2013-2014 高数 B2 期中

一、（40 分）试解下列各题：

1.（8 分）设$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=2$，求$\left[\left(\vec{a}+\vec{b}\right)\times\left(\vec{b}+\vec{c}\right)\right]\cdot\left(\vec{c}+\vec{a}\right)$。

解：$\left[\left(\vec{a}+\vec{b}\right)\times\left(\vec{b}+\vec{c}\right)\right]\cdot\left(\vec{c}+\vec{a}\right)=\left[\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}\right]\cdot\left(\vec{c}+\vec{a}\right)$

$=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}+\left(\vec{b}\times\vec{c}\right)\cdot\vec{a}=2\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=4$。

2.（8 分）求过直线$\dfrac{x-2}{5}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$且与平面$x+4y-3z+1=0$垂直的平面方程。

解：直线$\dfrac{x-2}{5}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$的一般方程$\begin{cases}\dfrac{x-2}{5}=\dfrac{y+1}{2}\\ \dfrac{y+1}{2}=\dfrac{z-2}{4}\end{cases}$即

$\begin{cases}2x-5y-9=0\\2y-z+4=0\end{cases}$。用平面束方法。设所求平面为

$$\pi_\lambda:2x-5y-9+\lambda(2y-z+4)=0$$

即

$$\pi_\lambda:2x+(2\lambda-5)y-\lambda z+4\lambda-9=0$$

$$\vec{n}_\lambda=\{2,2\lambda-5,-\lambda\}$$

平面$x+4y-3z+1=0$的法向量$\vec{n}=\{1,4,-3\}$。$\vec{n}_\lambda\perp\vec{n},\vec{n}_\lambda\cdot\vec{n}=0$，

$2+4(2\lambda-5)+3\lambda=0$解得$\lambda=\dfrac{18}{11}$。所求平面为

$$\pi_{\frac{18}{11}}:2x-5y-9+\frac{18}{11}(2y-z+4)=0$$

即

$$\pi_{\frac{18}{11}}:22x-19y-18z-27=0$$

3.（8 分）求由曲线$\begin{cases}3x^2+2y^2=12\\ z=0\end{cases}$绕$y$轴转一周得到的旋转曲面在点$\left(0,\sqrt{3},\sqrt{2}\right)$处的

指向外侧的单位法向量。

解：旋转曲面$F = 3x^2 + 3z^2 + 2y^2 - 12 = 0$。在点$\left(0,\sqrt{3},\sqrt{2}\right)$处的指向外侧的法向量$\vec{n} = \left\{F_x, F_y, F_z\right\}\Big|_{\left(0,\sqrt{3},\sqrt{2}\right)} = \left\{6x, 4y, 6z\right\}\Big|_{\left(0,\sqrt{3},\sqrt{2}\right)} = \left\{0, 4\sqrt{3}, 6\sqrt{2}\right\}$。单位化得到指向外侧的单位法向量

$$\vec{e}_{\vec{n}} = \left\{0, \frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right\}$$

4.（8分）设直线$L: \begin{cases} x + y + b = 0 \\ x + ay - z - 3 = 0 \end{cases}$在平面$\pi$上，且平面$\pi$又与曲面$z = x^2 + y^2$相切于点$P\left(1, -2, 5\right)$，求$a, b$。

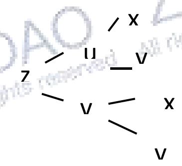
解：曲面$z = x^2 + y^2$为$F = x^2 + y^2 - z = 0$。π的法向量$\vec{n} = \left\{F_x, F_y, F_z\right\}\Big|_P = \left\{2x, 2y, -1\right\}\Big|_P = \left\{2, -4, -1\right\}$。用平面束方法。设$\pi$为

$$\pi: x + y + b + \lambda(x + ay - z - 3) = 0$$

法向量$\vec{n}_\lambda = \left\{\lambda + 1, a\lambda + 1, -\lambda\right\}$。$\vec{n}_\lambda // \vec{n}, \dfrac{\lambda + 1}{2} = \dfrac{a\lambda + 1}{-4} = \lambda$。解得$\lambda = 1, a = -5$。把$P\left(1, -2, 5\right)$代入$\pi: 2x - 4y - z + b - 3 = 0$得$b = -2$。

5.（8分）设变换$\begin{cases} u = x + a\sqrt{y} \\ v = x + 2\sqrt{y} \end{cases}$把方程$\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{2}\dfrac{\partial z}{\partial y} = 0$化为$\dfrac{\partial^2 z}{\partial u \partial v} = 0$，试确定$a$。

解：按照函数图



求导。

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\frac{\partial u}{\partial x}\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial x}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\left(\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}\frac{a}{2\sqrt{y}}+\frac{\partial z}{\partial v}\frac{1}{\sqrt{y}},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{\partial u}{\partial y}\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial y}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\left(\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{a}{2\sqrt{y}}\left(\frac{\partial^2 z}{\partial u^2}\frac{a}{2\sqrt{y}}+\frac{\partial^2 z}{\partial u\partial v}\frac{1}{\sqrt{y}}\right)-\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{3}{2}}}+\frac{1}{\sqrt{y}}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{\sqrt{y}}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2\sqrt{y}}\right)-\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{3}{2}}}$$

$$=\frac{\partial^2 z}{\partial u^2}\frac{a^2}{4y}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2y}-\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{3}{2}}}+\frac{\partial^2 z}{\partial v^2}\frac{1}{y}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2y}-\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial^2 z}{\partial u^2}\frac{a^2}{4}-\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2}+\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{1}{2}}}-\frac{\partial^2 z}{\partial v^2}-\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2}+\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{1}{2}}}$$

$$-\frac{\partial z}{\partial u}\frac{a}{4\sqrt{y}}-\frac{\partial z}{\partial v}\frac{1}{2\sqrt{y}}$$

$$=\left(1-\frac{a^2}{4}\right)\frac{\partial^2 z}{\partial u^2}+(2-a)\frac{\partial^2 z}{\partial u\partial v}$$

令$1-\dfrac{a^2}{4}=0,2-a\neq 0$解得$a=-2$。

二、(12 分) 证明函数$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$在点$(0,0)$连续且偏导数存在，但在此点不可微。

解：$0 \le \left|\frac{x^2y}{x^2+y^2}\right| \le |x|$ 又 $\lim\limits_{\substack{x\to 0\\ y\to 0}}|x| = 0$，所以 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left|\frac{x^2y}{x^2+y^2}\right| = 0$，从而

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^2+y^2} = 0 = f(0,0)$$

所以 $f(x,y)$ 在点 $(0,0)$ 连续。

$\varphi(x) = f(x,0) \equiv 0, \psi(y) = f(0,y) \equiv 0$，

$f_x(0,0) = \varphi'(0) = 0, f_y(0,0) = \psi'(0) = 0$ 都存在。

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y) - f(0,0) - [f_x(0,0)x + f_y(0,0)y]}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

当 $y = kx(k \ne 0)$ 时，$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{\left(x^2+y^2\right)^{\frac{3}{2}}} = \frac{k}{\left(1+k^2\right)^{\frac{3}{2}}}$ 与 k 有关，$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$ 不存在。故 $f(x,y)$ 在点 $(0,0)$ 不可微。

三、（12 分）设 $z = f(2x-y) + g(x,xy)$，其中函数 $f(t)$ 二阶可导，$g(u,v)$ 有连续二阶偏导数，求 $\frac{\partial^2 z}{\partial x\partial y}$。

解：$\frac{\partial z}{\partial x} = 2f'(2x-y) + g_1(x,xy) + yg_2(x,xy)$，

$$\frac{\partial^2 z}{\partial x\partial y} = -2f''(2x-y) + xg_{12}(x,xy) + g_2(x,xy) + xyg_{22}(x,xy)。$$

四、（，所以 12 分）设 $u = f(x,y,z), \varphi(x^2, e^y, z) = 0, y = \sin x$，其中 f,φ 都具有一阶连续偏导数且 $\frac{\partial\varphi}{\partial z} \ne 0$，求 $\frac{du}{dx}$。

解：$\varphi(x^2, e^{\sin x}, z) = 0$ 两边对 x 求导

$$2x\varphi_1(x^2, e^{\sin x}, z) + e^{\sin x}\cos x\varphi_2(x^2, e^{\sin x}, z) + \varphi_3(x^2, e^{\sin x}, z)\frac{dz}{dx} = 0，$$

$$\frac{dz}{dx} = -\frac{2x\varphi_1(x^2, e^{\sin x}, z) + e^{\sin x}\cos x\varphi_2(x^2, e^{\sin x}, z)}{\varphi_3(x^2, e^{\sin x}, z)}。$$

$$\frac{du}{dx}=f_1+\cos xf_2-\frac{2x\varphi_1(x^2,e^{\sin x},z)+e^{\sin x}\cos x\varphi_2(x^2,e^{\sin x},z)}{\varphi_3(x^2,e^{\sin x},z)}f_3。$$

五、（12 分）设 $f(x,y,z)=x^3-xy^2-z$。

（1）求 $f(x,y,z)$ 在点 $P_0(1,1,0)$ 处的梯度及方向导数的最大值；

（2）问：$f(x,y,z)$ 在哪些点的梯度垂直于 x 轴？

解：（1）$\overrightarrow{grad}f(P_0)=\{f_x(P_0),f_y(P_0),f_z(P_0)\}=\{3x^2-y^2,-2xy,-1\}\big|_{P_0}=\{2,-2,-1\}$。

在点 $P_0(1,1,0)$ 处方向导数的最大值 $=\left|\overrightarrow{grad}f(P_0)\right|=3$。

（2）$\overrightarrow{grad}f(x,y,z)=\{f_x,f_y,f_z\}=\{3x^2-y^2,-2xy,-1\}$ 在两条相交直线 $3x^2-y^2=0$ 上点的梯度垂直于 x 轴。

六、（12 分）在椭球面 $4x^2+y^2+z^2=4$ 的第一卦限部分上求一点，使得椭球面在该点的切平面、椭球面及三个坐标平面所围成在第一卦限部分的立体的体积最小。

解：$F=4x^2+y^2+z^2-4=0$ 在点 (x,y,z) 的切平面

$$4x(X-x)+y(Y-y)+z(Z-z)=0$$

化简为

$$4xX+yY+zZ=4$$

三个截距分别是 $\frac{1}{x},\frac{4}{y},\frac{4}{z}$。四面体的体积 $V_4=\frac{8}{3xyz}$。

椭球第一卦限部分的体积 V_1 显然是常数。所给立体的体积=V_4-V_1。因此归结为条件极值问题

$$\begin{cases} f=\dfrac{1}{xyz} \\ 4x^2+y^2+z^2-4=0 \end{cases}(x>0,y>0,z>0)$$

$L=\frac{1}{xyz}+\lambda(4x^2+y^2+z^2-4)$，令

$$\begin{cases} L_x = -\dfrac{1}{x^2yz} + 8\lambda x = 0 \\ L_y = -\dfrac{1}{xy^2z} + 2\lambda y = 0 \\ L_z = -\dfrac{1}{xyz^2} + 2\lambda z = 0 \\ L_\lambda = 4x^2 + y^2 + z^2 - 4 = 0 \end{cases}$$

由前三个方程有$\lambda \neq 0, 4x^2 = y^2 = z^2$。代入最后方程解得$x = \dfrac{\sqrt{3}}{3}$，$y = z = \dfrac{2\sqrt{3}}{3}$。

根据问题的实际，这就是所给立体体积最小的点。