In Method 1 [17], we find first the conditional reference prior $\pi_R(\sigma \,|\, \epsilon, \mu)$ and then multiply by the evidence-based prior $\pi(\epsilon, \mu)$ to construct the full prior $\pi(\sigma, \epsilon, \mu)$. As will be illustrated in Sec. IV, the single-count model is regular enough to warrant using Jeffreys' rule in the first step of the calculation of $\pi_R(\sigma \,|\, \epsilon, \mu)$. We therefore apply Eq. (15) to the σ dependence of the likelihood (16), while holding ϵ and μ constant; this yields:

$$\pi_J(\sigma \,|\, \epsilon, \mu) \;\propto\; \sqrt{\mathbb{E}\left[-\frac{\partial^2}{\partial\sigma^2}\ln p(n|\sigma, \epsilon, \mu)\right]} \;\propto\; \frac{\epsilon}{\sqrt{\epsilon\,\sigma + \mu}}. \tag{19}$$

This prior is clearly improper with respect to σ and is therefore only defined up to a proportionality constant. However, this constant could very well depend on ϵ and μ, since we kept these parameters fixed in the calculation. It is important to obtain this dependence correctly, as examples have shown that otherwise inconsistent Bayes estimators may result. Reference [17] proposes a compact subset normalization procedure. One starts by choosing a nested sequence $\Theta_1 \subset \Theta_2 \subset \cdots$ of compact subsets of the parameter space $\Theta = \{(\sigma, \epsilon, \mu)\}$, such that $\cup_\ell \Theta_\ell = \Theta$ and the integral $K_\ell(\epsilon, \mu)$ of $\pi_J(\sigma \,|\, \epsilon, \mu)$ over $\Omega_\ell \equiv \{\sigma : (\sigma, \epsilon, \mu) \in \Theta_\ell\}$ is finite. The conditional reference prior for σ on Ω_ℓ is then

$$\pi_{R,\ell}(\sigma \,|\, \epsilon, \mu) \;=\; \frac{\pi_J(\sigma \,|\, \epsilon, \mu)}{K_\ell(\epsilon, \mu)}. \tag{20}$$

To obtain the conditional reference prior on the whole parameter space, one chooses a fixed point $(\sigma_0, \epsilon_0, \mu_0)$ within that space and takes the limit of the ratio

$$\pi_R(\sigma \,|\, \epsilon, \mu) \;\propto\; \lim_{\ell\to\infty} \frac{\pi_{R,\ell}(\sigma \,|\, \epsilon, \mu)}{\pi_{R,\ell}(\sigma_0 \,|\, \epsilon_0, \mu_0)}. \tag{21}$$

By taking the limit in this ratio form, one avoids problems arising from $K_\ell(\epsilon, \mu)$ becoming infinite as $\ell \to \infty$.

The theory of reference priors currently does not provide guidelines for choosing the compact sets Θ_ℓ, other than to require that the resulting posterior be proper. In most cases this choice makes no difference and one is free to base the choice of compact sets on considerations of simplicity and convenience. However, we have found that some care is required with the single-count model. Indeed, suppose we make the plausible choice

$$\Theta_\ell \;=\; \Big\{(\sigma, \epsilon, \mu) : \; \sigma \in [0, u_\ell], \; \epsilon \in [0, v_\ell], \; \mu \in [0, w_\ell]\Big\}, \tag{22}$$

where $\{u_\ell\}$, $\{v_\ell\}$, and $\{w_\ell\}$ are increasing sequences of positive constants. If we use these sets in applying Eqs. (20) and (21) to the prior (19), we obtain:

$$\pi_R(\sigma \,|\, \epsilon, \mu) \;\propto\; \sqrt{\frac{\epsilon}{\epsilon\,\sigma + \mu}}. \tag{23}$$