Math 4242 Homework 3

(1) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}_{\leq 2}[x]$. Let $T(a, b, c) = a + b(x-1) + c(x-1)^2$. Is T linear? If so, identify a basis for V and W and write down the matrix $\mathcal{M}(T)$.

(2) Consider the linear map $T : M_{2,2}(\mathbb{R}) \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - b, c + d)$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Img}(T)$.

(3) Suppose $T \in \mathrm{End}(V)$ is an invertible map. Prove that if $v_1, \cdots, v_n$ is a basis, then $Tv_1, \cdots, Tv_n$ is also a basis.

(4) Prove that (a) $(U + W)^0 = U^0 \cap W^0$ (b) $(U \cap W)^0 = U^0 + W^0$.

(5) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x + 3y + 4x, 3x + 4y + 5z)$. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$ and f_1, f_2 denote the standard basis of $\mathbb{R}^2$. (a) Describe the linear functionals $T^*(f_1^*)$ and $T^*(f_2^*)$. (b) Write $T^*(f_1^*)$ and $T^*(f_2^*)$ as linear combinations of e_1^*, e_2^*, e_3^*.

(6) Suppose U is a subspace of V, and $\pi : V \to V/U$ the quotient map. Consider the dual of the quotient map $\pi^* \in \mathrm{Hom}((V/U)^*, V^*)$. Show that $\mathrm{Img}(\pi^*) = U^0$ and π^* is an isomorphism $(U/V)^* \cong U^0$.

(7) OS 3.1.9

(8) OS 3.1.17