Homework 6

1. (book # 4.4)
Let X denote the number of heads when flipping a fair coin n times, i.e., $X \sim \mathrm{Bin}(n, p)$ with $p = 1/2$. Find a Chernoff bound for $\Pr(X \geq a)$. Find the sharpest (i.e., smallest) Chernoff bound.

Evaluate your answer for $n = 100$ and $a = 68$.

$\boxed{\Pr(X \geq a) \leq \qquad}$

2. Prove Theorem 4.3.

Let $S = X + Y$, and let $M_X(t), M_Y(t)$ and $M_S(t)$ denote the m.g.f. of X, Y, and S, respectively. Similarly, let $p_X(x)$, $p_Y(y)$ and $p_S(s)$ denote the probability functions.
If $X \perp Y$ then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

Which of the following arguments is a valid proof?

(a) $M_S(t) = E(e^{tS}) = e^{E(tS)} = e^{tX+tY}$
(b) $M_S(t) = E[e^{tS}] = E[e^{tX+tY}] = E[e^{tX}] \cdot E[e^{tY}]$
(c) $M_S(t) = E(e^{tS}) \geq E(1 + tS) = E(1 + tX + tY) \geq E(e^{tX})E(e^{tY})$
(d) $M_S(t) = E(e^{tS}) = \sum_s e^{ts} p_S(s) \geq 1 + \sum_s (ts) p_S(s) = \sum_x e^{tx} p_X(x) \cdot \sum_y e^{ty} p_Y(y)$
(e) none of these

Hint: For problem 3, read Section 4.2.1 in the book.

3. Let $X_i \sim \mathrm{Bern}(p_i)$, $i = 1, \ldots, n$, be a sequence of indepdent Bernoulli (i.e., binary) r.v.'s. Let $X = \sum_{i=1}^n X_i$ and let $\mu = E(X) = \sum p_i$.

3a. Show

$$M_{X_i}(t) \leq e^{p_i(e^t-1)}. \tag{1}$$

Hint: use $1 + x \leq e^x$.

Which of the following arguments shows the claim?

(a) $M_{X_i}(t) = E(e^{X_i t}) \leq 1 + E(X_i t) \leq e^{p_i(e^t-1)}$
(b) $M_{X_i}(t) = E(e^{X_i t}) = p_i e^t + (1 - p_i) = 1 + p_i(e^t - 1) \leq e^{p_i(e^t-1)}$
(c) $X_i \sim \mathrm{Bin}(1, p) \implies M_{X_i}(t) \leq 1 + E(X_i t) = e^{p_i(e^t-1)}$
(d) $\Pr(X_i = 1) = p_i > 0 \implies E(e^{X_i t}) \leq e^t \leq e^{p_i(e^t-1)}$
(e) none of these

3b. Conclude $M_X(t) \leq e^{(e^t-1)\mu}$.

Nothing to turn in.

3c. In the same setup, show that for any $\delta > 0$

$$\Pr[X \geq (1+\delta)\mu] < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu. \tag{2}$$

Hint: Use a Chernoff bound substituting (1) for M_X, and evaluate it for $t = \log(1 + \delta)$.

Nothing to turn in.

3d. For $0 < \delta \leq 1$ show

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\mu\delta^2/3}.$$

Hint: Use the bound (2), and then use basic calculus (with $f'(\delta)$ and $f''(\delta)$ over $0 < \delta \leq 1$) to show $f(\delta) = \delta - (1+\delta)\log(1+\delta) + \delta^2/3 \leq 0$. The latter can then be used to show the inequality on the log scale, after substituting (2).

Nothing to turn in.

3e. Similarly, for $0 < \delta < 1$ it can be shown that $\Pr(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/3}$ and therefore

$$\Pr(|X - \mu| \geq \delta\mu) \leq 2e^{-\mu\delta^2/3} \tag{3}$$

(no need to show). Show then that for $p_i = 0.5$, $i = 1, \ldots, n$, we have

$$\Pr\left(|X - n/2| \geq \frac{1}{2}\sqrt{6n\log(n)}\right) \leq \frac{2}{n}. \tag{4}$$

To prove (4), use (3) with appropriate choice for δ. Which of the following is the correct choice of δ to prove the claim? Mark the right choice.

(a) $\delta = 3\log(n)/n$
(b) $\delta = M_X(t)/e^{at}$
(c) $\delta = \sqrt{6\log(n)/n}$
(d) $\delta = \mathrm{Var}(X)/a^2$, with $a = \frac{1}{2}\sqrt{6n\log(n)}$
(e) none of these

In equation (3) we established a useful tail inequality for a sum of Poisson trials (sum of independent 0-1 r.v.'s X_i with success probabilities p_i). In equation (3) we found a bound for the special case of $p_i = 0.5$, i.e., independent coin flips.