PACE Problem 1

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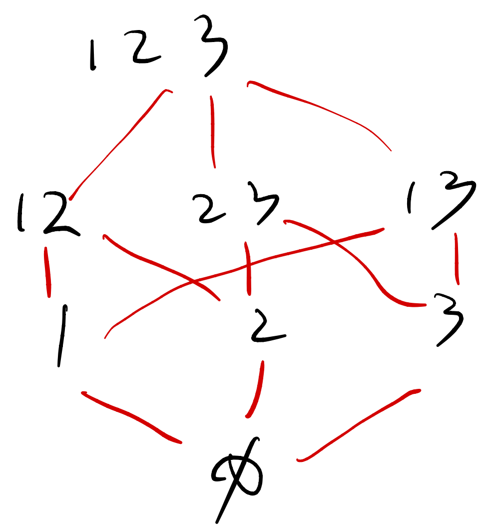
Dynamics on Trapezoid Posets

Posets = partially ordered set.

e.g. $[n]:=\{1,\cdots,n\}$ $\quad 2^{[n]}=\{\text{subsets of } [n]\}$

$A,B\in 2^{[n]}$ $\quad A\leq B \iff A\subseteq B$

Hasse diagram. $\quad 2^{[3]}$



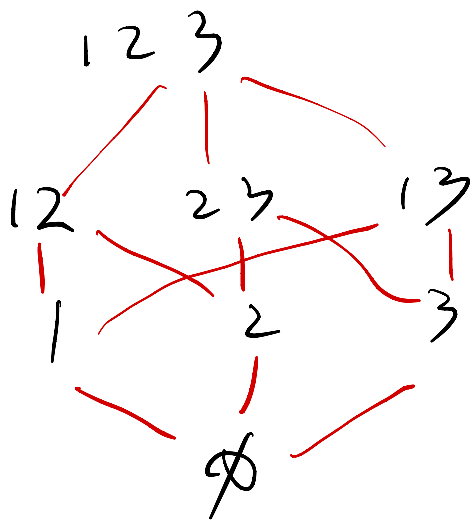
$$\boxed{\begin{array}{ll} b & \\ | & a\leq b \ \& \\ a & \neg\exists\, c,\ a\leq c\leq b \end{array}}$$

covering relation $a\lessdot b$

order ideal (down set)

for poset $P, \leq$ $\quad I \subseteq P$ is an order ideal

if downward close.



$\{1, 2, 12\}$ is not an order id.

$\{\phi, 1, 2\}$ is an order ideal.

$J(P) :=$ set of all order ideals of P.

Motivation.

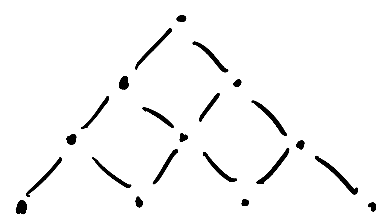
	Type A	Type B
G Lie group.	$GL_n(\mathbb{C})$	

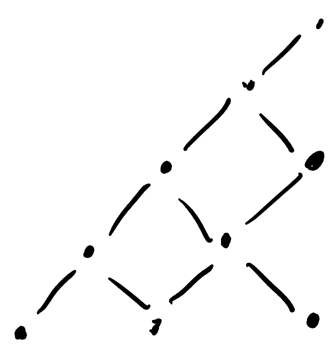
$GL_n(\mathbb{C})$ general linear group.

$= n \times n$ invertible matrices over $\mathbb{C}$

root system, $\supset$ positive roots. Φ_G^+ $\downarrow$ is a poset.

$x < y$ when $y - x \in \Phi_G^+$.

In type A_n  $= A_4$

 $= B_3$

Minuscule poset. $\Lambda_{G/P}$

e.g. $\emptyset \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$

Flag variety. $G/B = \{\emptyset \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n : \dim V_i = i\}$

B $\shortparallel$ Borel subgroup Type A $B = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

partial flag variety.

$$G/P = \{ \phi \subseteq V_1 \subseteq V_2 \cdots \}$$

parabolic subgroup.

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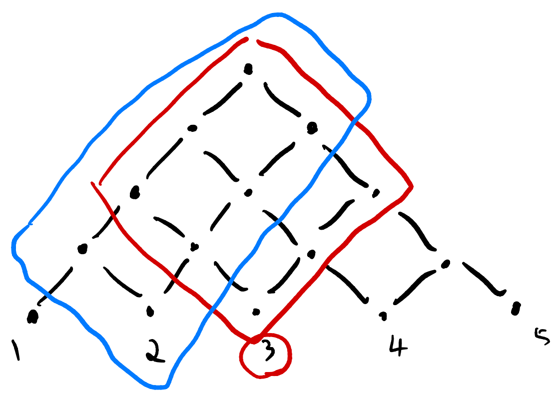
miniscule flag variety.

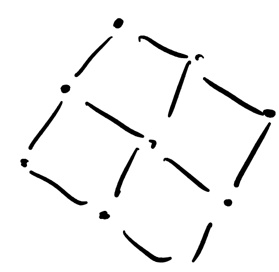
Type A : $\quad Gr(k,n) := \{ V \subset \mathbb{C}^n : \dim V = k \}$

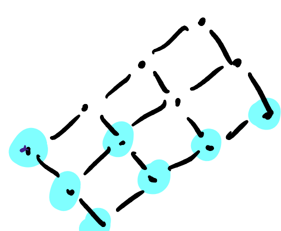
G/P miniscule flag variety

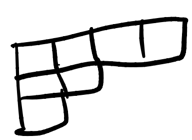
$\Lambda_{G/P}$ knows the schubert decomposition of G/P.

Schubert cells of $G/P \iff J(\Lambda_{G/P})$

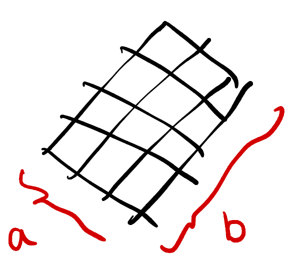


$\Lambda_{Gr(3,6)} =$ 

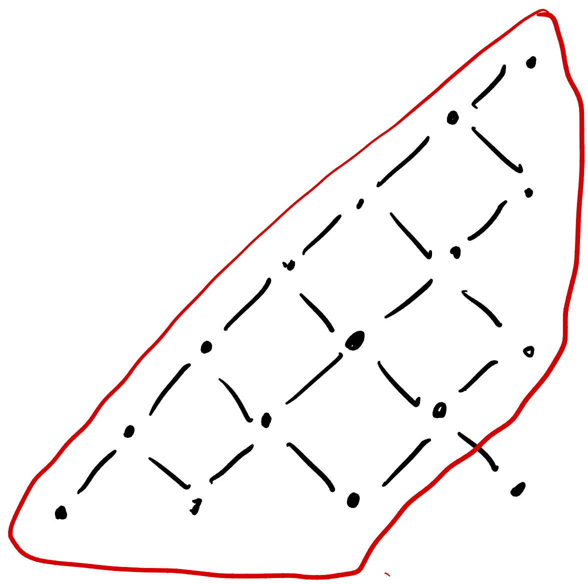


 partition Ferrer diagram.

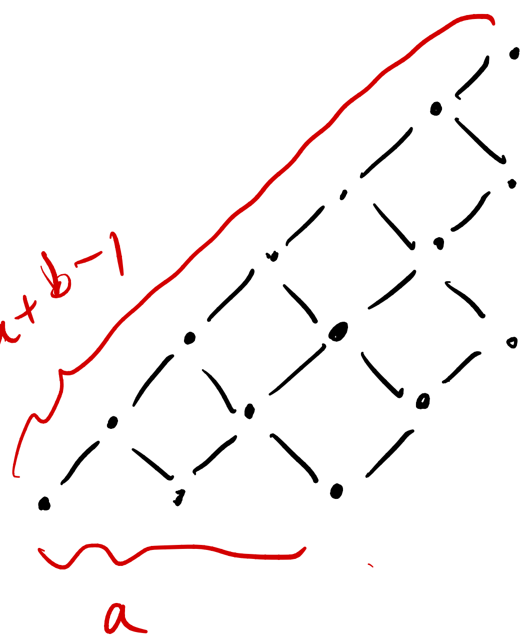
$R_{a,b}$ rectangle poset.



$$\Lambda_{Gr(k,n)} \approx \Phi^+_{B_{k,n}}$$ (Proctor)



Trapezoid poset. $T_{a,b}$

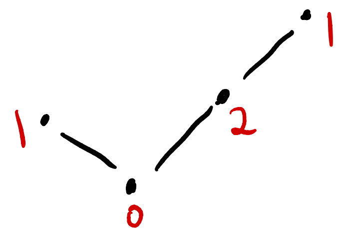


$T_{3,5}$

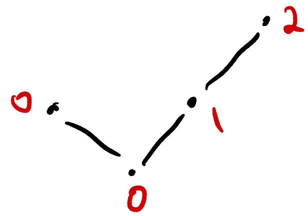
$$R_{ab} \approx T_{a,b}$$

Def. (P-partition) P poset.

a P-partition of P is an $\boxed{\text{order-preserving}}$ φ map from P to $\mathbb{N}$. $a \leq_P b \Rightarrow \varphi(a) \leq \varphi(b)$



is not a P-partition.



is a P-partition.

$PP^m(P) :=$ set of all P-partitions of P with height m.

$\shortparallel$

maps from P to $[m]$.

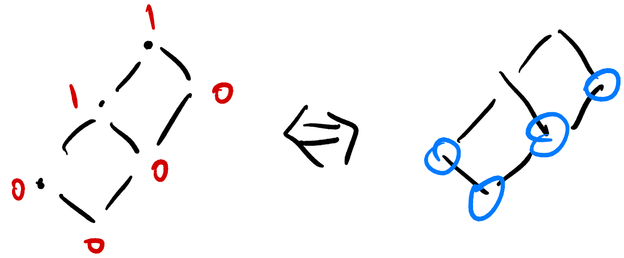
order polynomial of P $\quad \Omega_P(m) = \# PP^{m-1}$

Thm. $\Omega_{R_{ab}}(m) = \Omega_{T_{ab}}(m)$

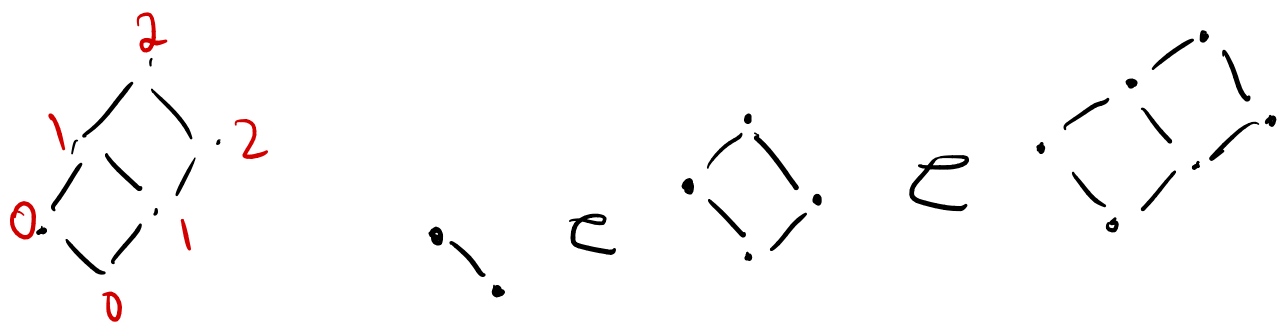
(Proctor 1983)

(bijective. HPPW 2016) $\# PP^m R_{ab} = \# PP^m T_{a,b} \quad \forall\, a, b.$

Remark. $PP^1(P) = J(P)$



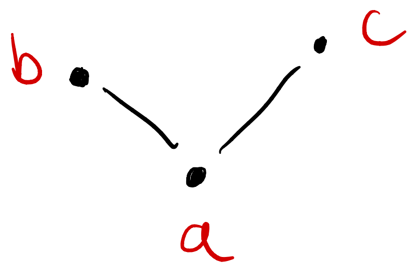
In general $PP^n(P)$ are sequences of order ideals.



Theorem (MacMahon 1915).

$$\Omega_{R_{a,b}}(m) = \prod_{j=1}^{a} \prod_{i=1}^{b} \frac{i+j+m-2}{i+j-1}$$

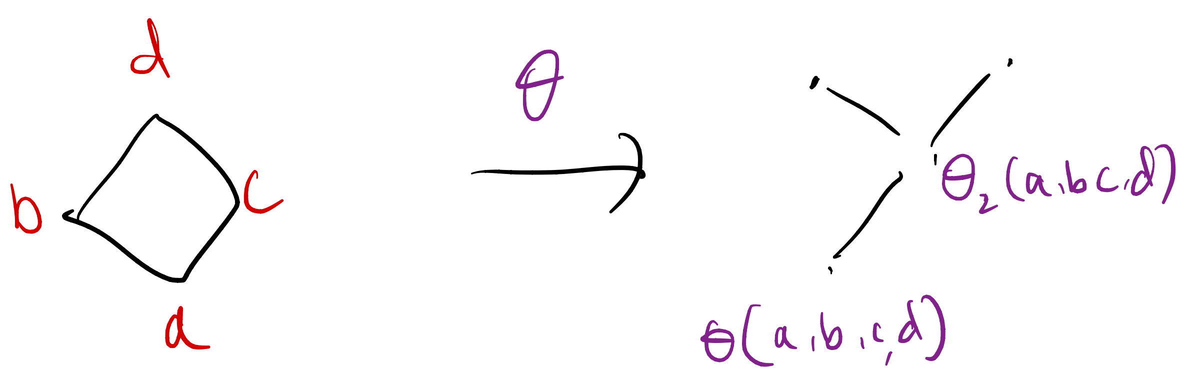
Order polytope



$$\left.\begin{array}{l} 0 \le a \le b \\ 0 \le a \le c \\ a \le b, c \le x \end{array}\right\} \Rightarrow \text{order polytope}$$

Conj (work-in-pagress of Liu & ?)

order polytope of R_{ab} & T_{ab} are isomorphic.



Dynamics

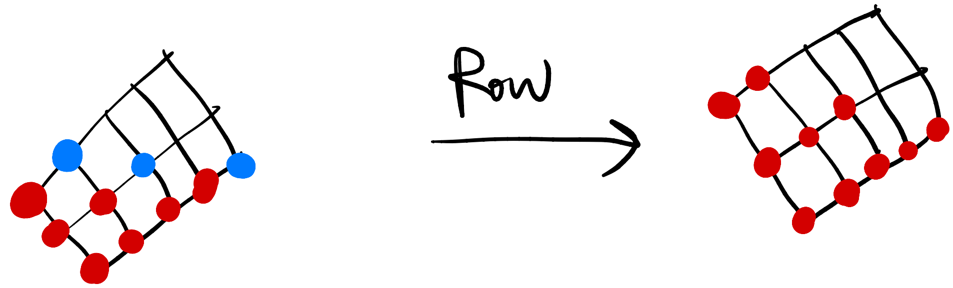
Heuristic of S. Hopkins.

$\Omega_{P_1} = \Omega_{P_2} \iff P_1 P_2$ have same <u>rowmotion</u> orbit.

<u>Rowmotion</u>. on order ideals.

$I \subset J(P)$

$\mathrm{Row}(I) \overset{\text{def}}{=\!=}$
① find $P - I$
② $\min(P-I)$
③ order ideal generated by

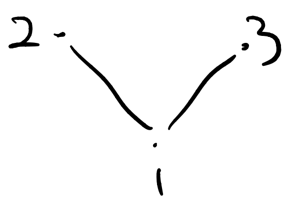


Claim: Row is invertible.

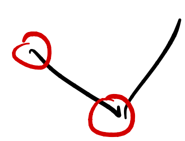
Rowmotion via toggles.

P poset, $I \subset J(P)$, $q \in P$

$$\tau_q(I) = \begin{cases} I \cup q & \text{if } q \notin I \ \& \ I \cup q \in J(P) \\ I \setminus q & \text{if } q \in I \ \& \ I \setminus q \in J(P) \\ I & \text{otherwise.} \end{cases}$$



$I = \{1, 2\}$



$\tau_3(I) = \{1, 2, 3\}$

$\tau_2(I) = \{1\}$

$\tau_1(I) = I$

Remark. τ_i is an involution.

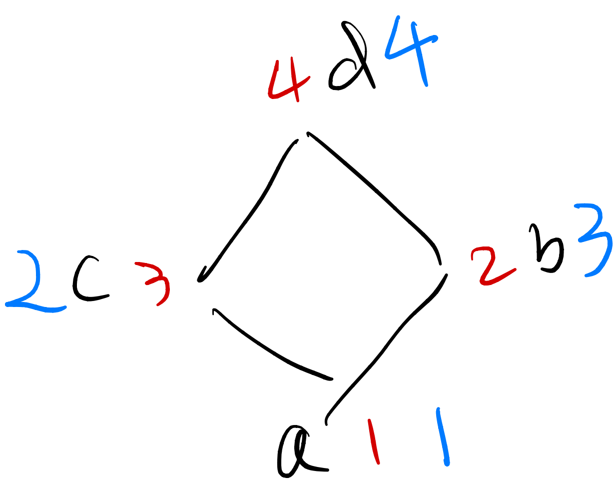
Thm (Cameron — Fon-der-Flaass).

Take any strick numbering of P.

linear extension.

then $\mathrm{Row} = \tau_{p_1} \tau_{p_2} \tau_{p_3} \cdots \tau_{p_k}$.

from top to bottom based on this numbering.



$$\mathrm{Row} = \tau_a \tau_b \tau_c \tau_d$$

$$= \tau_a \tau_c \tau_b \tau_d$$

<u>Exerise</u>. 1.0

prove that $\tau_a \circ \tau_b = \tau_b \circ \tau_a$

when a & b are not comparable.

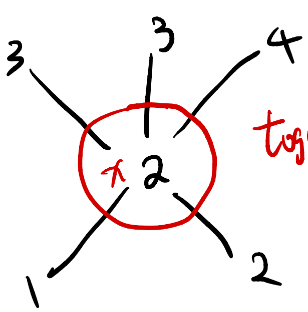
Piecewise linear Toggle.

$\varphi \in PP^m(P)$

$\tau_q(\varphi) = \varphi' \overset{?}{\in} PP^m(P)$

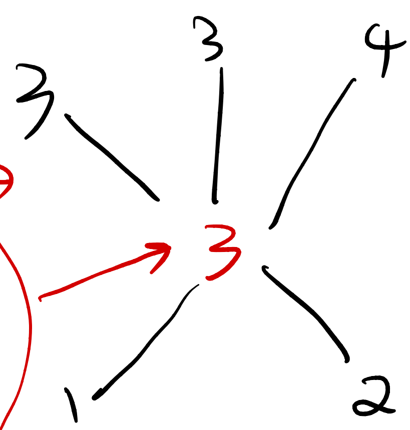
$$\varphi'(t) = \begin{cases} \varphi(t) & \text{if } \quad t \neq q \\ \min\{\varphi(i) \mid q \lessdot i\} + \max\{\varphi(j) \mid j \lessdot q\} - \varphi(q) & t = q \end{cases}$$

$t \in P$

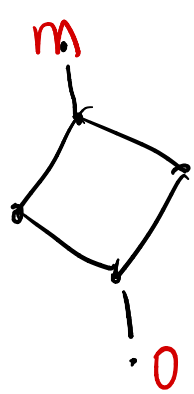


toggle at x $\longrightarrow$

min{3 3 4}
+ max{1, 2}
− 2



$\mathrm{Row}_{PL}(\varphi) \overset{\text{def}}{:=}$ PL toggle from to to bottom



Exercise 1.1: PL toggle preserves inequalities.

Exercise 1.2 PL Rowmotion on $PP^1(P)$ is the same as orderideal rowmotion on $J(P)$

Conj (Hopkins)
$PP^m(R_{ab})$ & $PP^m(T_{ab})$ have the same rowmotion orbit.

$Row^{a+b} = id$.

This is only known for $m=1$ for T_{ab}.

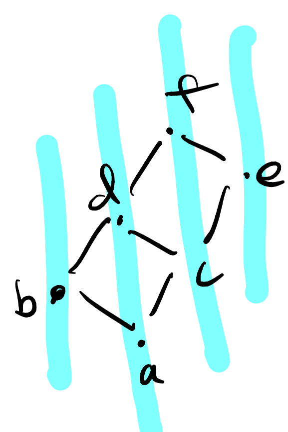
Question Find a bijection

$$\overline{\Psi} : PP^m R_{ab} \longrightarrow PP^m T_{ab}$$

$$\begin{tikzcd}
PP^m R_{ab} \arrow[r, "\mathrm{Row}"] \arrow[d, "\overline{\Psi}"'] & PP^m R_{ab} \arrow[d, "\overline{\Psi}"] \\
PP^m T_{ab} \arrow[r, "\mathrm{Row}"'] & PP^m T_{ab}
\end{tikzcd}$$

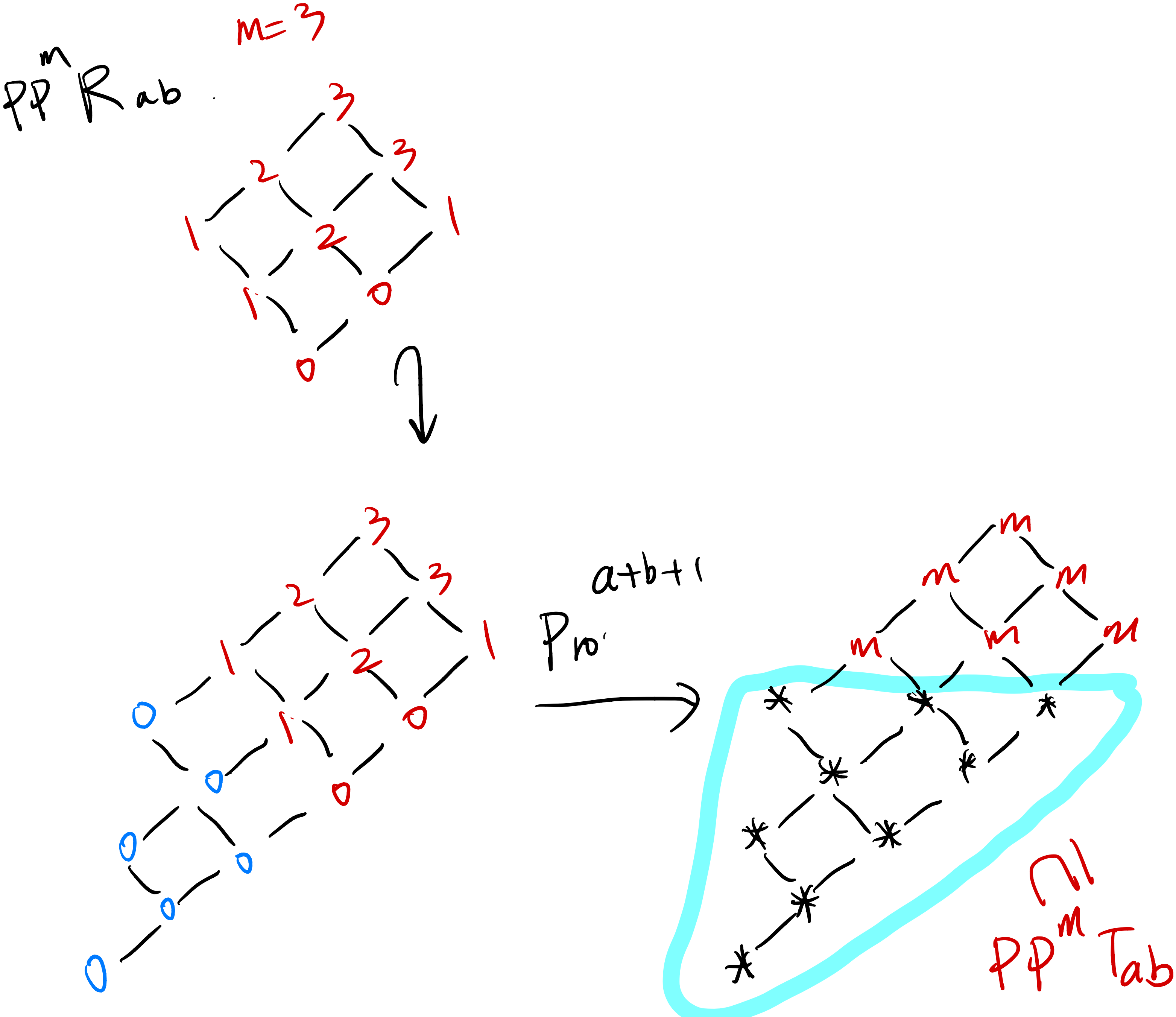
Conjecture: Rowmotion $\rightsquigarrow$ $\boxed{\text{Promotion}}$

toggle from left to right



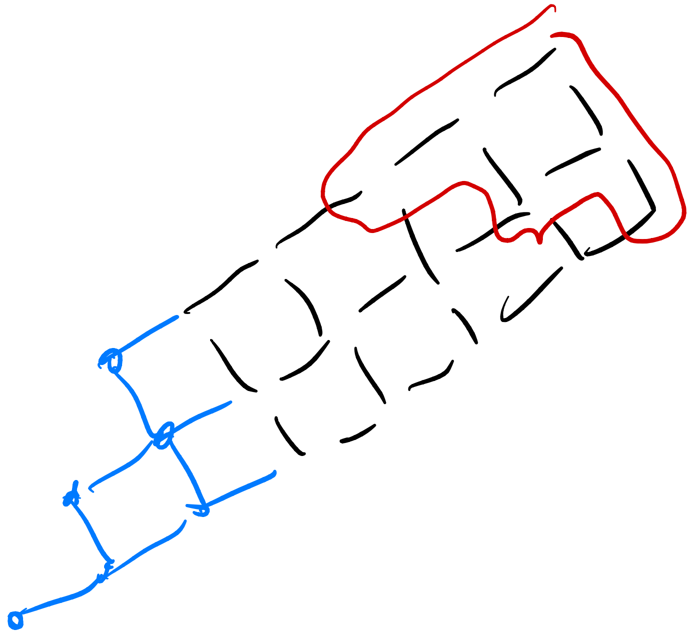
$$\mathrm{Pro} = \tau_e \tau_f \tau_c \tau_d \tau_a \tau_b$$

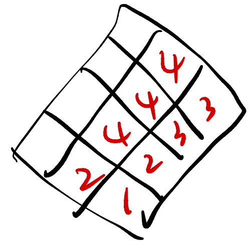
Fact: have same orbit.



Top entries will all end up be $\underline{m}$.

If this is true the induces a bijection commutes w/ $\underline{\mathrm{Pro}}$.





9:00

Beijing ~~8:30~~ Pm Office Hour

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10:00 Pm