hence limit the applicability of many methods. Also, one could argue that the neglect of available information besides historic data is rather a short-coming than an advantage. A good overview of these issues is given in [1].

This paper is organized as follows. We start by exposing short-comings of the traditional approach in section 2. In section 3 we present our approach and discuss the interpretation of futures and risk. In section 4 we discuss the interpretation of Black-Scholes pricing in our context. In section 5 we present applications of the probabilistic framework to pricing under an uncertain second moment (uncertain uncertainty), which naturally leads to an implied volatility skew, derivative exposure and risk management.

2 The Traditional Approach and its Problems

The standard approach to derivatives pricing based on stochastic models and risk-neutral measures (see appendix A for a short review) raises a number of issues. Firstly, it does not allow for subjectivity as it assumes the existence of an objective "real distribution", and hence does not take into account that people have different information and believes[2]. This effectively leads to the idea that there is a "correct" value for a derivative, independently of its current trading price — it would possibly be "mispriced".

We think that this is misleading. Every market participant will assign a different value according to his state of knowledge and beliefs. If there are sellers and buyers with different *subjective* valuations, it might result in trades. These in turn might result in observed market prices. One should note that the market price then does not necessarily represent the *subjective* valuation of neither the buyer nor the seller[3].

Secondly, even if we assume this awkward "underlying real distribution", we find it difficult to follow the risk-neutral hedging argument, as the eliminated term does not only contain uncertainty and risk. Consider for example the standard log-normal case with

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}X, \tag{1}$$

where X is a "normally distributed random variable"[4]. It is clear that $\mathrm{d}S$ becomes 'increasingly deterministic' as σ goes to zero — for $\sigma = 0$ it just expresses the deterministic drift. Hence we want to argue that $\mathrm{d}S$ is really a mixture of a certain and uncertain component. As one cannot eliminate the dependence of the portfolio value change on the different components of $\mathrm{d}S$ separately, it is not possible to follow a hedging strategy which keeps the drift, but eliminates the randomness.

Now, by eliminating $\mathrm{d}S$ one also discards the non-random drift term and the resulting hedging is generally (except for a drift equal to the risk free rate) suboptimal. This is most easily illustrated in extreme (hypothetical) cases with huge (compared to the volatility) positive or negative drift.

It appears as if it is often not understood what it really *means* to have a drift in the distribution. This is not just something observed in historical data, but something we claim to *know* about the possibilities in the future. If we use a log-normal distribution with a 1000% drift rate and 10% volatility then we also say that it is practically impossible to observe a draw-down over the next year (and even to see a return of less than 900%) — and we claim to *know* this.

Yet risk-neutral pricing tells us to ignore the *knowledge* of the extreme drift and to hedge as if there is a 50/50 chance of a draw-dawn (below the risk-free rate). Having a short put-delta position will lead us with near certainty to buy back the asset at higher prices.

Figure 1 illustrates this. The x-axis shows the underlying asset value, with $x = 1$ being the current spot price. The solid line in the upper graph is a log-normal distribution with an expected return equal to the risk-free rate (here 10%). The lower graph shows the corresponding Black-Scholes put-delta position for a 110% strike. For this case it makes sense

[2] A good pricing theory should be able to value an asset even for an inside-trader.

[3] See appendix E.

[4] It is questionable whether the concept of a random variable can be consistently defined [1]. One can argue that "randomness" is a result of insufficient information about the state of the system. A random variable results then from uncontrolled (and hence unknown) initial conditions.