PACE Problem 12.

Intro to Cluster Algebras

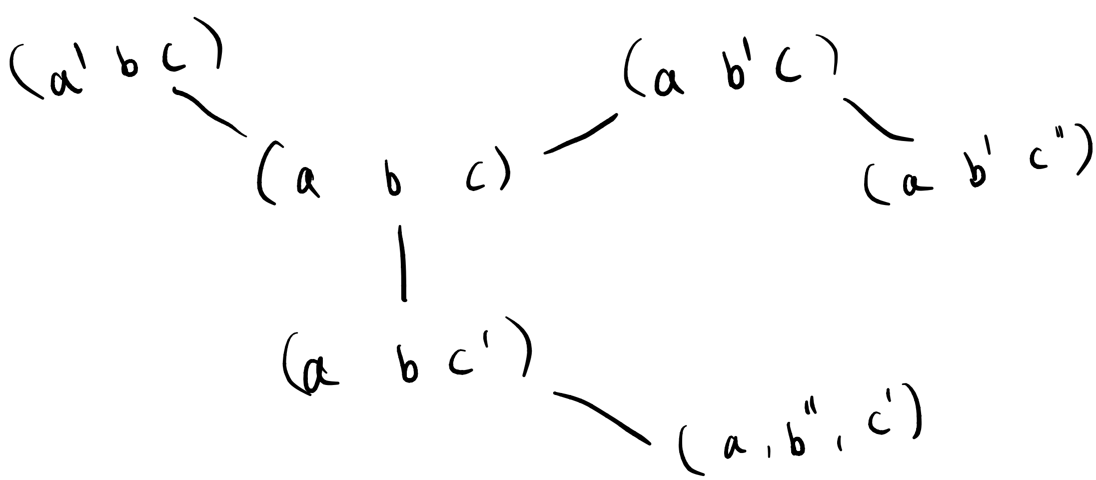
Roughly speaking.

Commutative ring. w/ additional Combi. structures.

- distinguished set of variables —— Cluster var.
- Cl. var. are grouped into sets of equal cardinality
- Some clusters are related by "mutations".

$$(a \quad b \quad c) \xleftrightarrow{\text{mutation}} (a \quad b' \quad c)$$

$$b\,b' = F_1 + F_2.$$ "exchange relation"



Totally positive matrix.

$$X = (x_{ij})_{\substack{i \in [m] \\ j \in [n]}} \in M_{m \times n}$$

$I \subset [m]$, $J \subset [n]$ $\quad |I| = |J|$

$$\Delta_{I,J}(X) = \det (x_{ij})_{\substack{i \in I \\ j \in J}}.$$

$x \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$ $\quad I = (1,2) \quad J = (1,3)$

$$\Delta_{IJ}(x) = \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Def A matrix X is totally positive if all minor are > 0

(Totally non-negative $\geq$)

☆ How to check if X is TP?

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad$ 5 minors: $a, b, c, d, \underset{e}{\underset{\|}{ad - bc}}.$

$$e = ad - bc$$

$$ad = bc + e$$

$$\boxed{d = \frac{bc + e}{a}}$$

an example of exchange relation.

$\Rightarrow$ Only needs to check 4 minors $a\ b\ c\ e$.

dim Space of all $M_{2\times 2}^{>0} = 4$.

Lustig's canonical basis (Kashiwara global basis)

Quantum Group $U_q(\mathfrak{g})$

TP

Lustig

dual canonical basis.

Fomin & Zelevinsky studied parametrization of TP matrices.

Lustig generalized this to other Lie types.

FZ's cluster algebras

Connection to
- Quiver rep theory.
- Hyperbolic geometry.
- number theory.
- integrable system
- math. phys.
- knot theory ......

<u>Grassmanian of planes</u> $\quad Gr(2,n) = \{ V \subset \mathbb{C}^n : \dim V = 2 \}$

$Gr(2,4)$. $\quad X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \longleftrightarrow \text{row span}(X) \in Gr(2,4)$

Plücker embedding. $\quad Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$

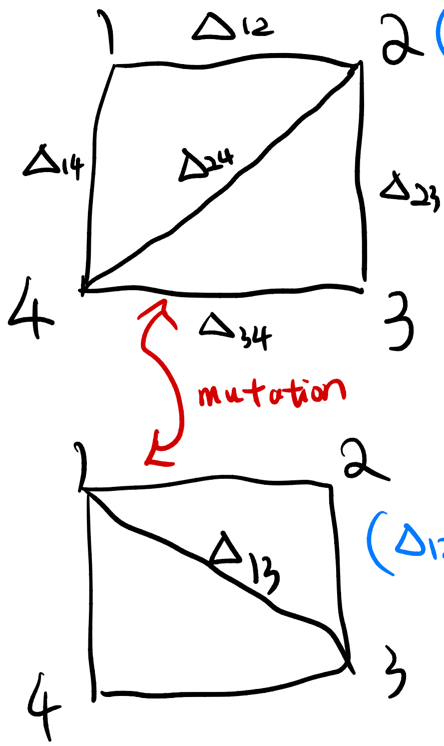
$\Delta_{ij}(X) = \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix}$ gives coordinates in $\mathbb{P}^{\binom{n}{2}-1}$!

Plücker relations $\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$

"exchange relation"

plücker coord. Δ_{ij} are **cluster variables**.

n-gon.



$(\Delta_{12}\,\Delta_{14}\,\Delta_{23}\,\Delta_{34}\,\Delta_{24})$

Triangulations are **clusters**.

$$\Delta_{13}\,\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{23}\,\Delta_{14}$$

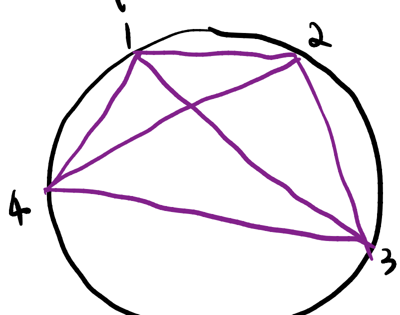
$(\Delta_{12}\,\Delta_{14}\,\Delta_{23}\,\Delta_{34}\,\Delta_{13})$

mutations are "flips" of quadrilaterals.

The coordinate ring of $Gr(2,n)$ is a cluster algebra.

"paper presentation topic" : cluster structure of $Gr(k,n)$.

Ptolemy's theorem



$$d_{13}\cdot d_{24} = d_{12}\,d_{34} + d_{14}\,d_{23}$$

Formal definition of cluster algebras. of rank n over a UFD; (any number field).

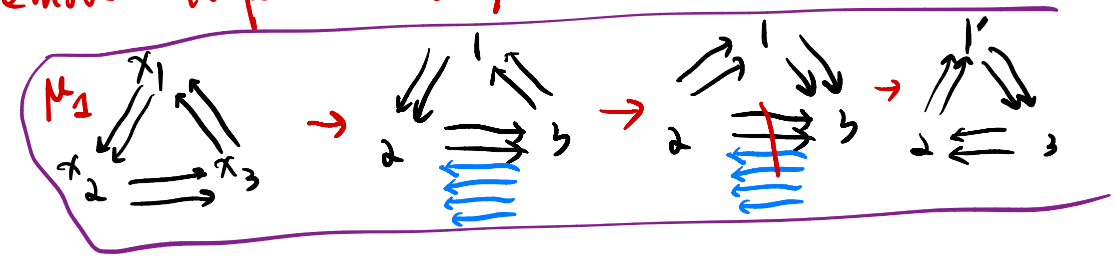
- Seed (X, Q)

X → cluster. $(x_1, x_2, \cdots, x_n)$

Q → "quiver" directed graph on n vertices. without loops or 2-cycles.

- for any $k \in [n]$, $\mu_k(Q)$ mutation of Q.

1. for any $i \to k \to j$ add an arrow $i \to j$
2. reverse all arrows incident to k.
3. remove ~~loops and~~ 2-cycles.



- Seed mutation.

$$((x_1 \cdots\cdots x_k \cdots\cdots x_n), Q) \xleftrightarrow{\mu_k} ((x_1 \cdots x_k' \cdots x_n), \mu_k(Q))$$

$$x_k' x_k = \prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j$$

$$\boxed{x_1' x_1 = x_3^2 + x_2^2}$$

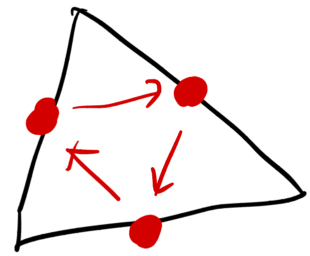
Fact: mutations are involutions. $\mu_k^2 = id$.

Exercise 12.1.

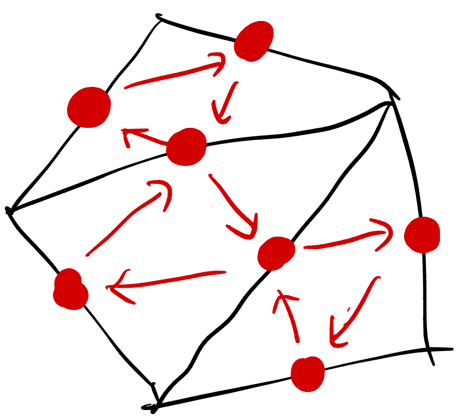
• Finally. Start from an intial seed and mutate at all possible directions generates the entire cluster alg.

Triangulations and Quiver mutations.

- T a triangulation.
- to every edge of T, assign a vertex.
- to every triangle of T. put

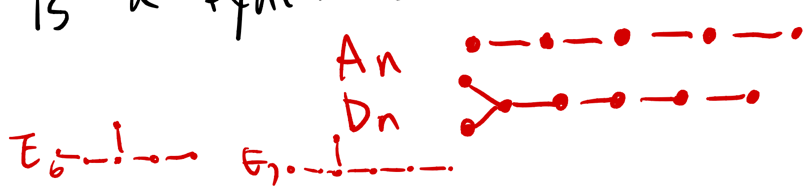


Exercise 12.2.
Quiver mutation = Quadrilateral flip.
Exchange relation = Ptolemy relation
(Plücker rel. $Gr(2,n)$)



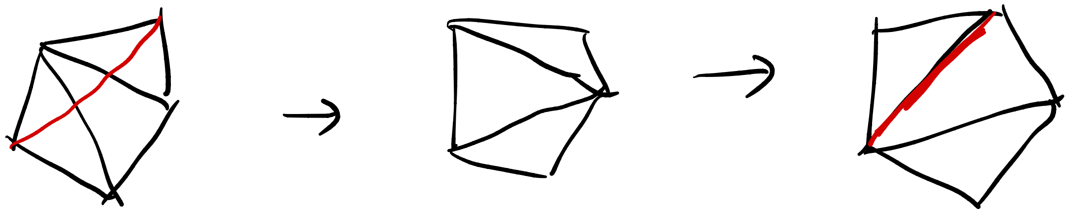
Finite Type Classification (Fomin - Zelevinsky).

A cluster alg. has finitely many cluster variables iff the quiver is a Dynkin Quiver of ADE.



For any cluster var. θ, and any cluster $(x_1 \dots x_n)$.

Can get to θ from $(x_1 \dots x_n)$ via a seq of mutations.



Can write θ in terms of $x_1 \dots x_n$.

Thm (Fomin-Zelevinsky).

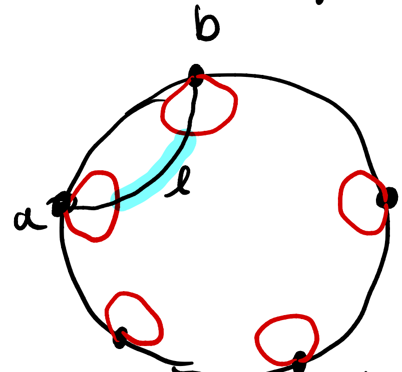
θ must be a Laurent polynomial. $\dfrac{\text{polynomial}}{\text{monomial}}$

& coefficients are positive.

Q give a combinatorial interpretation of θ.

Teichmuller space. $T(F) = \mathrm{Hom}(\pi_1(F), PSL_2) / PSL_2$.

Def (Penner coordinate)

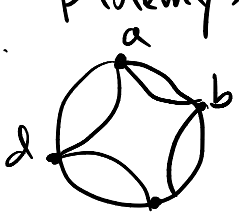


Hyperbolic disk.

- Decorated T--- space $\tilde{T}$

$\lambda_{ab} := e^{\frac{l}{2}}$ λ-length.

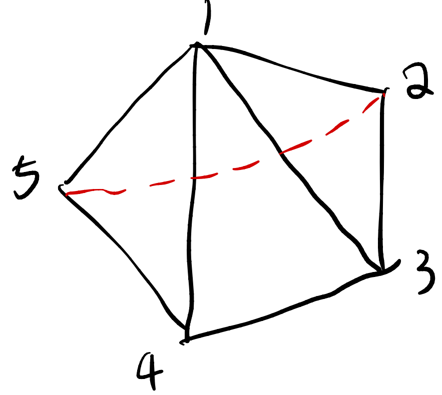
Thm Ptolemy's theorem holds for λ-lengths.



$\lambda_{ab}\lambda_{cd} + \lambda_{ad}\lambda_{bc} = \lambda_{ac}\lambda_{bd}$.

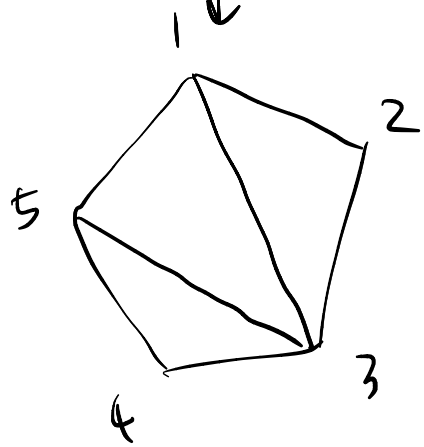
$\tilde{T}(F)$ is a cluster algebra. when F is a marked disk,

$\cong$ cl. alg. of $Gr(2,n)$

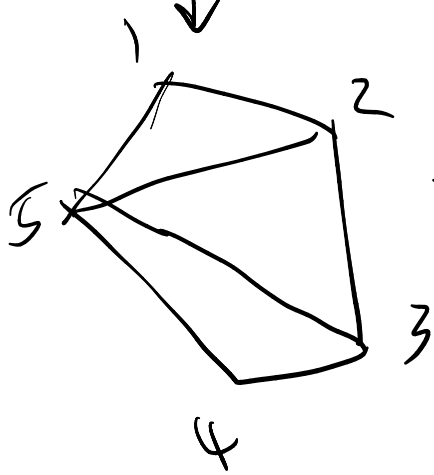


known. λ_{12} λ_{13} λ_{23} ....

what is λ_{25}?



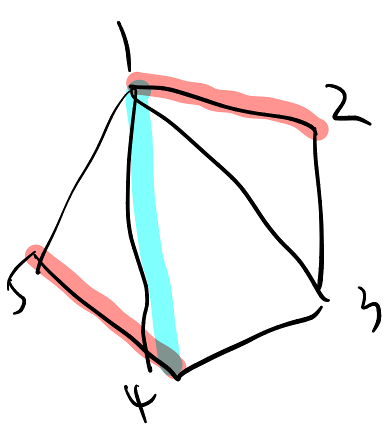
$$\lambda_{35} = \frac{\lambda_{13}\lambda_{45} + \lambda_{15}\lambda_{34}}{\lambda_{14}}$$

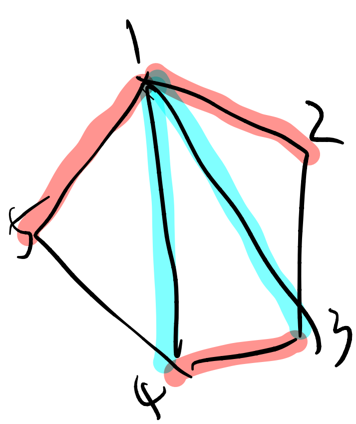


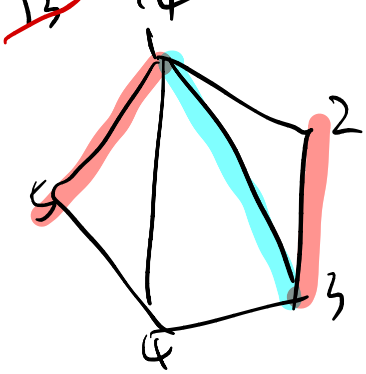
$$\lambda_{25} = \frac{\lambda_{12}\lambda_{35} + \lambda_{15}\lambda_{23}}{\lambda_{13}}$$

$$= \frac{\lambda_{12}}{\lambda_{13}}\left(\frac{\lambda_{13}\lambda_{45} + \lambda_{15}\lambda_{34}}{\lambda_{14}}\right) + \frac{\lambda_{15}\lambda_{23}}{\lambda_{13}}$$

$$= \frac{12\ \cancel{13}\ 45}{\cancel{13}\ 14} + \frac{12\ 15\ 34}{13\ 14} + \frac{15\ 23}{13}$$

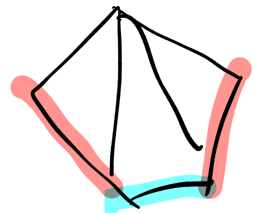






<u>Observation</u>. λ_{ab} expansion seems to be paths from a to b on T.

However some paths don't work



<u>Exercise</u> 12.3 (Schiffler's T-path formula)

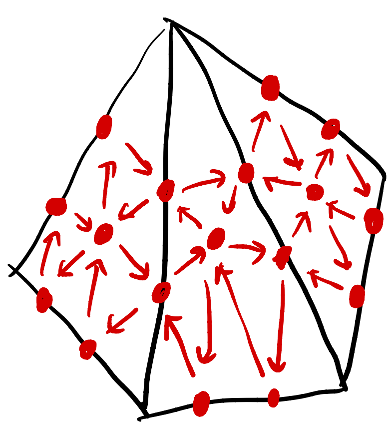
Find out the rules for such paths, such that λ_{ab} is a weighted sum of them. (induction).

Higher Teichmüller space. (Fock-Goncharov).

$$\mathrm{Hom}(\pi_1(F), GL_n) / GL_n.$$

This has a cluster structure. • Clusters are triangulations

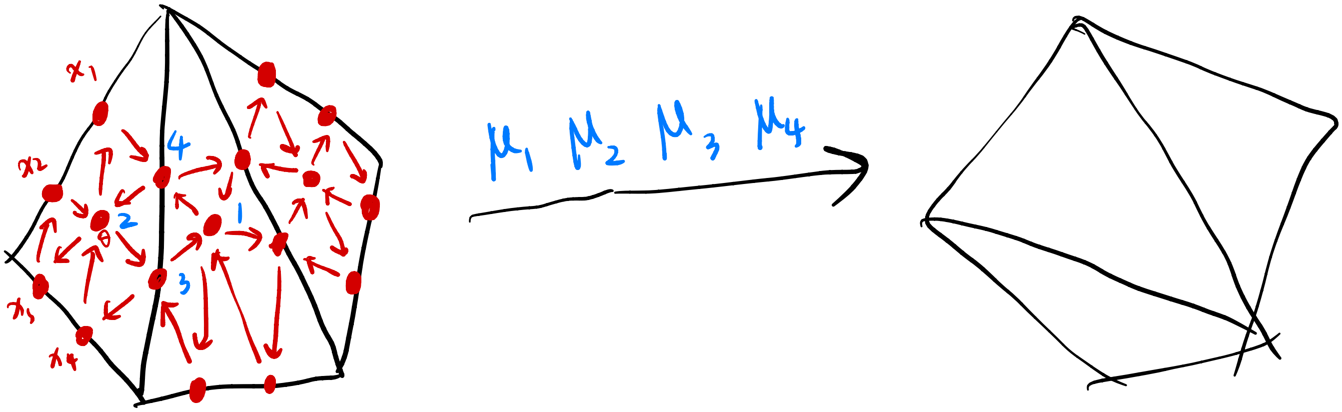
$n=3$



• put $n-1$ vertex on edges.

• put vertices on the faces to form little triangles.

Exercise 12.4 generalize 12.2 in GL_3 case.



PACE Problem 12.

Find T-path formulas for cluster variables (i.e. generalize Exercise 12.3.)

---

they are paths on