

# MATH 4242 Applied Linear Algebra

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## 1. Systems of Linear Equations

A  $m \times n$  system of linear equation is of the form

$$\begin{aligned}a_{11}x_1 + \cdots + a_{n1}x_n &= b_1 \\a_{21}x_1 + \cdots + a_{n2}x_n &= b_n \\&\cdots \quad \cdots \quad \cdots \\a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n\end{aligned}$$

Such equation can be represented using product of matrices.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or by an augmented matrix.

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$$\left[ \begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{m1} & b_1 \\ a_{21} & a_{22} & \cdots & a_{m2} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array} \right]$$

*Definition 1.1.* We have three types of elementary row operations.

- (1) Multiply the  $i$ -th equation (or the  $i$ -th row of the augmented matrix), then add it to the  $j$ -th equation (or the  $j$ -th row of the augmented matrix).
- (2) Permute the equations (or the rows of the augmented matrix)
- (3) Multiply one equation (or one row of the augmented matrix) by a non-zero number.

**1.1. Systems of  $n \times n$  Equations.** Matrices considered in this sections are all  $n \times n$ .

*Definition 1.2.* A matrix is *regular* if it can be turned into a upper triangular matrix such that every entry on the diagonal is non-zero.

**Proposition 1.3.** Let  $E$  be the matrix with 1's on the diagonal and  $E_{ij} = k \neq 0$  is the only other non-zero entry in the lower triangular part. Then for any matrix  $M$ ,  $EM$  is the matrix obtained by multiplying the  $j$ -th row of  $M$  then adding to the  $i$ -th row of  $M$ .

**Proposition 1.4.** A matrix  $A$  is regular if and only if it has an  $LU$  factorization, i.e.

$$A = LU$$

where  $L$  is a lower uni-triangular matrix, and  $U$  is a upper triangular matrix with non-zero diagonal entries.

*Definition 1.5.* Let  $w \in S_n$  be a permutation, then define  $P_w = \{a_{ij}\}$  to be the matrix such that

$$a_{i,j} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 1.6.** For any matrix  $M$ ,  $P_w M$  is the matrix obtained by permuting the rows of  $M$  according to the permutation  $w$ .

*Definition 1.7.* A matrix  $A$  is called *non-singular* if it can be turned into a upper triangular matrix without non-zero diagonal entry via row operations of the first two types.

**Proposition 1.8.** A matrix  $A$  is non-singular if and only if it has a permuted  $LU$  factorization:  $PA = LU$  where  $P$  is some permutation matrix.

*Definition 1.9.* Let  $A = (a_{ij})$ , defined transpose of  $A$  to be  $A^t := (a_{ji})$ .

**Proposition 1.10.** Denote  $A^t$  the transpose of  $A$ . We have that  $AB = (BA)^t$ .

**Proposition 1.11.** A matrix  $A$  is regular iff it admits an  $LDV$  factorization,  $A = LDU$  where  $L$  is lower-unitriangular matrix,  $D$  is a diagonal matrix, and  $U$  is a uni-upper triangular matrix.

*Definition 1.12.* Let  $A$  be an  $n \times n$  matrix. Suppose  $X$  is a matrix such that  $XA = AX = I$  where  $I$  is the identity matrix. Then  $X$  is called the inverse of  $A$  and denoted by  $A^{-1}$ . A matrix is called *invertible* if  $A^{-1}$  exists.

**Proposition 1.13.** A matrix is invertible if and only if it is non-singular.

*Remark 1.14.* Inverse of a matrix can be found using Gauss-Jordan Elimination — see chapter 1 of Olver-Shakiban.

## 1.2. Systems of $m \times n$ Equations.

*Definition 1.15.* A matrix is in row echelon form if it looks like,

$$\begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where  $\bullet$ 's are non-zero entries (called *pivots*) and  $*$  represent generic entries. The pivots are the first non-zero entries in each rows. We require the pivots occupy the first several rows consecutively.

**Proposition 1.16.** *Every matrix can be turned into a row echelon form using elementary row operations of type I and II. In other words, every matrix  $A$  has a factorization  $PA = LU$  where  $P$  is a permutation matrix,  $L$  is a lower uni-triangular matrix, and  $U$  a matrix in row-echelon form.*

*Definition 1.17.* Since every matrix can be turned in to row-echelon form using elementary row operations, we define its *rank* to be the number of pivots.

**Proposition 1.18.** *A square  $n \times n$  matrix is non-singular if its rank is  $n$  (full-rank).*

## 2. Vector Spaces

### 2.1. Some Basic Setup.

*Definition 2.1.*<sup>1</sup> A field is a set  $\mathbb{F}$  with two binary operations  $\times$  (multiplication) and  $+$  (addition), satisfying the following axioms.

- $a + b = b + a$  and  $a \times b = b \times a$  for all  $a, b \in \mathbb{F}$ .
- There exists an additive identity  $0$  such that  $0 + a = a + 0 = a$  for all  $a \in \mathbb{F}$ .
- There exists a multiplication identity  $1$  such that  $1 \times a = a \times 1 = a$  for all  $a \in \mathbb{F}$ .
- For every  $a \in \mathbb{F}$ , there exists an element denoted  $-a$ , such that  $a + (-a) = 0$ .
- $0 \neq 1$ .
- For every  $a \in \mathbb{F}$  and  $a \neq 0$ , there exists an element denoted  $a^{-1}$ , such that  $a \times (a^{-1}) = 1$ .
- For every  $a, b, c \in \mathbb{F}$ ,  $a \times (b + c) = ab + ac$ .

For most part of this class, we will take  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}$ .

*Definition 2.2.* For a field  $\mathbb{F}$ , denote  $\mathbb{F}[x]$  the ring<sup>2</sup> of polynomials over  $\mathbb{F}$ .

$$\mathbb{F}[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_0, \dots, a_n \in \mathbb{F}, n \geq 0, x^m x^n = x^{m+n}\}$$

**Proposition 2.3.** *Every polynomial  $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$  with complex coefficient has at least one complex solution. Note that this is not true for real polynomials.*

*Definition 2.4.* A field  $\mathbb{F}$  is called *algebraically closed* if every polynomial in  $\mathbb{F}[x]$  has a solution in  $\mathbb{F}$ . (By Proposition 2.3,  $\mathbb{C}$  is algebraically closed).

<sup>1</sup>You don't need to worry too much about the abstract structures of a field. The purpose of this definition is to make everything self-contained. You can basically think of a field as a set on which you can do some sort of arithmetic.

<sup>2</sup>A ring is a field, where multiplication need not to be commutative, and multiplicative identity ( $0$ ) need not exists.

**Proposition 2.5.** *The field of complex numbers  $\mathbb{C}$  is the algebraic closure of  $\mathbb{R}$ . In other words,  $\mathbb{C}$  is the smallest algebraically closed field that contains  $\mathbb{R}$ .*

## 2.2. Vector Spaces and Subspaces.

Let  $\mathbb{F}$  be a field.

*Definition 2.6.* A set  $V$  is called a vector space over  $\mathbb{F}$  if there exists an addition map

$$add : V \times V \rightarrow V$$

and a scalar multiplication map

$$mult : \mathbb{F} \times V \rightarrow V$$

(Here  $\times$  denote the Cartesian product of sets<sup>3</sup>.) We will abbreviate them by  $a(v_1, v_2) = v_1 + v_2$  and  $mult(a, v) = av$ .

Note that this definition (implicitly) requires that a vector space  $V$  is closed under addition and scalar multiplication, i.e.  $v_1 + v_2 = add(v_1, v_2) \in V$  and  $av = mult(a, v) \in V$ .

Elements of a vector spaces are called *vectors*.

*Definition 2.7.* Let  $V$  be a vector space over  $\mathbb{F}$ . A subset  $U$  of  $V$  is a *subspace* if it is closed under addition and scalar multiplication. (In other words, a subspace is a subset that is a vector space itself.)

*Definition 2.8.* Let  $U_1, \dots, U_m$  be subspaces of  $V$ . Then define their sum to be

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m \mid u_1 \in U_1, \dots, u_m \in U_m\}$$

**Proposition 2.9.** *Let  $U_1, \dots, U_m$  be subspaces of  $V$ . Then  $U_1 + \dots + U_m$  is also a subspace of  $V$ , furthermore, it's the smallest subspace of  $V$  that contain all of  $U_1, \dots, U_m$ .*

*Definition 2.10.* A sum of subspaces  $U_1 + \dots + U_m$  of  $V$  is a *direct sum* if every vector  $v \in U_1 + \dots + U_m$  can be uniquely written as  $v = u_1 + \dots + u_m$  where  $u_i \in U_i$  for each  $i$ . When a summation is direct, we denote it as  $U_1 \oplus \dots \oplus U_m$ .

## 2.3. Linear Combination, Span, and Dimension.

Let  $V$  be a vector space over  $\mathbb{F}$ .

*Definition 2.11.* Let  $v_1, v_2, \dots, v_n \in V$ , a vector  $v \in V$  is a linear combination of  $\{v_1, \dots, v_n\}$  if there exists  $a_1, \dots, a_n \in \mathbb{F}$  such that

$$v = a_1v_1 + \dots + a_nv_n$$

*Definition 2.12.* Let  $v_1, v_2, \dots, v_n$  be a list of vectors in  $V$ , define their span to be the set of all linear combinations of  $v_1, \dots, v_n$ .

$$\text{span}(v_1, \dots, v_n) = \{a_1v_1 + \dots + a_nv_n \mid a_1, \dots, a_n \in \mathbb{F}\}$$

**Proposition 2.13.** *For a list of vectors  $v_1, \dots, v_n \in V$ ,  $\text{span}(v_1, \dots, v_n)$  is a subspace of  $V$ . Furthermore, it's the smallest subspace containing all of  $v_1, \dots, v_n$ .*

*Definition 2.14.* A vector space  $V$  is said to be *finite dimensional* if it is the span of a finitely many vectors.

*Definition 2.15.*  $v_1, \dots, v_m \in V$  are *linearly independent* if the only way to write 0 as a linear combination of  $v_1, \dots, v_n$  is

$$0 = 0v_1 + 0v_2 + \dots + 0v_n.$$

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<sup>3</sup>For sets  $A$  and  $B$ , defined  $A \times B = \{(a, b) \mid a \in A, b \in B\}$

**Proposition 2.16.**  $v_1, \dots, v_m \in V$  are linearly independent if and only if any vector  $v \in \text{span}(v_1, \dots, v_m)$  can be uniquely written as a linear combination of  $v_1, \dots, v_m$ .

**Definition 2.17.** A list of vectors  $v_1, \dots, v_n$  is a basis of  $V$  if

- $V = \text{span}(v_1, \dots, v_n)$
- $v_1, \dots, v_n$  are linearly independent.

**Proposition 2.18.**  $v_1, \dots, v_n$  is a basis of  $V$  iff every vector  $v \in V$  can be uniquely written as a linear combination of  $v_1, \dots, v_n$ .

**Lemma 2.19.** Let  $v_1, \dots, v_m \in V$  be a list of vectors that spans  $V$ , i.e.  $\text{span}(v_1, \dots, v_m) = V$ . Then  $\{v_1, \dots, v_m\}$  can be reduced to a basis of  $V$ . In other words, there exists a basis  $\{w_1, \dots, w_n\}$  of  $V$  such that  $w_i \in \{v_1, \dots, v_m\}$  for all  $i$  and  $n \leq m$ .

**Lemma 2.20.** Let  $v_1, \dots, v_k \in V$  be linearly independent. Then there exists a basis of  $V$  in the form

$$\{v_1, \dots, v_k, w_1, \dots, w_m\}$$

Note that it's possible that  $m = 0$ , in the case when  $\{v_1, \dots, v_k\}$  is already a basis.

**Corollary 2.21.** If  $U$  is a subspace of  $V$ , then there exists another subspace  $W$  such that  $V = U \oplus W$ .

**Proposition 2.22.** If  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is another basis of  $V$ . Then  $n = m$ .

**Definition 2.23.** Define the dimension of a vector space to be the size of its basis.

**Proposition 2.24.** If  $\{v_1, \dots, v_n\}$  linearly independent and  $n = \dim(V)$ , then  $\{v_1, \dots, v_n\}$  is a basis.

**Proposition 2.25.** If  $U$  is a subspace of  $V$ , then  $\dim(U) \leq \dim(V)$ . Furthermore,  $\dim(U) = \dim(V)$  iff  $U = V$ .

**Proposition 2.26.** If  $\text{span}(v_1, \dots, v_n) = V$  and  $n = \dim(V)$ , then  $\{v_1, \dots, v_n\}$  is a basis.

**Theorem 2.27.** Let  $V$  be a finite dimensional vector space and  $V_1, V_2$  subspaces. Then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

**Corollary 2.28.** If  $V_1 + V_2$  is a direct sum, then  $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$ .<sup>4</sup>

### 3. Linear Maps and Matrices

**3.1. Linear Maps.** Let  $V, W$  be vector spaces over  $\mathbb{F}$ .

**Definition 3.1.** A map  $T : V \rightarrow W$  is linear if

- (1)  $T(u + v) = T(u) + T(v)$  for all  $u, v \in V$ .
- (2)  $T(\lambda v) = \lambda T(v)$  for all  $\lambda \in \mathbb{F}$  and  $v \in V$ .

**Definition 3.2.** We denote the set of all linear maps from  $V \rightarrow W$  by  $\text{Hom}(V, W)$ . And define  $\text{End}(V) = \text{Hom}(V, V)$ .

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<sup>4</sup>We will see later that the converse is also true.

**Lemma 3.3.** Let  $v_1, \dots, v_n$  be a basis for  $V$  and  $w_1, \dots, w_n$  a basis for  $W$  (i.e.  $V, W$  same dimension). Then there exists a unique linear map  $T \in \text{Hom}(V, W)$  such that  $T(v_i) = w_i$  for all  $i$ . The map is given by  $T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$ .

**Proposition 3.4.** The set  $\text{Hom}(V, W)$  is a vector space over  $\mathbb{F}$ , with addition and scalar multiplication given as follows.

$$\begin{aligned}(\varphi + \psi)(v) &:= \varphi(v) + \psi(v) \\ (\lambda\varphi)(v) &:= \lambda\varphi(v)\end{aligned}$$

**Lemma 3.5.** Let  $T \in \text{Hom}(V, W)$ , then  $T(0_V) = 0_W$ .

Let  $T \in \text{Hom}(V, W)$ .

**Definition 3.6.** The kernel (or null space) of  $T$  is  $\text{Ker}(T) = \{v \in V : Tv = 0\}$

**Proposition 3.7.**  $\text{Ker}(T)$  is a subspace of  $V$ .

**Proposition 3.8.**  $\text{Ker}(T) = \{0\}$  if and only if  $T$  is injective.

**Definition 3.9.** The image (or range) of  $T$  is  $\text{Img}(T) = \{Tv | v \in V\}$

**Proposition 3.10.**  $\text{Img}(T)$  is a subspace of  $W$ .

**Proposition 3.11.**  $T$  is surjective iff  $\text{Img}(T) = W$ .

**Theorem 3.12.**  $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Img}(T))$ .

**Proposition 3.13.** (1) if  $\dim(V) > \dim(W)$ , then any  $T \in \text{Hom}(V, W)$  is not injective.  
 (2) if  $\dim(V) < \dim(W)$ , then any  $T \in \text{Hom}(V, W)$  is not surjective.  
 (3) if there exists a bijective  $T \in \text{Hom}(V, W)$ , then  $\dim(V) = \dim(W)$ .

**3.2. Matrices from Linear Maps.** Denote the set of all  $m \times n$  matrix with entries in  $\mathbb{F}$  by  $M_{m \times n}(\mathbb{F})$ . Let  $V, W$  be finite dimensional vector spaces over  $\mathbb{F}$ .

**Definition 3.14.** Suppose  $V$  has basis  $v_1, \dots, v_n$  and  $W$  has basis  $w_1, \dots, w_m$ . Let  $T \in \text{Hom}(V, W)$ . Then define  $\mathcal{M}(T)$  to be the matrix  $[a_{ij}]$  such that

$$T(v_k) = a_{1k}w_1 + a_{2k}w_2 + \dots + a_{mk}w_m.$$

**Remark 3.15.** Note that the usage of  $\mathcal{M}$  requires a choice of basis for  $V$  and  $W$ . In general we shall denote  $\mathcal{M}_{B_1, B_2}(T)$  where  $B_1$  is the basis for  $V$  and  $B_2$  the basis for  $W$ . However in most case we will omit the subscript when the context is clear.

**Proposition 3.16.** Let  $S, T \in \text{Hom}(V, W)$ ,  $\mathcal{M}(S) + \mathcal{M}(T) = \mathcal{M}(S + T)$

Let  $S \in \text{Hom}(U, W)$  and  $T \in \text{Hom}(V, U)$ , then  $\mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$ .

**Definition 3.17.** For any  $A \in M_{m \times n}$ , Let  $A_{\bullet, k}$  denote the  $k$ -th column vector and  $A_{k, \bullet}$  denote the  $k$ -th row vector.

**Proposition 3.18.**  $(AB)_{\bullet, k} = A(B_{\bullet, k})$

**Theorem 3.19.** For any  $A \in M_{m \times n}$ , we have

$$\dim(\text{span}(A_{1, \bullet}, \dots, A_{m, \bullet})) = \dim(\text{span}(A_{\bullet, 1}, \dots, A_{\bullet, n}))$$

**Proposition 3.20.** The dimension of column span or row span of a matrix equals to its rank (see Definition 1.17).

**Definition 3.21.** A linear map  $T \in \text{Hom}(V, W)$  is invertible if there exists a linear map  $S \in \text{Hom}(W, V)$  such that  $TS = \text{id}_W$  and  $ST = \text{id}_V$ .

**Proposition 3.22.** If a map  $T$  is invertible, then its inverse is unique, denote it by  $T^{-1}$ .

**Proposition 3.23.** A map  $T$  is invertible iff  $\mathcal{M}(T)$  is non-singular Definition 1.7. Furthermore,  $\mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$ .

**Definition 3.24.** A linear map is an *isomorphism* if it's invertible. Two vector spaces are *isomorphic* if there exists an isomorphism between them.

**Theorem 3.25.** Two vector spaces over  $\mathbb{F}$  is isomorphic if and only if they have the same dimension. (In other words, vector spaces are classified by  $\mathbb{N}$ )

**Corollary 3.26.** Let  $\dim V = n$  and  $\dim W = m$ . The vector space  $\text{Hom}(V, W) \cong M_{m \times n}(\mathbb{F})$  are isomorphic, with the map  $\mathcal{M}$  being the isomorphism.

**Theorem 3.27.** Let  $V$  be a vector space with basis  $B_1 = \{v_1, \dots, v_n\}$ . Suppose it has another basis  $B_2 = \{w_1, \dots, w_n\}$ . Let  $C = \mathcal{M}_{B_1, B_2}(\text{id})$  where  $\text{id} \in \text{Hom}(V, V)$  is the identity map. Then change of basis corresponds to conjugation by  $C$ .

In particular, let  $T \in \text{Hom}(V, V)$  and  $A = \mathcal{M}_{B_1, B_1}(T)$  and  $B = \mathcal{M}_{B_2, B_2}(T)$ . Then we have

$$A = C^{-1}BC$$

### 3.3. Quotient and Dual spaces.

**Definition 3.28.** Let  $v \in V$  and  $U \subseteq V$ . Define  $v + U := \{v + u | u \in U\}$ . This is called a *coset*.

**Definition 3.29.** Let  $U \subseteq V$ . Define the quotient space  $V/U$  to be  $\{v + U | v \in V\}$ , with addition and scalar multiplication given by

$$\begin{aligned} (v_1 + U) + (v_2 + U) &= (v_1 + v_2) + U \\ \lambda(v + U) &= \lambda v + U \end{aligned}$$

**Definition 3.30** (alternative definition). Let  $\sim$  be an equivalence relation on  $V$ . Define  $[v]_{\sim} := \{u \in V | u \sim v\}$  the equivalence class generated by  $v$ . Then we can define quotient space  $V/\sim := \{[v]_{\sim} | v \in V\}$ .

**Remark 3.31.** For  $U \subset V$ , define an equivalence relation  $\sim_U$  by  $v \sim_U u \iff v - u \in U$ . Then Definitions 3.29 and 3.30 agree, i.e.  $V/U = V/\sim_U$ .

**Definition 3.32.** For  $U \subset V$ , define the quotient map  $\pi : V \rightarrow V/U$  by  $\pi(v) = v + U$ . Note that  $\text{Ker}(\pi) = U$ .

**Proposition 3.33.**  $\dim V/U = \dim V - \dim U$ .

**Theorem 3.34.** For any  $T \in \text{Hom}(V, W)$ , define  $\tilde{T} \in \text{Hom}(V/\text{Ker}(T), W)$  by  $\tilde{T}(v + \text{Ker}(T)) = Tv$ . Then  $\tilde{T}\pi = T$ , and defines an isomorphism between  $V/\text{Ker}(T)$  and  $\text{Im}(T)$ .

**Definition 3.35.** A linear map from  $V$  to  $\mathbb{F}$  is called a *linear functional*. Denote  $\text{Hom}(V, \mathbb{F})$  the set of all linear functionals on  $V$ .

**Proposition 3.36.**  $\text{Hom}(V, \mathbb{F})$  is a vector space, with addition and multiplication given by  $(f + g)(v) = f(v) + g(v)$  and  $(\lambda f)(v) = \lambda f(v)$ . This is called the dual space of  $V$ , and is denoted by  $T^*$ .

**Proposition 3.37.**  $\dim V = \dim V^*$ .

*Definition 3.38.* Let  $v_1, \dots, v_n$  be a basis of  $V$ . Then define  $v_i^* \in V^*$  to be the linear functional  $v_i^*(v_j) = \delta_{i,j}$ <sup>5</sup>. For any  $v = a_1v_1 + \dots + a_nv_n \in V$ , define  $v^* = a_1v_1^* + \dots + a_nv_n^*$ .

**Proposition 3.39.** Let  $v_1, \dots, v_n$  be a basis. Then  $v = v_1^*(v)v_1 + \dots + v_n^*(v)v_n$  for all  $v \in V$ .

**Proposition 3.40.**  $v_1^*, \dots, v_n^*$  is a basis for  $V^*$ .

*Definition 3.41.* Suppose  $T \in \text{Hom}(V, W)$ . Define the *dual linear map*  $T^* \in \text{Hom}(W^*, V^*)$  to be

$$T^*(f) = f \circ T$$

**Proposition 3.42.** •  $(S + T)^* = S^* + T^*$

- $(\lambda T)^* = \lambda T^*$ .
- $(ST)^* = T^*S^*$ .

*Definition 3.43.* For any subspace  $U \subseteq V$ , define its annihilator  $U^0 := \{f \in V^* : f(u) = 0 \text{ for all } u \in U\}$ .

**Proposition 3.44.**  $U^0$  is a subspace of  $V^*$ .

**Proposition 3.45.**  $\dim U^0 = \dim V - \dim U$ . Recall that this is also the dimension of  $V/U$ . In particular, there is an isomorphism  $(V/U)^* \cong U^0$  given by  $\pi^*$ .

**Proposition 3.46.** (a)  $U^0 = \{0\} \iff U = V$  (b)  $U^0 = V^* \iff U = \{0\}$ .

**Theorem 3.47.** (a)  $(\text{Im } T)^0 = \text{Ker } T^*$  (b)  $(\text{Ker } T)^0 = \text{Im } T^*$

**Corollary 3.48.**  $T$  is injective iff  $T^*$  is surjective.  $T$  is surjective iff  $T^*$  is injective.

**Theorem 3.49.** Let  $T \in \text{Hom}(V, W)$ , and  $T^* \in \text{Hom}(W^*, V^*)$ . Then  $\mathcal{M}(T)^t = \mathcal{M}(T^*)$ .

## 4. Inner Product Spaces

Throughout this section, let  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ .

### 4.1. Inner Products and Norms.

*Definition 4.1.* The dot product of two vectors in  $\mathbb{R}^n$  is a map from  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{F}$ , defined by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1y_1 + \dots + x_ny_n$$

*Definition 4.2.* The dot product of two vectors in  $\mathbb{C}^n$  is a map from  $\mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{F}$ , defined by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1\bar{y}_1 + \dots + x_n\bar{y}_n$$

where  $\overline{a + bi} = a - bi$  is the complex conjugate.

*Definition 4.3.* Let  $V$  be vector space over  $\mathbb{F}$  ( $\mathbb{C}$  or  $\mathbb{R}$ ). A inner product on  $V$  is a map  $V \times V \rightarrow \mathbb{F}$  which sends  $(v, u)$  to  $\langle v, u \rangle$  such that

- (1)  $\langle v, v \rangle \geq 0$ .
- (2)  $\langle v, v \rangle = 0 \iff v = 0$ .
- (3)  $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- (4)  $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ .

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<sup>5</sup>Here  $\delta_{i,j}$  is the Kronecker delta symbol:  $\delta_{i,j} = 1$  if  $i = j$  and 0 otherwise.



$$(5) \langle u, v \rangle = \overline{\langle v, u \rangle}^6$$

**Proposition 4.4** (Bilinearity). *A inner product  $\langle \cdot, \cdot \rangle$  satisfy  $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$  and  $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ .*

*Remark 4.5.* A pairing satisfying (4) and (6) of Definition 4.3 and Proposition 4.4 together is known as being *bilinear*. Usually a inner product is defined to be bilinear, however, as we see here one-sided linearity is enough to imply bilinearity.

**Proposition 4.6.** *An inner product  $\langle \cdot, \cdot \rangle$  on  $V$  satisfy*

- (1) *Fix any  $u \in V$ , the map  $v \mapsto \langle u, v \rangle$  is a linear functional.*
- (2)  *$\langle v, 0 \rangle = 0 = \langle 0, v \rangle$  for any  $v \in V$ .*

*Definition 4.7.* Given an inner product  $\langle \cdot, \cdot \rangle$ , define the norm  $\| \cdot \|$  to be the positive square-root  $\|v\| = \sqrt{\langle v, v \rangle}$ .

**Proposition 4.8.** *Let  $I = [a, b] \subset \mathbb{R}$  be an closed interval on  $\mathbb{R}$ . Let  $V = C^0(I)$  denote all continuous  $\mathbb{R}$ -valued functions defined on  $I$  (domain is  $I$ ). Then*

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

*defines an inner product on  $V$ . The norm  $\|f\| = \sqrt{\langle f, f \rangle}$  is called the  $L_2$  norm on  $C^0(I)$ .*

*Definition 4.9.* For  $z \in \mathbb{C}$ , define the complex modulus to be  $|z| = \sqrt{z\bar{z}}$ . Note that when  $z$  is real (i.e. no imaginary part), then  $|z|$  is the absolute value.

**Theorem 4.10** (Cauchy-Schwartz inequality). *Let  $V$  be an inner product space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Then for any  $u, v \in V$ , we have*

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

*Moreover, the equality occurs only when  $u, v$  are linearly independent.*

*Remark 4.11.* The Cauchy-Schwartz inequality tells us that the ratio  $\frac{|\langle u, v \rangle|}{\|u\| \|v\|}$  is in between  $-1$  and  $1$ . Therefore we can define the ‘abstract’ angle between two vectors  $v, u$  to be

$$\theta_{u,v} = \arccos \frac{|\langle u, v \rangle|}{\|u\| \|v\|}$$

*Definition 4.12.* We say two vectors  $u, v$  are orthogonal if  $\langle u, v \rangle = 0$ .

**Proposition 4.13.** *If  $v, u$  orthogonal in  $V$ , then  $\|v\|^2 + \|u\|^2 = \|v + u\|^2$ .*

*Remark 4.14.* Definition 4.12 generalizes the usual notion of orthogonality in  $\mathbb{R}^2$  in a sense that when two vectors are orthogonal, then the angle between them is  $\theta_{u,v} = \pi/2$ .

**Theorem 4.15.** *Let  $V$  be an inner product space, and  $u, v \in V$ . Then  $\|u + v\| \leq \|u\| + \|v\|$ .*

*Definition 4.16.* We can define norms more generally without requiring an inner product. A norm on  $V$  is a map  $\| \cdot \| : V \rightarrow \mathbb{R}_{\geq 0}$  such that

- $\|v\| \geq 0$  and  $\|v\| = 0$  only when  $v = 0$ .
- $\|\lambda v\| = |\lambda| \|v\|$
- $\|v + u\| \leq \|v\| + \|u\|$

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<sup>6</sup>When  $\mathbb{F} = \mathbb{R}$ , the ‘complex’ conjugate of a real number is just itself.

**Proposition 4.17.** Let  $K = (k_{ij})$  be the matrix whose entries are the inner product of the basis vectors, i.e.  $k_{ij} = \langle e_i, e_j \rangle$ . Then for  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  we have

$$\langle x, y \rangle = \left\langle \sum_i x_i e_i, \sum_j y_j e_j \right\rangle = \sum_{i,j} x_i y_j \langle e_i, e_j \rangle = x^t K y$$

*Definition 4.18.* A  $n \times n$  matrix is positive-definite if  $K^t = K$  and satisfy  $x^t K x > 0$  for all  $0 \neq x \in \mathbb{F}^n$ . More generally, we say  $K$  is positive semi-definite if  $K^t = K$  and  $x^t K x \geq 0$  for all  $x$ .

**Theorem 4.19.** Every inner product is given by  $\langle x, y \rangle = x^t K y$  where  $K$  is a positive-definite matrix.

**Proposition 4.20.** Positive-definite matrices are non-singular (invertible).

*Definition 4.21.* Given any  $v_1, \dots, v_n \in V$ , we define the Gram matrix to be  $K = (k_{ij})$  where  $k_{ij} = \langle v_i, v_j \rangle$ . In particular, Let  $A$  be the matrix whose column vectors are  $v_1, \dots, v_n$ , then the Gram matrix is  $K = A^t C A$ , where  $C$  is the symmetric positive definite matrix defining the inner product.

**Proposition 4.22.** A Gram matrix is always positive semi-definite. A Gram matrix is positive-definite if and only if  $v_1, \dots, v_n$  are linearly independent.

**Proposition 4.23.** Let  $A$  be an  $m \times n$  matrix ( $m \geq n$ ), then TFAE:

- $K = A^t A$  is positive-definite;
- $\text{Ker}(A) = 0$ ;
- $A$  has linearly independent columns;
- $\text{rank}(A) = n$ .

**Theorem 4.24.** Suppose  $A \in M_{m \times n}$  and  $K = A^t A$  is positive-definite. Then for any symmetric positive definite matrix  $C \in M_{m \times m}$ , the matrix  $K' = A^t C A$  is also positive-definite.

**Proposition 4.25.** For  $K = A^t C A$ , we have  $\text{Ker}(K) = \text{Ker}(A)$ , and hence  $\text{rank}(K) = \text{rank}(A)$ .

## 4.2. Orthonormal Basis.

*Definition 4.26.* Let  $V$  be a real or complex inner product space. A basis  $v_1, \dots, v_n$  of  $V$  is called *orthogonal* if  $\langle v_i, v_j \rangle = \delta_{i,j}$  for all  $i$ . An orthogonal basis of unit vectors is called *orthonormal*.

**Proposition 4.27.** Let  $v_1, \dots, v_n \in V$  be pair-wise orthogonal, then they must be linearly independent.

**Corollary 4.28.** Let  $v_1, \dots, v_n \in V$  be pair-wise orthogonal, then they form a basis for  $\text{span}(v_1, \dots, v_n)$ .

**Theorem 4.29.** Given any basis  $w_1, \dots, w_n$  of  $V$ , one can construct an orthogonal basis  $v_1, \dots, v_n$  using the Gram-Schmidt process:

- $v_1 = w_1$ .
- $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$

- $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$
- $\dots$
- $v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i$
- $\dots$ .

**Corollary 4.30.** *Every finite dimensional inner-product space has an orthonormal basis.*

*Definition 4.31.* A matrix  $A$  is orthogonal if  $A^t A = I = A A^t$ , or equivalently  $A^t = A^{-1}$ .

**Proposition 4.32.** *A matrix is orthogonal if and only if its column vectors form an orthonormal basis of  $\mathbb{F}^n$  w.r.t the dot product.*