

Math 2263 Problem Sets

Sylvester W. Zhang

Spring 2022

CONTENTS

1. Vectors and the Three-Dimensional Space (12.1 – 12.3)	2
2. Cross Product, Lines and Planes (12.4 – 12.5)	4
3. Multivariable Functions, Limits and Partial Derivatives (14.1 - 14.3)	6
4. Chain Rule and Directional Derivatives	9
A. Additional Problems I	11
5. Maxima and Minima (14.7)	12
6. Lagrange Multipliers	15
7. Basic Double Integrals	17
8. More on Double Integrals	18
9. Double Integral with Polar Coordinates	19
10. Triple integrals	20
11. Cylindrical, spherical coordinates, and change of variables.	22
B. Additional Problems II	23

1. Vectors and the Three-Dimensional Space (12.1 – 12.3)

Problem 1.1. Determine if the given three points are co-linear (i.e. lie on one line).

(1) $A = (2, 0, -1)$, $B = (1, -1, -2)$ and $C = (-3, 1, 0)$

(2) $A = (-1, 4, 3)$, $B = (-2, 4, 1)$ and $C = (2, 0, 1)$

Solution. Three points A, B, C are co-linear if and only if the two vectors \overrightarrow{AB} and \overrightarrow{BC} have the same direction (or equivalently, \overrightarrow{AB} and \overrightarrow{AC} , or \overrightarrow{BC} and \overrightarrow{AC}). Recall two vectors have the same direction if and only if one is a scalar multiple of another.

(1) We calculate that $\overrightarrow{AB} = B - A = \langle -1, -1, -1 \rangle$ and $\overrightarrow{BC} = C - B = \langle -4, 2, 2 \rangle$. \overrightarrow{AB} is not a scalar multiple of \overrightarrow{BC} , therefore A, B, C are not co-linear.

(2): Similarly, $\overrightarrow{AB} = B - A = \langle -3, 0, -2 \rangle$ and $\overrightarrow{BC} = \langle 4, -4, 0 \rangle$. So \overrightarrow{AB} is not a scalar multiple of \overrightarrow{BC} , therefore A, B, C are not co-linear. \square

Problem 1.2. Describe and find the equation of the set of all points that are equidistant to the two points $A = (-1, 5, 3)$ and $B = (6, 2, -2)$.

Solution. It is a plane that is perpendicular to the line AB and contains the middle point of A and B .

Algebraically, it has all the points (x, y, z) which satisfies the following equation

$$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2},$$

namely, the distance to point A (LHS) equals the distance to point B (RHS).

Now we simplify the above equation.

$$\begin{aligned} (x+1)^2 + (y-5)^2 + (z-3)^2 &= (x-6)^2 + (y-2)^2 + (z+2)^2 \\ x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6y + 9 &= x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \\ 14x - 6y - 10z - 9 &= 0 \end{aligned}$$

where we end up with a linear equation, which is plane in \mathbb{R}^3 . \square

Problem 1.3. For each of the vectors given below, find a unit vector that has the same direction.

$$\mathbf{v} = \langle 2, 1, -2 \rangle \quad \mathbf{w} = \langle -4, 0, 3 \rangle$$

Further, find vectors of length 2 with the same direction.

Solution. To scale a vector \mathbf{v} into a unit vector, we simply divide by its magnitude: $\frac{1}{|\mathbf{v}|}\mathbf{v}$.

So the unit vector for \mathbf{v} is

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}}\langle 2, 1, -2 \rangle = \frac{1}{3}\langle 2, 1, -2 \rangle = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle$$

And similarly

$$\frac{1}{|\mathbf{w}|}\mathbf{w} = \frac{1}{\sqrt{(-4)^2 + 0^2 + 3^2}}\langle -4, 0, 3 \rangle = \frac{1}{5}\langle -4, 0, 3 \rangle = \left\langle -\frac{4}{5}, 0, \frac{3}{5} \right\rangle$$

To find the vectors with length 2, we simply multiply the unit vectors by 2.

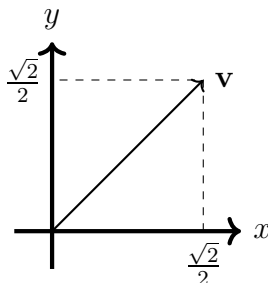
$$\frac{2}{|\mathbf{v}|}\mathbf{v} = 2 \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle = \left\langle \frac{4}{3}, \frac{2}{3}, -\frac{4}{3} \right\rangle$$

$$\frac{2}{|\mathbf{w}|}\mathbf{w} = 2 \left\langle -\frac{4}{5}, 0, \frac{3}{5} \right\rangle = \left\langle -\frac{8}{5}, 0, \frac{6}{5} \right\rangle$$

□

Problem 1.4. In \mathbb{R}^2 , \mathbf{v} is a unit vector which lies in the first quadrant. Suppose the angle between \mathbf{v} and the positive y -axis is $\pi/4$, find \mathbf{v} in component form.

Solution. We may assume that \mathbf{v} starts at the origin.



The \mathbf{v} forms an angle of $\pi/4 = 45^\circ$ with the y -axis, as depicted in the diagram above. Since the length of \mathbf{v} is 1, it follows that the ‘head’ of \mathbf{v} is $(\sqrt{2}/2, \sqrt{2}/2)$, therefore $\mathbf{v} = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$. □

Problem 1.5. Let $\mathbf{a} = \langle 2, 1, 1 \rangle$ and $\mathbf{b} = \langle -1, x, 3 \rangle$. Find the value of x such that \mathbf{a} is orthogonal to \mathbf{b} .

Solution. Two vectors are orthogonal if and only if their dot product is zero. Therefore we need to find the x such that

$$\langle 2, 1, 1 \rangle \cdot \langle -1, x, 3 \rangle = -2 + x + 3 = 0$$

Solving for x we get $x = -1$. □

2. Cross Product, Lines and Planes (12.4 – 12.5)

Problem 2.1. Find a non-zero vector that is orthogonal to the plane containing the three points

$$A = (2, -3, 4) \quad B = (-1, -2, 2) \quad C = (3, 1, -3)$$

Solution. We first calculate the vectors \overrightarrow{AB} and \overrightarrow{BC} .

$$\overrightarrow{AB} = B - A = \langle -3, 1, -2 \rangle$$

$$\overrightarrow{BC} = C - B = \langle 4, 3, -5 \rangle$$

A vector that is perpendicular to both \overrightarrow{AB} and \overrightarrow{BC} will be perpendicular to the plane of ABC . We find such a vector using the cross product.

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -2 \\ 4 & 3 & -5 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -3 & -2 \\ 4 & -5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} = \langle 1, 23, -13 \rangle \quad \square$$

Problem 2.2. Determine whether the following points are co-planer.

$$A = (1, 3, 2) \quad B = (3, -1, 6) \quad C = (5, 2, 0) \quad D = (3, 6, -4)$$

Solution. We use the triple product method. Consider the vectors

$$\overrightarrow{AB} = \langle 2, -4, 4 \rangle \quad \overrightarrow{AC} = \langle 4, -1, -2 \rangle \quad \overrightarrow{AD} = \langle 2, 3, -6 \rangle.$$

The four points are coplaner if and only if the volume of the parallelepiped determined by these three vectors is zero. Said volume is given by the triple product

$$\begin{aligned} & \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) \\ &= \overrightarrow{AB} \cdot \begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} \\ &= \langle 2, -4, 4 \rangle \cdot \langle 12, 20, 14 \rangle \\ &= 0 \end{aligned}$$

Therefore the four points are indeed coplaner. \square

Problem 2.3. Use equations of lines to determine whether the following three points are collinear.

$$A = (2, 4, -3) \quad B = (3, -1, 1) \quad C = (1, 9, 1)$$

Hint: Find the equation of the line through AB and check if C is on the line.

Solution. The equation of a line through two points \mathbf{r}_0 and \mathbf{r}_1 is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

We use this to calculate the equation of \overline{AB} :

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)\langle 2, 4, -3 \rangle + t\langle 3, -1, 1 \rangle \\ &= \langle 2(1 - t) + 3t, 4(1 - t) - t, -3(1 - t) + t \rangle \\ &= \langle 2 + t, 4 - 5t, -3 + 4t \rangle\end{aligned}$$

If C is on \overline{AB} , then we need to have $2 + t = 1 \implies t = -1$ in order for the first component to match up.

$$\mathbf{r}(-1) = (1, 9, -7) \neq C$$

Therefore C does not lie on the line \overline{AB} , hence A, B and C are not co-linear. \square

Problem 2.4. Find the equation of the plane through $A = (2, 4, -3)$, $B = (3, -1, 1)$, and $C = (1, 9, 1)$.

Solution. We first calculate the vectors $\overrightarrow{AB} = \langle 1, -5, 4 \rangle$ and $\overrightarrow{AC} = \langle -1, 5, 4 \rangle$. Their cross product is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -40, -8, 0 \rangle$. This is a vector that is orthogonal to both AB and AC , hence is orthogonal to the plane. Therefore it is a normal vector. Hence the equation of the plane is

$$-40(x - 2) - 8(y - 4) + 0(z + 3) = 0$$

which can be simplified to

$$5x + y - 14 = 0$$

\square

Problem 2.5. Find the equation of the line through $(3, 2, -4)$ with direction $\langle -1, 2, 5 \rangle$. Find its intersection with the plane from Problem 2.4.

Solution. The line has parametric equation

$$\mathbf{r}(t) = \langle 3 - t, 2 + 2t, -4 + 5t \rangle,$$

and the equation of the plane from previous problem is $5x + y = 14$. Substitute the *parametric* equation of the line to the *standard* equation of the plane

$$5(3 - t) + (2 + 2t) = 14.$$

Solving for t we get $t = 1$. Therefore the intersection is $\mathbf{r}(1) = (2, 4, 1)$. \square

3. Multivariable Functions, Limits and Partial Derivatives (14.1 - 14.3)

Problem 3.1. Find the domains and level curves of the functions

$$f(x, y) = \sqrt{4 - x^2 - y^2} \quad \text{and} \quad f(x, y) = x + \sqrt{y},$$

and sketch their graphs.

Solution.

- (1) The domain for $f(x, y)$ is the points where $4 - x^2 - y^2 \geq 0$, i.e. $x^2 + y^2 \leq 4$, which is the set of points inside the circle centered at $(0, 0)$ with radius 2 (including boundary).

The level curves are

$$f(x, y) = 0 \implies x^2 + y^2 = 4$$

$$f(x, y) = 1 \implies x^2 + y^2 = 3$$

$$f(x, y) = 2 \implies x^2 + y^2 = 0$$

There are no level curves for $L > 2$ or $L < 0$. (Why?) The level curves are circles. And the graph is a sphere.

- (2) We only need $y \geq 0$ for the domain, so it is the upper half of the plane.

The level curves are

$$x + \sqrt{y} = -1 \implies y = (x + 1)^2, \quad x \leq -1$$

$$x + \sqrt{y} = 0 \implies y = x^2, \quad x \leq 0$$

$$x + \sqrt{y} = 1 \implies y = (x - 1)^2, \quad x \leq 1$$

$$x + \sqrt{y} = 2 \implies y = (x - 2)^2, \quad x \leq 2$$

These are (half) parabolas, so the graph of $f(x, y)$ is a parabolic cylinder. □

Problem 3.2. Find the following limits, or demonstrate if not exists.

$$(1) \lim_{(x,y) \rightarrow (2,-1)} \frac{x^2 y + x y^2}{x^2 - y^2}$$

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{x y^3}{x^4 + y^4}$$

$$(3) \lim_{(x,y) \rightarrow (0,0)} \frac{5 y^2 \cos^2 x}{x^2 + y^2}$$

Solution. (1) This is a rational function, which is continuous everywhere in its domain. (Recall that the domain of a rational function is the set of points where the denominator is non-zero.) $(2, -1)$ is in the domain, so the limit is

$$\lim_{(x,y) \rightarrow (2,-1)} f(x, y) = f(2, -1) = \frac{2^2 \cdot (-1) + 2 \cdot (-1)^2}{2^2 - (-1)^2} = -\frac{2}{3}$$

(2) Taking the limit in the direction of $y = 0$, we have

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} \frac{x \cdot 0}{x^2 + 0} = 0$$

And taking the limit through $y = x$ we have

$$\lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{x \cdot x^3}{x^4 + x^4} = \frac{1}{2}$$

Since $0 \neq 1/2$, the limit DNE.

(3) With $x = 0$, the limit is

$$\lim_{y \rightarrow 0} f(0, y) = \lim_{y \rightarrow 0} \frac{5y^2 \cos^2(0)}{y^2} = \lim_{y \rightarrow 0} \frac{5y^2}{y^2} = 5.$$

For $y = 0$, the limit is

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} \frac{5 \cdot 0 \cdot \cos(x)}{x^2} = 0$$

Since $0 \neq 5$, the limit DNE. □

Problem 3.3. Determine the set of points where the function is continuous.

$$(1) f(x, y) = \frac{2x^2 + y}{1 - x^2 - y^2}$$

$$(2) f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2 + xy} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution. (1) The function is a rational function, which is continuous everywhere in its domain. The domain of the function is $\{(x, y) \in \mathbb{R}^2 \mid 1 - x^2 - y^2 \neq 0\}$.

(2) the function $\frac{2xy}{x^2 + y^2 + xy}$ is continuous whenever the denominator is non-zero. First we show that the denominator $x^2 + y^2 + xy$ equals 0 only when $(x, y) = (0, 0)$, by solving the equation $x^2 + y^2 + xy = 0$.

$$\begin{aligned} x^2 + y^2 + xy &= 0 \\ 4x^2 + 4y^2 + 4xy &= 0 \\ (4x^2 + 4xy + y^2) + 3y^2 &= 0 \\ (2x + y)^2 + 3y^2 &= 0 \end{aligned}$$

Since both $(2x + y)^2$ and $3y^2$ are non-negative, it follows that the solution will satisfy both

$$(2x + y)^2 = 0 \quad \text{and} \quad 3y^2 = 0.$$

Clearly then the only solution is $x = 0, y = 0$. Therefore the rational function $\frac{2xy}{x^2 + y^2 + xy}$ is not continuous only at $(0, 0)$.

Now the function $f(x, y)$ is defined to be 0 at $(0, 0)$. So it would be continuous if

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2 + xy} = 0$$

This is false because, the limit with direction $y = 0$ is

$$\lim_{x \rightarrow 0} \frac{0}{x^2 + 0 + 0} = 0$$

while the limit with direction $y = x$ is

$$\lim_{x \rightarrow 0} \frac{2x^2}{x^2 + x^2 + x \cdot x} = \frac{2}{3} \neq 0.$$

Therefore the limit DNE, so the function $f(x, y)$ is continuous at $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$. \square

Problem 3.4. Evaluate the following second partial derivatives.

$$(1) \frac{\partial^2}{\partial x \partial y} \ln(x + y)$$

$$(2) \frac{\partial^2}{\partial x \partial y} e^{xy} \sin(x)$$

Solution. (1) $\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \ln(x + y) \right) = \frac{\partial}{\partial x} \left(\frac{1}{x + y} \right) = -\frac{1}{(x + y)^2}$

(2)

$$\frac{\partial}{\partial y} (e^{xy} \sin x) = \sin x \left(\frac{\partial}{\partial y} e^{xy} \right) = \sin x \cdot \frac{\partial e^{xy}}{\partial(xy)} \cdot \frac{\partial xy}{\partial y} = \sin x \cdot e^{xy} \cdot x$$

$$\begin{aligned} & \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} e^{xy} \sin x \right) \\ &= \frac{\partial}{\partial x} x e^{xy} \sin x \\ &= \sin x \left(\frac{\partial}{\partial x} x e^{xy} \right) + x e^{xy} \left(\frac{\partial}{\partial x} \sin x \right) \\ &= \sin x \left(e^{xy} + x \left(\frac{\partial}{\partial x} e^{xy} \right) \right) + x e^{xy} \cos x \\ &= \sin x (e^{xy} + x (e^{xy} y)) + x e^{xy} \cos x \end{aligned}$$

\square

4. Chain Rule and Directional Derivatives

Problem 4.1. Find dz/dt for $z = \sqrt{xy+1}$, $x = \tan t$ and $y = \arctan(t)$.

Solution. We use chain rule.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= \left(\frac{y}{2\sqrt{xy+1}} \right) \cdot \sec^2(t) + \left(\frac{x}{2\sqrt{xy+1}} \right) \cdot \left(\frac{1}{t^2+1} \right)\end{aligned}\quad \square$$

Problem 4.2. Find $\partial u/\partial s$ and $\partial u/\partial t$ for

$$u = ze^{xy} \quad x = s+t \quad y = s-t \quad z = st$$

Solution. Use chain rule.

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s} \\ &= (yz \cdot e^{xy}) \cdot 1 + (xz \cdot e^{xy}) \cdot 1 + e^{xy} \cdot t \\ &= e^{xy}(yz + xz + t) \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \\ &= (yz \cdot e^{xy}) \cdot 1 + (xz \cdot e^{xy}) \cdot (-1) + e^{xy} \cdot s \\ &= e^{xy}(yz - xz + s)\end{aligned}\quad \square$$

Problem 4.3. Find $\partial z/\partial x$ and $\partial z/\partial y$, where

$$x^2 + 4y^2 + z^2 - 2z = 6$$

Solution. We use chain rule and implicit differentiation. The above equation can be written as

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 2z - 6 = 0.$$

Therefore,

$$\begin{aligned}\frac{\partial z}{\partial x} &= - \frac{\partial F}{\partial x} \bigg/ \frac{\partial F}{\partial z} = - \frac{2x}{2z-2} \\ \frac{\partial z}{\partial y} &= - \frac{\partial F}{\partial y} \bigg/ \frac{\partial F}{\partial z} = - \frac{8y}{2z-2}\end{aligned}\quad \square$$

Problem 4.4. For each function f , find the gradient ∇f and the directional derivative $D_{\mathbf{u}}f$.

(1) $f(x, y, z) = x^2z + xyz + yz^2$, $\mathbf{u} = \langle 1, -1, 1 \rangle$.

(2) $f(x, y) = e^x \sin(xy)$, $\mathbf{u} = \langle 2, 1 \rangle$.

(3) $f(x, y, z) = xe^y - y^2e^{xz}$, $\mathbf{u} = \langle -1, 0, 2 \rangle$.

Solution. (1) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2xz + yz, xz + z^2, x^2 + xy + 2yz)$ We turn \mathbf{u} into a unit vector by dividing by its magnitude $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 1} = \sqrt{3}$. Then

$$D_{\mathbf{u}}f = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{3}}(2xz + yz - (xz + z^2) + x^2 + xy + 2yz)$$

(2) $\nabla f = (e^x(\sin(xy) + y \cos(xy)), e^x x \cos(xy))$

$$D_{\mathbf{u}}f = \frac{1}{|\mathbf{u}|} \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{5}}(2e^x(\sin(xy) + y \cos(xy)) + e^x x \cos(xy))$$

(3) $\nabla f = \langle e^y - y^2ze^{xz}, xe^y - 2ye^{xz}, -xy^2e^{xz} \rangle$. $D_{\mathbf{u}}f = \frac{1}{|\mathbf{u}|} \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{5}}(-e^y + y^2ze^{xz} - 2xy^2e^{xz})$ \square

Problem 4.5. Find the maximal rate of change of $f(x, y, z) = xe^y - y^2e^{xz}$ at the point $P(1, 0, -1)$. In what direction does that occur?

Solution. $\nabla f(x, y, z) = \langle e^y - y^2ze^{xz}, xe^y - 2ye^{xz}, -xy^2e^{xz} \rangle$. The gradient vector at P is $\nabla f(1, 0, -1) = \langle 1, 1, 0 \rangle$. So the maximal rate of change is $|\nabla f(P)| = |\langle 1, 1, 0 \rangle| = \sqrt{2}$, which happens in the direction of the gradient vector $\langle 1, 1, 0 \rangle$. \square

Problem 4.6. Find the tangent plane and normal line to $xy^2 = 2ze^{x+y} + 3$ at $(1, -1, -1)$.

Solution. Let $F(x, y, z) = xy^2 - 2ze^{x+y} - 3$. We first calculate the gradient vector

$$\nabla F(x, y, z) = \langle y^2 - 2e^{x+y}z, 2xy - 2e^{x+y}z, -2e^{x+y} \rangle \quad \nabla F(1, -1, -1) = \langle 3, 0, -2 \rangle$$

Then the tangent plane is

$$3(x - 1) + 0(y + 1) - 2(z + 1) = 0 \implies 3x - 2z - 5 = 0$$

The normal line is

$$r(t) = \langle 1, -1, -1 \rangle + t\langle 3, 0, -2 \rangle = \langle 1 + 3t, -1, -1 - 2t \rangle$$

\square

A. Additional Problems I

Problem A.1. Show that the following limits do not exist.

$$(1) \lim_{(x,y) \rightarrow (0,0)} \frac{x \sin y}{y^2}$$

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{x^6 + y^4}$$

Solution. (1) We find two paths, $x = 0$ and $y = x$, which produce different limits as follows.

$$\lim_{x=0, y \rightarrow 0} \frac{x \sin y}{y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0$$

$$\lim_{x \rightarrow 0, y=x} \frac{x \sin y}{y^2} = \lim_{x \rightarrow 0} \frac{x \sin x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

(2) Use the two paths $x = 0$ (or $y = 0$) and $y = x^{3/2}$.

$$\lim_{x=0, y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{0 \cdot y^2}{y^4} = 0$$

$$\lim_{x \rightarrow 0, y=x^{3/2}} f(x, y) = \lim_{x \rightarrow 0} \frac{x^3 (x^{3/2})^2}{x^6 + (x^{3/2})^4} = \lim_{x \rightarrow 0} \frac{x^6}{x^6 + x^6} = \frac{1}{2} \quad \square$$

Problem A.2. Find the limit or show that it doesn't exist.

$$(1) \lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

$$(2) \lim_{(x,y) \rightarrow (0,1)} \frac{y - 1}{x^2 + y - 1}$$

$$(3) \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y + x^2 y^2}{2x^6 + y^3}$$

Solution. (1) The denominator is zero at $(2, 1)$, however, since the numerator also vanishes at $(2, 1)$, we can factor and simplify the rational function:

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y) \rightarrow (2,1)} \frac{x(x - 2y)}{(x + 2y)(x - 2y)} = \lim_{(x,y) \rightarrow (2,1)} \frac{x}{x + 2y} = \frac{1}{2}$$

(2) Along $x = 0$ we have $\lim_{y \rightarrow 1} \frac{y - 1}{y - 1} = 1$. But when $y = 1$, $\lim_{x \rightarrow 0} \frac{0}{x^2 + 0} = 0$. So DNE.

(3) Along the path $x = 0$ or $y = 0$, the limit is zero (verify this). But along the path $y = x^2$, we have $\lim_{x \rightarrow 0} \frac{x^4 x^2 + x^2 (x^2)^2}{2x^6 + (x^2)^3} = \lim_{x \rightarrow 0} \frac{x^6}{2x^6 + x^6} = \frac{1}{3} \quad \square$

5. Maxima and Minima (14.7)

Problem 5.1. Find the local maxima/minima and saddle points of the function.

$$f(x, y) = x^2 + y - 2xy \quad \text{and} \quad f(x, y) = \frac{x^2 + y^2}{e^x}$$

Solution. (1) $f_x(x, y) = 2x - 2y$, $f_y(x, y) = 1 - 2x$. So $f_y(x, y) = 0 \implies 1 - 2x = 0 \implies x = 1/2$. Then $f_x(x, y) = 2x - 2y = 2\frac{1}{2} - 2y = 0 \implies x = 1/2$. So the only critical point is $(1/2, 1/2)$. Next we use the second derivative test:

$$f_{xx} = 2, f_{yy} = 0, f_{xy} = -2$$

Therefore $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -4$, which is a constant function. So the critical point must be a saddle point.

(2) Taking the partial derivatives

$$f_x(x, y) = -(x^2 + y^2 - 2x)e^{-x}$$

$$f_y(x, y) = 2ye^{-x}$$

We first find the critical points, if $f_x(x, y) = 2ye^{-x} = 0$, then since $e^{-x} \neq 0$, we must have $y = 0$. Going from here, we have $f_x(x, y) = -(x^2 + 0 - 2x)e^{-x} = 0$, which (for the same reason that $e^{-x} = 0$) implies that $x^2 - 2x = 0$. Then $x(x - 2) = 0$, which yields two solutions $x = 0$ and $x = 2$. Therefore there are two critical points $(2, 0)$ and $(0, 0)$.

Next we use 2nd derivative test to determine the types of the critical points. We have

$$f_{xx}(x, y) = e^{-x}(2 - 4x + x^2 + y^2)$$

$$f_{yy}(x, y) = 2e^{-x}$$

$$f_{xy}(x, y) = f_{yx}(x, y) = -2e^{-x}y$$

At the point $(0, 0)$, we have $f_{xx}(0, 0) = 2 > 0$, and

$$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 \times 2 - 0 = 4 > 0$$

Therefore $(0, 0)$ is a local minimum. For $(2, 0)$ we have

$$f_{xx}(2, 0) = -2e^{-2} < 0$$

$$D(2, 0) = -2e^{-2} \cdot 2e^{-2} - 0 < 0$$

Therefore it's a saddle point. □

Problem 5.2. Find the shortest distance from the plane $x - 2y - z - 3 = 0$ to the origin.

Solution. A point on the plane has the form $(x, y, x - 2y - 3)$. Let

$$f(x, y) = \text{distance}^2 = x^2 + y^2 + (x - 2y - 3)^2$$

And we would like to find the local minimum (if any) of f . We first find its critical points. We have $f_x(x, y) = 4x - 4y - 6$ and $f_y(x, y) = -4x + 10y + 12$. Thus we have to solve for a 2×2 system of linear equations:

$$\begin{cases} 4x - 4y = 6 & (1) \\ 4x - 10y = 12 & (2) \end{cases}$$

$eq.(1) - eq.(2)$ gives $6y = -6$, thus $y = -1$. And plug this back in $eq.(1)$ we get $4x = 2$, thus $x = 1/2$. So the only critical point is $(\frac{1}{2}, -1)$.

Now let's check if this indeed is a local minimum.

The second derivatives are

$$f_{xx}(x, y) = 4 \quad f_{yy}(x, y) = 10 \quad f_{xy} = -4$$

And

$$D(x, y) = 4 \times 10 - (-4)^2 = 24$$

(Note that all the second derivatives are constant functions.) Since $f_{xx} > 0$ and $D > 0$, the critical point is a local minimum. Therefore, the shortest distance is

$$\sqrt{f\left(\frac{1}{2}, -1\right)} = \sqrt{(1/2)^2 + (-1)^2 + (1/2 + 2 - 3)^2} = \frac{\sqrt{6}}{2}$$

□

Problem 5.3. Find the absolute minima of the function $f(x, y) = x^2 - 4xy + y^2 + 3y$ in the quadrilateral given by the four points $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(2, 3)$.

Solution. First, we find all the critical points.

$$f_x(x, y) = 2x - 4y = 0 \quad f_y(x, y) = 2y - 4x + 3 = 0$$

This yields one solution: $(1, \frac{1}{2})$. Second, we examine the values of $f(x, y)$ at the boundary of the region, i.e. the four sides of the quadrilateral.

(i) $y = 0, 0 \leq x \leq 2$. In this case, $f(x, y)|_{y=0} = x^2$, which is an increasing function of x for $x \in [0, 2]$. (What is the vertex of a parabola?) Thus the minimum along this boundary is $f(0, 0) = 0$.

(ii) If $y = 3, 0 \leq x \leq 2$. In this case, $f(x, y)|_{y=3} = x^2 - 12x + 18$. For $x \in [0, 2]$, this is a decreasing function in x , thus the minimum is $f(2, 3) = -2$.

(iii) If $x = 0, 0 \leq y \leq 3$. Here we have $f(x, y)|_{x=0} = y^2 + 3y$, which is increasing for $y \in [0, 3]$. Therefore the minimum is $f(0, 0) = 0$.

(iv) If $x = 2, 0 \leq y \leq 3$, we have $f(x, y)|_{x=2} = y^2 - 5y + 4$. The minimum is attained when $y = 5/2$. (Why? Try sketching the graph.) So the minimum is $f(2, 5/2) = -9/4$.

Finally we compare the value of critical points and the minima at the boundary:

$$f(1, \frac{1}{2}) = \frac{3}{4} \quad f(0, 0) = 0 \quad f(2, 3) = -2 \quad f(2, \frac{5}{2}) = -\frac{9}{4}$$

Hence the minimum is $-9/4$ which is attained at the boundary with $(x, y) = (2, 5/2)$. \square

Problem 5.4. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + 2xy + y$ in the region bounded by $y = 1 - x^2$, $y = x - 1$, the y -axis and $x \geq 0$.

6. Lagrange Multipliers

Problem 6.1. Find the extreme values of $f(x, y, z) = e^{xyz}$ with constraint $2x^2 + y^2 + z^2 = 24$

Solution. Let $g(x, y, z) = 2x^2 + y^2 + z^2$, and we need to solve for $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 24$.

$$\begin{cases} yze^{xyz} = 4x\lambda & (1) \\ xze^{xyz} = 2y\lambda & (2) \\ zye^{xyz} = 2z\lambda & (3) \\ 2x^2 + y^2 + z^2 = 24 & (4) \end{cases}$$

Take the ratio of equation (1) and equation (2), we get

$$(5) \quad \frac{yze^{xyz}}{xze^{xyz}} = \frac{4x\lambda}{2y\lambda} \implies \frac{y}{x} = \frac{2x}{y} \implies y^2 = 2x^2$$

Take the ratio of equation (1) and equation (3), we get

$$(6) \quad \frac{yze^{xyz}}{xye^{xyz}} = \frac{4x\lambda}{2z\lambda} \implies \frac{z}{x} = \frac{2x}{z} \implies z^2 = 2x^2$$

Now substitute (5) and (6) into (4) we get

$$2x^2 + 2x^2 + 2x^2 = 24 \implies x^2 = 4 \implies x = \pm 2$$

Plug $x^2 = 4$ into (5) and (6) we get $y^2 = 8$ and $z^2 = 8$, hence $y = \pm\sqrt{8}$ and $z = \pm\sqrt{8}$.

So extreme value is attained at 8 points $(\pm 2, \pm\sqrt{8}, \pm\sqrt{8})$. But there are only two extreme values, $f(\pm 2, \pm\sqrt{8}, \pm\sqrt{8}) = e^{\pm 16}$. \square

Problem 6.2. Find the shortest distance from the plane $x - 2y - z - 3 = 0$ to the origin. Problem 5.2 once again, this time use Lagrange multiplier.

Solution. Let (x, y, z) be an arbitrary point in the 3-space, its distance to the origin is $\sqrt{x^2 + y^2 + z^2}$. Let $f(x, y, z)$ be the square of said distance: $f(x, y, z) = x^2 + y^2 + z^2$.

We would like to find the extreme (minimum) value of $f(x, y, z)$, when (x, y, z) is on the plane, i.e. with constraint that $x - 2y - z - 3 = 0$. So set $g(x, y, z) = x - 2y - z$. The system of equations is

$$\begin{cases} 2x = \lambda & (1) \\ 2y = -2\lambda & (2) \\ 2z = -\lambda & (3) \\ x - 2y - z = 3 & (4) \end{cases}$$

Equations (1) to (3) can be rewritten as $x = \frac{\lambda}{2}$, $y = -\lambda$, and $z = -\frac{\lambda}{2}$. Substitute these to equation (4) we get

$$\frac{\lambda}{2} - 2(-\lambda) - (-\frac{\lambda}{2}) = 3$$

which yields $\lambda = 1$. Now plug this back in to the equations (1) to (3) we found $x = \frac{1}{2}$, $y = -1$ and $z = -\frac{1}{2}$. So $(\frac{1}{2}, -1, -\frac{1}{2})$ is the point on the plane that is closest to the origin. Thus the shortest distance is $\sqrt{(\frac{1}{2})^2 + (-1)^2 + (-\frac{1}{2})^2} = \frac{\sqrt{6}}{2}$ \square

Problem 6.3. Find the extreme value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x - y = 1$ and $y^2 - z^2 = 1$.

Solution. Set up the system of equations for Lagrange multipliers:

$$\begin{cases} 2x = \lambda & (1) \\ 2y = -\lambda + 2y\mu & (2) \\ 2z = -2z\mu & (3) \\ x - y = 1 & (4) \\ y^2 - z^2 = 1 & (5) \end{cases}$$

First observe that equation (3) can be simplified to $2z(\mu + 1) = 0$ which has two possible solutions: $\mu = -1$ or $z = 0$. We break into two cases.

(i) Suppose $\mu = -1$. Substitute $\mu = -1$ into eq.(2) gives $2y = -\lambda - 2y \implies \lambda = -4y$. Combining this with eq.(4) we get $\lambda = -4(x-1)$. Now use eq. (1) we get $2x = \lambda = -4(x-1)$, which implies $x = \frac{2}{3}$, thus by eq.(4) $y = -\frac{1}{3}$. Then by eq.(5), $z^2 = y^2 - 1 = \frac{1}{9} - 1 = -\frac{8}{9}$ which has no real solutions. (But there are two complex solutions $z = \pm \frac{\sqrt{8}}{3}i$. So in this cases there are two complex solutions of the equations: $(x, y, z) = (\frac{2}{3}, -\frac{1}{3}, \pm \frac{\sqrt{8}}{3}i)$.)

(ii) Now suppose $z = 0$. Then by equation (5) we know $y^2 = 1$ which means $y = \pm 1$. If $y = 1$, by eq.(4) we have $x = 2$, thus we have $(x, y, z) = (2, 1, 0)$. In case of $y = -1$, by eq. (4) we have $x = 0$, giving the other solution $(x, y, z) = (0, -1, 0)$.

Finally, in \mathbb{R}^3 the function f attains extreme value at two points $(2, 1, 0)$ and $(0, -1, 0)$. The extreme values are $f(2, 1, 0) = 2^2 + 1^2 = 5$ (the maximum) and $f(0, -1, 0) = 1$ (the minimum). \square

7. Basic Double Integrals

Problem 7.1. Evaluate the following integrals.

$$(1) \int_0^\pi \int_0^1 2x + \sin(y) \, dx \, dy$$

$$(2) \int_1^3 \int_1^{\frac{1}{3}} \frac{\ln y}{xy} \, dy \, dx$$

$$(3) \iint_R \frac{2xy^2}{x^2 + 1} \, dA, \text{ where } R = [0, 1] \times [-3, 3]. \text{ (i.e. } 0 \leq x \leq 1, -3 \leq y \leq 3.)$$

Solution.

$$\begin{aligned} (1) \quad & \int_0^\pi \left(\int_0^1 2x + \sin(y) \, dx \right) dy \\ &= \int_0^\pi \left([x^2 + x \sin(y)]_0^1 \right) dy = \int_0^\pi (1 + \sin y) \, dy = [y - \cos(y)]_0^\pi = 2 + \pi \\ (2) \quad & \int_1^3 \int_1^{\frac{1}{3}} \frac{\ln y}{xy} \, dy \, dx = \left(\int_1^3 \frac{1}{x} \, dx \right) \left(\int_1^{\frac{1}{3}} \frac{\ln y}{y} \, dy \right) = \ln 3 \cdot \left[\frac{\ln(y)^2}{2} \right]_1^{\frac{1}{3}} = \frac{\ln(3)^3}{2} \\ (3) \quad & \iint_R \frac{2xy^2}{x^2 + 1} \, dA = \int_0^1 \int_{-3}^3 \frac{2xy^2}{x^2 + 1} \, dy \, dx \\ &= \int_0^1 \frac{2x}{x^2 + 1} \, dx \cdot \int_{-3}^3 y^2 \, dy = [\ln(x^2 + 1)]_0^1 \cdot \left[\frac{y^3}{3} \right]_{-3}^3 = 18 \ln(2) \quad \square \end{aligned}$$

Problem 7.2. Fill in the boxes so that the following equality holds

$$\int_0^2 \int_{-1}^{x^2-1} xy \, dy \, dx = \int_{\square}^{\square} \int_{\square}^{\square} xy \, dx \, dy.$$

Then evaluate the integral using one of the above.

Solution. The region is given by $D = \{0 \leq x \leq 2, -1 \leq y \leq x^2 - 1\}$. We rewrite these inequalities: $y \leq x^2 - 1 \implies y - 1 \leq x^2 \implies \sqrt{y - 1} \leq x$. Plug in $x = 2$ to $y \leq x^2 - 1$ we get $y \leq 3$. Thus $D = \{\sqrt{y - 1} \leq x \leq 2, -1 \leq y \leq 3\}$. Therefore we have

$$\begin{aligned} & \int_0^2 \int_{-1}^{x^2-1} xy \, dy \, dx = \int_{-1}^3 \int_{\sqrt{y-1}}^2 xy \, dx \, dy. \\ & \int_{-1}^3 \int_{\sqrt{y-1}}^2 xy \, dx \, dy = \int_{-1}^3 \left[\frac{yx^2}{2} \right]_{\sqrt{y-1}}^2 dy = \int_{-1}^3 \frac{4y - y(y+1)}{2} dy = \frac{4}{3} \quad \square \end{aligned}$$

8. More on Double Integrals

Problem 8.1. Evaluate the following double integrals.

$$(1) \int_0^{\pi/2} \int_0^x x \sin y \, dy \, dx$$

$$(2) \iint_D e^{y^2} \, dA, \text{ where } D = \{(x, y) : 0 \leq y \leq 1, 0 \leq x \leq y\}$$

Solution. (1) $= \int_0^{\pi/2} [-x \cos y]_0^x \, dx = \int_0^{\pi/2} (-x \cos x + x) \, dx = \left[-x \sin x - \cos x + \frac{x^2}{2} \right]_0^{\pi/2} = 1 - \frac{\pi}{2} + \frac{\pi^2}{8}$. (Need to use integration by part for the integrand $x \cos x$.)

$$(2) = \int_0^1 \int_0^y e^{y^2} \, dx \, dy = \int_0^1 [xe^{y^2}]_0^y \, dy = \int_0^1 ye^{y^2} \, dy = \left[\frac{e^{y^2}}{2} \right]_0^1 = \frac{e-1}{2} \quad \square$$

Problem 8.2. Evaluate the following integrals.

$$(1) \iint_D (x^2 + 2y) \, dA, \text{ where } D \text{ is bounded by } y = x, y = x^3, x \geq 0.$$

$$(2) \iint_D (2x - y) \, dA, \text{ where } D \text{ is the circle centered at the origin with radius 2.}$$

$$\text{Solution. (1) } \int_0^1 \int_{x^3}^x (x^2 + 2y) \, dy \, dx = \int_0^1 [x^2 y + y^2]_{x^3}^x \, dx = \int_0^1 (x^3 + x^2 - x^5 - x^6) \, dx = \frac{23}{84}.$$

$$(2) \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x - y) \, dx \, dy = \int_{-2}^2 [x^2 - xy]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \, dy = \int_{-2}^2 2y\sqrt{4-y^2} \, dy = 0. \quad \square$$

Problem 8.3. Find the volume of the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

Solution. First we find the volume above the xy -plane.

$$\begin{aligned} & \int_{-r}^r \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2} \, dx \, dy \\ &= \int_{-r}^r \left[x\sqrt{r^2-y^2} \right]_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \, dy = \int_{-r}^r 2(r^2-y^2) \, dy = 2 \left[r^2 y - \frac{y^3}{3} \right]_{-r}^r = \frac{8}{3} r^3 \end{aligned}$$

Finally by symmetry, we multiply by 2 to get the volume of the solid, $\frac{16}{3} r^3$. \square

9. Double Integral with Polar Coordinates

Problem 9.1 (Problems 8.2 (2)). Evaluate $\iint_D (2x - y) \, dA$, where D is the circle centered at the origin with radius 2.

Solution.

$$\begin{aligned} &= \int_0^{2\pi} \int_0^2 r(2r \cos(\theta) - r \sin(\theta)) \, dr \, d\theta \\ &= \int_0^{2\pi} r^2 \, dr \int_0^{2\pi} (2 \cos \theta - \sin \theta) \, d\theta \\ &= \left[\frac{r^3}{3} \right]_0^2 \cdot [2 \sin(\theta) + \cos(\theta)]_0^{2\pi} = \frac{8}{3} \cdot 0 = 0 \end{aligned}$$

□

Problem 9.2. Find the following integral using polar coordinates.

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} xy^2 \, dx \, dy$$

Solution. $\int_0^{\frac{\pi}{2}} \int_0^a r(r \cos(\theta) r^2 \sin^2(\theta)) \, dr \, d\theta = \left(\int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos(\theta) \, d\theta \right) \left(\int_0^a r^4 \, dr \right)$

$$\left[\frac{\sin^3(\theta)}{3} \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^5}{5} \right]_0^a = \frac{1}{3} \cdot \frac{a^5}{5} = \frac{a^5}{15}$$

□

Problem 9.3. Find the $\iint_R (x^2 + y^2) \, dA$ where R is in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $y = x$ and $y = 0$.

Solution. $\iint_R (x^2 + y^2) \, dA = \int_0^{\pi/4} \int_1^3 r^2 \cdot r \, dr \, d\theta = \int_0^{\pi/4} \left[\frac{r^4}{4} \right]_1^3 \, d\theta = 5\pi$

□

10. Triple integrals

Problem 10.1. Evaluate the integral $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dz \, dx \, dy$

Solution.

$$\begin{aligned}
 &= \int_0^1 \int_y^{2y} [6xyz]_0^{x+y} \, dx \, dy \\
 &= \int_0^1 \int_y^{2y} 6xy(x+y) \, dx \, dy \\
 &= \int_0^1 \left[6y \left(\frac{x^3}{3} + \frac{x^2y}{2} \right) \right]_y^{2y} \, dy \\
 &= \int_0^1 23y^4 \, dy = \frac{23}{5}
 \end{aligned}$$

□

Problem 10.2. Evaluate the integral $\iiint_E e^{z/y} \, dV$, where E is bounded by $E = \{(x, y, z) | 0 \leq y \leq 1, y \leq x \leq 1, 0 \leq z \leq xy\}$.

Solution.

$$\begin{aligned}
 &\int_0^1 \int_y^1 \int_0^{xy} e^{\frac{z}{y}} \, dz \, dx \, dy \\
 &= \int_0^1 \int_y^1 \left[ye^{\frac{z}{y}} \right]_0^{xy} \, dx \, dy \\
 &= \int_0^1 \int_y^1 (ye^x - y) \, dx \, dy \\
 &= \int_0^1 [ye^x - yx]_y^1 \, dy \\
 &= \int_0^1 (ey - y - ye^y + y^2) \, dy \\
 &= \left[\frac{y^3}{3} + \frac{(e-1)y^2}{2} - e^y(y-1) \right]_0^1 \\
 &= \frac{e}{2} - \frac{7}{6}
 \end{aligned}$$

□

Problem 10.3. Evaluate $\iiint_E x^2 dV$ where E is the solid bounded by $x^2 + y^2 = 4$, $x + z = 2$, and $z = 0$. (Hint: You may use the fact that $\int_0^{2\pi} \cos^3(\theta) d\theta = 0$.)

Solution. We can rewrite the integral as $\iint_D \int_0^{2-x} x^2 dz dA$, where D is the the region given by $x^2 + y^2 = 4$ (the circle). Going from here, we have

$$\iint_D [x^2 z]_0^{2-x} dA = \iint_D [x^2 z]_0^{2-x} dA = \iint_D x^2(2-x) dA$$

From here we switch to polar coordinates¹:

$$\begin{aligned} & \int_0^{2\pi} \int_0^2 (2r^2 \cos^2(\theta) - r^3 \cos^3(\theta)) r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{2r^4}{4} \cos^2(\theta) - \frac{r^5}{5} \cos^3(\theta) \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left(8 \cos^2 \theta - \frac{2^5}{5} \cos^3 \theta \right) d\theta = \int_0^{2\pi} 8 \cos^2 \theta d\theta \\ &= \int_0^{2\pi} 8 \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \\ &= 8\pi + 4 \int_0^{2\pi} \cos(2\theta) d\theta = 8\pi \end{aligned}$$

□

Problem 10.4. Find the volume of the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$.

Solution.

$$\begin{aligned} & \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dz dy dx \\ &= 8 \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2}} dz dy dx \\ &= 8 \int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2} dy dx \\ &= 8 \int_0^r r^2 - x^2 dx = 8 \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 8 \cdot \frac{2}{3} r^3 = \frac{16}{3} r^3 \end{aligned}$$

□

¹Note that this is essentially using cylindrical coordinates (in the next section)

11. Cylindrical, spherical coordinates, and change of variables.

Problem 11.1. Set up the integral to calculate the volume bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{3(x^2 + y^2)}$ using Cartesian coordinates, cylindrical coordinates and spherical coordinates respectively.

Problem 11.2. Rewrite the integral $\iiint_E x e^{x^2 + y^2 + z^2} dV$ where E is the portion of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Problem 11.3.

B. Additional Problems II