MATH 4242 Applied Linear Algebra

Sylvester W. Zhang[♮]

Summer 2024

Contents

1. Systems of Linear Equations

A $m \times n$ system of linear equation is of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{n1}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{n2}x_n &= b_n \\ \cdots \quad \cdots & \quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

Such equation can be represented using product of matrices.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or by an augmented matrix.

$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{m1} & b_1 \\ a_{21} & a_{22} & \cdots & a_{m2} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Definition 1.1. We have three types of elementary row operations.

(1) Multiply the i-th equation (or the i-th row of the augmented matrix), then add it to the j-th equation (or the j-th row of the augmented matrix).
(2) Permute the equations (or the rows of the augmented matrix)

[♮] swzhang@umn.edu University of Minnesota.

(3) Multiply one equation (or one row of the augmented matrix) by a non-zero number.

1.1. **Systems of $n \times n$ Equations.**

Matrices considered in this sections are all $n \times n$.

Definition 1.2. A matrix is *regular* if it can be turned into a upper triangular matrix such that every entry on the diagonal is non-zero.

Proposition 1.3. *Let E be the matrix with 1's on the diagonal and $E_{ij} = k \neq 0$ is the only other non-zero entry in the lower triangular part. Then for any matrix M, EM is the matrix obtained by multiplying the j-th row of M then adding to the i-th row of M.*

Proposition 1.4. *A matrix A is regular if and only if it has an LU factorization, i.e.*

$$A = LU$$

where L is a lower uni-triangular matrix, and U is a upper triangular matrix with non-zero diagonal entries.

Definition 1.5. Let $w \in S_n$ be a permutation, then define $P_w = \{a_{ij}\}$ to be the matrix such that

$$a_{i,j} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.6. *For any matrix M, $P_w M$ is the matrix obtained by permuting the rows of M according to the permutation w.*

Definition 1.7. A matrix A is called *non-singular* if it can be turned into a upper triangular matrix without non-zero diagonal entry via row operations of the first two types.

Proposition 1.8. *A matrix A is non-singular if and only if it has a permuted LU factorization: $PA = LU$ where P is some permutation matrix.*

Proposition 1.9. *Denote A^T the transpose of A. We have that $AB = (BA)^T$.*

Proposition 1.10. *A matrix A is regular if it admits an LDV factorization, $A = LDU$ where L is lower-unitriangular matrix, D is a diagonal matrix, and U is a uni-upper triangular matrix.*

1.2. **Systems of $m \times n$ Equations.**

Definition 1.11. A matrix is in row echelon form if it looks like,

$$\begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\bullet$'s are non-zero entries (called *pivots*) and $*$ represent generic entries. The pivots are the first non-zero entries in each rows. We require the pivots occupy the first several rows consecutively.

Proposition 1.12. *Every matrix can be turned into a row echelon form using elementary row operations of type I and II. In other words, every matrix A has a factorization $PA = LU$ where P is a permutation matrix, L is a lower uni-triangular matrix, and U a matrix in row-echelon form.*

Definition 1.13. Since every matrix can be turned in to row-echelon form using elementary row operations, we define its *rank* to be the number of pivots.

Proposition 1.14. *A square $n \times n$ matrix is non-singular if its rank is n (full-rank).*

2. Vector Spaces

2.1. **Some basic setup.**

Definition 2.1. [1] A field is a set $\mathbb{F}$ with two binary operations $\times$ (multiplication) and $+$ (addition), satisfying the following axioms.

- $a + b = b + a$ and $a \times b = b \times a$ for all $a, b \in \mathbb{F}$.
- There exists an additive identity 0 such that $0 + a = a + 0 = a$ for all $a \in \mathbb{F}$.
- There exists a multiplication identity 1 such that $1 \times a = a \times 1 = a$ for all $a \in \mathbb{F}$.
- For every $a \in \mathbb{F}$, there exists an element denoted $-a$, such that $a + (-a) = 0$.
- $0 \neq 1$.
- For every $a \in \mathbb{F}$ and $a \neq 0$, there exists an element denoted a^{-1}, such that $a \times (a^{-1}) = 1$.
- For every $a, b, c \in \mathbb{F}$, $a \times (b + c) = ab + ac$.

For most part of this class, we will take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}$.

Definition 2.2. For a field $\mathbb{F}$, denote $\mathbb{F}[x]$ the ring[2] of polynomials over $\mathbb{F}$.

$$\mathbb{F}[x] = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n | a_0, \cdots, a_n \in \mathbb{F}, n \geqslant 0, x^m x^n = x^{m+n}\}$$

Proposition 2.3. *Every polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ with complex coefficient has at least one complex solution. Note that this is not true for real polynomials.*

Definition 2.4. A field $\mathbb{F}$ is called *algebraically closed* if every polynomial in $\mathbb{F}[x]$ has a solution in $\mathbb{F}$. (By Proposition 2.3, $\mathbb{C}$ is algebraically closed).

Proposition 2.5. *The field of complex numbers $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$. In other words, $\mathbb{C}$ is the smallest algebraically closed field that contains $\mathbb{R}$.*

2.2. **Vector spaces.** Let $\mathbb{F}$ be a field.

Definition 2.6. A set V is called a vector space over $\mathbb{F}$ if there exists an commutative addition map

$$a : V \times V \to V$$

and a scalar multiplication map

$$m : \mathbb{F} \times V \to V$$

(Here $\times$ denote the Cartesian product of sets[3].)

3. Linear Maps and Matrices

[1] You don't need to worry too much about the abstract structures of a field. The purpose of this definition is to make everything self-contained. You can basically think of a field as a set on which you can do some sort of arithmetic.

[2] A ring is a field, where multiplication need not to be commutative, and multiplicative identity (0) need not exists.

[3] For sets A and B, defined $A \times B = \{(a, b) | a \in A, b \in B\}$