configuration C_t. For the RSU we define $\rho(C)$ as the average density of active sites in the steady state of the master equation (3) if the system evolves from the configuration C. Obviously, $\rho(C) = 0$ irrespective of update rules if $S(C) = 0$.

Although $S(C) = 1$ implies $\rho(C) > 0$, $\rho(C)$ generally depends on the update rule. For example, consider a configuration (0202) for the TM $\tilde{F}$ in Eq. (16) with $V = 4$. Obviously, $\rho(C) = 1/2$ for the PU. In the RSU, there are two possible patterns, (0202) and (1012), up to translation. Since the average waiting time to the next jump of the first pattern is 1/2 while that of the second pattern is 1, the probability that the first (second) pattern is found in the steady state is 1/3 (2/3). Thus, $\rho(C) = 1/3$ when the RSU is employed.

Now, we define the second order parameter ϕ_2 as

$$\phi_2 \equiv \sum_C \rho(C) P_0(C; \zeta, V), \tag{27}$$

where P_0 is the initial distribution and ζ satisfies Eq. (9). As we have shown, ϕ_2 depends on the update rule, which should be contrasted with ϕ_1. Since $\rho(C) \le S(C)$ for any C, ϕ_2 cannot be larger than ϕ_1. Thus, $\phi_1 = 0$ implies $\phi_2 = 0$. For finite V, $\phi_1 \neq 0$ should imply $\phi_2 \neq 0$. Thus, a phase transition point for finite V, if it exists, is the same irrespective of whether ϕ_1 or ϕ_2 is used as an order parameter.

Although transition points for finite V do not depend on which order parameter is used, it is nontrivial to answer whether or not the infinite-size limit affects this conclusion. This is because there are configurations such that $\rho(C) \to 0$ under the infinite V limit, while $S(C) = 1$ for any V. For example, if $C = \cdots 1111201111 \cdots$ for the TM (16), $\rho(C) = 1/V \to 0$ while $S(C) = 1$ for any V. In Appendix A, we show that it is always possible to construct such a configuration with $E = Z_m$ for any symmetric TM. If such configurations exist for any E in the range $Z_m \le E \le Z_M$, we cannot exclude the possibility that ϕ_1 and ϕ_2 for the same initial condition can give different transition points in the infinite-size limit. Presumably, an ignorant preparation for the initial condition as independent and identical Poisson distributions would not generate such complications. This might be an interesting question, but we will not pursue the difference in ϕ_1 and ϕ_2 any further in this paper.

C. Order parameter ϕ_3

In the previous two sections, the infinite-size limit, when necessary, is preceded by the infinite-time limit. Now we discuss the consequence of changing the order of the two limits.

If the infinite-size limit is taken first, we cannot assign a unique value to $S(C)$, introduced in Sec. III A, irrespective of the update rule. For example, consider the TM

$$
\begin{aligned}
t = 0 &: 0000\underline{22222222222222222222}0000\\
t = 1 &: 00011\underline{222222222222222222}11000\\
t = 2 &: 000121\underline{2222222222222222}121000\\
t = 3 &: 0002031\underline{22222222222222}1302000\\
t = 4 &: 00102131\underline{222222222222}13120100\\
t = 5 &: 001103131\underline{2222222222}131301100
\end{aligned}
$$

FIG. 2. Time evolution of the DFES with the TM (16) for the initial configuration (32) with $V = 80$. The PU is employed. At every time step, the length of the middle string of 2's (underlined) decreases by 2.

(16) with the following initial configuration

$$z_i = \begin{cases} 2, & \text{if } i \pmod 3 = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

which resembles $\cdots 002002002002 \cdots$ in a one-dimensional infinite lattice. If the PU is employed, the system falls into an absorbing state in one unit time even if the system size is infinite. However, if we employ the RSU, that is, the master equation, the probability that this configuration evolves into an absorbing state in finite time is zero. Still, the density of active sites is zero in the infinite-time limit (the average density of active sites at time t in the RSU is $e^{-t}/3$).

As this example suggests, it is appropriate to study the time dependence of the density of active sites. From the above consideration, we define the third order parameter ϕ_3 as

$$\rho_a(t) = \limsup_{V \to \infty} \sum_C \frac{n(C, V; t)}{V} P_0(C; \zeta, V), \tag{29}$$

$$\phi_3 = \lim_{t \to \infty} \rho_a(t), \tag{30}$$

where $n(C, V; t)$ is the number of active sites averaged over ensembles at time t when the system evolves from the initial configuration C. For convenience, we will exclusively refer to $\rho_a(t)$ as the *activity density* (at time t). Although we did not mention it explicitly, the infinite-size limit above should be understood as lim sup to guarantee the existence of the limit for any sequence of P_0.

It may be tempting to claim that if $S(C) = 0$ for any finite V, the limit of $n(C, V; t)$ should be

$$\lim_{t \to \infty} \lim_{V \to \infty} \frac{n(C, V; t)}{V} = 0. \tag{31}$$

However, it is not true in general. For example, consider again the TM (16) with an initial configuration C_0,

$$z_i = \begin{cases} 2, & \text{if } 1 \le i \le V/4, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$