MATH 4242 Applied Linear Algebra

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1. Systems of Linear Equations

A $m \times n$ system of linear equation is of the form

$$a_{11}x_1 + \dots + a_{n1}x_n = b_1$$

$$a_{21}x_1 + \dots + a_{n2}x_n = b_n$$

$$\dots \dots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_n$$

Such equation can be represented using product of matrices.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or by an augmented matrix.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & b_1 \\ a_{21} & a_{22} & \cdots & a_{m2} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$

Definition 1.1. We have three types of elementary row operations.

- (1) Multiply the *i*-th equation (or the *i*-th row of the augmented matrix), then add it to the *j*-th equation (or the *j*-th row of the augmented matrix).
- (2) Permute the equations (or the rows of the augmented matrix)
- (3) Multiply one equation (or one row of the augmented matrix) by a non-zero number.
- 1.1. Systems of $n \times n$ Equations. Matrices considered in this sections are all $n \times n$.

Definition 1.2. A matrix is regular if it can be turned into a upper triangular matrix such that every entry on the diagonal is non-zero.

Proposition 1.3. Let E be the matrix with 1's on the diagonal and $E_{ij} = k \neq 0$ is the only other non-zero entry in the lower triangular part. Then for any matrix M, EM is the matrix obtained by multiplying the j-th row of M then adding to the i-th row of M.

Proposition 1.4. A matrix A is regular if and only if it has an LU factorization, i.e.

$$A = LU$$

where L is a lower uni-triangular matrix, and U is a upper triangular matrix with non-zero diagonal entries.

Definition 1.5. Let $w \in S_n$ be a permutation, then define $P_w = \{a_{ij}\}$ to be the matrix such that

$$a_{i,j} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.6. For any matrix M, P_wM is the matrix obtained by permuting the rows of M according to the permutation w.

Definition 1.7. A matrix A is called non-singular if it can be turned into a upper triangular matrix without non-zero diagonal entry via row operations of the first two types.

Proposition 1.8. A matrix A is non-singular if and only if it has a permuted LU factorization: PA = LU where P is some permutation matrix.

Definition 1.9. Let $A = (a_{ij})$, defined transpose of A to be $A^t := (a_{ji})$.

Proposition 1.10. Denote A^t the transpose of A. We have that $AB = (BA)^t$.

Proposition 1.11. A matrix A is regular iff it admits an LDV factorization, A = LDU where L is lower-unitriangular matrix, D is a diagonal matrix, and U is a uni-upper triangular matrix.

Definition 1.12. Let A be an $n \times n$ matrix. Suppose X is a matrix such that XA = AX = I where I is the identity matrix. Then X is called the inverse of A and denoted by A^{-1} . A matrix is called *invertible* if A^{-1} exists.

Proposition 1.13. A matrix is invertible if and only if it is non-singular.

Remark 1.14. Inverse of a matrix can be found using Gauss-Jordan Elimination — see chapter 1 of Olver-Shakiban.

1.2. Systems of $m \times n$ Equations.

Definition 1.15. A matrix is in row echelon form if it looks like,

$$\begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where \bullet 's are non-zero entries (called pivots) and * represent generic entries. The pivots are the first non-zero entries in each rows. We require the pivots occupy the first several rows consecutively.

Proposition 1.16. Every matrix can be turned into a row echelon form using elementary row operations of type I and II. In other words, every matrix A has a factorization PA = LU where P is a permutation matrix, L is a lower uni-triangular matrix, and U a matrix in row-echelon form.

Definition 1.17. Since every matrix can be turned in to row-echelon form using elementary row operations, we define its rank to be the number of pivots.

Proposition 1.18. A square $n \times n$ matrix is non-singular if its rank is n (full-rank).

2. Vector Spaces

2.1. Some Basic Setup.

Definition 2.1. 1 A field is a set \mathbb{F} with two binary operations \times (multiplication) and + (addition), satisfying the following axioms.

- a + b = b + a and $a \times b = b \times a$ for all $a, b \in \mathbb{F}$.
- There exists an additive identity 0 such that 0 + a = a + 0 = a for all $a \in \mathbb{F}$.
- There exists a multiplication identity 1 such that $1 \times a = a \times 1 = a$ for all $a \in \mathbb{F}$.
- For every $a \in \mathbb{F}$, there exists an element denoted -a, such that a + (-a) = 0.
- $0 \neq 1$.
- For every $a \in \mathbb{F}$ and $a \neq 0$, there exists an element denoted a^{-1} , such that $a \times (a^{-1}) = 1$
- For every $a, b, c \in \mathbb{F}$, $a \times (b+c) = ab + ac$.

For most part of this class, we will take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}.$

Definition 2.2. For a field \mathbb{F} , denote $\mathbb{F}[x]$ the ring² of polynomials over \mathbb{F} .

$$\mathbb{F}[x] = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n | a_0, \dots, a_n \in \mathbb{F}, n \geqslant 0, x^m x^n = x^{m+n}\}\$$

Proposition 2.3. Every polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ with complex coefficient has at least one complex solution. Note that this is not true for real polynomials.

Definition 2.4. A field \mathbb{F} is called algebraically closed if every polynomial in $\mathbb{F}[x]$ has a solution in \mathbb{F} . (By Proposition 2.3, \mathbb{C} is algebraically closed).

¹You don't need to worry too much about the abstract structures of a field. The purpose of this definition is to make everything self-contained. You can basically think of a field as a set on which you can do some sort of arithmetic.

 $^{^{2}}$ A ring is a field, where multiplication need not to be commutative, and multiplicative identity (0) need not exists.

Proposition 2.5. The field of complex numbers \mathbb{C} is the algebraic closure of \mathbb{R} . In other words, \mathbb{C} is the smallest algebraically closed field that contains \mathbb{R} .

2.2. Vector Spaces and Subspaces. Let \mathbb{F} be a field.

Definition 2.6. A set V is called a vector space over \mathbb{F} if there exists an addition map

$$add: V \times V \rightarrow V$$

and a scalar multiplication map

$$mult: \mathbb{F} \times V \to V$$

(Here \times denote the Cartesian product of sets³.) We will abbreviate them by $a(v_1, v_2) = v_1 + v_2$ and mult(a, v) = av.

Note that this definition (implicitly) requires that a vector space V is closed under addition and scalar multiplication, i.e. $v_1 + v_2 = add(v_1, v_2) \in V$ and $av = mult(a, v) \in V$.

Elements of a vector spaces are called *vectors*.

Definition 2.7. Let V be a vector space over \mathbb{F} . A subset U of V is a subspace if it is closed under addition and scalar multiplication. (In other words, a subspace is a subset that is a vector space itself.)

Definition 2.8. Let U_1, \dots, U_m be subspaces of V. Then define their sum to be

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m | u_1 \in U_1, \dots, u_m \in U_m\}$$

Proposition 2.9. Let U_1, \dots, U_m be subspaces of V. Then $U_1 + \dots + U_m$ is also a subspace of V, furthermore, it's the smallest subspace of V that contain all of U_1, \dots, U_m .

Definition 2.10. A sum of subspaces $U_1 + \cdots + U_m$ of V is a direct sum if every vector $v \in U_1 + \cdots + U_m$ can be uniquely written as $v = u_1 + \cdots + u_m$ where $u_i \in U_i$ for each i. When a summation is direct, we denote it as $U_1 \oplus \cdots \oplus U_m$.

2.3. Linear Combination, Span, and Dimension. Let V be a vector space over \mathbb{F} .

Definition 2.11. Let $v_1, v_2, \dots, v_n \in V$, a vector $v \in V$ is a linear combination of $\{v_1, \dots, v_n\}$ if there exists $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \dots + a_n v_n$$

Definition 2.12. Let v_1, v_2, \dots, v_n be a list of vectors in V, define their span to be the set of all linear combinations of v_1, \dots, v_n .

$$\operatorname{span}(v_1, \dots, v_n) = \{a_1v_1 + \dots + a_nv_n | a_1, \dots, a_n \in \mathbb{F}\}\$$

Proposition 2.13. For a list of vectors $v_1, \dots, v_n \in V$, $\operatorname{span}(v_1, \dots, v_n)$ is a subspace of V. Furthermore, it's the smallest subspace containing all of v_1, \dots, v_n .

Definition 2.14. A vector space V is said to be finite dimensional it it is the span of a finitely many vectors.

Definition 2.15. $v_1, \dots, v_m \in V$ are linearly independent if the only way to write 0 as a linear combination of v_1, \dots, v_n is

$$0 = 0v_1 + 0v_2 + \dots + 0v_n.$$

³For sets A and B, defined $A \times B = \{(a,b) | a \in A, b \in B\}$

Proposition 2.16. $v_1, \dots, v_m \in V$ are linearly independent if and only if any vector $v \in \text{span}(v_1, \dots, v_m)$ can be uniquely written as a linear combination of v_1, \dots, v_n .

Definition 2.17. A list of vectors v_1, \dots, v_n is a basis of V if

- $V = \operatorname{span}(v_1, \cdots, v_n)$
- v_1, \dots, v_n are linearly independent.

Proposition 2.18. v_1, \dots, v_n is a basis of V iff every vector $v \in V$ can be uniquely written as a linear combination of v_1, \dots, v_n .

Lemma 2.19. Let $v_1, \dots, v_m \in V$ be a list of vectors that spans V, i.e. $\operatorname{span}(v_1, \dots, v_m) = V$. Then $\{v_1, \dots, v_m\}$ can be reduced to a basis of V. In other words, there exists a basis $\{w_1, \dots, w_n\}$ of V such that $w_i \in \{v_1, \dots, v_m\}$ for all i and $n \leq m$.

Lemma 2.20. Let $v_1, \dots, v_k \in V$ be linearly independent. Then there exists a basis of V in the form

$$\{v_1,\cdots,v_k,w_1,\cdots,w_m\}$$

Note that it's possible that m = 0, in the case when $\{v_1 \cdots v_k\}$ is already a basis.

Corollary 2.21. If U is a subspace of V, then there exists another subspace W such that $V = U \oplus W$.

Proposition 2.22. If v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is another basis of V. Then n = m.

Definition 2.23. Define the dimension of a vector space to be the size of its basis.

Proposition 2.24. If $\{v_1, \dots, v_n\}$ linearly independent and $n = \dim(V)$, then $\{v_1, \dots, v_n\}$ is a basis.

Proposition 2.25. If U is a subspace of V, then $\dim(U) \leq \dim(V)$. Furthermore, $\dim(U) = \dim(V)$ iff U = V.

Proposition 2.26. If span $(v_1, \dots, v_n) = V$ and $n = \dim(V)$, then $\{v_1, \dots, v_n\}$ is a basis.

Theorem 2.27. Let V be a finite dimensional vector space and V_1, V_2 subspaces. Then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Corollary 2.28. If $V_1 + V_2$ is a direct sum, then $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

3. Linear Maps and Matrices

3.1. Linear Maps. Let V, W be vector spaces over \mathbb{F} .

Definition 3.1. A map $T: V \to W$ is linear if

- (1) T(u+v) = T(u) + T(v) for all $u, v \in V$.
- (2) $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.

Definition 3.2. We denote the set of all linear maps from $V \to W$ by Hom(V, W). And define End(V) = Hom(V, V).

⁴We will see later that the converse is also true.

Lemma 3.3. Let v_1, \dots, v_n be a basis for V and w_1, \dots, w_n a basis for W (i.e. V, W same dimension). Then there exists a unique linear map $T \in \text{Hom}(V, W)$ such that $T(v_i) = w_i$ for all i. The map is given by $T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$.

Proposition 3.4. The set Hom(V, W) is a vector space over \mathbb{F} , with addition and scalar multiplication given as follows.

$$(\varphi + \psi)(v) := \varphi(v) + \psi(v)$$
$$(\lambda \varphi)(v) := \lambda \varphi(v)$$

Lemma 3.5. Let $T \in \text{Hom}(V, W)$, then $T(0_V) = 0_W$.

Let $T \in \text{Hom}(V, W)$.

Definition 3.6. The kernal (or null space) of T is $Ker(T) = \{v \in V : Tv = 0\}$

Proposition 3.7. Ker(T) is a subspace of V.

Proposition 3.8. Ker $(T) = \{0\}$ if and only if T is injective.

Definition 3.9. The image (or range) of T is $Img(T) = \{Tv | v \in V\}$

Proposition 3.10. $\operatorname{Img}(T)$ is a subspace of W.

Proposition 3.11. T is surjective iff Img(T) = W.

Theorem 3.12. $\dim(V) = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Img}(T)).$

Proposition 3.13. (1) if $\dim(V) > \dim(W)$, then any $T \in \operatorname{Hom}(V, W)$ is not injective.

- (2) if $\dim(V) < \dim(W)$, then any $T \in \operatorname{Hom}(V, W)$ is not surjective.
- (3) if there exists a bijective $T \in \text{Hom}(V, W)$, then $\dim(V) = \dim(W)$.

3.2. Matrices from Linear Maps. Denote the set of all $m \times n$ matrix with entries in \mathbb{F} by $M_{m \times n}(\mathbb{F})$. Let V, W be finite dimensional vector spaces over \mathbb{F} .

Definition 3.14. Suppose V has basis v_1, \dots, v_n and W has basis w_1, \dots, w_m . Let $T \in \text{Hom}(V, W)$. Then define $\mathcal{M}(T)$ to be the matrix $[a_{ij}]$ such that

$$T(v_k) = a_{1k}w_1 + a_{2k}w_2 + \dots + a_{m,k}w_m.$$

Remark 3.15. Note that the usage of \mathcal{M} requires a choice of basis for V and W. In general we shall denote $\mathcal{M}_{B_1,B_2}(T)$ where B_1 is the basis for V and B_2 the basis for W. However in most case we will omit the subscript when the context is clear.

Proposition 3.16. Let $S, T \in \text{Hom}(V, W)$, $\mathcal{M}(S) + \mathcal{M}(T) = \mathcal{M}(S + T)$ Let $S \in \text{Hom}(U, W)$ and $T \in \text{Hom}(V, U)$, then $\mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$.

Definition 3.17. For any $A \in M_{m \times n}$, Let $A_{\bullet,k}$ denote the k-th column vector and $A_{k,\bullet}$ denote the k-th row vector.

Proposition 3.18. $(AB)_{\bullet,k} = A(B_{\bullet,k})$

Theorem 3.19. For any $A \in M_{m \times n}$, we have

$$\dim(\operatorname{span}(A_{1,\bullet},\cdots,A_{m,\bullet})) = \dim(\operatorname{span}(A_{\bullet,1},\cdots,A_{\bullet,n}))$$

Proposition 3.20. The dimension of column span or row span of a matrix equals to its rank (see Definition 1.17).

Definition 3.21. A linear map $T \in \text{Hom}(V, W)$ is invertible if there exists an linear map $S \in \text{Hom}(W, V)$ such that $TS = \text{id}_W$ and $ST = \text{id}_V$.

Proposition 3.22. If a map T is invertible, then its inverse is unique, denote it by T^{-1} .

Proposition 3.23. A map T is invertible iff $\mathcal{M}(T)$ is non-singular Definition 1.7. Furthermore, $\mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

Definition 3.24. A linear map is an *isomorphism* if it's invertible. Two vectors spaces are *isomorphic* if there exists an isomorphism between them.

Theorem 3.25. Two vector spaces over \mathbb{F} is isomorphic if and only if they have the same dimension. (In other words, vector spaces are classified by \mathbb{N})

Corollary 3.26. Let dim V = n and dim W = m. The vector space $\text{Hom}(V, W) \cong M_{m \times n}(\mathbb{F})$ are isomorphic, with the map \mathcal{M} being the isomorphism.

Theorem 3.27. Let V be a vector space with basis $B_1 = \{v_1, \dots, v_n\}$. Suppose it has another basis $B_2 = \{w_1, \dots, w_n\}$. Let $C = \mathcal{M}_{B_1, B_2}(\mathrm{id})$ where $\mathrm{id} \in \mathrm{Hom}(V, V)$ is the identity map. Then change of basis corresponds to conjugation by C.

In particular, let $T \in \text{Hom}(V, V)$ and $A = \mathcal{M}_{B_1, B_1}(T)$ and $B = \mathcal{M}_{B_2, B_2}(T)$. Then we have

$$A = C^{-1}BC$$

3.3. Quotient and Dual spaces.

Definition 3.28. Let $v \in V$ and $U \subseteq V$. Define $v + U := \{v + u | u \in U\}$. This is called a coset

Definition 3.29. Let $U \subseteq V$. Define the quotient space V/U to be $\{v + U | v \in V\}$, with addition and scalar multiplication given by

$$(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$$

 $\lambda(v + U) = \lambda v + U$

Definition 3.30 (alternative definition). Let \sim be an equivalence relation on V. Define $[v]_{\sim} := \{u \in V | u \sim v\}$ the equivalence class generated by v. Then we can define quotient space $V/\sim:=\{[v]_{\sim}|v\in V\}$.

Remark 3.31. For $U \subset V$, define an equivalence relation \sim_U by $v \sim_U u \iff v - u \in U$. Then Definitions 3.29 and 3.30 agree, i.e. $V/U = V/\sim_U$.

Definition 3.32. For $U \subset V$, define the quotient map $\pi: V \to V/U$ by $\pi(v) = v + U$. Note that $\operatorname{Ker}(\pi) = U$.

Proposition 3.33. $\dim V/U = \dim V - \dim U$.

Theorem 3.34. For any $T \in \operatorname{Hom}(V,W)$, define $\tilde{T} \in \operatorname{Hom}(V/\operatorname{Ker}(T),W)$ by $\tilde{T}(v + \operatorname{Ker}(T)) = Tv$. Then $\tilde{T}\pi = T$, and defines an isomorphism between $V/\operatorname{Ker}(T)$ and $\operatorname{Img}(T)$.

Definition 3.35. A linear map from V to \mathbb{F} is called a *linear functional*. Denote $\operatorname{Hom}(V,\mathbb{F})$ the set of all linear functionals on V.

Proposition 3.36. Hom (V, \mathbb{F}) is a vector space, with addition and multiplication given by (f+g)(v) = f(v) + g(v) and $(\lambda f)(v) = \lambda f(v)$. This is called the dual space of V, and is denoted by T^* .

Proposition 3.37. $\dim V = \dim V^*$.

Definition 3.38. Let v_1, \dots, v_n be a basis of V. Then define $v_i^* \in V^*$ to be the linear functional $v_i^*(v_j) = \delta_{i,j}^{5}$. For any $v = a_1v_1 + \dots + a_nv_n \in V$, define $v^* = a_1v_1^* + \dots + a_nv_n^*$.

Proposition 3.39. Let v_1, \dots, v_n be a basis. Then $v = v_1^*(v)v_1 + \dots + v_n^*(v)v_n$ for all $v \in V$.

Proposition 3.40. v_1^*, \dots, v_n^* is a basis for V^* .

Definition 3.41. Suppose $T \in \text{Hom}(V, W)$. Define the dual linear map $T^* \in \text{Hom}(W^*, V^*)$ to be

$$T^*(f) = f \circ T$$

Proposition 3.42. • $(S+T)^* = S^* + T^*$

- $(\lambda T)^* = \lambda T^*$.
- $(ST)^* = T^*S^*$.

Definition 3.43. For any subspace $U \subseteq V$, define its annihilator $U^0 := \{ f \in V^* : f(u) = 0 \text{ for all } u \in U \}$.

Proposition 3.44. U^0 is a subspace of V^* .

Proposition 3.45. dim $U^0 = \dim V - \dim U$. Recall that this is also the dimension of V/U. In particular, there is an isomorphism $(V/U)^* \cong U^0$ given by π^* .

Proposition 3.46. (a) $U^0 = \{0\} \iff U = V$ (b) $U^0 = V^* \iff U = \{0\}.$

Theorem 3.47. (a) $(\operatorname{Img} T)^0 = \operatorname{Ker} T^*$ (b) $(\operatorname{Ker} T)^0 = \operatorname{Img} T^*$

Corollary 3.48. T is injective iff T^* is surjective. T is surjective iff T^* is injective.

Theorem 3.49. Let $T \in \text{Hom}(V, W)$, and $T^* \in \text{Hom}(W^*, V^*)$. Then $\mathcal{M}(T)^t = \mathcal{M}(T^*)$.

4. Inner Product Spaces

Throughout this section, let $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

4.1. Inner Products and Norms.

Definition 4.1. The dot product of two vectors in \mathbb{R}^n is a map from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{F}$, defined by

$$(x_1, \cdots, x_n) \cdot (y_1, \cdots, y_n) = x_1 y_1 + \cdots + x_n y_n$$

Definition 4.2. The dot product of two vectors in \mathbb{C}^n is a map from $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{F}$, defined by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 \overline{y}_1 + \dots + x_n \overline{y}_n$$

where $\overline{a+bi}=a-bi$ is the complex conjugate.

Definition 4.3. Let V be vector space over \mathbb{F} (\mathbb{C} or \mathbb{R}). A inner product on V is a map $V \times V \to \mathbb{F}$ which sends (v, u) to $\langle v, u \rangle$ such that

- (1) $\langle v, v \rangle \geqslant 0$.
- (2) $\langle v, v \rangle = 0 \iff v = 0.$
- (3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- (4) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$.

⁵Here $\delta_{i,j}$ is the Kronecker delta symbol: $\delta_{i,j} = 1$ if i = j and 0 otherwise.

(5)
$$\langle u, v \rangle = \overline{\langle v, u \rangle^6}$$

Proposition 4.4 (Bilinearity). A inner product $\langle \ , \ \rangle$ satisfy $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ and $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$.

Remark 4.5. A pairing satisfying (4) and (6) of Definition 4.3 and Proposition 4.4 together is known as being *bilinear*. Usually a inner product is defined to be bilinear, however, as we see here one-sided linearity is enough to imply bilinearity.

Proposition 4.6. An inner product \langle , \rangle on V satisfy

- (1) Fix any $u \in V$, the map $v \mapsto \langle u, v \rangle$ is a linear functional.
- (2) $\langle v, 0 \rangle = 0 = \langle 0, v \rangle$ for any $v \in V$.

Definition 4.7. Given an inner product \langle , \rangle , define the norm $\| \|$ to be the positive square-root $\|v\| = \sqrt{\langle v, v \rangle}$.

Proposition 4.8. Let $I = [a, b] \subset \mathbb{R}$ be an closed interval on \mathbb{R} . Let $V = C^0(I)$ denote all continuous \mathbb{R} -valued functions defined on I (domain is I). Then

$$\langle f, g \rangle = \int_{a}^{b} f(x)g(x) \ dx$$

defines an inner product on V. The norm $||f|| = \sqrt{\langle f, g \rangle}$ is called the L_2 norm on $C^0(I)$.

Definition 4.9. For $z \in \mathbb{C}$, define the complex modulus to be $|z| = \sqrt{z\overline{z}}$. Note that when z is real (i.e. no imaginary part), then |z| is the absolute value.

Theorem 4.10 (Cauchy-Schwartz inequality). Let V be an inner product space with inner product \langle , \rangle and norm $\| \|$. Then for any $u, v \in V$, we have

$$|\langle u, v \rangle| \leqslant ||u|| ||v||$$

Moreover, the equality occurs only when u, v are linearly independent.

Remark 4.11. The Cauchy-Schwartz inequality tells us that the ratio $\frac{|\langle u,v\rangle|}{\|u\|\|v\|}$ is in between -1 and 1. Therefore we can define the 'abstract' angle between two vectors v, u to be

$$\theta_{u,v} = \arccos \frac{|\langle u, v \rangle|}{\|u\| \|v\|}$$

Definition 4.12. We say two vectors u, v are orthogonal if $\langle u, v \rangle = 0$.

Proposition 4.13. If v, u orthogonal in V, then $||v||^2 + ||u||^2 = ||v + u||^2$.

Remark 4.14. Definition 4.12 generalizes the usual notion of orthogonality in \mathbb{R}^2 in a sense that when two vectors are orthogonal, then the angle between then is $\theta_{u,v} = \pi/2$.

Theorem 4.15. Let V be an inner product space, and $u, v \in V$. Then $||u+v|| \leq ||u|| + ||v||$.

Definition 4.16. We can define norms more generally without requiring an inner product. A norm on V is a map $\|\cdot\|: V \to \mathbb{R}_{\geq 0}$ such that

- $||v|| \ge 0$ and ||v|| = 0 only when v = 0.
- $\bullet \|\lambda v\| = |\lambda| \|v\|$
- $||v + u|| \le ||v|| + ||u||$

⁶When $\mathbb{F} = \mathbb{R}$, the 'complex' conjugate of a real number is just itself.

Proposition 4.17. Let $K = (k_{ij})$ be the matrix whose entries are the inner product of the basis vectors, i.e. $k_{ij} = \langle e_i, e_j \rangle$. Then for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we have

$$\langle x, y \rangle = \left\langle \sum_{i} x_{i} e_{i}, \sum_{j} y_{j} e_{j} \right\rangle = \sum_{i,j} x_{i} y_{j} \langle e_{i}, e_{j} \rangle = x^{t} K y$$

Definition 4.18. A $n \times n$ matrix is positive-definite if $K^t = K$ and satisfy $x^t K x > 0$ for all $0 \neq x \in \mathbb{F}^n$. More generally, we say K is positive semi-definite if $K^t = K$ and $x^t K x \ge 0$ for all x.

Theorem 4.19. Every inner product is given by $\langle x,y\rangle=x^tKy$ where K is a positive-definite matrix.

Proposition 4.20. Positive-definite matrices are non-singular (invertible).

Definition 4.21. Given any $v_1, \dots, v_n \in V$, we define the Gram matrix to be $K = (k_{ij})$ where $k_{ij} = \langle v_i, v_j \rangle$. In particular, Let A be the matrix whose column vectors are v_1, \dots, v_n , then the Gram matrix is $K = A^t C A$, where C is the symmetric positive definite matrix defining the inner product.

Proposition 4.22. A Gram matrix is always positive semi-definite. A Gram matrix is positive-definite if and only if v_1, \dots, v_n are linearly independent.

Proposition 4.23. Let A be an $m \times n$ matrix $(m \ge n)$, then TFAE:

- $K = A^t A$ is positive-definite;
- $\operatorname{Ker}(A) = 0$;
- A has linearly independent columns;
- $\operatorname{rank}(A) = n$.

Theorem 4.24. Suppose $A \in M_{m \times n}$ and $K = A^T A$ is positive-definite. Then for any symmetric positive definite matrix $C \in M_{m \times m}$, the matrix $K' = A^t CA$ is also positivedefinite.

Proposition 4.25. For $K = A^tCA$, we have Ker(K) = Ker(A), and hence rank(K) =rank(A).

4.2. Orthonormal Basis.

Definition 4.26. Let V be a real or complex inner product space. A basis v_1, \dots, v_n of V is called orthogonal if $\langle v_i, v_j \rangle = \delta_{i,j}$ for all i. An orthogonal basis of unit vectors is called orthonormal.

Proposition 4.27. Let $v_1, \dots, v_n \in V$ be pair-wise orthogonal, then they must be linearly independent.

Corollary 4.28. Let $v_1, \dots, v_n \in V$ be pair-wise orthogonal, then they form a basis for $\operatorname{span}(v_1,\cdots,v_n)$.

Theorem 4.29. Given any basis w_1, \dots, w_n of V, one can construct an orthogonal basis v_1, \cdots, v_n using the Gram-Schmidt process:

•
$$v_1 = w_1$$
.

•
$$v_1 = w_1$$
.
• $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$

•
$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

• \cdots
• $v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i$

Corollary 4.30. Every finite dimensional inner-product space has an orthonormal basis.

Definition 4.31. A matrix A is orthogonal if $A^tA = I = AA^t$, or equivalently $A^t = A^{-1}$.

Proposition 4.32. A matrix is orthogonal if and only if its column vectors form an orthonormal basis of \mathbb{F}^n w.r.t the dot product.