9 The Normal Distribution

9.1 The Standard Normal

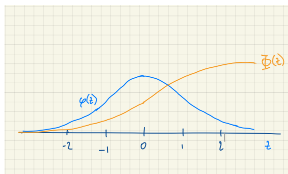
Slide 1

N(0, 1) mean centered at zero and with variance one.

The Standard Normal Distribution $N(0,1)$

Density function: a bell shape, centered at 0 (± 1)

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



$\phi(z)$ and $\Phi(z)$

and corresponding c.d.f. $\Phi(z) = \int_{-\infty}^{z} \phi(t)dt$

Moments: $E(Z) = 0$, by symmetry; and

$$\mathrm{Var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx = \ldots = 1$$

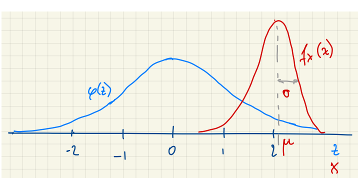
by integration by parts (using $u = x$ and $dv = xe^{-x^2/2}dx$).

9.2 The Univariate Normal Distribution

Slide 2

The Univariate Normal Distribution

Univariate normal: We say $X \sim N(\mu, \sigma^2)$ if X is a shifted and scaled standard normal r.v. Z, i.e.,



$$X = \mu + \sigma Z,$$

with $Z \sim N(0,1)$,

or

$$Z = \frac{X - \mu}{\sigma}.$$

Moments: by construction,

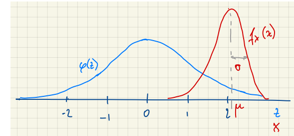
$$\begin{aligned} E(X) &= E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu \\ \mathrm{Var}(X) &= \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2. \end{aligned}$$

μ: mu
σ: sigma

Slide 3

Density:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$



$\phi(z)$, $f_X(x)$

This is true because

$$F_X(x) = \Pr(X \le x) = \Pr(Z \le (x-\mu)/\sigma) = \int_{-\infty}^{(x-\mu)/\sigma} \phi(z)dz$$

and by the fundamental theorem of calculus

$$\Longrightarrow f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{\sigma}\phi\{(x-\mu)/\sigma\}$$

9.3 Examples

Slide 4

Examples

Let $X \sim N(\mu, \sigma^2)$ with $\mu = 3$ and $\sigma = 3$. Find $\Pr(2 < X < 5)$.

Solution:

$$\Pr(2 < X < 5) = \Pr\left(\frac{2-\mu}{\sigma} < \underbrace{\frac{X-\mu}{\sigma}}_{Z} < \frac{5-\mu}{\sigma}\right) =$$

$$\Pr(-1/3 < Z < 2/3) = \Phi(2/3) - \Phi(-1/3) = .38$$

9.4 Example: Signal Detection

Slide 5

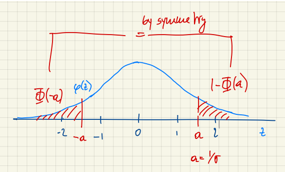
Example: Signal Detection

- A signal $S \in \{-1, 1\}$ is transmitted with noise, as $S + Y$, where $Y \sim N(0, \sigma^2)$.
- The signal is received and decoded as $R = \mathrm{sgn}(S + Y)$ (i.e., $R = 1$ if $S + Y > 0$ and $R = -1$ if $S + Y < 0$).
- Find $\Pr(R \neq S)$:
  - When $S = 1$, then $R \neq S$ if $Y \le -1$

    $$\Pr(Y \le -1) = \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{-1-\mu}{\sigma}\right) = \Phi\left(-\frac{1}{\sigma}\right)$$

    [1] using $\mu = 0$.
  - Similarly, when $S = -1$, then $R \neq S$ if $Y \ge 1$, $\Pr(Y \ge 1) = \ldots = 1 - \Phi\left(\frac{1}{\sigma}\right)$.

[1] there was a missing "-" in the slides.



By symmetry

$$\Phi\left(-\frac{1}{\sigma}\right) = 1 - \Phi\left(\frac{1}{\sigma}\right) \implies$$

$\Pr(\text{wrong transmission}) = \Phi\left(-\frac{1}{\sigma}\right)$.

9.5 The Central Limit Theorem (CLT)

Slide 6

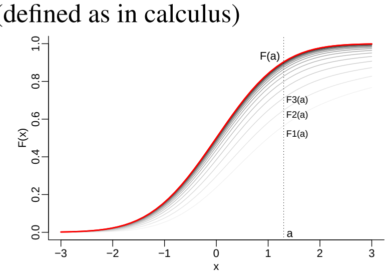
The Central Limit Theorem

In words, averages of i.i.d. r.v's $X_1, \ldots, X_n$ are asymptotically (as $n \to \infty$) normal distributed.

Definition 9.1: Convergence in Distribution. $F_1, F_2, \ldots$ converges in distribution to F, we write $F_n \xrightarrow{D} F$ if for any a

$$\lim_{n\to\infty} F_n(a) = F(a)$$

This is a lim of the sequence $F_n(a) \in \Re$ (defined as in calculus)



If $Z_n \sim F_n$, we also write $Z_n \xrightarrow{D} F$.

Slide 7

Theorem 9.5: CLT. Let $X_1, X_2, \ldots$ be i.i.d. r.v's with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$, and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then for any a

$$\lim_{n\to\infty} \Pr\left(\underbrace{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}_{Z_n} \le a\right) = \Phi(a)$$

i.e.,

$$Z_n \equiv \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1),$$

or

$$\bar{X}_n \overset{\text{approx}}{\sim} N(\mu, \sigma^2/n), \text{ for large } n.$$

or

$$n\bar{X}_n = \sum X_i \overset{\text{approx}}{\sim} N(n\mu, n\sigma^2), \text{ for large } n.$$

Slide 8

Plan of proof: Assume $\mu = 0$, $\sigma = 1$, and X_i has a mgf, $M(t)$.

In that case $Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \bar{X}\sqrt{n}$ and CLT becomes

$$\underbrace{\bar{X}\sqrt{n}}_{Z_n} \xrightarrow{D} N(0,1), \text{ or } M_{Z_n}(t) \to e^{t^2/2} \text{ or } \underline{\log[M_{Z_n}(t)] \to t^2/2}$$

We will (i) set up $\log M_{Z_n}(t)$ as a function of $L(t) \equiv \log M(t)$;

(iii) Take $\lim_{n\to\infty}$ – this will need L'Hopital's rule (twice!)

(ii) show $L''(0) = 1$, in preparation for (iii). that's all!!

(i): Note $Z_n = \bar{X}\sqrt{n} = \sqrt{n}\frac{1}{n}\sum X_i = \sum \frac{X_i}{\sqrt{n}}$, $\implies$

$$M_{X_i/\sqrt{n}}(t) = E\left[\exp\left(t X_i/\sqrt{n}\right)\right] = M(t/\sqrt{n}).$$

Therefore

$$M_{Z_n}(t) = \left[M(t/\sqrt{n})\right]^n \text{ or } \log M_{Z_n}(t) = \underline{\log M(t/\sqrt{n}) \cdot n}$$

need $\underline{\ldots \to \frac{t^2}{2}}$. *remember* $\underline{\frac{\log M(t/\sqrt{n})}{n^{-1}} \to \frac{t^2}{2}}$ *for next slide*

Slide 9

(ii) Let $L(t) = \log M(t)$, and therefore $L(0) = 0$,

$$L'(0) = M'(0)/M(0) = \mu = 0$$

$$L''(0) = \frac{M(0)M''(0) - [M'(0)]^2}{[M(0)]^2} = E(X^2) = 1$$

(iii) Recall, we need: $\lim\left[\underline{\log M(t/\sqrt{n}) \cdot n}\right] =$

$$\underline{\lim \frac{L(t/\sqrt{n})}{n^{-1}}} = \lim \frac{-L'(t/\sqrt{n})\frac{1}{2}n^{-3/2}t}{-n^{-2}} = \lim \frac{L'(t/\sqrt{n})t}{2n^{-1/2}}$$

$$= \lim \frac{L''(t/\sqrt{n})n^{-3/2}t^2}{-2n^{-3/2}} = \lim L''(t/\sqrt{n})\frac{t^2}{2} = \underline{\frac{t^2}{2}}$$

9.6 Example

Slide 10

Example

Fifty numbers are rounded off to the nearest integer and then summed. The individual roundoff errors U_i are uniformly distributed over $(-.5, .5)$, i.e., $U_i \sim \text{Unif}[(-.5, .5)]$.

Approximate the probability that the resultant sum differs from the exact sum by more than 0.3.

Solution: Note $E(U_i) = 0$ and $\text{Var}(U_i) = \frac{1}{12}$. Let $X = \sum U_i$.

$\implies \mu = E(X) = nE(U_i) = 0$, $\sigma^2 = \text{Var}(X) = 20\text{Var}(U_i) = \frac{20}{12}$.

By the CLT

$$\Pr(-.3 < X < 0.3) = \Pr\left(\frac{-.3-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{-.3-\mu}{\sigma}\right) \approx$$

$$\approx \Pr(-.3/\sigma < Z < .3/\sigma) = \Phi(b) - \Phi(-b) = 2\Phi(b) - 1$$

with $b = 0.3/\sqrt{20/12} = 0.23$ and $\Phi(b) = 0.59$ and therefore $\Pr(-0.3 < X < 0.3) \approx 0.18$ and the probability of X differing from the exact sum by more than 0.3 is $1 - 0.18 = 0.82$.

9.7 Use of the Normal Distribution in Statistics

Slide 11

Use of the Normal Distribution in Statistics

Normal distribution: plays an important role in probability theory, due to the CLT and other results.

Statistical inference: similarly, normal r.v.'s play a central role in statistical inference, among other uses as

- (approximate) sampling distribution for means or proportions,
- differences of means, or
- test statistic, and as
- residual distribution in regression.