Preface to Homework 13

Why are exponential families included in this semester's work?
Answer In some later courses, you will sometimes hear "I can do this calculation for this type of distribution because it is an exponential family."

What are some of those things we can do with exponential families that we can't do with distributions in general? **What are the two most important "pieces"** of the exponential family characterization?

Answers:

1. Immediately look at the distribution in its exponential family notation and pick out the most important form of the way the data is generally used – is it as x, or log x, or 1/x, etc.
2. There are various other uses that are more complicated than we will be able to do this week. (But you can find, on the web, various descriptions of these with differing levels of details and complexity.) What all of those have in common is that you must be able to look at the pdf in the "exponential family" notation.
3. The $\omega(\theta)$ and the $t(x)$ have more important roles in statistical theory than the other functions you will identify.

Goals:

1. Understand the difference between a distribution and a family of distributions.
2. Understand that members of a family of distributions have various things in common, such as the "structure" of calculations, similarities of graphs, etc.
3. Many of the types of families of dist'ns you have seen are exponential families, but there are some families of distributions you have seen that are NOT exponential families. This week you should identify at least two types which are NOT. More to the point, you should notice WHY they are not exponential families. That is, notice what "goes wrong" when you try to write them in the form of exponential families.
4. Learn to correctly identify these three important characteristics of our expression of the exponential family
    a. The size of the sum, *k*, is as small as possible. (The Binomial dist'n example shows how you can sometimes make *k* smaller than you originally thought.)
    b. The function $t(x)$ is important, because it leads to finding "sufficient statistics."
    c. The function $\omega(\theta)$ is important – it is considered a more "natural" parameter than the original parameter(s).
5. Become skilled enough with the notation to notice when a family of distribution cannot be written in this form, and thus, is not an exponential family.

Lecture topics. The numbers here correspond to subfolders for this topic's Discussion Board. This may be more than necessary, but it's my attempt to keep the discussion organized in a way you'll find helpful.

1. What makes it a family of dist'ns rather than just a distribution?
2. Indicator functions.
3. Characterization of full one-parameter exponential family. (Only one term in the summation instead of k terms.)
4. Important algebra result about exponents and logarithms.
    a. Do you agree with this algebra? $e^{n \cdot \log m} = e^{\log m^n} = m^n$.
    b. It is usually used in the other order, to take an expression that does NOT include an exponential factor and turn it into an expression that DOES include an exponential factor.

    That is, $m^n = e^{\log m^n} = e^{n \cdot \log m}$
5. Pareto dist'n, only one parameter β . Main solution.
6. Pareto dist'n with only one parameter β . Specific questions about choices
7. Pareto dist'n only one parameter α . VERY IMPORTANT. Why is this NOT an exponential family?
8. Exponential Families, Part II (allows for more than one parameter) Comments about definition and its parts.
9. Specific example of Gamma distribution with two parameters (Needs two functions of the data and two functions of the parameters.)
10. Binomial dist'n with only one parameter. How do we avoid needing k = 2 here? (Do you see that, if we use k=2, we have, basically, two terms in that exponent with the same parameter? So using k = 2 is making it unnecessarily complicated.)

Homework 13.

First, ask questions about the lecture material. (From the list of topics above.) That's very important. Then, as you go through the material below, try your best to relate it to something in the lecture material on your own. And when you ask for help, try to describe where you are in it and the immediate thing you need help with.

1. Consider this description: $\text{Gamma}(\alpha, \theta)$. Choose the most appropriate answer.
    a. Not a family of dist'ns
    b. a one-parameter family of dist'ns
    c. a two-parameter family of dist'ns.
2. Consider this description: $\text{Gamma}(3, \theta)$. Choose the most appropriate answer.
    a. Not a family of dist'ns
    b. a one-parameter family of dist'ns
    c. a two-parameter family of dist'ns.
3. Consider this description: $\text{Gamma}(3, 4)$. Choose the most appropriate answer.
    a. Not a family of dist'ns
    b. a one-parameter family of dist'ns
    c. a two-parameter family of dist'ns.

For problems 4 – 10 , attempt to write the pdf in the form of an exponential family with k as small as possible. To prepare for work in the future, it is important that you write ALL the parts (factors) of the pdf in the appropriate form for an exponential family. **For the purpose of grading this homework**, you are asked to focus on the *t(x)* and make the appropriate choice from the list below. In any other context you need to consider exponential families, you must be able to identify all of the parts.

Here are the choices (A) - (J) for the answers on the online graded homework on problems 4 -- 10.

(A) t(x) = log(x) (B) t(x) = log(1-x)

(C) t(x) = e^x (D) t(x) = e^{-x}

(E) t(x) = x (F) t(x) = 1/x

(G) $t(x) = x^r$ where the exponent r here is a number besides 1 or -1.

(H) Can be written as an exponential family but **requires** k > 1, so there two or more expressions for t(x).

(I) Can be written in the form of an exponential family with k = 1 but the statistic that is different from any of these and **cannot be modified to be one of these** by changing the form of the function of the parameter. (Recall the example from the Pareto dist'n in which we modified the form of the function of x.)

(J) Not an exponential family (Cannot be written in the form of an exponential family.)

4. Consider a distribution with pdf $f(x) = p(1-p)^x$ where $x = 1, 2, 3, \ldots$ and $0 < p < 1$
5. Consider a $\text{Normal}(\mu, \theta)$ distribution where the variance θ is known.
6. Consider a $\text{Normal}(\mu, \theta)$ distribution where μ is known and the variance θ is unknown.
7. Consider a $\text{Normal}(\mu, \theta)$ distribution where both the mean μ and variance θ are unknown.
8. Consider a distribution with pdf $f(x \mid \theta) = \dfrac{2x}{\theta^2}$ where $0 < x < \theta$ and $\theta > 0$
9. Consider a distribution with pdf

$$f(x \mid \theta) = \frac{\Gamma(7+\theta)}{\Gamma(\theta)\Gamma(7)}(1-x)^\theta x^5 \quad \text{for} \quad 0 \le x \le 1 \text{ and } \theta > 0$$

10. Consider a distribution with pdf

$$f(x \mid \theta, \sigma) = \frac{1}{\pi\sigma} \cdot \frac{1}{1+\left(\dfrac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \theta < \infty, \quad \sigma > 0$$