since $\varepsilon_n/h_n \to_{n\to\infty} \infty$ by assumption. Furthermore, $t \in N(s_{i,n})$ and the fact that K is decreasing implies

$$\max_{j:\|s_{j,n}-t\|<2\varepsilon_n} K\left(\|t-s_{j,n}\|/h_n\right) = K\left(\|t-s_{i,n}\|/h_n\right).$$

Thus,

$$\begin{aligned}
1 \le B_{i,n}(t) &= \frac{1}{K\left(\|t-s_{i,n}\|/h_n\right)}\Bigg(E\bigg(\max_{j:\|s_{j,n}-t\|<2\varepsilon_n} K\left(\|t-s_{j,n}\|/h_n\right) Z_{s_{j,n}} \\
&\qquad - \max_{j:\|s_{j,n}-t\|<2\varepsilon_n} K\left(\|t-s_{j,n}\|/h_n\right) Z_{s_{i,n}}\bigg)\Bigg) + 1 \\
&\le \frac{E\left(\max_{j:\|s_{j,n}-t\|<2\varepsilon_n} K\left(\|t-s_{j,n}\|/h_n\right)\left|Z_{s_{j,n}} - Z_{s_{i,n}}\right|\right)}{K\left(\|t-s_{i,n}\|/h_n\right)} + 1 \\
&\le E\left(\max_{j:\|s_{j,n}-t\|<2\varepsilon_n} \left|Z_{s_{j,n}} - Z_{s_{i,n}}\right|\right) + 1 \\
&\le E\left(\sup_{\|r-s\|<3\varepsilon_n} |Z_r - Z_s|\right) + 1 \\
&\to_{n\to\infty} 1,
\end{aligned}$$

because of Lemma 3.2. Note that $\|s_{j,n}-t\| < 2\varepsilon_n$ and $t \in N(s_{i,n})$ imply $\|s_{j,n}-s_{i,n}\| < 3\varepsilon_n$. □

We have now gathered the tools to prove convergence of the mean squared error to zero.

Theorem 3.4. *Define $\hat{\boldsymbol{\eta}}_n$ and ε_n as above, $n \in \mathbb{N}$. Then for every $t \in [0,1]^k$*

$$\mathrm{MSE}\left(\hat{\eta}_{t,n}\right) \to_{n\to\infty} 0,$$

and

$$\mathrm{IMSE}\left(\hat{\eta}_{t,n}\right) := \int_{[0,1]^k} \mathrm{MSE}\left(\hat{\eta}_{t,n}\right)\, dt \to_{n\to\infty} 0,$$

if $\varepsilon_n \to_{n\to\infty} 0$, $h_n \to_{n\to\infty} 0$, $\varepsilon_n/h_n \to_{n\to\infty} \infty$.

Proof. Denote by

$$\hat{Z}_{t,n} = \max_{j=1,\dots,d}\left(g_{j,n}(t) Z_{s_{j,n}}\right), \qquad t \in [0,1]^k,$$