

Conformal Field Theory

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This is a note of [Gab00], and we focus on the 2D conformal theory.

1 General structure of a local conformal theory

1.1 The space of states

Generally speaking, the space of states is a Hilbert space \mathcal{H} and the correlation functions are defined for some dense subspace \mathcal{F} of \mathcal{H} . Let Ω be the vacuum state in \mathcal{F} and $(-, -)$ be the inner product of \mathcal{H} . A 2D conformal field theory is defined on a Riemann surface with the coordinate z , we will assign an smooth operator field $V(\psi, z)$ to each state $\psi \in \mathcal{F}$ such that

$$\psi = V(\psi, z_0)\Omega \quad (1.1)$$

for some z_0 which we assume to be 0 usually. The correlation function is defined by

$$\langle V(\psi_1, z_1) \cdots V(\psi_n, z_n) \rangle := (\Omega, V(\psi_1, z_1) \cdots V(\psi_n, z_n)\Omega). \quad (1.2)$$

The point z_0 is called the "past infinity" in physics. If we fix a future infinity z_∞ , we will get some sense of time evolution.

Given a state $\psi \in \mathcal{F}$, if for any state $\psi_i \in \mathcal{F}$, the correlation functions

$$\langle V(\psi, z) V(\psi_1, z_1) \cdots V(\psi_n, z_n) \rangle \quad (1.3)$$

is a meromorphic function, then ψ is called a meromorphic state. The space \mathcal{F}_0 consisting of all meromorphic states is called a meromorphic subtheory of \mathcal{H} . Similarly, there is an anti-meromorphic subtheory $\bar{\mathcal{F}}_0$ with a similar definition.

The correlation functions of the theory determine the *operator product expansion* (OPE)

$$V(\psi_i, z_1) V(\psi_2, z_2) = \sum_i (z_1 - z_2)^{\Delta_i} (\bar{z}_1 - \bar{z}_2)^{\bar{\Delta}_i} \sum_{r,s \geq 0} V(\phi_{r,s}^i, z_2) (z_1 - z_2)^r (\bar{z}_1 - \bar{z}_2)^s, \quad (1.4)$$

eq:OPE

where $\Delta_i, \bar{\Delta}_i \in \mathbb{R}$ and $r, s \in \mathbb{N}$ and $\psi_1, \psi_2, \phi_{r,s}^i \in \mathcal{F}$. The OPE (1.4) defined so-called a vertex operator algebra of meromorphic fields and anti-meromorphic fields in [Bor92, Bor86]. The OPE can be read off from

$$\begin{aligned} & \langle V(\psi_i, z_1) V(\psi_2, z_2) V(\phi_1, w_1) \cdots V(\phi_n, w_n) \rangle \\ &= \sum_i (z_1 - z_2)^{\Delta_i} (\bar{z}_1 - \bar{z}_2)^{\bar{\Delta}_i} \sum_{r,s \geq 0} (z_1 - z_2)^r (\bar{z}_1 - \bar{z}_2)^s \langle V(\phi_{r,s}^i, z_2) V(\psi_1, w_1) \cdots V(\psi_n, w_n) \rangle. \end{aligned} \quad (1.5)$$

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Since the OPE is associative, \mathcal{F} is a representation of two vertex operator algebras. \mathcal{F} can be decomposed into indecomposable representations

$$\mathcal{H} = \bigoplus_{(j, \bar{j})} \mathcal{H}_{(j, \bar{j})}. \quad (1.6) \quad \boxed{\text{eq:indecomposable}}$$

If the decomposition (1.6) is finite, then we call this theory a finite theory.

1.2 Modular invariance

To introduce the modular invariance, we consider torus, i.e. the Riemann surface of genus 1. Recall that complex structures of torus is parametrized by $\tau \in \{z \in \mathbb{C} | \text{Im} z > 0\}$ modular the action

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d}, \quad (1.7)$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M} := SL(2, \mathbb{Z})/\mathbb{Z}_2. \quad (1.8)$$

If we cut the torus along a non-trivial cycle, we can obtain an annulus, on which there is a propagator along the annulus:

$$\mathcal{O}(q, \bar{q}) = q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \quad (1.9)$$

The vacuum correlator (or the partition function) on the torus is describe by the trace of $\mathcal{O}(q, \bar{q})$:

$$\sum_{(j, \bar{j})} \text{Tr}_{\mathcal{H}_{(j, \bar{j})}} \mathcal{O}(q, \bar{q}), \quad (1.10)$$

where L_0 and \bar{L}_0 are the scaling operators of two vertex algebras and c and \bar{c} their central charges. In our case of torus, $q = e^{2\pi i \tau}$. Since τ and $A\tau$ describe the same torus, the partition function is invariant under the transformation $\tau \mapsto A\tau$.

In most of cases,

$$\mathcal{H}_{(j, \bar{j})} = \mathcal{H}_j \otimes \mathcal{H}_{\bar{j}} \quad (1.11)$$

where \mathcal{H}_j is an irreducible representation of meromorphic vertex operator algebra and $\mathcal{H}_{\bar{j}}$ is an irreducible representation of anti-meromorphic vertex operator algebra. The partition function on the torus is given by

$$\sum_{(j, \bar{j})} \chi_j(\tau) \bar{\chi}_{\bar{j}}(\bar{\tau}), \quad (1.12)$$

where $\chi_j(\tau) = \text{Tr}_{\mathcal{H}_j}(q^{L_0 - \frac{c}{24}})$ and the $\bar{\chi}_{\bar{j}}(\bar{\tau})$ is defined by a similar formula. A remarkable fact about a big class of vertex operator algebras is that characters transform into one another under the modular group $\mathcal{M} = SL(2, \mathbb{Z})/\mathbb{Z}_2$

$$\chi_j\left(-\frac{1}{\tau}\right) = \sum_k S_j^k \chi_k(\tau), \quad \chi_j(\tau + 1) = \sum_k T_j^k \chi_k(\tau). \quad (1.13)$$

And we define $\bar{S}_{\bar{j}}^{\bar{k}}$ and $\bar{T}_{\bar{j}}^{\bar{k}}$ by a similar method for anti-meromorphic vertex operator algebra. Assume that

$$\mathcal{H} = \bigoplus_{i, \bar{j}} M^{i\bar{j}} \mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{j}}. \quad (1.14)$$

Then we have

$$\sum_{i, \bar{j}} S_i^l M^{i\bar{j}} \bar{S}_{\bar{j}}^{\bar{k}} = \sum_{i, \bar{j}} T_i^l M^{i\bar{j}} \bar{T}_{\bar{j}}^{\bar{k}} = M^{l\bar{k}}. \quad (1.15)$$

This provides powerful constrains for matrix $M^{i\bar{j}}$. In the case of finite theory, these conditions allow one to obtain finite solution and a nice classification. See [CIZ87b, CIZ87a, Gan00] and [Gan, Gan97].

2 Meromorphic conformal field theory on the sphere

In this section, our main objects are the meromorphic fields in \mathcal{F}_0 . For each meromorphic field ψ , there is a vertex operator $V(\psi, z)$ which create ψ from the vacuum state Ω . The operators are assume to be local in the sense that

$$V(\psi, z)V(\phi, w) = \varepsilon V(\phi, w)V(\psi, z) \quad (2.1)$$

for $z \neq w$. If $\varepsilon = -2$, then ψ and ϕ are both fermionic and $\varepsilon = 1$ otherwise. The meromorphic states space \mathcal{F}_0 can be decomposed as

$$\mathcal{F}_0 = \mathcal{F}_0^B \oplus \mathcal{F}_0^F, \quad (2.2)$$

where \mathcal{F}_0^B is the space of bosonic states and \mathcal{F}_0^F is the space of fermionic states. In the following text, we assume that a state is either fermionic or bosonic.

We consider the transformation of states under Möbius group \mathcal{M} . The generators of \mathcal{M} are

$$e^{\lambda \mathcal{L}_{-1}}(z) = z + \lambda, \quad e^{\lambda \mathcal{L}_0}(z) = e^\lambda z, \quad e^{\lambda \mathcal{L}_1}(z) = \frac{z}{1 - \lambda z}. \quad (2.3)$$

In the language of $SL(2, \mathbb{C})$, we have

$$e^{\lambda \mathcal{L}_{-1}} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad e^{\lambda \mathcal{L}_0} = \begin{pmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{pmatrix}, \quad e^{\lambda \mathcal{L}_1} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \quad (2.4)$$

And the Lie algebra of \mathcal{M} is generated by

$$\mathcal{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{L}_0 = \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (2.5)$$

They satisfy the commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n}, \quad m, n = 0, \pm 1. \quad (2.6)$$

In physics, \mathcal{F}_0 forms a representation of this algebra, and we associate the operator L_m to \mathcal{L}_m . L_0 can be identified with the energy operator, hence the spectrum of L_0 is bounded from below. \mathcal{F}_0 can be decomposed into irreducible representations, in a given irreducible highest weight representation, we denote by ψ the highest weight vector with weight h . Since

$$L_0 L_1 \psi = (h - 1) L_1 \psi, \quad (2.7)$$

we have $L_1 \psi = 0$. States with property

$$L_1 \psi = 0, \quad L_0 \psi = h \psi \quad (2.8)$$

are called *quasiprimary*, and h is called the conformal weight of ψ . Each quasiprimary state ψ generate a irreducible representation of $sl(2, \mathbb{C})$ that consists of L_{-1} -descendants of ψ . Notice that

$$L_1 L_{-1}^n \psi = 2n \left(h + \frac{1}{2}(n - 1) \right) L_{-1}^{n-1} \psi, \quad (2.9)$$

if h is an half integer, the above representation is given by $\text{Span}_{\mathbb{C}}\{L_{-1}^n \psi | n \geq 0\}$ moduli the subrepresentation $\text{Span}_{\mathbb{C}}\{L_{-1}^n \psi | n \geq 1 - 2h\}$. And we obtain a finite-dimensional irreducible representation. Then we have

$$L_{-1} \Omega = L_0 \Omega = L_1 \Omega = 0. \quad (2.10)$$

Next, we consider the action of \mathcal{M} on correlation functions

$$\left\langle \prod_{i=1}^n V(\psi_i, z_i) \right\rangle \quad (2.11)$$

where ψ_i are all quasiprimary states with conformal weight h_i . The action of $\gamma \in \mathcal{M}$ on it is defined by

$$\left\langle \prod_{i=1}^n V(\psi_i, z_i) \right\rangle = \prod_{i=1}^n \left(\frac{d\gamma(z_i)}{dz_i} \right)^{h_i} \left\langle \prod_{i=1}^n V(\psi_i, \gamma(z_i)) \right\rangle \quad (2.12)$$

2.1 Some examples

2.1.1 The free boson

Here we consider a single free boson. By factorization, the states space reduces to a vector space V of dimension 1. Suppose it is generated by J of conformal weight 1, and the corresponding vertex operator is

$$J(z) := V(J, z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}. \quad (2.13)$$

The Fock space is given by the linear combination of

$$J(z_1)J(z_2) \cdots J(z_m)\Omega. \quad (2.14)$$

The amplitudes is given by

$$\langle J(z_1)J(z_2) \cdots J(z_{2n}) \rangle = k^n \sum_{\pi \in S'_n} \prod_{j=1}^n \frac{1}{(z_{\pi(j)} - z_{\pi(j+n)})^2}, \quad (2.15)$$

where S'_n is a subgroup of S_n defined by

$$S'_n = \{\sigma = S_n | \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1), \sigma(2k-1) < \sigma(2k) \forall k = 1, \dots, n\}. \quad (2.16)$$

And the amplitudes with odd number of $J(z)$ is 0. We emphasize that this does not mean $\langle V(\psi, z) \rangle = 0$ for any $\psi \in \mathcal{F}_0$.

When $n = 1$, we have

$$\langle J(z)J(w) \rangle = \frac{k}{(z-w)^2}.$$

OPE tells us that

$$J(z)J(w) = \sum_{n \leq 1} V(J_n J, w)(z-w)^{-n-1}. \quad (2.17)$$

Recall that $L_0 J_n J = (1-n)J_n J$, so $J_0 J = cJ$ for some $c \in \mathbb{C}$ and $J_1 J = k\Omega$, hence

$$\langle V(J_0 J, w) \rangle = 0, \quad \langle V(J_1 J, w) \rangle = k.$$

Hence we have

$$J(z)J(w) \sim \frac{k}{(z-w)^2}. \quad (2.18)$$

The symbol \sim means that the both sides are equal up to a non-singular part at $z = w$. Since

$$[J_m, J_n] = V_{m+n}(J_0 J) + mV_{m+n}(J_1 J) = cJ_{m+n} + mk\delta_{m+n,0}. \quad (2.19)$$

References

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| [Bor86] | Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. <i>Proceedings of the National Academy of Sciences</i> , 83(10):3068–3071, May 1986. |
| [Bor92] | Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. <i>Inventiones Mathematicae</i> , 109(1):405–444, December 1992. |
| [CIZ87a] | A. Cappelli, C. Itzykson, and J. B. Zuber. The A-D-E classification of minimal and $A_1(1)$ conformal invariant theories. <i>Communications in Mathematical Physics</i> , 113(1):1–26, March 1987. |
| [CIZ87b] | A. Cappelli, C. Itzykson, and J. B. Zuber. Modular invariant partition functions in two dimensions. <i>Nuclear Physics B</i> , 280:445–465, January 1987. |
| [Gab00] | Matthias R Gaberdiel. An introduction to conformal field theory. <i>Reports on Progress in Physics</i> , 63(4):607–667, April 2000. |
| [Gan] | Terry Gannon. The classification of $SU(3)$ modular invariants revisited. |
| [Gan97] | Terry Gannon. $U(1)_m$ modular invariants, $N = 2$ minimal models, and the quantum Hall effect. <i>Nuclear Physics B</i> , 491(3):659–688, May 1997. |
| [Gan00] | T. Gannon. The Cappelli-Itzykson-Zuber A-D-E classification. <i>Reviews in Mathematical Physics</i> , 12(05):739–748, May 2000. |