

MATH 4242 Applied Linear Algebra

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1. Systems of Linear Equations

A $m \times n$ system of linear equation is of the form

$$\begin{aligned}a_{11}x_1 + \cdots + a_{n1}x_n &= b_1 \\a_{21}x_1 + \cdots + a_{n2}x_n &= b_n \\&\cdots \quad \cdots \quad \cdots \\a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n\end{aligned}$$

Such equation can be represented using product of matrices.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or by an augmented matrix.

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$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{m1} & b_1 \\ a_{21} & a_{22} & \cdots & a_{m2} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array} \right]$$

Definition 1.1. We have three types of elementary row operations.

- (1) Multiply the i -th equation (or the i -th row of the augmented matrix), then add it to the j -th equation (or the j -th row of the augmented matrix).
- (2) Permute the equations (or the rows of the augmented matrix)
- (3) Multiply one equation (or one row of the augmented matrix) by a non-zero number.

1.1. Systems of $n \times n$ Equations. Matrices considered in this sections are all $n \times n$.

Definition 1.2. A matrix is *regular* if it can be turned into a upper triangular matrix such that every entry on the diagonal is non-zero.

Proposition 1.3. Let E be the matrix with 1's on the diagonal and $E_{ij} = k \neq 0$ is the only other non-zero entry in the lower triangular part. Then for any matrix M , EM is the matrix obtained by multiplying the j -th row of M then adding to the i -th row of M .

Proposition 1.4. A matrix A is regular if and only if it has an LU factorization, i.e.

$$A = LU$$

where L is a lower uni-triangular matrix, and U is a upper triangular matrix with non-zero diagonal entries.

Definition 1.5. Let $w \in S_n$ be a permutation, then define $P_w = \{a_{ij}\}$ to be the matrix such that

$$a_{i,j} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.6. For any matrix M , $P_w M$ is the matrix obtained by permuting the rows of M according to the permutation w .

Definition 1.7. A matrix A is called *non-singular* if it can be turned into a upper triangular matrix without non-zero diagonal entry via row operations of the first two types.

Proposition 1.8. A matrix A is non-singular if and only if it has a permuted LU factorization: $PA = LU$ where P is some permutation matrix.

Definition 1.9. Let $A = (a_{ij})$, defined transpose of A to be $A^t := (a_{ji})$.

Proposition 1.10. Denote A^t the transpose of A . We have that $AB = (BA)^t$.

Proposition 1.11. A matrix A is regular iff it admits an LDV factorization, $A = LDU$ where L is lower-unitriangular matrix, D is a diagonal matrix, and U is a uni-upper triangular matrix.

Definition 1.12. Let A be an $n \times n$ matrix. Suppose X is a matrix such that $XA = AX = I$ where I is the identity matrix. Then X is called the inverse of A and denoted by A^{-1} . A matrix is called *invertible* if A^{-1} exists.

Proposition 1.13. A matrix is invertible if and only if it is non-singular.

Remark 1.14. Inverse of a matrix can be found using Gauss-Jordan Elimination — see chapter 1 of Olver-Shakiban.

1.2. Systems of $m \times n$ Equations.

Definition 1.15. A matrix is in row echelon form if it looks like,

$$\begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where \bullet 's are non-zero entries (called *pivots*) and $*$ represent generic entries. The pivots are the first non-zero entries in each rows. We require the pivots occupy the first several rows consecutively.

Proposition 1.16. *Every matrix can be turned into a row echelon form using elementary row operations of type I and II. In other words, every matrix A has a factorization $PA = LU$ where P is a permutation matrix, L is a lower uni-triangular matrix, and U a matrix in row-echelon form.*

Definition 1.17. Since every matrix can be turned in to row-echelon form using elementary row operations, we define its *rank* to be the number of pivots.

Proposition 1.18. *A square $n \times n$ matrix is non-singular if its rank is n (full-rank).*

2. Vector Spaces

2.1. Some Basic Setup.

*Definition 2.1.*¹ A field is a set \mathbb{F} with two binary operations \times (multiplication) and $+$ (addition), satisfying the following axioms.

- $a + b = b + a$ and $a \times b = b \times a$ for all $a, b \in \mathbb{F}$.
- There exists an additive identity 0 such that $0 + a = a + 0 = a$ for all $a \in \mathbb{F}$.
- There exists a multiplication identity 1 such that $1 \times a = a \times 1 = a$ for all $a \in \mathbb{F}$.
- For every $a \in \mathbb{F}$, there exists an element denoted $-a$, such that $a + (-a) = 0$.
- $0 \neq 1$.
- For every $a \in \mathbb{F}$ and $a \neq 0$, there exists an element denoted a^{-1} , such that $a \times (a^{-1}) = 1$.
- For every $a, b, c \in \mathbb{F}$, $a \times (b + c) = ab + ac$.

For most part of this class, we will take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}$.

Definition 2.2. For a field \mathbb{F} , denote $\mathbb{F}[x]$ the ring² of polynomials over \mathbb{F} .

$$\mathbb{F}[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_0, \dots, a_n \in \mathbb{F}, n \geq 0, x^m x^n = x^{m+n}\}$$

Proposition 2.3. *Every polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ with complex coefficient has at least one complex solution. Note that this is not true for real polynomials.*

Definition 2.4. A field \mathbb{F} is called *algebraically closed* if every polynomial in $\mathbb{F}[x]$ has a solution in \mathbb{F} . (By Proposition 2.3, \mathbb{C} is algebraically closed).

¹You don't need to worry too much about the abstract structures of a field. The purpose of this definition is to make everything self-contained. You can basically think of a field as a set on which you can do some sort of arithmetic.

²A ring is a field, where multiplication need not to be commutative, and multiplicative identity (0) need not exists.

Proposition 2.5. *The field of complex numbers \mathbb{C} is the algebraic closure of \mathbb{R} . In other words, \mathbb{C} is the smallest algebraically closed field that contains \mathbb{R} .*

2.2. Vector Spaces and Subspaces.

Let \mathbb{F} be a field.

Definition 2.6. A set V is called a vector space over \mathbb{F} if there exists an addition map

$$add : V \times V \rightarrow V,$$

a scalar multiplication map

$$mult : \mathbb{F} \times V \rightarrow V,$$

and a zero vector 0 such that $v + 0 = v$ for all $v \in V$ and $\lambda 0 = 0$ for all $\lambda \in \mathbb{F}$. (Here \times denote the Cartesian product of sets³.) We will abbreviate them by $a(v_1, v_2) = v_1 + v_2$ and $mult(a, v) = av$.

Note that this definition (implicitly) requires that a vector space V is closed under addition and scalar multiplication, i.e. $v_1 + v_2 = add(v_1, v_2) \in V$ and $av = mult(a, v) \in V$.

Elements of a vector spaces are called *vectors*.

Definition 2.7. Let V be a vector space over \mathbb{F} . A subset U of V is a *subspace* if it is closed under addition and scalar multiplication, and contains the zero vector. (In other words, a subspace is a subset that is a vector space itself.)

Definition 2.8. Let U_1, \dots, U_m be subspaces of V . Then define their sum to be

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m \mid u_1 \in U_1, \dots, u_m \in U_m\}$$

Proposition 2.9. *Let U_1, \dots, U_m be subspaces of V . Then $U_1 + \dots + U_m$ is also a subspace of V , furthermore, it's the smallest subspace of V that contain all of U_1, \dots, U_m .*

Definition 2.10. A sum of subspaces $U_1 + \dots + U_m$ of V is a *direct sum* if every vector $v \in U_1 + \dots + U_m$ can be uniquely written as $v = u_1 + \dots + u_m$ where $u_i \in U_i$ for each i . When a summation is direct, we denote it as $U_1 \oplus \dots \oplus U_m$.

2.3. Linear Combination, Span, and Dimension.

Let V be a vector space over \mathbb{F} .

Definition 2.11. Let $v_1, v_2, \dots, v_n \in V$, a vector $v \in V$ is a linear combination of $\{v_1, \dots, v_n\}$ if there exists $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \dots + a_n v_n$$

Definition 2.12. Let v_1, v_2, \dots, v_n be a list of vectors in V , define their span to be the set of all linear combinations of v_1, \dots, v_n .

$$\text{span}(v_1, \dots, v_n) = \{a_1 v_1 + \dots + a_n v_n \mid a_1, \dots, a_n \in \mathbb{F}\}$$

Proposition 2.13. *For a list of vectors $v_1, \dots, v_n \in V$, $\text{span}(v_1, \dots, v_n)$ is a subspace of V . Furthermore, it's the smallest subspace containing all of v_1, \dots, v_n .*

Definition 2.14. A vector space V is said to be *finite dimensional* if it is the span of a finitely many vectors.

Definition 2.15. $v_1, \dots, v_m \in V$ are *linearly independent* if the only way to write 0 as a linear combination of v_1, \dots, v_n is

$$0 = 0v_1 + 0v_2 + \dots + 0v_n.$$

³For sets A and B , defined $A \times B = \{(a, b) \mid a \in A, b \in B\}$

Proposition 2.16. $v_1, \dots, v_m \in V$ are linearly independent if and only if any vector $v \in \text{span}(v_1, \dots, v_m)$ can be uniquely written as a linear combination of v_1, \dots, v_m .

Definition 2.17. A list of vectors v_1, \dots, v_n is a basis of V if

- $V = \text{span}(v_1, \dots, v_n)$
- v_1, \dots, v_n are linearly independent.

Proposition 2.18. v_1, \dots, v_n is a basis of V iff every vector $v \in V$ can be uniquely written as a linear combination of v_1, \dots, v_n .

Lemma 2.19. Let $v_1, \dots, v_m \in V$ be a list of vectors that spans V , i.e. $\text{span}(v_1, \dots, v_m) = V$. Then $\{v_1, \dots, v_m\}$ can be reduced to a basis of V . In other words, there exists a basis $\{w_1, \dots, w_n\}$ of V such that $w_i \in \{v_1, \dots, v_m\}$ for all i and $n \leq m$.

Lemma 2.20. Let $v_1, \dots, v_k \in V$ be linearly independent. Then there exists a basis of V in the form

$$\{v_1, \dots, v_k, w_1, \dots, w_m\}$$

Note that it's possible that $m = 0$, in the case when $\{v_1, \dots, v_k\}$ is already a basis.

Corollary 2.21. If U is a subspace of V , then there exists another subspace W such that $V = U \oplus W$.

Proposition 2.22. If v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is another basis of V . Then $n = m$.

Definition 2.23. Define the dimension of a vector space to be the size of its basis.

Proposition 2.24. If $\{v_1, \dots, v_n\}$ linearly independent and $n = \dim(V)$, then $\{v_1, \dots, v_n\}$ is a basis.

Proposition 2.25. If U is a subspace of V , then $\dim(U) \leq \dim(V)$. Furthermore, $\dim(U) = \dim(V)$ iff $U = V$.

Proposition 2.26. If $\text{span}(v_1, \dots, v_n) = V$ and $n = \dim(V)$, then $\{v_1, \dots, v_n\}$ is a basis.

Theorem 2.27. Let V be a finite dimensional vector space and V_1, V_2 subspaces. Then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Corollary 2.28. If $V_1 + V_2$ is a direct sum, then $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.⁴

3. Linear Maps and Matrices

3.1. Linear Maps. Let V, W be vector spaces over \mathbb{F} .

Definition 3.1. A map $T : V \rightarrow W$ is linear if

- (1) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
- (2) $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.

Definition 3.2. We denote the set of all linear maps from $V \rightarrow W$ by $\text{Hom}(V, W)$. And define $\text{End}(V) = \text{Hom}(V, V)$.

⁴We will see later that the converse is also true.

Lemma 3.3. Let v_1, \dots, v_n be a basis for V and w_1, \dots, w_n a basis for W (i.e. V, W same dimension). Then there exists a unique linear map $T \in \text{Hom}(V, W)$ such that $T(v_i) = w_i$ for all i . The map is given by $T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$.

Proposition 3.4. The set $\text{Hom}(V, W)$ is a vector space over \mathbb{F} , with addition and scalar multiplication given as follows.

$$\begin{aligned}(\varphi + \psi)(v) &:= \varphi(v) + \psi(v) \\ (\lambda\varphi)(v) &:= \lambda\varphi(v)\end{aligned}$$

Lemma 3.5. Let $T \in \text{Hom}(V, W)$, then $T(0_V) = 0_W$.

Let $T \in \text{Hom}(V, W)$.

Definition 3.6. The kernel (or null space) of T is $\text{Ker}(T) = \{v \in V : Tv = 0\}$

Proposition 3.7. $\text{Ker}(T)$ is a subspace of V .

Proposition 3.8. $\text{Ker}(T) = \{0\}$ if and only if T is injective.

Definition 3.9. The image (or range) of T is $\text{Img}(T) = \{Tv | v \in V\}$

Proposition 3.10. $\text{Img}(T)$ is a subspace of W .

Proposition 3.11. T is surjective iff $\text{Img}(T) = W$.

Theorem 3.12. $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Img}(T))$.

Proposition 3.13. (1) if $\dim(V) > \dim(W)$, then any $T \in \text{Hom}(V, W)$ is not injective.
 (2) if $\dim(V) < \dim(W)$, then any $T \in \text{Hom}(V, W)$ is not surjective.
 (3) if there exists a bijective $T \in \text{Hom}(V, W)$, then $\dim(V) = \dim(W)$.

3.2. Matrices from Linear Maps. Denote the set of all $m \times n$ matrix with entries in \mathbb{F} by $M_{m \times n}(\mathbb{F})$. Let V, W be finite dimensional vector spaces over \mathbb{F} .

Definition 3.14. Suppose V has basis v_1, \dots, v_n and W has basis w_1, \dots, w_m . Let $T \in \text{Hom}(V, W)$. Then define $\mathcal{M}(T)$ to be the matrix $[a_{ij}]$ such that

$$T(v_k) = a_{1k}w_1 + a_{2k}w_2 + \dots + a_{mk}w_m.$$

Remark 3.15. Note that the usage of \mathcal{M} requires a choice of basis for V and W . In general we shall denote $\mathcal{M}_{B_1, B_2}(T)$ where B_1 is the basis for V and B_2 the basis for W . However in most case we will omit the subscript when the context is clear.

Proposition 3.16. Let $S, T \in \text{Hom}(V, W)$, $\mathcal{M}(S) + \mathcal{M}(T) = \mathcal{M}(S + T)$

Let $S \in \text{Hom}(U, W)$ and $T \in \text{Hom}(V, U)$, then $\mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$.

Definition 3.17. For any $A \in M_{m \times n}$, Let $A_{\bullet, k}$ denote the k -th column vector and $A_{k, \bullet}$ denote the k -th row vector.

Proposition 3.18. $(AB)_{\bullet, k} = A(B_{\bullet, k})$

Theorem 3.19. For any $A \in M_{m \times n}$, we have

$$\dim(\text{span}(A_{1, \bullet}, \dots, A_{m, \bullet})) = \dim(\text{span}(A_{\bullet, 1}, \dots, A_{\bullet, n}))$$

Proposition 3.20. The dimension of column span or row span of a matrix equals to its rank (see Definition 1.17).

Definition 3.21. A linear map $T \in \text{Hom}(V, W)$ is invertible if there exists a linear map $S \in \text{Hom}(W, V)$ such that $TS = \text{id}_W$ and $ST = \text{id}_V$.

Proposition 3.22. If a map T is invertible, then its inverse is unique, denote it by T^{-1} .

Proposition 3.23. A map T is invertible iff $\mathcal{M}(T)$ is non-singular Definition 1.7. Furthermore, $\mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

Definition 3.24. A linear map is an *isomorphism* if it's invertible. Two vector spaces are *isomorphic* if there exists an isomorphism between them.

Theorem 3.25. Two vector spaces over \mathbb{F} is isomorphic if and only if they have the same dimension. (In other words, vector spaces are classified by \mathbb{N})

Corollary 3.26. Let $\dim V = n$ and $\dim W = m$. The vector space $\text{Hom}(V, W) \cong M_{m \times n}(\mathbb{F})$ are isomorphic, with the map \mathcal{M} being the isomorphism.

Theorem 3.27. Let V be a vector space with basis $B_1 = \{v_1, \dots, v_n\}$. Suppose it has another basis $B_2 = \{w_1, \dots, w_n\}$. Let $C = \mathcal{M}_{B_1, B_2}(\text{id})$ where $\text{id} \in \text{Hom}(V, V)$ is the identity map. Then change of basis corresponds to conjugation by C .

In particular, let $T \in \text{Hom}(V, V)$ and $A = \mathcal{M}_{B_1, B_1}(T)$ and $B = \mathcal{M}_{B_2, B_2}(T)$. Then we have

$$A = C^{-1}BC$$

3.3. Quotient and Dual spaces.

Definition 3.28. Let $v \in V$ and $U \subseteq V$. Define $v + U := \{v + u | u \in U\}$. This is called a *coset*.

Definition 3.29. Let $U \subseteq V$. Define the quotient space V/U to be $\{v + U | v \in V\}$, with addition and scalar multiplication given by

$$\begin{aligned} (v_1 + U) + (v_2 + U) &= (v_1 + v_2) + U \\ \lambda(v + U) &= \lambda v + U \end{aligned}$$

Definition 3.30 (alternative definition). Let \sim be an equivalence relation on V . Define $[v]_{\sim} := \{u \in V | u \sim v\}$ the equivalence class generated by v . Then we can define quotient space $V/\sim := \{[v]_{\sim} | v \in V\}$.

Remark 3.31. For $U \subset V$, define an equivalence relation \sim_U by $v \sim_U u \iff v - u \in U$. Then Definitions 3.29 and 3.30 agree, i.e. $V/U = V/\sim_U$.

Definition 3.32. For $U \subset V$, define the quotient map $\pi : V \rightarrow V/U$ by $\pi(v) = v + U$. Note that $\text{Ker}(\pi) = U$.

Proposition 3.33. $\dim V/U = \dim V - \dim U$.

Theorem 3.34. For any $T \in \text{Hom}(V, W)$, define $\tilde{T} \in \text{Hom}(V/\text{Ker}(T), W)$ by $\tilde{T}(v + \text{Ker}(T)) = Tv$. Then $\tilde{T}\pi = T$, and defines an isomorphism between $V/\text{Ker}(T)$ and $\text{Im}(T)$.

Definition 3.35. A linear map from V to \mathbb{F} is called a *linear functional*. Denote $\text{Hom}(V, \mathbb{F})$ the set of all linear functionals on V .

Proposition 3.36. $\text{Hom}(V, \mathbb{F})$ is a vector space, with addition and multiplication given by $(f + g)(v) = f(v) + g(v)$ and $(\lambda f)(v) = \lambda f(v)$. This is called the dual space of V , and is denoted by T^* .

Proposition 3.37. $\dim V = \dim V^*$.

Definition 3.38. Let v_1, \dots, v_n be a basis of V . Then define $v_i^* \in V^*$ to be the linear functional $v_i^*(v_j) = \delta_{i,j}$ ⁵. For any $v = a_1v_1 + \dots + a_nv_n \in V$, define $v^* = a_1v_1^* + \dots + a_nv_n^*$.

Proposition 3.39. Let v_1, \dots, v_n be a basis. Then $v = v_1^*(v)v_1 + \dots + v_n^*(v)v_n$ for all $v \in V$.

Proposition 3.40. v_1^*, \dots, v_n^* is a basis for V^* .

Definition 3.41. Suppose $T \in \text{Hom}(V, W)$. Define the *dual linear map* $T^* \in \text{Hom}(W^*, V^*)$ to be

$$T^*(f) = f \circ T$$

Proposition 3.42. • $(S + T)^* = S^* + T^*$

- $(\lambda T)^* = \lambda T^*$.
- $(ST)^* = T^*S^*$.

Definition 3.43. For any subspace $U \subseteq V$, define its annihilator $U^0 := \{f \in V^* : f(u) = 0 \text{ for all } u \in U\}$.

Proposition 3.44. U^0 is a subspace of V^* .

Proposition 3.45. $\dim U^0 = \dim V - \dim U$. Recall that this is also the dimension of V/U . In particular, there is an isomorphism $(V/U)^* \cong U^0$ given by π^* .

Proposition 3.46. (a) $U^0 = \{0\} \iff U = V$ (b) $U^0 = V^* \iff U = \{0\}$.

Theorem 3.47. (a) $(\text{Img } T)^0 = \text{Ker } T^*$ (b) $(\text{Ker } T)^0 = \text{Img } T^*$

Corollary 3.48. $\text{Ker } T^* \cong (V/\text{Img } T)^*$ and $\text{Img } T^* \cong (V/\text{Ker } T)^*$.

Corollary 3.49. T is injective iff T^* is surjective. T is surjective iff T^* is injective.

Theorem 3.50. Let $T \in \text{Hom}(V, W)$, and $T^* \in \text{Hom}(W^*, V^*)$. Then $\mathcal{M}(T)^t = \mathcal{M}(T^*)$.

4. Inner Product Spaces

Throughout this section, let $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

4.1. Inner Products and Norms.

Definition 4.1. The dot product of two vectors in \mathbb{R}^n is a map from $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{F}$, defined by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1y_1 + \dots + x_ny_n$$

Definition 4.2. The dot product of two vectors in \mathbb{C}^n is a map from $\mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{F}$, defined by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1\bar{y}_1 + \dots + x_n\bar{y}_n$$

where $\overline{a + bi} = a - bi$ is the complex conjugate.

Definition 4.3. Let V be vector space over \mathbb{F} (\mathbb{C} or \mathbb{R}). A inner product on V is a map $V \times V \rightarrow \mathbb{F}$ which sends (v, u) to $\langle v, u \rangle$ such that

- (1) $\langle v, v \rangle \geq 0$.
- (2) $\langle v, v \rangle = 0 \iff v = 0$.
- (3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

⁵Here $\delta_{i,j}$ is the Kronecker delta symbol: $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

$$(4) \langle \lambda u, v \rangle = \lambda \langle u, v \rangle.$$

$$(5) \langle u, v \rangle = \overline{\langle v, u \rangle}^6$$

Proposition 4.4 (Bilinearity). *A inner product $\langle \cdot, \cdot \rangle$ satisfy $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ and $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$.*

Remark 4.5. A pairing satisfying (4) and (6) of Definition 4.3 and Proposition 4.4 together is known as being *bilinear*. Usually a inner product is defined to be bilinear, however, as we see here one-sided linearity is enough to imply bilinearity.

Proposition 4.6. *An inner product $\langle \cdot, \cdot \rangle$ on V satisfy*

(1) *Fix any $u \in V$, the map $v \mapsto \langle u, v \rangle$ is a linear functional.*

(2) *$\langle v, 0 \rangle = 0 = \langle 0, v \rangle$ for any $v \in V$.*

Definition 4.7. Given an inner product $\langle \cdot, \cdot \rangle$, define the norm $\| \cdot \|$ to be the positive square-root $\|v\| = \sqrt{\langle v, v \rangle}$.

Proposition 4.8. *Let $I = [a, b] \subset \mathbb{R}$ be an closed interval on \mathbb{R} . Let $V = C^0(I)$ denote all continuous \mathbb{R} -valued functions defined on I (domain is I). Then*

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

defines an inner product on V . The norm $\|f\| = \sqrt{\langle f, f \rangle}$ is called the L_2 norm on $C^0(I)$.

Definition 4.9. For $z \in \mathbb{C}$, define the complex modulus to be $|z| = \sqrt{z\bar{z}}$. Note that when z is real (i.e. no imaginary part), then $|z|$ is the absolute value.

Theorem 4.10 (Cauchy-Schwartz inequality). *Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Then for any $u, v \in V$, we have*

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

Moreover, the equality occurs only when u, v are linearly independent.

Remark 4.11. The Cauchy-Schwartz inequality tells us that the ratio $\frac{|\langle u, v \rangle|}{\|u\| \|v\|}$ is in between -1 and 1 . Therefore we can define the ‘abstract’ angle between two vectors v, u to be

$$\theta_{u,v} = \arccos \frac{|\langle u, v \rangle|}{\|u\| \|v\|}$$

Definition 4.12. We say two vectors u, v are orthogonal if $\langle u, v \rangle = 0$.

Proposition 4.13. *If v, u orthogonal in V , then $\|v\|^2 + \|u\|^2 = \|v + u\|^2$.*

Remark 4.14. Definition 4.12 generalizes the usual notion of orthogonality in \mathbb{R}^2 in a sense that when two vectors are orthogonal, then the angle between them is $\theta_{u,v} = \pi/2$.

Theorem 4.15. *Let V be an inner product space, and $u, v \in V$. Then $\|u + v\| \leq \|u\| + \|v\|$.*

Definition 4.16. We can define norms more generally without requiring an inner product. A norm on V is a map $\| \cdot \| : V \rightarrow \mathbb{R}_{\geq 0}$ such that

- $\|v\| \geq 0$ and $\|v\| = 0$ only when $v = 0$.
- $\|\lambda v\| = |\lambda| \|v\|$

⁶When $\mathbb{F} = \mathbb{R}$, the ‘complex’ conjugate of a real number is just itself.

- $\|v + u\| \leq \|v\| + \|u\|$

Proposition 4.17. Let $K = (k_{ij})$ be the matrix whose entries are the inner product of the basis vectors, i.e. $k_{ij} = \langle e_i, e_j \rangle$. Then for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we have

$$\langle x, y \rangle = \left\langle \sum_i x_i e_i, \sum_j y_j e_j \right\rangle = \sum_{i,j} x_i y_j \langle e_i, e_j \rangle = x^t K y$$

Definition 4.18. A $n \times n$ matrix is positive-definite if $K^t = K$ and satisfy $x^t K x > 0$ for all $0 \neq x \in \mathbb{F}^n$. More generally, we say K is positive semi-definite if $K^t = K$ and $x^t K x \geq 0$ for all x .

Theorem 4.19. Every inner product is given by $\langle x, y \rangle = x^t K y$ where K is a positive-definite matrix.

Proposition 4.20. Positive-definite matrices are non-singular (invertible).

Definition 4.21. Given any $v_1, \dots, v_n \in V$, we define the Gram matrix to be $K = (k_{ij})$ where $k_{ij} = \langle v_i, v_j \rangle$. In particular, Let A be the matrix whose column vectors are v_1, \dots, v_n , then the Gram matrix is $K = A^t C A$, where C is the symmetric positive definite matrix defining the inner product.

Proposition 4.22. A Gram matrix is always positive semi-definite. A Gram matrix is positive-definite if and only if v_1, \dots, v_n are linearly independent.

Proposition 4.23. Let A be an $m \times n$ matrix ($m \geq n$), then TFAE:

- $K = A^t A$ is positive-definite;
- $\text{Ker}(A) = 0$;
- A has linearly independent columns;
- $\text{rank}(A) = n$.

Theorem 4.24. Suppose $A \in M_{m \times n}$ and $K = A^t A$ is positive-definite. Then for any symmetric positive definite matrix $C \in M_{m \times m}$, the matrix $K' = A^t C A$ is also positive-definite.

Proposition 4.25. For $K = A^t C A$, we have $\text{Ker}(K) = \text{Ker}(A)$, and hence $\text{rank}(K) = \text{rank}(A)$.

4.2. Orthonormal Basis.

Definition 4.26. Let V be a real or complex inner product space. A basis v_1, \dots, v_n of V is called *orthogonal* if $\langle v_i, v_j \rangle = \delta_{i,j}$ for all i . An orthogonal basis of unit vectors is called *orthonormal*.

Proposition 4.27. Let $v_1, \dots, v_n \in V$ be pair-wise orthogonal, then they must be linearly independent.

Corollary 4.28. Let $v_1, \dots, v_n \in V$ be pair-wise orthogonal, then they form a basis for $\text{span}(v_1, \dots, v_n)$.

Proposition 4.29. If e_1, \dots, e_n is an orthonormal basis, then for any $v \in V$, we have that

- $v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$
- $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$

Proposition 4.30. *If e_1, \dots, e_n is an orthonormal basis, then*

Theorem 4.31. *Given any basis w_1, \dots, w_n of V , one can construct an orthogonal basis v_1, \dots, v_n using the Gram-Schmidt process:*

- $v_1 = w_1.$
- $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$
- $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$
- \dots
- $v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i$
- $\dots.$

Corollary 4.32. *Every finite dimensional inner-product space has an orthonormal basis.*

Theorem 4.33. *Suppose V is finite-dimensional and T is a linear functional on V . Then there is a unique vector $v \in V$ such that $T(u) = \langle u, v \rangle$ for every $u \in V$.*

Definition 4.34. A matrix A is orthogonal if $A^t A = I = A A^t$, or equivalently $A^t = A^{-1}$.

Proposition 4.35. *A matrix is orthogonal if and only if its column vectors form an orthonormal basis of \mathbb{F}^n w.r.t the dot product.*

Definition 4.36. Let $W \subset V$ be a subspace. A vector $v \in V$ is said to be orthogonal to W is $\langle v, w \rangle = 0$ for all $w \in W$.

Definition 4.37. Two subspaces $W, U \subset V$ are said to be orthogonal if $\langle w, u \rangle = 0$ for all $w \in W, u \in U$.

Definition 4.38. The orthogonal complement of a subset $W \subset V$, denoted W^\perp , is the set of all vectors in V that are orthogonal to W .

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

Proposition 4.39. *Let U^\perp be the orthogonal complement of $U \subset V$.*

- (1) U^\perp is always a subspace of V ⁷.
- (2) $\{0\}^\perp = V$.
- (3) $V^\perp = \{0\}$
- (4) $U^\perp \cap U \subseteq \{0\}$.
- (5) If $W \subset U \subset V$, then $U^\perp \subset W^\perp$.

Proposition 4.40. *Let U be a finite dimensional subspace of V , then*

$$V = U^\perp \oplus U$$

And $\dim U^\perp = \dim V - \dim U$.

Proposition 4.41. $U = (U^\perp)^\perp$

⁷Even if U is not a subspace.

Definition 4.42. The orthogonal projection of v onto W , denoted $\text{Proj}_W(v)$ is the element $w \in W$ such that $v - w$ is orthogonal to W .

In other words, if we write v in the direct sum of $V = W^\perp \oplus W$ as $v = w' + w$ with $w \in W$ and $w' \in W^\perp$, then $\text{Proj}_W(v) = w$.

Theorem 4.43. Let w_1, \dots, w_n be an orthogonal basis for a subspace $W \subset V$. Then the orthogonal projection of v onto W is

$$\text{Proj}_W(v) = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_n \rangle u_n$$

Definition 4.44. Define the cokernel of a linear map $T \in \text{Hom}(V, W)$ to be the quotient space $\text{coKer}(T) = W / \text{Img}(T)$ and the coimage to be $\text{coImg}(T) = V / \text{Ker}(T)$.

Theorem 4.45. We have $\text{Ker}(T) = \text{coImg}(T)^\perp$ and $\text{Img}(T) = \text{coKer}(T)^\perp$

Proposition 4.46. The equation $Ax = b$ has a solution if b is orthogonal to the cokernel of A .

5. Eigenvalues and Eigenvectors

Definition 5.1. Let A be an $n \times n$ complex or real matrix, then λ is an eigenvalue of A if there exists a non-zero vector v , called an eigenvector, such that $Av = \lambda v$.

Proposition 5.2. λ is an eigenvalue of A if and only if $A - \lambda I$ is singular, i.e. there exist solutions to the equation $(A - \lambda I)v = 0$.

Definition 5.3. The characteristic polynomial of A , denoted $P_A(x)$, is defined to be

$$P_A(x) = \det(A - xI)$$

Remark 5.4. A matrix A and its transpose A^t have the same characteristic polynomial.

Proposition 5.5. Over the complex numbers, the characteristic polynomial can be factored into

$$P_A(x) = (-1)^n (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

where $\lambda_1, \dots, \lambda_n$ are the complex eigenvalues of A .

Proposition 5.6. $\lambda_1 + \dots + \lambda_n = \text{tr}(A)$ and $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$.

Definition 5.7. The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of $P_A(x)$.

Proposition 5.8. If v_1, \dots, v_k are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then v_1, \dots, v_k are linearly independent.

Definition 5.9. Let λ be an eigenvalue of A , then the eigenspace corresponding to λ , denoted by V_λ is the space $\text{Ker}(A - \lambda I)$. The dimension of V_λ is called the geometric multiplicity of λ .

Definition 5.10. A matrix A is called complete if the algebraic multiplicity and geometric multiplicity of all eigenvalues equal.

Proposition 5.11. A $n \times n$ matrix with n distinct eigenvalues is complete.

Theorem 5.12. *Every complete matrix A can be diagonalized as follows*

$$A = SDS^{-1}$$

where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ and S is the matrix whose columns are linearly independent eigenvectors v_1, \dots, v_n .

Remark 5.13. We will more often call complete matrices diagonalizable.

Theorem 5.14. *If A is a diagonalizable (complete) matrix, then all the k dimensional complex invariant subspace of A are spanned by linearly independent eigenvectors of A .*

Proposition 5.15. *Every real symmetric matrix is diagonalizable.*

Theorem 5.16 (spectral decomposition). *Let A be a symmetric matrix, then $A = QDQ^t$ where D is the diagonal matrix of eigenvalues of A , and Q is an orthogonal matrix whose columns are the orthonormal eigenvectors of A .*

Corollary 5.17. *A symmetric real matrix A is positive definite if and only if all of its eigenvalues are positive. (Note that when A is not necessary symmetric, being positive definite implies having positive eigenvalues, but the converse is not always true). A symmetric matrix A is positive semi-definite if and only if all of its eigenvalues are non-negative (possibly zero).*

Definition 5.18. Let λ be an eigenvalue of A . Then a Jordan chain of λ is a list of vectors w_1, \dots, w_k such that

$$Aw_1 = \lambda w_1, \quad Aw_2 = \lambda w_2 + w_1, \dots, Aw_k = \lambda w_k + w_{k-1}$$

Definition 5.19. A non-zero vector w is called a generalized eigenvector of A if $(A - \lambda I)^k w = 0$ for some finite number k .

Definition 5.20. A Jordan basis of a square matrix A is a basis of \mathbb{C}^n (or \mathbb{R}^n) consisting of Jordan chains of A .

Theorem 5.21. *Every square matrix has a Jordan basis.*

Definition 5.22. A Jordan block is a matrix such that the diagonal entries are the same number, the super-diagonal entries are either 1 or 0, and the other entries are zero.

Definition 5.23. A Jordan canonical form of a square matrix A , is the block-diagonal matrix where each diagonal block is a Jordan block with eigenvalues on the diagonal. Denote J_A the Jordan canonical form of A .

Theorem 5.24. *Any square matrix A can be written as $A = SJ_AS^{-1}$ where J_A is the Jordan canonical form of A , and S is the matrix whose columns form the Jordan basis.*

Remark 5.25. The number of Jordan blocks in J_A equals to the number of Jordan chains in the Jordan basis of A .

Definition 5.26. The singular values of a general (non-square) matrix are the square roots of the eigenvalues of the Gram matrix $K = A^t A$.

Theorem 5.27. *Every $m \times n$ matrix of rank r can be written as*

$$A = PAQ^t$$

where Λ is a $r \times r$ diagonal matrix with the singular values of A . The columns of A form an orthogonal basis for $\text{Im}(A)$ and the columns of Q form an basis for $\text{coIm}(A)$. In particular, the columns of Q are the normalized eigenvectors of the Gram matrix $K = A^t A$ corresponding to the non-zero eigenvalues.

Remark 5.28. If A has no zero eigenvalue, then the SVD of A is can give us the spectral decomposition of $K = A^t A$ in the following way $K = A^t A = (Q\Lambda P^t)(P\Lambda Q^t) = Q\Lambda^2 Q^t$.