# Pre-Program Exercises for PACE 2023 Participants

April 26, 2023

## 1 The symmetric group and the Bruhat orders

**Definition 1.1.** A *permutation* is a bijection from $[n] := \{1, 2, \ldots, n\}$ to $[n]$.

We typically write a permutation via *one-line notation*. For example, $w = 3142$ means $w(1) = 3$, $w(2) = 1$, $w(3) = 4$ and $w(4) = 2$.

**Definition 1.2.** The *symmetric group* $S_n$ is the group of all permutations on $[n]$ with the group operation being composition of maps. Let $s_i = (i\ i{+}1)$ be the permutation that swaps $i$ and $i+1$, called a *simple transposition* or a *simple generator*, for $i = 1, \ldots, n-1$.

It is clear that $\{s_1, \ldots, s_{n-1}\}$ generates $S_n$.
For example, the permutation 3142 can be written as $s_2 s_1 s_3$.

**Exercise 1.3.** Show that $S_n = \langle s_1, \ldots, s_{n-1}\rangle / \mathfrak{I}$ where $\mathfrak{I}$ is generated by three types of relations:

- $s_i^2 = 1$ for $i = 1, \ldots, n-1$;
- $s_i s_j = s_j s_i$ for $|i - j| \geq 2$;
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i = 1, \ldots, n-2$.

**Definition 1.4.** The *length* of a permutation $w$ is defined to be the smallest $\ell$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for some simple generators $s_{i_1}, \ldots, s_{i_\ell}$. Denote the length by $\ell(w)$.

For example, the permutation 3142 has length 3 as $3142 = s_2 s_1 s_3$.

**Exercise 1.5.** Let $I(w) = \{(i, j) \mid i < j,\ w(i) > w(j)\}$ be the set of *inversions* of $w$. Show that $\ell(w) = |I(w)|$.

For example, $I(3142) = \{(1, 2), (1, 4), (3, 4)\}$ has size 3, which equals $\ell(3142)$.

**Exercise 1.6.** Show that there exists a unique permutation with maximal length, which we denote as $w_0$. Show that $\ell(w_0 w) = \ell(w w_0) = \ell(w_0) - \ell(w)$ for any $w \in S_n$.

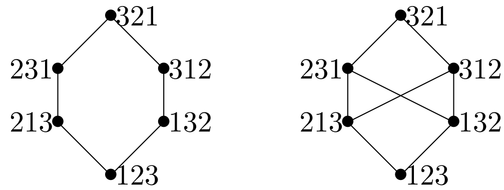



Figure 1: The weak and strong order on $S_3$.

**Definition 1.7.** The (right) *weak Bruhat order* is a partial order defined on $S_n$ such that $w \lessdot_R ws_i$ if $\ell(w) < \ell(ws_i)$ for some $i$. The left weak Bruhat order is defined by $w \lessdot_L s_i w$ if $\ell(w) < \ell(s_i w)$.

**Exercise 1.8.** Show that $u \leq_L w$ in the left weak Bruhat order if and only if $I(u) \subset I(w)$.

**Exercise 1.9.** Show that the weak Bruhat order is a graded lattice graded by length $\ell$.

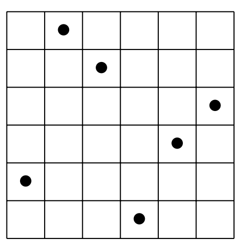
Now define the *transpositions* $T$ to be all the conjugates of $\{s_1, \ldots, s_{n-1}\}$. In other words, $T = \{(i\ j) \mid 1 \leq i < j \leq n\}$ contains all the 2-cycles. Write $t_{ij} := (i\ j)$.

**Definition 1.10.** The *(strong) Bruhat order* is defined by $w < wt_{ij}$ if $\ell(w) < \ell(wt_{ij})$.

**Exercise 1.11.** Why do we not define a left version of the strong Bruhat order?

We are now going to discuss criteria characterizing the strong Bruhat order.

**Exercise 1.12.** For a permutation $w \in S_n$, write $w[i,j] = \#\{k \mid k \leq i,\ w(k) \leq j\}$ for all $i, j \in [n]$. This is called the *rank-matrix* associated with $w$. See Figure 2 for an example. Show that $x \leq y$ in



| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 2 | 2 | 2 | 2 |
| 0 | 1 | 2 | 2 | 2 | 3 |
| 0 | 1 | 2 | 2 | 3 | 4 |
| 1 | 2 | 3 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 6 |

Figure 2: A rank matrix for permutation $w = 236514$

the (strong) Bruhat order if and only if $x[i,j] \geq y[i,j]$ for all $i, j \in [n]$.

**Exercise 1.13.** For two subsets $A$ and $B$ of the same size, we say that $A \leq B$ in the *Gale order* if $a_i \leq b_i$ for all $i$ where $A = \{a_1 < \cdots < a_k\}$, $B = \{b_1 < \cdots < b_k\}$. Show that $x \leq y$ in the (strong) Bruhat order if $\{x(1), \ldots, x(i)\} \leq \{y(1), \ldots, y(i)\}$ in the Gale order for all $i = 1, \ldots, n$. This is called the *tableaux criterion*.

**Exercise 1.14.** Fix a reduced word for some $w = s_{i_1} \cdots s_{i_\ell}$. Show that if $u \leq w$ in the (strong) Bruhat order, then there exists a subword of $i_1, \ldots, i_\ell$ which is a reduced word for $u$. This is called the *subword property*. For example, fix $3241 = s_1 s_2 s_3 s_1$. Then $3214 = 321 < 3241$ and we can choose $s_1 s_2 s_1 = 321$ as a subword for 321.

**Exercise 1.15.** Is it true that for any reduced word $u = s_{j_1} \cdots s_{j_k}$ and $w \geq u$, there is a reduced word for $w$ that contains $j_1, \ldots, j_k$ as a subword?

**Exercise 1.16.** Show that the strong Bruhat order is a graded poset. Furthermore, show that any length 2 interval $[u, v]$ in the strong Bruhat order is isomorphic to a diamond. This is called the *diamond rule.*

**Exercise 1.17.** Show that if $u \leq v$ in the strong Bruhat order, and simple reflection $s_i$ satisfies $\ell(u) < \ell(us_i)$ and $\ell(v) > \ell(vs_i)$, then $u \leq vs_i$ and $us_i \leq v$. This is called the *lifting property.*

**Exercise 1.18.** Show that the strong Bruhat order is *top-heavy*, meaning that for any $w \in S_n$ and any $k \leq \ell(w)/2$,

$$|\{u \in S_n \mid u \leq w,\ \ell(u) = k\}| \leq \{u \in S_n \mid u \leq w,\ \ell(u) = \ell(w) - k\}.$$

For example, when $w = 3412$ and $k = 1$, we have 2134, 1324, 1243 on the left hand side and 1432, 2413, 3142, 3214 on the right hand side, and indeed $3 \leq 4$.

# 2 Matroids

**Definition 2.1.** A *matroid* $M$ is a pair $(E, \mathfrak{B})$, where $E$ is a finite set (called the *ground set* of $M$) and $\mathfrak{B}$ is a collection of subsets of $E$ (called *bases* of $M$), such that

(B1) $\mathfrak{B} \neq \emptyset$;

(B1) If $B_1, B_2 \in \mathfrak{B}$, and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathfrak{B}$. This is called the *basis exchange property.*

**Exercise 2.2.** Show that any two bases of $M$ have the same carndinality, called the *rank* of $M$.

**Exercise 2.3.** Let $E = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a collection of vectors in $\mathbb{R}^k$ that spans $\mathbb{R}^k$. Let $\mathfrak{B}$ be the collection of subsets in $E$ that form a basis of $\mathbb{R}^k$. Show that $M = (E, \mathfrak{B})$ is a matroid. This is called a *realizable matroid (over $\mathbb{R}$).*

**Exercise 2.4.** Let $G = (V, E)$ be a finite connected graph, and $\mathfrak{B}$ be the collection of spanning trees in $G$. Show that $M_G = (E, \mathfrak{B})$ is a matroid. This is called a *graphical matroid.*

**Exercise 2.5.** Let $M = (E, \mathfrak{B})$ be a matroid. Define $\mathfrak{B}^* = \{E \setminus B : B \in \mathfrak{B}\}$. Show that $M^* = (E, \mathfrak{B}^*)$ is also a matroid, called the *dual matroid* of $M$.

**Definition 2.6.** A set $A \subset E$ is an *independent set* if $A$ is contained in some basis $B \in \mathfrak{B}$. A *circuit* $C \subset E$ is a minimal dependent set. The *rank function* $r : 2^E \to \mathbb{N}$ is a function that assigns every subset $A \subset E$ a number $r(A) =$ the maximal number of elements in an independent subset of $E$ contained in $A$. A *flat* is a subset $F \subset E$ which is maximal for its rank, meaning that the addition of any other element to the set would increase the rank.

**Exercise 2.7.** What are the independent sets, circuits, rank functions, and flats in a realizable matroid/graphical matroid?