A Brief Look at the Mathematical Models of Long-run Economics Growth

Xiaoteng Zhang

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1 Introduction

Economic growth is a very important concept in macroeconomics. It can be defined as the increase or improvement in the inflation-adjusted market value of the goods and services produced by an economy (also known as *GDP*) over a certain period of time. Since high economic growth would generally make a nation more prosperous,[1], which in turn would generally improve its citizens' standards of living. It is for this reason that in the modern history, increasing the rate of economic growth is almost the universal preoccupation of almost every country's government regardless of the form of governance.

However, the rate of economic growth can vary quite widely from one year to the other, which poses a problem for government planning. Therefore, if we can predict the rate of economic growth, it would be undoubtedly helpful for the government in terms of planning its government expenditure, taxation, etc. Currently, one of the most classic models that is used to predict long-run economic growth is the **Solow–Swan model** or **exogenous growth model.** Mathematically, this is an nonlinear system consisting of a single ordinary differential equation that models the evolution of the per capita stock of capital. In the following sections, I will firstly introduce the model before going on to explore the nonlinear dynamics of the system for two different versions of the model (with increasing complexity)[2] as well as trying to compare the model to some real-world data that I was able to collect online.

2 The Solow-Swan model with AK technology and constant labour growth rate

It is worth mentioning before we go any further that we are measuring the rate of economic growth by rate of capital accumulation here. It is important to note that these two concepts are not the same yet they are still deeply connected in the sense that the accumulation of capital goods translates to investment and the production of more goods and services, which should boost the income of the population and stimulate demand. (In the other words, economic growth)

This version of the model is arguably the most basic form of the model, but it will serve as a basis

[1] When we say high economic growth here, we are referring strictly to the amount of growth within an economy's capacity. When the economic growth is too rapid, the economy will start to *"overheat"*, which will cause high rate of inflation, among many other undesirable effects

[2] Here, it is important to keep in mind that the versions of model explored would not be exhaustive

for our investigation going forward. This version of the model is set up based on the following assumptions/conditions:

1. Tbere is one good (Y_t) produced by only one factor of production (K_t). Therefore the resulting production function is $Y_t = F(K_t) = AK_t$, where A is an exogenous positive constant that represents the level of technology.

2. Saving is proportional to the aggregate income i.e. $S_t = sY_t$ where s denotes the exogeneous saving rate.

3. Since the economy is closed, the change in the capital stock could be represented as [3] $K_t\prime = sY_t - \delta K_t$, where δ is the rate of depreciation

4. In the context of the model, we consider population = labour. That is to say that $n = \frac{L\prime_t}{L_t}$, the constant population growth rate would equate to labour growth rate

Since we are more interested in per-capita terms, we define a new variable called the **capital-labour ratio** $k_t = \frac{K_t}{L_t}$. Then we have $\frac{k_t\prime}{k_t} = \frac{K_t\prime}{K_t} - \frac{L_t\prime}{L_t}$. Then by some simple calculations that I would omit here, we get:

$$k_t\prime = (N - n)k_t, N \equiv sA - \delta$$

This first order differential equation in k_t is the fundamental equation of the Solow-Swan model with a production function which is linear in physical capital. From this equation, we can plot the evolution of the capital of stock with respect to time as shown in Figure 1 when the quantity $(N - n)$ takes different signs. We can notice that:

- For $N = n$, regardless the initial condition, steady state has **zero growth** in per-worker capital stock

- For $N < n$, we notice negative growth throughout, and the economy will converge in the long-run to a per-worker capital stock equal to zero

- For $N > n$,we notice positive growth throughout, and the economy will approach an infinitely large level of per-worker capital stock in the long run

Obviously, the third senario will be the most favourable for an economy. When we translate the condition $N > n$ into plain english, it will state that the savings rate (adjusted by the level of technology,sA) minus the depreciation rate of physical capital (δ) should be greater than the constant population growth rate (n).

3 The Solow-Swan model with AK technology and logistic labour growth rate

In this section, we are going to be modifying the model for population growth to complicate things a little bit. The problem that we had with constant labour growth rate before is that population would grow exponentially without any constraint. Namely, the solution to $n = \frac{L\prime_t}{L_t}$ is $L_t = L_0 * e^{nt}$. Obviously, this

[3] $K_t\prime$ here is the same as $\dot{K}_t$, representing differentiation with respect to time

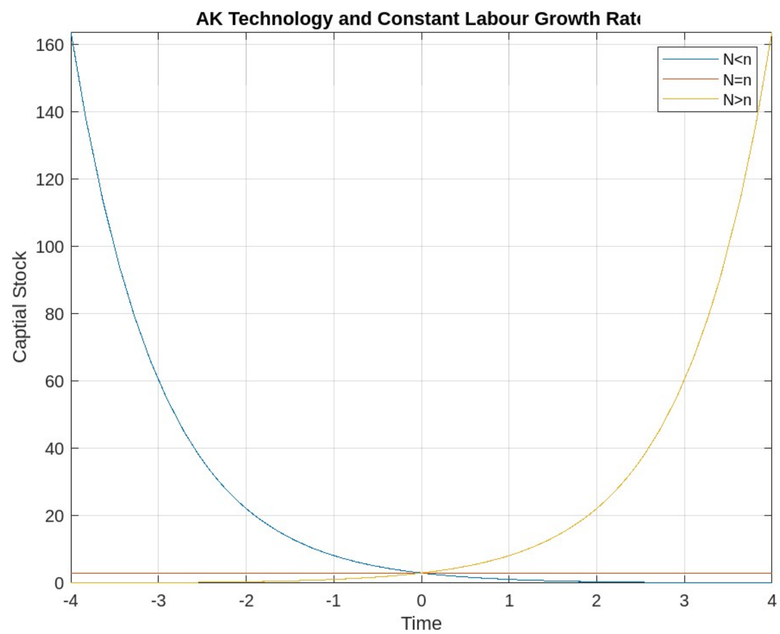


Figure 1: Evolution of Capital for AK Technology and Constant Labour Growth

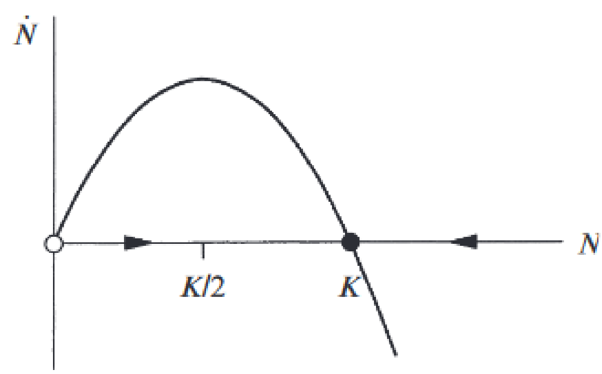


Figure 2: Dynamics of Logistic Population Growth

model of population growth would not be realistic, which is the reason why we are introducing the logistic equation for labour/population growth:[4]

$$\frac{L_t\prime}{L_t} = n - bL_t$$

where n is population growth rate as usual, b is a constant satisfying $n - bL_t > 0$. This equation can also be written as $\frac{L_t\prime}{L_t} = n(1 - \frac{L_t}{\mathbf{K}})$ where **K** denotes **carrying capacity**. (i.e., the maximum population size of a biological species that can be sustained by that specific environment.) Instead of solving this differential equation analytically, we plot $L_t\prime$ against L_t like we have done many times as shown in Figure 2. This figure shows that no matter where the population starts from, the population converges eventually to the carry capacity as t goes to infinity. (In the other words, K in the graph is a stable fixed point.) Therefore, incorporating the symbols and equations that we have introduced a bit earlier, we get the following system

[4]This population growth model is discussed in depth at Page 22 in Strogatz. To save the repetitiveness, I will just summarize the results here.

of equation to represent the AK technology model with logistic labour growth rate:

$$\begin{cases} k_t\prime &= [N-(n-bL_t)]\,k_t \\ L_t\prime &= L_t(n-bL_t) \end{cases} (*)$$

where if we are given the initial condition $(k_0 > 0, L_0 > 0)$ we can get a unique solution (k_t, L_t). In the next section, we are going to investigate the steady state (see the definition a bit later) and some of the dynamics of the model after we made the switch to the logistic labour growth.

4 Steady state and dynamics of model with logistic labour growth rate

We start off by defining what is the steady state of the model. A steady state in an economy refers to the situation where the growth of population/labour and per capita physical capital are both zero. After reading this definition, it is natural to make a connection with something we learned early on in class. And yes indeed, steady state in this context is just a fancier name for fixed point of two-dimension system. [5]We denote the steady state values of this system as $k_*$and L_*. From the first glance, it is evident that $(k_t, L_t) = (0, 0)$ is always a fixed point. However, in this context, we will **disregard** this solution as this is economically meaningless. After all, who would like to study the dynamics of an economy that has 0 workers in it ? We would probably be better off spending our time elsewhere. After excluding this not-so-exciting solution, we have another pair of solution that satisfies:

$$\begin{cases} Nk_t &= 0 \\ n-bL_t &= 0 \end{cases}$$

Solving this system of the equation, we see that if $N \neq 0$, we wouldn't obtain any non-trivial steady state as the only way to satisfy the first equation is to make $k_t = 0$. On the other hand, we see that if $N = 0$, k_t can take on any value to satisfy the first equation and along with $L_t = \frac{n}{b}$, a non-trivial fixed point could be obtained. To summarize, the economy has **no** steady state when $N \neq 0$ and has **infinite** stead state when $N = 0$. Now, the legitimate question that we can ask ourselves is that what kind of fixed point do we have when $N \neq 0$. To find out, we employ linearization:

$$\begin{bmatrix} \frac{dk_t'}{dk_t} & \frac{dk_t'}{dL_t} \\ \frac{dL_t'}{dk_t} & \frac{dL_t'}{dL_t} \end{bmatrix} = \begin{bmatrix} N-n+bL_t & bk_t \\ 0 & n-2bL_t \end{bmatrix} = \begin{bmatrix} N & bk_t \\ 0 & -n \end{bmatrix}$$

To look for the eigenvalue, set the determinant to zero at the fixed point $(k_t, L_t = \frac{n}{b})$:

$$det \begin{bmatrix} N-\lambda & bk_t \\ 0 & -n-\lambda \end{bmatrix} = 0$$

which gives:

$$(N-\lambda)(-n-\lambda) = 0; (N-\lambda)(n+\lambda) = 0$$

Then, we have two eigenvalues $\lambda_1 = N$ and $\lambda_2 = -n$. In the other word, depending on the signs of N and

[5]For more detailed discussion on this, please refer to Section 6.3 in Strogatz.

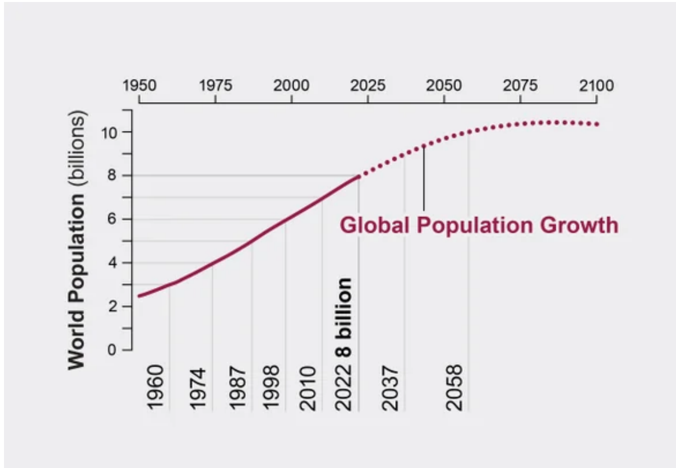


Figure 3: Population growth trend

$-n$, we will have either saddle, unstable, stable node locally around the steady state (fixed point). These are all hyperbolic points, so according to the *Hartman-Grobman Theory* [6], our use of linearization to determine the stability of steady state is justified.

Remark: Even though theoretically we can take any value for the pair (N, n) as long as $N \neq 0$, we would typically want to take both as positive in reality. If one recalls the definition from an earlier section, the variable n represents population/labour growth rate, which has the range $0 \leq \frac{L_t'}{L_t} \leq n - bL_t$. This range confirms what we are observing in the real world.[7] With this observation, this fixed point would be either a stable node or saddle point.

From system $(*)$ from the last section, we can solve for the explicit expression of k_t:

$$k_t = k_0 e^{\int_0^t (N - \frac{L_t'}{L_t})dt} = \frac{k_0 L_0 e^{Nt}}{L_t} (**)$$

However, from the equation we got under the expoential population growth rate assumption:

$$k_t\prime = (N - n)k_t$$

We can also solve k_t to have the same expression as $(**)$. That is an interesting observation, meaning that even under our new population growth assumption, the path of per-worker capital is still the same. In particular, given the positive initial condition $k_0 > 0; L_0 > 0$, we have k_t negatively on the population/labour size L_t at time t. [8] Figure 3 does a good job illustrating the time path of the population growth under logistic model. We finish this section can make some simple observations on the relationship between the economic growth $(\frac{k_t'}{k_t})$ and net adjusted saving rate (N). For simplicity define $\frac{k_t'}{k_t} \equiv \gamma_{k_t}$ and $\frac{L_t'}{L_t} \equiv \gamma_{L_t}$. Note that we have the following from the first equation of the system(*):

$$k_t\prime = [N - (n - bL_t)]\, k_t = [N - \frac{L_t'}{L_t}]k_t = [N - \gamma_{L_t}]k_t$$

[6]For more detailed discussion on this, please refer to page 156 in Strogatz.

[7]For more intuitive discussion on this claim, please read this excellent article by the scientific american.

[8]That conclusion is fairly intuitive when we draw an analogy with dissolving some fixed amount of particle in increasing larger quantity of solutions (dilution).

Rearranging terms to get:

$$\gamma_{k_t} = N - \gamma_{L_t}$$

With this we have the following:

- If $N = 0$, $\gamma_{k_t} = -\gamma_{L_t} < 0$. The last inequality sign is from the range that is given earlier in the section. Therefore, if $N = 0$, we will have negative economic growth. However, as $t \to \infty$, economic growth would tend to zero.[9]

- If $N < 0$, $\gamma_{k_t} = N - \gamma_{L_t} < 0$. Therefore, if $N < 0$, we will also have negative economic growth. However, as $t \to \infty$, economic growth would stay negative.

- If $N \geq n - bL_0 > 0$, $\gamma_{k_t} \geq 0$ for each $t \geq 0$. Therefore, in this situation, we will have positive economic growth.

5 Solow growth model in real life

With all these above being said, one might want very naturally that whether the economic growth in the real life can be predicted to some degree of accuracy by the basic Solow model described above. That is precisely what we are going to investigate in this section. From the expression ($**$) that we obtained in the previous section, we see that the value k_t will increase as the value of N_t increases. However, as $N_t \equiv sA - \delta$, we can derive that the value k_t increases as saving's rate (s) increases. In the same expression, we also see that as the quantity L_t increases, the value k_t would decrease. Equivalently, we can say that the value k_t decreases as the population growth rate n increases. Recall that at the very beginning, we assumed that we are using so-called **AK Technology production function** (i.e. $Y_t = F(K_t) = AK_t$). Since Y_t in this equation is proportional to K_t, Y_t would also have the same relationship with respect to saving's rate (s) and population growth rate (n). In summary:

- Income Y_t (therefore income per capita) is **increasing** in the saving's rate

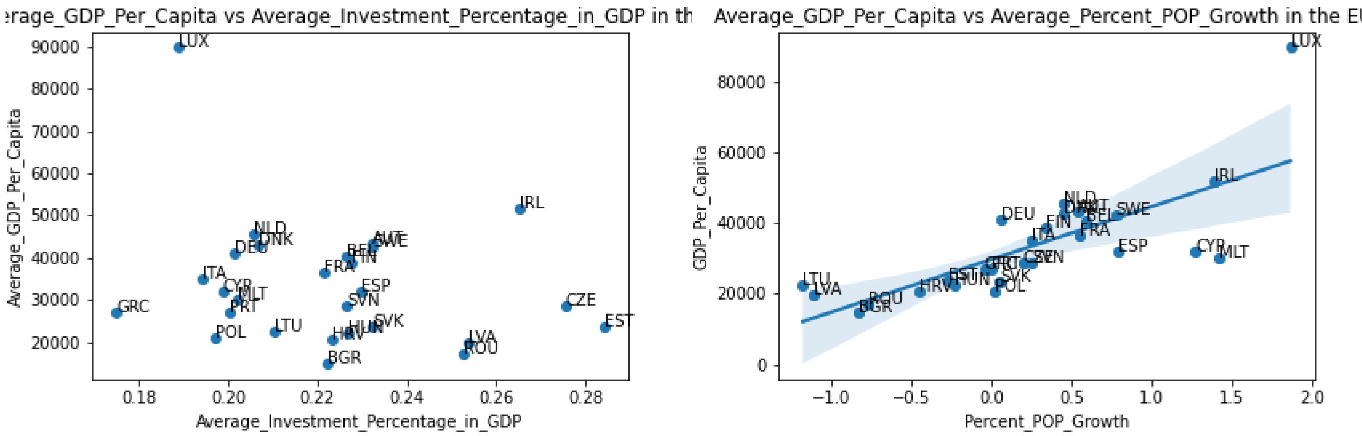
- Income Y_t (therefore income per capita) is **decreasing** in the population growth rate

The above two relationships given by the Solow model are what we are going to test on the real-life dataset. Before jumping into the testing, there is one remark to be made regarding to the saving's rate (s): due to the difficulties that I encountered in getting saving's rate (s) statistics for different countries online, this parameter would be replaced by the percentage of GDP a country spends in investment. In fact, this choice is justified as it has been shown [10]that national saving and investment rates are highly positively correlated in virtually all countries.

The data that I am going to use comes from this website and I am investigating the yearly data from 2001-2019 for 27 current European Union member states. Firstly, I plotted Average percentage of country's GDP spent on investment over the 19 years vs Average GDP per capita over the 19 years (on the left). From the graph, we can see the points follow approximately the prediction given by the model in that there is a weak positive correlation between the percentage of GDP spent in investment and GDP per capita in

[9] We can see from Figure 3 shown above that the population growth tends to flatten out. In the context of logistic population growth model, this happens when the population of a territory/habiat reaches the carry capacity K.

[10] One of the examples is this paper written by Marianne Baxter and Mario J. Crucini published on The American Economic Review in 1993



this time period if we just exclude some outliers such as Luxembourg. Then secondly, I plotted Average population growth rate over the 19 years vs. Average GDP per capita over the 19 years (on the right).

As one can see from the line fitted in the graph, we have exactly the reverse as what the model would predict: we see higher GDP per capita for countries that have the most rapid population growth rate. Even though this situation is not completely unexpected especially given the simplicity of the model that we investigated here, we would at least expect that the model gives the approximately correct qualitative predictions, which once again doesn't seem to be the case for the second assumption/plot. In the next section, I will attempt to give some plausible explanations into why this seemingly "incorrect" conclusion raised.

6 Limitations of the simple Solow growth model

In this section, I will attempt to analyze why especially the second graph produced seemingly completely incorrect conclusions with respect to the model prediction. Firstly, this discrepancy may stem from the fact that we used population growth rate data instead of labour population growth rate data. The reason why I chose to use general population growth data here is due to the scarcity of the statistics on labour population growth rate as well as the assumption that we used throughout this report that labour population and general population can be considered the same. It is not unimaginable that a country's labour force grows slower than that country's population growth due to factors such as aging population accompanied by better healthcare. That is actually the case for the US economy where during the next 10 years the labour force is projected to grow 0.5 percent annually, slower than the 0.8-percent annual growth projected for the population. [11] Therefore, it is entirely possible in this context that countries such as Luxembourg experiences slower labour population growth while having the highest general percentage of population growth over the last twenty years. Therefore, the assumptions that we had about labour population = general population might have to be reconsidered if we are to seek a more reliable and accurate result from the model.

Secondly, Solow's growth model is established on the theory of a closed economy (as indicated in the second section), which simply means that a group of nations does not have any kind of relationship. In the case that we are considering here about 27 current EU member states, this assumption could not have been more inaccurate. One of the defining characteristics of the European Union is that these 27 member states

[11] For more details, please consult here.

form an Economic Union, meaning that the member states have both common policies on product regulation, freedom of movement of goods, services and the factors of production (capital and labour) and a common external trade policy. Therefore, just on the contrary of the assumption, these 27 countries considered might just have closer economic ties with each other than any other countries in the world. On this basis, I would imagine that if we do the same thing on more isolated economies such as these in sub-Sahara Africa, we would get closer results to the prediction than what we currently have. However, just to be fair to Solow, when he first proposed his model back in the 1950s, a time that largely predated the creation of international trade associations such as WTO (1995), he probably hardly imagined that the global economy nowadays would be so closely intertwined.

Thirdly, in the third assumption that we presented in the second section, we have the following equation:

$$K_t\prime = sY_t - \delta K_t$$

Implicitly, this equation doesn't differentiate between human capitals (skills acquired through schooling, training, practice, and socialization) and physical capital (tools, machinery, structures employed by human beings as means of production). As their names suggest, these two kinds of capitals are very different hence it would be extremely misleading to model the depreciation of both using a single δ . Sometimes physical capital can be lost or damaged in ways that human capitals cannot: as long as a company manages to retain its personnel, the company's investment in human capital would be secure since the human capital lies in personnel's capabilities and skills; on the other hand, even if a company manages to keep its machinery, they might be subjected to damage or lost hence replacement. The difference that I just mentioned is one of the many differences between human and physical capital. The key takeaway is that not differentiating between those two would in some ways make this model impractical.

7 Conclusion

As we see from the last section, there are severe limitations to the Solow growth model, which might explain partially the discrepancies we observe when plotting the real world data. Admittedly, we can add many further terms to the existing Solow model in the hope of capturing as many variables as possible while sticking to realistic assumptions. However, since economics is such an inherently complex and unpredictable subject, there will never be a perfect model. I think the lesson that we can take from analyzing this model is that the point of models such as that of Solow's is not to capture accurately every single detail of the real world; instead, the main point is for it to provide us with some qualitative understandings of the complex world.