MATH 4242
Summer 2024
Exam 2

Name: ______________
Student ID: ______________

- Exam 2 contains 7 problems. Please check to see if any page is missing. There are 9 pages in total.
- Time limit: July 25 10:10 am — 12:05 pm. (115 min)
- Work individually without reference to a textbook, notes, the internet, or a calculator.
- The lecture notes available from the course website is allowed. This is the only resource that is allowed during the exam. You are encouraged to refer to the theorem number in the lecture notes when you use them in your solution.
- Show your work on each problem. Specifically
  - Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
  - Unsupported answers will not receive credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will likely receive partial credit.
  - Circle your final answer for problems involving a series of computations.
  - Do NOT put answers on the back of the pages.

Problem 1 ($3 \times 4 = 12$ pts)**.** For each of the following, determine whether the statement is true or false. For each false statement, provide a counterexample or prove that the statement is false. You do not need to prove true statements.

1) **T F** If the matrix A has eigenvalue λ, then A^2 has eigenvalue λ^2.

2) **T F** Suppose A and B are matrices such that $AB = I_m$ where I_m is the $m \times m$ identity matrix. Then A and B must be square matrices.

3) **T F** A matrix with all distinct eigenvalues is always diagonalizable.

4) **T F** If a $n \times n$ real matrix A has n distinct eigenvalues, then A is invertible.

Problem 2 $(3 \times 4 = 12$ pts$)$**.** Below are 4 statements about $\mathbb{R}^n$ where the inner product is taken to be the Euclidean dot product. Circle T for true and F for false statements. Provide a counter-example or a brief explanation.

1) **T F** If u is orthogonal to v, then $u + v$ is orthogonal to $u - v$.

2) **T F** A set of non-zero orthogonal vectors in $\mathbb{R}^n$ must be linearly dependent.

3) **T F** Suppose u and v are orthogonal, then $||u + v||$ is bigger than $||u||$ or $||v||$.

4) **T F** If A and B are matrices with orthogonal columns, then $A + B$ must also have orthogonal columns.

Problem 3 ($3 \times 4 = 12$pts)**.** For each of the following, determine whether the statement is true or false. For each false statement, provide a counterexample or prove that the statement is false. You do not need to prove true statements.

1) **T** **F** A matrix with repeated eigenvalues cannot be diagonalizable.

2) **T** **F** Every square matrix has a Jordan canonical form.

3) **T** **F** If all the eigenvalues of a matrix are distinct, then the Jordan canonical form of the matrix is a diagonal matrix.

4) **T** **F** If A^2 is diagonalizable, then A must be diagonalizable.

Problem 4 (10 pts)**.** Is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

positive definite? Explain your answer.

Problem 5 (20 pts)**.** Define an inner product on $\mathbb{R}^2$ by $\langle u, v\rangle = 2u_1v_2 + 2u_2v_2 + u_1v_2 + u_2v_1$, where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

(a) Find the matrix K such that $\langle u, v\rangle = u^T K v$. (10 pts)

(b) Find an orthogonal basis for $\mathbb{R}^2$ with respect to the above inner product. (10 pts)

Problem 6 (14 pts)**.** Find a Jordan basis and the Jordan canonical form of the matrix A.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Problem 7 (20 pts)**.** Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

(a) Find the eigenvalues and eigenvectors of A. (8 pts)

(b) Determine whether A is positive definite. (4 pts)

(c) Fine the spectral decomposition of A if possible. If not, state why. (8 pts)