

## Math 206A Problem Sets

- Submit a genuine attempt for each assigned problem set. Perfect solutions are not required for full credit, but your work must be clearly typeset.
- Printed submissions are preferred. Email submissions are allowed in special circumstances.
- Collaboration is encouraged. However, write up your solutions independently and acknowledge any collaborators.
- **AI policy:** You may use LLMs to find references or write codes. Do not use AI to directly solve or write solutions for this problem set.
- [Add.] Here's a more rigorous definition of a genuine attempt : $\geq$  perfect solutions for half of each problem set.
- [Add.] Numbers after each problem shows difficulty,  $3+ > 3 > 3- > 2+ > \dots$ . Grades don't depend on difficulty.

### 1. Homework Assignment 1 (Due Oct 10 F)

**Problem 1.1** (2+). Recall that  $\mathbb{Y} = \mathbb{Q}\text{-span}(\mathbf{Y})$ , where  $\mathbf{Y}$  is the Young's lattice, and the up/down operators  $U, D$  defined by  $U(\lambda) = \sum_{\lambda \lessdot \mu} \mu$  and  $D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$ . Prove that

$$[D, U] = \text{id}$$

**Problem 1.2** (2-). Recall the Weyl algebra  $\mathcal{W}$  is the  $\mathbb{Z}$ -algebra with unit 1 and generators  $U, D$  with relations  $[D, U] = 1$ . Rewrite  $D^n U^n$  as an  $\mathcal{W}$ -element such that no  $D$  appears before an  $U$ . What is the identity coefficient?

For example,  $D^2 U^2 = D(UD + 1)U = DUDU + DU = (UD + 1)(UD + 1) + (UD + 1) = UDUD + 3UD + 2 = U(UD + 1)D + 3UD + 2 = U^2 D^2 + 4UD + 2$ . The identity coefficient is 2.

**Problem 1.3** (2). Prove that  $\lambda \leq \mu$  if and only if  $\mu' \leq \lambda'$ . Here  $\lambda'$  is the conjugate of  $\lambda$ .

**Problem 1.4** (2). Prove that

$$\sum_{k \geq 0} h_k t^k = \prod_{i \geq 0} \frac{1}{1 - x_i t}$$

**Problem 1.5** (2+). Is the power series  $f = \prod_{i \geq 1} (1 + x_i + x_i^2)$  symmetric? If so, expand in the  $e$ -basis. [Hint: You can use (without proof) the change of basis matrix between  $m_\lambda$  and  $e_\lambda$ .]

**Problem 1.6** (2). Expand  $h_3 e_4$  in the Schur basis.

**Problem 1.7** (2+). [Bonus] Expand  $h_m e_n$  in the Schur basis. You may start experimenting with SageMath to make a conjecture.

## 2. Homework Assignment 2 (Due Oct 20 M)

**Problem 2.1** (2+). Prove that if  $\lambda, \mu \vdash n$  and  $\lambda \leq \mu$ , then  $K_{\lambda\mu} \neq 0$ .

**Problem 2.2** (2). Prove that  $s_\lambda \cdot s_\square = \sum_{\lambda \lessdot \mu} s_\mu$ . Here  $s_\square = m_1 = x_1 + x_2 + \dots$ , and  $\lessdot$  denotes the covering relation in Young's lattice.

Rmk: Can you define a  $\mathcal{W}$ -action on  $\Lambda_{\mathbb{Q}}$ ? By P 2.2, we get an action of  $U$  on  $\Lambda$  via multiplication by  $s_\square$ . What should be the action of  $D$  on  $\Lambda_{\mathbb{Q}}$  such that the map  $\lambda \mapsto s_\lambda$  is an isomorphism from  $\mathbb{Y}$  to  $\Lambda_{\mathbb{Q}}$  as  $\mathcal{W}$ -modules?

**Problem 2.3** (2+). Let  $u_i : \mathbb{Y} \rightarrow \mathbb{Y}$  be the operator of adding a box to the  $i$ -th row when possible. Define

$$H_k = h_k(u_1, u_2, \dots) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

The operators  $u_i$  satisfy the relations

$$\begin{aligned} u_i u_j &= u_j u_i & \text{if } |i - j| \geq 2 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_i \\ u_{i+1} u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ u_{i+1} u_{i+2} u_{i+1} u_i &= u_{i+1} u_{i+2} u_i u_{i+1} \end{aligned}$$

Classify the relations of  $\{H_i \mid i \in \mathbb{N}\}$ .

(Bonus) What about

$$E_k = e_k(u_1, u_2, \dots) = \sum_{1 \leq i_1 < i_2 < \dots < i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

[Hint: How does  $H_k$  act on  $\mathbb{Y}$ ? You may or may not need to use the relations of the  $u_i$ 's.]

**Problem 2.4** (3+). Define  $H'_k = h_k(d_1, d_2, \dots)$ . What relations do  $\{H_i, H'_j : i, j \in \mathbb{N}\}$  satisfy?

**Problem 2.5** (3). Let  $P(w)$  denote the row-insertion tableau, and  $P'(w)$  denote the column-insertion tableau. Prove that  $P(w) = P'(w_0 w)$ , where  $w_0 w = (w_n, w_{n-1}, \dots, w_2, w_1)$ .

## 3. Homework Assignment 3 (Due Oct 29 W)

**Problem 3.1** (2). Define  $A(x) = \dots (1 + xu_3)(1 + xu_2)(1 + xu_1)$  and  $B(x) = (1 + xd_1)(1 + xd_2)(1 + xd_3) \dots$ . Prove that  $B(y)A(x) = A(x)B(y)(1 - xy)^{-1}$ . [Hint: first show that  $(1 + a)(1 - ba)^{-1}(1 + b) = (1 + b)(1 - ab)^{-1}(1 + a)$  for any non-commutative  $a, b$ .]

**Problem 3.2** (2+). Let  $P$  be a finite poset, and  $a_k(P)$  the size of maximal  $k$ -antichain of  $P$ . Prove that

$$a_k(P) = a_1(P \times [k])$$

where  $[k]$  is the chain of  $k$  elements.

**Problem 3.3** (Bonus, 3+). Let  $P$  be a finite poset. Let  $C = (C_1, C_2, \dots, C_l)$  be a partition of  $P$  into chains.

(i) Show that for any  $C$ , we have

$$a_k \leq \sum_i \min(|C_i|, k)$$

A chain-partition  $C$  is called *k-saturated* if the equality holds.

(ii) Prove that for any  $P$  there exists a  $k$ -saturated chain partition for all  $k$ .

[Hints: (1) The case  $k = 1$  is the Dilworth theorem, which you can use without proving (2) use Problem 3.3]

**Problem 3.4** (2+). Denote  $\text{SYT}_{\lambda \setminus \mu}$  the set of all skew standard Young tableaux of shape  $\lambda \setminus \mu$ . Fix a  $\mu \in \mathbf{Y}$  and define  $D_\mu = \{(T_1, T_2) \in \text{SYT}_{\lambda \setminus \mu} : |\lambda \setminus \mu| = n\}$ . What is  $\#D_\mu$ ? Note that the RS bijection implies that  $\#D_\emptyset = n!$ .

**correction: You only need to find a lower bound for  $\#D_\lambda$ .**

[Hint: use growth diagram.]

**Problem 3.5** (Bonus, 5). Recall that a permutation is an involution, i.e.  $w = w^{-1}$ , if and only if  $P(w) = Q(w)$ . The Bruhat order on  $S_n$  induces a partial order on involutions, thus a partial order on SYT's. Give a description of this partial order on SYT's.

#### 4. Homework Assignment 4 (Due Nov 7 F)

**Problem 4.1** (2). Consider

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 5 & 0 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 4 & 4 \\ 5 & -3 & -4 \\ 5 & 5 & -4 \end{pmatrix}.$$

And let  $E_A = \langle A_1 \rangle \subset \langle A_1, A_2 \rangle \subset \langle A_1, A_2, A_3 \rangle^3$  and let  $E_B = \langle B_1 \rangle \subset \langle B_1, B_2 \rangle \subset \langle B_1, B_2, B_3 \rangle^3$ . Compute the relative position of  $E_A, E_B$ .

**Problem 4.2** (2+). (i) Compute  $\mathfrak{S}_{321654}$ .

(ii) Which Schubert polynomials are monomials?

**Problem 4.3** (3). Let  $\text{GP}_{k,n}$  denote the set of all  $k$ -Grassmannian permutations in  $S_n$ . For  $w \in \text{GP}_{k,n}$ , denote  $\lambda(w)$  the partition associated to it. Prove that  $\mathfrak{S}_w = s_{\lambda(w)}(x_1, \dots, x_k)$ .

**Problem 4.4** (Bonus, 4). Let  $\tau : S_n \rightarrow S_{n+1}$  be the map that sends  $w$  to  $u$  where  $u_1 = 1$  and  $u_i = w_{i-1} + 1$ . For any  $w \in S_n$ , define  $F_w = \lim_{k \rightarrow \infty} \mathfrak{S}_{\tau^k(w)}$ . Prove that  $F_w$  is symmetric

## 5. Homework Assignment 5 (Due Nov 17 M)

**Problem 5.1 (5+).** Read Chapter 1 of Sagan's book

**Problem 5.2 (1).** Let  $w \in S_n$  and  $w' \in S_{n+1}$  such that  $w'(i) = w(i)$  for  $i \in [n]$  and  $w'(n+1) = n+1$ . Show that  $\mathfrak{S}_w = \mathfrak{S}_{w'}$ .

**Problem 5.3 (1).** Use Growth diagram to compute the inverse RSK of  $(T_1, T_2)$  where  $T_1 =$

1	1	3	4
2	2		
3			

and  $T_2 =$

1	2	2	3
2	3		
3			

**Problem 5.4 (2).** Can you construct a permutation representation for the dihedral group  $D_{2n} = \{s^2 = r^n = 1, srs = r^{-1}\}$ . Is it faithful?

**Problem 5.5 (Bonus, 2).** Recall the action of the (right) divided difference operator on the vector space  $\text{span}\langle S_n \rangle$ :

$$\partial_i(w) = \begin{cases} ws_i & \text{if } ws_i < w \\ 0 & \text{otherwise} \end{cases}$$

We also define the left divided difference operator  $\sigma_i$ :

$$\sigma_i(w) = \begin{cases} s_i w & \text{if } s_i w < w \\ 0 & \text{otherwise} \end{cases}$$

Note that both of these operators satisfy the Nil-Coxeter relations.

Can you describe the action of these operators on (the vector space of)  $\mathbf{Y}_{k \times (n-k)} \cong GP_{k,n} \subset S_n$ . Here  $\mathbf{Y}_{k \times (n-k)}$  is the set of all partitions that fit inside a  $k \times (n-k)$  rectangle.

## 6. Homework Assignment 6 and 7 (Due Dec 5 F)

**Problem 6.1.** Find all representations of  $D_{2n}$  up to equivalence. (Hint:  $C_n$  is a normal subgroup of  $D_{2n}$ ).

**Problem 6.2.** For any  $w \in S_n$ , prove that  $\chi^{(n-1,1)}(w) = \#\{\text{fixed points of } \pi\} - 1$

**Problem 6.3.** Suppose we have groups  $K \leq H \leq G$  and  $\rho : K \rightarrow \text{GL}(\mathbb{F}^n)$  a representation of  $K$ . Prove that

$$\text{Ind}_K^G \rho = \text{Ind}_H^G \text{Ind}_K^H \rho$$

**Problem 6.4.** Compute the Specht module for  $\lambda = (n-1, 1)$ . What's its dimension?

**Problem 6.5.**

**Problem 6.6.**