Program Structure and Algorithms

Sid Nath

Lecture 12

Agenda

- Administrative
  - HW5, PA2 will be out after today's class
- Lecture
- Quiz

P5: Rod Cutting

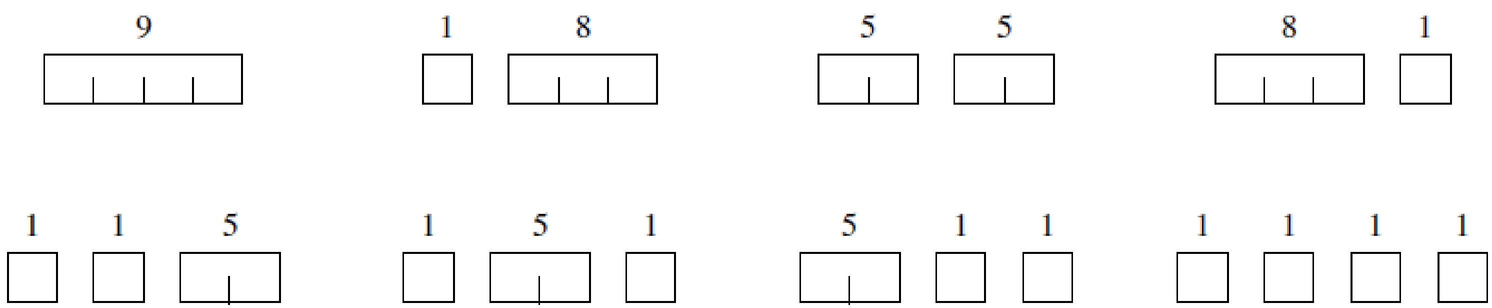
- How to cut steel rods into pieces in order to maximize the revenue you can get? Each cut is free. Rod lengths are always an integer number of inches.
- Input: A length n (inches) and table of prices $p_i, i = 1, 2, \ldots, n$
- Output: Maximum revenue obtainable from rods whose lengths sum to n, computed as the sum of the prices for the individual rods

- Trivial solution: If p_n is very large, optimal solution needs no cuts, i.e., leave the rod as n inches long

Example

length i	1	2	3	4	5	6	7	8
price p_i	1	5	8	9	10	17	17	20

- After first $n-1$ inches, we can either cut or not .. 2^{n-1}ways
- 8 ways to cut a rod of length 4
  - best two 2 inches that yields a revenue of 5+5 =10



Formulation

Let r_i be the max revenue for a rod of length i

By inspection

For n = 7, one opt sol makes a cut at 3in → 2 subproblems of lengths 3 and 4 → solve both optimally.

The opt sol of length 4 → 2 + 2 → used in opt sol of length 7

i	r_i	optimal solution
1	1	1 (no cuts)
2	5	2 (no cuts)
3	8	3 (no cuts)
4	10	$2+2$
5	13	$2+3$
6	17	6 (no cuts)
7	18	$1+6$ or $2+2+3$
8	22	$2+6$

Optimal revenue r_n by taking max of

- p_n: revenue from not making a cut
- $r_1 + r_{n-1}$: max revenue from a rod 1in and rod $(n-1)$in
- $r_2 + r_{n-2}$: max revenue from a rod 2in and rod $(n-2)$in, …
- $r_{n-1} + r_1$

$$\boldsymbol{r_n = \max\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\}}$$

Formulation

Intuition: Every opt sol has a leftmost cut, i.e., there's some cut that gives a first piece of length i cut off the left end, and a remaining piece of length $n - i$ on the right

Need to divide only the remainder, not the first piece

Sol with no cuts has first piece $i = n$ with revenue p_n and remaining size of length 0 has revenue $r_0 = 0$ // base case

$$\boldsymbol{r_n = \max\{ p_i + r_{n-i}, \forall i = 1 \ldots n\}}$$

Subproblem: Max revenue achievable by cutting a rod of length i off the left end

If you think of the rod demarcated at intervals of unit length, then entire rod is a sequence $x[1{:}n]$

Then, subproblem will be max revenue achievable by cutting $x[1{:}i]$

Naïve Recursive Top-Down Solution

What's the problem?

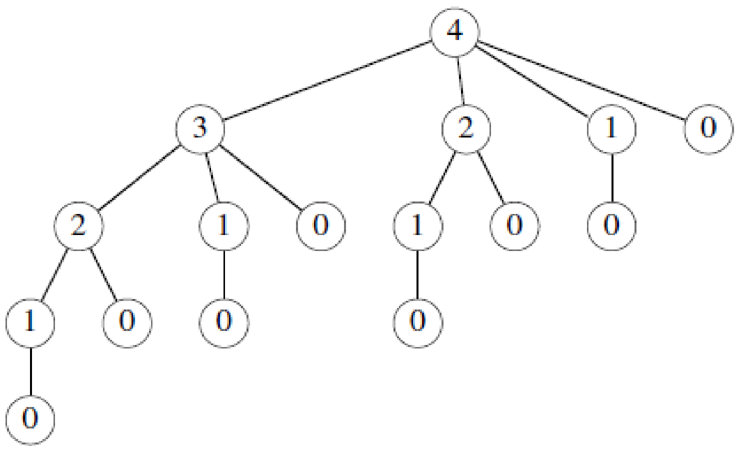
CUT-ROD calls itself repeatedly on solved subproblems!!!

```
CUT-ROD(p, n)
  if n == 0
      return 0
  q = −∞
  for i = 1 to n
      q = max {q, p[i] + CUT-ROD(p, n − i)}
  return q
```

Let's study for n = 4

Many repeated subproblems

Solves for subproblem for size 2 twice, for size 1 four times, for size 0 eight times!!!



$$T(n) = \begin{cases} 1, & if\ n = 0 \\ 1 + \sum_{j=0}^{n-1} T(j), & if\ n \geq 1 \end{cases} = O(2^n)$$

Subproblem Graphs

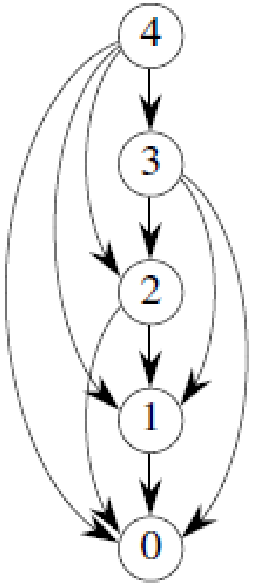
How to understand the dependency of subproblems? Directed graph

One vertex for each distinct subproblem

Directed edge (x, y) if computing an opt sol to subproblem x ***directly*** requires knowing an opt sol to subproblem y

Running time: Sum of times needed to solve each subproblem

Time to compute a subproblem is typically linear in the out-degree of its vertex



DP Solution

Arrange to solve each subproblem just once by saving solution in a table

Pick solution from table when revisiting solved subproblems

Turns exponential-time solution to a polynomial-time solution

Two basic approaches:

- Top-down with memoization
- Bottom-up

Top-Down with Memoization

Memorize what has been computed previously

Store solution to subproblem of length i in an array entry $r[i]$

```
MEMOIZED-CUT-ROD(p, n)
  let r[0 : n] be a new array        // will remember solution values in r
  for i = 0 to n
      r[i] = −∞
  return MEMOIZED-CUT-ROD-AUX(p, n, r)

MEMOIZED-CUT-ROD-AUX(p, n, r)
  if r[n] ≥ 0                  // already have a solution for length n?
      return r[n]
  if n == 0
      q = 0
  else q = −∞
      for i = 1 to n      // i is the position of the first cut
          q = max {q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r)}
  r[n] = q                     // remember the solution value for length n
  return q
```

Bottom-Up

Sort the subproblems by size and solve the smaller ones first

```
BOTTOM-UP-CUT-ROD(p, n)
  let r[0 : n] be a new array        // will remember solution values in r
  r[0] = 0
  for j = 1 to n                     // for increasing rod length j
      q = −∞
      for i = 1 to j                 // i is the position of the first cut
          q = max {q, p[i] + r[j − i]}
      r[j] = q                       // remember the solution value for length j
  return r[n]
```

How to Reconstruct a Solution?

Extend bottom-up approach to output opt values AND opt choices

Save opt choices in a separate table and print it

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
  let r[0 : n] and s[1 : n] be new arrays
  r[0] = 0
  for j = 1 to n                   // for increasing rod length j
      q = −∞
      for i = 1 to j               // i is the position of the first cut
          if q < p[i] + r[j − i]
              q = p[i] + r[j − i]
              s[j] = i             // best cut location so far for length j
      r[j] = q                     // remember the solution value for length j
  return r and s
```

```
PRINT-CUT-ROD-SOLUTION(p, n)
  (r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
  while n > 0
      print s[n]          // cut location for length n
      n = n − s[n]        // length of the remainder of the rod
```

Example of Output

Example: PRINT-CUT-ROD-SOLUTION(p , 8)

length i	1	2	3	4	5	6	7	8
price p_i	1	5	8	9	10	17	17	20

Example of Output

Example: PRINT-CUT-ROD-SOLUTION(p , 8)

length i	1	2	3	4	5	6	7	8
price p_i	1	5	8	9	10	17	17	20

i	0	1	2	3	4	5	6	7	8
$r[i]$	0	1	5	8	10	13	17	18	22
$s[i]$		1	2	3	2	2	6	1	2

Runtime and Space Complexity

Top-down: To solve a subproblem of size n, the for loop iterates n times → total #iters form an arithmetic series

Bottom-up: #Iters of inner for loop forms an arithmetic series

Running time: $\theta(n^2)$

Space complexity: $\theta(n)$

Chain of Thought

- What are my subproblems?
- What are the decisions to solve each subproblem?
- Recursive formulation
  - Base case
- How many subproblems? What is the running time per subproblem?
- What is the overall running time?

P6: Knapsack w/ Repetition

Given a "knapsack" that can hold a maximum weight of W $pounds$, a robber has n $items$ to pick from. Each item has weight $\{w_1, w_2, \ldots, w_n\}$ and dollar value $\{v_1, v_2, \ldots, v_n\}$.

What's the ***most valuable*** combination of items the robber can fit into his bag / knapsack?

Generalizes to a wide variety of resource-constrained selection problems!

Example

Item	Weight (lbs)	Value ($)
1	7	12
2	3	2
3	20	41

W = 58 lbs

Option 1: $2 \times Item3 + 1 \times Item2 + 2 \times Item1$

Weight: $40 + 3 + 14 = \ 57\ lbs \leq 58\ lbs$

Value: $82 + 2 + 24 = \$108$

Option 2: $2 \times Item3 + 6 \times Item2 + 0 \times Item1$

Weight: $40 + 18 + 0 = \ 58\ lbs \leq 58\ lbs$

Value: $82 + 12 + 0 = \$94$

A Brute-Force Approach

- Try all possible combinations of items
- Compute weight and profits; eliminate combinations with total weight $> W$
- Find max value

Runtime: $\sum_{k=0}^{n} C(n,k) = \Omega(2^n)$

DP: Structure of an optimal solution

Idea: Look at smaller knapsack capacities $w \leq W$

Let $K(w)$ be the max value achievable with a knapsack capacity $\mathrm{w} \leq W$.

We want $K(W)$.

If the optimal solution includes item i,

- removing this item leaves an optimal solution to $K(w - w_i)$
- Add value of item i, which is v_i

Need to consider all possible values of i

DP: A Recursive Solution

$$K(w) = \max\{K(w - w_i) + v_i\}\ \forall i\colon\ w_i \le w$$

Base case: $K(0) = 0$

DP: Pseudocode (Tabular, bottom-up)

$K(0) = 0$
$\texttt{for } w = 1 \texttt{ to } W\texttt{:}$
$\quad K(w) = \max\{K(w - w_i) + v_i : w_i \leq w\}$
$\texttt{return } K(W)$

O(W) subproblems
Each entry can take up to $O(n)$ time to compute
Runtime: $\mathrm{O}(nW)$

If $W \sim 2^8 or\ 2^{20}$, i.e., 8bits to 20bits, complexity increases exponentially with #bits

P7: Knapsack w/o Repetition

If repetitions are not allowed to our knapsack problem. How can we solve it using DP?

$K(j, w)$ = max value achievable using knapsack of capacity w and items $1, \ldots, j$

We need $K(n, W)$

$$K(j, w) = \max\{K(j-1, w-w_j) + v_j, K(j-1, w)\}$$

P5: Example

Item	1	2	3	4	5
W (lbs)	1	2	1	3	5
V ($)	10	40	20	20	60

W = 10 lbs

j = 1, w = 1, w_j=1, v_j=10
K[1, 1] = max(K[0,0]+10, K[0,1]) = 10
K[1,w] = max(K[0, w-w_1]+10, K[0,w]); K[0, w] = 0; K[0, w-w_1] = 0 $\forall 2 \leq w \leq 10$

j = 2, w = 1, w_j=2, v_i=40; $w_j > w$, so K[2,1] = K[1,1]
j = 2, w = 2, w_j=2, v_i=40; $w_j \leq w$, so K[2,2] = max(K[1,0]+40, K[1,2]) = max(40,10) = 40
j = 2, w = 3, w_j=2, v_i=40; $w_j \leq w$, so K[2,3] = max(K[1,1]+40, K[1,3]) = max(10+40,10) = 50
K[2,w] = max(K[1, w-w_2]+40, K[1,w]) = 50

weight capacity →			0	1	2	3	4	5	6	7	8	9	10
weights	values	0	0	0	0	0	0	0	0	0	0	0	0
1	10	1	0	10	10	10	10	10	10	10	10	10	10
2	40	2	0	10	40	50	50	50	50	50	50	50	50
1	20	3	0	20	40	60	70	70	70	70	70	70	70
3	20	4	0	20	40	60	70	70	80	90	90	90	90
5	60	5	0	20	40	60	70	70	80	100	120	130	$\boxed{130}$

K[n,W] = K[5,10] = 130
= max knapsack value

P8: Weighted Interval Scheduling

Given a set of n jobs, each job $j \in n$ starts at s_j and finishes at f_j and has a weight or value v_j. Two jobs are compatible if they don't overlap.

Goal: Find maximum weight subset of mutually compatible jobs

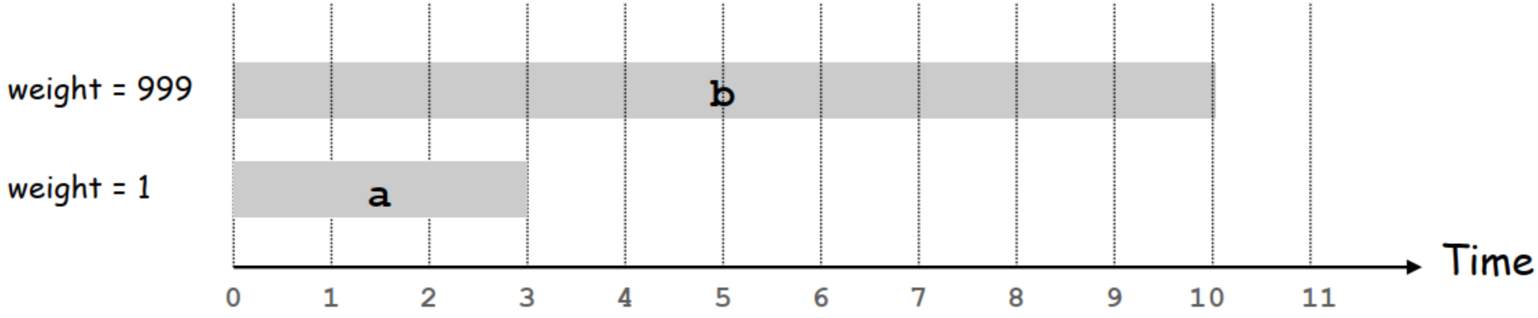
Weighted Interval Scheduling

Given a set of n jobs, each job $j \in n$ starts at s_j and finishes at f_j and has a weight or value v_j. Two jobs are compatible if they don't overlap.

Goal: Find maximum weight subset of mutually compatible jobs

When does greedy work? $v_j = 1, \forall j = 1, 2, \dots, n$

-- When all weights are 1 (or, the same)



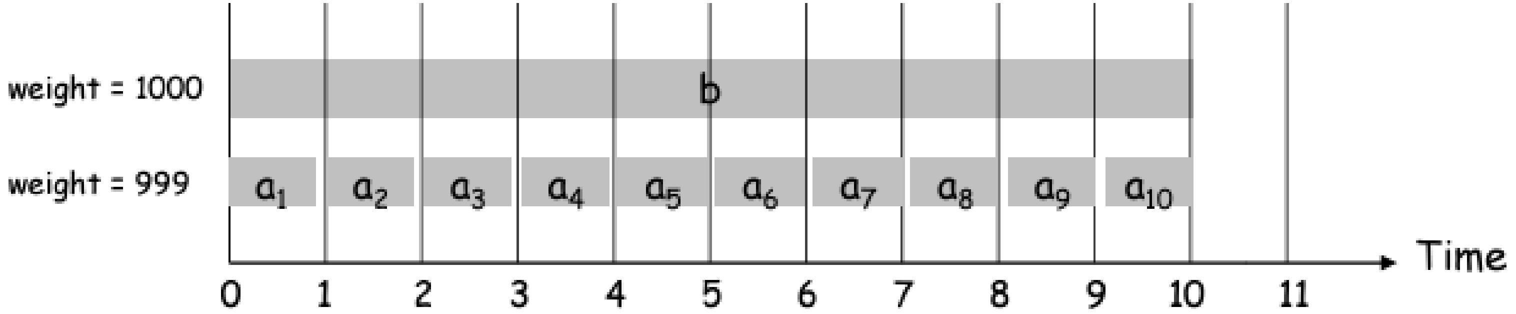
Weighted Interval Scheduling

Given a set of n jobs, each job $j \in n$ starts at s_j and finishes at f_j and has a weight or value v_j. Two jobs are compatible if they don't overlap.

Goal: Find maximum weight subset of mutually compatible jobs

When does greedy work? $v_j = 1, \forall j = 1, 2, \ldots, n$

- When all weights are 1 (or, the same)
- Sort by weight?



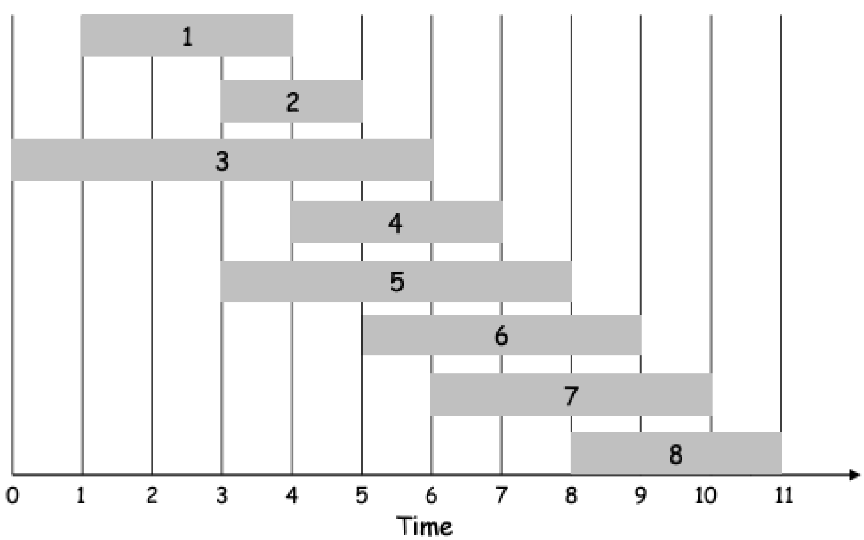
Preprocessing Step (Something new!)

<u>Idea:</u>

Sort by finish times: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Maintain a table $\mathrm{p}(j) = \mathrm{i}$, where i is the largest index $i < j$ s.t. job i and j are m.c.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



j	p(j)
0	-
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

DP Approach

Subproblems: $WIS(j)$= max weight subset of jobs start at 1 and ending at j

Decisions:

- Include j: add profit v_j, include optimal solution to previous set of m.c. jobs, i.e., 1, …, p(j)
- Do not include j: include optimal solution to previous jobs, 1, …, j-1

$$WIS(j) = \begin{Bmatrix} 0, & if\ j = 0 \\ \max\{v_j + WIS\big(p(j)\big), WIS(j-1)\}, & otherwise \end{Bmatrix}$$

Running time:

$O(n \log n)\ to\ sort.\ O(n)\ to\ compute\ WIS(j) \forall j = 1, \dots, n$

P9: Share Trading

We're given the price of a stock over n consecutive days $i = 1, 2, \ldots, n$. For each day i, we're given a price $p(i)$ per share for the stock on that day. (Assume that the price is fixed per day.)

How should we choose a day i on which to buy the stock and a later day $j > i$ on which to sell it so that we maximize the profit per share, $p(j) - p(i)$? If there is no way to make money during the n days, we should conclude this instead.

Share Trading

Subproblem: Let $OPT(j)$ $(j = 1, \ldots, n)$ = max possible return if investor sells share on day j .. We want $OPT(n)$

Decisions: On day j , investor either holding it on day $j - 1$ or weren't.

- If not, $OPT(j) = 0$.
- If yes, then $OPT(j) = OPT(j - 1) + \left(\mathrm{p(j)} - \mathrm{p(j-1)}\right)$.

Recursion:

$$OPT_j = \max\{\, OPT(j) \quad + \left(\mathrm{p(j)} - \mathrm{p(j-1)}\right), 0 \,\}$$

Base case: $OPT(1) = 0$

Running time: O(n) subproblems, O(1) per subproblem, so O(n)

Problems So Far …

Prob #	Definition	Opt Type	Template	Running Time (only DP part)
1	Shortest paths in DAGs	Min	#1	O(|V|+|E|) x O(1)
2	Bellman Ford	Min	#1	O(|V|) x O(|E|)
3	Floyd Warshall	Min	#2	$O(n^2) \times O(n)$
4	Transitive Closure of Graph	Min	#2	$O(n^2) \times O(n)$
5	Rod Cutting	Max	#1	O(n) x O(n)
6	Knapsack w/ repetition	Max	#1	O(W) x O(n)
7	Knapsack w/o repetition	Max	#1	O(W) x O(n)
8	Weighted Interval Scheduling	Max	#1	O(n) x O(1)
9	Share trading	Max	#1	O(n) x O(1)

Lecture 12 summary

- Rod cutting
- Knapsack w/ repetition
- Knapsack w/ repetition
- Weighted Interval Scheduling
- Share trading