Problems for PACE 2023

Sylvester W. Zhang

June 3, 2023

1. Introduction

Recall that an *order ideal* of a poset is a downward closed subposet. For P a poset, denote $J(P)$ the set of its order ideals. We consider an action on $J(P)$ called *rowmotion*.

Rowmotion, denoted Row, is an invertible operator on the order ideals of any partially ordered set. For an order ideal I, $\mathrm{Row}(I)$ is the order ideal generated by the minimal elements of the complement of I. Rowmotion was first introduced by Brouwer and Schrijver [BS74] and has been extensively studied by many different authors, including [Duc74, FDF93, CFdF95, Pan09, SW12]. The name 'rowmotion' is due to Striker and Williams [SW12].

The action of Rowmotion turned out to be extremely nice on certain posets, in particular, the *rectangle poset* $\mathscr{R}(a,b) := \{(i,j) \in \mathbb{Z}^2 : 1 \le i \le a, 1 \le j \le b\}$. Notably, rowmotion on $\mathscr{R}(a,b)$ is a canonical example of the cyclic sieving phenomenon. In particular, it is periodic of order $a+b$, i.e. $\mathrm{Row}^{a+b}(I) = I$ for $I \in J(\mathscr{R}(a,b))$.

It has been observed and conjectured (N. Williams) that a seemingly not related poset, the *trapezoid poset* $\mathscr{T}(a,b) := \{(i,j) \in \mathbb{Z}^2 : 1 \le i \le a, i \le j \le a+b-i\}$ also has periodic orbit under rowmotion.

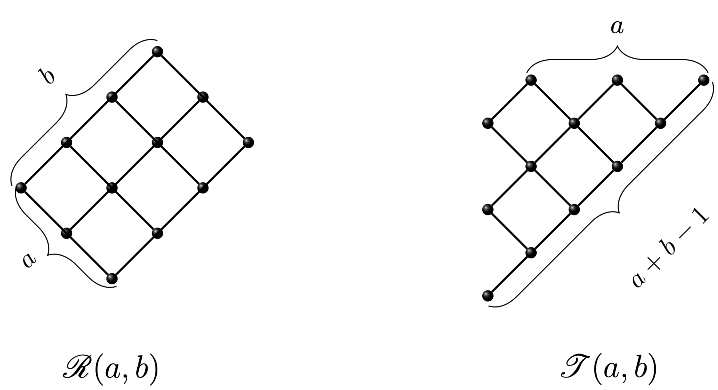

FIGURE 1. The rectangle poset $\mathscr{R}(3,4)$ and trapezoid poset $\mathscr{T}(3,4)$.

Hopkins [Hop20] conjectured, and Dao, Wellman, Yost–Wolff and myself [DWYWZ20] proved that, Row and $J(\mathscr{R}(a,b))$ and $J(\mathscr{T}(a,b))$ have the same orbit structure. Hopkins further conjectured that, rowmotion on P-partitions (generalization of order ideals) of $\mathscr{R}$ and $\mathscr{T}$ also have the same orbit structure.

In this project we propose a conjectural bijection to attack Hopkin's conjecture, but before that, we first define P-partitions and rowmotion on them.

2. More on Rowmotion

As we briefly mentioned in the introduction, the original definition of rowmotion is defined as follows. Take $I \in J(P)$ an order ideal, and find the minimal non-elements of I, then $\mathrm{Row}(I)$ is the order ideal generated by (below) those elements. See Figure 2 for example.

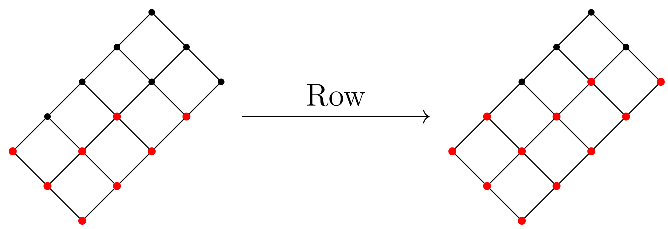

FIGURE 2. Rowmotion acting on order ideals.

From this definition it is unclear whether Row is invertible or not. It become clear, however, from the following alternative definition of Cameron and Fon-der-Flaass [CFdF95].

Proposition 2.1. *For $p \in \mathcal{P}$ and order ideal I, we define the toggle operation of p on I as follows.*

$$\tau_p(I) = \begin{cases} I \cup p & \text{if } p \notin I \text{ and } I \cup p \in J(\mathcal{P}), \\ I \setminus p & \text{if } p \in I \text{ and } I \setminus p \in J(\mathcal{P}), \\ I & \text{otherwise.} \end{cases}$$

Then rowmotion is performed by toggles "row by row" from the largest to smallest, i.e.

$$\mathrm{Row}(I) = \tau_{p_1} \circ \tau_{p_{n-1}} \circ \cdots \circ \tau_{p_n}(I)$$

where $p_1 \leq \cdots \leq p_n$ is any linear extension of the poset $\mathcal{P}$.

Exercise 2.2. Prove the above proposition, i.e. show that the toggle definition of Row agrees with the original definition.

Exercise 2.3. Identify the orbits of Row on the rectangle poset $\mathscr{R}(2,2)$ and the trapezoid poset $\mathscr{T}(2,2)$. And speculate a bijection which commutes with rowmotion.

3. P-Partitions.

For a poset P, a P-partition is an order preserving map φ from P to $\mathbb{N}$, i.e.

$$a \leq_P b \iff \varphi(a) \leq \varphi(b)$$

They can be thought of as increasing labelings of the poset. A P-partition is said to have *height* if all the labelings are smaller than or equal to m. We denote $\mathcal{P}^m(P)$ the set of all P-partitions of height m. Note that $J(P) = \mathcal{P}^1(P)$.

Stanley defined the *order polynomial* of a poset as follows

$$\Omega_P(m) = \#\mathcal{P}^{m-1}(m).$$

For rectangle posets, MacMahon (1915) has a very nice formula

$$\Omega_{\mathscr{R}(a,b)}(m) = \prod_{i=1}^{a} \prod_{i=1}^{b} \frac{i+j+m-2}{i+j-1}$$

And furthermore, Proctor (1983) proved that,

$$\Omega_{\mathscr{R}(a,b)}(m) = \Omega_{\mathscr{T}(a,b)}(m)$$

suggesting that P-partitions of $\mathscr{R}$ are equinumerous with those of $\mathscr{T}$. Proctor's proof was not bijective, and the first bijective proof is due to [HPPW18], using K-theoretic tableaux combinatorics.

4. Rowmotion on P-partitions

There is a natural translation for rowmotion on the level of P-partitions, using *piecewise linear toggles.*

Definition 4.1. For $p \in P$ and $\varphi : P \to \mathbb{N} \leq \gtrdot$ and P-partition of height m, we define the piecewise-linear toggle operation of p on φ as follows. First extend the poset P to $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ where $\hat{0}$ is below everything and $\hat{1}$ is above everything, and set $\varphi(\hat{0}) = 0$ and $\varphi(\hat{1}) = m$. Then,

$$\tau_p(\varphi)(t) := \begin{cases} t & \text{if } t \neq p \\ \min\{\varphi(i) : p \lessdot i\} + \max\{\varphi(i) : i \lessdot p\} - \varphi(p) & \text{if } t = p \end{cases}$$

We then define piecewise linear rowmotion in a similar way.

$$\mathrm{Row}_{pl}(\varphi) = \tau_{p_1} \circ \tau_{p_{n-1}} \circ \cdots \circ \tau_{p_n}(\varphi)$$

where $p_1 \leq \cdots \leq p_n$ is any linear extension of the poset $\mathcal{P}$.

Exercise 4.2. (1) Show that the piecewise linear toggle is a well-defined action on P-partitions. In other words, $\tau_p(\varphi)$ will still be a P-partition.

(2) Show that when height $m = 1$, piecewise linear rowmotion agrees with order ideal rowmotion.

Exercise 4.3. Identify the orbits of rowmotion on $\mathcal{P}^3(\mathscr{R}(2,2))$ and $\mathcal{P}^3(\mathscr{T}(2,2))$ and speculate a bijection.

Hopkins [Hop20] further conjectured that $\mathscr{R}$ and $\mathscr{T}$ also have the same rowmotion orbit structure on P-partitions. This suggests that there should be a bijection between $\mathcal{P}^m(\mathscr{R})$ and $\mathcal{P}^m(\mathscr{T})$ equivarient with respect to rowmotion. The [HPPW18] bijection, however, only works in the case of $m = 1$ (result of [DWYWZ20]).

Problem 4.4. Find a bijection between $\mathcal{P}^m(\mathscr{R}(a,b))$ and $\mathcal{P}^m(\mathscr{T}(a,b))$ which commutes with piecewise linear rowmotion. In particular, we have a conjectural bijection which we know to commute with rowmotion, but we aren't able to show that it's actually a bijection.

5. Birational Rowmotion

One can de-tropicalize the notion of piecewise linear toggle, and lift it to a birational map. Precisely, for any labelling φ of the poset, define birational toggling to be

$$\tau_p(\varphi)(t) := \begin{cases} \varphi(t) & \text{if } t \neq p \\ \dfrac{1}{\varphi(p)} \dfrac{\sum_{p \lessdot i} \varphi(i)}{\sum_{i \lessdot p} \frac{1}{\varphi(i)}} & \text{if } t = p \end{cases}$$

We can then define birational rowmotion in a similar maner. Grinberg-Roby conjectured that birational rowmotion on the trapezoid poset $\mathscr{T}(a,b)$ is periodic of $a+b$.

Problem 5.1. Does the conjectural bijection in Problem 4.4 list to a bijaction map which is equivarient with respect to bijectional rowmotion?

References

[BS74] Andries E Brouwer and Alexander Schrijver. On the period of an operator, defined on antichains. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, (ZW 24/74), 1974.

[CFdF95] Peter J Cameron and Dmitry G Fon-der Flaass. Orbits of antichains revisited. *European Journal of Combinatorics*, 16(6):545–554, 1995.

[Duc74] Pierre Duchet. Sur les hypergraphes invariants. *Discrete Mathematics*, 8(3):269–280, 1974.

[DWYWZ20] Quang Vu Dao, Julian Wellman, Calvin Yost-Wolff, and Sylvester W Zhang. Rowmotion orbits of trapezoid posets. *arXiv preprint arXiv:2002.04810*, 2020.

[FDF93] Dmitry G Fon-Der-Flaass. Orbits of antichains in ranked posets. *European journal of combinatorics*, 14(1):17–22, 1993.

[Hop20] Sam Hopkins. Minuscule doppelgängers, the coincidental down-degree expectations property, and rowmotion. *Experimental Mathematics*, pages 1–29, 2020.

[HPPW18] Zachary Hamaker, Rebecca Patrias, Oliver Pechenik, and Nathan Williams. Doppelgängers: Bijections of Plane Partitions. *International Mathematics Research Notices*, 03 2018.

[Pan09] Dimitri Panyushev. On orbits of antichains of positive roots. *European Journal of Combinatorics*, 30:586–594, 2009.

[SW12] Jessica Striker and Nathan Williams. Promotion and rowmotion. *European Journal of Combinatorics*, 33(8):1919–1942, 2012.