

## MATH 4242 Exam 1 Practice Problems

- (1) Compute the LU factorization of the following matrix. (Hint: Use Gaussian elimination)

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & 7 & 11 \\ 0 & -4 & 2 \end{bmatrix}$$

*Proof.* Use Gaussian elimination to turn the matrix in to  $U$ , and  $L$  is the product of the matrices corresponding to the elementary row operations.  $\square$

- (2) Let  $V = \mathbb{F}_{\leq m}[x]$  be the vector space of polynomials with degree at most  $m$ .

(a) Let  $U = \{f \in V \mid f(1) = 0\}$ . Is  $U$  a subspace of  $V$ ?

(b) Let  $W = \{f \in V \mid \deg(f) \text{ is even}\}$ . Is  $W$  a subspace of  $V$ ?

*Proof.* (a) If  $f, g \in U$ , then  $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$  and  $(kf)(1) = kf(1) = k \cdot 0 = 0$ , thus  $U$  is closed under addition and scalar multiplication. So  $U$  is a subspace

(b) Sums of even degree polynomials are still even degree, and scalar multiplication doesn't change the degree of a polynomial, so  $W$  is also a subspace.  $\square$

- (3) Let  $V = M_{n \times n}(\mathbb{R})$  the vector space of all  $n \times n$  matrices. Let  $B$  be the set of all upper triangular matrices. Is  $B$  a subspace of  $V$ ?

*Proof.* Yes.  $\square$

- (4) Suppose  $v_1, \dots, v_4$  are some vectors in  $\mathbb{R}^4$ , and  $A$  is the matrix whose columns are  $v_1, \dots, v_4$ . Suppose the row echelon form of  $A$  is

$$\begin{bmatrix} 3 & 1 & 7 & 1 \\ 0 & -4 & 8 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Does  $v_1, \dots, v_4$  form a basis of  $\mathbb{R}^4$ ?

(b) Is  $v_3$  in  $\text{span}(v_1, v_2, v_4)$ ? If so, write  $v_3$  as a linear combination of them.

(Hint: You may use: permuted LU factorization; some properties about matrix multiplication and invertible maps. )

*Proof.* (a) No, the rank of the matrix is 3 (from the number of pivots), therefore the column span has dimension 3 which is smaller than  $\dim(\mathbb{R}^4) = 4$ .

(b) Let  $w_1, \dots, w_4$  be the column vector of the row echelon form  $U$  of  $A$ . Use LU factorization, we know that  $PA = LU$ , where  $U$  is the row echelon form above. In other words,  $A = P^{-1}LU$ , and set  $P^{-1}L = K$ . We know that  $P$  and  $L$  are all products of invertible matrices so  $K$  must be also invertible. By properties of matrix product, we have that  $w_i = Kv_i$ . Since  $w_1, \dots, w_4$  are in row echelon form, we know that  $w_1, w_2, w_4$  (the columns that contain a pivot) form a basis of the column space. Therefore, by invertibility of  $K$ ,  $v_1 = K^{-1}w_1, v_2 = K^{-1}w_2, v_4 = K^{-1}w_4$  also form a basis of the column space, thus  $v_3 \in \text{span}(v_1, v_2, v_4)$ .

From the row echelon form matrix, we know that  $w_3 = 3w_1 - 2w_2$ , thus  $K^{-1}w_3 = 3K^{-1}w_1 - 2K^{-1}w_2$ , thus  $v_3 = 3v_1 - 2v_2$ .  $\square$

- (5) Let  $V = \mathbb{R}^2$  with basis  $v_1 = (1, 2)$  and  $v_2 = (0, 1)$ . Let  $T$  be the map  $T(a, b) = (b, a)$  for all  $a, b \in \mathbb{R}$ .

(a) Show that  $T$  is linear;

(b) Find the matrix of  $T$ ;

(c) Find a basis for  $\ker(T)$ . (Hint: use row echelon form of  $\mathcal{M}(T)$ ).

*Proof.* (a) omitted (b)  $T(v_1) = T(1, 2) = (2, 1) = 2v_1 - 3v_2$  and

$$T(v_2) = T(0, 1) = T(1, 0) = v_1 - 2v_2. \text{ Thus } \mathcal{M}(T) = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}.$$

(c) A row echelon form of  $M(T)$  is  $\begin{pmatrix} 2 & 1 \\ 0 & 7/2 \end{pmatrix}$  thus  $\text{rank}(T) = 2$  and thus  $\dim \ker(T) = 2 - 2 = 0$ , therefore  $\ker(T) = \{0\}$   $\square$

- (6) Let  $V = \text{End}(\mathbb{R}^2)$ . Let  $W$  be the set of non-invertible linear maps in  $V$ . Show that  $W$  is not a vector space.

(Hint: give an example of two non-invertible maps that sums to an invertible map. You might try to use some theorems to do this in terms of matrices).

*Proof.* Consider  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . Both  $A, B$  are singular (non-invertible), but  $A + B = I$  is invertible.

So the set of non-invertible maps is not closed under addition.  $\square$

- (7) Let  $V = \mathbb{R}^2$  under the standard basis and  $U = \{(x, 0) : x \in \mathbb{R}\}$  a subspace of  $V$ . Describe the quotient space  $V/U$ .

*Proof.* Suppose  $v, u \in \mathbb{R}^2$  such that  $v + U = u + U$ , i.e.  $\{(x + v_1, v_2) | x, v_1, v_2 \in \mathbb{R}\} = \{(x + u_1, u_2) | x, u_1, u_2 \in \mathbb{R}\}$  which is true iff  $v_2 = u_2$ . Therefore two vectors in  $V$  belong to the same coset in  $V/U$  if they have the same second entry. Therefore the elements of  $V/U$  are parametrized by  $\mathbb{R}$  according the second entry of the vectors.

In particular,  $V/U = \{U(k) | k \in \mathbb{R}\} \cong \mathbb{R}$  where  $U(k) := \{(x, k) | x \in \mathbb{R}\}$ .  $\square$

(Hint: What is the map whose kernel is  $U$ ?)

- (8) Is the following matrix positive definite?

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & 12 \end{pmatrix}$$

Does  $\langle x, y \rangle = x^T A y$  define an inner product on  $\mathbb{R}^3$ ?

- (9) Find the value for  $a, b$  such that the matrix  $A$  is orthogonal.

$$A = \frac{1}{3} \begin{pmatrix} 2 & -2 & b \\ 1 & 2 & 2 \\ a & 1 & 2 \end{pmatrix}$$