Higher Dimer Covers on Snake Graphs and Multi-dimensional Continued Fractions

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joint work w/

Gregg Musiker UMN

Nick Ovenhouse Yale

Ralf Schiffler UConn

① Fibonacci numbers

Golden Ratio

Higher dim Fibonacci #s

G. Raney 1966

n-dim GR

② Continued fractions

Schiffler
Canakci 2016

n-dim Continued fractions

③ Dimer covers on
Snake graphs

Higher dimer covers

④ reverse plane partitions
quasi sym polynomials
rank gen. fns

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \cdots$$

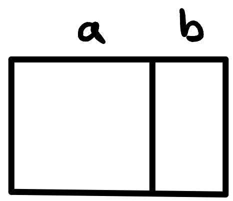
"Golden Ratio"

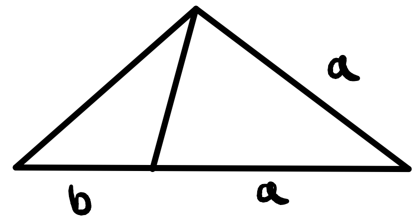
$$1 \quad 2 \quad \frac{3}{2} \quad \frac{5}{2} \quad \frac{8}{5} \quad \frac{13}{8} \quad \cdots \quad \varphi$$

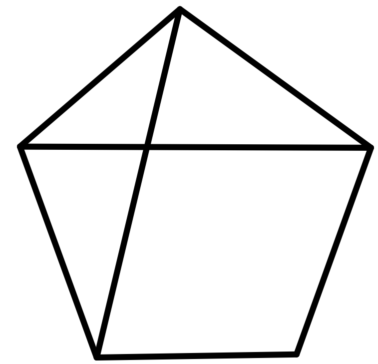
$$1 \quad 1+1 \quad 1+\frac{1}{1+1} \quad 1+\frac{1}{1+\frac{1}{1+1}} \quad \cdots \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}$$

The golden ratio

$$\frac{b}{a} = \frac{a}{a+b}$$







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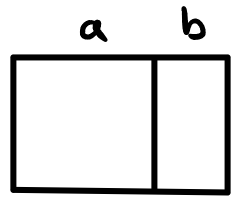
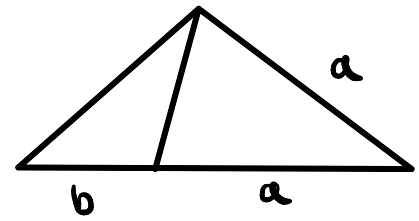
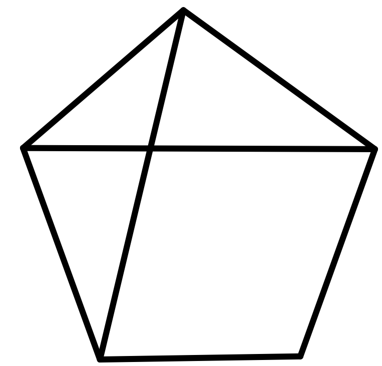


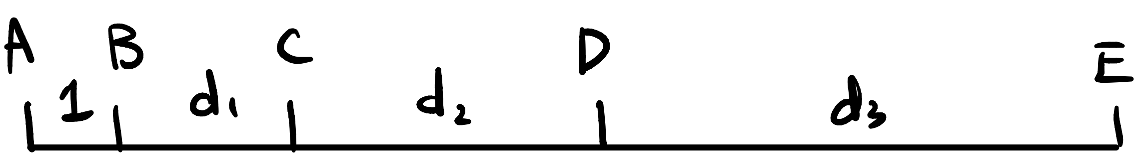


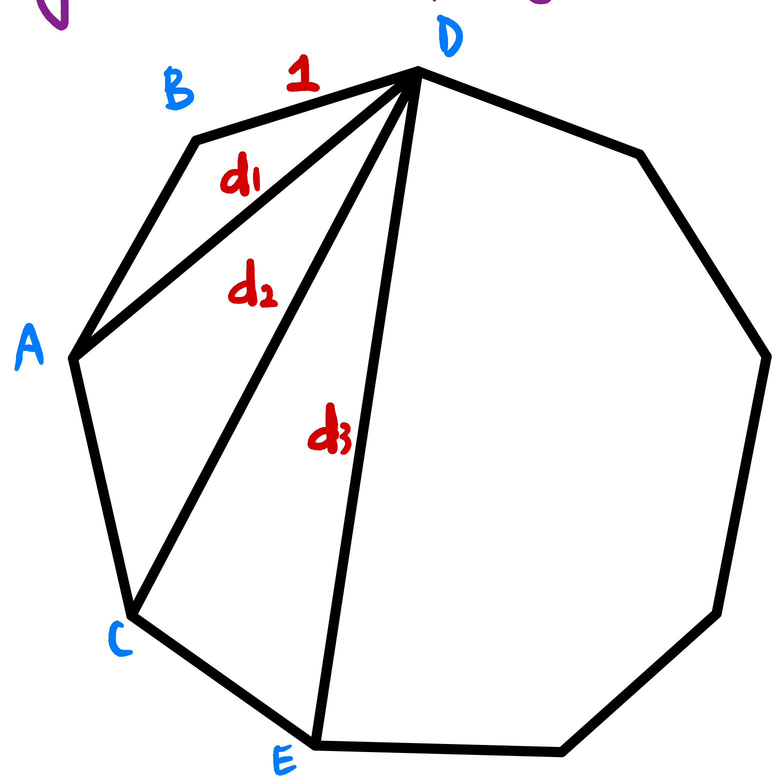


Higher dim?

regular 2n+1 polygon

$$\begin{aligned}
&1 : d_1 : d_2 : d_3 \\
&= d_1 : 1+d_2 : d_1+d_3 : d_2+d_3 \\
&= d_2 : d_1+d_3 : 1+d_2+d_3 : d_1+d_2+d_3 \\
&= d_3 : d_2+d_3 : d_1+d_2+d_3 : 1+d_1+d_2+d_3
\end{aligned}$$





Golden portions / n-dim Golden Ratios.

$\varphi_i^{(n)} :=$ i-th diag of regular $(2n+3)$-gon

where $\varphi_0^{(n)} = 1$.

$$\left(\varphi_1^{(1)} = \frac{1+\sqrt{5}}{2} \quad \text{the classical GR} \right)$$

n-dim GR : $\left(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_n^{(n)} \right)$

n- dim Fibonacci #s. (to be defined later)

TABLE I

	$d = 0$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
$n = 1$	(1)	(1)	(1)	(1)	(1)	(1)	(1)
$n = 2$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}5\\3\end{pmatrix}$	$\begin{pmatrix}8\\5\end{pmatrix}$	$\begin{pmatrix}13\\8\end{pmatrix}$
$n = 3$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\\1\end{pmatrix}$	$\begin{pmatrix}6\\5\\3\end{pmatrix}$	$\begin{pmatrix}14\\11\\6\end{pmatrix}$	$\begin{pmatrix}31\\25\\14\end{pmatrix}$	$\begin{pmatrix}70\\56\\31\end{pmatrix}$
$n = 4$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}4\\3\\2\\1\end{pmatrix}$	$\begin{pmatrix}10\\9\\7\\4\end{pmatrix}$	$\begin{pmatrix}30\\26\\19\\10\end{pmatrix}$	$\begin{pmatrix}85\\75\\56\\30\end{pmatrix}$	$\begin{pmatrix}246\\216\\160\\85\end{pmatrix}$

<u>Prop</u>. n-dim Fibonacci converges to n-dim GR.
as pt in a proj space.

Continued Fractions

$$f_n : 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots \cfrac{1}{1}}} = [\overbrace{1, 1, \cdots, 1}^{n}]$$

$$\varphi_1^{(1)} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = [1, 1, \cdots\cdot]$$

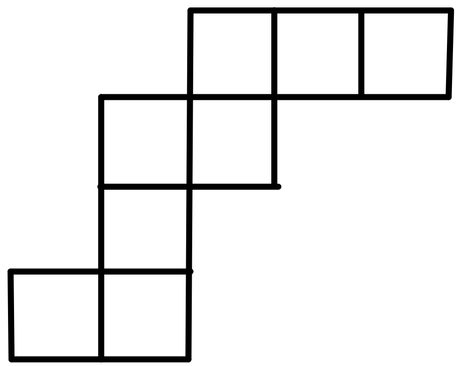
General continued fraction

$$[a_1, a_2, \cdots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_n}}}$$

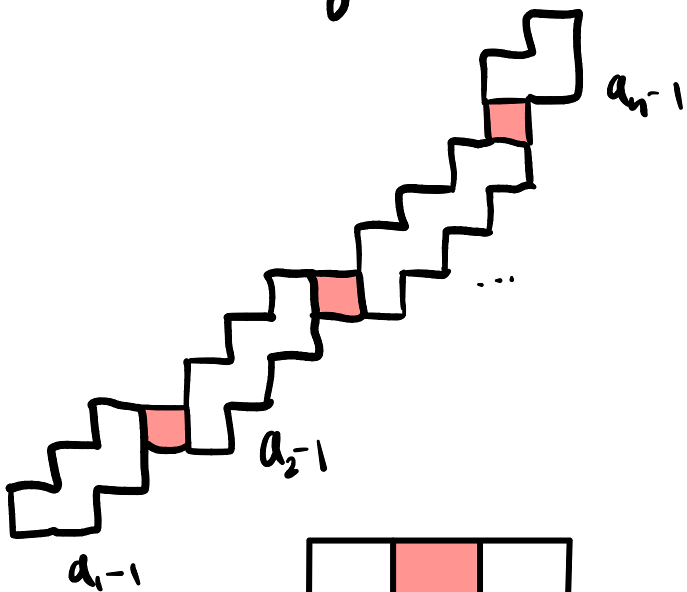
$$
\begin{tikzcd}
\text{Golden Ratio \& Fibonacci \#s} \arrow[r] & \text{n-dim GR \& Fibonacci \#s.} \\
\text{Continued fractions} \arrow[u] \arrow[r, "?"] & \text{n-dim continued fractions ??} \arrow[u, "?"']
\end{tikzcd}
$$

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}$$

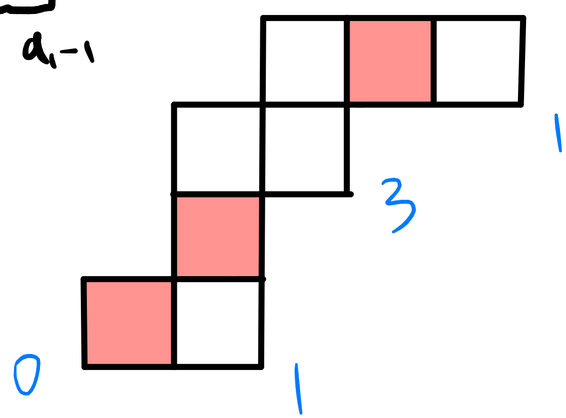
Introducing Snake Graphs



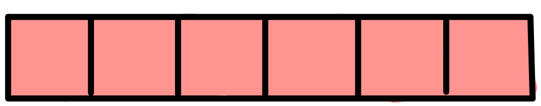
Introducing Snake Graphs



$$= \mathcal{G}[a_1, a_2, \cdots, a_n]$$



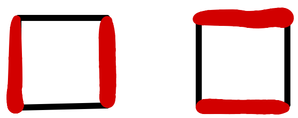
$$= \mathcal{G}[1, 2, 4, 2]$$

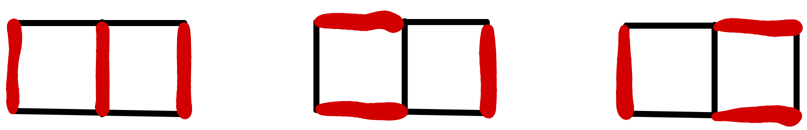


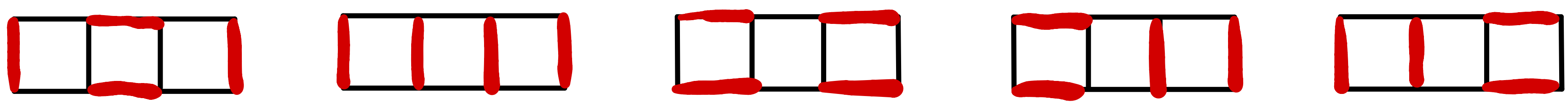
$$= \mathcal{G}[1, 1, \cdots, 1]$$

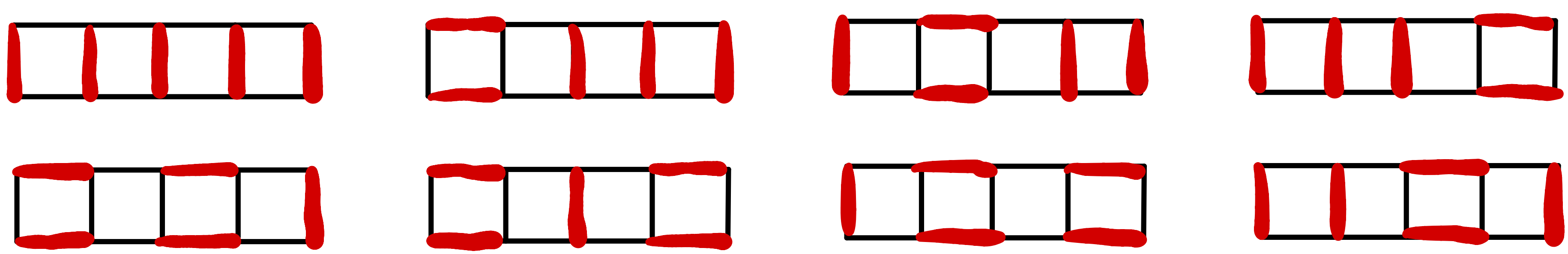
Perfect Matchings (Dimer covers)

Denote $\Omega[a_1 \cdots a_n]$ the # of dimer covers on $\mathcal{G}[a_1 \cdots a_n]$

2 

3 

5 

8 

Continued fractions via Snake Graphs.

Thm. Schiffler - Canakci 2016

$$[a_1, a_2, \dots, a_n] = \frac{\Omega[a_1, a_2, \dots, a_n]}{\Omega[a_2, \dots, a_n]}$$

Define $\Lambda(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ then...

Thm (old)

$$\Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_n) = \begin{pmatrix} \Omega[a_1 \cdots a_n] & * \\ \Omega[a_2 \cdots a_n] & * \end{pmatrix}$$

Can be further decomposed....

More on matrix formula

Define $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then...

$$R^{a_1} L^{a_2} R^{a_3} \cdots = \Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_n)$$ "up to sign"

$$= \begin{pmatrix} \Omega[a_1 \cdots a_n] & * \\ \Omega[a_2 \cdots a_n] & * \end{pmatrix}$$

Denote this matrix $\triangle(a_1, a_2, \ldots, a_n)$

Remark L, R and hence $\triangle \in SL_2(\mathbb{Z})$

Our motivation

In Musiker-Schiffler 08 & Musiker-Schiffler-Williams 09, dimers on Snake graphs are used to give formula for Type A Cluster algebras.

“$Gr(2,n)$ or $T(\ )$”

In Musiker-Ovenhouse-Z. 21, we found that certain super cluster variables correspond to double dimer covers of the same snake graph.

Question [Schiffler. OPAC 2022]

Find Continued fraction / Number theory interpretation for double dimers.

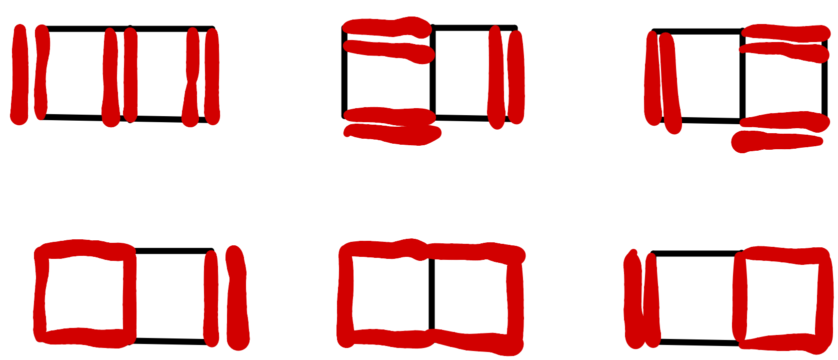
Answer [Musiker-Ovenhouse-Schiffler-Z. 2023]

What are <u>Higher Dimers</u>?

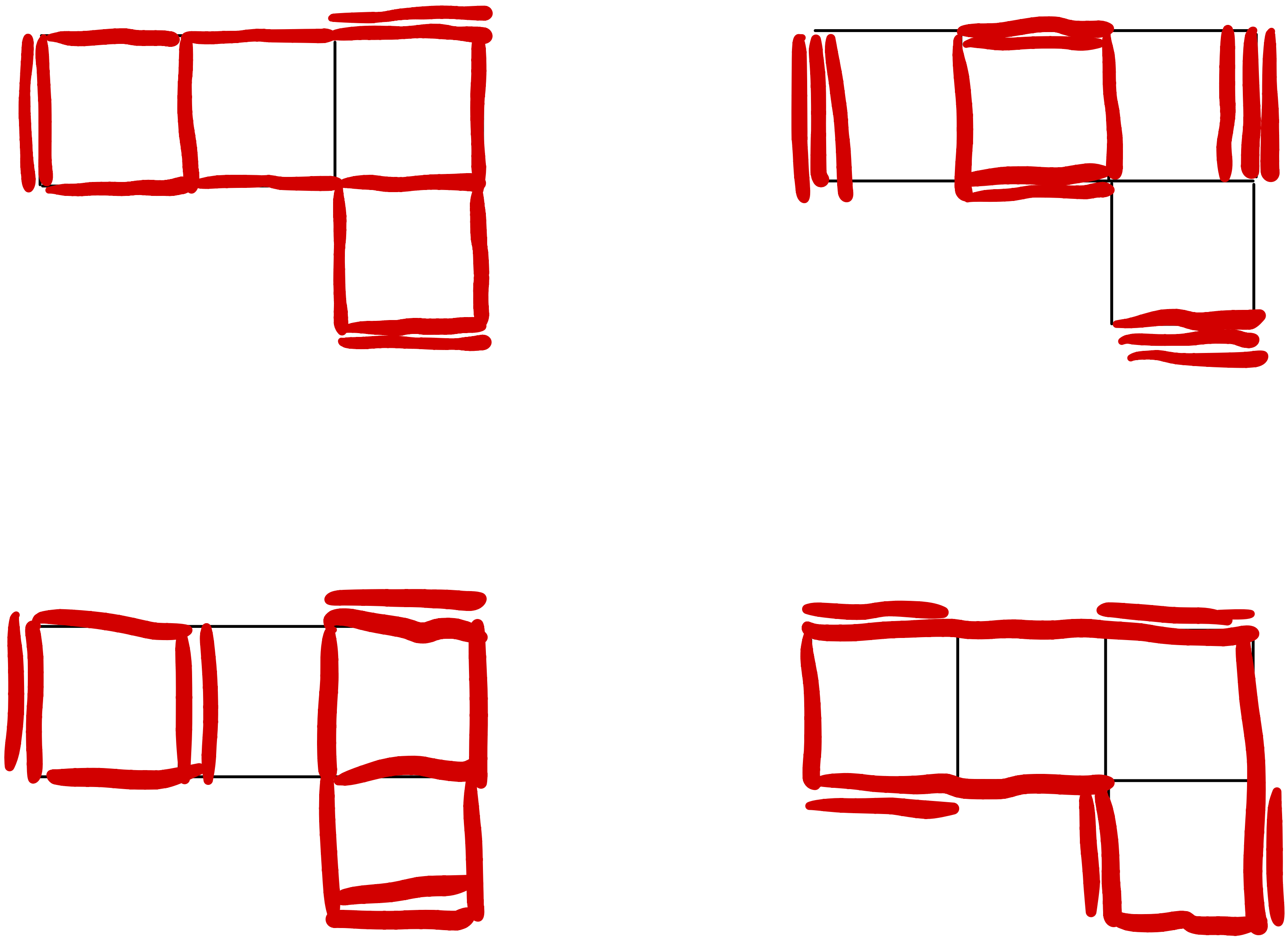
Def An <u>m-dimer cover</u> is a multi-collection of edges such that every vertex is incident to m of them.

i.e. overlay m different perfect matchings (dimers)

Eg. Double dimers on $G[1,1,1]$.



E.g. Some 3-dimers.



Define $\Omega^m[a_1, \cdots, a_n]$ the # of m-dimers on $G[a_1 \cdots a_n]$

Then define $\Omega^m[\underbrace{1, \cdots, 1}_{n}]$ to be the n-th m-dim Fibb #.

In Raney's language : $\begin{pmatrix} \Omega^m[1^{n-1}] \\ \vdots \\ \Omega^m[1^n] \end{pmatrix}$

How to count them ?

Matrix formula upgraded.

$$\Lambda_m(a) := \begin{pmatrix} \genfrac{[}{]}{0pt}{}{a}{m} & \genfrac{[}{]}{0pt}{}{a}{m-1} & \cdots & \genfrac{[}{]}{0pt}{}{a}{m} & 1 \\ \vdots & \vdots & \iddots & \iddots & \\ \genfrac{[}{]}{0pt}{}{a}{3} & \genfrac{[}{]}{0pt}{}{a}{2} & \genfrac{[}{]}{0pt}{}{a}{1} & 1 & \\ \genfrac{[}{]}{0pt}{}{a}{2} & \genfrac{[}{]}{0pt}{}{a}{1} & 1 & & \\ \genfrac{[}{]}{0pt}{}{a}{1} & 1 & & & \\ 1 & & & & \end{pmatrix}$$

where

$$\genfrac{[}{]}{0pt}{}{a}{i} = \left(\!\!\binom{a}{i}\!\!\right) = \binom{a+i-1}{i}$$

Thm [MOSZ23]

$$\Lambda_m(a_1) \cdots \Lambda_m(a_n) = \begin{pmatrix} \Omega^m[a_1 \cdots a_n] & * & * & \cdots & * \\ & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & * & * & \cdots & * \\ \Omega^m[a_2 \cdots a_n] & * & * & \cdots & * \end{pmatrix} := \Lambda_m(a_1 \cdots a_n)$$

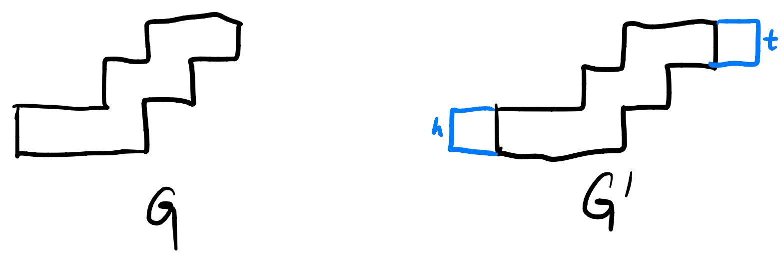
def $L_m = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & & \ddots & & \\ \underbrace{1 & 1 & \cdots & 1 & 1}_{m+1} \end{pmatrix}$ $\quad R_m = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$

Thm [MOSZ23]

$$\Delta_m(a_1 \cdots a_n) = R_m^{a_1} L_m^{a_2} R_m^{a_3} \cdots\cdots$$ "up to sign"

<u>Cor</u> Raney's def of m-dim Fib. #. is the first column of $\Delta_m(a_1 \cdots a_n)$.

Thm [MOSZ23] Let $X = \Lambda_m(a_1)\Lambda_m(a_2)\cdots\Lambda_m(a_n)$ ang $G = \mathcal{G}[a_1 \cdots a_n]$. Let $G' = \mathcal{G}[1, a_1, a_2, \cdots a_n, 1]$ and Call the first and last edge of G' h and t. then,



$$X_{ij} = \#\text{ of } m\text{-dimer covers of } G' \text{ such that}$$

h is matched $m+1-i$ times

t is matched $m+1-j$ times.

Continued fractions

Recall that $[a_1, \cdots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$ has recurrance.

$$[a_1\ a_2 \cdots a_n] = a_1 + \frac{1}{[a_2 \cdots a_n]}$$

m-dim Cont. Frac.

$$[\![a_1 \cdots a_n]\!]^m = \left(r_1(a_1 \cdots a_n), r_2(a_1 \cdots a_n), \cdots, r_m(a_1 \cdots a_n)\right)$$

$$r_k(a_1 \cdots a_n) = \frac{\sum \begin{bmatrix} a_1 \\ i \end{bmatrix} \cdot r_{i+m-k}(a_2 \cdots a_n)}{r_m(a_2 \cdots a_n)}$$

Thm (MOSZ 23)

$$r_m(a_1 \cdots a_n) = \frac{\Omega_m[a_1 \cdots a_n]}{\Omega_m[a_2 \cdots a_n]}$$

and more generally

$$r_i(a_1 \cdots a_n) = \frac{X_{1,m+1-i}}{X_{1,m+1}}$$

where $X = \Delta(a_1 \cdots a_n)$

<u>Cor</u> $[\![1,1,1, \ldots \ldots]\!]^m = (\ell_1, \ell_2, \cdots \ell_n)$

where ℓ_i are lengths of diagonals in a regular $(2m+3)$ gon

Thm For any infinite sequence $a_1 \; a_2 \; a_3 \cdots\cdots$

and any $m \geq 1$ the limit

$$r_m(a_1)\,,\; r_m(a_1\, a_2)\,,\; r_m(a_1, a_2\, a_3) \cdots\cdots$$

is a real number.

Hermite's problem

Find continued fraction algorithm $[a_1, a_2, \cdots]$ which is periodic percisely when it's cubic.

We can't solve Hermite's problem but...

thm When $m=2$ and periodic $a_1 a_2 a_3 \cdots\cdots$

$$\lim_{n\to\infty} r_2(a_1, a_2, \ldots, a_n) \in \mathbb{Q}(\lambda)$$

where λ is a cubic irrational.

More questions.

- Which subset of $\mathbb{R}^m$ is the image of $[\![\cdots]\!]^m$?

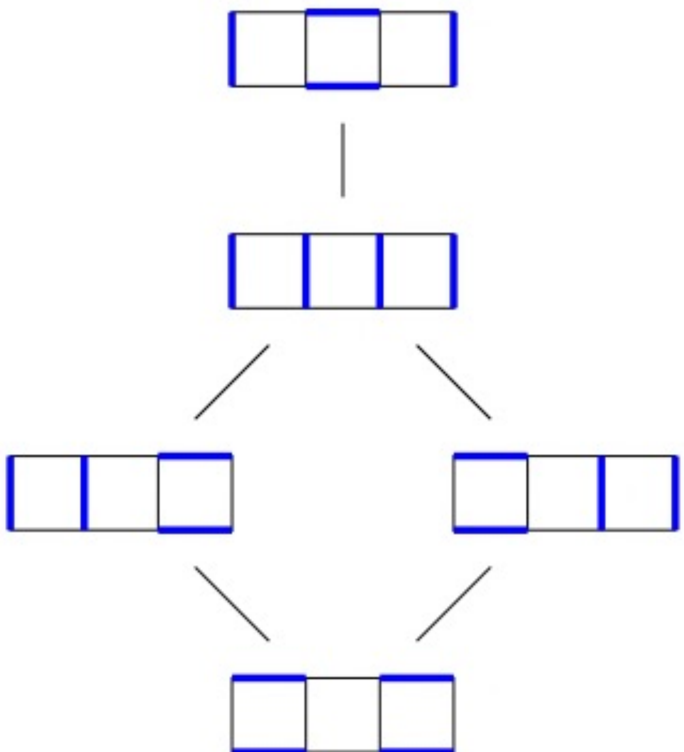
- Conj The map

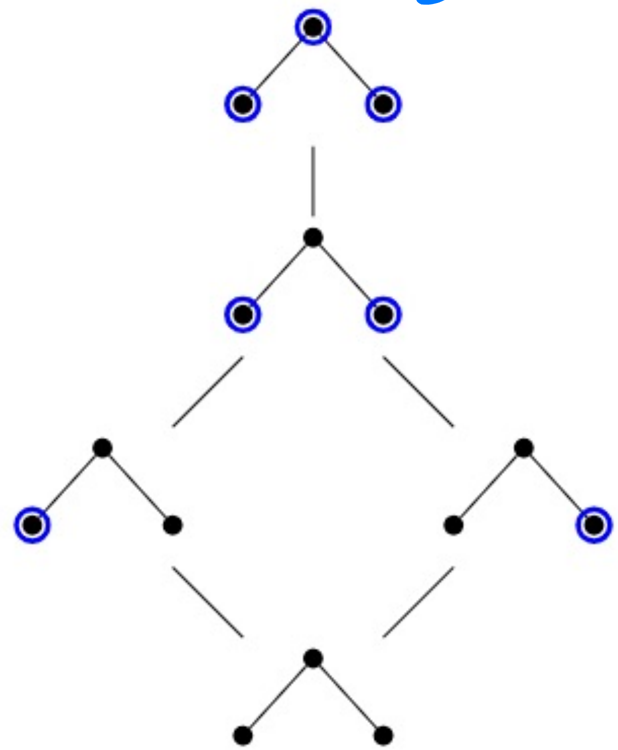
$$[a_1, a_2, \cdots] = x \longmapsto r_m(a_1, a_2 \cdots)$$

is continous and monotonically increasing function.

Distributive Lattice of Dimer covers.

The fence poset

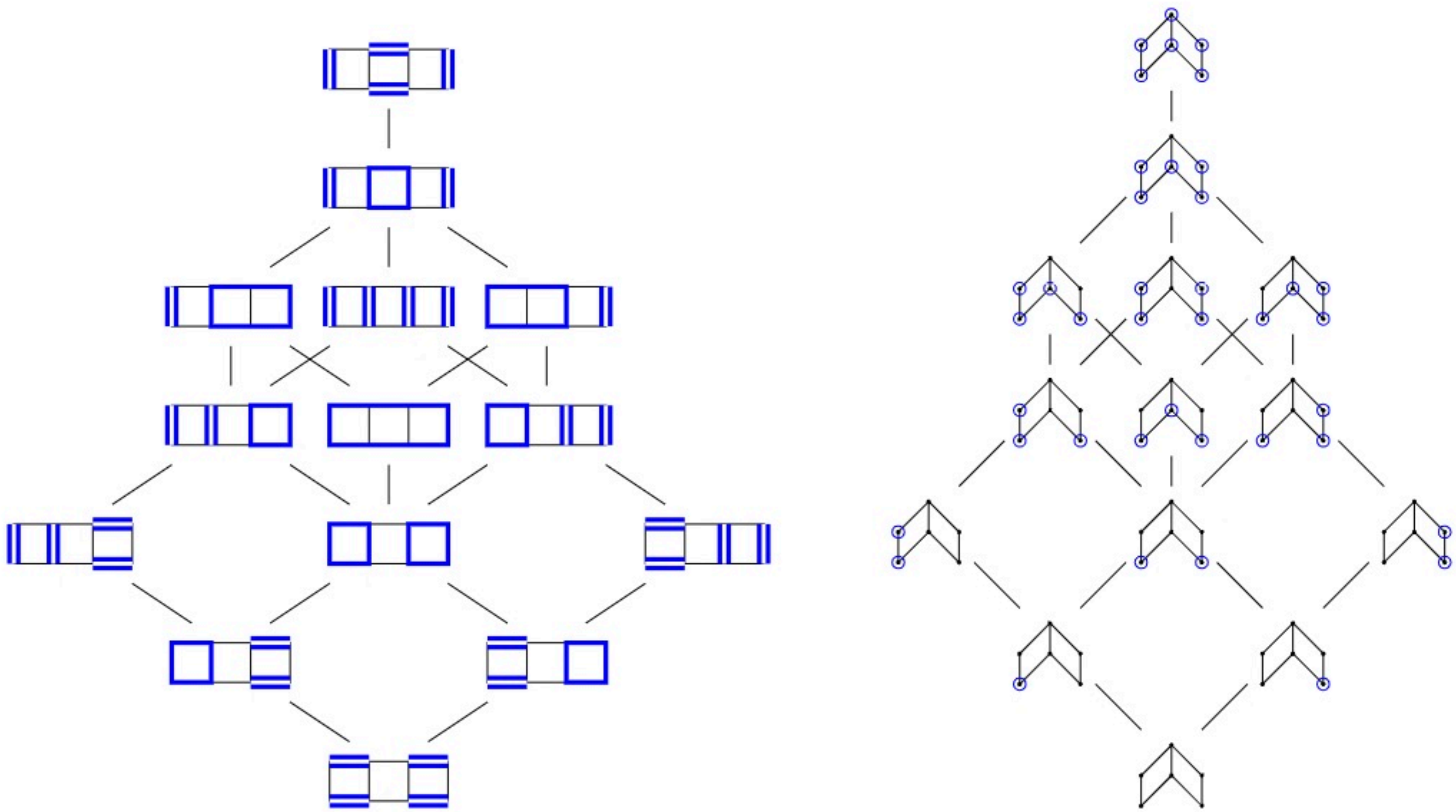




minimal matching

covering relation: $|\ | \leftrightarrow =$

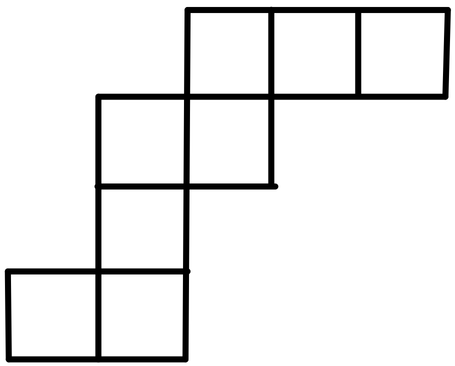
The same story is true for m-dimers.



Denote this poset $\mathcal{P}_G^{(m)}$ for a snake graph G.

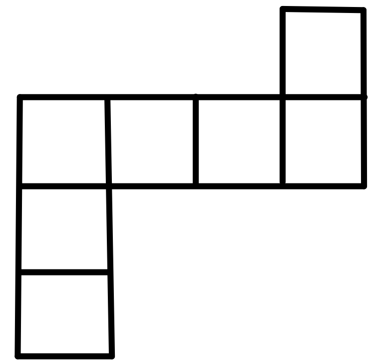
Dual Snake Graph.

G



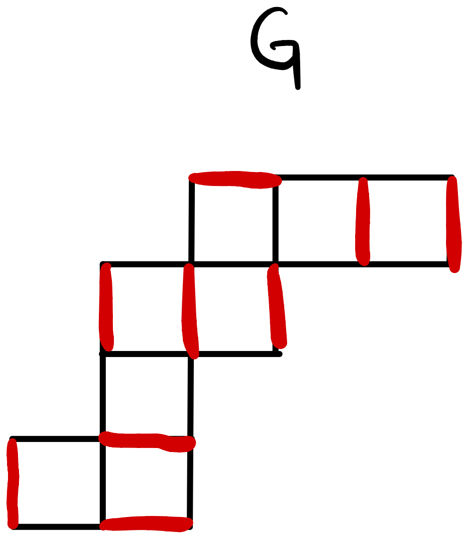
RUURURR

G^*



UURRRRU

Dual snake graph. (Propp, 2005)



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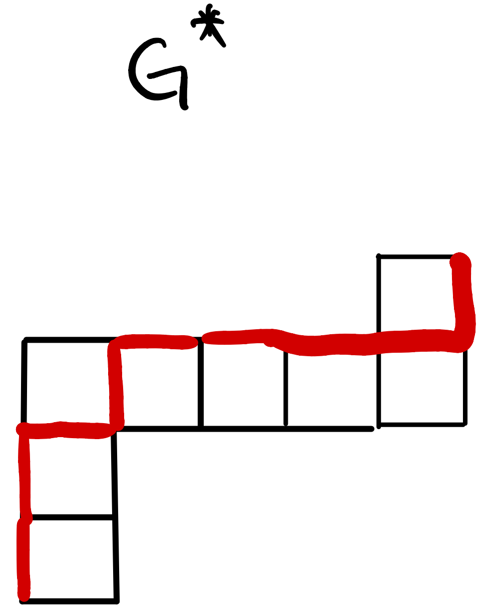
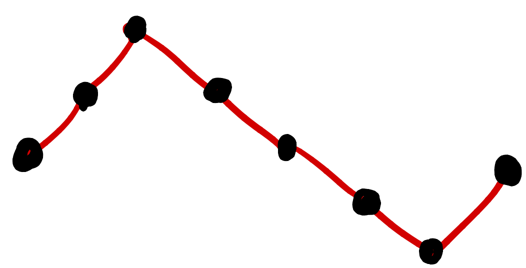


UURRRRU

The fence of G is the "shape" of G^*

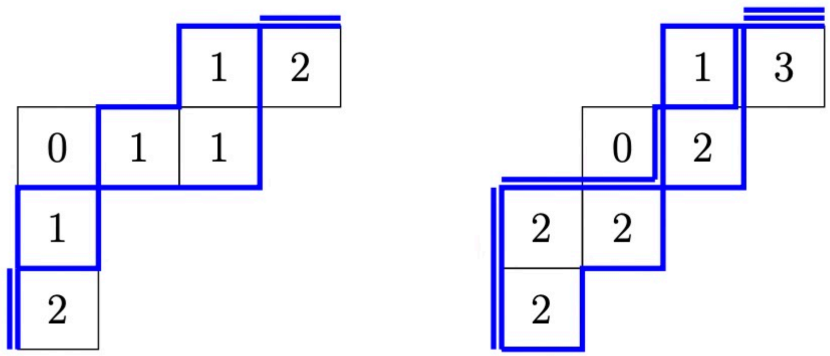


$\Omega^m(G)$ m-dimers of G = m-lattice paths of G^* $\mathcal{L}^m(G^*)$

= reverse plane partitions

= P-partitions of fence poset.

label entries by the # of paths above.



$$\sum_{\ell \in \mathcal{L}(G^*)} \prod x_i^{\# i}$$ is a quasi-symmetric polynomial.

as an instance of the wave Schur functions (Lam-Pylyavskyy)

Thus $\Omega^m(G)$ can be counted by a Jacobi-Trudi formula.

Rank generating functions of $\mathcal{P}_G^{(m)}$:= poset of m-dimers of G.

Let $U_{\mathcal{P}_G^m}(q) := \sum_{p \in \mathcal{P}_G^m} q^{rk(p)}$ Question: Find a formula.

Partial answer by Stanley ↴

Thm (Stanley's thesis Prop 8.3)

$$F_G(q,x) = \sum_{m=1}^{\infty} U_{\mathcal{P}_G^m}(q)\, x^m = \frac{\sum_{\text{linear extensions } \pi} q^{maj(\pi)} x^{des(\pi)}}{(1-x)(1-qx)\cdots(1-q^{N-1})}$$

Question remains: What is $U_{P,m}(q) = \langle F_P(q,x), x^m \rangle$?

Rmk: By Morales Pak Panova 2015 $F_P(q,1) = \sum_{S \text{ pleasant diag}} \prod_{e \in S} \frac{q^{h(e)}}{1-q^{h(e)}}$ using "pleasant diagrams".

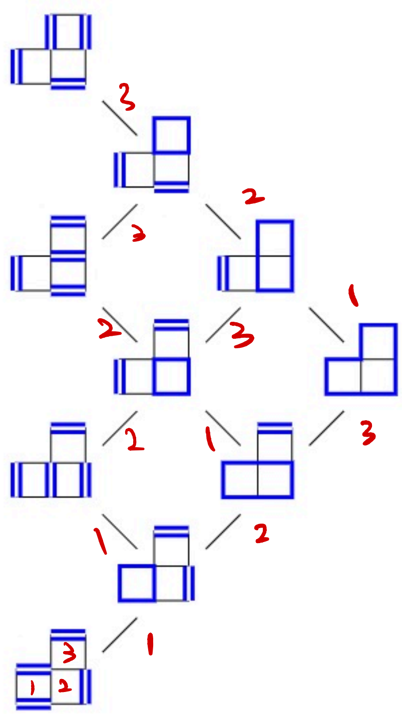
$U_{P_G^1}(q)$ is the numerator of the <u>q-rational</u> number of Morier-Genoud — Ovsienko.

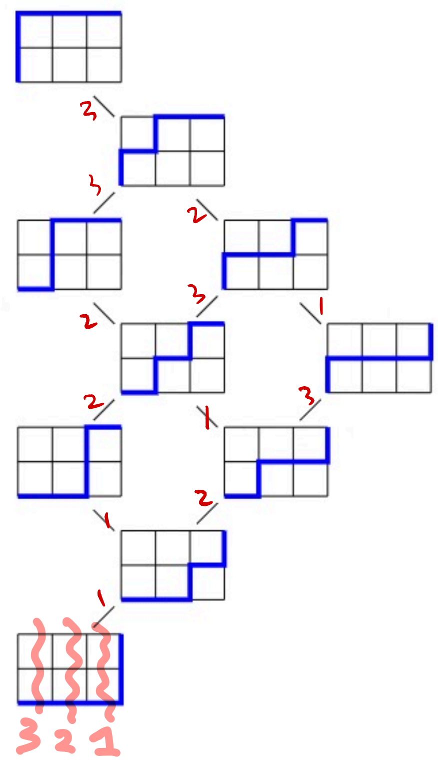
and they conjectured (proved by Oğuz-Ravichandran) that $U_{P_G^1}(q)$ is <u>unimodal</u>.

<u>Conj</u> $U_{P_G^n}(q)$ is unimodal for all n.

A special case, when $G = \mathfrak{g}[n]$

$$U_{P_G^m}(q) = \binom{n+m}{m}_q \quad \text{the } q\text{-binomial coefficient.}$$





$m = 2 \quad n = 3$

$$\sum_p q^{rk(p)} = \binom{5}{2}_q$$

thank you !