

1 Events & Probability

1.1 Example: Comparing Polynomials

Slide 1

Power also called Exponents or Indices

Example 1: Comparing Polynomials

Problem: compare two polynomials, $F(x) \stackrel{?}{=} G(x)$? e.g.
 $(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \stackrel{?}{=} x^6 - 7x^3 + 25?$
 $d = 6$ $d = 6$

Solution: trying to avoid converting both to canonical form;
instead: Let $d =$ degree of the polynomial

canonical form: $x^2 + x + 30$

1. Select a value r uniformly in the range $\{1, \dots, 100d\}$;
 2. Evaluate $F(r)$ and $G(r)$.
 3. If $F(r) \neq G(r)$, then surely report $F(\cdot) \neq G(\cdot)$
 4. If $F(r) = G(r)$, what should we report?
- we could have by chance found a root of $F(x) - G(x) = 0$.
 - If $F(\cdot) \neq G(\cdot) \implies$ at most d solutions of $F(x) - G(x) = 0$;
 - \implies prob to select one of these roots by chance $\leq 1/100$.

Should we report $F(\cdot) = G(\cdot)$? Allowing for a possible error.

A polynomial of degree n can have n real roots

a cubic polynomial can have ≤ 3 roots

a quadratic polynomial can have ≤ 2 roots

1.2 Axioms of Probability

Slide 2

Axioms of Probability

In general, a chance experiment is characterized by:

Definition 1.1.: Probability space Little Omega (ω)

1. Sample space $\Omega =$ all possible outcomes.
In the example, pairs of numbers $(F(r), G(r))$.
The elements $\omega \in \Omega$ are the “elementary outcomes”.
2. Allowable events, a family \mathcal{F} of sets $E \subseteq \Omega$.
E.g., $E =$ a pair of matching values.
The sets that $r =$ root of $F(x) - G(x)$
3. A probability function $\Pr : \mathcal{F} \rightarrow R$.
E.g., any of the pairs $(F(1), G(1)), \dots, (F(100d), G(100d))$ is equally likely (“uniform”).

R: The set of real numbers

Family of sets: also known as a collection of sets or a set of sets

1.3 Interpreting probabilities

Slide 3

Interpreting Probabilities

$|\Omega|$ represents the cardinality or the number of elements in the set Ω

Equally likely outcomes: e.g. in the example, $E =$ “selecting one of the roots”, let $N = |\Omega|$ and $n_E = |E|$. We used

$$n_E \text{ is number of } E \quad \Pr(E) = \frac{n_E}{N}$$

and we argued $n_E \leq d \implies \Pr(E) \leq d/(100d)$

Subjective probabilities: $E =$ “nuclear war in 21st century”.

$\Pr(E) = 0.001 (\implies (E^c) = 0.999, \text{ for } E^c = \text{not } E)$.

Here $\Pr(E)$ is my subjective judgement.

Long run frequencies: Weather in Austin, on 12/31 12pm

- $\Omega = \{ \text{“sunny”, “cloudy”, “rain”, “snow”}\}$.
- $\Pr(\text{sunny}) = 80\%$ (based on many past years)

here $\Pr(\cdot)$ is a long run frequency.

All interpretation fit into the same formal framework ...

Slide 4

Probability Function

In general, the probability function needs to satisfy some constraints. For example, $\Pr(\text{“something happening”})$ better be 1 (some weather will surely happen!), be ≥ 0 etc.

Definition 1.2: Probability function $\Pr : \mathcal{F} \rightarrow R$, with

A1. $0 \leq \Pr(E) \leq 1$, for any $E \in \mathcal{F}$;

A2. $\Pr(\Omega) = 1$;

A3. For any (finite or countably infinite) sequence of pairwise mutually exclusive events E_1, E_2, \dots

$$\Pr\left(\bigcup_{i \geq 1} E_i\right) = \sum_{i \geq 1} \Pr(E_i)$$

1.4 Addition Rule

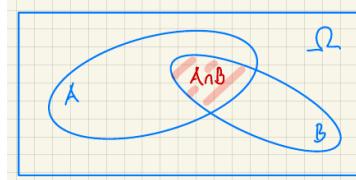
Slide 5

Addition Rule

Lemma 1.1: Addition Rule. For any two events A and B

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Why is this true? Use a Venn diagram to see



(this is not a proof - just a visualization)

→ book for a proof (using

$E_1 = A \cap B^c, E_2 = B \cap A^c,$

$E_3 = A \cap B$)

Since all $\Pr(\cdot)$ are non-negative, we have

Multiplication rule: immediate corollary

Lemma 1.2: Union bound. For any sequence of events E_1, E_2, \dots

$$\Pr\left(\bigcup_{i \geq 1} E_i\right) \leq \sum_{i \geq 1} \Pr(E_i)$$

$$\begin{aligned}\Pr(E \cap F) &= \Pr(F) \Pr(E | F) \\ &= \Pr(E) \Pr(F | E)\end{aligned}$$

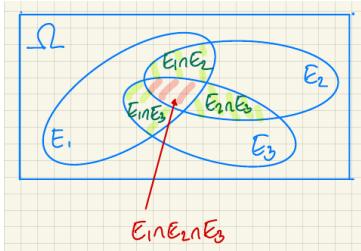
Careful: $\Pr(E | F) \neq \Pr(F | E)$ is not symmetric.

Slide 6

Lemma 1.3: Inclusion-exclusion principle. For any events E_1, \dots, E_n ,

$$\begin{aligned}\Pr\left(\bigcup_{i=1}^n E_i\right) &= \sum_{i=1}^n \Pr(E_i) - \sum_{i < j} \Pr(E_i \cap E_j) + \\ &\quad + \sum_{i < j < k} \Pr(E_i \cap E_j \cap E_k) - \dots \\ &\quad + (-1)^{\ell-1} \sum_{i_1 < i_2 < \dots < i_\ell} \Pr(E_{i_1} \cap \dots \cap E_{i_\ell})\end{aligned}$$

Why is this true?

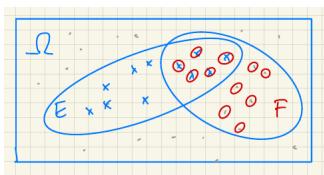


1.5 Conditional Probability

Slide 7

Conditional Probability

Equally likely outcomes: we define $\Pr(E | F) \equiv$ the relative # of outcomes that favor E , among all those that favor F :



N: cardinality of sample space

$$\Pr(E | F) = \frac{n_{E \cap F}}{n_F}$$

$$\dots = \frac{n_{E \cap F} / N}{n_F / N} = \frac{\Pr(E \cap F)}{\Pr(F)}$$

General: this motivates the general definition

numerator
denominator

$$\Pr(E | F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

Careful: $\Pr(E | F) \neq \Pr(F | E)$ is not symmetric.

Slide 8

1.6 Independence

Slide 9

Independence

Example ① (ctd.): Recall, we draw $r_1 \in \{1, \dots, 100d\}$.

If $F(r_1) - G(r_1) \neq 0 \rightarrow$ report $F \neq G$ – easy.

If $F(r_1) - G(r_1) = 0 \rightarrow$ report $F = G$, with error prob $\leq \frac{1}{100}$.

Improved error bound: repeat with more random draws:

$r_2 \in \{1, \dots, 100d\}$ (and r_3 , etc.)

Let $E_i = \{F(r_i) - G(r_i) = 0\}$ and $d_0 = \# \text{ roots}$.

If in fact $F(\cdot) \neq G(\cdot)$, then **d0: denote number of roots of (F-G)**

$$\begin{aligned}\Pr(E_1 \cap E_2) &= \frac{n_{E_1 \cap E_2}}{N} = \frac{d_0 \cdot d_0}{100d \cdot 100d} \\ &= \frac{d_0}{100d} \cdot \frac{d_0}{100d} = \Pr(E_1) \Pr(E_2) \leq \left(\frac{1}{100}\right)^2\end{aligned}$$

Independence: In general we define E, F are independent if

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$$

Note when $\Pr(F) > 0$, then $\dots \iff \Pr(E | F) = \Pr(E)$.

“Knowing F does not change judgment on E .”

Knowing F does not change our judgement about E

Slide 10

Independence (ctd.)

Extend the same definition to multiple events:

k-events: E_1, \dots, E_k are *mutually independent* if for any subset

$$I \subseteq \{1, \dots, k\}$$

$$\Pr\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} \Pr(E_i)$$

Note: pairwise ind. for all pairs $E_i, E_j \Rightarrow$ mutual ind.

For example, tossing two fair coins, $E_1 = \{HT, HH\}$ (head on 1st), $E_2 = \{TH, HH\}$ (H on 2nd), and $E_3 = \{HH, TT\}$ (same on both).

Then E_1, E_2, E_3 are pairwise, but not mutually independent.

Slide 11

Example ① (ctd.)

We can improve the error bound even more:

- Sample $r_i \in \{1, \dots, 100d\}$, with replacement, $i = 1, \dots, k$
- If in fact $F(\cdot) - G(\cdot) \neq 0$, then

$$\Pr(E_1 \cap \dots \cap E_k) = \prod_{i=1}^k \Pr(E_i) \leq \left(\frac{1}{100}\right)^k$$

k: times to meet $\text{Let } E_i = \{F(r_i) - G(r_i) = 0\}$ and $d0 = \#\text{roots}$.

- When $F(r_i) = G(r_i)$, $i = 1 \dots k \rightarrow$ report “ $F(\cdot) = G(\cdot)$ ”, with error probability $\leq (1/100)^k$.

#1.3(e): Poker is played with a deck of $52 = 13 \times 4$ cards, with 13 different denominations in each of 4 suits (diamonds, clubs, hearts and spades).

A 5-card poker hand is said to be a *full house* if it consists of 3 cards of the same denomination and 2 other cards of another denomination, i.e., a pattern “aaabb” where “a” and “b” are two different denominations. Let C denote the event that one is dealt a full house. Find $\Pr(C)$.

Solution: $\Pr(C) = n_C/N$, with $N = \binom{52}{5}$ and $n_C = \binom{4}{3} \cdot 13 \cdot \binom{4}{2} \cdot 12$, and therefore

$$\Pr(C) = \frac{\frac{4!}{3!} \cdot 13 \cdot \frac{4!}{2!} \cdot 12}{\frac{52!}{47!5!}} = 0.0014$$

1.7 Examples

Slide 12

Examples

1.2: Rolling two dice. Let A = “both dice show the same number”, $B = \{a_1 > a_2\}$, where a_j = number on j -th die, and $C = \{a_1 + a_2 \geq 10\}$.

Find $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(C | A)$ and $\Pr(A | C)$. Are A and C independent?

Solution: There are $N = 36$ outcomes, $N_A = 6$ favor A
 $\implies \Pr(A) = 6/36$.

$N_B = 6$ outcomes with $a_1 = a_2$, and therefore (by symmetry)
 $N_B = (36 - 6)/2 = 15$ with $a_1 > a_2 \implies \Pr(B) = 15/36$.

$$n_C = 6 \implies \Pr(C) = \frac{6}{36} = \frac{1}{6},$$

$$n_{A \cap C} = 2 \implies \Pr(C | A) = \frac{2}{6} = \frac{1}{3}, \Pr(A | C) = \frac{2}{6} = \frac{1}{3}.$$

A and C are not independent because $\Pr(C) \neq \Pr(C | A)$.

$\Pr(C | A) = \frac{1}{3} = \Pr(A | C)$ is a coincidence here – not true in general.

1.8 Example: Verifying Matrix Multiplication

Slide 15

Example ②: Verifying Matrix Multiplication

Problem: Given $(n \times n)$ matrices A, B, C , verify $AB = C$?
 Simple matrix multiplication needs order n^3 operations
 $(\Theta(n^3))$

A randomized algorithm: for a faster verification:

- Select a random vector $r = (r_1, \dots, r_n) \in \{0, 1\}^n$;
- compute Br , $A(Br)$ and Cr ($\Theta(n^2)$ operations);
- if $A(Br) \neq Cr \Rightarrow$ report $AB \neq C$.
 if $A(Br) = Cr$ report $AB = C$ (and we might be wrong ...).

some x can make $ABx = Cx$ even though $AB \neq C$ (x is vector)

Slide 13

Slide 16

Example ② (ctd.)

Before the next example, two useful results to count # outcomes:

1. **Permutation:** $n! = \#$ of ordered arrangements of n items (in n positions). **n! = ordered arrangements of n items**
2. **Combinations:** $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \#$ of un-ordered subsets of size k of n elements ($k \leq n$).
 Note $\binom{n}{k} = \binom{n}{n-k}$ (since not selecting $(n - k)$ elements = selecting k elements).
3. **Basic principle of counting:**
 if experiment (step) 1 can result in n_1 outcomes, and
 experiment 2 in n_2 outcomes
 $\implies (n_1 \cdot n_2)$ outcomes of the joint experiment.

!: factorial
#: number

Slide 14

Result: Sampling r uniformly in $\{0, 1\}^n$ is equivalent to sampling r_i independently, uniformly from $\{0, 1\}$

(we write $r_i \sim \text{Unif}\{0, 1\}$, i.i.d.) **0 or 1, equal probabilities**

Proof: $\Pr(r = x) = \Pr(r_1 = x_1, \dots, r_n = x_n) = \prod_{i=1}^n \Pr(r_i = x_i) = \prod_{i=1}^n \frac{1}{2} = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$

Prove that we can randomly pick binary vector by selecting each digit by coin flip

1.9 Law of Total Prob

Slide 17

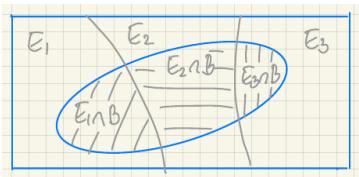
Law of Total Probability

Theorem 1.6: Law of Total Probability: If E_1, \dots, E_n are **mutually disjoint events in Ω** and $\bigcup_i E_i = \Omega$ (i.e., $\Omega = \bigcup E_i$ is a **partition**), then

$$\Pr(B) = \sum_{i=1}^n \Pr(B \cap E_i) = \sum_{i=1}^n \Pr(B | E_i) \Pr(E_i).$$

In words: the (marginal) probability $\Pr(B)$ is the average of the conditional probabilities $\Pr(B | E_i)$ under scenarios E_1, \dots, E_n .

Why is that true?



Proof: → book (easy)

- Now we are ready for the proof: Assume $AB \neq C$. We will use the law of total probability.

Slide 20 Ej: it represents an event where these subvectors of vectors r and x have the same values.

Assume $AB \neq C$. We use the law of total probability, with $E_j = \{r_{2:n} = x_{2:n}\}$ for $\Pr(ABr = Cr) = \sum_j \Pr((ABr = Cr) \cap E_j)$:

$$\begin{aligned} \Pr(ABr = Cr) &= \sum_{x_{2:n}} \Pr \left((ABr = Cr) \cap (\underbrace{r_{2:n} = x_{2:n}}_{E_j}) \right) \leq \\ &\leq \sum_{x_{2:n}} \Pr \left(\left(r_1 = -\frac{\sum_{j=2}^n d_{1j} r_j}{d_{11}} \right) \cap (r_{2:n} = x_{2:n}) \right) \\ &= \sum_{x_{2:n}} \Pr \left(r_1 = -\frac{\sum_{j=2}^n d_{1j} r_j}{d_{11}} \right) \cdot \Pr(r_{2:n} = x_{2:n}) \\ &\leq \sum_{x_{2:n}} \frac{1}{2} \Pr(r_{2:n} = x_{2:n}) = \frac{1}{2} \end{aligned}$$

1.10 Verifying Matrix Mult (ctd.)

Slide 18

Example ② (ctd.)

Recall the problem to verify $AB = C$. We sample r uniformly from $\{0, 1\}^n$ (in short, $r \sim \text{Unif}(\{0, 1\}^n)$).

- If $(\underbrace{AB - C}_D)r \neq 0 \rightarrow$ easy – report $AB \neq C$.

- If $Dr = 0$, should we report $AB = C$?

But even if $AB \neq C$, we could by chance have generated r as a solution of $Dr = 0$.

Next, we evaluate the chance of this happening.

Slide 19 Textbook Page08

Result: If in fact $AB \neq C$, and $r = (r_1, r_2, \dots, r_n) \sim \text{Unif}(\{0, 1\}^n)$, then

$$\Pr(ABr = Cr) \leq \frac{1}{2}$$

Proof: first consider r_1 assuming we know $r_{2:n} \equiv (r_2, \dots, r_n)$.

We can then bound $\Pr(ABr - Cr = 0)$ using the law of total probability.

- Let $D = AB - C$. Then $ABr = Cr \iff Dr = 0$. If $D \neq 0$ then some elements of D are $\neq 0$ – assume that's $d_{11} \neq 0$ (w.l.o.g.).

- Then $Dr = 0 \implies \sum_{j=1}^n d_{1j} r_j = 0$ or $r_1 = \frac{\sum_{j=2}^n d_{1j} r_j}{d_{11}}$.

That is, for given $r_{2:n} = x_{2:n}$ we need $r_1 = \frac{\sum_{j=2}^n d_{1j} r_j}{d_{11}}$.

But $r_1 \sim \text{Unif}(\{0, 1\}) \implies \Pr(r_1 = \dots | r_2, \dots, r_n) \leq \frac{1}{2}$.

r_1 can be 0 or 1, assume result of the
orange can only become 0 or 1, so the P of
'r1 = orange' = 1/2. But the orange can be
other values, so P of 'r1 = orange' <= 1/2

1.11 Example: The Birthday Paradox

Slide 21

Example ③: The Birthday Paradox

Note: this is §5.1 in the book.

Question: In a class of $m = 30$, is

$$\Pr(2 \text{ same b-day}) \stackrel{?}{>} \Pr(\text{no two same b-days})$$

It's easier to find $\Pr(\bar{A})$ for \bar{A} = “no 2 same b-days” (and then use $\Pr(A) = 1 - \Pr(\bar{A})$).

Solution: assume that (i) b-days are selected uniform over $n = 365$ days, (ii) independently across the m people.

Count $N_{\bar{A}} = |\bar{A}| = \# \text{ outcomes that favor } \bar{A}$, and $N = \# \text{ all possible outcomes}$, to get $\Pr(\bar{A}) = N_{\bar{A}}/N$.

- # ways to chose $m = 30$ birthdays = $\binom{365}{30}$, and # ways to assign those = $30!$; therefore $N_{\bar{A}} = \binom{365}{30} \cdot 30!$
- $N = 365^{30}$ (365 choices for each of the 30 b-days)
- $\Pr(\bar{A}) = N_{\bar{A}}/N = 0.29 \implies \Pr(A) = 0.71$.

Slide 22

An Alternative Solution

Let $E_i = (i\text{-th b-day} \neq \text{the first } i-1)$,

$$\Pr(E_i | E_1 \cap \dots \cap E_{i-1}) = \frac{365 - (i-1)}{365} = 1 - \frac{i-1}{365}$$

p of not the every of i-1 b-day

Multiplication rule

$$\Pr(E \cap F) = \Pr(F) \Pr(E | F)$$

Assume upfront the probability of correct is 1/2

Using Bayes' theorem we can find $\Pr(E | B)$.

$$\begin{aligned}\Pr(\bar{A}) &= \Pr\left(\bigcap_{i=2}^m E_i\right) = \prod_{i=2}^m \Pr(E_i | E_2 \cap \dots \cap E_{i-1}) \\ &= \prod_{i=2}^m \left(1 - \frac{i-1}{365}\right) = 0.29.\end{aligned}$$

$$\Pr(A^\perp) = (1 - 1/365) * (1 - 2/365) * \dots * (1 - 29/365)$$

- Start with $\Pr(E) = \frac{1}{2}$ (before the computer experiment with r).

- We know $\Pr(B | E) = 1$ and just found $\Pr(B | \bar{E}) \leq \frac{1}{2}$.

$$\begin{aligned}\Rightarrow \Pr(E | B) &= \frac{\Pr(B | E) \Pr(E)}{\Pr(B | E) \Pr(E) + \Pr(B | \bar{E}) (1 - \Pr(E))} \\ &\geq \frac{1 \cdot 1/2}{1 \cdot 1/2 + 1/2 \cdot 1/2} = \frac{2}{3}\end{aligned}$$

When B is the experiment (evidence), we refer to

- $\Pr(E)$ as “prior probability”, and

- $\Pr(E | B)$ as “posterior probability”

know all $\Pr(B | E)$, want to know $\Pr(E | B)$, use Bayes Theorem

Note: know all $\Pr(B | E)$ mean that know $\Pr(B | E)$ and $\Pr(B | E \text{ complement})$ or

know all $\Pr(B | E_1), \Pr(B | E_2), \Pr(B | E_3) \dots$

1.12 Bayes Theorem

Slide 23

Bayes Theorem

In Example ②, let $E_1 = \{AB = C\}$, $E_2 = \{AB \neq C\}$ and event $B = \{ABr = Cr\}$, i.e., “works for r ”.

Did you notice that we answered the wrong question? We found $\Pr(B | E_2)$. But if we already know that the identity is wrong, it's silly to test it! We really want $\Pr(E_j | B)$!

Theorem 1.7: Bayes Theorem. Let $\bigcup_{i=1}^k E_i$ be a partition of the sample space. Then

$$\Pr(E_j | B) = \frac{\Pr(B | E_j) \Pr(E_j)}{\sum_i \Pr(B | E_i) \Pr(E_i)}$$

Bayes' Theorem “turns around the conditioning”.

Proof: Use the definition of cond prob, the multiplication rule, and the law of total prob

$$\Pr(E_j | B) = \frac{\Pr(B \cap E_j)}{\Pr(B)} = \frac{\Pr(B | E_j) \Pr(E_j)}{\sum_i \Pr(B | E_i) \Pr(E_i)} \quad \begin{matrix} \text{Multiplication rule} \\ \text{Total Probability} \end{matrix}$$

that's all!

Slide 24 Probability of B given A(给定A) = $\Pr(B | A)$

Let \bar{E} = not E denote the complement to E .

Bayes Theorem is often useful for the partition (E, \bar{E}) :

$$\Pr(E | B) = \frac{\Pr(B | E) \Pr(E)}{\Pr(B | E) \Pr(E) + \Pr(B | \bar{E}) (1 - \Pr(E))}.$$

Next, we will use this for Example ②, using

- E = identity is correct, $AB = C$,
- B = “works for r ” (i.e., $B = \{ABr = Cr\}$).

Slide 25

Example ② (ctd.)

- E = identity is correct, $AB = C$.
- B = “works for r ” (i.e., $B = \{ABr = Cr\}$).

- Start with $\Pr(E) = \frac{1}{2}$ (before the computer experiment with r).

- We know $\Pr(B | E) = 1$ and just found $\Pr(B | \bar{E}) \leq \frac{1}{2}$.

$$\begin{aligned}\Rightarrow \Pr(E | B) &= \frac{\Pr(B | E) \Pr(E)}{\Pr(B | E) \Pr(E) + \Pr(B | \bar{E}) (1 - \Pr(E))} \\ &\geq \frac{1 \cdot 1/2}{1 \cdot 1/2 + 1/2 \cdot 1/2} = \frac{2}{3}\end{aligned}$$

When B is the experiment (evidence), we refer to

- $\Pr(E)$ as “prior probability”, and

- $\Pr(E | B)$ as “posterior probability”

know all $\Pr(B | E)$, want to know $\Pr(E | B)$, use Bayes Theorem

Note: know all $\Pr(B | E)$ mean that know $\Pr(B | E)$ and $\Pr(B | E \text{ complement})$ or

know all $\Pr(B | E_1), \Pr(B | E_2), \Pr(B | E_3) \dots$

1.13 Example

Slide 26

Example: Monty Hall Problem

#1.12: game show,

- 3 doors: Behind one door is a car, behind the other two doors a goat. $E_1, E_2, E_3, \text{car behind door } 1, \text{door } 2, \text{door } 3$
Let E_j denote the event “car behind door j ”.
- Initially $\Pr(E_j) = \frac{1}{3}$, and you randomly guess one door to have the car, say you guess E_1 .
- I open one of the other two doors, say door 2, to show you a goat. This is the evidence, B .

You now have a chance to change your guess. Assuming you want the door with the car, should you change your guess?

Find $\Pr(E_1 | B)$ and $\Pr(E_3 | B)$.

$\Pr(H | D)$ is the posterior probability of the hypothesis H given the evidence D .
 D is the evidence in this case

Slide 27

Solution:

We use Bayes theorem. Let B = “I show you a goat behind door 2”. We assume

- $\Pr(B | E_1) = \frac{1}{2}$, since I could open door 2 or 3 to show you a goat,
- $\Pr(B | E_3) = 1$, since I can only open door 2 now, assuming that I don't want to show you the car behind door 3!

And we already have $\Pr(E_i) = 1/3$. Then

$$\Pr(E_1 | B) = \frac{\Pr(B | E_1) \Pr(E_1)}{\Pr(B | E_1) \Pr(E_1) + \Pr(B | E_3) \Pr(E_3)} = \frac{1/2}{1/2 + 1} = \frac{1}{3}$$

and therefore $\Pr(E_3 | B) = \frac{2}{3}$. Change your guess to E_3 !

know $\Pr(B | E_1)$ and $\Pr(B | E_3)$, want to get $\Pr(E_1 | B)$ and $\Pr(E_3 | B)$