Matlab Experiment 4 Report: Sequence and Series

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Abstract

The primary concern of mathematicians in dealing with sequences or series is their convergent properties. Even when a sequence or series is divergent, one is eager to figure out its growth rate to infinity. In this experiment we are supposed to analyze the convergent properties of some well-known sequences or series, Fabonacci sequence, Harmonic series, and the (3n+1)-problem (or Collatz conjecture).

Contents

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1 Introduction and Purpose

In Mathematics, a sequence is an enumerated collection of objects in which repetitions are allowed.[1] A series is, roughly speaking, a description of the operation of adding infinitely many quantities, one after the other, to a given starting quantity.[2] More precisely, a series can be viewed as a cumulative summation over all terms in a particular sequence. From this point of view, the partial sum of a series is indeed an infinite sequence.

The paramount concern in dealing with sequences and series must be their convergent properties. Given a numerical sequence $\{a_n\}_{n=1}^{\infty} \in \mathbb{R}$, it is said to be convergent if for any $\epsilon > 0$, there exists an $N > 0$ such that

$$|a_n - L| < \epsilon \quad \text{when } n > N,$$

where L is a real number. Then we denote the preceding definition by $\lim\limits_{n\to\infty} a_n = L$. This formal definition can be generalized to sequences in any euclidean spaces, or even in metric spaces.

For a numerical series, it is said to be convergent if its partial sum is convergent and we write $\sum\limits_{n=1}^{\infty} a_n = s$, where s is called the sum of the series.[3] Any sequence or series that does not converge is said to diverge.

In this experiment, we aim to analyze the convergent properties of some well-known sequences or series, the change tendency of a sequence or series as the index n tends to infinity, and some asymptotic properties of divergent sequences or series. As examples, we focus on the scrutiny of the Fabonacci sequence, harmonic series, and the famous Collatz conjecture (or so-called (3n+1)-problem). All our experiment is based on computer programming on the platform of Matlab 2016a, which may approximately give us the overview of these properties. Essentially, the main theme of our experiments will follow the Chapter 4 of the book "Mathematical Experiments" written by Shangzhi Li et.al[4].

2 Methods and Results

2.1 Fabonacci Sequence

The pervasive Fabonacci sequence, originating from Indian Mathematics, can be defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n,\ n = 1, 2, ...,$$

where $F_1 = F_2 = 1$.

To investigate the growth rate of Fabonacci sequence, we make line plots of the points $(n, F_n), n = 1, 2, ..., N$, where $N = 20, 50, 100, 200, 500$, respectively. See Figure 1 for details.

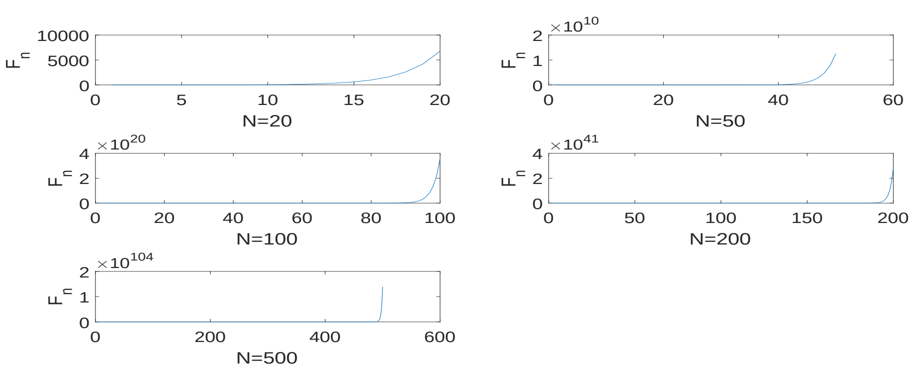

Figure 1: The Growth Rate of Fabonacci Sequence

Based on the recurrence formula of Fabonacci sequence, we know that $\frac{3}{2}F_{n+1} < F_{n+2} = F_{n+1} + F_n < 2F_{n+1}$. Hence the growth rate of Fabonacci sequence should be in the range of $(\frac{3}{2})^n$ and 2^n. This inspires us to make line plots of the points $(n, log(F_n)), n = 1, 2, ..., N$, where $N = 2000, 5000, 10000$ in this scenario. See Figure 2 for details.

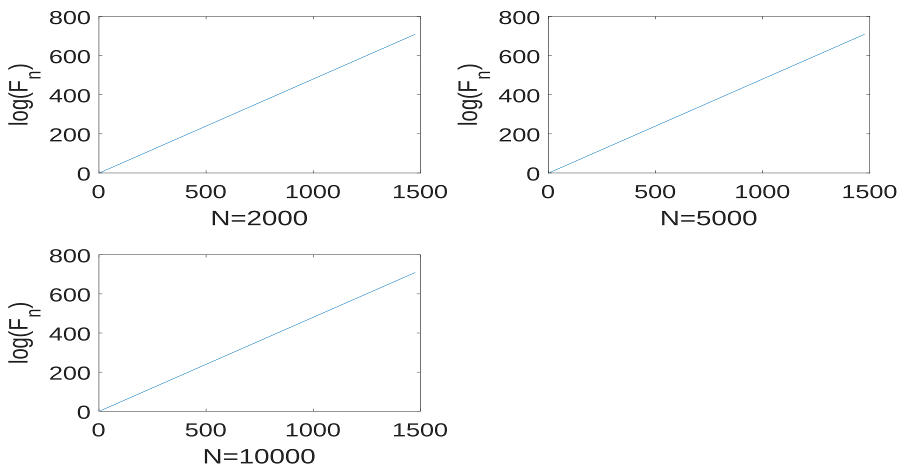

Figure 2: The Growth Rate of the Logarithmic Fabonacci Sequence

We also fit the coefficients of $log(F_n)$ in terms of n when $N = 200, 500, 1000$, which are $0.48118x - 0.80062, 0.48121x - 0.80308, 0.48121x - 0.8039$. The cases $N = 2000, 5000, 10000$ in the reference book[4] are not applicable when it comes to the first order polynomial fittings, since the magnitude of the terms in the sequences has excelled the float-point format in Matlab.

From previous analysis, we may guess that the general formula of Fabonacci sequence could be in the form of $F_n = cr^n$. By plugging it in the recurrence formula, we can obtain that $F_n = c(\frac{1+\sqrt{5}}{2})^n$. However, it turns out that this intuitive formula is not the desired general formula for Fabonacci sequence. With the assumptions that $F_1 = F_2 = 1$, we know that $c = \frac{2}{1+\sqrt{5}}$. Nevertheless, it does provide an reasonable approach to figure out the general formula. By assuming $b_n = F_n - cr^n$, we again guess $b_n = \bar{c}\bar{r}^n$ and obtain that

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

. Consequently, the growth rate of Fabonacci sequence is $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$. When conducting experiments via Matlab to verify this results, we cannot exhaust all the natural number $\mathbb{N}$. Thus we simply take $N = 1000$ and justify this general formula for the first 1000 terms. See codes in the Appendix for details.

```
norm(log(F)-log(F_r))<1e-5

ans =

     1
```

Interestingly, Fabonacci sequence can not only be seen tangibly but also listened to by our ears. The Fabonacci sequence module an integer m yields a periodic sequence. Therefore, we can arbitrarily take some values from the resulted periodic sequence as frequencies of waves and plot them via Matlab. Here we use the sine function as the base function. See Figure 3 for details.

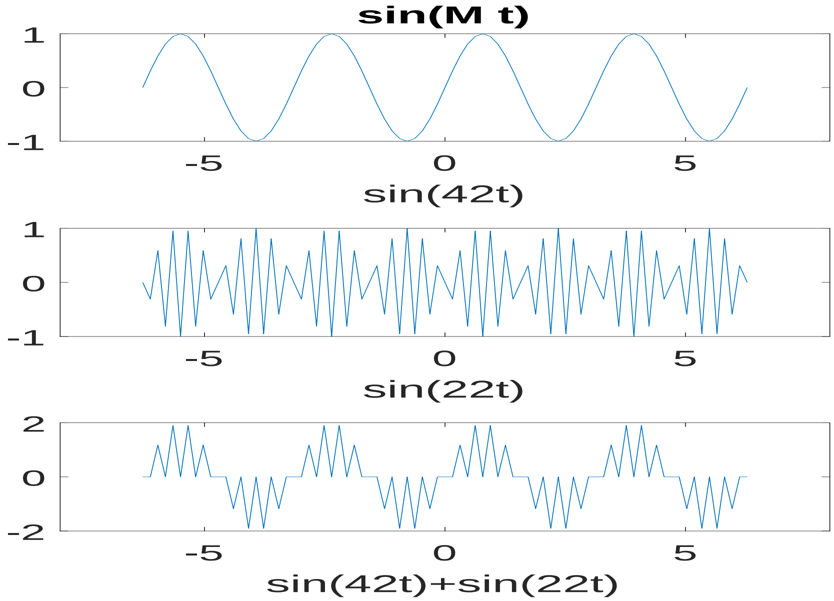

Figure 3: Wave Representations with Different Frequencies

In general, we hope to obtain the general formula for

$$a_{n+k} = \mu_{k-1}a_{n+k-1} + \mu_{k-2}a_{n+k-2} + \cdots + \mu_0 a_n.$$

This recurrence formula has the characteristic function $r^k = \mu_{k-1}r^{k-1} + \mu_{k-2}r^{k-2} + \cdots + \mu_0$ and therefore its general formula is of the form

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

where $r_1, r_2, ..., r_k$ are the roots of the characteristic function.

2.2 Harmonic Series

As is known to us, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

is convergent when $\alpha > 1$ and divergent otherwise. In particular, we are concerned the growth rate of the divergent series when $\alpha = 1$. One obvious method is to plot all the points (n, S_n), where S_n is the partial sum of the so-called harmonic series. Meanwhile, we draw the graphs of $y = x$, $y = \sqrt{x}$, and $y = \sqrt[4]{x}$ to compare the growth rate of the harmonic series. See Figure 4 for details.

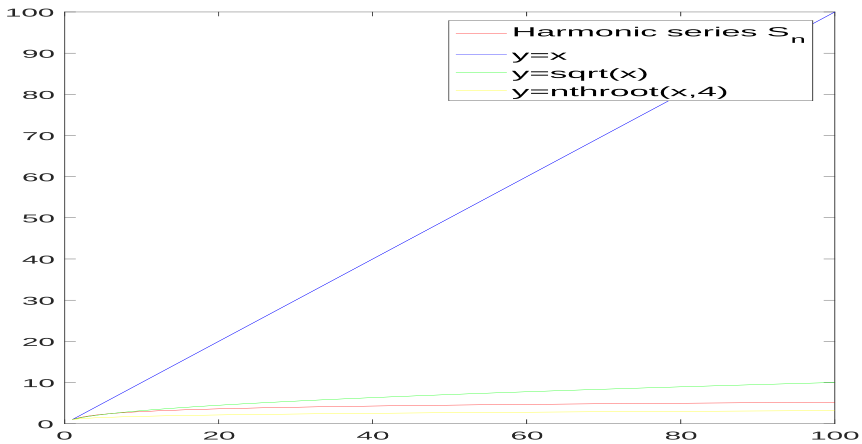

Figure 4: The Growth Trend of the Harmonic Series

To further examine the growth rate of the harmonic series, we consider the limit of $S_{2n} - S_n$ as $n \to \infty$, where $S_n = \sum_{k=1}^{n} \frac{1}{n}$. In reality, one can obtain the limit $log2$ with the inequality $log(x+1) \leq x$. In general, $S_{2^k n} - S_n$ tends to $k \cdot log2$ with k fixed and $n \to \infty$. All these results can be justified via Matlab given the number of terms in the series is sufficiently large. This result motivates us to form a more straight-forward interpretation, i.e., plot the points $(k, S_{2^k n}), k = 1, 2, ...$ given a fixed n. Moreover, we apply a linear function to fit the points. See Figure 5 for details.

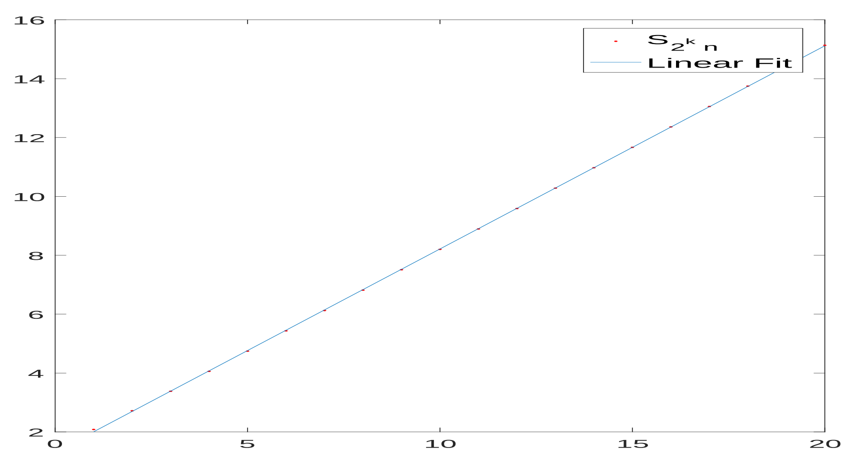

Figure 5: Plotting of the Points $(k, S_{2^k n})$ and the Linear Fitting

Therefore, $S_{2^k n}$ tends to infinity with the same speed as $O(k)$, which in turn means that S_n tends to infinity in the speed of $O(log(n))$. More precisely, Leonhard Euler proved that

$$\sum_{n=1}^{k} \frac{1}{n} = log(k) + \gamma + \epsilon_k,$$

where γ is the Euler-Mascheroni constant and $\epsilon_k \sim \frac{1}{2k}$ which approaches 0 as k goes to infinity.[5]

Next we are supposed to approach the same result from a different angle. Let $J(n)$ be be the least integer that is greater than S_n. Then by computing $J(2n)-J(n)$, we know that for any $N>0$, there exists an $n_1, n_2 > N$ such that $J(2n_1)-J(n_1)=1$ and $J(2n_2)-J(n_2)=0$. Hence $J(2n)-J(n)$ is divergent, which in turn indicates the divergent property of harmonic series. For each $n \in \mathbb{N}$, let $J(m)=J(n)+1$. Then $\frac{m}{n}$ ranges from 1 to 3.

For each $m=1,2,...,15$, let n be the maximal integer such that $J(n)=m$, which is denoted by $L(m)$. By calculating the ratio $\frac{L(m+1)}{L(m)}$, we may hypothesize that its limit is 0. In addition, since we are interested in the growth rate of $L(m)$, we make a line plot of $L(m)$ in terms of m. See Figure 6 for details.

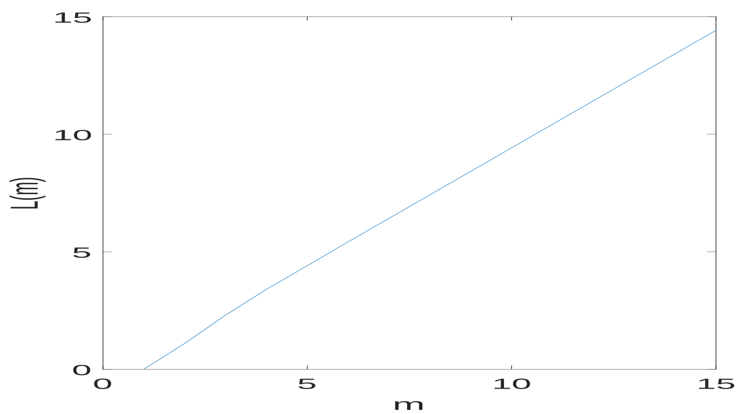

Figure 6: Plot of $L(m)$ and m

Based on the linear relation between $L(m)$ and m, we conclude that $S_n = O(n)$ as $n \to \infty$.

Furthermore, if $\alpha < 1$, we can prove that the growth rate of the partial sum is $O(\frac{n^{1-\alpha}}{1-\alpha})$ with almost the same argument and experiments. To give readers a more clear interpretation, we make the line plot of S_n versus $\frac{n^{1-\alpha}}{1-\alpha}$ in this case. See Figure 7 for details.

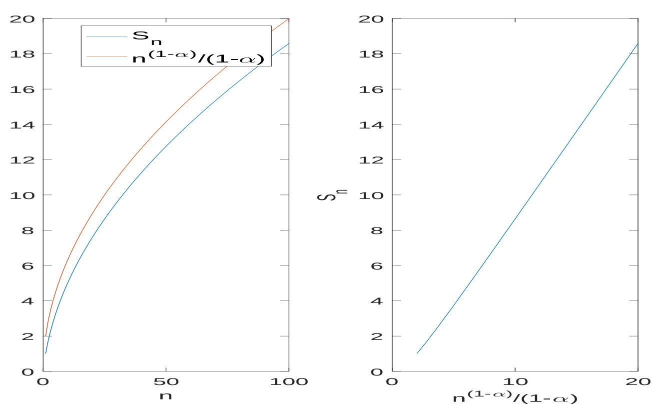

Figure 7: Plots of S_n versus $\frac{n^{1-\alpha}}{1-\alpha}$

2.3 (3n+1)-Problem

Now we move on to the last main problem of our experiments, the Collatz conjecture. The problem can be described again by a recurrence sequence,

$$a_{n+1} = \begin{cases} 3a_n + 1 & \text{if } a_n \text{ is odd,} \\ a_n/2 & \text{otherwise.} \end{cases}$$

We first encapsulate the computation of the defined sequence into a function called *"Pro3nPlus1"*, which takes an integer as the input and returns a sequence according to the preceding definition. Given a positive integer as the input, it seems that the returned sequence will eventually fall in the cycle $4 \to 2 \to 1$. Another interesting observation is that the conjecture holds if the corresponding sequence of any integer n contains a number less than n. Thus, we carry out some experiments for integers of the form $4k+1, 4k+3, 8k+1, 8k+5, 8k+7, 16k+1, 16k+3$.

```
m=zeros(1,length(n));
for k=1:length(n)
      a=Pro3nPlus1(n(k));
      m(k)=any(a<n(k));
end
all(m)

ans =

      1
```

These experiments, though rather intuitive, are not sufficient to confirm the conjecture. To further analyze the problem, we should introduce some notations.

First, we view the sequence as a flight and let $F(n)$ denote the length of flight, namely, the number of operations before the sequence reaches 1. It turns out that the upper bound of $F(n)$ can be dominated by n, i.e., $F(n) < n$ when n is large enough. See Figure 8 for details.

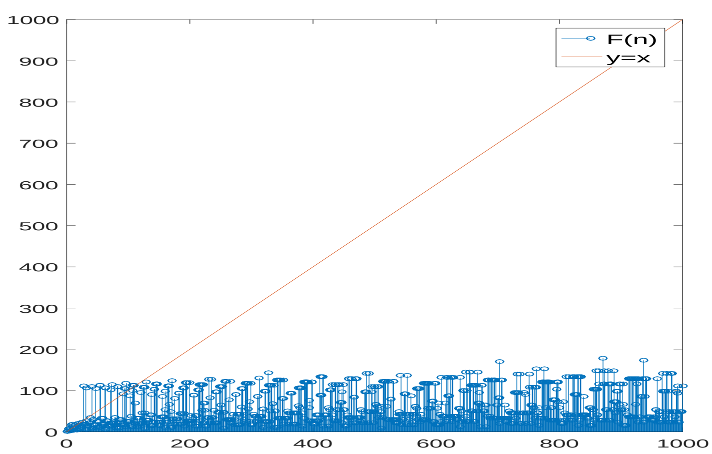

Figure 8: Plot of $F(n)$ versus n

Second, let $G(n)$ denote the length of the flight retaining its height. By plotting $\frac{G(n)}{log(n)}$ versus n, we notice that $G(n)$ is dominated by $log(n)$. See Figure 9 for details.

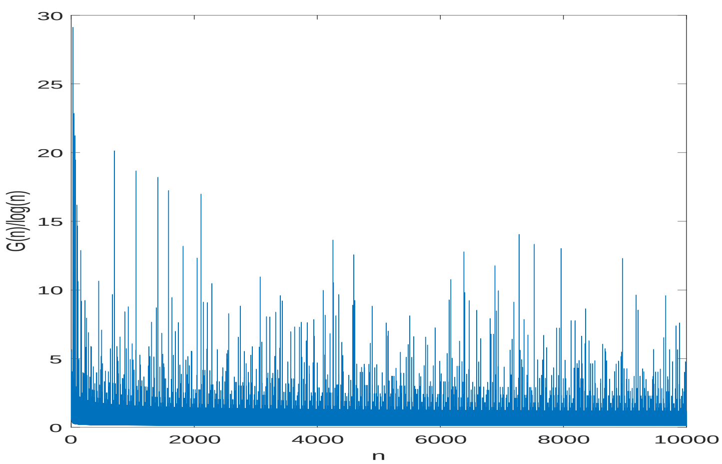

Figure 9: Plot of $\frac{G(n)}{log(n)}$ versus n

Third, let $T(n)$ be the maximal height of the flight. It turns out $T(n)$ can be exceptionally large for some particular n. Nevertheless, it can still be dominated by n^2. See Figure 10 for details.

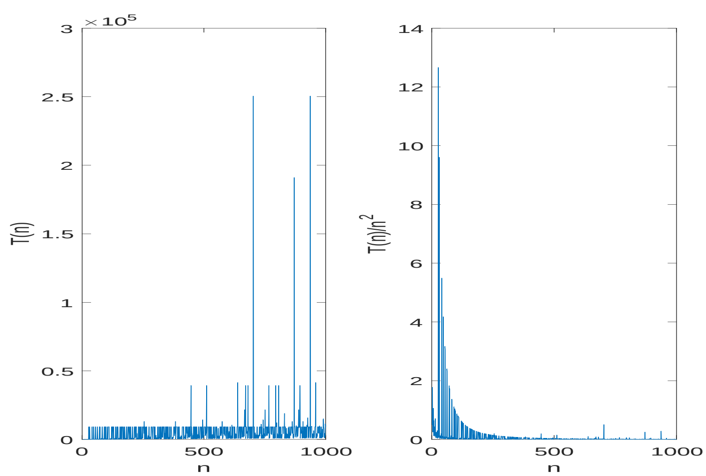

Figure 10: Plots of $T(n)$ versus n and $\frac{T(n)}{n^2}$ versus n

Finally, we denote the number of odd operations in the sequence by $O(n)$ while the number of even operations by $E(n)$. It seems that the upper bound of $\frac{O(n)}{E(n)}$ is 0.6, which also serves as its limit as $n \to \infty$. See Figure 11 for details.

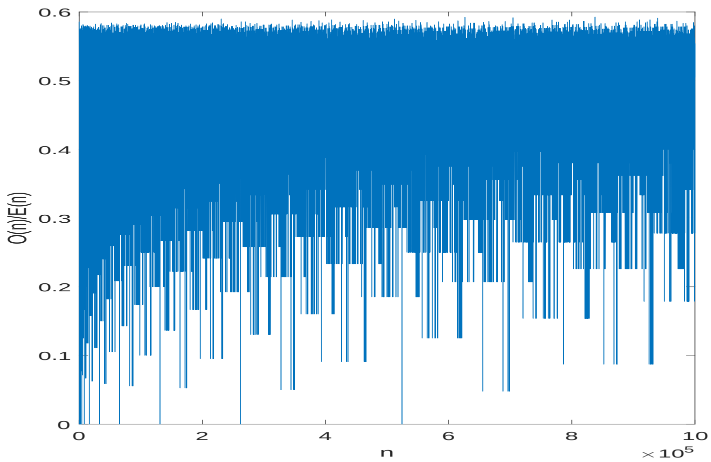

Figure 11: Plot of $\frac{O(n)}{E(n)}$ versus n

3 Conclusion and Discussion

From the whole experiments, we scrutinize the general formula of Fabonacci sequence and its growth rate, deal with the divergent properties of harmonic series from different angles, and approach the verification of the Collatz conjecture by building a flight model. All the results are coded via Matlab language and displayed by some informative figures. However, as the Collatz conjecture has not yet been mathematically proved, we only justify some of the results on a superficial level. With the rapid development of algebraic geometry and number theory, I believe this conjecture can be proved in the near future.

4 Reference

[1] Wikipedia ***Sequence (Mathematics).*** URL: `https://en.wikipedia.org/wiki/Sequence` Retrieved 13 December, 2017.

[2] Wikipedia ***Series (Mathematics).*** URL: `https://en.wikipedia.org/wiki/Series_(mathematics)` Retrieved 13 December, 2017.

[3] Walter Rudin (1976) ***Principles of Mathematical Analysis, Third Edition.*** McGraw-Hill Companies, Inc.

[4] Shangzhi Li, Falai Chen, Yaohua Wu, and Yunhua Zhang (1999) ***Mathematical Experiments.*** Textbook Seris for 21st Century, Higher Education Press.

[5] Wikipedia ***Harmonic Series (Mathematics).*** URL: `https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)` Retrieved 13 December, 2017.

5 Appendix

```
% Exercise 1
N=[20,50,100,200,500];
figure('PaperSize',[15 10])
hold on
for i=1:5
    F=zeros(1,N(i));
    F([1,2])=[1,1];
    for j=2:(N(i)-1)
```

```
        F(j+1)=F(j)+F(j-1);
    end
    t=1:N(i);
    subplot(3,2,i), plot(t,F)
    ylabel('F_n'), xlabel(['N=',num2str(N(i))])
end
hold off
print('Ex1','-dpdf','-fillpage')

% Exercise 2
N=[200,500,1000];
figure('PaperSize',[15 10])
s=char();
hold on
for i=1:3
    F=zeros(1,N(i));
    F([1,2])=[1,1];
    for j=2:(N(i)-1)
        F(j+1)=F(j)+F(j-1);
    end
    t=1:N(i);
    subplot(2,2,i), plot(t,log(F))
    ylabel('log(F_n)'), xlabel(['N=',num2str(N(i))])
    linearCoef=polyfit(t,log(F),1);
    s=char(s,poly2str(linearCoef,'x'));
end
s
exp(linearCoef)
print('Ex2','-dpdf','-fillpage')

% Exercise 3
c=2/(1+sqrt(5));
norm(c*((1+sqrt(5))/2)^2-1)>0.1

% Exercise 4
N=1000;
F=zeros(1,N);
F_real=@(n) 1/sqrt(5).*(((1+sqrt(5))/2).^n-((1-sqrt(5))/2).^n);
F_r=F_real(1:N);
F([1,2])=[1,1];
 for j=2:(N-1)
     F(j+1)=F(j)+F(j-1);
 end
norm(log(F)-log(F_r))<1e-5

% Exercise 5
m=51;N=1000;
M=mod(F,m);
r=randi([1 N],1,2);
Mran=M(r);
t=-2*pi:pi/20:2*pi;
subplot(3,1,1), plot(t,sin(Mran(1)*t))
title('sin(M_t)'), xlabel(['sin(',num2str(Mran(1)),'t)'])
subplot(3,1,2), plot(t,sin(Mran(2)*t))
xlabel(['sin(',num2str(Mran(2)),'t)'])
subplot(3,1,3), plot(t,sin(Mran(1)*t)+sin(Mran(2)*t))
xlabel(['sin(',num2str(Mran(1)),'t)+sin(',num2str(Mran(2)),'t)'])
print('Ex5','-dpdf','-fillpage')
```

```
% Exercise 6

% Exercise 7
N=100;
n=1:N;
S=cumsum(1./n);
plot(n,S,'r')
hold on
plot(n,n,'b')
plot(n,sqrt(n),'g')
plot(n,nthroot(n,4),'y')
legend('Harmonic series S_n','y=x','y=sqrt(x)','y=nthroot(x,4)')
hold off
print('Ex7','-dpdf','-fillpage')

% Exercise 8
N=10000;n=1:N;
k=2;
S_2n=cumsum(1./(1:2^k*N));
S_2n=S_2n(~mod(1:2^k*N,2^k));
S_n=cumsum(1./n);
a_n=S_2n-S_n;
norm(a_n(end)-k*log(2))<1e-4

N=20;S_2kn=zeros(1,N);t=zeros(1,N);n=2;
for k=1:N
    S_2kn(k)=sum(1./(1:2.^k*n));
end
plot(1:N,S_2kn,'.r')
hold on
p=polyfit(1:N,S_2kn,1);
plot(1:N,polyval(p,1:N))
legend('S_{2^k n}','Linear Fit')
hold off
print('Ex8','-dpdf','-fillpage')

% Exercise 9 (1)
N=10000000;
n=1:N;
S_n=cumsum(1./n);
J_n=ceil(S_n);
S_2n=cumsum(1./(1:2*N));
S_2n=S_2n(~mod(1:2*N,2));
J_2n=ceil(S_2n);
b_n=J_2n-J_n;
m=zeros(1,100);
for i=1:100
    k=i;
    while J_n(k)==J_n(i)
        k=k+1;
    end
    m(i)=k;
end
m./n(1:100)
% Exercise 9 (2)
L_m=zeros(1,15);
for m=1:15
    L=find(J_n==m);
    L_m(m)=L(end);
```

```
end
L_m((1:14)+1)./L_m(1:14)
plot(1:15,log(L_m)),ylabel('L(m)'),xlabel('m')
print('Ex9','-dpdf','-fillpage')

% Exercise 10
alpha=1/2;N=100;n=1:N;
S_n=cumsum(1./n.^alpha);
subplot(1,2,1),plot(n,S_n),xlabel('n')
hold on
plot(n,n.^(1-alpha)./(1-alpha))
legend('S_n','n^{(1-\alpha)}/(1-\alpha)','interpreter','latex')
subplot(1,2,2),plot(n.^(1-alpha)./(1-alpha),S_n),xlabel('n^{(1-\alpha)}/(1-\alpha)')
hold off
print('Ex10','-dpdf','-fillpage')

% Exercise 11

function a = Pro3nPlus1(n)

a(1)=n;i=1;
while a(i)~=1
    if mod(a(i),2)==0
        a(i+1)=a(i)/2;
    else
        a(i+1)=3*a(i)+1;
    end
    i=i+1;
end


n=input('Please input a positive integer:');i=1;
a=Pro3nPlus1(n)

% Exercise 12
k=randi([1,1000],1);
n=[4*k+1,4*k+3,8*k+1,8*k+3,8*k+5,8*k+7,16*k+1,16*k+3];
m=zeros(1,length(n));
for k=1:length(n)
    a=Pro3nPlus1(n(k));
    m(k)=any(a<n(k));
end
all(m)

% Exercise 13
N=1000;
F_n=zeros(1,N);
for n=1:N
    F_n(n)=length(Pro3nPlus1(n))-1;
end
stem(1:N,F_n),xlabel('n'),
hold on
plot(1:N,1:N),legend('F(n)','y=x')
hold off
print('Ex13','-dpdf','-fillpage')

% Exercise 14
N=10000;
G_n=zeros(1,N);
```

```
G_n(1)=1;
for n=2:N
    index=find(Pro3nPlus1(n)<n);
    G_n(n)=index(1)-1;
end
plot(1:N,G_n./log(1:N)),xlabel('n'),ylabel('G(n)/log(n)')
print('Ex14','-dpdf','-fillpage')

% Exercise 15
N=1000;
T_n=zeros(1,N);
for n=1:N
    T_n(n)=max(Pro3nPlus1(n));
end
subplot(1,2,1),plot(1:N,T_n),xlabel('n'),ylabel('T(n)')
subplot(1,2,2),plot(1:N,T_n./(1:N).^2),xlabel('n'),ylabel('T(n)/n^2')
print('Ex15','-dpdf','-fillpage')

% Exercise 16
N=1000000;
O_n=zeros(1,N);E_n=zeros(1,N);
for n=1:N
    a=n;
    while a~=1
        if mod(a,2)==0
            a=a/2;
            E_n(n)=E_n(n)+1;
        else
            a=3*a+1;
            O_n(n)=O_n(n)+1;
        end
    end
end
max(O_n./E_n)
plot(1:N,O_n./E_n),xlabel('n'),ylabel('O(n)/E(n)')
print('Ex16','-dpdf','-fillpage')
```