MATH 4242
Summer 2024
Time limit: 115 minites

Name: __________
Student ID: __________

Problem 1 (10 pts)**.** Below are 5 statements about $n \times n$ invertible matrices A and B over $\mathbb{R}$. Circle T for true and F for false statements. Provide a counter-example for the false statements, and a brief explanation for the correct statements (not necessarily a rigorous proof.)

1) T F $(AB)^{-1} = B^{-1}A^{-1}$

2) T F $(A + B)$ must be non-singular.

3) T F $\ker(A) = 0$.

4) T F The columns of A are linearly independent.

5) T F $(\lambda A)^{-1} = \lambda(A^{-1})$.

Problem 2. Below are 5 statements about finite dimensional vector spaces. Circle T for true and F for false statements. Provide a counter-example for the false statements, and an explanation for the correct statements.

(1) There exists subspaces 2-dimensional subspace   of $\mathbb{R}^4$.

(1) Compute the LU factorization of the following matrix. (Hint: Use Gaussian elimination)

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & 7 & 11 \\ 0 & -4 & 2 \end{bmatrix}$$

(2) Let $V = \mathbb{F}_{\leq m}[x]$ be the vector space of polynomials with degree at most m.
(a) Let $U = \{f \in V | f(1) = 0\}$. Is U a subspace of V?
(b) Let $W = \{f \in V | \deg(f) \text{ is even}\}$. If W a subspace of V?

(3) Let $V = M_{n\times n}(\mathbb{R})$ the vector space of all $n \times n$ matrices. Let B be the set of all upper triangular matrices. Is B a subspace of V?

(4) Suppose $v_1, \cdots, v_4$ are some vectors in $\mathbb{R}^4$, and A is the matrix whose columns are $v_1, \cdots, v_4$. Suppose the row echelon form of A is

$$\begin{bmatrix} 3 & 1 & 7 & 1 \\ 0 & -4 & 8 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Does $v_1, \cdots, v_4$ form a basis of $\mathbb{R}^4$?
(b) Is v_3 in $\operatorname{span}(v_1, v_2, v_4)$? If so, write v_3 as a linear combination of them.
(Hint: You may use: permuted LU factorization; some properties about matrix multiplication and invertible maps.)

(5) Let $V = \mathbb{R}^2$ with basis $v_1 = (1, 2)$ and $v_2 = (0, 1)$. Let T be the map $T(av_1 + bv_2) = bv_1 + av_2$ for all $a, b \in \mathbb{R}$.
(a) Show that T is linear;
(b) Find the matrix of T;
(c) Find a basis for $\ker(T)$. (Hint: use row echelon form of $\mathcal{M}(T)$.

(6) Let $V = \operatorname{End}(\mathbb{R}^2)$. Let W be the set of non-invertible linear maps in V. Show that W is not a vector space.
(Hint: give an example of two non-invertible maps that sums to an invertible map. You might try to use some theorems to do this in terms of matrices).

(7) Let $V = \mathbb{R}^2$ under the standard basis and $U = \{(x, 0) : x \in \mathbb{R}\}$ a subspace of V. Describe the quotient space V/U.
(Hint: What is the map whose kernel is U?)

(8) Is the following matrix positive definite?

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & 12 \end{pmatrix}$$

Does $\langle x, y\rangle = x^T Ay$ define an inner product on $\mathbb{R}^3$?

(9) Find the value for a, b such that the matrix A is orthogonal.

$$A = \frac{1}{3}\begin{pmatrix} 2 & -2 & b \\ 1 & 2 & 2 \\ a & 1 & 2 \end{pmatrix}$$