Math 206A Final Exam

- The exam is take-home and due on **December 12th at 11:59 PM** (Submit via email).
- Open-book/open-note; you may use any resources (including online) **except** for **AI tools** or **human consultation/collaboration**.
- Any external result or theorem used **must be properly cited**.
- Your solution must be **self-contained**. Any theorem not proven in class **must be proven** in your submission.
- Your Final Score $= \min(100, \text{Your Total Score})$, where $0 \leqslant \text{Your Total Score} \leqslant 200$.[1]

Problem 1 (20pts)**.** Let $\lambda(w)$ denote the corresponding partition of a Grassmanian permutation w. Prove that the reduced words of w are in bijection with Standard Young Tableaux of shape $\lambda(w)$.

Problem 2 (40 pts)**.** Given a partition λ and $j \geqslant 1$, let m_j denote the multiplicity of the parts of j in λ, so that $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots)$

(a) For each $j \in [n]$, prove that

$$(*) \qquad \sum_{\lambda} q^{|\lambda|} m_j(\lambda) = \frac{q^j}{1-q^j} \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

$$(**) \qquad \sum_{\lambda \vdash n} q^{|\lambda|} \cdot \#\{k : m_k \geqslant j\} = \frac{q^j}{1-q^j} \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

(b) Deduce that for fixed j, one has

$$\sum_{\lambda \vdash n} m_j(\lambda) = \sum_{\lambda \vdash n} \#\{k : m_k \geqslant j\}$$

(c) From (b), deduce that for any $n \geqslant 0$, one has

$$\prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i = \prod_{\lambda \vdash n} \prod_{j=1}^{\infty} m_j(\lambda)!$$

(d) From (c), deduce that

$$\prod_{\lambda \vdash n} z_\lambda = \left(\prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i \right)^2$$

[1] This means you don't need to submit all problems, but I do think each of the problems is very interesting to work out :)

Problem 3 (40pts)**.** Consider the basis $b_\lambda = \sum_{\mu \trianglelefteq \lambda, |\mu|=|\lambda|} m_\mu$ of Λ. Does each Schur function s_λ expand non-negatively into this basis?

Problem 4 (40pts)**.** Consider the generating function

$$H(t) = \frac{1}{\sum_{n \geqslant 0} (-1)^n e_n(x) t^n}.$$

Expand $H(t)$ in e_n, h_n and usual x_i variables. (Hint: use the Cauchy identity $\prod_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1-x_i y_j}$.)

Problem 5 (60pts)**.** Denote the trivial representation of S_4 by $\rho_{(4)}$ and the sign representation by $\rho_{(1,1,1,1)}$.

(a) Show that the defining representation (which permutes the coordinates in $\mathbb{C}^4$) decomposes

$$\rho_{def} = \rho_{(4)} \oplus \rho_{(1,1,1,1)}$$

(b) Show that the representation ρ_p which permutes all unordered pairs $(i,j) \in \binom{[4]}{2}$ decomposes

$$\rho_p = \rho_{(4)} \oplus \rho_{(1,1,1,1)} \oplus \rho_{(2,2)}$$

where $\rho_{(2,2)}$ is an irreducible representation of degree 2.

(e) Set $\rho_{(2,1,1)} = \rho_{(1,1,1,1)} \otimes \rho_{(3,1)}$. Show that it's irreducible, and that $\rho_{(4)}, \rho_{(3,1,1)}, \rho_{(2,2)}, \rho_{(2,1,1)}, \rho_{(1,1,1,1)}$ give a complete list of irreducible representations of S_4.

(f) Write down the conjugacy classes and character table for S_4.