Math 206A Problem Sets

- Submit a genuine attempt for each assigned problem set. Perfect solutions are not required for full credit, but your work must be clearly typesetted.
- Printed submissions are preferred. Email submissions are allowed in special circumstances.
- Collaboration is encouraged. However, write up your solutions independently and acknowledge any collaborators.
- **AI policy:** You may use LLMs to find references or write codes. Do not use AI to directly solve or write solutions for this problem set.
- Add. Here's a more rigorous definition of a genuine attempt :$\geqslant$ perfect solutions for half of each problem set.

1. Homework Assignment 1 (Due Oct 10 F)

Problem 1.1 (2+)**.** Recall that $\mathbb{Y} = \mathbb{Q}$-span($\mathbf{Y}$), where $\mathbf{Y}$ is the Young's lattice, and the up/down operators U, D defined by $U(\lambda) = \sum_{\lambda \lessdot \mu} \mu$ and $D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$. Prove that

$$[D, U] = \mathrm{id}$$

Problem 1.2 (2-)**.** Recall the Weyl algebra $\mathcal{W}$ is the $\mathbb{Z}$-algebra with unit 1 and generators U, D with relations $[D, U] = 1$. Rewrite $D^n U^n$ as an $\mathcal{W}$-element such that no D appears before an U. What is the identity coefficient?

For example, $D^2U^2 = D(UD + 1)U = DUDU + DU = (UD + 1)(UD + 1) + (UD + 1) = UDUD + 3UD + 2 = U(UD + 1)D + 3UD + 2 = U^2D^2 + 4UD + 2$. The identity coefficient is 2.

Problem 1.3 (2)**.** Prove that $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$. Here λ' is the conjugate of λ.

Problem 1.4 (2)**.** Prove that

$$\sum_{k \geqslant 0} h_k t^k = \prod_{i \geqslant 0} \frac{1}{1 - x_i t}$$

Problem 1.5 (2+)**.** Is the power series $f = \prod_{i \geqslant 1}(1 + x_i + x_i^2)$ symmetric? If so, expand in the e-basis. [Hint: You can use (without proof) the change of basis matrix between m_λ and e_λ.]

Problem 1.6 (2)**.** Expand $h_3 e_4$ in the Schur basis.

Problem 1.7 (2+)**.** [Bonus] Expand $h_m e_n$ in the Schur basis. You may start experimenting with `SageMath` to make a conjecture.

2. Homework Assignment 2 (Due Oct 20 M)

Problem 2.1. changed Prove that if $\lambda, \mu \vdash n$ and $\lambda \trianglelefteq \mu$, then $K_{\lambda\mu} \neq 0$.

Problem 2.2.

Problem 2.3 (3+)**.** Let $u_i : \mathbb{Y} \to \mathbb{Y}$ be the operator of adding a box to the i-th row when possible. Define

$$H_k = h_k(u_1, u_2, \cdots) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

The operators u_i satisfy the relations

$$\begin{aligned} u_i u_j &= u_j u_i \quad \text{if} |i-j| \geqslant 2 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_i \\ u_{i+1} u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ u_{i+1} u_{i+2} u_{i+1} u_i &= u_{i+1} u_{i+2} u_i u_{i+1} \end{aligned}$$

Classify the relations of $\{H_i | i \in \mathbb{N}\}$.

[Hint: How does H_k act on $\mathbb{Y}$? You may or may not need to use the relations of the u_i's.]

Problem 2.4 (Bonus)**.** What about

$$E_k = e_k(u_1, u_2, \cdots) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

3. Homework Assignment 3 (Due Oct 29 W)

Problem 3.1.

Problem 3.2.

Problem 3.3.

Problem 3.4. Recall that a permutation is an involution, i.e. $w = w^{-1}$, if and only if $P(w) = Q(w)$. The Bruhat order on S_n induces a partial order on involutions, thus a partial order on SYT's. Give a description of this partial order on SYT's.

4. Homework Assignment 4 (Due Nov 7 F)

5. Homework Assignment 5 (Due Nov 17 M)

6. Homework Assignment 6 (Due Nov 26 W)

7. Homework Assignment 7 (Due Dec 8 F)