Homework 7

1. (book #8.1)
Let $X \perp Y$ be independent r.v's, uniform on $[0,1]$. Find $f_W(w)$ for $W = X - Y$.

Using your solution for $f_W(w)$, evaluate $f_W(-0.2)$.

$f_W(-0.2) =$

2. (book #8.3)
Let $X \sim \text{Unif}(0,1)$.

2.a. Find $p = \Pr(X \leq 1/2 \mid 1/4 \leq X \leq 3/4)$.

$p =$

2.b. Find $p = \Pr(X \leq 1/4 \mid (X \leq 1/3) \cup (X \geq 2/3))$.

$p =$

3. (book #8.4)
A and B agree to try to meet between 12 and 1pm for lunch. Assuming that the arrival times T_A and T_B of A and B are uniform between 12 and 1pm, independently. If whoever comes first waits 15 minutes and then leaves, what is the probability p that they actually meet for lunch?

$p =$

Hint: Best to record T_A and T_B on $[0, 60]$, as minutes after 12pm. To see the probability of the event M ="they meet" it is helpful to make a figure with $0 < T_A < 60$ on the x-axis and $0 < T_B < 60$ on the y-axis, and mark M in the diagram.

4. (book #8.14)
Let $X_i \sim \text{Exp}(1)$, $i = 1, 2, \ldots$, independently.

4a. Show that $Y = X_1 + X_2$ is *not* an exponential r.v.
Hint: Recall from class when we solved #8.2. Let $f_1(x_1)$ and $f_2(x_2)$ denote the exponential p.d.f. for X_1 and X_2, respectively.

Which of the following statements is a valid argument? Mark the correct choice.

(a) $f_1(x_1) = e^{-x_1}$ and $f_2(x_2) = e^{-x_2} \implies f_Y(y) = e^{-x_1} + e^{-x_2}$
(b) $f_Y(y) = e^{f_1(x_1)+f_2(x_2)} = e^{f_1(y-x_2)+f_2(y-x_1)}$
(c) $f_Y(y) = \int_0^y e^{-(y-x_1)}\, e^{-x_1} dx_1 = ye^{-y}$
(d) Use the change of variable formula, $f_Y(y) = f_1(y - x_2)|\frac{dx_1}{d_y}| = e^{-(y-x_2)}$
(e) none of these

4b. Let $N \sim \text{Geom}(p)$. Show that $W = \sum_{i=1}^N X_i \sim \text{Exp}(p)$.
Hint: from #8.2. we get by induction the p.d.f. for fixed $N = k$, as $f_{W|N}(w \mid N = k) = \frac{1}{(k-1)!} w^{k-1} e^{-w}$ (using by a slight abuse of notation $f_{W|N}(\cdot)$ for the density of W under $N = k$). Start with the law of total probability $\Pr(W \leq w) = \sum_k \Pr(W \leq w \mid N = k) \Pr(N = k)$. Then recognize $\Pr(W \leq w) = F_W(w)$ as the c.d.f. for W, and therefore $f_W(w) = \frac{d}{dw} F_W(w) = (*) \ldots$

Which of the following arguments is a valid proof of the claim? Mark the correct choice.

(a) $(*) = \sum_k 1/(k-1)! w^{k-1} e^{-w} = pe^{-pw}$.
(b) $(*)$ proves the memoryless property of $W \implies W \sim \text{Exp}(p)$.
(c) $F_W(w) = 1/(k-1)! w^{k-1} e^{-w} \implies f_W(w) = e^{-w}$, i.e., $W \sim \text{Exp}(p)$
(d) $(*) = \sum_{k=1}^{\infty} \frac{w^{k-1}}{(k-1)!} e^{-w} \, (1-p)^{k-1} p = pe^{-w} \sum_{\ell=0}^{\infty} \frac{(w(1-p))^{\ell}}{\ell!} = p \, e^{-pw}$
(e) none of these

5. Let $X > 0$ denote a random variable with p.d.f. $f_X(x)$ and c.d.f. $F_X(x)$. Assume $F_X(\cdot)$ is monotone increasing, and let $Y = F_X(X)$.
That is, Y is a random variable that takes the value $F_X(x)$ when $X = x$. Find $f_Y(y)$.
Mark the correct answer

(a) $f_Y(y) = 1,\ 0 < y < 1$
(b) $f_Y(y) = \frac{d}{dx} F_X(x) = f(x)$
(c) $f_Y(y) = f_X(F_X^{-1}(y))$
(d) $f_Y(y) \propto y(1-y)$
(e) none of these

6. For $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$, independently, find the conditional distribution of X given that $S = X + Y = n$.

Hint: Note that $(X = x, S = n) = (X = x, Y = n - x)$.
Also, recall from Lecture 2.16 (Sec 5.3 in the book) that

$$p_X(x) = \frac{e^{-\lambda_1} \lambda_1^x}{x!}$$

and similarly for $p_Y(y)$, and because of independence $p_{X,Y}(x, y) = p_X(x) \, p_Y(y)$. Also recall from Lecture 2.18 (Lemma 5.2), $S \sim \text{Poi}(\lambda)$ with $\lambda = \lambda_1 + \lambda_2$.

Mark the right answer

(a) $\text{Poi}(\lambda_1 + \lambda_2)$
(b) $\text{Bin}(n, p)$ with $p = \lambda_1/(\lambda_1 + \lambda_2)$
(c) $\text{Poi}(\mu)$ with $\mu = \min\{\lambda_1, \lambda_2\}$
(d) $\text{Bin}(n, p)$ with $p = 1/\lambda_1/(1/\lambda_1 + 1/\lambda_2)$
(e) none of these

(Note, this problem is in preparation of the next few problems, when we will use the Poisson distribution again).

7. We prove the claim from Lecture 7.13, Slide 25 (using T_n instead of W for the sum, in anticipation of the next problems).
Consider a sequence X_i, $i = 1, 2, \ldots$, of independent exponentially distributed r.v.'s, $X_i \sim \text{Exp}(\lambda)$, $i \geq 1$. Let $T_n = \sum_{i=1}^{n} X_i$. If X_i were the waiting time for the i-th event (for example, the i-th customer in a shop; or the eruption of a vulcano, etc.), then T_n would be the arrival time of the n-th event.

Show that T_n is a r.v. with p.d.f.

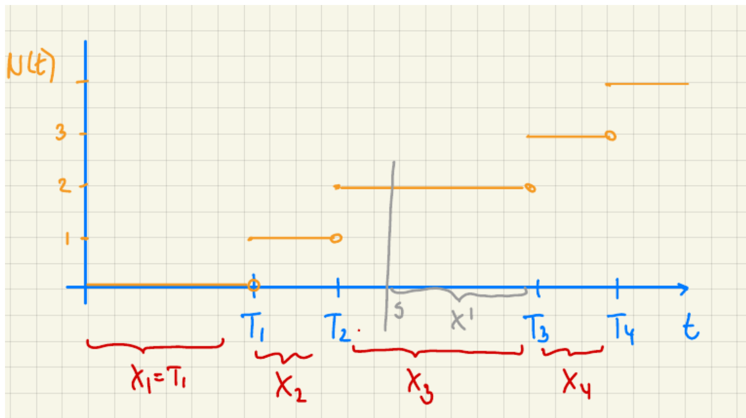
$$f_{T_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0.$$

Hint: use a proof by induction. The formula is right for $n = 1$, with $T_1 = X_1$. Assuming the formula is right for T_n, show it is true for $n + 1$. Use the formula for f_{X+Y} that we derived as part of Example #8.2 in class, substituting $X = T_n$ and $Y = X_{n+1}$.

Nothing to turn in.

Nothing to turn in for the following problem. The problem introduces an important stochastic process known as the Poisson process. For each question, please try to work out the requested step in the proof, and then only see the solution.

8. Consider again the arrival times T_n and inter-arrival times X_n defined in the previous problem. Define $N(t) = \max_n\{\sum_{i=1}^n X_i \le t\}$. Then $N(t)$ is simply the # customers (or, in general, arrivals) until time t.



$N(t)$, X_n and T_n, and (in grey) the remaining waiting time X' for the next event after time s (see question 4b, below).

8a. Show that $N(t) \sim \text{Poi}(\lambda t)$.

Hint: Use $\{N(t) = n\} = \{T_n \le t, X_{n+1} > t - T_n\}$.

Nothing to turn in.

8b. Show that $N(t+s) - N(s) \sim \text{Poi}(\lambda t)$.

Hint: Let $D(t) = N(t+s) - N(s)$, and let $X' = X_{N(s)+1} - s$ denote the remaining waiting time for the first arrival after time s. Then $\{D(t) = n\}$ can be characterized in terms of X' and the next n waiting times. What can you say about the distribution of X', and the distribution of the next n waiting times? No calculus needed.

Nothing to turn in.

The process $N(t)$, or equivalently X_n or T_n, is known as a *Poisson process*. See section 8.4 in the book for more discussion.