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MA677 Homework 5

1. 70 purchases, 58 by women, 12 by men. Find MLE of p

$$f(x) = \begin{cases} p^{x_i}(1-p)^{1-x_i} & , x = 0, 1 \\ 0 & , \text{o.w.} \end{cases}$$

$$f(x|p) = \prod_{i=1}^{70} p^{x_i}(1-p)^{1-x_i}$$

$$= p^{x_1}(1-p)^{1-x_1}\left(p^{x_2}(1-p)^{1-x_2}\right) \ldots \left(p^{x_{70}}(1-p)^{1-x_{70}}\right)$$

$$= p^{\sum_{i=1}^{70} x_i}(1-p)^{70-\sum_{i=1}^{70} x_i}$$

$$\ln(f(x|p)) = \sum_{i=1}^{70} x_i \ln(p) + \left(70 - \sum_{i=1}^{70} x_i\right)\ln(1-p)$$

$$\frac{d\ln(f)}{dp} = 0 = \sum_{i=1}^{70} x_i\left(\frac{1}{p}\right) + \left(70 - \sum_{i=1}^{70} x_i\right)\left(-\frac{1}{1-p}\right)$$

$$\frac{1}{1-p}\left(70 - \sum_{i=1}^{70} x_i\right) = \sum_{i=1}^{70} x_i\left(\frac{1}{p}\right)$$

$$p\left(70 - \sum_{i=1}^{70} x_i\right) = \sum_{i=1}^{70} x_i(1-p)$$

$$70p - p\sum_{i=1}^{70} x_i = \sum_{i=1}^{70} x_i - p\sum_{i=1}^{70} x_i$$

$$70p = \sum_{i=1}^{70} x_i$$

$$\boxed{p = \frac{1}{70}\sum_{i=1}^{70} x_i}$$

2. show MLE does not exist if every observed value is 0 or 1.

$$f(x|p) = \theta^{x_i}(1-\theta)^{1-x_i}$$

$$f(x|p) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$\ln(f(x|p)) = \sum_{i=1}^{n} x_i \ln(\theta) + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-\theta)$$

From here, if $\theta = 1$ or $\theta = 0$, we would get an $\ln(0)$ from $\ln(\theta)$ or $\ln(1-\theta)$ which is undefined. Therefore, the MLE of θ cannot exist if $\theta = 0$ or 1.

3. a. $f(\lambda) = \dfrac{\lambda^k(e^{-\lambda})}{k!}$

$$f(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{k_i}(e^{-\lambda})}{k_i!} = \frac{\lambda^{k_1}(e^{-\lambda})}{k_1!}\left(\frac{\lambda^{k_2}(e^{-\lambda})}{k_2!}\right) \ldots \left(\frac{\lambda^{k_n}(e^{-\lambda})}{k_n!}\right)$$

$$= \lambda^{\sum_{i=1}^{n} k_i}\left(\frac{e^{-n\lambda}}{\prod_{i=1}^{n} k_i!}\right)$$

$$\ln(f(\lambda)) = \sum_{i=1}^{n} k_i \ln(\lambda) + (-n\lambda)\cancel{\ln(e)} - \ln\left(\prod_{i=1}^{n} k_i!\right)$$

$$\frac{d\ln}{d\lambda} = \sum_{i=1}^{n} k_i\left(\frac{1}{\lambda}\right) - n = 0$$

$$\sum_{i=1}^{n} k_i = \lambda n$$

$$\boxed{\lambda = \frac{1}{n}\sum_{i=1}^{n} k_i}$$

b. Show that the MLE of λ does not exist if every observed value is 0.

From above, we can see that:

$$\ln(f(\lambda)) = \sum_{i=1}^{n} k_i \ln(\lambda) + (-n\lambda) - \ln\left(\prod_{i=1}^{n} k_i!\right)$$

If $\lambda = 0$, we get $\ln(0)$ which doesn't exist. Therefore, the MLE of λ cannot exist if every observed value is 0.

4. $f(x|\theta^2) = \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x-\mu)^2}{2\theta^2}}$

$f(x|\theta^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x_i-\mu)^2}{2\theta^2}}$

$= \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x_1-\mu)^2}{2\theta^2}} \left(\frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x_2-\mu)^2}{2\theta^2}}\right) \cdots \left(\frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x_n-\mu)^2}{2\theta^2}}\right)$

$= \frac{1}{(2\pi\theta^2)^{n/2}} e^{\frac{\sum_{i=1}^{n} -(x_n-\mu)^2}{2\theta^2}}$

$\ln(f(x|\theta^2)) = -\frac{n}{2}\ln(2\pi\theta^2) + \frac{1}{2\theta^2}\sum_{i=1}^{n} -(x_n-\mu)^2 \ln(e)$

$\frac{d\ln}{d\theta^2} = -\frac{n}{2}\left(2\pi\left(\frac{1}{2\pi\theta^2}\right)\right) + \left(-(2)(2\theta^2)^{-2}\sum_{i=1}^{n}(-x_n-\mu)^2\right) = 0$

$-\frac{n}{2}\left(\frac{1}{\theta^2}\right) - \frac{1}{2\theta^4}\sum_{i=1}^{n}(x_n-\mu)^2 = 0$

$-\frac{n}{2}\left(\frac{1}{\theta^2}\right) = \frac{1}{2\theta^4}\sum_{i=1}^{n}(x_n-\mu)^2$

$\boxed{\theta^2 = -\frac{1}{n}\sum_{i=1}^{n}(x_n-\mu)^2}$