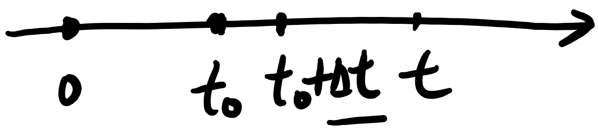
1.引例：变速直线运动 $S(t)$

求 t_0 时刻的瞬时速度

分析：



$$\Delta S = S(t_0+\Delta t) - S(t_0)$$

$$\frac{\Delta S}{\Delta t} = \bar{v}$$

$$V(t_0) = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{S(t_0+\Delta t) - S(t_0)}{\Delta t} = S'(t_0)$$

2.导数的定义：

设 $y=f(x)$ 在 x_0 邻域内有定义 $((x_0-\delta, x_0+\delta))$

$$x: x_0 \to x_0+\Delta x$$

$$\Delta y = f(x_0+\Delta x) - f(x_0)$$

若 $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

则称极限值为 $f(x)$ 在 x_0 点导数

记作 $f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} = y'|_{x=x_0} = \left.\frac{dy}{dx}\right|_{x=x_0}$

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$\overset{x_0+\Delta x = x}{=} \lim \frac{f(x) - f(x_0)}{x - x_0}$$

$\qquad x\to x_0 \quad \underline{x-x_0}$

3. 导数的物理意义：

$S'(t_0)$ 表示 瞬时速度

$v'(t_0)$ 表示 t_0 时刻的加速度

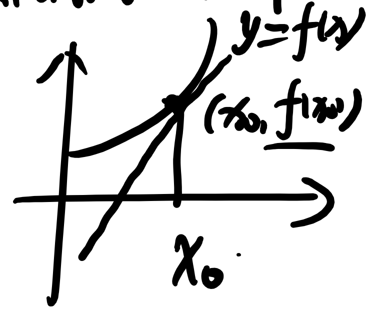
4. 导数的数学意义：

$f'(x_0)$ 表示 $f(x)$ 在 x_0 点变化率

5. 导数的几何意义：

$f'(x_0)$ 表示曲线 $y=f(x)$ 在 x_0 点切线的斜率

切线方程为 $y-f(x_0)$
$=f'(x_0)(x-x_0)$



6. 导函数.

设 $y=f(x)$ 在 (a,b) 内每一点均可导

则 $\forall x\in(a,b)$ 称 $f'(x)$ 为 $f(x)$ 在 (a,b) 上导函数.

且 $\boxed{f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}}$

例1. $y=x^2$ 则 $y'=2x$ $\quad =x^2+2x\Delta x+(\Delta x)^2$

分析：$(x^2)'=\lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^2-x^2}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

例2. $y = x^n$ 则 $y' = n \cdot x^{n-1}$ $\quad (a+b)^n = C_n^0 a^0 b^n + C_n^1 a b^{n-1} + \cdots + C_n^n a^n b^0$

分析: $$(x^n)' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\underbrace{\boxed{C_n^0 x^n \cdot (\Delta x)^0}}_{}^{x^n} + C_n^1 x^{n-1} (\Delta x) + \cdots + C_n^n x^0 (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{n \cdot x^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} (\Delta x)^2 + \cdots + (\Delta x)^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(n \cdot x^{n-1} + \frac{n(n-1)}{2} x^{n-2} \cdot \Delta x + \cdots + (\Delta x)^{n-1} \right)$$

$$= n x^{n-1}$$

例3: $y = e^x$ 则 $y' = e^x$

分析: $y = a^x$. ($a > 0$ 且 $a \neq 1$) 则 $y' =$

$$(a^x)' = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x \cdot \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \qquad a^b = e^{\ln a^b} = e^{b \ln a}$$

$a^{\Delta x \cdot \ln a}$

$$= a^x \cdot \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$a^b = e^{\ln a^b} = e^{b \ln a}$$

$$= a^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x \cdot \ln a} - 1}{\Delta x}$$

$$e^{\square} - 1 \sim \square$$

$$= a^x \lim_{\Delta x \to 0} \frac{\cancel{\Delta x} \cdot \ln a}{\cancel{\Delta x}}$$

$$(a^x)' = a^x \cdot \ln a$$

例4. $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

7、导数四则运算

设 $f(x), g(x)$ 均为可导的函数

则
$$\boxed{\begin{aligned} &(f(x) \pm g(x))' = f'(x) \pm g'(x) \\ &\boxed{(f(x) \cdot g(x))' = f'(x) g(x) + f(x) \cdot g'(x)} \\ &\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) g'(x)}{g^2(x)} \end{aligned}}$$

证明: $(f(x) g(x))' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) g(x+\Delta x) - f(x) g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) g(x+\Delta x) - f(x) g(x+\Delta x)) + (f(x) g(x+\Delta x) - f(x) g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x+\Delta x) + f(x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$$

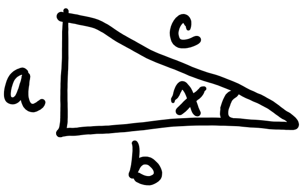
$$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

例5、令 $y=\tan x$ 则 $y'=$ $\quad (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1}$

分析：$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x\cdot(\cos x)'}{\cos^2 x}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \quad \text{(正割)}$$



$\sin x = \frac{a}{c}$ ✓ $\quad \cos x = \frac{b}{c}$ ✓

$\tan x = \frac{a}{b}$ ✓ $\quad \cot x = \frac{b}{a}$ ✓

$\csc x = \frac{c}{a} = \frac{1}{\sin x} \quad \sec x = \frac{c}{b} = \frac{1}{\cos x}$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{(1)'\cos x - 1\cdot(\cos x)'}{(\cos x)^2}$$

$$= \frac{\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x\cdot\sec x$$

8、高阶导

称 $y'', y^{(3)}, \cdots y^{(n)}$ 为 y 的高阶导

例1：$y = x^n$,

分析1 $y' = n\cdot x^{n-1}$, $y'' = (n\cdot x^{n-1})'$

$$= n\cdot(x^{n-1})'$$

$$= n\cdot(n-1)x^{n-2}$$

$$y''' = n\cdot(n-1)\cdot(x^{n-2})' = n(n-1)(n-2)x^{n-3}$$

$$\cdots\ y^{(100)} = (x^n)^{(100)} = n(n-1)\cdots(n-99)\cdot x^{n-100}\ (n>100)$$

例2. $y = a^x$, $\boxed{y' = a^x\cdot\ln a}$

分析: $y'' = (a^x\cdot\ln a)' = \ln a\cdot(a^x)' = \ln a\cdot a^x\cdot\ln a$

$$= (\ln a)^2\cdot a^x$$

$$\cdots\ y^{(n)} = (\ln a)^n\cdot a^x$$

例3: $y = \sin x$, $y' = \cos x = \sin(x+\frac{\pi}{2})$

$$y'' = (\sin(x+\tfrac{\pi}{2}))' = \cos(x+\tfrac{\pi}{2})$$

$\cos\square = \sin(\square+\frac{\pi}{2})$

$$= \sin(x+\tfrac{\pi}{2}+\tfrac{\pi}{2})$$

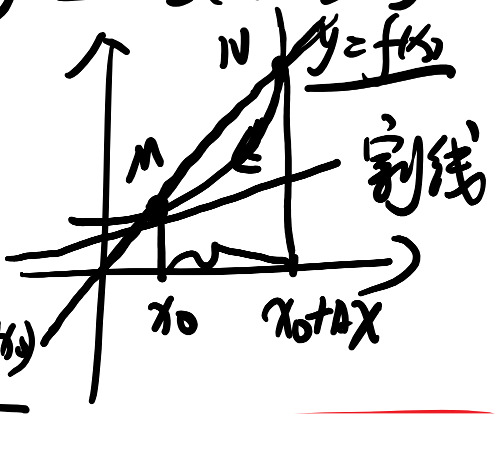
$$= \sin(x+\tfrac{\pi}{2}\cdot 2)$$

$$\cdots\ y^{(n)} = (\sin x)^{(n)} = \sin(x+\tfrac{\pi}{2}\cdot n)$$

$$(\cos x)^{(n)} = \cos(x+\tfrac{\pi}{2}n)$$

$M(x_0, f(x_0))$

$N(x_0+\Delta x, f(x_0+\Delta x))$



$k = \dfrac{f(x_0+\Delta x)-f(x_0)}{\ }$

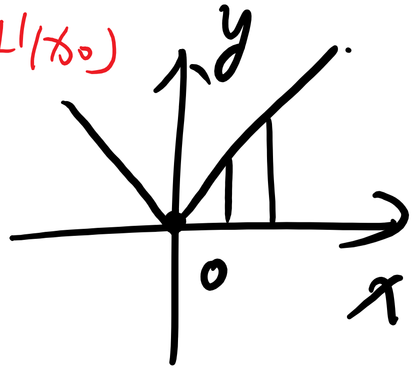
$$k_{割} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$k_{切} = \lim_{\Delta x\to 0} k_{割} = \boxed{\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}} = f'(x_0)$$

$$y-f(x_0)=\underset{f'(x_0)}{\underset{\parallel}{k}}(x-x_0)$$

例 $y=|x|$

在$x=0$处不可导



9. 复合函数求导. $y=\cos x^2$

①分解: $y=\cos u$ $u=x^2$ $(\cos x)'_x=-\sin x$

②分别求导: $\frac{dy}{du}=(\cos u)'_u=-\sin u$

$$\frac{du}{dx}=(x^2)'_x=2x$$

③相乘: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=-\sin u\cdot 2x$

导数—微商 $\frac{a}{b}=\frac{a}{c}\cdot\frac{c}{b}$

④还代: $\frac{dy}{dx}=-\sin x^2\cdot 2x$

$$=-2x\sin x^2$$

例11. $y=e^{x^2+\cos x}$ $(e^x)'_x=e^x$

例1. $y=e^{x^2+\cos x}$ $\qquad (e^x)'_x=e^x$

解：① $y=e^u \qquad u=x^2+\cos x$

② $\frac{dy}{du}=e^u \qquad \frac{du}{dx}=(x^2+\cos x)'$

$$=(x^2)'+(\cos x)'$$

$$=2x-\sin x$$

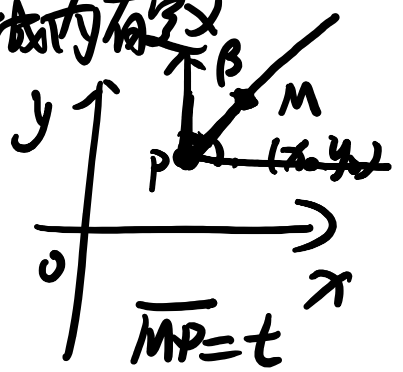
③ $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot(2x-\sin x)$

$$=e^{x^2+\cos x}\cdot(2x-\sin x)$$

$y=f(x)$

10. 方向导数.

设 $z=f(x,y)$ 在 (x_0,y_0) 点邻域内有定义

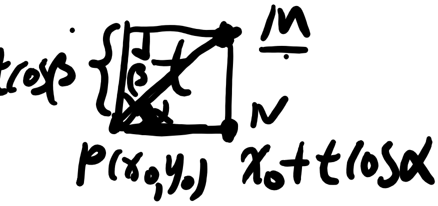


设 l 以 (x_0,y_0) 为起点，

以 $(\cos\alpha,\cos\beta)$ 为方向的射线

$\overline{MP}=t$

则 $l:\begin{cases}x=x_0+t\cos\alpha\\ y=y_0+t\cos\beta\end{cases}$



若 $\lim\limits_{t\to 0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta)-f(x_0,y_0)}{t}=A$

则称 A 为 $f(x,y)$ 在 (x_0,y_0) 点沿着 l 方向上的方向导数. 记作 $\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}=\lim\limits_{t\to 0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta)-f(x_0,y_0)}{t}$

数学意义即为 $f(x,y)$ 在 (x_0,y_0) 沿着 l 方向上的变化率

11、求方向导数

设 $z=f(x,y)$ 在 (x_0,y_0) 可微，

则 $f(x,y)$ 在 (x_0,y_0) 沿着 $l\,(\cos\alpha,\cos\beta)$ 的方向导数

为 $$\left.\frac{\partial f}{\partial l}\right|_{(x_0,y_0)}=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\cos\beta$$

例：设 $f(x,y)=x^2+y^2$，l 以 $(1,1)$ 为起点以 $(1,2)$ 为方向向量射线 则 $\left.\frac{\partial f}{\partial l}\right|_{(1,1)}=$

解：
$$\frac{\partial f}{\partial x}=\frac{\partial(x^2+y^2)}{\partial x}=\frac{\partial x^2}{\partial x}=2x$$

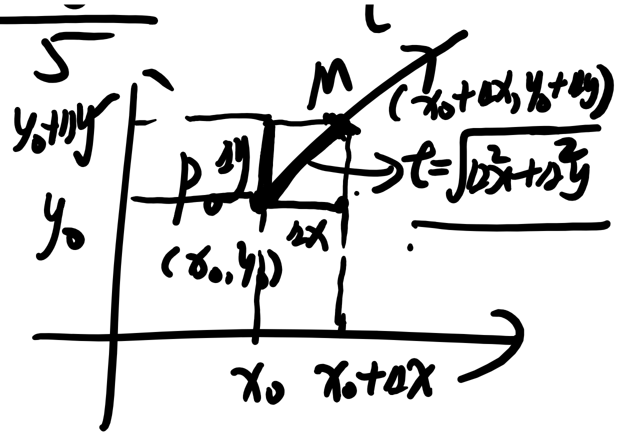
$$\frac{\partial f}{\partial y}=\frac{\partial(x^2+y^2)}{\partial y}=2y$$

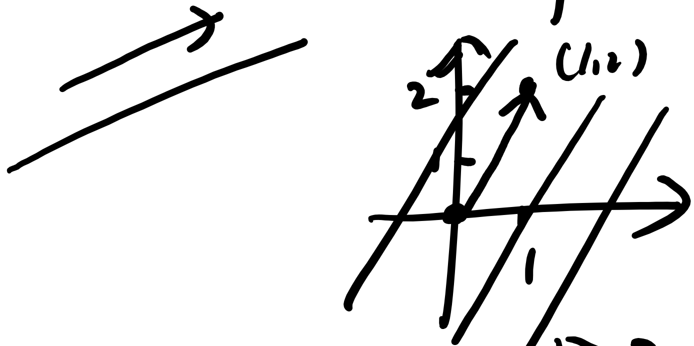
$$\cos\alpha=\frac{1}{\|(1,2)\|}=\frac{1}{\sqrt{1^2+2^2}}=\frac{\sqrt5}{5}$$

$$\cos\beta=\frac{2}{\|(1,2)\|}=\frac{2\sqrt5}{5}$$

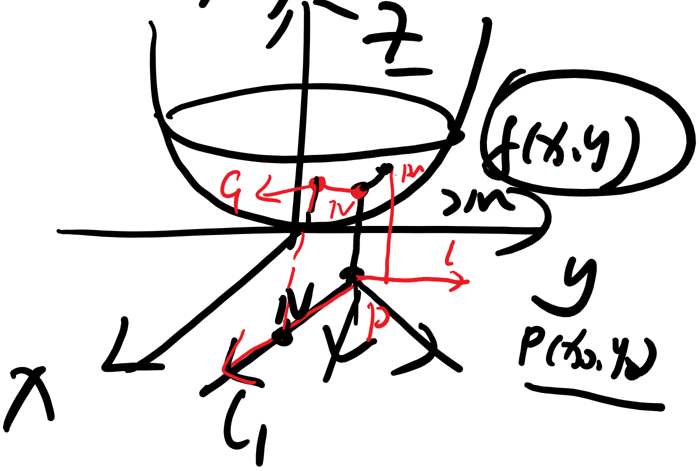
故 $$\left.\frac{\partial f}{\partial l}\right|_{(1,1)}=\left.\frac{\partial f}{\partial x}\right|_{(1,1)}\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_{(1,1)}\cos\beta$$
$$=2\cdot\frac{\sqrt5}{5}+2\cdot\frac{2\sqrt5}{5}$$
$$=\frac{6\sqrt5}{5}$$

$= \frac{\quad}{5}$





$z=f(x,y)=x^2+y^2$



$\frac{f(x_0,y_0)}{f(x,y)}$

12. $\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\cos\alpha + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\cos\beta$

$= \left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\right)\cdot(\cos\alpha,\cos\beta)$

$= \left\|\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\right)\right\|\cdot\|(\cos\alpha,\cos\beta)\|\cos\theta$

（θ：两向量夹角 $0\le\theta\le\pi$）

当 $\theta=0$ 时 $\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}$ 最大

即 $\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}$ 点沿着 $\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\right)$

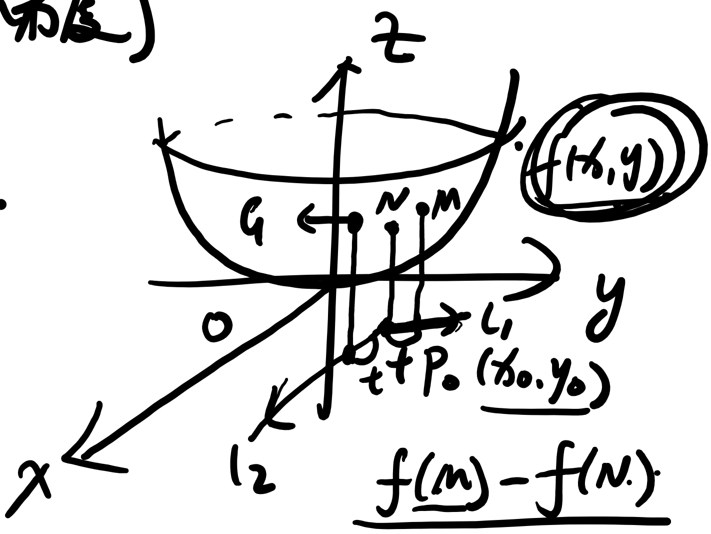
即 $\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\right)$ 点沿着 $\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\right)$

的方向导数最大

此时 $\mathrm{grad} f\big|_{(x_0,y_0)} = \boxed{\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\right)}$

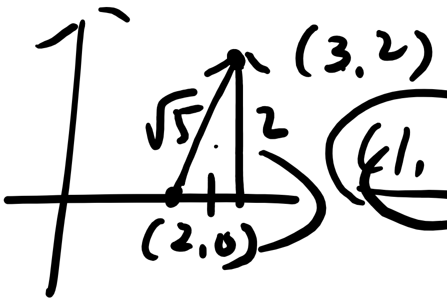
(梯度)

$z = x^2 + y^2$.



$$\frac{f(M) - f(N)}{t}$$

$$\frac{f(G) - f(N)}{t}$$



$(1, 2)$ $\sqrt{1^2+2^2}$

$y = f(x)$. 可微 $\Leftrightarrow$ 可导

$z = f(x, y)$. 可微(全微分存在) $\Rightarrow$ 偏导存在