stationary states of the system. Branches (4) and (5) have no linear counterparts since at $\mu \simeq 2.828 \times 10^{-12}$ eV they "collide" and disappear. The states that belong to these branches are asymmetric, exhibiting two nodes. The state with larger number of atoms (for a fixed chemical potential) has one node approximately at the barrier and one node in one well. The other state has both nodes in one well. Once again, the occupation number in the well with two nodes is less than the one of the well with one node, so as to balance the chemical potential.

Branch (6) starts from the second excited state of the linear problem, hence $\mu \to \omega_2$ as $N \to 0$. Therefore, the density of the respective state is symmetric and there is one node in each well.

Branch (7) starts from the third excited state of the linear problem, and $\mu \to \omega_3$ as $N \to 0$. The states belonging to this branch have three nodes, one located in each well and one at the barrier, and are symmetric. Close to the linear limit, at $\mu \simeq 2.795 \times 10^{-12}$ eV, two asymmetric three-soliton branches, namely branch (8) and its mirror image with respect to the $z = 0$ axis, bifurcate from this state. This state has two nodes in one well and one in the other. The occupation number in the well with more nodes is smaller than in the well with just one node in order to balance the chemical potential. Notice that there exist two more three-soliton branches without linear counterparts, but they only occur at higher chemical potentials — where the BEC has occupied four-wells rather than two — so they will not be considered here.

Next, let us study the stability of the states belonging to the above mentioned branches by considering the respective excitation spectra, shown in Fig. 2. In this figure, the (blue) $\divideontimes$ symbol denotes a positive Krein sign mode (or a zero one for a vanishing frequency), the (green) $\times$ symbol a negative Krein sign mode, i.e., a negative energy *anomalous mode*, and the (red) (+) symbol a vanishing Krein sign, i.e., an eigenmode associated with complex/imaginary eigenfrequency. Generally speaking, there are two different kinds of instability, corresponding to the cases of either a purely imaginary eigenfrequency, or a genuinely complex eigenfrequency. The latter case gives rise to the so-called oscillatory instability, stemming from the collision of a negative energy mode with one of positive energy.

The first panel of Fig. 2 shows the BdG spectra along the first branch. The imaginary part is zero for every value of the chemical potential, a fact reflecting the stability of this state. The absence of negative energy modes is expected, as this state is actually the ground