Program Structure and Algorithms

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Lecture 11

Agenda

- Administrative
  - HW5 and PA2 will be announced soon..
- Lecture
- Quiz

Algorithm Design Key Ideas

- For SOME problems, the following property enables an efficient solution:
  - **"Principle of Optimality": Any subsolution of an optimal solution is itself an optimal solution**
- For such problems, solution strategies can include:
  - **"Tabulate (cache) Subproblem Solutions":**
    - **Avoid recomputation by creating a table of subproblem solutions**
  - **"Relaxation / Successive Approximation":**
    - **Make multiple passes, each time solving a less restricted version of the original problem**
      - **Eventually, solve completely unrestricted = original problem instance**

Dynamic Programming (DP)

- Powerful technique to solve optimization (and some non-optimization) problems in polynomial time

- Basic idea is to somehow break a problem down into a polynomial number of subproblems

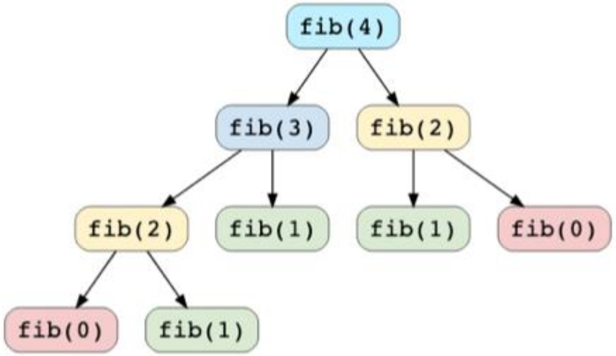
- Solutions to each of these subproblems can be used to construct the optimal solution

- Ordering is important (bring in induction mindset)
  - Ordering from the first index or an interval?

DP is a Promising Approach if…

- **Optimal substructure**: A problem exhibits optimal substructure if an optimal solution to the problem contains *within* it optimal solutions to subproblems
  - True for DP and greedy

- **Overlapping subproblems**: A recursive algorithm for the problem would end up solving the same subproblems over and over, rather than always generating new subproblems



DQ vs. DP

- In both cases, we start by formulating the problem recursively, in terms of subproblems

DQ

- Problem of size n can be decomposed into a few subproblems that are significantly smaller (e.g., n/2, 3n/4, etc.)
- Size of subproblems decreases geometrically
- Use a recursive algorithm

DP

- Problem of size n can be expressed in terms of subproblems that are not much smaller (e.g., n-1, n-2, etc.)
  - Recursive algorithm takes exponential time
  - But only polynomially many subproblems in total
- Avoid recursion, and solve subproblems one by one, saving answers in a table

DP Problem Examples

- Rod cutting to maximize revenue
- Longest increasing subsequence
- Minimum edit distance
- Knapsack
- Matrix chain product
- All pairs shortest path (Floyd-Warshall)
- Independent sets in trees
- Vertex cover in trees
- Longest common subsequence
- Transitive closure
- Change making
- ….

DP: Common Subproblems

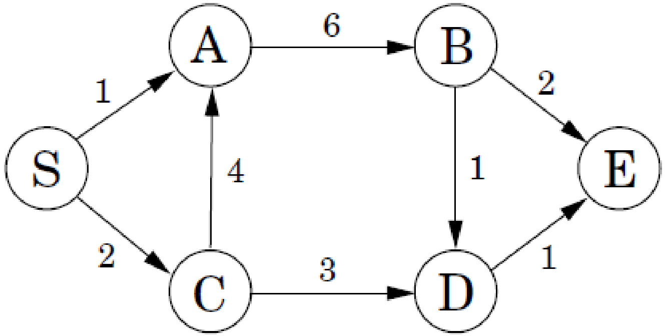
Finding the right subproblem takes creativity and experimentation. Some standard choices below

- Template #1: Input is $x_1, x_2, \ldots, x_n$, ($or, x[1{:}n]$) and a subproblem is $x_1, x_2, \ldots, x_i$ (or, $x[1{:}i]$)
  - Number of subproblems is linear
- Template #2: Input is $x_1, x_2, \ldots, x_n$, and a subproblem is $x_i, x_{i+1}, \ldots, x_j$ (or, $x[i{:}j]$)
  - #subproblems is quadratic $O(n^2)$
- Template #3: Inputs are $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_m$; a subproblem is $x_1, x_2, \ldots, x_i$ and $y_1, y_2, \ldots, y_j$
  - #subproblems is quadratic $O(mn)$

DP Steps

- State subproblem (must include optimization intent)
- Articulate decisions and the cost/value for each decision
  - Focus on how you'd solve ONE subproblem
- Recursion to solve all subproblems
  - Left to right, or right to left
- Identify the number of subproblems, n_{sub}
- Identify the running time per subproblem, t_{sub}
- DP running time: $n_{sub} \times t_{sub}$

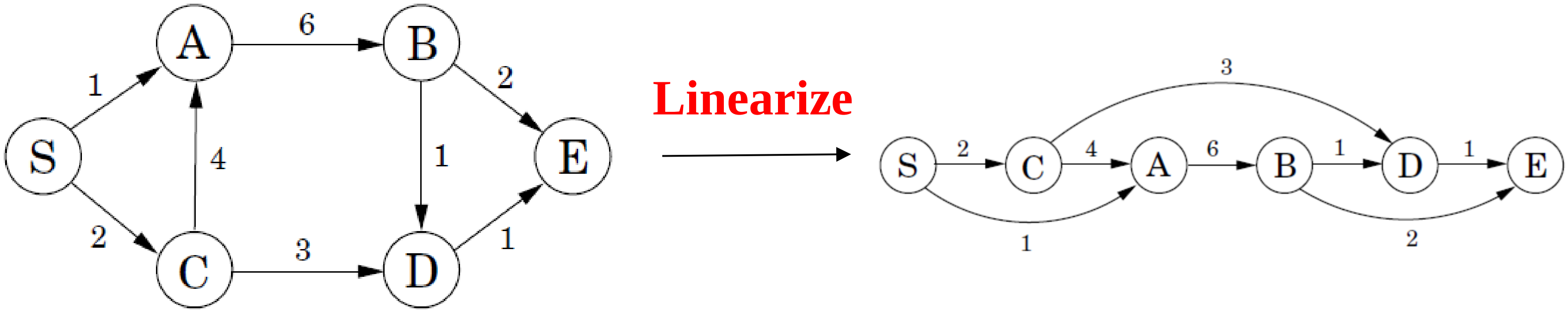
P1: Shortest Paths in DAGs



Given DAG G = (V; E), what is the shortest path from S to any node in G?

E.g., D

P1: Shortest Paths in DAGs



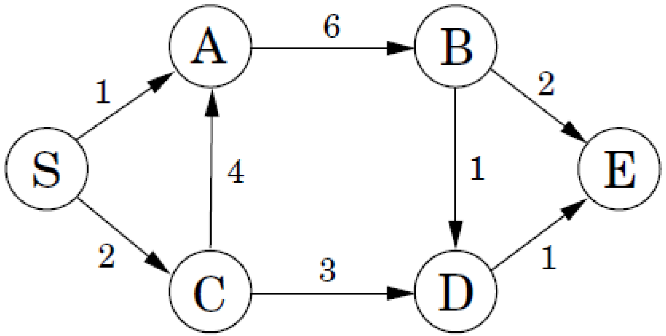
Given DAG G = (V; E), what is the shortest path from S to any node?

E.g., D

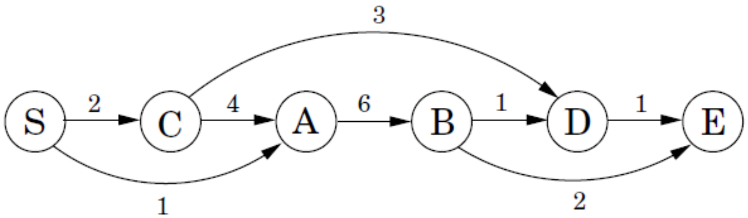
Intuition: What if we knew the shortest paths from S to the parents of D?

$$dist(D) = \min\{dist(B) + 1, dist(C) + 3\}$$

P1: Shortest Paths in DAGs



Linearize



Intuition: What if we knew the shortest paths from S to the parents of D? $dist(D) = \min\{dist(B) + 1, dist(C) + 3\}$

Does it work for all nodes?

dist(S) = 0

dist(C) = dist(S) + 2 = 2

dist(A) = min{dist(C) + 4, dist(S)+1} = min(2+4, 0+1) = 1 //2 decisions

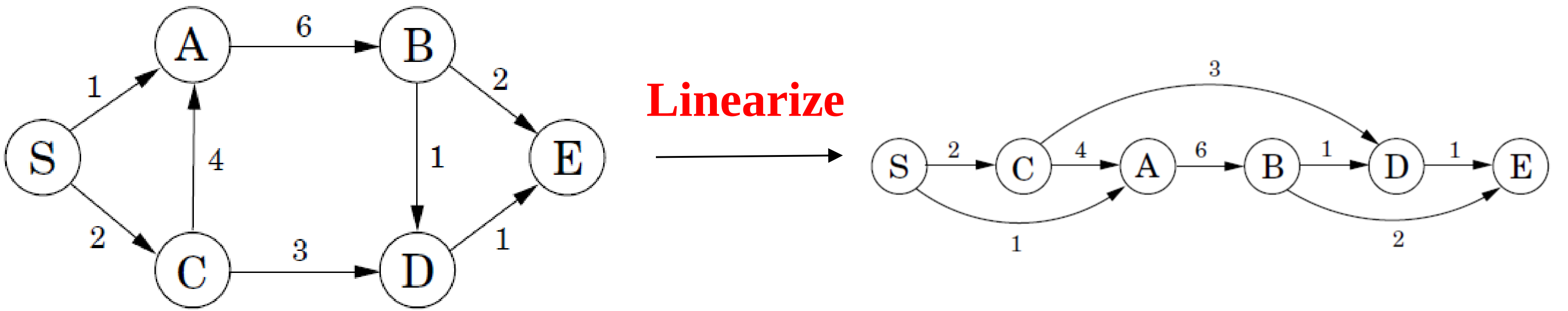
dist(B) = dist(A) + 6 = 1 + 6 = 7

dist(D) = min{dist(B) + 1, dist(C) + 3} = min(7+1, 2+3) = 5 // 2 decisions

dist(E) = min{dist(B) + 2, dist(D) + 1} = min(7+2, 5+1) = 6 // 2 decisions

Without linearization/topo sort, this idea will NOT work!!!

Shortest Paths in DAGs

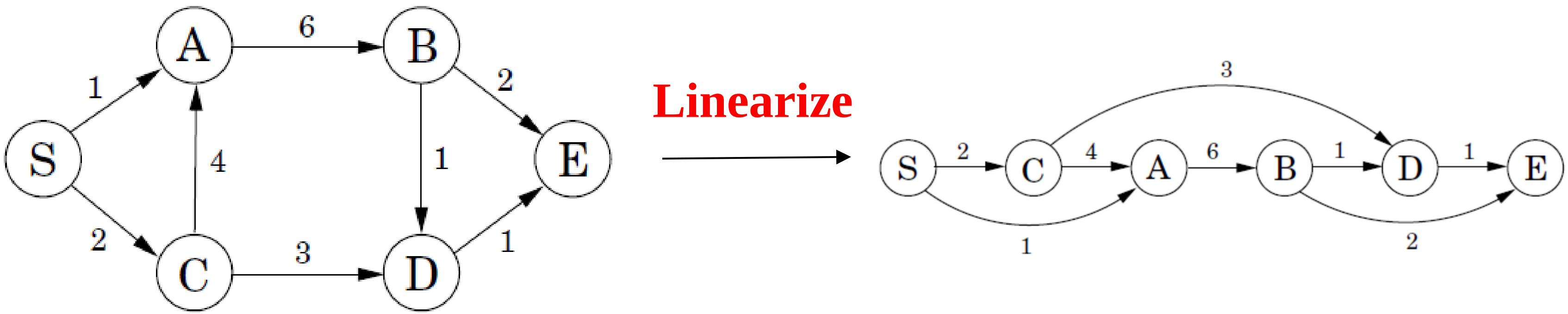


Given DAG G = (V; E), what is the shortest path from S to any node?

For any DP, we need to

- Identify and articulate subproblems
- Articulate the decisions
- Describe the recursion
- Analyze running time: #subproblems x time to solve each subproblem

P1: Shortest Paths in DAGs



Given DAG G = (V; E), what is the shortest path from S to any node?

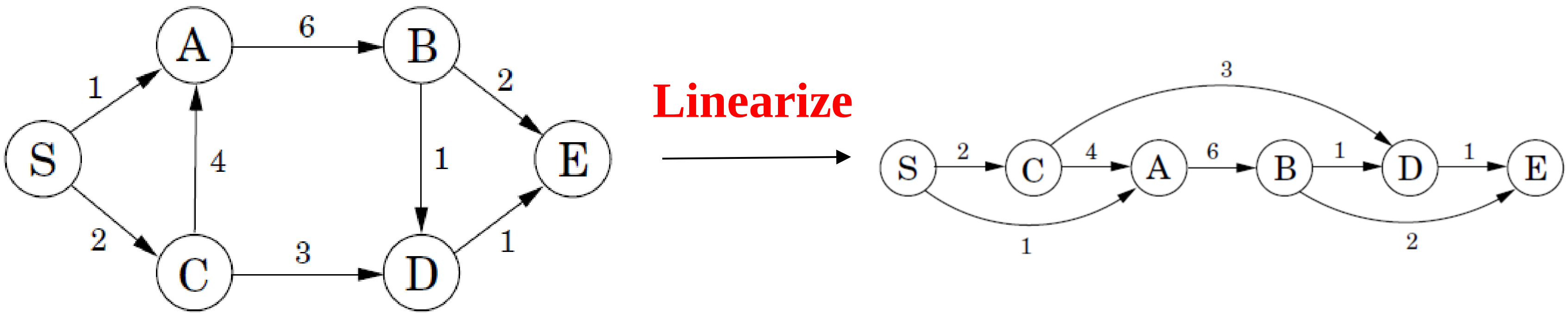
Subproblem: Let dist(v) denote the shortest path to any node v in G from S

- When we linearize, we get an order of vertices x, if src = 1, v = i are indices of the source and v
- We can say subproblems are to look at the min distance x[1:i], i.e., starting from the source and ending at vertex index i

Decisions: Consider dist(u) to all parents of v and pick the min of dist(u) + l(u,v)

Recursion: dist(v) = min{dist(u) + l(u,v) | for all (u,v) in linearized G}

P1: Shortest Paths in DAGs



Given DAG G = (V; E), what is the shortest path from S to any node?

$$\texttt{initialize all dist}(\cdot)\ \texttt{values to}\ \infty$$
$$\texttt{dist}(s) = 0$$
$$\texttt{for each}\ v \in V \backslash \{s\},\ \texttt{in linearized order:}$$
$$\quad \texttt{dist}(v) = \min_{(u,v)\in E}\{\texttt{dist}(u) + l(u,v)\}$$

How many subproblems? O(|V|+|E|)

Running time per subproblem? O(1) (one min operation)

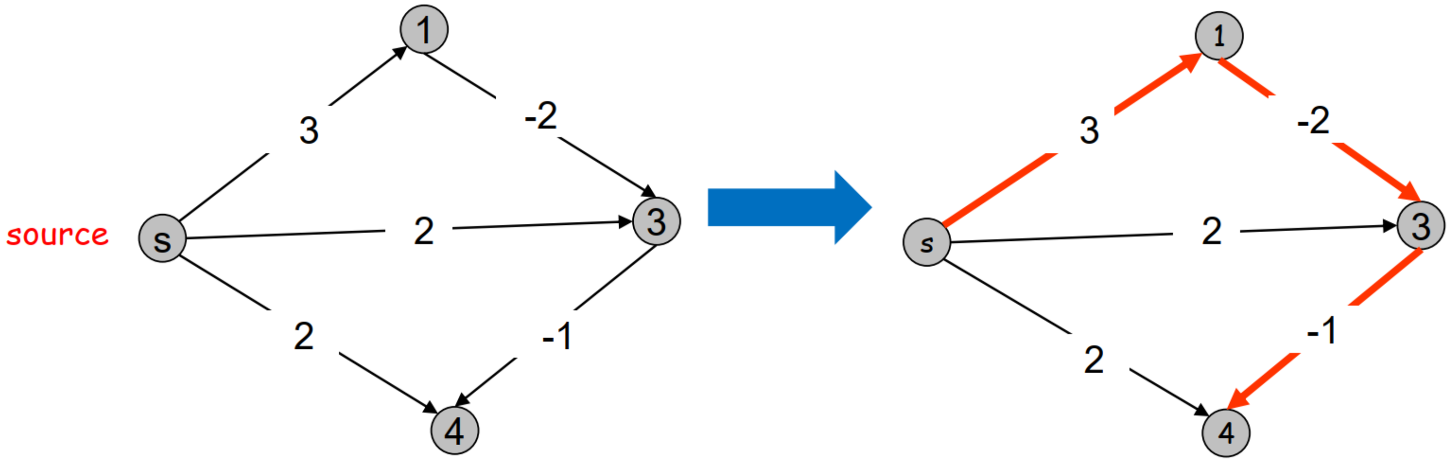
Total Running time: O(|V|+|E|)

P2: Shortest Paths w/ Negative Edge Weights

(aka, Bellman-Ford)

Given $G = (V, E)$, a source vertex $s \in V$ and real-valued edge weights for each edge in E as $c_{u,v}$, find the shortest path from s to all other vertices in G

Why does Dijkstra fail?



Shortest Paths w/ Negative Edge Weights

We cannot have negative cycles in G

Let $d(v)$ be the length of the shortest $s - v$ path

$$d(v) = \begin{cases} 0 & if\ v = s \\ \min_{u:(u.v)\in E} d(u) + c_{u,v} \end{cases}$$

No ordering, so how do we compute?

Shortest Paths w/ Negative Edge Weights

Subproblem: Let $d(v, i)$ be the length of the shortest $s - v$ path with at most i edges

Intuition: Solve all intermediate routes from s to v

Decisions:

Case 1: i$^{\text{th}}$ edge is not used, then $d(v, i) = d(v, i - 1)$

Case 2: i$^{\text{th}}$ edge is used

Let $s, v_1, v_2, \ldots, v_{i-1}$ be the $d(v, i)$ path with i edges

Then, $s, v_1, v_2, \ldots, v_{i-1}$ MUST be the shortest $s - v_{i-1}$ path with at most $(i - 1)$ edges

That is, $d(v, i) = d(v_{i-1}, i - 1) + c_{v_{i-1},v}$

Shortest Paths w/ Negative Edge Weights

Let $d(v,i)$ be the length of the shortest $s - v$ path with at most i edges

Recursion:

$$d(v,i) = \begin{cases} 0 & if\ v = s \\ \infty & if\ v \neq s, i = 0 \\ \min(d(v,i-1), \min\limits_{u:(u.v)\in E} d(u,i-1) + c_{u,v} \end{cases}$$

This is Bellman-Ford algorithm

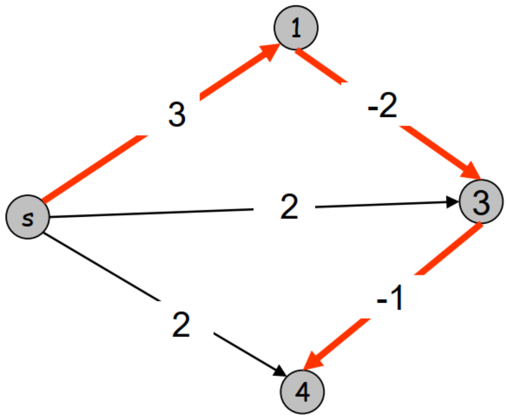
Bellman Ford

```
for v=1 to n
      if v≠s then
            M[v,0]=∞
      M[s,0]=0

for i=1 to n-1
      for v=1 to n
            M[v,i]= M[v,i-1]
            for every edge (u,v)
                  M[v,i]=min(M[v,i], M[u,i-1]+c[u,v])
```

Running time: $O(|V||E|)$

Working of Bellman Ford



P3: All Pairs Shortest Path

Given directed graph $G = (V, E)$ and real-valued edge weights, create $n \times n$ matrix of shortest distances $\delta(u, v)$ between all pairs of vertices; $|V| = n$

Adjacency matrix representation of G

- $n \times n$ matrix $W = \left(w_{ij}\right)$ of edge weights (can be –ve)
- $w_{ii} = 0\ \forall i$, i.e., SP to self has no edges, as long as there are no negative cycles

Brute-Force Approach

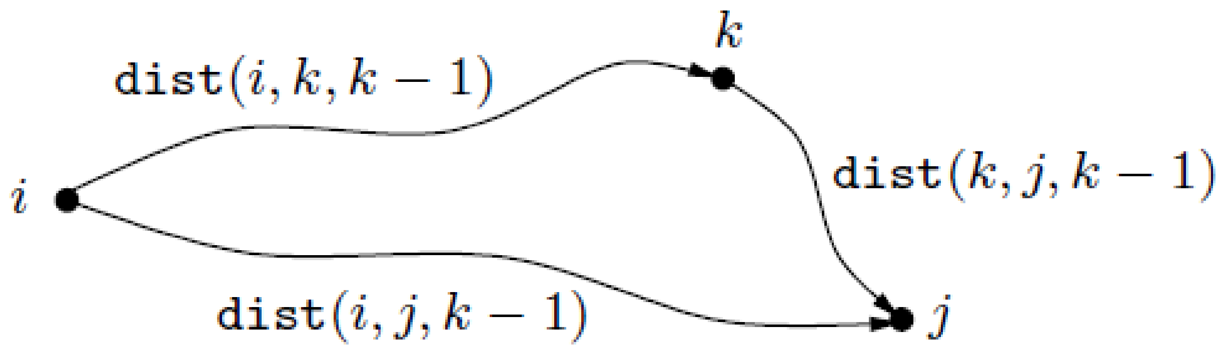
- Run Bellman-Ford once from each vertex (since edge weights may be negative)
- $O(|V||E|) \times O(|V|) = O(|V|^4)$ for dense graphs

DP: Structure of an optimal solution

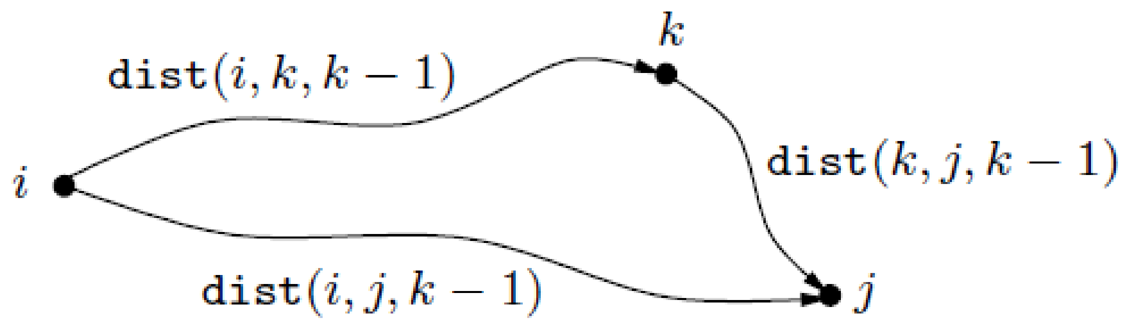
"Relaxation" idea: Similar to Bellman Ford

$Shortest\ dist\ between\ vertices\ i\ and\ j, \delta(i,j)\ with \leq m\ edges$ (**relax** $0 \leq m \leq n-1$)

- Equivalent to saying shortest path between i and j (or, SP $\delta(i,j)$) uses some of $\{1, 2, \ldots, k\}$ intermediate nodes
- That is, $i \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_k \rightarrow j$



DP: Structure of an optimal solution



Subproblems: Let $dist(i, j, k)$ be the cost of the shortest path $i \rightarrow j$ with intermediate vertices in the set $\{1, 2, \ldots, k\}$

Intuition: In BF, s was given. Here, the source can be any vertex, so need a variable for it.

Decisions

- Either w_k is an intermediate vertex in SP $\delta(i, j) \rightarrow \delta(i, j) = \delta(i, w_k) + \delta(w_k, v)$
- Or, w_k is not an intermediate vertex, so consider $\delta(i, j)$ only involving $\{w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_{k-1}\}$

DP: Structure of an optimal solution

Decisions

- Either w_k is an intermediate vertex in SP $\delta(i, j) \Rightarrow \delta(i, j) = \delta(i, w_k) + \delta(w_k, v)$
- Or, w_k is not an intermediate vertex, so consider $\delta(i, j)$ only involving $\{w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_{k-1}\}$

Case 1: If $i = j, d_{ij}^0 = 0$ (base case #1)

Case 2: If $i \neq j$, $dist(i, j, k)$ is the $\delta(i, j)$ in which only nodes $\{1, 2, \dots, k\}$ can be used as intermediates.

Case 3: $dist(i, j, 0)$ is the length of direct edge between i and j, i.e., $dist(i, j, 0) = w_{ij}$ (base case #2)

DP: A Recursive Solution

Let $dist(i, j, k)$ be the cost of the shortest path $i \rightarrow j$ with intermediate vertices in the set $\{1, 2, \dots, k\}$

Try for all k and pick the min cost one.

Recursion:

$$dist(i, j, k) = min \begin{cases} w_{ij} & if\ k = 0 \\ dist(i, j, k-1), & if\ k\ is\ not\ an\ intermediate\ node \\ dist(i, k, k-1) + dist(k, j, k-1), & otherwise \end{cases}$$

This is Floyd-Warshall algorithm!

DP: Pseudocode (Bottom-up)

$\texttt{for } i = 1 \texttt{ to } n\texttt{:}$
$\quad \texttt{for } j = 1 \texttt{ to } n\texttt{:}$
$\quad\quad \texttt{dist}(i, j, 0) = \infty$
$\texttt{for all } (i, j) \in E\texttt{:}$
$\quad \texttt{dist}(i, j, 0) = \ell(i, j)$
$\texttt{for } k = 1 \texttt{ to } n\texttt{:}$
$\quad \texttt{for } i = 1 \texttt{ to } n\texttt{:}$
$\quad\quad \texttt{for } j = 1 \texttt{ to } n\texttt{:}$
$\quad\quad\quad \texttt{dist}(i, j, k) = \min\{\texttt{dist}(i, k, k-1) + \texttt{dist}(k, j, k-1),\ \texttt{dist}(i, j, k-1)\}$

Runtime: $\Theta(n^3)$ due to
n^2 *subproblems, each takes* $\Theta(n)$ *to solve*

DP: Pseudocode (Top-down)

FW-RECUR($V, i, j, k, w, memo$)

if $memo[i, j, k]$
  return $memo[i, j, k]$
if $k = 0$
  return $w(v_i, v_j)$
$memo[i, j, k] \leftarrow$
$\min [\text{FW-RECUR}(V, i, j, k-1, w), \text{FW-RECUR}(V, i, k, k-1, w) + \text{FW-RECUR}(V, k, j, k-1, w)]$
return $memo[i, j, k]$

FLOYD-WARSHALL(V, w)

$memo \leftarrow$ Empty memo
for $v_i \in V$
  for $v_j \in V$
    $\delta(v_i, v_j) \leftarrow \text{FW-RECUR}(V, i, j, |V| - 1, w, memo)$
return δ

Similar: Transitive Closure of Graph

Transitive closure of G = (V,E) is a graph G* = (V, E*)

$(i, j) \in E^* \; iff \; \exists \; path \; i \rightarrow j \; in \; G$

Input: Adjacency matrix elements with elements in {0, 1}

Run Floyd-Warshall with

- "min" → "OR"
- "+" → "AND"

Runtime: $\Theta(n^3)$

Applications

- Dependencies in SW modules
- Reachability in social / transportation networks, node importance
- Speedup database queries by clustering related data
- Feature learning in ML

Summary of DP Problems

Prob #	Definition	Opt Type	Template	Running Time (only DP part)
1	Shortest paths in DAGs	Min	#1	O(|V|+|E|) x O(1)
2	Bellman Ford	Min	#1	O(|V|) x O(|E|)
3	Floyd Warshall	Min	#2	O(n^2) x O(n)
4	Transitive Closure	Min	#2	O(n^2) x O(n)

Lecture 11 summary

- DP

- P1: Shortest path in a DAG
- P2: SSSP in a graph (Bellman Ford)
- P3: APSP
- P4: Transitive closure

- Please practice writing code given the pseudocodes