Let $f(x) = \frac{x^2+x}{\sqrt{x^2+x^3}}$. Find $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, and $\lim_{x\to 0} f(x)$, if they exist.

Solution: First we simplify our function.

$$\begin{aligned} f(x) &= \frac{x^2+x}{\sqrt{x^2+x^3}} \\ &= \frac{x(x+1)}{\sqrt{x^2(x+1)}} \\ &= \frac{x(x+1)}{|x|\sqrt{x+1}}. \end{aligned}$$

The reason we can do this last step is because when you square a number and take the square root, you get the same number back, but always positive. So if $x = 2$, then $x^2 = 4$, and $\sqrt{x^2} = 2$ again. But if $x = -2$, then $x^2 = 4$ and $\sqrt{x^2} = 2$ as well. In this case, we don't get x back, but we do get $\sqrt{x^2} = |x|$.

Now if x is positive, $|x|$ is just x, so

$$\begin{aligned} f(x) &= \frac{x(x+1)}{x\sqrt{x+1}} \\ &= \frac{x+1}{\sqrt{x+1}} = \sqrt{x+1}, \end{aligned}$$

and so

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x+1} = 1.$$

On the other hand, if x is negative, then $-x$ is positive, and so $|x| = -x$ (which is positive). In this case,

$$\begin{aligned} f(x) &= \frac{-x(x+1)}{x\sqrt{x+1}} \\ &= \frac{x+1}{-\sqrt{x+1}} = -\sqrt{x+1}, \end{aligned}$$

and so

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -\sqrt{x+1} = -1.$$

Since we're getting different limits from the left and from the right, we see that

$$\lim_{x\to 0} f(x) \text{ does not exist.}$$