Report of Project4

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Problem 1

a.

With **forward algorithm**, we have:

$$
\begin{aligned}
\mathbb{P}(c_2 \mid D_2=0) &= \alpha \mathbb{P}(D_2=0 \mid c_2) \sum_{c_1 \in \{0,1\}} \mathbb{P}(c_2 \mid c_1) \mathbb{P}(c_1) \\
&= \alpha \langle 1-\eta, \eta \rangle * (\epsilon * 0.5 + (1-\epsilon) * 0.5) \\
&= 0.5\alpha \langle 1-\eta, \eta \rangle \\
Normalization: & \\
& 0.5\alpha(1-\eta) + 0.5\alpha\eta = 1 \\
& \Rightarrow \alpha = 2 \\
& \Rightarrow \mathbb{P}(C_2 = 1 \mid D_2 = 0) = \eta
\end{aligned}
$$

b.

According to **forward-backward algorithm**,

$$
\begin{aligned}
\mathbb{P}(D_3=1 \mid c_2) &= \sum_{c_3 \in \{0,1\}} \mathbb{P}(c_3 \mid c_2) \mathbb{P}(D_3=1 \mid c_3) \mathbb{P}(\mid c_3) \\
&= \langle 1-\epsilon, \epsilon \rangle * \eta * 1 + \langle \epsilon, 1-\epsilon \rangle * (1-\eta) * 1 \\
&= \langle \epsilon + \eta - 2\epsilon\eta, 1 - \epsilon - \eta + 2\epsilon\eta \rangle \\
\mathbb{P}(c_2 \mid D_2=0, D_3=1) &= \alpha \mathbb{P}(c_2 \mid D_2=0) \mathbb{P}(D_3=1 \mid c_2) \\
&= 0.5\alpha' \langle 1-\eta, \eta \rangle \times \langle \epsilon + \eta - 2\epsilon\eta, 1 - \epsilon - \eta + 2\epsilon\eta \rangle \\
&= 0.5\alpha' \left\langle \epsilon + \eta - 3\epsilon\eta - \eta^2 + 2\epsilon\eta^2, \eta - \epsilon\eta - \eta^2 + 2\epsilon\eta^2 \right\rangle \\
Normalization: & \\
& \Rightarrow \alpha' = \frac{2}{\epsilon + 2\eta - 4\epsilon\eta - 2\eta^2 + 4\epsilon\eta^2} \\
& \Rightarrow \mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1) = \frac{\eta - \epsilon\eta - \eta^2 + 2\epsilon\eta^2}{\epsilon + 2\eta - 4\epsilon\eta - 2\eta^2 + 4\epsilon\eta^2}
\end{aligned}
$$

c.

- i.

  Take $\epsilon = 0.1, \eta = 0.2$ into the fomula above,

$$\mathbb{P}(C_2 = 1 \mid D_2 = 0) = 0.2$$
$$\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1) \approx 0.4157$$

- **ii.**

  The probability of $C_2 = 1$ has increased after adding the second sensor reading $D_3 = 1$. The reason for this is that $\mathbb{P}(C_3 = 1 \mid D_3 = 1) > \mathbb{P}(C_3 = 0 \mid D_3 = 1)$. In other words, the evidence of $D_3 = 1$ indicates that C_3 are more likely to be 1 than 0, thus increasing the posterior probability of $C_2 = 1$ with the help of $P(C_2 = 1 \mid C_3 = 1)$. This is how future evidences influence current probability in smoothing.

  Formally put,

$$\begin{aligned}\mathbb{P}(C_k \mid D_{2:t}) &= \alpha f_{2:k} b_{k+1:t} \\ &= \frac{1}{\mathbb{P}(D_{2:k})} \mathbb{P}(C_k \mid D_{2:k}) \mathbb{P}(D_{k+1:t} \mid C_k)\end{aligned}$$

  Specifically for this problem,

$$\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1) = \mathbb{P}(C_2 = 1 \mid D_2 = 0) \frac{\mathbb{P}(D_3 = 1 \mid C_2 = 1)}{\mathbb{P}(D_3 = 1)}$$
$$where \quad \frac{\mathbb{P}(D_3 = 1 \mid C_2 = 1)}{\mathbb{P}(D_3 = 1)} > 1$$

- **iii.**

  Intuitively, I would set $\eta = 0.5$ so that no matter what the ture position of a car is, we have the same probability to get all of the observations. Therefore, our observations would have no effect on the probability of car position.

  Formally,

$$\alpha b_{k+1:t} = \frac{\mathbb{P}(D_{k+1:t} \mid C_k)}{\mathbb{P}(D_{2:k})} = 1$$
$$\frac{\mathbb{P}(D_3 = 1 \mid C_2 = 1)}{\mathbb{P}(D_3 = 1)} = 1$$