集合与函数的基础知识

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1 集合

2 映射

- X, Y是集合,$R \subset X \times Y$是X与Y间的关系.$(x, y) \in R$可记作xRy.
  - $\mathrm{Dom}(R) = \{x \in X : \exists y \in Y : xRy\}, \mathrm{Ran}(R) = \{y \in Y : \exists x \in X : xRy\}$.
  - $A \subset X$,R在A上的限制$R|_A = A \times Y \cap R$.
  - $R^{-1} \subset Y \times X, xRy \iff yR^{-1}x$.
  - $R(A) = \{y \in Y : \exists x \in A : xRy\} = \mathrm{Ran}(R|_A), R^{-1}(B) = \{x \in X : \exists y \in B : xRy\}$.
  - 关系的复合:$R_1 \subset X \times Y, R_2 \subset Y \times Z, R_2 \circ R_1 \subset X \times Z. x(R_2 \circ R_1)z \iff (\exists y \in Y : xR_1y \wedge yR_2z)$
- 特殊的关系
  - 等价关系$\sim \subset X^2$:①自反性 $x \sim x$;②传递性 $x \sim y \wedge y \sim z \implies x \sim z$;③对称性 $x \sim y \iff y \sim x$.
  - 偏序关系$\leq \subset X^2$:①自反性;②传递性;③反对称性 $x \leq y \wedge y \leq x \iff x = y$.
- 一个映射$f : X \to Y$是指一个X到Y的关系f,满足$\forall x \in X \exists! y \in Y : xfy$,记$y = f(x)$.
  - $B \subset Y, f^{-1}(B) := \{x \in X : f(x) \in B\}$.
- $f : X \to Y$是单射 $\iff \forall y \in \mathrm{Im} f \exists! x : y = f(x) \iff f^{-1}(y)$是单点集.
  $f : X \to Y$是满射 $\iff \forall y \in Y \exists x \in X : y = f(x) \iff f(X) = Y \iff f^{-1}(Y) = X$.
  f是双射 $\iff f$是单射且是满射.
- $f : X \to Y$和$g : Y \to X$ $\begin{matrix}\text{单}\\\text{满}\end{matrix} \implies g \circ f \begin{matrix}\text{单}\\\text{满}\end{matrix} \implies \begin{matrix}f\text{单}\\g\text{满}\end{matrix} \iff \exists \begin{matrix}g : Y \to X\\f : X \to Y\end{matrix} : g \circ f = \mathrm{id}_X$,但仅当$f \circ g = \mathrm{id}_Y$时两者互逆.
- f^{-1}不一定是映射,但$\{f^{-1}(y) : y \in Y\}$是X上分划.
- f单时$f^{-1}|_{\mathrm{Im} f}$是映射.特别的,f是双射时存在唯一的逆f^{-1}作为映射.但f的左右逆不一定唯一.

以下设$f : X \to Y, A \subset X, B \subset Y$.蓝色符号表示$f$单时取等,红色符号表示$f$满时去等.

1. $A_1 \subset A_2 \implies f(A_1) \subset f(A_2), B_1 \subset B_2 \implies f^{-1}(B_1) \subset f^{-1}(B_2)$.
2. $f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i), \bigcup_{i \in I} f(A_i) = f\left(\bigcup_{i \in I} A_i\right). f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i), \bigcup_{i \in I} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i \in I} B_i\right)$.
3. $f(A_1 - A_2) \supset f(A_1) - f(A_2), f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.
4. $f^{-1}(f(A)) \supset A, f(f^{-1}(B)) \subset B, f(f^{-1}(B) \cap A) = B \cap f(A)$.

3 拓扑学

红色符号表示I有限时取等.

$$\overline{\bigcap_{i \in I} E_i} \subset \bigcap_{i \in I} \overline{E_i}, \bigcup_{i \in I} \overline{E_i} \subset \overline{\bigcup_{i \in I} E_i}, \qquad \left(\bigcap_{i \in I} E_i\right)^\circ \subset \bigcap_{i \in I} E_i^\circ, \bigcup_{i \in I} E_i^\circ \subset \left(\bigcup_{i \in I} E_i\right)^\circ$$