

## 9 The Normal Distribution

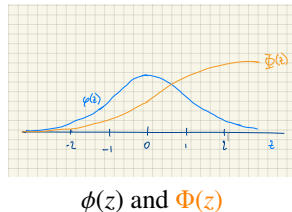
### 9.1 The Standard Normal

Slide 1

The Standard Normal Distribution  $N(0, 1)$

**Density function:** a bell shape, centered at 0 ( $\pm 1$ )

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



and corresponding c.d.f.  $\Phi(z) = \int_{-\infty}^z \phi(t) dt$

**Moments:**  $E(Z) = 0$ , by symmetry; and

$$\text{Var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx = \dots = 1$$

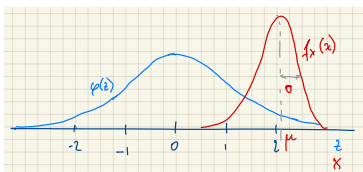
by integration by parts (using  $u = x$  and  $dv = x e^{-x^2/2} dx$ ).

### 9.2 The Univariate Normal Distribution

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The Univariate Normal Distribution

**Univariate normal:** We say  $X \sim N(\mu, \sigma^2)$  if  $X$  is a shifted and scaled standard normal r.v.  $Z$ , i.e.,



$$X = \mu + \sigma Z,$$

with  $Z \sim N(0, 1)$ ,

or

$$Z = \frac{X - \mu}{\sigma}.$$

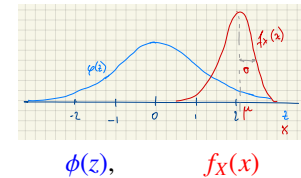
**Moments:** by construction,

$$\begin{aligned} E(X) &= E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu \\ \text{Var}(X) &= \text{Var}(\mu + \sigma Z) = \sigma^2 \text{Var}(Z) = \sigma^2. \end{aligned}$$

Slide 3

Density:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$



This is true because

$$F_X(x) = \Pr(X \leq x) = \Pr(Z \leq (x - \mu)/\sigma) = \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz$$

and by the fundamental theorem of calculus

$$\Rightarrow f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{\sigma} \phi\{(x - \mu)/\sigma\}$$

### 9.3 Examples

Slide 4

Examples

Let  $X \sim N(\mu, \sigma^2)$  with  $\mu = 3$  and  $\sigma = 3$ . Find  $\Pr(2 < X < 5)$ .

**Solution:**

$$\Pr(2 < X < 5) = \Pr\left(\frac{2 - \mu}{\sigma} < \underbrace{\frac{X - \mu}{\sigma}}_Z < \frac{5 - \mu}{\sigma}\right) =$$

$$\Pr(-1/3 < Z < 2/3) = \Phi(2/3) - \Phi(-1/3) = .38$$

### 9.4 Example: Signal Detection

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Example: Signal Detection

- A signal  $S \in \{-1, 1\}$  is transmitted with noise, as  $S + Y$ , where  $Y \sim N(0, \sigma^2)$ .
- The signal is received and decoded as  $R = \text{sgn}(S + Y)$  (i.e.,  $R = 1$  if  $S + Y > 0$  and  $R = -1$  if  $S + Y < 0$ ).
- Find  $\Pr(R \neq S)$ :

– When  $S = 1$ , then  $R \neq S$  if  $Y \leq -1$

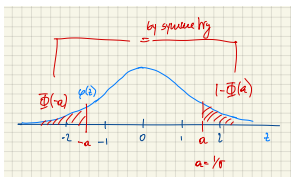
$$\Pr(Y \leq -1) = \Pr\left(\frac{Y - \mu}{\sigma} \leq \frac{-1 - \mu}{\sigma}\right) = \Phi\left(-\frac{1}{\sigma}\right)$$

<sup>1</sup> using  $\mu = 0$ .

– Similarly, when  $S = -1$ , then  $R \neq S$  if  $Y \geq 1$ ,

$$\Pr(Y \geq 1) = \dots = 1 - \Phi\left(\frac{1}{\sigma}\right).$$

<sup>1</sup> there was a missing “-” in the slides.



By symmetry

$$\Phi\left(-\frac{1}{\sigma}\right) = 1 - \Phi\left(\frac{1}{\sigma}\right) \Rightarrow$$

$$\Pr(\text{wrong transmission}) = \Phi\left(-\frac{1}{\sigma}\right).$$

## 9.5 The Central Limit Theorem (CLT)

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### The Central Limit Theorem

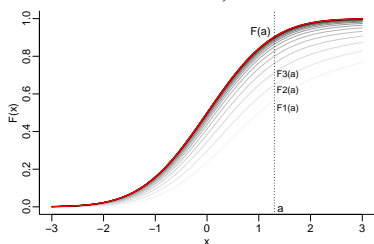
In words, averages of i.i.d. r.v's  $X_1, \dots, X_n$  are asymptotically (as  $n \rightarrow \infty$ ) normal distributed.

**Definition 9.1: Convergence in Distribution.**  $F_1, F_2, \dots$

converges in distribution to  $F$ , we write  $F_n \xrightarrow{D} F$  if for any  $a$

$$\lim_{n \rightarrow \infty} F_n(a) = F(a)$$

This is a lim of the sequence  $F_n(a) \in \mathbb{R}$  (defined as in calculus)



If  $Z_n \sim F_n$ , we also write  $Z_n \xrightarrow{D} F$ .

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**Theorem 9.5: CLT.** Let  $X_1, X_2, \dots$  be i.i.d. r.v's with  $E(X_i) = \mu$  and  $\text{Var}(X_i) = \sigma^2$ , and  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . Then for any  $a$

$$\lim_{n \rightarrow \infty} \Pr\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq a\right) = \Phi(a)$$

i.e.,

$$Z_n \equiv \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0, 1),$$

or

$$\bar{X}_n \overset{\text{approx}}{\sim} N(\mu, \sigma^2/n), \text{ for large } n.$$

or

$$n\bar{X}_n = \sum X_i \overset{\text{approx}}{\sim} N(n\mu, n\sigma^2), \text{ for large } n.$$

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**Plan of proof:** Assume  $\mu = 0, \sigma = 1$ , and  $X_i$  has a mgf,  $M(t)$ .

In that case  $Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \bar{X} \sqrt{n}$  and CLT becomes

$$\underbrace{\bar{X} \sqrt{n}}_{Z_n} \xrightarrow{D} N(0, 1), \text{ or } M_{Z_n}(t) \rightarrow e^{t^2/2} \text{ or } \log[M_{Z_n}(t)] \rightarrow t^2/2$$

We will (i) set up  $\log M_{Z_n}(t)$  as a function of  $L(t) \equiv \log M(t)$ ;

(iii) Take  $\lim_{n \rightarrow \infty}$  – this will need L'Hopital's rule (twice!)

(ii) show  $L''(0) = 1$ , in preparation for (iii). that's all!!

(i): Note  $Z_n = \bar{X} \sqrt{n} = \sqrt{n} \frac{1}{n} \sum X_i = \sum \frac{X_i}{\sqrt{n}}, \Rightarrow$

$$M_{X_i/\sqrt{n}}(t) = E\left[\exp\left(t X_i / \sqrt{n}\right)\right] = M(t/\sqrt{n}).$$

Therefore

$$M_{Z_n}(t) = \left[M(t/\sqrt{n})\right]^n \text{ or } \log M_{Z_n}(t) = \log M(t/\sqrt{n}) \cdot n$$

need  $\dots \rightarrow \frac{t^2}{2}$ . remember  $\frac{\log M(t/\sqrt{n})}{n^{-1}} \rightarrow \frac{t^2}{2}$  for next slide

Slide 9

(ii) Let  $L(t) = \log M(t)$ , and therefore  $L(0) = 0$ ,

$$L'(0) = M'(0)/M(0) = \mu = 0$$

$$L''(0) = \frac{M(0)M''(0) - [M'(0)]^2}{[M(0)]^2} = E(X^2) = 1$$

(iii) Recall, we need:  $\lim_{n \rightarrow \infty} \left[\log M(t/\sqrt{n}) \cdot n\right] =$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{L(t/\sqrt{n})}{n^{-1}} &= \lim_{n \rightarrow \infty} \frac{-L'(t/\sqrt{n}) \frac{1}{2} n^{-3/2} t}{-n^{-2}} = \lim_{n \rightarrow \infty} \frac{L'(t/\sqrt{n}) t}{2n^{-1/2}} \\ &= \lim_{n \rightarrow \infty} \frac{L''(t/\sqrt{n}) n^{-3/2} t^2}{-2n^{-3/2}} = \lim_{n \rightarrow \infty} L''(t/\sqrt{n}) \frac{t^2}{2} = \frac{t^2}{2} \end{aligned}$$

## 9.6 Example

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### Example

Fifty numbers are rounded off to the nearest integer and then summed. The individual roundoff errors  $U_i$  are uniformly distributed over  $(-.5, .5)$ , i.e.,  $U_i \sim \text{Unif}[-.5, .5]$ .

Approximate the probability that the resultant sum differs from the exact sum by more than 0.3.

**Solution:** Note  $E(U_i) = 0$  and  $\text{Var}(U_i) = \frac{1}{12}$ . Let  $X = \sum U_i$ .  
 $\Rightarrow \mu = E(X) = nE(U_i) = 0, \sigma^2 = \text{Var}(X) = 20\text{Var}(U_i) = \frac{20}{12}$ .  
 By the CLT

$$\begin{aligned} \Pr(-.3 < X < .3) &= \Pr\left(\frac{-.3 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{.3 - \mu}{\sigma}\right) \approx \\ &\approx \Pr(-.3/\sigma < Z < .3/\sigma) = \Phi(b) - \Phi(-b) = 2\Phi(b) - 1 \end{aligned}$$

with  $b = 0.3 / \sqrt{20/12} = 0.23$  and  $\Phi(b) = 0.59$  and therefore  
 $\Pr(-.3 < X < .3) \approx 0.18$  and the probability of  $X$  differing from the exact sum by more than 0.3 is  $1 - 0.18 = 0.82$ .

## 9.7 Use of the Normal Distribution in Statistics

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*Slide 11*

### *Use of the Normal Distribution in Statistics*

**Normal distribution:** plays an important role in probability theory, due to the CLT and other results.

**Statistical inference:** similarly, normal r.v.'s play a central role in statistical inference, among other uses as

- (approximate) sampling distribution for means or proportions,
- differences of means, or
- test statistic, and as
- residual distribution in regression.