

Math 206A Problem Sets

- Submit a genuine attempt for each assigned problem set. Perfect solutions are not required for full credit, but your work must be clearly typesetted.
- Printed submissions are preferred. Email submissions are allowed in special circumstances.
- Collaboration is encouraged. However, write up your solutions independently and acknowledge any collaborators.
- **AI policy:** You may use LLMs to find references or write codes. Do not use AI to directly solve or write solutions for this problem set.
- [Add.] Here's a more rigorous definition of a genuine attempt : \geq perfect solutions for half of each problem set.
- [Add.] Numbers after each problem shows difficulty, $3+ > 3 > 3- > 2+ > \dots$. Grades don't depend on difficulty.

1. Homework Assignment 1 (Due Oct 10 F)

Problem 1.1 (2+). Recall that $\mathbb{Y} = \mathbb{Q}\text{-span}(\mathbf{Y})$, where \mathbf{Y} is the Young's lattice, and the up/down operators U, D defined by $U(\lambda) = \sum_{\lambda \lessdot \mu} \mu$ and $D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$. Prove that

$$[D, U] = \text{id}$$

Problem 1.2 (2-). Recall the Weyl algebra \mathcal{W} is the \mathbb{Z} -algebra with unit 1 and generators U, D with relations $[D, U] = 1$. Rewrite $D^n U^n$ as an \mathcal{W} -element such that no D appears before an U . What is the identity coefficient?

For example, $D^2 U^2 = D(UD + 1)U = DUDU + DU = (UD + 1)(UD + 1) + (UD + 1) = UDU D + 3UD + 2 = U(UD + 1)D + 3UD + 2 = U^2 D^2 + 4UD + 2$. The identity coefficient is 2.

Problem 1.3 (2). Prove that $\lambda \leq \mu$ if and only if $\mu' \leq \lambda'$. Here λ' is the conjugate of λ .

Problem 1.4 (2). Prove that

$$\sum_{k \geq 0} h_k t^k = \prod_{i \geq 0} \frac{1}{1 - x_i t}$$

Problem 1.5 (2+). Is the power series $f = \prod_{i \geq 1} (1 + x_i + x_i^2)$ symmetric? If so, expand in the e -basis. [Hint: You can use (without proof) the change of basis matrix between m_λ and e_λ .]

Problem 1.6 (2). Expand $h_3 e_4$ in the Schur basis.

Problem 1.7 (2+). [Bonus] Expand $h_m e_n$ in the Schur basis. You may start experimenting with SageMath to make a conjecture.

2. Homework Assignment 2 (Due Oct 20 M)

Problem 2.1 (2+). Prove that if $\lambda, \mu \vdash n$ and $\lambda \leq \mu$, then $K_{\lambda\mu} \neq 0$.

Problem 2.2 (2). Prove that $s_\lambda \cdot s_\square = \sum_{\lambda \leq \mu} s_\mu$. Here $s_\square = m_1 = x_1 + x_2 + \cdots$, and \leq denotes the covering relation in Young's lattice.

Rmk: Can you define a \mathcal{W} -action on $\Lambda_{\mathbb{Q}}$? By P 2.2, we get an action of U on Λ via multiplication by s_\square . What should be the action of D on $\Lambda_{\mathbb{Q}}$ such that the map $\lambda \mapsto s_\lambda$ is an isomorphism from \mathbb{Y} to $\Lambda_{\mathbb{Q}}$ as \mathcal{W} -modules?

Problem 2.3 (2+). Let $u_i : \mathbb{Y} \rightarrow \mathbb{Y}$ be the operator of adding a box to the i -th row when possible. Define

$$H_k = h_k(u_1, u_2, \dots) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

The operators u_i satisfy the relations

$$\begin{aligned} u_i u_j &= u_j u_i \quad \text{if } |i - j| \geq 2 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_i \\ u_{i+1} u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ u_{i+1} u_{i+2} u_{i+1} u_i &= u_{i+1} u_{i+2} u_i u_{i+1} \end{aligned}$$

Classify the relations of $\{H_i | i \in \mathbb{N}\}$.

(Bonus) What about

$$E_k = e_k(u_1, u_2, \dots) = \sum_{1 \leq i_1 < i_2 < \dots < i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

[Hint: How does H_k act on \mathbb{Y} ? You may or may not need to use the relations of the u_i 's.]

Problem 2.4 (3+). Define $H'_k = h_k(d_1, d_2, \dots)$. What relations do $\{H_i, H'_j : i, j \in \mathbb{N}\}$ satisfy?

Problem 2.5 (3). Let $P(w)$ denote the row-insertion tableau, and $P'(w)$ denote the column-insertion tableau. Prove that $P(w) = P'(w_0 w)$, where $w_0 w = (w_n, w_{n-1}, \dots, w_2, w_1)$.

3. Homework Assignment 3 (Due Oct 29 W)

Problem 3.1 (2). Define $A(x) = \cdots (1 + xu_3)(1 + xu_2)(1 + xu_1)$ and $B(x) = (1 + xd_1)(1 + xd_2)(1 + xd_3) \cdots$. Prove that $B(y)A(x) = A(x)B(y)(1 - xy)^{-1}$. [Hint: first show that $(1 + a)(1 - ba)^{-1}(1 + b) = (1 + b)(1 - ab)^{-1}(1 + a)$ for any non-commutative a, b .]

Problem 3.2 (2+). Let P be a finite poset, and $a_k(P)$ the size of maximal k -antichain of P . Prove that

$$a_k(P) = a_1(P \times [k])$$

where $[k]$ is the chain of k elements.

Problem 3.3 (Bonus, 3+). Let P be a finite poset. Let $C = (C_1, C_2, \dots, C_l)$ be a partition of P into chains.

(i) Show that for any C , we have

$$a_k \leq \sum_i \min(|C_i|, k)$$

A chain-partition C is called k -saturated if the equality holds.

(ii) Prove that for any P there exists a k -saturated chain partition for all k .

[Hints: (1) The case $k = 1$ is the Dilworth theorem, which you can use without proving (2) use Problem 3.3]

Problem 3.4 (2+). Denote $\text{SYT}_{\lambda \setminus \mu}$ the set of all skew standard Young tableaux of shape $\lambda \setminus \mu$. Fix a $\mu \in \mathbf{Y}$ and define $D_\mu = \{(T_1, T_2) \in \text{SYT}_{\lambda \setminus \mu} : |\lambda \setminus \mu| = n\}$. What is $\#D_\mu$? Note that the RS bijection implies that $\#D_\emptyset = n!$.

correction: You only need to find a lower bound for $\#D_\lambda$.

[Hint: use growth diagram.]

Problem 3.5 (Bonus, 5). Recall that a permutation is an involution, i.e. $w = w^{-1}$, if and only if $P(w) = Q(w)$. The Bruhat order on S_n induces a partial order on involutions, thus a partial order on SYT's. Give a description of this partial order on SYT's.

4. Homework Assignment 4 (Due Nov 7 F)

Problem 4.1 (2). Consider

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 5 & 0 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 4 & 4 \\ 5 & -3 & -4 \\ 5 & 5 & -4 \end{pmatrix}.$$

And let $E_A = \langle A_1 \rangle \subset \langle A_1, A_2 \rangle \subset \langle A_1, A_2, A_3 \rangle^3$ and let $E_B = \langle B_1 \rangle \subset \langle B_1, B_2 \rangle \subset \langle B_1, B_2, B_3 \rangle^3$. Compute the relative position of E_A, E_B .

Problem 4.2 (2+). (i) Compute \mathfrak{S}_{321654} .

(ii) Which Schubert polynomials are monomials?

Problem 4.3 (3). Let $\text{GP}_{k,n}$ denote the set of all k -Grassmanian permutations in S_n . For $w \in \text{GP}_{k,n}$, denote $\lambda(w)$ the partition associated to it. Prove that $\mathfrak{S}_w = s_{\lambda(w)}(x_1, \dots, x_k)$.

Problem 4.4 (Bonus, 4). Let $\tau : S_n \rightarrow S_{n+1}$ be the map that sends w to u where $u_1 = 1$ and $u_i = w_{i-1} + 1$. For any $w \in S_n$, define $F_w = \lim_{k \rightarrow \infty} \mathfrak{S}_{\tau^k(w)}$. Prove that F_w is symmetric

5. Homework Assignment 5 (Due Nov 17 M)

Problem 5.1 (5+). Read Chapter 1 of Sagan's book

Problem 5.2 (1). Let $w \in S_n$ and $w' \in S_{n+1}$ such that $w'(i) = w(i)$ for $i \in [n]$ and $w'(n+1) = n+1$. Show that $\mathfrak{S}_w = \mathfrak{S}_{w'}$.

Problem 5.3 (1). Use Growth diagram to compute the inverse RSK of (T_1, T_2) where $T_1 =$

1	1	3	4
2	2		
3			

and $T_2 =$

1	2	2	3
2	3		
3			

Problem 5.4 (2). Can you construct a permutation representation for the dihedral group $D_{2n} = \{s^2 = r^n = 1, srs = r^{-1}\}$. Is it faithful?

Problem 5.5 (Bonus, 2). Can you describe the action of divided difference operator ∂_i on $\mathbf{Y}_{k \times (n-k)} \cong GP_{k,n} \subset S_n$. Here $\mathbf{Y}_{k \times (n-k)}$ is the set of all partitions that fit inside a $k \times (n-k)$ rectangle.

6. Homework Assignment 6 (Due Nov 26 W)

7. Homework Assignment 7 (Due Dec 8 F)