# Conformal Field Theory

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This is a note of [Gab00], and we focus on the 2D conformal theory.

# 1 General structure of a local conformal theory

## 1.1 The space of states

Generally speaking, the space of states is a Hilbert space $\mathcal{H}$ and the correlation functions are defined for some dense subspace $\mathcal{F}$ of $\mathcal{H}$. Let $\Omega$ be the vacuum state in $\mathcal{F}$ and $(-,-)$ be the inner product of $\mathcal{H}$. A 2D conformal field theory is defined on a Riemann surface with the coordinate $z$, we will assign an smooth operator field $V(\psi, z)$ to each state $\psi \in \mathcal{F}$ such that

$$\psi = V(\psi, z_0)\Omega \tag{1.1}$$

for some $z_0$ which we assume to be 0 usually. The correlation function is defined by

$$\langle V(\psi_1, z_1)\cdots V(\psi_n, z_n)\rangle := (\Omega, V(\psi_1, z_1)\cdots V(\psi_n, z_n)\Omega). \tag{1.2}$$

The point $z_0$ is called the "past infinity" in physics. If we fix a future infinity $z_\infty$, we will get some sense of time evolution.

Given a state $\psi \in \mathcal{F}$, if for any state $\psi_i \in \mathcal{F}$, the correlation functions

$$\langle V(\psi, z)V(\psi_1, z_1)\cdots V(\psi_n, z_n)\rangle \tag{1.3}$$

is a meromorphic function, then $\psi$ is called a meromorphic state. The space $\mathcal{F}_0$ consisting of all meromorphic states is called a meromorphic subtheory of $\mathcal{H}$. Similarly, there is an anti-meromorphic subtheory $\overline{\mathcal{F}}_0$ with a similar definition.

The correlation functions of the theory determine the *operator product expansion*(OPE)

$$V(\psi_i, z_1)V(\psi_2, z_2) = \sum_i (z_1-z_2)^{\Delta_i}(\bar z_1-\bar z_2)^{\overline{\Delta}_i}\sum_{r,s\geq 0} V(\phi^i_{r,s}, z_2)(z_1-z_2)^r(\bar z_1-\bar z_2)^s, \tag{1.4}$$ eq:OPE

where $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$ and $r, s \in \mathbb{N}$ and $\psi_1, \psi_2, \phi^i_{r,s} \in \mathcal{F}$. The OPE (1.4) defined so-called a vertex operator algebra of meromorphic fields and anti-meromorphic fields in [Bor92, Bor86]. The OPE can be read off from

$$\begin{aligned}&\langle V(\psi_i, z_1)V(\psi_2, z_2)V(\phi_1, w_1)\cdots V(\phi_n, w_n)\rangle\\ =&\sum_i (z_1-z_2)^{\Delta_i}(\bar z_1-\bar z_2)^{\overline{\Delta}_i}\sum_{r,s\geq 0}(z_1-z_2)^r(\bar z_1-\bar z_2)^s\langle V(\phi^i_{r,s}, z_2)V(\psi_1, w_1)\cdots V(\psi_n, w_n)\rangle.\end{aligned} \tag{1.5}$$

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Since the OPE is associative, $\mathcal{F}$ is a representation of two vertex operator algebras. $\mathcal{F}$ can be decomposed into indecomposable representations

$$\mathcal{H} = \bigoplus_{(j,\bar{j})} \mathcal{H}_{(j,\bar{j})}. \tag{1.6}$$

eq:indecomposable

If the decomposition (1.6) is finite, then we call this theory a finite theory.

## 1.2 Modular invariance

To introduce the modular invariance, we consider torus, i.e. the Riemann surface of genus 1. Recall that complex structures of torus is parametrized by $\tau \in \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$ modular the action

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d}, \tag{1.7}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M} := SL(2,\mathbb{Z})/\mathbb{Z}_2. \tag{1.8}$$

If we cut the torus along a non-trivial cycle, we can obtain a annulus, on which there is a propagator along the annulus:

$$\mathcal{O}(q,\bar{q}) = q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \tag{1.9}$$

The vacuum correlator(or the partition function) on the torus is describe by the trace of $\mathcal{O}(q,\bar{q})$:

$$\sum_{(j,\bar{j})} \mathrm{Tr}_{\mathcal{H}_{(j,\bar{j})}} \mathcal{O}(q,\bar{q}), \tag{1.10}$$

where $L_0$ and $\bar{L}_0$ are the scaling operators of two vertex algebras and $c$ and $\bar{c}$ their central charges. In our case of torus, $q = e^{2\pi i \tau}$. Since $\tau$ and $A\tau$ describe the same torus, the partition function is invariant under the transformation $\tau \mapsto A\tau$.

In most of cases,

$$\mathcal{H}_{(j,\bar{j})} = \mathcal{H}_j \otimes \mathcal{H}_{\bar{j}} \tag{1.11}$$

where $\mathcal{H}_j$ is an irreducible representation of meromorphic vertex operator algebra and $\mathcal{H}_{\bar{j}}$ is an irreducible representation of anti-meromorphic vertex operator algebra. The partition function on the torus is given by

$$\sum_{(j,\bar{j})} \chi_j(\tau) \bar{\chi}_{\bar{j}}(\bar{\tau}), \tag{1.12}$$

where $\chi_j(\tau) = \mathrm{Tr}_{\mathcal{H}_j}(q^{L_0 - \frac{c}{24}})$ and the $\bar{\chi}_{\bar{j}}(\bar{\tau})$ is defined by a similar formula. A remarkable fact about a big class of vertex operator algebras is that characters transform into one another under the modular group $\mathcal{M} = SL(2,\mathbb{Z})/\mathbb{Z}_2$

$$\chi_j\left(-\frac{1}{\tau}\right) = \sum_k S_j^k \chi_k(\tau), \quad \chi_j(\tau+1) = \sum_k T_j^k \chi_k(\tau). \tag{1.13}$$

And we define $\bar{S}_{\bar{j}}^{\bar{k}}$ and $\bar{T}_{\bar{j}}^{\bar{k}}$ by a similar method for anti-meromorphic vertex operator algebra. Assume that

$$\mathcal{H} = \bigoplus_{i,\bar{j}} M^{i\bar{j}} \mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{j}}. \tag{1.14}$$

Then we have

$$\sum_{i,\bar{j}} S_i^l M^{i,\bar{j}} \bar{S}_{\bar{j}}^{\bar{k}} = \sum_{i,\bar{j}} T_i^l M^{i,\bar{j}} \bar{T}_{\bar{j}}^{\bar{k}} = M^{l\bar{k}}. \tag{1.15}$$

This provides powerful constrains for matrix $M^{i\bar{j}}$. In the case of finite theory, these conditions allow one to obtain finite solution and a nice classification. See [CIZ87b, CIZ87a, Gan00] and [Gan, Gan97].

## 2 Meromorphic conformal field theory on the sphere

In this section, our main objects are the meromorphic fields in $\mathcal{F}_0$. For each meromorphic field $\psi$, there is a vertex operator $V(\psi, z)$ which create $\psi$ from the vacuum state $\Omega$. The operators are assume to be local in the sense that

$$V(\psi,z)V(\phi,w) = \varepsilon V(\phi,w)V(\psi,z) \tag{2.1}$$

for $z \neq w$. If $\varepsilon = -2$, then $\psi$ and $\phi$ are both fermionic and $\varepsilon = 1$ otherwise. The meromorphic states space $\mathcal{F}_0$ can be decomposed as

$$\mathcal{F}_0 = \mathcal{F}_0^B \oplus \mathcal{F}_0^F, \tag{2.2}$$

where $\mathcal{F}_0^B$ is the space of bosonic states and $\mathcal{F}_0^F$ is the space of fermionic states. In the following text, we assume that a state is either fermionic or bosonic.

We consider the transformation of states under Möbius group $\mathcal{M}$. The generators of $\mathcal{M}$ are

$$e^{\lambda\mathcal{L}_{-1}}(z) = z+\lambda, \quad e^{\lambda\mathcal{L}_0}(z) = e^{\lambda}z, \quad e^{\lambda\mathcal{L}_1}(z) = \frac{z}{1-\lambda z}. \tag{2.3}$$

In the language of $SL(2,\mathbb{C})$, we have

$$e^{\lambda\mathcal{L}_{-1}} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad e^{\lambda\mathcal{L}_0} = \begin{pmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{pmatrix}, \quad e^{\lambda\mathcal{L}_1} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \tag{2.4}$$

And the Lie algebra of $\mathcal{M}$ is generated by

$$\mathcal{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{L}_0 = \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{2.5}$$

They satisfy the commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad m,n = 0, \pm 1. \tag{2.6}$$

In physics, $\mathcal{F}_0$ forms a representation of this algebra, and we associate the operator $L_m$ to $\mathcal{L}_m$. $L_0$ can be identified with the energy operator, hence the spectrum of $L_0$ is bounded from below. $\mathcal{F}_0$ can be decomposed into irreducible representations, in a given irreducible highest weight representation, we denote by $\psi$ the highest weight vector with weight $h$. Since

$$L_0 L_1 \psi = (h-1)\, L_1\psi, \tag{2.7}$$

we have $L_1\psi = 0$. States with property

$$L_1\psi = 0, \quad L_0\psi = h\psi \tag{2.8}$$

are called *quasiprimary*, and $h$ is called the conformal weight of $\psi$. Each quasiprimary state $\psi$ generate a irreducible representation of $sl(2,\mathbb{C})$ that consists of $L_{-1}$-descendants of $\psi$. Notice that

$$L_1 L_{-1}^n \psi = 2n\left(h + \frac{1}{2}(n-1)\right) L_{-1}^{n-1}\psi, \tag{2.9}$$

if $h$ is an half integer, the above representation is given by $\mathrm{Span}_{\mathbb{C}}\{L_{-1}^n\psi | n \geq 0\}$ moduli the subrepresentation $\mathrm{Span}_{\mathbb{C}}\{L_{-1}^n\psi | n \geq 1-2h\}$. And we obtain a finite-dimensional irreducible representation. Then we have

$$L_{-1}\Omega = L_0\Omega = L_1\Omega = 0. \tag{2.10}$$

Next, we consider the action of $\mathcal{M}$ on correlation functions

$$\left\langle \prod_{i=1}^{n} V\left(\psi_i, z_i\right) \right\rangle \tag{2.11}$$

where $\psi_i$ are all quasiprimary states with conformal weight $h_i$. The action of $\gamma \in \mathcal{M}$ on it is defined by

$$\left\langle \prod_{i=1}^{n} V\left(\psi_i, z_i\right) \right\rangle = \prod_{i=1}^{n} \left( \frac{\mathrm{d}\gamma\left(z_i\right)}{\mathrm{d}z_i} \right)^{h_i} \left\langle \prod_{i=1}^{n} V\left(\psi_i, \gamma\left(z_i\right)\right) \right\rangle \tag{2.12}$$

## 2.1 Some examples

### 2.1.1 The free boson

Here we consider a single free boson. By factorization, the states space reduces to a vector space $V$ of dimension 1. Suppose it is generated by $J$ of conformal weight 1, and the corresponding vertex operator is

$$J(z) := V(J, z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}. \tag{2.13}$$

The Fock space is given by the linear combination of

$$J(z_1)J(z_2)\cdots J(z_m)\Omega. \tag{2.14}$$

The amplitudes is given by

$$\langle J(z_1)J(z_2)\cdots J(z_{2n})\rangle = k^n \sum_{\pi \in S'_n} \prod_{j=1}^{n} \frac{1}{(z_{\pi(j)} - z_{\pi_{j+n}})^2}, \tag{2.15}$$

where $S'_n$ is a subgroup of $S_n$ defined by

$$S'_n = \{\sigma = S_n | \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1), \sigma(2k-1) < \sigma(2k) \forall k = 1, \ldots, n\}. \tag{2.16}$$

And the amplitudes with odd number of $J(z)$ is 0. We emphasize that this does not mean $\langle V(\psi, z)\rangle = 0$ for any $\psi \in \mathcal{F}_0$.

When $n = 1$, we have

$$\langle J(z)J(w)\rangle = \frac{k}{(z-w)^2}.$$

OPE tells us that

$$J(z)J(w) = \sum_{n \leq 1} V(J_n J, w)(z-w)^{-n-1}. \tag{2.17}$$

Recall that $L_0 J_n J = (1-n) J_n J$, so $J_0 J = cJ$ for some $c \in \mathbb{C}$ and

$$\langle V(J_0 J, w)\rangle = 0.$$

Hence we have

$$J(z)J(w) \sim \frac{k}{(z-w)^2}. \tag{2.18}$$

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