Topological Quantum Field Theory

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This note is based on [CR18].

1 Motivation and definition of TQFT

1.1 Motivation: path integrals

A path integral is an integral on infinite dimensional space

$$\mathcal{Z} = \int D\Phi \, e^{-S[\Phi]}, \tag{1.1}$$

The notations in (1.1) are as follows.

- A *field* $\Phi : M \to X$ is a smooth map between two Riemannian manifold M and X.
- The *action functional* $S[\Phi]$ depends on fields Φ and its first derivatives

  $$S[\Phi] = \int_M \mathcal{L}(\Phi, \Phi_\mu)(x)\sqrt{\det g}\, \mathrm{d}^n x,$$

  where g is the metric on M and $n = \dim M$.
- The integral $\int D\Phi$ means to integral over all fields Φ, which makes no sense in mathematics.

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be some observables, i.e. functionals from the set of configurations $\{\Phi : M \to X\}$ to $\mathbb{C}$. The *correlation* function of $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is

$$\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle_g = \frac{1}{\mathcal{Z}} \int D\Phi \, \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S[\Phi]},$$

where g is the metric on M. If the correlation functions is independent of g, we get a topological field theory.

Example 1.1 (Chern-Simons theory)**.** Let M be a compact 3-dimensional oriented Riemannian manifold and G be a compact Lie group . $\mathfrak{g}$ is the Lie algebra of G. Assume that $A \in \Omega^1(M, \mathfrak{g})$ is a connection 1-form on a principle G-bundle $P \to M$. Then the action functional is given by

$$S[A] = \gamma \int_M \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right), \tag{1.2}$$

where γ is a constant and tr is the matrix trace. The (1.2) is independent of metrics, hence Chern-Simons theory is a topological theory. For details, see [Wit89].

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Let us illustrate the essential properties of path integral in TQFT of dimension n that we want:

(1) For each $(n-1)$-manifold E, there is a state space $\mathcal{H}_E$ which consists all functionals on the classical fields on E. We emphasize that state spaces are all Hilbert spaces.

(2) For each oriented n-manifold M with boundary E, we assign M a functional $\mathcal{Z}(M) \in \mathcal{H}_E$ on the space of classical fields,

$$\mathcal{Z}(M)(\varphi) = \int_{\{\Phi|\Phi_E=\varphi\}} D\Phi\, e^{-S[\Phi]}.$$

(3) For a $(n-1)$-manifold $E = E_1 \sqcup E_2$, we expect $\mathcal{H}_E = \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2}$ since the linear space $\mathcal{F}(E)$ of functionals from the space $\mathcal{M}(E)$ of classical field to $\mathbb{C}$ would satisfy $\mathcal{F}(E) = \mathcal{F}(E_1) \otimes \mathcal{F}(E_2)$ when $\mathcal{M}(E_1)$ and $\mathcal{M}(E_2)$ are both finite sets.

(4) For cylinder $M = E \times [0,1]$, the path integral produces an element $\mathcal{Z}(M) \in \mathcal{H}_E \otimes \mathcal{H}_{\bar{E}}$ where $\bar{E}$ is E with reversed orientation. Let h and $\bar{h}$ be the metric on $\mathcal{H}_E$ and $\mathcal{H}_{\bar{E}}$ respectively and denote that

$$\mathcal{Z}(M) = \sum_i e_i \otimes f^i$$

where $e_i \in \mathcal{H}_E$ and $f^i \in \mathcal{H}_{\bar{E}}$, then $\mathcal{Z}(M)$ define an anti-linear map from $\mathcal{H}_{\bar{E}}$ to $\mathcal{H}_E$ by

$$v \mapsto \sum_i \bar{h}(f^i, v) e_i. \tag{1.3}$$

We want this map to be injective since we can distinguish two states in $\mathcal{H}_E$ by testing state in $\mathcal{H}_{\bar{E}}$ if (1.3) is injective.

(5) Given an n-dimensional manifold M with non-empty boundary $E = \partial M$ and an $(n-1)$-dimensional submanifold $U \in M$, we cut M along U and obtain a new manifold N, then $\partial N = E \sqcup U \sqcup \bar{U}$. Suppose that $\{u_i\}$ is a orthonormal basis of $\mathcal{H}_U$ and $\bar{u}^i$ is the preimage of u_i when (1.3) is surjective, then we want

$$\mathcal{Z}(M) = \sum_i h(u_i, u_i') \bar{h}(\bar{u}_i, \bar{u}_i') v_i \in \mathcal{H}_E, \tag{1.4}$$

where

$$\mathcal{Z}(N) = \sum_i u_i' \otimes \bar{u}_i' \otimes v_i \in \mathcal{H}_U \otimes \mathcal{H}_{\bar{U}} \otimes \mathcal{H}_E$$

for the reason that

$$\mathcal{Z}(M) = \int_\psi D\psi \int_{\{\Phi|\Phi|_E=\varphi, \Phi|_U=\psi\}} D\Phi\, e^{-S[\Phi]}.$$

1.2 TQFTs as functors

For general QFTs, we need a functor from geometry to algebra to describe it. It's really difficult. For TQFT, the functor we need is given by Atiya and Segal[Ati88, Seg88].

Definition 1.2. An n-dimensional oriented closed TQFT is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{K}}. \tag{1.5}$$

At first, I will explain what is a symmetric monoidal structure. And then, I will compare Definition 1.2 with the definition using path integrals.

Roughly speaking, the words 'symmetry' and 'monoidal' mean that there is a multiplication operation between the objects in the given category $\mathcal{C}$. For $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the multiplication $\otimes_{\mathcal{C}} : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ satisfies the following conditions:

- Symmetry: $A \otimes_{\mathcal{C}} B \cong B \otimes_{\mathcal{C}} A$,
- Associativity: $(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \cong A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$,
- unit: there is $1 \in \mathrm{Obj}(\mathcal{C})$ such that $1 \otimes_{\mathcal{C}} A \cong A \otimes_{\mathcal{C}} 1 \cong A$.

For details and explicit definition see [EGNO15].

In $\mathrm{Vect}_{\mathbb{K}}$, the objects are $\mathbb{K}$ vector spaces and the morphisms are $\mathbb{K}$-linear map. The multiplication in the monoidal structure is given by the tensor product with identity $\mathbb{K}$.

In Bord_n, the objects are oriented closed $(n-1)$-dimensional real manifolds E for $n \in \mathbb{N}_+$. To understand the morphisms in Bord_n, we should talk about bordisms first.

Definition 1.3. Let $E, F \in \mathrm{Obj}(\mathrm{Bord}_n)$. Then a *bordism* $E \to F$ is an oriented compact n-dimensional manifold M with smooth maps $\iota_{\text{in}} : E \to M$ and $\iota_{\text{out}} : F \to M$ such that

$$\bar{\iota}_{\text{in}} \sqcup \iota_{\text{out}} : \bar{E} \sqcup F \to \partial M \tag{1.6}$$

is an orientation-preserving diffeomorphism where $\bar{E}$ is E with reverse orientation.

Remark 1.3.1. An oriented compact n-dimensional manifold represents more than one morphism without specifying the source and the target, for example, the oriented compact manifold M given in Definition 1.3 is a bordism from E to F, but it can also be regarded as a bordism from $E \sqcup \bar{F}$ to $\emptyset$. So we should specify the source and the target when we mention a morphism in Bord_n.

Define the equivalent relation between bordisms $(M, \iota_{\text{in}}, \iota_{\text{out}}), (M', \iota'_{\text{in}}, \iota'_{\text{out}}) : E \to F$ by that if there is an orientation-preserving diffeomorphism $\psi : M \to M'$ such that

$$\begin{tikzcd}
& M \arrow[dd, "\psi"] & \\
E \arrow[ur, "\iota_{\text{in}}"] \arrow[dr, "\iota'_{\text{in}}"'] & & F \arrow[ul, "\iota_{\text{out}}"'] \arrow[dl, "\iota'_{\text{out}}"] \\
& M' &
\end{tikzcd} \tag{1.7}$$

commutes. Morphisms in $\mathrm{Obj}(M)$ are equivalent classes of bordisms. And composition of morphisms $M_1 : E \to F$ and $M_2 : F \to G$ in Bord_n is given by gluing M_1 and M_2 along F. The multiplication in the monoidal stucture is the disjoint product $\sqcup$ and the unit is $\emptyset$. For details, see [Koc04].

We can see that there are lots of difference between the motivation illustrated in §1.1 and the Definition 1.2. For example, we do not need $\mathcal{Z}(M)$ to be a Hilbert space in Definition 1.2. But the condition (4) and (5) can also be satisfied in another sense without metrics. Actually, for $E \in \mathrm{Obj}(\mathrm{Bord}_n)$ and $v \in \mathcal{H}_E$ and $\bar{v} \in \mathcal{H}_{\bar{E}}$, we can define a pairing

$$d_E : \mathcal{H}_E \otimes \mathcal{H}_{\bar{E}} \to \mathbb{K} \tag{1.8}$$

by acting the functor $\mathcal{Z}$ on the morphism

$$E \times [0,1] : E \sqcup \bar{E} \to \emptyset,$$

and using the isomorphism $\mathcal{Z}(E) \otimes \mathcal{Z}(\bar{E}) \cong \mathcal{Z}(E \sqcup \bar{E})$. For a condition similar to (4) in §1.1, we can regard $\mathcal{Z}(M) \in \mathcal{Z}(\bar{E}) \otimes \mathcal{Z}(F)$ as a linear map from $\mathcal{Z}(E)$ to $\mathcal{Z}(F)$ by

$$\mathcal{Z}(E) \xrightarrow{\otimes \mathcal{Z}(M)} \mathcal{Z}(E) \otimes \mathcal{Z}(M) \xrightarrow{d_E \otimes \mathrm{id}_{\mathcal{Z}(F)}} \mathcal{Z}(F). \tag{1.9}$$

If we cut an n-manifold M with boundary E along an $(n-1)$-submanifold $U \subseteq M$ and obtain a new manifold N, we can demand that

$$\mathcal{Z}(M) = d_U \otimes \mathrm{id}_E(\mathcal{Z}(N)), \tag{1.10}$$

as in (5) in §1.1.

Now, we present the most crucial property of TQFT.

Proposition 1.4. *Let* $\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{K}}$ *be a TQFT and* $E \in \mathrm{Obj}(\mathrm{Bord}_n)$. *Then*

(1) $\mathcal{Z}(E \times [0,1] : E \to E) = \mathrm{id}_{\mathcal{Z}(E)}$,

(2) $\dim \mathcal{Z}(E) < \infty$ *and* $\mathcal{Z}(\bar{E}) \cong \mathcal{Z}(E)^*$.

Proof. Denote the morphism $b := E \times [0,1] : E \to E$ in Bord_n, then for any morphism $b' \in \mathrm{Hom}_{\mathrm{Bord}_n}(E, F)$ and $b'' \in \mathrm{Hom}_{\mathrm{Bord}_n}(F, E)$, we have

$$b \circ b'' = b'', \quad b' \circ b = b',$$

since gluing with a cylinder does not change the smooth structure. Hence $b = \mathrm{id}_E$.

Consider the following morphisms in Bord_n

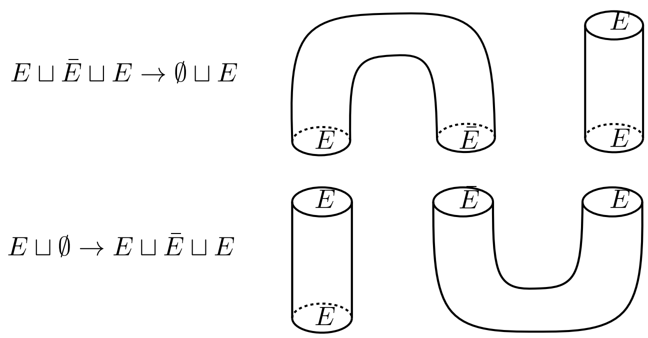


Figure 1.1: 2 morphisms in Bord_n

By gluing them, we got a morphism 1.2

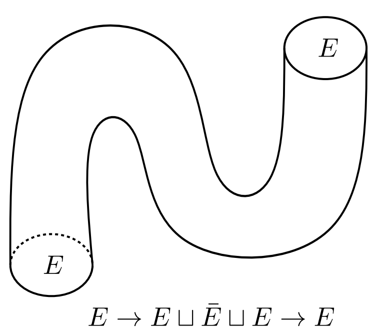


Figure 1.2: composition of 2 morphism

Denote $\gamma = \mathcal{Z}(E \times [0,1] : \emptyset \to \bar{E} \sqcup E)$. Notice that

$$\mathcal{Z}(E \times [0,1] \sqcup E \times [0,1] : E \to E \sqcup \bar{E} \sqcup E) = \mathrm{id}_{\mathcal{Z}(E)} \otimes \gamma, \tag{1.11}$$

$$\mathcal{Z}(E \times [0,1] \sqcup E \times [0,1] : E \sqcup \bar{E} \sqcup E \to E) = d_E \otimes \mathrm{id}_{\mathcal{Z}(E)}, \tag{1.12}$$

hence

$$(d_E \otimes \mathrm{id}_{\mathcal{Z}(E)}) \circ (\mathrm{id}_{\mathcal{Z}(E)} \otimes \gamma) = \mathrm{id}_{\mathcal{Z}(E)}. \tag{1.13}$$

Assume that

$$\gamma(1) = \sum_{i \in I} \bar{v}_i \otimes v_i,$$

where $\{v_i | i \in I\}$ is linear independent, then for $v \in \mathcal{Z}(E)$, we have

$$v \cong v \otimes 1 \mapsto v \otimes \left(\sum_{i \in I} \bar{v}_i \otimes v_i \right) \mapsto \sum_{i \in I} d_E(v, \bar{v}_i) v_i = v, \tag{1.14}$$

(1.14) shows that $\mathcal{Z}(E)$ is spanned by a finite set $\{v_i | i \in I\}$ and the pairing d_E is non-degenerated, and it gives a canonical isomorphism $\mathcal{Z}(\bar{E}) \cong \mathcal{Z}(E)^*$ defined by

$$\bar{v} \mapsto d_E(-, \bar{v}). \tag{1.15}$$

□

Remark 1.4.1. In the language of category $\mathcal{DP}$ of dual pairs, (1.13) is called the Zorro moves, and the linear maps γ and d_E are called birth and death, respectively, which we will explain in §2.1.

Corollary 1.4.1. *Let E be an object in* Bord_n. *Then for $k \in \mathbb{K}$, we have*

$$\mathcal{Z}(E \times S^1)(k) = \dim \mathcal{Z}(E) \cdot k.$$

$E \times [0,1] : E \sqcup \bar{E} \to \emptyset$

$E \times [0,1] : \emptyset \to E \sqcup \bar{E}$

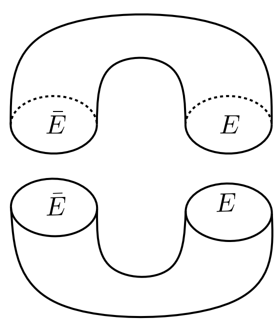


Figure 1.3: Trace

Proof. $E \times S^1$ is a morphism from $\emptyset$ to $\emptyset$ without ambiguity, and it is the composition of two morphisms in the figure 1.3. Hence $\mathcal{Z}(E \times S^1)$ is a morphism from $\mathbb{K}$ to $\mathbb{K}$. By taking $v = v_j$ in (1.14), we get

$$d_E(v_j, \bar{v}_i) = \delta_{i,j},$$

hence

$$\mathcal{Z}(E \times S^1)(1) = d_{\bar{E}} \circ \gamma(1) = \sum_{i \in I} d_{\bar{E}}(\bar{v}_i \otimes v_i) = |I| = \dim \mathcal{Z}(E).$$

□

Given a n-dimensional TQFT $\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{K}}$ and an oriented compact r-dimensional manifold X with $r < n$, we have the following reduced TQFT of dimension $n - r$

$$\mathcal{Z}^{\mathrm{red}} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{K}}, \quad \mathcal{Z}^{\mathrm{red}}(E \xrightarrow{M} F) := \mathcal{Z}(E \times X \xrightarrow{M \times X} F \times X). \tag{1.16}$$

We have the following commutative diagram of symmetric monoidal categories

$$\begin{tikzcd} \mathrm{Born}_{n-r} \arrow[rr, "- \times X"] \arrow[dr, "\mathcal{Z}"'] & & \mathrm{Born}_n \arrow[dl, "\mathcal{Z}^{\mathrm{red}}"] \\ & \mathrm{Vect}_{\mathbb{K}} & \end{tikzcd} .$$

From this one can see the reason why a higher dimensional TQFT is more complicated than a lower dimensional one.

Sometimes we want to see if two TQFTs $\mathcal{Z}$ and $\mathcal{Z}'$ of dimensional n are equivalent. For this purpose, it is enough to show that they are equivalent for connected manifolds since the results for disconnected manifolds can be obtained from multiplications in categories.

2 Lower dimensional examples

2.1 TQFT of dimension 1

There are only two objects in $\mathrm{Obj}(\mathrm{Bord}_n)$:

$$\bullet_+, \bullet_-, \tag{2.1}$$

i.e. two points with different orientations. And it can be proved by using Morse theory that every morphism can be obtained by composing and tensoring the following 6 elements:

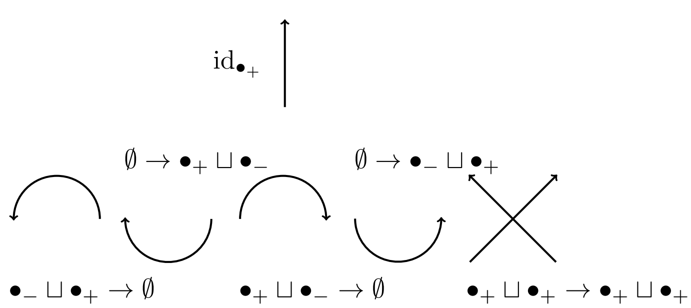


Figure 2.1: identity and generators

TQFT of dimension 1 is simple since there is a bijection $(\mathcal{Z} : \mathrm{Bord}_1 \to \mathrm{Vect}_{\mathbb{K}}) \mapsto \mathcal{Z}(\bullet_+)$ between TQFT of dimension 1 and finite dimensional vector spaces over $\mathbb{K}$. To make this precisely, we need the concepts of the category $\mathcal{DP}$ of *dual pairs*:

- The objects in $\mathcal{DP}$ are 4 tuples (U, V, b, d) where the birth $b : \mathbb{K} \to U \otimes V$ and death $d : V \otimes U \to \mathbb{K}$ are linear maps and U, V are linear spaces dual to each other in the sense that they satisfy the following Zorro moves

$$\begin{aligned} (d \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes b) = \mathrm{id}_V, \\ (\mathrm{id}_U \otimes d) \circ (b \otimes \mathrm{id}_U) = \mathrm{id}_U. \end{aligned} \tag{2.2}$$

- A morphism from (U, V, b, d) to (U', V', b', d') is a pair (f, g) of linear maps where $f : U \to U'$ and $g : V \to V'$ such that

$$b' = (f \otimes g) \circ b, \quad d' = d \circ (g \otimes f).$$

Theorem 2.1. *The functor $\mathcal{Z} \to (\mathcal{Z}(\bullet_+), \mathcal{Z}(\bullet_-), \mathcal{Z}(\bullet_+ \sqcup \bullet_- \to \emptyset), \mathcal{Z}(\bullet_- \sqcup \bullet_+ \to \emptyset))$ is an equivalence of groupoid between 1-dimensional TQFT and $\mathcal{DP}_{\mathbb{K}}$.*

Remark 2.1.1. TQFQ of dimension n itself is a groupoid (See [CR18, §2.5]), but $\mathrm{Vect}_{\mathbb{K}}$ is not a groupoid. In the above statement, we just throw out the non-invertible linear maps in $\mathrm{Vect}_{\mathbb{K}}$ to get a groupoid whose objects consisting of vector spaces.

If we denote $V = \mathcal{Z}(\bullet_+)$, then we have

$$\mathcal{Z}(\bullet_+^{\sqcup m} \sqcup \bullet_-^{\sqcup n}) = V^{\otimes m} \otimes_{\mathbb{K}} (V^*)^{\otimes n} \tag{2.3}$$

$$\mathcal{Z}(\bullet_- \sqcup \bullet_+ \to \emptyset) : (V^*) \otimes V \to \mathbb{K}, \qquad f \otimes v \mapsto f(v), \tag{2.4}$$

$$\mathcal{Z}(\emptyset \to \bullet_+ \sqcup \bullet_-) : \mathbb{K} \to V \otimes V^*, \qquad k \mapsto k \sum_{i \in I} e_i \otimes f^i, \tag{2.5}$$

$$\mathcal{Z}(\bullet_+ \sqcup \bullet_- \to \emptyset) : V \otimes (V^*) \to \mathbb{K}, \qquad v \otimes f \mapsto f(v), \tag{2.6}$$

$$\mathcal{Z}(\emptyset \to \bullet_- \sqcup \bullet_+) : \mathbb{K} \to V^* \otimes V, \qquad k \mapsto k \sum_{i \in I} f_i \otimes e^i, \tag{2.7}$$

$$\mathcal{Z}(\bullet_+ \sqcup \bullet_+ \to \bullet_+ \sqcup \bullet_+) : V \otimes V \to V \otimes V, \qquad u \otimes v \mapsto v \otimes u. \tag{2.8}$$

As the end of this section, we mention that TQFT of dimension 1 is freely generated as a symmetric monoidal category by the objects

$$\bullet_+, \quad \bullet_-, \tag{2.9}$$

and the morphisms

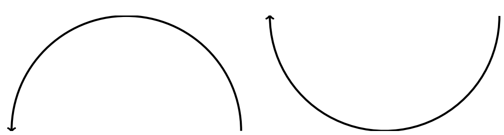


Figure 2.2: Generators of morphisms

subject to the relations

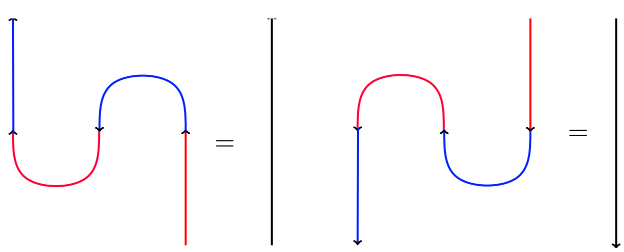


Figure 2.3: Relations of morphisms

For details of freely generated symmetric monoidal categories and the explicit construction of these categories from generators and relations, see [CR18, §3.2].

2.2 TQFT of dimension 2

At first, we consider the source category Bord_2. Objects in Bord_2 is generated by

$$G_0 = \{S^1\}, \tag{2.10}$$

and morphism in Bord_2 can be obtained by composing and tensoring the following elements

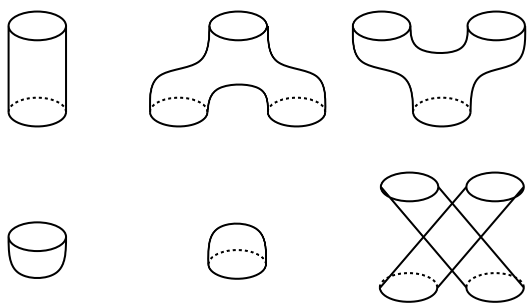


Figure 2.4: Elementary morphisms in Bord_2

Here the first one is the identity on S^1 and the last one is the braiding bordism. One can drop the identity and the braiding bordism to get the set G_1 of generators of morphisms. And the set G_2 relations are given by

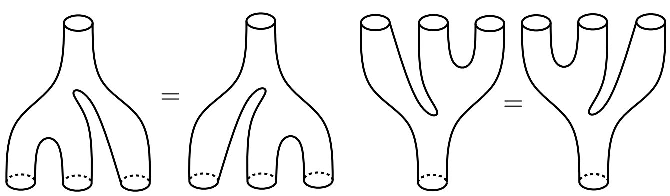


Figure 2.5: Associative law

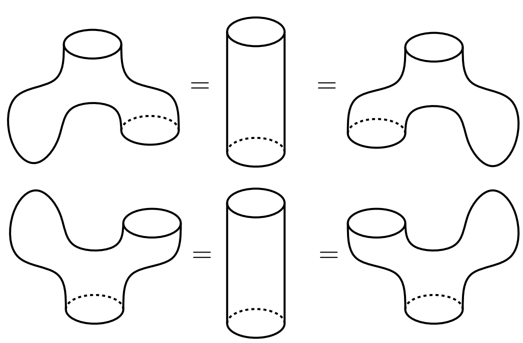


Figure 2.6: Unit

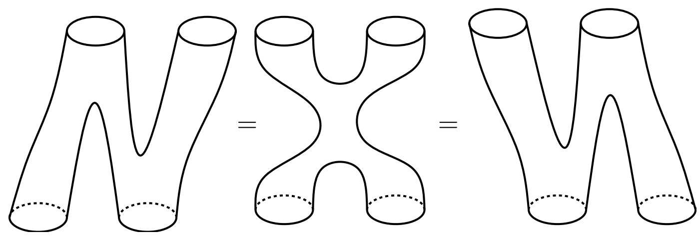


Figure 2.7: Zorro moves

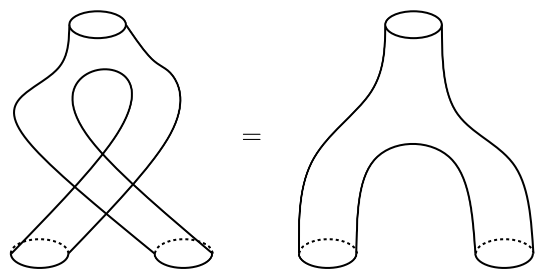
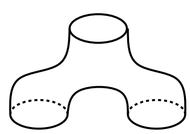


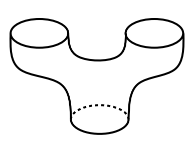
Figure 2.8: Commutative law

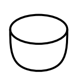
There are some redundant relations in above figures, and we do not try to find a set of minimal relations. The main theorem of Bord_2 is follows.

Theorem 2.2. Bord_2 *is freely generated as a symmetric monoidal category by* G_0, G_1, G_2.

Set $A = \mathcal{Z}(S^1)$, and we denote the action of a given TQFT $\mathcal{Z} : \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{K}}$ on generators by

Multiplication:  $\mapsto (\mu : A \otimes_{\mathbb{K}} A \to A)$

Comultiplication:  $\mapsto (\Delta : A \to A \otimes_{\mathbb{K}} A)$

Unit:  $\mapsto (\eta : \mathbb{K} \to A)$

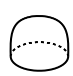
Counit:  $\mapsto (\varepsilon : A \to \mathbb{K})$

Figure 2.9: Actions of $\mathcal{Z}$ on G_1

According the relations 2.5-2.8, we can define the Frobenius algebra as follows.

Definition 2.3. A *Frobenius algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space with

(1) an associative unital algebra structure (A, μ, η), i.e.

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu), \tag{2.11}$$
$$\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta), \tag{2.12}$$

(2) a coassociative counital coalgebra structure (A, Δ, ε), i.e.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{2.13}$$
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{2.14}$$

such that

$$(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \Delta \circ \mu = (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}). \tag{2.15}$$

The equations (2.11) and (2.13) correspond to figure 2.5, (2.12) and (2.14) correspond to figure 2.6, and (2.8) corresponds to figure 2.7.

A morphism between Frobenius algebras $\psi : A \to A$ is a $\mathbb{K}$ linear maps keeping the algebra structures and the coalgebra structures, i.e.

$$\mu'(\psi \otimes \psi) = \psi \circ \mu, \quad \eta' = \psi \circ \eta, \quad (\psi \otimes \psi) \circ \Delta = \Delta' \circ \psi, \quad \varepsilon = \varepsilon' \circ \psi. \tag{2.16}$$

Frobenius algebras with their morphisms form a category, and we denote the full subcategory of commutative Frobenius algebras by $\mathrm{comFrob}_{\mathbb{K}}$, then we have

Theorem 2.4. *The functor* $\{\mathcal{Z} : \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{K}} | \mathcal{Z} \textit{ is a TQFT}\} \to \mathrm{comFrob}_{\mathbb{K}}$ *defined by*

$$\mathcal{Z} \to \mathcal{Z}(S^1) \tag{2.17}$$

is an equivalence of groupoid.

Remark 2.4.1. The proof of Theorem 2.4 can be generalized to symmetric monoidal functor $\mathcal{Z}$: $\mathrm{Bord}_2 \to \mathcal{C}$ for any symmetric monoidal category $\mathcal{C}$. An example is given in Example 2.7.

The Definition 2.3 of Frobenius algebra is a bit cumbersome, so we will use the simple one in the following which is equivalent to Definition 2.3.

Definition 2.5. A Frobenius algebra $\mathbb{K}$ is an unital associative $\mathbb{K}$-algebra with a nondegenerate and invariant bilinear form $\langle -, - \rangle : A \times A \to A$.

Remark 2.5.1. The proof of Theorem

In the definition above, the invariance of $\langle -, - \rangle$ means

$$\langle a, b \cdot c \rangle = \langle a \cdot b, c \rangle \,.$$

The correspondence between Definition 2.5 and 2.3 is given by the follows.

- To produce a Frobenius algebra in Definition 2.3 by using Definition 2.5, we can define the algebra structure by

  $$\mu(a \otimes b) = a \cdot b, \quad \eta(k) = k \cdot e$$

  for $a, b \in A$ and e is the unit in A. To obtain the coalgebra structure, we need a pair of dual basis $\{e_i\}, \{f_i\}$ in the sense that

  $$\langle e_i, f_j \rangle = \delta_{i,j},$$

  then we can define the copairing by

  $$c(\lambda) = \lambda \sum_i e_i \otimes f_i. \tag{2.18}$$

  And the comultiplication and co unit is give by

  $$\Delta(a) = (\mu \otimes \mathrm{id}) \circ (a \otimes c(1)) = \sum_i (a \cdot e_i) \otimes f_i, \quad \varepsilon(a) = \langle a, e \rangle \,. \tag{2.19}$$

- To produce a Frobenius algebra in Definition 2.5 by using Definition 2.3, the unit is given by $e = \eta(1)$, the multiplication is given by $a \cdot b = \mu(a \otimes b)$ and the pairing is given by

$$\langle a, b\rangle = \varepsilon(a \cdot b) \tag{2.20}$$

One can check by calculation directly that the above structure is what we need. For details, see [Koc04, Chapter 2].

Here are some interesting examples of TQFT of dimension 2.

Example 2.6 (Affine Landau-Ginzburg models)**.** Let $W \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial with only isolated singularity. Then the quotient

$$A := \mathbb{C}[x_1, \dots, x_n]/(\partial_{x_1} W, \dots, \partial_{x_n} W) \tag{2.21}$$

is a finite-dimensional over $\mathbb{C}$. The unit and the multiplication in A is obvious, and the pairing is given by

$$\langle P, Q\rangle = \mathrm{Res}\left[\frac{P \cdot Q \mathrm{d}x}{\partial_{x_1} W \cdot \dots \cdot \partial_{x_n} W}\right], \tag{2.22}$$

then A is a Frobenius algebra. TQFTs associated to Frobenius algebras of this type is called *affine Landau-Ginzburg models*[Vaf91, HL05, CM16].

Example 2.7. Let X be a real compact oriented n-dimensional manifold, and denote the de Rham cohomology of X by

$$\tilde{A} := H^*_{\mathrm{dR}}(X) = \bigoplus_{k=0}^{n} H^k_{\mathrm{dR}}(X). \tag{2.23}$$

Then $\tilde{A}$ is a Frobenius algebra which is only graded commutative with unit $1 \in H^0_{\mathrm{dR}}(X)$, multiplication $\wedge$ and the Poincaré pairing

$$\langle \alpha, \beta\rangle = \int_X \alpha \wedge \beta. \tag{2.24}$$

By taking the even part $A := H^{\mathrm{ev}}_{\mathrm{dR}}(X)$ of $H^*_{\mathrm{dR}}(X)$ we get a commutative Frobenius algebra, hence a TQFT of dimension 2.

As we remark in 2.4.1, we can change the target category $\mathrm{Vect}_{\mathbb{K}}$ to the category $\mathrm{Vect}_{\mathbb{K}}^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$ graded vector spaces. It makes sense because the symmetry condition

$$A \otimes_{\mathcal{C}} B \cong B \otimes_{\mathcal{C}} A$$

in the definition of symmetric monoidal category is only an isomorphism, which can not imply the symmetry of multiplication if we change the braiding isomorphism in $\mathrm{Vect}_{\mathbb{K}}$. If we take the braiding isomorphisms in $\mathrm{Vect}_{\mathbb{K}}^{\mathbb{Z}_2}$ to be the graded isomorphisms

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x, \tag{2.25}$$

where $x \in A, y \in B$ are homogeneous elements and $|x|, |y|$ denotes the degrees of x and y respectively, $\mathrm{Vect}_{\mathbb{K}}^{\mathbb{Z}_2}$ is also a symmetric monoidal category, and we can also regard $\tilde{A}$ as a TQFT of dimension 2 corresponding to $\tilde{\mathcal{Z}} : \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{K}}^{\mathbb{Z}_2}$ which generalizes the concepts of TQFT.

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