Math 1271 Midterm 2 Preparation (Partial) Solution

Fall 2020

Question 1. Find the derivative of the following functions.

$$y = \cos(x^2+1)\sin^2(x)$$

$$y = \sin(\cos^2(x)+1)$$

Hint: Section 3.3 and 3.4, Trig derivatives and Chain rule.

(1) $2\sin(x)\cos(x)\cos(x^2+1) - 2x\sin^2(x)\sin(x^2+1)$.

(2) $-2\cos(\cos^2 x + 1)\cos(x)\sin(x)$

Question 2. Evaluate the following limits.

$$\lim_{x\to 0}\frac{\sin(x^2)}{x}$$

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{2\theta^2}$$

Hint: Try not to use l'Hôpital's Rule.

(1) 0.

(2) To avoid using l'Hôpital's rule, we need to use the trig-formula: $\cos(2x) = 1 - 2\sin^2(x)$. Rearrange this identity: $\sin^2(x) = \frac{1-\cos(2x)}{2}$, then substitute $x = \theta/2$ we get $\frac{1-\cos(\theta)}{2} = \sin^2(\frac{\theta}{2})$.

Using this, we have

$$\begin{aligned}
&\lim_{\theta\to 0}\frac{\cos\theta - 1}{2\theta^2}\\
&= \lim_{\theta\to 0} -\frac{1-\cos\theta}{2}\cdot\frac{1}{\theta^2}\\
&= \lim_{\theta\to 0} -\frac{\sin^2(\theta/2)}{\theta^2}\\
&= \lim_{\theta\to 0} -\left(\frac{\sin(\theta/2)}{\theta}\right)^2\\
&= -\left(\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta}\right)^2
\end{aligned}$$

We knew that $\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta} = \frac{1}{2}\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta/2} = \frac{1}{2}$, so the original limit equals $-(1/2)^2 = -\frac{1}{4}$.

Question 3. Evaluate the following derivatives.

$$y = \sin(e^{x^2+1}\cos(x^2))$$

$$y = x^2\ln(e^x(x^2+1))$$

Hint: Use chain rule with product rule.

Question 4. Find dy/dx by implicit differentiation.

$$xe^y = x - y$$

$$y\cos x = x^2 + y^2$$

(1) $\dfrac{dy}{dx} = \dfrac{1-e^y}{1+xe^y}$ (2) $\dfrac{dy}{dx} = \dfrac{2x + y\sin x}{\cos x - 2y}$

Question 5. Find y'' by implicit differentiation.

$$x^2 + xy + y^2 = 3$$

Question 6. Find the derivative of the following functions.

$$y = \sin^{-1}(x^2+1)$$

$$y = \arccos(\sqrt{x})$$

Hint: Turn the inverse trig functions into normal trig functions, then use implicit differentiation.

(1)

$$\begin{aligned}
y &= \sin^{-1}(x^2+1) && (1)\\
\sin(y) &= x^2+1 && (2)\\
\frac{d}{dx}\sin(y) &= \frac{d}{dx}(x^2+1) && (3)\\
\frac{d\sin(y)}{dy}\frac{dy}{dx} &= 2x && (4)\\
\cos(y)y' &= 2x && (5)\\
y' &= \frac{2x}{\cos(y)} && (6)
\end{aligned}$$

Question 7. Find dy/dx using logarithmic differentiation.

$$y = x^{x^2+1}$$

$$x^y = y^x$$

Question 8. Find dy/dx:

$$y = 2\ln(\sin(xe^x))$$

Hint: We can use log-differentiation the other way around — start with $e^y = e^{2\ln(\sin(xe^x))}$.

First notice that, using $b\ln(a) = \ln(a^b)$, we have $2\ln(\sin(xe^x)) = \ln(\sin^2(xe^x))$.

We use a reverse log-differentiation, buy taking the exponential of two sides (instead of logarithm).

$$e^y = e^{2\ln(\sin(xe^x))}$$

$$e^y = e^{\ln(\sin^2(xe^x))}$$

$$e^y = \sin^2(xe^x)$$

Then we use implicit differentiation

$$\frac{d}{dx}e^y = \frac{d}{dx}\sin^2(xe^x)$$
$$\frac{de^y}{dy}\frac{dy}{dx} = \frac{d\sin^2(xe^x)}{d\sin(xe^x)}\frac{d\sin(xe^x)}{d(xe^x)}\frac{d(xe^x)}{dx}$$
$$e^y y' = 2\sin(xe^x)\cos(xe^x)(xe^x + e^x)$$

Therefore

$$y' = \frac{2\sin(xe^x)\cos(xe^x)(xe^x + e^x)}{e^y}$$

Question 9. Suppose $y = \sqrt{x^2+1}$, y and x are both functions of t.

1. If $dx/dt = 3$, find dy/dt when $y = 4$.
2. If $dy/dt = 5$, find dx/dt when $x = 12$

Take derivative of $y = \sqrt{x^2+1}$ with respect to t, using implicit differentiation and chain rule:

$$\frac{dy}{dt} = \frac{d\sqrt{x^2+1}}{dt}$$
$$\frac{dy}{dt} = \frac{d\sqrt{x^2+1}}{d(x^2+1)}\frac{d(x^2+1)}{dx}\frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x\frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{x}{\sqrt{x^2+1}}\frac{dx}{dt}$$

(1) When $y = 4$, we have $\sqrt{x^2+1} = 4$. And then $x^2 + 1 = 16$, meaning that $x = \pm\sqrt{15}$. Plug in $\frac{dy}{dx} = 3$, $\sqrt{x^2+1} = 4$ and $x = \pm\sqrt{15}$, we get

$$\frac{dy}{dt} = \frac{\pm\sqrt{15}}{4} \times 3 = \pm\frac{3\sqrt{15}}{4}$$

(2) If $dy/dx = 5$ and $x = 12$, we have

$$5 = \frac{12}{\sqrt{12^2+1}}\frac{dx}{dt}$$

Solving this we get

$$\frac{dx}{dt} = \frac{5\sqrt{12^2+1}}{12} = \frac{5\sqrt{145}}{12}$$

Question 10. Two cars start moving from the same point, one travels south at 60 mi/h and another travels west at 25 mi/h. At what rate is the distance between the two cars increasing two hours later?

The first car travels south, let y denote the distance between the first car and the origin. The second car travels west, let x denote the distance between the second car and the origin. Let D denote the distance between the tow cars. (D, y and x are both functions of t).

We have $D^2 = x^2 + y^2$. Take derivative with respect to t, we get

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Simplify:

$$D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Two hours later, the first car traveled 60 mi/h $\times$ 2 h $=$ 120 miles, so $y = 120$; the second car traveled $25 \times 2 = 50$ miles, so $x = 50$. Then the distance between two cars is $D = \sqrt{50^2 + 120^2} = 130$. Furthermore, we knew that $\frac{dx}{dt} = 25$ and $\frac{dy}{dt} = 60$ mi/h. So plugging these values into the above equation, we have

$$130 \cdot \frac{dD}{dt} = 50 \cdot 25 + 120 \cdot 60$$

which gives us

$$\frac{dD}{dt} = 65$$

the rate of change of the distance between two cars, after two hours.

Question 11. Approximate $\ln(\sqrt{1.001})$.

Hint: need to use chain rule.

Let $f(x) = \ln(\sqrt{x})$, then we want to approximate $f(1.001)$.

Find the linearization of $f(x)$ at $a = 1$ —— $L(x) = f(1) + f'(1)(x-1)$

Let's first calculate the derivative:

$$f'(x) = \frac{d\ln(\sqrt{x})}{d\sqrt{x}} \cdot \frac{d\sqrt{x}}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

Therefore $f'(1) = \frac{1}{2}$. Hence

$$L(x) = \ln(1) + \frac{1}{2}(x-1)$$

Notice that $\ln(1) = 0$, then

$$L(x) = \frac{1}{2}(x-1)$$

Thus $f(1.001) \approx L(1.001) = \frac{0.001}{2} = 0.0005$.

Question 12. Find the absolute maximum and minimum of the function $f(x) = x^3 - 6x + 5$ on the interval $[-3, 5]$.

Hint: Find all critical points, then evaluate the function at all critical points and the end points.

Question 13. Find all critical points of the function $y = |x-1|(x+1)$.

Hint: This is similar to what we did in class. Show where the function's derivative DNE by showing the limit doesn't exist.

Question 14. Show that $x^3 + e^x = 0$ has exactly one real root.

Hint: Using mean value theorem.

Let $f(x) = x^3 + e^x$. We first prove that $f(x) = 0$ has at most one root.

For sake of contradiction, assume there are more than one real roots, then we can find at least two real roots, call them x_1 and x_2. (say $x_1 < x_2$.)

Then by MVT, there exist a number c such that $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{0 - 0}{x_1 - x_2} = 0$$

Now let's take the derivative of f: $f'(x) = 3x^2 + e^x$. Notice that $3x^2 \geq 0$ and $e^x > 0$, therefore $f'(x) > 0$. This means that there doesn't exist a number c such that $f'(c) = 0$, a contradiction. So our original assumption is false. $f(x) = 0$ has at most one root.

Having at most one root doesn't necessary mean that it has a root. So we still need to show that $f(x) = 0$ has a root.

For this we will use intermediate value theorem: Find a number a such that $f(a) > 0$ and find a number b such that $f(b) < 0$. Clearly $f(1) = 1 + e > 0$. Try $f(-2) = -8 + e^{-2} \approx -7.86 < 0$. Therefore by IVT, $f(x) = 0$ has exactly one root.

Question 15. Verify the function $f(x) = x^3 - 3x + 2$ satisfy the hypothesis of Mean-Value-Theorem on the interval $[-2, 2]$. Find all numbers c that satisfy the conclusion of Mean-Value-Theorem.

Question 16. A local maximum/minimum must be a critical points, but not all critical points are local extrema. Find the critical points of $f(x) = x^3$, and show if they are local maximum/minimum using first derivative text.

Question 17. Let $f(x) = x^3 - 2x^2 - 4x + 1$.

1. Find the intervals on which f is increasing and decreasing.
2. Find the local maximum and minimum of f.
3. Find the intervals of concavity and the inflection points.

Question 18. Let $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$.

1. Find the vertical and horizontal asymptote.
2. Find the interval of increase and decrease.
3. Find the local maximum and minimum.
4. Find the interval of concavity and inflection points.
5. Sketch the curve.