Math 206A Problem Sets

- Submit a genuine attempt for each assigned problem set. Perfect solutions are not required for full credit, but your work must be clearly typesetted.
- Printed submissions are preferred. Email submissions are allowed in special circumstances.
- Collaboration is encouraged. However, write up your solutions independently and acknowledge any collaborators.
- **AI policy:** You may use LLMs to find references or write codes. Do not use AI to directly solve or write solutions for this problem set.
- [Add.] Here's a more rigorous definition of a genuine attempt :$\geqslant$ perfect solutions for half of each problem set.
- [Add.] Numbers after each problem shows difficulty, $3+ > 3 > 3- > 2+ > \cdots$. Grades don't depend on difficulty.

1. Homework Assignment 1 (Due Oct 10 F)

Problem 1.1 (2+)**.** Recall that $\mathbb{Y} = \mathbb{Q}\text{-span}(\mathbf{Y})$, where $\mathbf{Y}$ is the Young's lattice, and the up/down operators U, D defined by $U(\lambda) = \sum_{\lambda \lessdot \mu} \mu$ and $D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$. Prove that

$$[D, U] = \text{id}$$

Problem 1.2 (2-)**.** Recall the Weyl algebra $\mathcal{W}$ is the $\mathbb{Z}$-algebra with unit 1 and generators U, D with relations $[D, U] = 1$. Rewrite $D^n U^n$ as an $\mathcal{W}$-element such that no D appears before an U. What is the identity coefficient?

For example, $D^2U^2 = D(UD+1)U = DUDU + DU = (UD+1)(UD+1) + (UD+1) = UDUD + 3UD + 2 = U(UD+1)D + 3UD + 2 = U^2D^2 + 4UD + 2$. The identity coefficient is 2.

Problem 1.3 (2)**.** Prove that $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$. Here λ' is the conjugate of λ.

Problem 1.4 (2)**.** Prove that

$$\sum_{k \geqslant 0} h_k t^k = \prod_{i \geqslant 0} \frac{1}{1 - x_i t}$$

Problem 1.5 (2+)**.** Is the power series $f = \prod_{i \geqslant 1}(1 + x_i + x_i^2)$ symmetric? If so, expand in the e-basis. [Hint: You can use (without proof) the change of basis matrix between m_λ and e_λ.]

Problem 1.6 (2)**.** Expand $h_3 e_4$ in the Schur basis.

Problem 1.7 (2+)**.** [Bonus] Expand $h_m e_n$ in the Schur basis. You may start experimenting with `SageMath` to make a conjecture.

2. Homework Assignment 2 (Due Oct 20 M)

Problem 2.1 (2+). Prove that if $\lambda, \mu \vdash n$ and $\lambda \trianglelefteq \mu$, then $K_{\lambda\mu} \neq 0$.

Problem 2.2 (2). Prove that $s_\lambda \cdot s_\square = \sum_{\lambda \lessdot \mu} s_\mu$. Here $s_\square = m_1 = x_1 + x_2 + \cdots$, and $\lessdot$ denotes the covering relation in Young's lattice.

Rmk: Can you define a $\mathcal{W}$-action on $\Lambda_{\mathbb{Q}}$? By P 2.2, we get an action of U on Λ via multiplication by $s_\square$. What should be the action of D on $\Lambda_{\mathbb{Q}}$ such that the map $\lambda \mapsto s_\lambda$ is an isomorphism from $\mathbb{Y}$ to $\Lambda_{\mathbb{Q}}$ as $\mathcal{W}$-modules?

Problem 2.3 (2+). Let $u_i : \mathbb{Y} \to \mathbb{Y}$ be the operator of adding a box to the i-th row when possible. Define

$$H_k = h_k(u_1, u_2, \cdots) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

The operators u_i satisfy the relations

$$\begin{aligned} u_i u_j &= u_j u_i \quad \text{if} |i-j| \geqslant 2 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_i \\ u_{i+1} u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ u_{i+1} u_{i+2} u_{i+1} u_i &= u_{i+1} u_{i+2} u_i u_{i+1} \end{aligned}$$

Classify the relations of $\{H_i | i \in \mathbb{N}\}$.

(Bonus) What about

$$E_k = e_k(u_1, u_2, \cdots) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

[Hint: How does H_k act on $\mathbb{Y}$? You may or may not need to use the relations of the u_i's.]

Problem 2.4 (3+). Define $H'_k = h_k(d_1, d_2, \cdots)$. What relations do $\{H_i, H'_j : i, j \in \mathbb{N}\}$ satisfy?

Problem 2.5 (3). Let $P(w)$ denote the row-insertion tableau, and $P'(w)$ denote the column-insertion tableau. Prove that $P(w) = P'(w_0 w)$, where $w_0 w = (w_n, w_{n-1}, \cdots, w_2, w_1)$.

3. Homework Assignment 3 (Due Oct 29 W)

Problem 3.1 (2). Define $A(x) = \cdots (1+xu_3)(1+xu_2)(1+xu_1)$ and $B(x) = (1+xd_1)(1+xd_2)(1+xd_3)\cdots$. Prove that $B(y)A(x) = A(x)B(y)(1-xy)^{-1}$. [Hint: first show that $(1+a)(1-ba)^{-1}(1+b) = (1+b)(1-ab)^{-1}(1+a)$ for any non-commutative a, b.]

Problem 3.2 (2+). Let P be a finite poset, and $a_k(P)$ the size of maximal k-antichain of P. Prove that

$$a_k(P) = a_1(P \times [k])$$

where $[k]$ is the chain of k elements.

Problem 3.3 (Bonus, 3+)**.** Let P be a finite poset. Let $C = (C_1, C_2, \cdots, C_l)$ be a partition of P into chains.

(i) Show that for any C, we have

$$a_k \leqslant \sum_i \min(|C_i|, k)$$

A chain-partition C is called *k-saturated* if the equality holds.

(ii) Prove that for any P there exists a k-saturated chain partition for all k.

[Hints: (1) The case $k = 1$ is the Dilworth theorem, which you can use without proving (2) use Problem 3.3]

Problem 3.4 (2+)**.** Denote $\mathrm{SYT}_{\lambda\backslash\mu}$ the set of all skew standard Young tableaux of shape $\lambda\backslash\mu$. Fix a $\mu \in \mathbf{Y}$ and define $D_\mu = \{(T_1, T_2) \in \mathrm{SYT}_{\lambda\backslash\mu} : |\lambda\backslash\mu| = n\}$. What is $\#D_\mu$? Note that the RS bijection implies that $\#D_\varnothing = n!$.

correction: You only need to find a lower bound for $\#D_\lambda$.

[Hint: use growth diagram.]

Problem 3.5 (Bonus, 5)**.** Recall that a permutation is an involution, i.e. $w = w^{-1}$, if and only if $P(w) = Q(w)$. The Bruhat order on S_n induces a partial order on involutions, thus a partial order on SYT's. Give a description of this partial order on SYT's.

4. Homework Assignment 4 (Due Nov 7 F)

Problem 4.1 (2)**.** Consider

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 5 & 0 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 & 4 & 4 \\ 5 & -3 & -4 \\ 5 & 5 & -4 \end{pmatrix}.$$

And let $E_A = \langle A_1 \rangle \subset \langle A_1, A_2 \rangle \subset \langle A_1, A_2, A_3 \rangle^3$ and let $E_B = \langle B_1 \rangle \subset \langle B_1, B_2 \rangle \subset \langle B_1, B_2, B_3 \rangle^3$. Compute the relative position of E_A, E_B.

Problem 4.2 (2+)**.** (i) Compute $\mathfrak{S}_{321654}$.

(ii) Which Schubert polynomials are monomials?

Problem 4.3 (3)**.** Let $\mathrm{GP}_{k,n}$ denote the set of all k-Grassmanian permutations in S_n. For $w \in \mathrm{GP}_{k,n}$, denote $\lambda(w)$ the partition associated to it. Prove that $\mathfrak{S}_w = s_{\lambda(w)}(x_1, \cdots, x_k)$.

Problem 4.4 (Bonus, 4)**.** Let $\tau : S_n \to S_{n+1}$ be the map that sends w to u where $u_1 = 1$ and $u_i = w_{i-1} + 1$. For any $w \in S_n$, define $F_w = \lim_{k\to\infty} \mathfrak{S}_{\tau^k(w)}$. Prove that F_w is symmetric

5. Homework Assignment 5 (Due Nov 17 M)

Problem 5.1 (5+)**.** Read Chapter 1 of Sagan's book

Problem 5.2 (1)**.** Let $w \in S_n$ and $w' \in S_{n+1}$ such that $w'(i) = w(i)$ for $i \in [n]$ and $w'(n+1) = n+1$. Show that $\mathfrak{S}_w = \mathfrak{S}_{w'}$.

Problem 5.3 (1)**.** Use Growth diagram to compute the inverse RSK of (T_1, T_2) where $T_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$

Problem 5.4 (2)**.** Can you construct a permutation representation for the dihedral group $D_{2n} = \{s^2 = r^n = 1, srs = r^{-1}\}$. Is it faithful?

Problem 5.5 (Bonus, 2)**.** Recall the action of the (right) divided difference operator on the vector space span$\langle S_n \rangle$:

$$\partial_i(w) = \begin{cases} ws_i & \text{if } ws_i < w \\ 0 & \text{otherwise} \end{cases}$$

We also define the left divided difference operator σ_i:

$$\sigma_i(w) = \begin{cases} s_i w & \text{if } s_i w < w \\ 0 & \text{otherwise} \end{cases}$$

Note that both of these operators satisfy the Nil-Coxeter relations.

Can you describe the action of these operators on (the vector space of) $\mathbf{Y}_{k\times(n-k)} \cong GP_{k,n} \subset S_n$. Here $\mathbf{Y}_{k\times(n-k)}$ is the set of all partitions that fit inside a $k \times (n-k)$ rectangle.

6. Homework Assignment 6 and 7 (Due Dec 5 F)

Submit any 4 of the following

Problem 6.1 (2+)**.** Find all representations of D_{2n} up to equivalence. (Hint: C_n is a normal subgroup of D_{2n}).

Problem 6.2 (2)**.** (1) For any $w \in S_n$, prove that $\chi^{(n-1,1)}(w) = \#\{\text{fixed points of } \pi\} - 1$.

(2) Compute the Specht module for $\lambda = (n-1, 1)$.

Problem 6.3 (2+)**.** Suppose we have groups $K \leqslant H \leqslant G$ and $\rho : K \to \mathrm{GL}(\mathbb{F}^n)$ a representation of K. Prove that

$$\operatorname{Ind}_K^G \rho = \operatorname{Ind}_H^G \operatorname{Ind}_K^H \rho$$

Problem 6.4 (2)**.** Write down the character table of S_5.

Problem 6.5 (3-)**.** What's the determinant of the character table of S_n?

Problem 6.6 (3)**.** The Durfee square of a partition is the largest square contained in λ. Show that if $\chi_\lambda(w) = 0$ whenever the side length of the Durfee square is larger than the number of cycles of w.