m x^n = x^{m+n}\}$$

Proposition 2.3. *Every polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ with complex coefficient has at least one complex solution. Note that this is not true for real polynomials.*

Definition 2.4. A field $\mathbb{F}$ is called *algebraically closed* if every polynomial in $\mathbb{F}[x]$ has a solution in $\mathbb{F}$. (By Proposition 2.3, $\mathbb{C}$ is algebraically closed).

Proposition 2.5. *The field of complex numbers $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$. In other words, $\mathbb{C}$ is the smallest algebraically closed field that contains $\mathbb{R}$.*

2.2. **Vector Spaces and Subspaces.** Let $\mathbb{F}$ be a field.

Definition 2.6. A set V is called a vector space over $\mathbb{F}$ if there exists an addition map

$$add : V \times V \to V$$

and a scalar multiplication map

$$mult : \mathbb{F} \times V \to V$$

[1] You don't need to worry too much about the abstract structures of a field. The purpose of this definition is to make everything self-contained. You can basically think of a field as a set on which you can do some sort of arithmetic.

[2] A ring is a field, where multiplication need not to be commutative, and multiplicative identity (0) need not exists.

(Here $\times$ denote the Cartesian product of sets[3].) We will abbreviate them by $a(v_1, v_2) = v_1+v_2$ and $mult(a, v) = av$.

Note that this definition (implicitly) requires that a vector space V is closed under addition and scalar multiplication, i.e. $v_1 + v_2 = add(v_1, v_2) \in V$ and $av = mult(a, v) \in V$.

Elements of a vector spaces are called *vectors.*

Definition 2.7. Let V be a vector space over $\mathbb{F}$. A subset U of V is a *subspace* if it is closed under addition and scalar multiplication. (In other words, a subspace is a subset that is a vector space itself.)

Definition 2.8. Let $U_1, \cdots, U_m$ be subspaces of V. Then define their sum to be

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m | u_1 \in U_1, \cdots, u_m \in U_m\}$$

Proposition 2.9. *Let $U_1, \cdots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is also a subspace of V, furthermore, it's the smallest subspace of V that contain all of $U_1, \cdots, U_m$.*

Definition 2.10. A sum of subspaces $U_1 + \cdots + U_m$ of V is a *direct sum* if every vector $v \in U_1 + \cdots + U_m$ can be uniquely written as $v = u_1 + \cdots + u_m$ where $u_i \in U_i$ for each i. When a summation is direct, we denote it as $U_1 \oplus \cdots \oplus U_m$.

2.3. **Linear Combination, Span, and Dimension.** Let V be a vector space over $\mathbb{F}$.

Definition 2.11. Let $v_1, v_2, \cdots, v_n \in V$, a vector $v \in V$ is a linear combination of $\{v_1, \cdots, v_n\}$ if there exists $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$v = a_1v_1 + \cdots + a_nv_n$$

Definition 2.12. Let $v_1, v_2, \cdots, v_n$ be a list of vectors in V, define their span to be the set of all linear combinations of $v_1, \cdots, v_n$.

$$\operatorname{span}(v_1, \cdots, v_n) = \{a_1v_1 + \cdots + a_nv_n | a_1, \cdots, a_n \in \mathbb{F}\}$$

Proposition 2.13. *For a list of vectors $v_1, \cdots, v_n \in V$, $\operatorname{span}(v_1, \cdots, v_n)$ is a subspace of V. Furthermore, it's the smallest subspace containing all of $v_1, \cdots, v_n$.*

Definition 2.14. A vector space V is said to be *finite dimensional* it it is the span of a finitely many vectors.

Definition 2.15. $v_1, \cdots, v_m \in V$ are *linearly independent* if the only way to write 0 as a linear combination of $v_1, \cdots, v_n$ is

$$0 = 0v_1 + 0v_2 + \cdots + 0v_n.$$

Proposition 2.16. *$v_1, \cdots, v_m \in V$ are* linearly independent *if and only if any vector $v \in \operatorname{span}(v_1, \cdots, v_m)$ can be uniquely written as a linear combination of $v_1, \cdots, v_n$.*

Definition 2.17. A list of vectors $v_1, \cdots, v_n$ is a basis of V if

- $V = \operatorname{span}(v_1, \cdots, v_n)$
- $v_1, \cdots, v_n$ are linearly independent.

Proposition 2.18. *$v_1, \cdots, v_n$ is a basis of V iff every vector $v \in V$ can be uniquely written as a linear combination of $v_1, \cdots, v_n$.*

[3]For sets A and B, defined $A \times B = \{(a, b) | a \in A, b \in B\}$

Lemma 2.19. *Let $v_1, \cdots, v_m \in V$ be a list of vectors that spans V, i.e.* $\mathrm{span}(v_1, \cdots, v_m) = V$*. Then $\{v_1, \cdots, v_m\}$ can be reduced to a basis of V. In other words, there exists a basis $\{w_1, \cdots, w_n\}$ of V such that $w_i \in \{v_1, \cdots, v_m\}$ for all i and $n \leqslant m$.*

Lemma 2.20. *Let $v_1, \cdots, v_k \in V$ be linearly independent. Then there exists a basis of V in the form*

$$\{v_1, \cdots, v_k, w_1, \cdots, w_m\}$$

Note that it's possible that $m = 0$, in the case when $\{v_1 \cdots v_k\}$ is already a basis.

Corollary 2.21. *If U is a subspace of V, then there exists another subspace W such that $V = U \oplus W$.*

Proposition 2.22. *If $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is another basis of V. Then $n = m$.*

Definition 2.23. Define the dimension of a vector space to be the size of its basis.

Proposition 2.24. *If $\{v_1, \cdots, v_n\}$ linearly independent and $n = \dim(V)$, then $\{v_1, \cdots, v_n\}$ is a basis.*

Proposition 2.25. *If U is a subspace of V, then $\dim(U) \leqslant \dim(V)$. Furthermore, $\dim(U) = \dim(V)$ iff $U = V$.*

Proposition 2.26. *If $\mathrm{span}(v_1, \cdots, v_n) = V$ and $n = \dim(V)$, then $\{v_1, \cdots, v_n\}$ is a basis.*

Theorem 2.27. *Let V be a finite dimensional vector space and V_1, V_2 subspaces. Then*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Corollary 2.28. $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

3. Linear Maps and Matrices

3.1. **Linear Maps.** Let V, W be vector spaces over $\mathbb{F}$.

Definition 3.1. A map $T : V \to W$ is linear if

(1) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
(2) $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.

Definition 3.2. We denote the set of all linear maps from $V \to W$ by $\mathrm{Hom}(V, W)$. And define $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

Lemma 3.3. *Let $v_1, \cdots, v_n$ be a basis for V and $w_1, \cdots, w_n$ a basis for W (i.e. V, W same dimension). Then there exists a unique linear map $T \in \mathrm{Hom}(V, W)$ such that $T(v_i) = w_i$ for all i. The map is given by $T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n$.*

Proposition 3.4. *The set $\mathrm{Hom}(V, W)$ is a vector space over $\mathbb{F}$, with addition and scalar multiplication given as follows.*

$$(\varphi + \psi)(v) := \varphi(v) + \psi(v)$$
$$(\lambda\varphi)(v) := \lambda\varphi(v)$$

Proposition 3.5. *Compositions of linear maps have the following properties.*

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$
$$(T_1 + T_2) T_3 = T_1 T_3 + T_1 T_2$$

Lemma 3.6. *Let $T \in \mathrm{Hom}(V, W)$, then $T(0_V) = 0_W$.*

3.2. **Fundamental Subspaces.** Let $T \in \mathrm{Hom}(V, W)$.

Definition 3.7. The *kernal* (or null space) of T is $\mathrm{Ker}(T) = \{v \in V : Tv = 0\}$

Proposition 3.8. $\mathrm{Ker}(T)$ *is a subspace.*

Proposition 3.9. $\mathrm{Ker}(T) = \{0\}$ *if and only if* T *is injective.*

Definition 3.10. The *image* (or range) of T is $\mathrm{Img}(T) =$