Math 1271 Practice Midterm 3

April 21, 2021

Question 1 (section 4.4). Find the limits

(1) $\lim_{x\to 0} \dfrac{\cos 2x - 1}{e^{3x} - 3x + 2}$

solution. Use L'Hopital's rule. Taking derivative we get $\lim_{x\to 0} \dfrac{-2\sin(2x)}{3e^{3x} - 3}$ which has the form $\frac{0}{0}$. So use L'Hopital again: $\lim_{x\to 0} \dfrac{-4\cos(2x)}{9e^{3x}} = -\dfrac{4}{9}$ □

(2) $\lim_{x\to\infty} (2x)^{1/\ln x}$

solution. Let $y = \lim_{x\to\infty} (2x)^{1/\ln x}$, we have

$$\ln(y) = \lim_{x\to\infty} \ln\left((2x)^{1/\ln x}\right) = \lim_{x\to\infty} \frac{1}{\ln x} \ln(2x)$$

Then use L'Hopital's rule, take the derivative

$$\lim_{x\to\infty} \frac{\ln(2x)}{\ln x} = \lim_{x\to\infty} \frac{2\frac{1}{2x}}{\frac{1}{x}} = 1$$

Therefore $\ln(y) = 1 \implies y = e$. □

Question 2 (section 4.7).

(1) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

solution. Suppose the base of the cylinder has radius r, and the cylinder has hight h. Then we know that:

$$(2r)^2 + h^2 = 20^2$$

which means $r^2 = \frac{400-h^2}{4}$.

The volume of the cylinder is $V = \pi r^2 h = \pi \frac{100-h^2}{4} h$ as a function of h.

Take the derivative

$$\frac{dV}{dh} = 100\pi - \frac{3\pi}{4} h^2$$

Set this to zero, we get

$$100\pi = \frac{3\pi}{4} h^2 \implies h = \frac{20}{\sqrt{3}}$$

Then $r^2 = \frac{400-400/3}{4} \implies r = \frac{20}{\sqrt{6}}$. $\square$

(2) Find the shortest distance from the origin to the curve $x^2y^4 = 1$

Question 3 (section 4.8). Use the intermediate Value Theorem to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then by using the Newton's method find the root to four decimal places.

Question 4 (section 4.9).

(1) Find f if $f'(x) = 2x - 3\sin x + \dfrac{4}{1+x^2}$, $\quad f(0) = 5$

(2) A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

Question 5 (section 5.1-5.2). Consider the integral

$$\int_0^2 (x^2 + x)dx$$

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(1) Give a Riemann sum approximation of the integral with $n = 4$ rectangles, taking right endpoints of subintervals.

(2) Evaluate the integral using the definition of the definite integral (that is, taking a limit of Riemann sums).

solution.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^2 + \frac{2i}{n} \right) \\
=& \lim_{n\to\infty} \left(\frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} + \frac{2}{n} \sum_{i=1}^{n} \frac{2i}{n} \right) \\
=& \lim_{n\to\infty} \left(\frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{4}{n^2} \sum_{i=1}^{n} i \right) \\
=& \lim_{n\to\infty} \left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \right) \\
=& \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \lim_{n\to\infty} \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \\
=& \frac{8 \times 2}{6} + \frac{4}{2} = \frac{14}{3}
\end{aligned}$$

$\square$

(3) Now evaluate the integral using the Fundamental Theorem of Calculus.

solution.

$$\begin{aligned}&\int_0^2 (x^2+x)dx\\ =&\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^2\\ =&\left(\frac{8}{3}+\frac{4}{2}\right)-0\\ =&\frac{14}{3}\end{aligned}$$

□

For (2) the following identities may be useful:

$$\sum_{i=1}^{n} c = nc \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Question 6 (section 5.3). Evaluate $\int_1^4 \frac{1+\sqrt{x}e^x - x^3}{\sqrt{x}}dx$.

Question 7 (other questions from 5.1-3).

(1) Do question 5, but with the integral $\int_0^1 (1-2x^2)dx$

(2) Do question 5, but with the integral $\int_{-4}^5 (-3x+2)dx$

(3) Compute the derivative $\frac{d}{dx}\int_{-x^2}^{\sqrt{x}} \frac{\sin(t)}{1+t+t^2}dt$.

(4) Compute the derivative $\frac{d}{dx}\int_{\ln(x)}^{2-x} e^t\sin(t)dt$.

(5) Evaluate $\int_{-1}^3 |4-2x|dx$.

solution. Sketch the graph of the function and use geometry. □

(6) If $\displaystyle\int_0^{x^2} f(t)dt = x\sin(\pi x)$ and f is continuous, find $f(4)$.

solution. Hint: take the derivative of the left hand side: $\displaystyle\frac{d}{dx}\int_0^{x^2} f(t)dt$, then take the derivative of the right hand side $\displaystyle\frac{d}{dx}x\sin(\pi x)$, set them equal and solve for $f(x)$. □

Question 8 (section 5.2). Rewrite the following limit as definite integral, then evaluate it using FTC.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i^2/n}$$

solution.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i^2/n} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n(1+i^2/n^2)} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i^2/n^2}$$

This is by definition the definite integral

$$\int_0^1 \frac{1}{1+x^2}dx$$

□

Question 9 (section 5.4-5). Evaluate the following definite or indefinite integrals, some of which requires substitution method.

(1) $\displaystyle\int_0^{\pi/2}\frac{\cos^2\theta+1}{\cos^2\theta}\,d\theta$

(2) $\displaystyle\int\frac{x^2}{x^3+1}\,dx$

solution. $u = x^3$ □

(3) $\displaystyle\int_e^{e^4}\frac{1}{x\sqrt{\ln x}}\,dx$

solution. Let $u = \ln(x)$, then $\frac{du}{dx} = \frac{1}{x}$, i.e. $du = \frac{1}{x}dx$. Then

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx = \int_e^{e^4} \underbrace{\frac{1}{\sqrt{\ln x}}}_{=\frac{1}{\sqrt{u}}} \cdot \underbrace{\frac{1}{x}\,dx}_{=du} = \int_{\ln e}^{\ln(e^4)} \frac{1}{\sqrt{u}}du = \int_1^4 \frac{1}{\sqrt{u}}du$$

$$= 2\sqrt{u}\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$$

□

(4) $\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx$

solution. $u = x^2$ □

(5) $\displaystyle\int \sin(x)\sin(\cos(x))\ dx$

solution. $u = \cos(x)$ □

(6) $\displaystyle\int \frac{(\ln x)^2}{x}dx$

solution. $u = \ln(x)$ □

Question 10 (additional problems).

(1) Evaluate the following limit.

$$\lim_{x\to 2} \frac{e^x}{x-2}\int_2^x \frac{t-1}{t}dt$$

Hint: $\displaystyle\int_a^a f(t)dt = 0$, FTC and L'Hôpital's rule.

(2) Let $f(x) = \displaystyle\int_0^x \left(\int_1^{\sin\theta} \sqrt{1+t^4}\ dt\right) d\theta$. Find the second derivative of $f(x)$.