Tableaux Combinatorics and Symmetric Functions

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ABSTRACT. These are lecture notes for Math 206a (Algebraic combinatorics) at UCLA, Fall 2025.

CONTENTS

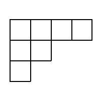
Let R be a commutative ring, usually taken to be $\mathbb{Q}$ or $\mathbb{Z}$.

Denote $\mathbf{x}_\mathrm{n} = (x_1, \cdots, x_n)$, and $\mathbf{x} = (x_1, x_2, \cdots)$

1. Partitions

An integer partition of $n \in \mathbb{N}$ is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ such that $\sum_i \lambda_i = n$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. We denote $\lambda \vdash n$, and $\ell(\lambda) = k$. For convenience, we set $\lambda_i = 0$ for $i > \ell(\lambda)$.

We will represent an integer partition using a *Young diagram*, which is a stack of boxes so that the i-th row has λ_i boxes. For example, the following is the Young diagram corresponds to $(4, 3, 1)$.



Denote $\mathbf{Y}_n$ the set of all partitions of n, and denote $\mathbf{Y} = \bigoplus_{i \in \mathbb{N}} \mathbf{Y}_i$.

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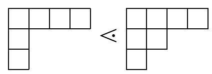
There are two natural partial orders equipped with $\mathbf{Y}$. The first one, called *Young's natural order*, is defined so that $\lambda \leqslant \mu$ when $\lambda_i \leqslant \mu_i$ for all i. In other words, the Young diagram of μ contains that of λ. In Young's order, λ is covered by μ precisely when μ has exactly one more box than λ. For example, $(4, 2, 1) \lessdot (4, 3, 1)$.

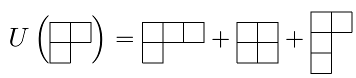


This partial order turns out to be a distributive lattices, hence called the *Young's lattice*.

Let $\mathbb{Y} = \mathbb{Q}\text{-span}(\mathbf{Y})$ be the vector space over $\mathbb{Q}$ whose basis are Young diagrams. Define two natural operators on $\mathbb{Y}$ as follows:

$$U(\lambda) = \sum_{\lambda \lessdot \mu} \mu \qquad D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$$

(The actions are defined on the basis then extend by linearity.) For example,



These operators satisfy the following identity.

Proposition 1.1. $DU - UD = \mathbf{1}$.

Proof. Exercise. Hint: given any partition λ, notice that there's always one more outer corner than inner corner. □

The algebra of the up/down operators is the *Weyl algebra*, denoted $\mathcal{W}$, which is the algebra of multiplication and differentiation on $k[x]$ since $[\frac{d}{dx}, x] = 1$.

Remark 1.2. For any poset one can define the operators U and D, but not all satisfy the relation of the Weyl algebra. In [Sta88], Stanley defined *differential posets* to be those that affords a combinatorial representation of $\mathcal{W}$, with $\mathbf{Y}$ being the canonical example.

Proposition 1.3. *Consider $D^n U^n$ as an element of $\mathcal{W}$, one can rewrite so that all U's appear before D's. Then the identity coefficient is $n!$.*

Proof. Exercise. □

Example 1.4. $D^2U^2 = D(UD + 1)U = DUDU + DU = (UD + 1)(UD + 1) + (UD + 1) = UDUD + 3UD + 2 = U(UD + 1)D + 3UD + 2 = U^2D^2 + 4UD + 2$.

The second partial order that we will introduce, is a partial order defined on $\mathbf{Y}_n$ (although extendable to $\mathbf{Y}$, it is more natural to consider as an order on $\mathbf{Y}_n$ for each n).

Definition 1.5. $\lambda \trianglelefteq \mu$ if and only if $\lambda_1 + \cdots + \lambda_k \leqslant \mu_1 + \cdots + \mu_k$ for all k[1]. This is called the *dominance order*.

[1]Recall that $\lambda_i = 0$ if $i > \ell(\lambda)$.

2. Symmetric polynomials

Let R be a commutative ring, usually taken to be $\mathbb{Q}$ or $\mathbb{Z}$. We say a polynomial $f \in R[x_1, \cdots, x_n]$ is *symmetric* to be all polynomials that are invariant under S_n, i.e.

$$f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

for all $\sigma \in S_n$.

Definition 2.1. Define $\Lambda_{R,n} := R[\mathbf{x}_\mathrm{n}]^{S_n} \subset R[\mathbf{x}_\mathrm{n}]$ the subring of all symmetric polynomials.

Given any monomial $\prod x_i^{\alpha_i}$, one can find the smallest symmetric polynomial that contains it. This is exactly the sum of all possible rearrangements of $\alpha = \{\alpha_1, \cdots, \alpha_l\}$. All symmetric polynomials that arise in this way are called *monomial symmetric polynomials*, defined as follows.

Definition 2.2. For $\lambda \in \mathbf{Y}$ such that $\ell(\lambda) \leqslant n$. Define *monomial symmetric polynomial* m_λ to be

$$m_\lambda = \sum_{\alpha \text{ rearrangement of } \lambda} {\mathbf{x}_\mathrm{n}}^\alpha$$

It is clear that these form a basis of $\Lambda_{R,n}$. At the same time, there's another natural (but less obvious) basis for the ring symmetric polynomials — the *elementary symmetric polynomials.*

Definition 2.3 (elementary symmetric polynomials)**.** For $k \in \mathbb{N}$, define

$$e_k(\mathbf{x}_\mathrm{n}) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

And for $\lambda \in \mathbf{Y}$, define $e_\lambda(\mathbf{x}_\mathrm{n}) = e_{\lambda_1}(\mathbf{x}_\mathrm{n}) \cdots e_{\lambda_l}(\mathbf{x}_\mathrm{n})$

That fact that e_λ's also generate $\Lambda_{R,n}$, is called the *Fundamental theorem of symmetric polynomials.*

Theorem 2.4 (Fundamental Theorem of Symmetric Polynomials)**.** *There is an isomorphism between the ring of symmetric polynomials $\Lambda_{R,n}$ and the polynomials ring of n variables $R[t_1, \cdots, t_n]$, via the map $e_n \mapsto t_n$.*

Proof. □

Now consider the ring of symmetric polynomials of differently many variables. There is an obvious projection map from $\Lambda_{R,j}$ to $\Lambda_{R,i}$ when $i \leqslant j$:

$$\rho_{i,j} : \Lambda_{R,j} \to \Lambda_{R,i} \quad f(x_1, \cdots, x_i, x_{i+1} \cdots, x_j) \mapsto f(x_1, \cdots, x_i, 0, \cdots, 0).$$

With the help of the fundamental theorem (Theorem 2.4), one can write down the maps $\phi_{i,j}$ of which $\rho_{i,j}$ are inverse to. For $i \leqslant j$, we define

$$\phi_{i,j} : \Lambda_{R,i} \to \Lambda_{R,j} \quad e_k(\mathbf{x}_n) \to e_k(\mathbf{x}_m)$$

It is easy to check that:

Lemma 2.5.
- $\phi_{i,i} = \mathbf{1}$ *for all* i.
- $\phi_{i,k} = f_{j,k} f_{i,j}$ *for all* $i \leqslant j \leqslant k$.

3. Symmetric Functions

4. Robinson-Schensted Correspondence

5. Greene-Kleitman Theory

References

[Sta88] Richard P Stanley, *Differential posets*, Journal of the American Mathematical Society **1** (1988), no. 4, 919–961. 2