ARBORESCENCES OF COVERING GRAPHS

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Abstract. An *arborescence* of a directed graph Γ is a spanning tree directed toward a particular vertex v. The arborescences of a graph rooted at a particular vertex may be encoded as a polynomial $A_v(\Gamma)$ representing the sum of the weights of all such arborescences. The arborescences of a graph and the arborescences of a covering graph $\tilde{\Gamma}$ are closely related. Using *voltage graphs* to construct arbitrary regular covers, we derive a novel explicit formula for the ratio of $A_v(\Gamma)$ to the sum of arborescences in the lift $A_{\tilde{v}}(\tilde{\Gamma})$ in terms of the determinant of Chaiken's voltage Laplacian matrix, a generalization of the Laplacian matrix. Chaiken's results on the relationship between the voltage Laplacian and vector fields on Γ are reviewed, and we provide a new proof of Chaiken's results via a deletion-contraction argument.

1. Introduction

In this paper, we examine the relationship between arborescences of a graph and the arborescences of its covering graph. An *arborescence* rooted at a vertex v in a directed graph Γ is a spanning tree of Γ that is directed towards v. We define $A_v(\Gamma)$ to be the sum of the weights of all arborescences in Γ rooted at v. Using the Matrix Tree Theorem [FS99, Theorem 5.6.8], we can compute $A_v(\Gamma)$ as a minor of the Laplacian matrix of Γ.

It is natural to ask to what extent the arborescences of a graph Γ characterize the arborescences of a covering graph $\tilde{\Gamma}$. Every arborescence of Γ lifts to a partial arborescence of $\tilde{\Gamma}$, and this lift is unique if the root of the arborescence in $\tilde{\Gamma}$ is fixed. Conversely, every arborescence of $\tilde{\Gamma}$ descends to a subgraph of Γ containing an arborescence. These properties lead us to ask whether there is a meaningful relationship between $A_v(\Gamma)$ and $A_{\tilde{v}}(\tilde{\Gamma})$, where $\tilde{v}$ is a lift of v. We show that $A_v(\Gamma)$ always divides $A_{\tilde{v}}(\tilde{\Gamma})$, meaning that each arborescence of Γ corresponds to a set of arborescences of $\tilde{\Gamma}$. The primary goals of this paper are to derive an explicit formula for the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ and to examine cases where this ratio is especially computationally nice.

The ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ first arose in Galashin and Pylyavskyy's study of *R-systems* [GP19]. The R-system is a discrete dynamical system on a edge-weighted strongly connected simple directed graph $\Gamma = (V, E, \mathrm{wt})$ whose state vector $X = (X_v)_{v\in V}$ evolves to its next state $X' = (X'_v)_{v\in V}$ according to the following relation:

$$\sum_{(u,v)\in E} \mathrm{wt}(u,v)\frac{X_v}{X'_u} = \sum_{(v,w)\in E} \mathrm{wt}(v,w)\frac{X_w}{X'_v} \tag{1}$$

This system is homogeneous in both X and X', so we consider solutions in projective space. Galashin and Pylyavskyy determined all solutions X' of this equation as a function of X:

Theorem 1.1. [GP19] *The system given by equation (1) has solution*

$$X'_v = \frac{X_v}{A_v(\Gamma)}.$$

This solution is unique up to scalar multiplication, yielding a unique solution to the R-system in $\mathbb{P}^{|V|}$.

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However, we can see the value of X'_v in equation (1) depends only on the neighborhood of the vertex v. Thus, in the case of a covering graph $\tilde{\Gamma}$, we may find two solutions to the R-system: one by applying the previous theorem directly, and one by treating each vertex of $\tilde{\Gamma}$ locally like a vertex of Γ, and then applying the theorem. The two respective solutions are

$$X'_{\tilde{v}} = \frac{X_{\tilde{v}}}{A_{\tilde{v}}(\tilde{\Gamma})} \qquad \text{and} \qquad X'_{\tilde{v}} = \frac{X_{\tilde{v}}}{A_v(\Gamma)}.$$

Therefore, uniqueness of the solution implies that the vectors

$$\left(\frac{X_{\tilde{v}}}{A_{\tilde{v}}(\tilde{\Gamma})}\right)_{\tilde{v}\in\tilde{V}} \qquad \text{and} \qquad \left(\frac{X_{\tilde{v}}}{A_v(\Gamma)}\right)_{\tilde{v}\in\tilde{V}}$$

are scalar multiples of each other, where $\tilde{v}$ is any lift of v. Equivalently:

Corollary 1.2. *When Γ is strongly connected and simple, the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is independent of the choice of vertex v and of the choice of lift $\tilde{v}$.*

The existence of this invariance motivates finding an explicit formula for this ratio. The following is the main theorem of this paper.

Theorem 1.3. *Let $\Gamma = (V, E, \mathrm{wt})$ be an edge-weighted multigraph, and let $\tilde{\Gamma}$ be a k-fold cover of Γ. Let $\mathscr{L}(\Gamma)$ be the voltage Laplacian of Γ. Then for any vertex v of Γ and any lift $\tilde{v}$ of v in $\tilde{\Gamma}$ of Γ, we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k}\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}.$$

If $\tilde{\Gamma}$ is a regular cover, it is a derived cover by a group G with $|G| = k$. In this case, in the above formula $\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$ is the determinant of $\mathscr{L}(\Gamma)$ as a $\mathbb{Z}[E]$-linear transformation. We can take this determinant by restriction of scalars (see Section 3 for details). For an arbitrary cover (including non-regular ones), the matrix $[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$ can be determined concretely from the covering graph (Definition 3.5).

When $\tilde{\Gamma}$ is a regular cover of prime order, we have the following refinement:

Corollary 1.4. *Let p be a prime, let $\Gamma = (V, E, \mathrm{wt}, \nu)$ be an edge-weighted $\mathbb{Z}/p\mathbb{Z}$-voltage directed multigraph, and let $\mathscr{L}(\Gamma)$ be its voltage Laplacian matrix. Then for any vertex v of Γ and any lift $\tilde{v}$ of v in the derived graph $\tilde{\Gamma}$ of Γ, we have*

$$\begin{aligned}\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} &= \frac{1}{|G|}N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\left[\det\mathscr{L}(\Gamma)\right]\\ &= \frac{1}{|G|}\prod_{i=1}^{p-1}\det[\sigma_i(\mathscr{L}(\Gamma))]\end{aligned}$$

where $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\left[\det\mathscr{L}(\Gamma)\right]$ denotes the field norm of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$, naturally extended to a norm on $\mathbb{Q}(\zeta_p)[E]$, and σ_i is the field automorphism on $\mathbb{Q}(\zeta_p)$ mapping $\zeta_p \mapsto \zeta_p^i$.

In the case $|G| = 2$, we obtain a conjecture by Galashin and Pylyavskyy:

Corollary 1.5. *Let $\Gamma = (V, E, \mathrm{wt}, \nu)$ be an edge-weighted $\mathbb{Z}/2\mathbb{Z}$-voltage directed multigraph, and let $\mathscr{L}(\Gamma)$ be its voltage Laplacian matrix. Then for any vertex v of Γ and any lift $\tilde{v}$ of v in the derived graph $\tilde{\Gamma}$ of Γ, we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{2}\det\mathscr{L}(\Gamma).$$

Note that Corollary 1.5 follows directly from Corollary 1.4 by setting $p = 2$ and noting that σ_1 is the identity.

Theorem 1.3 allows us to easily conclude nice properties about the ratio:

Corollary 1.6. *If the edge weights of Γ are indeterminants then the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is a homogeneous polynomial in the edge weights with integer coefficients.*

Proof. Since $\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]} \in \mathbb{Z}[E]$, Theorem 1.3 tells us that $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} \in \mathbb{Q}[E]$. Note that every coefficient of $A_v(\Gamma)$ is 1 since every edge of Γ has a different weight. Therefore, $A_v(\Gamma)$ is a primitive polynomial over the integers, so by Gauss' lemma, $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} \in \mathbb{Z}[E]$.

Homogeneity follows from the fact that every arborescence of a given graph has the same number of edges. ■

We furthermore conjecture the following (see Section 5):

Conjecture 1.7. *The ratio* $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ *has positive integer coefficients.*

This conjecture suggests there may be a combinatorial interpretation of $\det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma)$.

The rest of the paper will proceed as follows. Section 2 covers the background and conventions necessary to read this paper. In this section, we also discuss the Laplacian matrix and the Matrix Tree Theorem in greater detail, and give additional topological background on covering graphs. In particular, we introduce the *voltage graph*, a construction that allows us to compactly describe arbitrary regular covering graphs $\tilde{\Gamma}$ by assigning a group-valued voltage to each edge of Γ. In Section 3, we prove the main theorem. We also describe restriction of scalars and prove Corollary 1.4. Section 4 reviews some known results relating *vector fields* on voltage graphs to the voltage Laplacian. Vector fields are closely related to arborescences, and this discussion especially helps to frame the results of the case of 2-fold covers. We conclude with several open questions in Section 5.

2. Background and Definitions

2.1. **Arborescences.** Let $\Gamma = (V, E, \mathrm{wt})$ be an edge-weighted directed multigraph with a weight function on the edges $\mathrm{wt} : E \to R$, for some ring R. We will usually abbreviate "edge-weighted" to "weighted" and "directed multigraph" to "graph." We will consider the weights of the edges of G to be indeterminates, treating the weight $\mathrm{wt}(e)$ of an edge e as a variable. Let the set of such variables be denoted $\mathrm{wt}(E)$. We denote the source vertex of an edge e by e_s and target vertex of e by e_t. If an edge has source v and target w, we may write $e = (v, w)$. However, note that when Γ is not necessarily simple, there may be more than one edge satisfying these properties, so (v, w) may specify multiple edges. We denote the set of outgoing edges of a vertex v by $E_s(v)$, and the set of incoming edges of v by $E_t(v)$.

Definition 2.1. An *arborescence* T of Γ rooted at $v \in V$ is a spanning tree directed towards v. That is, for all vertices w, there exists a unique directed path from w to v in T. [1] We denote the set of arborescences of Γ rooted at vertex v by $\mathcal{T}_v(\Gamma)$. The *weight of an arborescence* $\mathrm{wt}(T)$ is the product of the weights of its edges:

$$\mathrm{wt}(T) = \prod_{e \in T} \mathrm{wt}(e)$$

We denote by $A_v(\Gamma)$ the sum of the weights of all arborescences of Γ rooted at v:

$$A_v(\Gamma) = \sum_{T \in \mathcal{T}_v(\Gamma)} \mathrm{wt}(T)$$

$A_v(\Gamma)$ is either zero or a homogeneous polynomial of degree $|V| - 1$ in the edge weights of G.

[1] In the literature, an arborescence rooted at v is usually defined to to be a spanning tree directed *away* from v, so that v is the unique source rather than the unique sink; see, for example, [KV06], [Cha82], and [GM89]. Our convention is consistent with the study of R-systems.

2.2. **The Laplacian matrix and the Matrix Tree Theorem.** The Matrix Tree Theorem, also known as Kirchoff's Theorem, yields a way of computing $A_v(\Gamma)$ through the *Laplacian matrix* of Γ.

Definition 2.2. Label the vertices of Γ as $v_1, v_2, \ldots$. The *Laplacian matrix* $L(\Gamma)$ of a graph Γ is the difference of the weighted degree matrix D and the weighted adjacency matrix A of Γ:

$$L(\Gamma) = D(\Gamma) - A(\Gamma).$$

Here, the weighted degree matrix is the diagonal matrix whose i-th entry is

$$d_{ii} = \sum_{e \in E_s(v_i)} \mathrm{wt}(e)$$

and the weighted adjacency matrix has entries defined by

$$a_{ij} = \sum_{e=(v_i,v_j)} \mathrm{wt}(e).$$

Since we will always be working with weighted graphs in this paper, we will usually drop the word "weighted" when talking about the Laplacian matrix. Note also the ordering of the rows and columns of the Laplacian. We will always assume that v_1 corresponds to the first row and column of $L(\Gamma)$, that v_2 corresponds to the second row and column of $L(\Gamma)$, and so on.

Theorem 2.3. (Matrix Tree Theorem) [Cha82] *The sum of the weights of arborescences rooted at v_i is equal to the minor of $L(\Gamma)$ obtained by removing the i-th row and column:*

$$A_{v_i}(\Gamma) = \det L_i^i(\Gamma).$$

2.3. **Covering graphs, voltage graphs, and derived graphs.**

Definition 2.4. A *k-fold cover* of $\Gamma = (V,E)$ is a graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ that is a k-fold covering space of G in the topological sense that preserves edge weight. In order to use this definition, we need to find a way to formally topologize directed graphs in a way that encodes edge orientation. To avoid this, we instead give a more concrete alternative definition of a covering graph. The graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ is a k-fold covering graph of $\Gamma = (V,E)$ if there exists a projection map $\pi : \tilde{\Gamma} \to \Gamma$ such that

(1) π maps vertices to vertices and edges to edges;
(2) $|\pi^{-1}(v)| = |\pi^{-1}(e)| = k$ for all $v \in V, e \in E$;
(3) For all $\tilde{e} \in \tilde{E}$, we have $\mathrm{wt}(\tilde{e}) = \mathrm{wt}(\pi(\tilde{e}))$;
(4) π is a local homeomorphism. Equivalently, $|E_s(\tilde{v})| = |E_s(\pi(\tilde{v}))|$ and $|E_t(\tilde{v})| = |E_t(\pi(\tilde{v}))|$ for all $\tilde{v} \in \tilde{V}$.

We do not require a covering graph to be connected—results about arborescences are trivial in the disconnected case anyways.

Definition 2.5. Let G be a finite group. A *weighted G-voltage graph* $\Gamma = (V, E, \mathrm{wt}, \nu)$ is a weighted directed multigraph with each edge e also labeled by an element $\nu(e)$ of G. This labeling is called the *voltage* of the edge e. Note that the voltage of an edge e is entirely distinct from the weight of e.

Definition 2.6. Given a G-voltage graph Γ, we may construct an $|G|$-fold covering graph of Γ

known as the *derived graph* $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \mathrm{wt})$. The derived graph of a voltage graph is a graph with vertex set $\tilde{V} = V \times G$ and edge set

$$\tilde{E} := \{[v \times x, w \times (gx)] : x \in G, e = (v,w) \in \Gamma, \nu(e) = g \in G\}.$$

Example 2.7. Let $G = \mathbb{Z}/3\mathbb{Z} = \{1, g, g^2\}$, and let Γ be the G-volted graph shown in Figure 1, where edges labeled (x,y) have edge weight x and voltage y. Then the derived graph $\tilde{\Gamma}$, with vertices $(v,y) = v^y$ and with edges labeled by weight, is shown in Figure 2.

While derived graphs might seem to be a very special subset of covering graphs, they in fact give rise to a broad class of covering graphs called *regular covering graphs*.

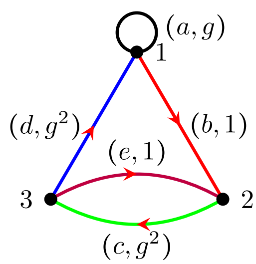


FIGURE 1. A $\mathbb{Z}/3\mathbb{Z}$-voltage graph Γ.

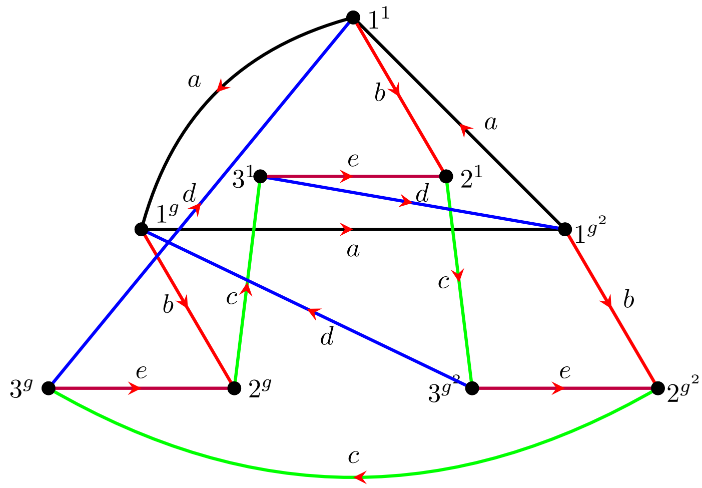


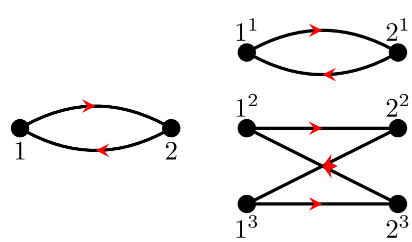
FIGURE 2. The derived covering graph $\tilde{\Gamma}$ of Γ in Figure 1. Edge colors denote correspondence to the edges of Γ via the quotient map.

Definition 2.8. Given a graph Γ and a covering graph $\tilde{\Gamma}$, the *deck group* $\mathrm{Aut}(\pi)$ of $\tilde{\Gamma}$ is the group of automorphisms on $\tilde{\Gamma}$ that preserve the fibers of the projection map π.

Definition 2.9. A *regular cover* $\tilde{\Gamma}$, sometimes known as a *Galois cover*, of a graph Γ is a covering graph whose deck group is transitive on each fiber $\pi^{-1}(v)$ for each $v \in V$.

Example 2.10. The derived graph in Example 2.7 is a regular cover because the cyclic permutation σ that sends each $v_{i,x}$ to $v_{i,gx}$ is in $\mathrm{Aut}(\pi)$, which shows that $\mathrm{Aut}(\pi)$ is transitive on each fiber $\pi^{-1}(v)$.

Example 2.11. The following is a simple example of a graph (left) and a non-regular covering graph (right): No automorphism maps vertex 1_1 to vertex 1_2, since, for example, 1_1 is part of a 2-cycle and 1_2 is not, so



$\mathrm{Aut}(\pi)$ is not transitive on $\pi^{-1}(1)$. Nevertheless, all criteria necessary to be a covering graph are met.

Theorem 2.12. (Theorems 3 and 4 in [GT75]) *Every regular cover $\tilde{\Gamma}$ of a graph Γ may be realized as a derived cover of Γ with voltage assignments in $\mathrm{Aut}(\pi)$. Conversely, every derived graph is a regular cover.*

The majority of this paper explores the relationship between the arborescences of a voltage graph Γ and the arborescences of its derived graph $\tilde{\Gamma}$. Theorem 2.12 allows us to deal with all regular covering graphs in the framework of a voltage. It turns out that regularity is not necessary for Theorem 1.3, which holds for all k-fold covers; however, the results of this main theorem have nice interpretations in terms of the voltage Laplacian in the regular case.

2.4. **The reduced group algebra.** We wish to define a matrix similar to the Laplacian matrix that tracks all the relevant information in an G-voltage graph. In order to do so in general, we need to extend our field of coefficients in order to codify the data given by the voltage function ν. Following the language of Reiner and Tseng in [RT14]:

Definition 2.13. The *reduced group algebra* of a finite group G over a ring R is the quotient

$$\overline{R[G]} = \frac{R[G]}{\left\langle \sum_{g\in G} h \right\rangle},$$

where $R[G]$ is the group algebra of G over R. That is, we quotient the group algebra by the all-ones vector with respect to the basis given by G.

For simplicity, in the remainder of this paper we take $R = \mathbb{Z}$. Note that if $G \cong \mathbb{Z}/2\mathbb{Z}$, then $\overline{\mathbb{Z}[G]} \cong \mathbb{Z}$, with the non-identity element of G identified with -1.

Similarly, if $G \cong \mathbb{Z}/p\mathbb{Z}$, with p prime, then $\overline{\mathbb{Z}[G]} \cong \mathbb{Z}(\zeta_p)$, where ζ_p is a primitive p-th root of unity and the generator g of G is identified with ζ_p. (To see this, note that both rings arise by adjoining to $\mathbb{Z}$ an element with minimal polynomial $\sum_{i=0}^{p-1} x^i$.) The fact that the reduced group alebgra of prime cyclic G lies in a field extension over $\mathbb{Q} \supseteq \mathbb{Z}$ will be important later in giving us nice formulas for the ratio of arborescences described in the introduction.

2.5. **The voltage Laplacian matrix.** We now define a generalization of the Laplacian matrix that takes into account voltages:

Definition 2.14. The *voltage adjacency matrix* $\mathscr{A}(G)$ has entries given by

$$a_{ij} = \sum_{e=(v_i,v_j)\in E} \nu(e)\mathrm{wt}(e),$$

where we consider $\nu(e)$ as an element of the reduced group algebra $\overline{\mathbb{Z}[G]}$. That is, the i,j-th entry consists of sum of the *volted* weights of all edges going from the i-th vertex to the j-th vertex. The *voltage Laplacian matrix* $\mathscr{L}(\Gamma)$ is defined as

$$\mathscr{L}(\Gamma) = D(\Gamma) - \mathscr{A}(\Gamma)$$

where $D(\Gamma)$ is the (unvolted) weighted degree matrix as described in Definition 2.2.

Note that when every edge has trivial voltage, then $\mathscr{L}(\Gamma) = L(\Gamma)$, so that the voltage Laplacian is indeed a generalization of the Laplacian. Since we consider the edge weights of Γ as indeterminates, we treat the entries of $\mathscr{L}(G)$ as elements of $\overline{\mathbb{Z}[G]}[\mathrm{wt}(E)]$—that is, the polynomial ring of edge weights with coefficients in the reduced group algebra. We will often abuse notation and refer to this ring as simply $\overline{\mathbb{Z}[G]}[E]$.

Example 2.15. Let Γ the $\mathbb{Z}/3\mathbb{Z}$-voltage graph in Figure 1. Under the identification $\overline{\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]} \cong \mathbb{Z}(\zeta_3)$, the voltage Laplacian of Γ is

$$\begin{aligned}\mathscr{L}(\Gamma) &= \begin{bmatrix} a+b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d+e \end{bmatrix} - \begin{bmatrix} \zeta_3 a & b & 0 \\ 0 & 0 & \zeta_3^2 c \\ \zeta_3^2 d & e & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-\zeta_3)a+b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{bmatrix}\end{aligned}$$

2.6. **Notation.** Before diving into the main result, we summarize the conventions and notation that will be used consistently throughout the rest of the paper. For a graph $\Gamma = (V, E, \mathrm{wt})$ and a covering graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \mathrm{wt})$,

- The parameter n refers to the number of vertices of Γ, and we write $V = \{v_i : i \in [n]\}$
- The parameter k refers to the order of the cover, i.e. $\tilde{\Gamma}$ is a k-fold covering graph of Γ. When the cover is regular, we have $k = |G|$.
- We write the vertices of $\tilde{\Gamma}$ as $\tilde{V} = \{v_i^j : i \in [n], j \in [k]\}$, where $v_i^1, \cdots, v_i^k$ are the lifts of v_i. For regular covers, we write $\tilde{V} = \{v_i^g : i \in [n], g \in G\}$.
- In the case of a regular G-cover, the edges in $\tilde{\Gamma}$ are $\tilde{E} = \{(v_i^h, v_j^{\nu(v_i)h}) : (v_i, v_j) \in E, h \in G\}$. When the cover is not necessarily regular, we assign a permutation in $\mathfrak{S}_k$ to each edge of Γ. Then an edge $e = (v_a, v_b)$ with permutation σ is lifted to the edges $\{(v_a^i, v_b^{\sigma(i)}) : i \in [k]\}$; by abuse of notation, we write $\nu(e) = \sigma$. Note the difference between $\sigma(i)$ and group multiplication. For example, in example 2.11, the permutations associated to the two edges are 123 and 132 in one-line notation.

3. Proof of the Main Theorem

3.1. Restriction of scalars.

Definition 3.1. Let R be a commutative ring, and let S be a free R-algebra of finite rank. Let T be an S-linear transformation on a free S-module M of finite rank. Then we may also consider M as a free R-module of finite rank, and T as an R-linear transformation; this is known as *restriction of scalars.* We write $\det_R T$ to denote the determinant of T as an R-linear transformation.

Recall that the voltage Laplacian $\mathscr{L}(\Gamma)$ has entries in the reduced group algebra augmented by edge weights: $S = \overline{\mathbb{Z}[G]}[E]$. Letting $R = \mathbb{Z}[E]$, we may also consider $\mathscr{L}(\Gamma)$ as an R-linear transformation on a R-module of rank $(k-1)n$. Note that due to the definition of the Laplacian (the fact that **row** entries sum to 0), we will consider our linear transformation to act on the right.

Example 3.2. Returning to Example 2.15, the voltage Laplacian $\mathscr{L}(\Gamma)$ is a matrix that represents a linear transformation on a $\mathbb{Z}(\zeta_3)[E]$-module with basis vectors indexed by the three vertices of Γ:

$$\mathscr{L}(\Gamma) = \begin{bmatrix} (1-\zeta_3)a + b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{bmatrix}$$

We may consider this same module as a $\mathbb{Z}[E]$-module instead, simply by disallowing scalar multiplication outside of the subring $\mathbb{Z}[E] \subseteq \mathbb{Z}(\zeta_3)[E]$. Now we look at the basis vectors of the $\mathbb{Z}[E]$-module. Since the $\mathbb{Z}[E]$-span of a set of vectors is smaller than its $\mathbb{Z}(\zeta_3)[E]$-span, however, we will need more basis vectors than before in order to span the entire module. One basis for this module has basis vectors doubly indexed by vertices and the two non-identity group elements of $\mathbb{Z}/3\mathbb{Z}$, which shows that the module has $\mathbb{Z}[E]$-rank 6. Ordering basis vectors as $v_1^g, v_2^g, v_3^g, v_1^{g^2}, v_2^{g^2}, v_3^{g^2}$, the voltage Laplacian may considered as a $\mathbb{Z}[E]$-linear transformation, with matrix

$$[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]} = \begin{bmatrix} a+b & -b & 0 & -a & 0 & 0 \\ 0 & c & c & 0 & 0 & c \\ d & -e & d+e & d & 0 & 0 \\ a & 0 & 0 & 2a+b & -b & 0 \\ 0 & 0 & -c & 0 & c & 0 \\ -d & 0 & 0 & 0 & -e & d+e \end{bmatrix}$$

and the $\mathbb{Z}[E]$-determinant of this transformation is

$$\begin{aligned} \det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma) &:= \det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]} \\ &= 3a^2c^2d^2 + 3b^2c^2d^2 + 6abc^2d^2 + 9a^2c^2e^2 + 3b^2c^2e^2 + 9abc^2e^2 + 9a^2c^2de + 3b^2c^2de + 12abc^2de \end{aligned}$$

3.2. **The prime cyclic case.** With restriction of scalars in hand, we can now prove our formula for the prime cyclic case (Corollary 1.4) follows from the main theorem (Theorem 1.3).

Proof of Corollary 1.4 given Theorem 1.3. The corollary follows from the theorem if we can show that $\det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}[\det \mathscr{L}(\Gamma)]$. Theorem 1 of [Sil00] states that if A is a commutative ring, if B is a commutative subring of $\mathrm{Mat}_n(A)$, and if $M \in \mathrm{Mat}_m(B)$, then

$$\det_A M = \det_A(\det_B M)$$

In this case, let $A := \mathbb{Q}[E]$. The reduced group algebra $B := \mathbb{Z}(\zeta_p)[E]$ may be realized as a subring of $\mathrm{Mat}_{p-1}(A)$, with an element α of $\mathbb{Z}(\zeta_p)[E]$ being identified with the $\mathbb{Z}[E]$-matrix corresponding to multiplication by α in $\mathbb{Z}(\zeta_p)[E]$, where we view $\mathbb{Z}(\zeta_p)[E]$ as an $\mathbb{Z}[E]$-module. Note that $\mathbb{Z}[E]$ and $\mathbb{Z}(\zeta_p)[E]$ are both commutative. Finally, we let $M = \mathscr{L}(\Gamma)$. But the field norm $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha)$ is defined as the determinant of the linear map $x \mapsto \alpha x$ as a $\mathbb{Q}$-linear transformation, or, equivalently in our case, a $\mathbb{Z}$-linear transformation. When extended to $\mathbb{Q}(\zeta_p)[E]$, this definition shows that

$$\det_{\mathbb{Z}[E]}\left(\det_{\mathbb{Z}(\zeta_p)[E]} \mathscr{L}(\Gamma)\right) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\left[\det_{\mathbb{Q}(\zeta_p)[E]} \mathscr{L}(\Gamma)\right]$$

as desired. ∎

Example 3.3. Let Γ be the graph from Figure 1. We compute $\det \mathscr{L}(\Gamma)$ in Example 4.3:

$$\det \mathscr{L}(\Gamma) = (1-\zeta_3)bcd + (1-\zeta_3)acd + (1-\zeta_3^2)bce + (1-\zeta_3)(1-\zeta_3^2)ace$$

Since voltage is given by $\mathbb{Z}/3\mathbb{Z}$, the reduced group algebra is $\mathbb{Z}(\zeta_3)[E] \subset \mathbb{Q}(\zeta_3)[E]$, which we treat as an extension over $\mathbb{Q}$. The Galois norm in this case is the same as the complex norm, since the Galois conjugate of an element of $\mathbb{Q}(\zeta_3)[E]$ is its complex conjugate. This norm is

$$\begin{aligned}\det \mathscr{L}(\Gamma) \det \overline{\mathscr{L}(\Gamma)} &= \big((1-\zeta_3)bcd + (1-\zeta_3)acd + (1-\zeta_3^2)bce + (1-\zeta_3)(1-\zeta_3^2)ace\big) \\ &\quad \cdot \big((1-\zeta_3^2)bcd + (1-\zeta_3^2)acd + (1-\zeta_3)bce + (1-\zeta_3^2)(1-\zeta_3)ace\big) \\ &= 3a^2c^2d^2 + 3b^2c^2d^2 + 6abc^2d^2 + 9a^2c^2e^2 + 3b^2c^2e^2 + 9abc^2e^2 + 9a^2c^2de + 3b^2c^2de + 12abc^2de\end{aligned}$$

which matches $\det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma)$ from Example 3.2.

3.3. **Triangularization.** In this section we build up the machinery to prove our main result, Theorem 1.3.

To prove Theorem 1.3, we carefully apply a change of basis to the Laplacian matrix of $\tilde{\Gamma}$. In order to do this, we need to fix an ordering on the basis of $\tilde{\Gamma}$. Most of the time we won't assume the covering graph to be regular, and using the notations in Section 2.6, we order the basis vectors of $\tilde{\Gamma}$ in a colexicographic order, i.e. $v_1^1 < \cdots < v_1^k < v_2^1 < \cdots < v_2^k < \cdots < v_n^1 < \cdots < v_n^k$.

We need to be a little more careful in the case of regular G-covers. The (ordinary) Laplacian matrix $L(\tilde{\Gamma})$ acts on the module $\mathbb{Z}[E]^{\tilde{V}}$. We have been implicitly writing this matrix with respect to the standard basis $\{v_i^g\}_{i\in[n],g\in G}$. We enforce an ordering on the vertices of $\tilde{\Gamma}$ and on the elements of G, and thus on these basis vectors. Take the standard ordering on the vertices V of Γ (i.e. $v_1 < v_2 < \cdots < v_n$) and fix an ordering of the elements of G such that the identity of G is first (a bijection from G to $[k]$). Then order the basis vectors in colexicographic order based on their labeling (v, g), so that the first n basis vectors are $v_1^{1_G}, \ldots, v_n^{1_G}$.

It will also be helpful to explicitly state a basis of the $\mathbb{Z}[E]$-module of rank $(k-1)n$ on which $\mathscr{L}(\Gamma)$ acts via restriction of scalars from $\overline{\mathbb{Z}[G]}[E]$ to $\mathbb{Z}[E]$. The $\overline{\mathbb{Z}[G]}[E]$-span of a vector m is equal to the $\mathbb{Z}[E]$-span of the set $\{gm\}_{g\in G,g\neq 1_G}$ (note that $m = -\sum_{g\in G,g\neq 1_G} gm$ by the definition of the reduced group algebra, so this set does indeed span over $\mathbb{Z}[E]$). Therefore, given the standard $\overline{\mathbb{Z}[G]}[E]$-basis $\{v_i\}_{i=1}^n$ corresponding to the vertices of Γ, we derive a standard $\mathbb{Z}[E]$-basis $\{v_i^g\}_{i\in[n],g\neq 1_G}$. Again, we order this basis in colexicographic order. Note that this basis corresponds to the last $(k-1)n$ vectors in the standard basis for $L(\tilde{\Gamma})$.

Having written $L(\tilde{\Gamma})$ with respect to our ordering of basis vectors, we wish to perform the change of basis described by the following lemma. Note that the following triangularization lemma holds for even non-regular covering graphs and does not rely on the algebraic structure of G when the cover is regular (see Definition 3.5).

Lemma 3.4. *Let G and Γ be as in Theorem 1.3. Write $L(\tilde{\Gamma})$ with basis vectors ordered as above. Let*

$$S = \begin{bmatrix} \mathbf{id}_n & \mathbf{id}_n & \dots & \mathbf{id}_n \\ 0_n & & & \\ \vdots & & \mathbf{id}_{(k-1)n} & \\ 0_n & & & \end{bmatrix}$$

Then the change of basis given by S yields the following block triangularization of $L(\tilde{\Gamma})$:

$$SL(\tilde{\Gamma})S^{-1} = \begin{bmatrix} L(\Gamma) & 0 \\ * & [\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]} \end{bmatrix} \tag{2}$$

Proof. Let $\beta_i = \sum_{j\in[k]} v_i^j$. Conjugation by S corresponds to a change of basis that maps $v_i^1 \mapsto \beta_i$ and $v_i^j \mapsto v_i^j$ when $j \neq 1$. Therefore, all we need to do is examine the action of the linear transformation corresponding to the matrix $L(\tilde{\Gamma})$ on this new basis. Denote this linear transformation as T.

First, we show that

$$T(\beta_i) = \sum_{j=1}^{n} \ell_{ij}\beta_j$$

where ℓ_{ij} is the (i,j)-entry of $L(\Gamma)$. To see this, consider the k rows of $L(\tilde{\Gamma})$ corresponding to the fiber $\{v_i^r\}_{r\in[k]}$ of v_i. The sum of these rows is equal to $T(\beta_i)$, expressed as a row vector with respect to the standard basis. Choose a column of $L(\tilde{\Gamma})$ corresponding to v_j^r. The entries of this column in the previously mentioned rows correspond (with negative sign) to edges to v_j^r from some element in the fiber of v_i. Since there should be one of these for each edge (v_i, v_j) in Γ, the sum of these values is ℓ_{ij}. Now choose the column of $L(\tilde{\Gamma})$ corresponding to v_i^r. The entries of this column in the previously mentioned rows on the off-diagonal correspond (with negative sign) to edges to v_i^r from some element in the fiber of v_i other than v_i^r. The diagonal entry of the column correspond (with negative sign) to edges from v_i^r to itself as well as (with positive sign) all edges out of v_i^r. Adding these values, we get ℓ_{ii}, since there is one edge into v_i^r from a vertex in the fiber of v_i for each edge from v_i to itself and one edge out of v_i^r for each edge out of v_i. Thus, the sum of the n rows of $L(\tilde{\Gamma})$ corresponding to the fiber $\{v_i^r\}_{r\in[k]}$ is precisely

$$T(\beta_i) = \sum_{j=1}^{n} \sum_{r\in[k]} \ell_{ij} v_j^r = \sum_{i=1}^{n} \ell_{ij}\beta_j$$

as desired. Therefore, the upper-left and upper-right blocks of (2) are correct.

The effect of our change of basis on the lower-right $(k-1)n \times (k-1)n$ block of $L(\tilde{\Gamma})$ is to subtract the i-th column of $L(\tilde{\Gamma})$, for $i \in [n]$, from columns $i+n, i+2n, \dots, i+(k-1)n$. When the covering graph is regular with a voltage group G present, this mimics the structure of the *reduced* group algebra; that is, what we have actually done in the lower-right hand block is to write $\mathscr{L}(\Gamma)$ as a $\mathbb{Z}$-matrix in the previously defined basis, as desired.

If the cover is not regular, then we simply define $[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$ to be the lower-right matrix; we give a concrete definition below. ∎

Definition 3.5. Let $\{v_1, \cdots, v_n\}$ be the set of vertices of our graph Γ, let $\tilde{\Gamma}$ be a k-fold cover of Γ, where vertex v_i is lifted to $v_i^1, \cdots, v_i^k$. Define $n(k-1) \times n(k-1)$ matrices D and A with basis $v_1^2, \cdots, v_n^2, v_1^3, \cdots, v_n^3, \cdots, v_1^k, \cdots, v_n^k$ as follows.

$$A[v_i^t, v_j^r] = \sum_{e=(v_i^t, v_j^r)} \mathrm{wt}(e) - \sum_{e=(v_j^r, v_i^1)} \mathrm{wt}(e)$$

$$D[v_i^t, v_i^t] = \sum_{e\in E_s(v_i^t)} \mathrm{wt}(e)$$

for $1 < t, r \le k$. Finally, we define

$$[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]} \coloneqq D - A.$$

Note that in the case of non-regular covers, the matrix cannot be interpreted as the $\mathbb{Z}$-linearization of a voltage Laplacian. Nevertheless, we can use this alternate description to extend the definition of $[\mathscr{L}(\Gamma)]_{\mathbb{Z}}$ to account for non-regular covers, which will make Lemma 3.4 true for arbitrary covers, and thus will make Theorem 1.3 true for arbitrary finite covers (the remaining lemmata do not make use of regularity).

With this definition in hand, we restate Theorem 1.3:

Theorem 1.3. *Let $\Gamma = (V, E, \mathrm{wt})$ be an edge-weighted multigraph, and let $\tilde{\Gamma}$ be a k-fold covering graph of Γ. Then for any vertex v of Γ and any lift $\tilde{v}$ of v, we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k}\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$$

with $[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$ given by definition 3.5.

The triangularization lemma is very close to giving us what we want for Theorem 1.3, but unfortunately taking minors and change of basis do not commute. Besides, we need to find a factor of k somewhere along the way.

Define $U = SL(\tilde{\Gamma})S^{-1}$. This is the matrix that we will use to connect the two sides of Theorem 1.3. Without loss of generality, assume that we want to root our arborescences of Γ at vertex v_1 and our arborescences of $\tilde{\Gamma}$ at vertex v_1^1. Then the following result is immediate from Lemma 3.4 and Theorem 2.3:

Corollary 3.6.

$$\det U_1^1 = A_{v_1}(\Gamma)\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}.$$

To complete the proof of Theorem 1.3, we need to show that

$$\det U_1^1 = k \cdot \det L_1^1(\tilde{\Gamma}).$$

3.4. **The two-step change of basis.** We will show the above equality by factoring S as QP for some matrices Q and P specified in Section 3.5. This means conjugation by S is the same as conjugation by P and then conjugation by Q. In this section we will prove two lemmas that will tell us how conjugation by each of these matrices affects minors.

Lemma 3.7. *Let L be the Laplacian matrix of some graph $\Gamma = (V, E, \mathrm{wt})$. Fix a basis vector v_i, and let P be the change of basis matrix that maps $v_i \mapsto \sum_{j=1}^n \alpha_j v_j$ with $\alpha_i \ne 0$. That is, P is the identity matrix but with α_j in entry (i, j) for each $j \in J$. Then*

$$\det(PLP^{-1})_i^i = \left(\sum_{j=1}^n \frac{\alpha_j}{\alpha_i}\right) A_{v_i}(\Gamma)$$

Proof. First note that we need $\alpha_i \ne 0$ in order for P to be invertible. Otherwise, column i would be all 0's. If $\alpha_j = 0$ for all $j \ne i$, then the statement holds trivially, because $(PLP^{-1})_i^i = L_i^i$ in this case. So, we can assume there is some $j \ne i$ such that $\alpha_j \ne 0$.

Without loss of generality, let $i = 1$ and $\alpha_2 \ne 0$. We can see that P^{-1} is the identity with $\frac{1}{\alpha_1}$ in the (1,1) entry and $\frac{-\alpha_i}{\alpha_1}$ in the $(1, i)$ entry. $L(\tilde{\Gamma})P^{-1}$ differs from $L(\tilde{\Gamma})$ in that the ith column of $L(\tilde{\Gamma})P^{-1}$ is the ith column of $L(\tilde{\Gamma})$ with $\frac{\alpha_i}{\alpha_1}$ times the first column of $L(\tilde{\Gamma})$ subtracted from it. $PL(\tilde{\Gamma})P^{-1}$ differs from $L(\tilde{\Gamma})P^{-1}$ only in the first row. However, since we are finding the determinant of $PL(\tilde{\Gamma})P^{-1}$ with the first row and column removed, we are only interested in the lower-right hand $(n-1) \times (n-1)$ submatrix and can ignore this operation.

Notice that P can be factored as $P_nP_{n-1}...P_2$ where P_2 is the identity but with α_1 in the $(1,1)$ entry and α_2 in the $(1,2)$ entry and the rest of the P_j's are the identity but with α_j in the $(1,j)$ entry. This also gives a factorization for P^{-1}. We will first focus on $P_2LP_2^{-1}$.

We may interpret $(P_2LP_2^{-1})_1^1$ as a submatrix of the Laplacian of a different graph, which we will denote as $\Gamma^{(2)}$. We construct $\Gamma^{(2)}$ as follows: the vertices of $\Gamma^{(2)}$ are $v_1^{(2)},...,v_n^{(2)}$. If there is an edge $v_r \to v_s$ in Γ, then there is an edge $v_r^{(2)} \to v_s^{(2)}$ in $\Gamma^{(2)}$, so $\Gamma^{(2)}$ contains Γ as a subgraph. For each edge $e = (v_2, v_1) \in \Gamma$, we add an additional edge $(v_2^{(2)}, v_1^{(2)})$ to $\Gamma^{(2)}$ with weight $\frac{\alpha_2}{\alpha_1}\,\mathrm{wt}(e)$; we call this an edge of type 1. Furthermore, for each such $e = (v_i, v_1) \in \Gamma$ where $i \neq 1, 2$, we add the edge $(v_i^{(2)}, v_2^{(2)})$ to $\Gamma^{(2)}$ with weight $\frac{-\alpha_2}{\alpha_1}\,\mathrm{wt}(e)$ and the edge $(v_i^{(2)}, v_1^{(2)})$ with weight $\frac{\alpha_2}{\alpha_1}\,\mathrm{wt}(e)$. The first of these edges will be called an edge of type 2 and the second an edge of type 3.

We can see that $L(\Gamma^{(2)})$ is the same as $L(\Gamma)$ except that (aside from the first row, which remains unchanged) $\frac{\alpha_2}{\alpha_1}$ times the first column is subtracted from first column and added to the second column. $L_1^1(\Gamma^{(2)}) = (P_2LP_2^{-1})_1^1$, so $\det(P_2LP_2^{-1})_1^1$ counts the arborescences of $\Gamma^{(2)}$ rooted at $v_1^{(2)}$.

We will divide the arborescences of $\Gamma^{(2)}$ into four categories (See Figure 4).

(1) Arborescences that do not contain any type 1, type 2, or type 3 edges. The weighted sum of these arborescences is counted by $A_{v_1}(\Gamma)$ because these are exactly the arborescences that use only edges in the subgraph Γ of $\Gamma^{(2)}$.

(2) Arborescences that contain a type 2 edge paired with arborescences that differ from these by replacing the type 2 edge with a type 3 edge of the same weight with opposite sign. For every type 2 edge, there is a type 3 edge of the same weight with opposite sign. This means that for every arborescence that contains a type 2 edge, there is an arborescence that is the same, except instead of the type 2 edge it has a type 3 edge of the same weight with opposite sign. The weights of these arborescences cancel out, so the weighted sum of all of these arborescences is 0.

(3) Arborescences that contain a type 1 edge not counted in the previous category (2). We claim that in such an arborescence, every edge lies in the subgraph Γ except for the unique type 1 edge. If such an arborescence contained an edge of type 2, then since vertex $2^{(2)}$ flows directly to the root the edge of type 2 could be replaced by its corresponding type 3 edge and still yield a valid arborescence; the same holds if we start with an edge of type 3. Thus, arborescences in this category correspond to arborescences in Γ where the edge out of 2 goes directly to 1. So, they contribute $\frac{\alpha_2}{\alpha_1}$ times the weight of such arborescences in Γ.

(4) Arborescences that contain an edge of type 3 that are not counted in category (2) either. These are arborescences where removing the edge $e = (v_j^{(2)}, v_1^{(2)})$ of type 3 and replacing it with the corresponding edge $e' = (v_j^{(2)}, v_2^{(2)})$ of type 2 does not give an arborescence. This only happens if adding e' would create a cycle, so we conclude that $v_j^{(2)}$ lies downstream from $v_2^{(2)}$ in the arborescence flow. This guarantees that there exists only one type 3 edge—if we have one type 3 edge out of $v_j^{(2)}$ and another one out of $v_{j'}^{(2)}$, then both of these vertices lie downstream of $v_2^{(2)}$ but both lead directly to $v_1^{(2)}$, which is a contradiction. Furthermore, such an arborescence cannot contain an edge of type 1—this would immediately contradict vertex $v_2^{(2)}$ lying upstream of $v_j^{(2)}$—or an edge of type 2, since type 2 edges can always be replaced by their corresponding type 3 edge and still yield a valid arborescence, which would again land us in category (2).
We conclude that the only "added" edge in this arborescence is e itself. Therefore, summing over $j \neq 1, 2$ we see that these arborescences in this category correspond bijectively to arborescences in Γ where the edge out of v_2 does not go to v_1. This means that in our sum, they contribute $\frac{\alpha_2}{\alpha_1}$ times the weight of such arborescences in Γ.

$$L(\Gamma) = \begin{bmatrix} a+b & -a & 0 & -b \\ -c & c+d & 0 & -d \\ -e & -g & e+g & 0 \\ 0 & 0 & -f & f \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_2 L(\Gamma) P_2^{-1} = \begin{bmatrix} a+b-c & -2a-b+2c+d & 0 & -b-d \\ -c & 2c+d & 0 & -d \\ -e & -g+e & g+e & 0 \\ 0 & 0 & -f & f \end{bmatrix}$$

FIGURE 3. A Laplacian matrix before and after applying the change of basis P_2 with $\alpha_1 = \alpha_2 = 1$. Note that $(P_2 L P_2^{-1})_1^1$ matches the corresponding submatrix of the Laplacian of $\Gamma^{(2)}$ (see Figure 4).

The last two categories combine to contribute $\frac{\alpha_2}{\alpha_1} A_{v_1}(\Gamma)$ to the arborescence count $A_{v_1^{(2)}}\Gamma^{(2)}$. Adding the weighted sums of the arborescences in these four categories, we find

$$A_{v_1^{(2)}}(\Gamma^{(2)}) = A_{v_1}(\Gamma) + \frac{\alpha_2}{\alpha_1} A_{v_1}(\Gamma) = \left(\frac{\alpha_1 + \alpha_2}{\alpha_1}\right) A_{v_1}(\Gamma).$$

From here, we proceed by induction—essentially, all we need to do is prove that continuing to iterate the previous construction over vertices other than v_2 continues to work the way we want. For $3 \leq k \leq n$, we will construct $\Gamma^{(k)}$ from $\Gamma^{(k-1)}$ in the same way we constructed $\Gamma^{(2)}$ from Γ. However, here the weights on our new edges will have a factor of $\frac{\alpha_k}{\sum_{j=1}^{k-1} \alpha_j}$ rather than $\frac{\alpha_2}{\alpha_1}$. We will show that $L(\Gamma^{(k)})$ is $L(\Gamma)$ except that (aside from the first row, which remains unchanged) $\frac{\alpha_j}{\alpha_1}$ times the first column of $L(\Gamma)$ is subtracted from first column and added to the jth column for $2 \leq j \leq k$. Note that this means that $L_1^1(\Gamma^{(k)}) = (P_k P_{k-1} ... P_2 L P_2^{-1} ... P_{k-1}^{-1} P_k^{-1})_1^1$, so $\det(P_k P_{k-1} ... P_2 L P_2^{-1} ... P_{k-1}^{-1} P_k^{-1})_1^1$ counts the arborescences of $\Gamma^{(k)}$ rooted at $v_1^{(k)}$. We will also show that

$$A_{v_1^{(k)}}(\Gamma^{(k)}) = \left(\sum_{j=1}^{k} \frac{\alpha_j}{\alpha_1}\right) A_{v_1}(\Gamma).$$

We begin with the Laplacian. We can see that $L(\Gamma^{(k)})$ is $L(\Gamma^{(k-1)})$ except that (aside from the first row, which remains unchanged) $\frac{\alpha_k}{\sum_{j=1}^{k-1} \alpha_j}$ times the first column of $L(\Gamma^{(k-1)})$ is subtracted from first column and added to the kth column. By our inductive hypothesis, $\frac{\alpha_k}{\sum_{j=1}^{k-1} \alpha_j}$ times the first column of $L(\Gamma^{(k-1)})$ is $\left(\frac{\alpha_k}{\sum_{j=1}^{k-1} \alpha_j}\right)\left(\frac{\sum_{j=1}^{k-1} \alpha_j}{\alpha_1}\right) = \frac{\alpha_k}{\alpha_1}$ times the first column of $L(\Gamma)$. This shows that $L(\Gamma^{(k)})$ is what we want.

Now we turn to the arborescences. The same method of counting arborescences in $\Gamma^{(2)}$ from arborescences in Γ applies for counting arborescences in $\Gamma^{(k)}$ from $\Gamma^{(k-1)}$. This means

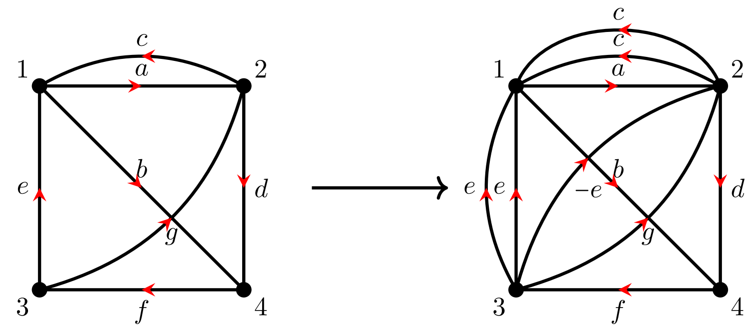
$$A_{v_1^{(k)}}(\Gamma^{(k)}) = A_{v_1^{(k-1)}}(\Gamma^{(k-1)}) + \frac{\alpha_k}{\sum_{j=1}^{k-1} \alpha_j} A_{v_1^{(k-1)}}(\Gamma^{(k-1)}),$$

which gives us what we want by the inductive hypothesis.

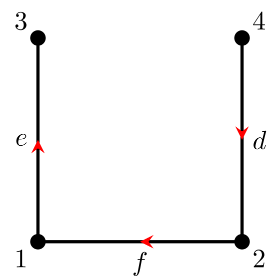
Thus, we have shown that

$$\det(PLP^{-1})_1^1 = \det(P_n P_{n-1} ... P_2 L P_2^{-1} ... P_{n-1}^{-1} P_n^{-1})_1^1 = \left(\sum_{j=1}^{n} \frac{\alpha_j}{\alpha_1}\right) A_{v_1}(\Gamma).$$

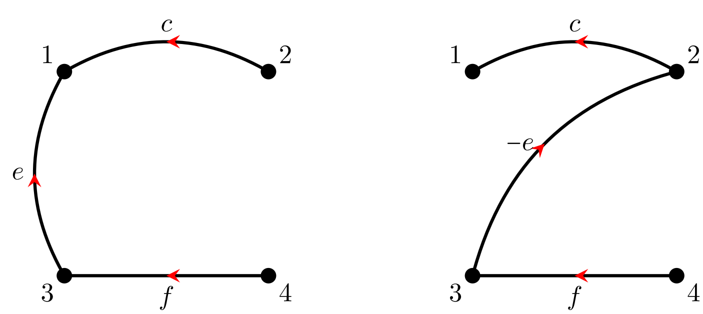
■



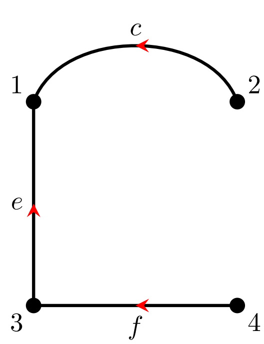
A graph Γ and the corresponding graph $\Gamma^{(2)}$.



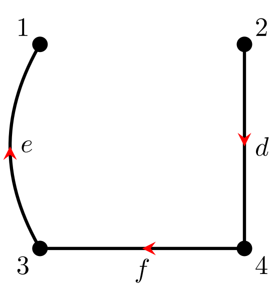
An arborescence of type 1.



A pair of arborescences of type 2.



An arborescence of type 3.



An arborescence of type 4.

FIGURE 4. Types of arborescences for $\Gamma^{(2)}$ with $\alpha_1 = \alpha_2 = 1$.

Here is the next lemma we need:

Lemma 3.8. *Let R be a commutative ring and let $M \in \operatorname{Mat}_n(R)$. Let $Q \in GL_n(R)$ such that the i-th row and column are each the i-th unit vector. Then*

$$\det(QMQ^{-1})_i^i = \det(M)_i^i$$

In other words, the change of basis given by Q commutes with taking the minor of M corresponding to removing the i-th row and column.

Proof. Without loss of generality $i = 1$. Write

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_1^1 \end{pmatrix}, \quad M = \begin{pmatrix} * & * \\ * & M_1^1 \end{pmatrix}$$

Thus,

$$QMQ^{-1} = \begin{pmatrix} * & * \\ * & (Q_1^1)(M_1^1)(Q_1^1)^{-1} \end{pmatrix}.$$

Therefore the desired minor is the same as the corresponding minor of M. ∎

3.5. **Proof of Theorem 1.3.**

Proof. Let P be the change of basis that maps $v_1^1 \mapsto \beta_1 := \sum_{s\in[k]} v_1^s$, and let Q be the change of basis that maps $v_i^1 \mapsto \sum_{r\in[k]} v_i^r$ for $i > 1$. Note that P satisfies the hypotheses for Lemma 3.7 with $i = 1$ and Q satisfies the hypotheses of Lemma 3.8 with $i = 1$. Letting S be the matrix from Lemma 3.4, we have $S = QP$. Thus, by Lemmas 3.7 and 3.8,

$$\begin{aligned} \det U_1^1 &= \det(QPL(\tilde{\Gamma})P^{-1}Q^{-1})_1^1 \\ &= \det(PL(\tilde{\Gamma})P^{-1})_1^1 \\ &= kA_{v_1^1}(\tilde{\Gamma}) \end{aligned}$$

However, from Corollary 3.6 we know that

$$\det U_1^1 = A_{v_1}(\Gamma) \det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma).$$

Therefore,

$$kA_{v_1^1}(\tilde{\Gamma}) = A_{v_1}(\Gamma) \det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma)$$

as desired. ∎

4. Vector fields and the voltage Laplacian

In this section, we discuss the connection between the voltage Laplacian and vector fields on voltage graphs, and its implications for positivity in the 2-fold cover case.

4.1. **Negative Vector Fields.**

Definition 4.1. A *vector field* γ of a directed graph Γ is a subgraph of Γ such that every vertex of γ has outdegree 1 in Γ. Similarly to arborescences, we define the *weight* of a vector field $\operatorname{wt}(\gamma) := \prod_{e\in\gamma} \operatorname{wt}(e)$ of a vector field be the product of its edge weights, so that $\operatorname{wt}(\gamma)$ is a degree n monomial with respect to the edge weights of Γ. Write $C(\gamma)$ for the set of cycles in a vector field γ, of which there is exactly one in each connected component. If G is abelian, and if c is a cycle of γ then we define the voltage of c as $\nu(c) := \prod_{e\in c} \nu(e)$—this product is well-defined when G is abelian.

The determinant of $\mathscr{L}(\Gamma)$ counts vector fields of Γ in the following way:

Theorem 4.2 (Chaiken)**.** *Let G be an abelian group, and let Γ be an edge-weighted G-voltage graph. Then*

$$\sum_{\gamma\subseteq\Gamma}\left[\operatorname{wt}(\gamma)\prod_{c\in C(\gamma)}(1-\nu(c))\right]=\det\mathscr{L}(\Gamma)$$

where the sum ranges over all vector fields γ of Γ.

Example 4.3. Let Γ be the $\mathbb{Z}/3\mathbb{Z}$-voltage graph of example 2.7. There are four distinct vector fields of Γ (see Figure 5).

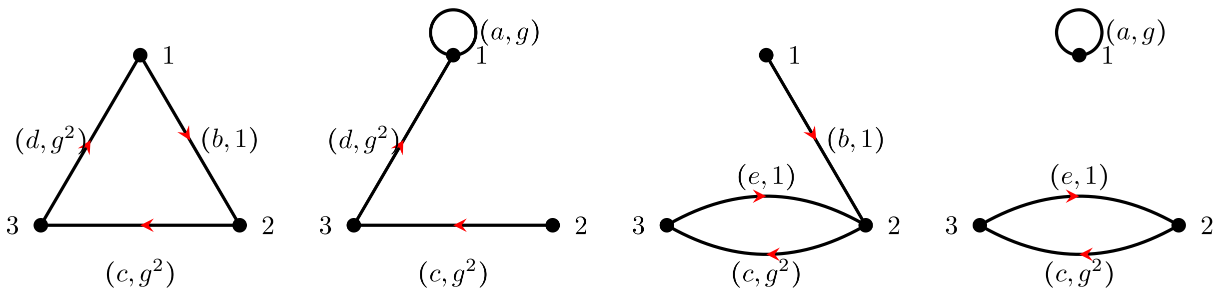


Figure 5. The four vector fields of Γ

The first three of these vector fields contain one cycle; from left to right, these unique cycles have weights ζ_3, ζ_3, and ζ_3^2. The rightmost vector field has two cycles, one with weight ζ and the other of weight ζ^2. From Example 2.15, we have

$$\det\mathscr{L}(\Gamma)=(1-\zeta_3)bcd+(1-\zeta_3)acd+(1-\zeta_3^2)bce+(1-\zeta_3)(1-\zeta_3^2)ace$$

The four terms in this expression correspond to the four vector fields of Γ as described by the theorem.

We briefly point out the special case $G=\mathbb{Z}/2\mathbb{Z}$, which is especially nice because the coefficients in Theorem 4.2 are nonnegative integers.

Definition 4.4. Suppose that Γ is a $\mathbb{Z}/2\mathbb{Z}$-voltage graph, also called a *signed graph.* A vector field γ of Γ is a *negative vector field* if every cycle c of γ has an odd number of negative edges, so that $\nu(c)=-1$.

Denote the set of negative vector fields of signed graph Γ by $\mathcal{N}(\Gamma)$. Then Theorem 4.2 may be written as:

Corollary 4.5.

$$\sum_{\gamma\in\mathcal{N}(\Gamma)}2^{\#C(\gamma)}\operatorname{wt}(\gamma)=\det\mathscr{L}(\Gamma)$$

Corollary 4.5 along with Corollary 1.5 has an immediate further corollary:

Corollary 4.6. *If $\tilde{\Gamma}$ is a 2–fold regular cover of Γ, then the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ has positive integer coefficients.*

Positivity for general covers is still unknown; see Conjecture 1.7.

4.2. **Proofs of Theorem 4.2.** We now present two proofs of Theorem 4.2. The first is new, and the second is essentially due to Chaiken.

The first proof proceeds by deletion-contraction, and requires the following lemma.

Lemma 4.7. *Let Γ be as in Theorem 4.2 with voltage function $\nu : E \to \overline{\mathbb{Z}[G]}$, let v be any vertex of Γ, and let $g \in G$. We define a new voltage function $\nu_{v,g}$ given by*

$$\nu_{v,g}(e) = \begin{cases} g\nu(e): & \text{if } e \in E_s(v), e \notin E_t(v) \\ g^{-1}\nu(e): & \text{if } e \in E_t(v), e \notin E_s(v) \\ \nu(e): & \text{else} \end{cases}$$

Then:

(a) *For any cycle c of Γ, we have $\nu(c) = \nu_{v,g}(c)$.*
(b) *The determinant of the voltage Laplacian of Γ with respect to the voltage ν is equal to the determinant of the voltage Laplacian of Γ with respect to $\nu_{v,g}$. That is,*

$$\det \mathscr{L}(V, E, \mathrm{wt}, \nu) = \det \mathscr{L}(V, E, \mathrm{wt}, \nu_{v,g})$$

Proof.

(a) If c does not contain the vertex v, or if c is a loop at v, then the voltages of all edges in c remain unchanged. Otherwise, c contains exactly one ingoing edge e of v and one outgoing edge f of v, so that

$$\begin{aligned} \nu_{v,h}(c) &= \frac{\nu(c)}{\nu(e)\nu(f)}[g\nu(e)][g^{-1}\nu(f)] \\ &= \nu(c) \end{aligned}$$

as desired.
(b) The matrix $\mathscr{L}(V, E, \mathrm{wt}, \nu)$ may be transformed into the matrix $\mathscr{L}(V, E, \mathrm{wt}, \nu_{v,g})$ by multiplying the row corresponding to v by g and multiplying the column corresponding to v by g^{-1}, so the determinant remains unchanged.

∎

This lemma will allow us some freedom to change the voltage of Γ as needed in the following proof.

First proof of Theorem 4.2. Denote the left-hand side of the theorem as

$$\Omega(\Gamma) := \sum_{\gamma \subseteq \Gamma} \left[\mathrm{wt}(\gamma) \prod_{c \in C(\gamma)} (1 - \nu(c)) \right]$$

We proceed by deletion-contraction. The base case is when the only edges of Γ are loops. When this happens, $\mathscr{L}(\Gamma)$ is diagonal, with

$$\ell_{ii} = \sum_{e=(v_i,v_i) \in E} (1 - \nu(e)) \, \mathrm{wt}(e).$$

Thus we have

$$\det \mathscr{L}(\Gamma) = \prod_{i=1}^{n} \left(\sum_{e=(v_i,v_i) \in E} [1 - \nu(e)] \, \mathrm{wt}(e) \right)$$

If we expand the product above, each term will correspond to a unique combination of one loop per vertex of Γ. But such combinations are precisely the vector fields of Γ, so we obtain

$$\det \mathscr{L}(\Gamma) = \Omega(\Gamma)$$

For the inductive step, assume that there exists at least one edge e between distinct vertices, and assume that the proposition holds for graphs with fewer non-loop edges than Γ. Using the lemma, we may change the voltage of Γ so that e has voltage 1 without changing either $\Omega(\Gamma)$ or $\det \mathscr{L}(\Gamma)$. Without loss of generality, let $v_1 = e_s$ and $v_2 = e_t$.

If γ is a vector field of Γ, then γ either contains e or it does not. In the latter case, γ is also a vector field of $\Gamma\backslash e$. Clearly all such γ arise uniquely from a vector field of $\Gamma\backslash e$. Therefore, there is a weight-preserving bijection between the vector fields of Γ not containing e and the vector fields of $\Gamma\backslash e$.

Otherwise, if $e \in \gamma$, then no other edge of the form (v_1, v_j) is in γ. We define a special type of contraction: let $\Gamma/_1 e := (\Gamma/e)\backslash E_s(v_1)$. That is, we contract along e, and delete all other edges originally in $E_s(v_1)$. Note that the contraction process merges vertices v_1 and v_2 into a "supervertex," which we denote v_{12}.

Then the vector field γ descends uniquely to a vector field $\overline{\gamma}$ on $\Gamma/_1 e$. Every vector field $\overline{\gamma}$ in $\Gamma/_1 e$ corresponds uniquely to a vector field of Γ containing e, obtained by letting the unique edge coming out the supervertex v_{12} in $\overline{\gamma}$ be the unique edge coming out of the vertex v_2 in γ, and letting e be the unique edge with source at v_1 in γ. This inverse map shows that the vector fields of Γ containing e are in bijection with the vector fields of $\Gamma/_1 e$. This bijection is weight-preserving up to a factor of $\mathrm{wt}(e)$. Finally, note that γ and its contraction $\overline{\gamma}$ have the same number of cycles, with the same voltages. If a cycle contains e in γ, then that cycle is made one edge shorter in $\overline{\gamma}$, but still has positive length since e is assumed to not be a loop. If c is a cycle containing e in Γ, then because e has voltage 1, the cycle voltage $\nu(c/e)$ of the contracted version of c is equal to the cycle voltage before contraction. Thus, we may write

$$\Omega(\Gamma) = \Omega(\Gamma\backslash e) + \mathrm{wt}(e)\Omega(\Gamma/_1 e)$$

By the inductive hypothesis, since $\Gamma\backslash e$ and $\Gamma/_1 e$ have strictly fewer non-loop edges than Γ, we have

$$\Omega(\Gamma\backslash e) + \mathrm{wt}(e)\Omega(\Gamma/_1 e) = \det \mathscr{L}(\Gamma\backslash e) + \mathrm{wt}(e) \det \mathscr{L}(\Gamma/_1 e)$$

Note that $\mathscr{L}(\Gamma\backslash e)$ is equal to $\mathscr{L}(\Gamma)$ with $\mathrm{wt}(e)$ deleted from both the $1,1$- and $1,2$-entries. Therefore, via expansion by minors, we obtain

$$\det \mathscr{L}(\Gamma\backslash e) + \mathrm{wt}(e) \det \mathscr{L}_1^1(\Gamma) + \mathrm{wt}(e) \det \mathscr{L}_1^2(\Gamma) = \det \mathscr{L}(\Gamma) \tag{3}$$

where $\mathscr{L}_i^j(\Gamma)$ is the submatrix of $\mathscr{L}(\Gamma)$ obtained by removing the i-th row and the j-th column.

To construct $\mathscr{L}(\Gamma/_1 e)$ from $\mathscr{L}(\Gamma)$, we disregard the first row of $\mathscr{L}(\Gamma)$, since the special contraction $\Gamma/_1 e$ simply removes the outgoing edges $E_s(v_1)$. Then, we combine the first two columns of $\mathscr{L}(\Gamma)$ by making their sum the first column of $\mathscr{L}(\Gamma/_1 e)$, since when we perform a contraction that merges v_1 and v_2 into v_{12}, we also have $E_t(v_1) \cup E_t(v_2) = E_t(v_{12})$. Thus $\mathscr{L}(\Gamma/_1 e)$ is a $(n-1) \times (n-1)$ matrix that agrees with both $\mathscr{L}_1^1(\Gamma)$ and $\mathscr{L}_1^2(\Gamma)$ on its last $n-2$ columns, and whose first column is the sum of the first columns of $\mathscr{L}_1^1(\Gamma)$ and $\mathscr{L}_1^2(\Gamma)$. Therefore,

$$\det \mathscr{L}(\Gamma/_1\, e) = \det \mathscr{L}_1^1(\Gamma) + \det \mathscr{L}_1^2(\Gamma)$$

Substituting into (3), we obtain

$$\begin{aligned}\det \mathscr{L}(\Gamma) &= \det \mathscr{L}(\Gamma\backslash e) + \mathrm{wt}(e) \det \mathscr{L}(\Gamma/_1\, e)\\ &= \Omega(\Gamma\backslash e) + \mathrm{wt}(e)\Omega(\Gamma/_1\, e)\\ &= \Omega(\Gamma)\end{aligned}$$

as desired. ∎

The second proof of the theorem follows a style similar to Chaiken's proof of the Matrix Tree Theorem in [Cha82]. Chaiken actually proves a more general identity, which he calls the "All-Minors Matrix Tree Theorem," that gives a combinatorial formula for any minor of the voltage Laplacian. We do not reproduce such generality here, but instead follow a simplified version of his proof, more along the lines of Stanton and White's version of Chaiken's proof of the Matrix Tree Theorem [SW86].

Second proof of Theorem 4.2 (Chaiken). Let Γ have n vertices. For simplicity, assume that Γ has no multiple edges, since we can always decompose $\det \mathscr{L}(\Gamma)$ into a sum of determinants of voltage Laplacians of simple subgraphs of Γ, which also partitions the sum given in the theorem. We also assume that Γ is a complete bidirected graph, since we can ignore edges not in Γ by just considering them to have edge weight 0. Write

$\mathscr{L}(\Gamma) = (\ell_{ij})$, write $D(\Gamma) = (d_{ij})$, and write $\mathscr{A}(\Gamma) = (a_{ij})$, so that $\ell_{ij} = \delta_{ij} d_{ii} - a_{ij}$. Then the determinant of $\mathscr{L}(\Gamma)$ may be decomposed as

$$\det \mathscr{L}(\Gamma) = \det(\delta_{ij} d_{ii} - a_{ij}) = \sum_{S \subseteq [n]} \left[\sum_{\pi \in P(S)} (-1)^{\#C(\pi)} \operatorname{wt}_\nu(\pi) \prod_{i \in [n]-S} d_{ii} \right]$$

where $P(S)$ denotes the set of permutations of S, the set $C(\pi)$ is set of cycles of π, and $\operatorname{wt}_\nu(\pi) := \prod_{i \in S} a_{i,\pi(i)}$. The product of the d_{ii} may be rewritten as a sum over functions $[n] - S \to [n]$, yielding

$$\begin{aligned} \det \mathscr{L}(\Gamma) &= \sum_{S \subseteq [n]} \sum_{\pi \in P(S)} (-1)^{c(\pi)} \operatorname{wt}_\nu(\pi) \sum_{f:[n]-S \to [n]} \operatorname{wt}(f) \\ &= \sum_{S \subseteq [n]} \sum_{\pi \in P(S)} \sum_{f:[n]-S \to [n]} (-1)^{c(\pi)} \operatorname{wt}_\nu(\pi) \operatorname{wt}(f) \end{aligned} \tag{4}$$

where $\operatorname{wt}(f)$ denotes the *unvolted* weight of the edge set corresponding to the function f, since this part of the product ultimately comes from the degree matrix. Thus, the determinant may be expressed as a sum of triples (S, π, f) of the above form—that is, we let S be an arbitrary subset of $[n]$, we let π be a permutation on S, and we let f be a function $[n] - S \mapsto [n]$.

The permutation π can always be decomposed into cycles, and f will sometimes have cycles as well—that is, sometimes we have $f^{(m)}(k) = k$ for some $k \in \mathbb{Z}$ and $k \in [n] - S$. We can "swap" cycles between π and f. Suppose c is a cycle of f that we want to swap into π. Let the subset of $[n]$ on which c is defined be denoted W. Then we may obtain from our old triple a new triple $(S \coprod W, \pi \coprod c, f|_{[n]-S-W})$, where $\pi \coprod c$ denotes the permutation on $S \coprod W$ given by $(\pi \coprod c)(v) = \pi(v)$ if $v \in S$ and $(\pi \coprod c)(v) = c(v)$ if $v \in W$. That is, we "move" C from f to π. Similarly, if c is a cycle of π, then we can obtain a new triple $(S - W, \pi|_{S-W}, f \coprod c)$. Note that these two operations are inverses.

This process is always weight-preserving: it does not matter whether c is considered as a part of π or as a part of f, since it will always contribute $\operatorname{wt}(c)$ to the product. However, one iteration of this map will swap the sign of $(-1)^{\#C(\pi)}$, and will also remove or add a factor from $\operatorname{wt}_\nu(\pi)$ corresponding to the voltage of c. If π and f have k cycles between both of them, then there are 2^k possibilities for swaps, yielding a free action of $(\mathbb{Z}/2\mathbb{Z})^k$. If we start from the case where π is the empty partition, then the sign $(-1)^{\#C(\pi)}$ starts at 1. Every time we choose to swap a cycle c into π from f, we flip this sign and multiply by $\nu(c)$, effectively multiplying by $-\nu(c)$. Thus, the sum of terms in (4) coming from the orbit of the action of $(\mathbb{Z}/2\mathbb{Z})^k$ on (S, f, π) is

$$\operatorname{wt}(\pi) \operatorname{wt}(f) \prod_{c \in C(\pi) \cup C(f)} (1 - \nu(c))$$

where $\operatorname{wt}(\pi)$ is now unvolted. This orbit class corresponds to the contribution of one vector field γ of Γ to the overall sum, where γ is the unique vector field such that $\operatorname{wt}(\gamma) = \operatorname{wt}(\pi) \operatorname{wt}(f)$. Thus, summing over all orbit classes, we obtain the desired formula:

$$\det \mathscr{L}(\Gamma) = \sum_{\gamma \subseteq \Gamma} \left[\operatorname{wt}(\gamma) \prod_{c \in C(\gamma)} (1 - \nu(c)) \right]$$

■

5. Conjectures and Future Directions

We end our paper by a discussion of several unanswered questions and possible future research directions.

5.1. **Interpreting the restriction-of-scalars determinant.** In the case where the voltage group G is prime cyclic, Corollary 1.4 yields a computationally nice interpretation of Theorem 1.3: the $\mathbb{Z}$-determinant is really a field norm, which may be computed in ways other than restriction of scalars—for example, as a product of Galois conjugates. This result could be extended if there existed an analogue to the field norm for arbitrary reduced group algebras, or indeed for general free algebras of finite rank. A good first step might be to consider abelian groups.

Problem 5.1. *Let R be a commutative ring, and let A be a free algebra over R of finite rank. Let $\alpha \in A$. Find an alternative expression or interpretation of $\det_R \alpha$, where the multiplicative action of α is viewed as a linear transformation on the R-module A, analogous to a field norm. Useful special cases include $R = \mathbb{Z}$ or $\mathbb{Q}$, when A is commutative, and/or when A is the group algebra or reduced group algebra of some finite group G.*

5.2. **Positivity of the ratio and possible combinatorial expression using vector fields.** By Corollary 1.6, the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is a homogeneous polynomial with integer coefficients. We further conjecture:

Conjecture 1.7. *The ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ has positive integer coefficients.*

Corollary 4.6 gives positivity for regular 2-fold cover, and the following proposition gives a way to extend that result to all regular covers by 2-groups. However, in the case of general regular covers, we do not have a concrete combinatorial interpretation of $\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}[E]}$. Such an interpretation would probably be the cleanest way to prove Conjecture 1.7.

Proposition 5.2. *Suppose we have the exact sequence of groups $1 \to N \to G \to H \to 1$, where N and H have the property that every regular N- (resp. H-) cover satisfies the positivity conjecture. Then every regular G-cover satisfies the positivity conjecture.*

Proof. Let Γ be a graph, let Γ_G be a regular G-cover of Γ, and let Γ_H be the image of Γ_G under the projection map $G \to H$. We will show that both the covers $\Gamma_H \to \Gamma$ and Γ_G to Γ_H are regular, and therefore that those arborescence ratios are positive. Therefore, the arborescence ratio for the cover $\Gamma_G \to \Gamma$ is the product of these ratios, and therefore positive as well.

Since G is a group extension of H by N, we will write elements of G as ordered pairs $(h, n), h \in H, n \in n$, where $n \mapsto (h, n) \mapsto h$ under our exact sequence. Multiplication of these elements involves Schreier theory:

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 h_2, f(h_1, h_2, n_1) n_2),$$

where $f(h_1, h_2, n_1)$ is an element of N that is independent of n_2 (see, for example, [Mor]).

To show that a given cover is regular, we need to demonstrate voltages for the edges of the base graph that give us the desired cover. First, we show that Γ_H is regular over Γ.

Let $e : v \to w$ be an edge in Γ, with G-voltage $g = (h_g, n_g)$. Then e lifts to the set of edges $\{e^{g'} : v^{g'} \to w^{gg'}\}$ in Γ_G, and these edges project to the set $\{e^h : v^h \to w^{h_g h}\}$ in Γ_H. Therefore, in the cover $\Gamma_H \to \Gamma$, we can set the voltage of e to be h_g, and this gives a regular H-cover.

Now we show that Γ_G is a regular N-cover of Γ_H. This is more challenging since H is not necessarily a subgroup of G. Consider the edge e of Γ from above, and let $h \in H$. The edge $e^h : v^h \to w^{h_g h}$ in Γ_H is covered by the edges $\{e^{(h,n)} | n \in N\}$ in Γ_G. Computations in G tell us that

$$e^{(h,n)} : v^{(h,n)} \to w^{(h_g, n_g)\cdot(h,n)} = w^{(h_g h, f(h_g, h n_g) n)}.$$

Set the voltage on e_h to be $f(h_g h n_g) \in N$. Then

$$(e^h)^n : (v^h)^n \to (w^{h_g h})^{f(h_g h n_g) n},$$

so identifying $(e^h)^n$ with $e^{(h,n)}$ and likewise for vertices gives us Γ_G as a regular cover of Γ_H. ∎

Corollary 5.3. *Let G be a 2-group. Then every regular G-cover satisfies the positivity conjecture.*

Proof. By Corollary 4.6, this result holds in the case of $\mathbb{Z}/2\mathbb{Z}$. Since G is a 2-group, $\mathbb{Z}/2\mathbb{Z}$ is a normal subgroup, and so the proposition can be applied inductively. ∎

Problem 5.4. *Find a combinatorial interpretation of the polynomial $\frac{1}{k}\det_{\mathbb{Z}[E]} \mathscr{L}(\Gamma) = \frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$, assuming Conjecture 1.7 is true.*

Vector fields are a potential source of a combinatorial interpretation for the arborescence ratio. We observed that in a k-fold cover, the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ always appears to be a product of $(k-1)$ weighted sums of vector fields.

Conjecture 5.5. *Let $\tilde{\Gamma}$ be a k-fold cover of Γ, then*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_{\tilde{v}}(\Gamma)} = \sum_{(\gamma_1,\cdots,\gamma_{k-1})\in\mathcal{V}^{k-1}} f(\gamma_1,\cdots,\gamma_{k-1}) \prod_{i=1}^{k-1} \mathrm{wt}(\gamma_i)$$

where $\mathcal{V}$ is the set of vector fields of Γ, and f is an $\mathbb{Z}_{\geq 0}$-valued function.

Moreover, as we look over all possible k-fold cover $\tilde{\Gamma}$, the ratios exhaust all possible $(k-1)$-tuples of vector fields of the base graph, which is only known in the 2-fold case. This observation motivates the following conjecture stated in terms of random covers.

Conjecture 5.6. *Let $\Gamma = (E, V)$ be a graph, fix a vertex v with non-trivial arborescence. Let Γ' be a random k-fold cover of Γ, assuming uniform distribution. Then the expected value of the ratio of arborescence is*

$$\mathbb{E}\left[\frac{\mathcal{A}_{v'}(\Gamma')}{\mathcal{A}_v(\Gamma)}\right] = \frac{1}{k}\left(\sum_{\gamma\in\mathcal{V}} \mathrm{wt}(\gamma)\right)^{k-1} = \frac{1}{k}\prod_{w\in V}\left(\sum_{\alpha\in E_s(w)} \mathrm{wt}(\alpha)\right)^{k-1}$$

where $\mathcal{V}$ is the set of vector fields of Γ.

Conjecture 5.6 is an alternative approach to positivity via a 'pigeon-hole' like argument: assuming the ratio of some covering graph has a negative coefficient, some cancellation shall happen as we sum over all possible covers; this might cause the expected value to not be 'large enough' to match Conjecture 5.6.

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