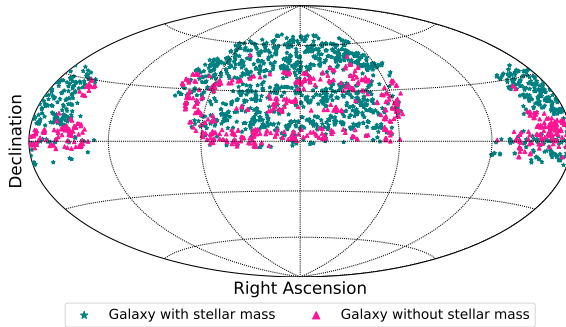# Efficient Inference on High-Dimensional Linear Models With Missing Outcomes

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Joint Work with *Alexander Giessing* and *Yen-Chi Chen*

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STATISTICS

W UNIVERSITY *of* WASHINGTON

Observed galaxies on the high redshift slice $0.4 \sim 0.401$.

▶ **Notes:** Sloan Digital Sky Survey (SDSS) observes millions of galaxies, but some (estimated) galactic stellar masses are missing in the associated value-added catalog (Comparat et al., 2017).
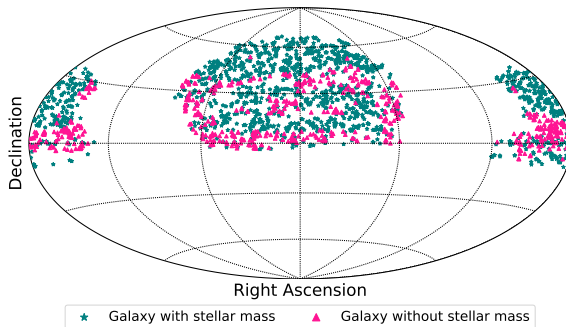
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▶ **Scientific Question:**

How can we quantify the uncertainty of the (estimated) stellar mass of a newly observed galaxy based on the spectroscopic and photometric properties?

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• Potential data contamination;

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▶ **Reasons for Missingness:**

● Limiting usage of the observational run in SDSS for galaxy targets;

● Potential data contamination;

● Misclassification of galaxies as stars.

▶ **Statistical Problem:**

How can we conduct valid and efficient inference on the regression function despite missing outcomes?

1. **Linearity:** The data $\{(Y_i, R_i, X_i)\}_{i=1}^n$ are i.i.d. observations from a sparse linear model

$$Y = X^T \beta_0 + \epsilon \quad \text{with} \quad \mathrm{E}\left(\epsilon | X\right) = 0 \quad \text{and} \quad \mathrm{E}\left(\epsilon^2 | X\right) = \sigma_\epsilon^2,$$

where $||\beta_0||_0 = s_\beta \ll d$ and $R \in \{0, 1\}$ when Y is missing or not.

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- Partially linear model (Müller and van de Geer, 2015).

2. **Missing At Random (MAR):** $Y_i \perp\!\!\!\perp R_i | X_i$ for $i = 1, ..., n$.

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1 Fully Observed Outcomes: Debiased Lasso (Zhang and Zhang, 2014; van de Geer et al., 2014; Javanmard and Montanari, 2014):

$$\widehat{\beta}^{\text{debias}} = \widehat{\beta}_\lambda + \frac{1}{n}\widehat{\Theta}\sum_{i=1}^{n} X_i(Y_i - X_i^T\widehat{\beta}_\lambda),$$

- $\widehat{\beta}_\lambda$ is a Lasso solution under the regularization parameter $\lambda > 0$;

- $\widehat{\Theta} \in \mathbb{R}^{d \times d}$ is an approximation to the matrix inverse $\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i^T\right)^{-1}$.

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2 **MAR Outcomes:** M-estimation framework with a Lasso-type debiased and doubly robust estimator (Chakrabortty et al., 2019).

▶ **Drawbacks of Existing Approaches:** Inference on $\beta_0 \in \mathbb{R}^d$.

1. Need to compute a $d \times d$ debiasing matrix $\widehat{\Theta}$.

2. Require sample splitting or cross fitting for valid inference.

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▶ **Our Focus:** Inference on $m_0(x) = x^T \beta_0$.

● *Computational efficiency:* Our debiasing program is convex and only needs to solve for an n-dimensional weight vector.

● *Statistical efficiency:* Our estimator is semi-parametrically efficient among all asymptotically linear estimators.

Methodology and Asymptotic Theory

- The debiased Lasso estimator is given by

$$\widehat{\beta}^{\text{debias}} = \widehat{\beta}_\lambda + \frac{1}{n} \sum_{i=1}^{n} R_i \widehat{\Theta} X_i \left(Y_i - X_i^T \widehat{\beta}_\lambda \right).$$

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▶ **Issue:** This naive estimator may not be asymptotically normal in general (van de Geer et al., 2014; Javanmard and Montanari, 2014)!

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▶ **Idea:** Introduce a weight vector $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{R}^n$ to replace

$$\frac{1}{\sqrt{n}} x^T \widehat{\Theta} X_i \implies w_i \quad \text{for} \quad i = 1, ..., n$$

and formulate a generic debiased estimator

$$\widehat{m}^{\text{debias}}(x; \boldsymbol{w}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i R_i \left(Y_i - X_i^T \widehat{\beta} \right). \tag{1}$$

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$$\widehat{m}^{\text{debias}}(x; w) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i R_i \left(Y_i - X_i^T \widehat{\beta} \right). \tag{1}$$

▶ **Question:** How do we estimate the weight vector $w = (w_1, ..., w_n)^T$?

The conditional mean squared error of $\sqrt{n}\, m^{\text{debias}}(x; \boldsymbol{w})$ is

$$\mathrm{E}\left[\left(\sqrt{n}\, m^{\text{debias}}(x; \boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \Big| X_1, ..., X_n\right]$$

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$$= \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi_i}_{\text{Main Conditional Variance}} + \underbrace{\left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x\right)^T \sqrt{n}\, (\beta_0 - \beta)\right]^2}_{\text{Conditional Bias}}$$

$$+ \underbrace{(\beta_0 - \beta)^T \left[\sum_{i=1}^n w_i^2 \pi_i (1 - \pi_i) X_i X_i^T\right] (\beta_0 - \beta)}_{\text{Asymptotically Negligible Conditional Variance}}.$$

▶ **Notes:** $\pi_i := \mathrm{P}(R_i = 1 | X_i)$ is the propensity score under the MAR condition.

$$E\left[\left(\sqrt{n}\, m^{\text{debias}}(x; \boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \Big| X_1, ..., X_n\right]$$

$$\asymp \quad \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi_i}_{\text{Main Conditional Variance}} \quad + \quad \underbrace{\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \pi_i X_i - x\right)^T \sqrt{n}\,(\beta_0 - \beta)\right]^2}_{\text{Conditional Bias}}.$$

- By Hölder's inequality,

$$\text{``Conditional Bias''} \leq \left[\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \pi_i X_i - x\right\|_\infty \sqrt{n}\, \|\beta_0 - \beta\|_1\right]^2.$$

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- We design our debiasing program as:

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} \sum_{i=1}^n w_i^2 \widehat{\pi}_i \quad \text{subject to} \quad \left\|x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \widehat{\pi}_i \cdot X_i\right\|_\infty \leq \frac{\gamma}{n}.$$

① Compute the Lasso pilot estimate $\widehat{\beta}_\lambda$ on the complete-case data

$$\widehat{\beta}_\lambda = \arg\min_{\beta \in \mathbb{R}^d} \left[\frac{1}{2n} \sum_{i=1}^{n} R_i (Y_i - X_i^T \beta)^2 + \lambda \, ||\beta||_1 \right].$$

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2. Obtain consistent propensity score estimates $\widehat{\pi}_i, i = 1, ..., n$ by *any machine learning method*.

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3. Solve the debiasing program defined as:

$$\min_{w \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \widehat{\pi}_i w_i^2 : \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \leq \frac{\gamma}{n} \right\}.$$

Our Debiasing Inference Procedure

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4. Define the debiased estimator for $m_0(x) = x^T \beta$ as:

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{w}_i R_i \left(Y_i - X_i^T \widehat{\beta} \right).$$

① How to select the tuning parameter $\gamma > 0$ for our debiasing program?

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \widehat{\pi}_i w_i^2 : \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \leq \frac{\gamma}{n} \right\}.$$

② Is our debiased estimator asymptotically normal?

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{w}_i R_i \left(Y_i - X_i^T \widehat{\beta} \right).$$

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▶ **Answer:** The above two questions can be addressed by the *dual formulation* of our debiasing program!

▶ **Primal Program:**

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \widehat{\pi}_i w_i^2 : \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \leq \frac{\gamma}{n} \right\}.$$

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▶ **Dual Program:**

$$\min_{\ell\in\mathbb{R}^d}\left\{\frac{1}{4n}\sum_{i=1}^n \widehat{\pi}_i \left(X_i^T \ell\right)^2 + x^T \ell + \frac{\gamma}{n}\left\|\ell\right\|_1\right\}.$$

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▶ **Primal-Dual Relation:** Under the strong duality,

$$\widehat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \widehat{\ell} \quad \text{for} \quad i = 1, ..., n.$$

▶ **Dual Program:**

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- It is an *unconstrained* optimization problem, and $\gamma > 0$ can be fine-tuned via cross-validation.

- Primal-dual relation $\widehat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \widehat{\ell}, \; i = 1, ..., n$ and dual consistency $\widehat{\ell} \xrightarrow{P} \ell_0$ reveal that

$$\sqrt{n} \left[\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) - m_0(x) \right] = \underbrace{-\frac{1}{2\sqrt{n}} \sum_{i=1}^{n} R_i \epsilon_i X_i^T \ell_0}_{\text{i.i.d. sum!}} + \underbrace{\text{``Bias terms''}}_{o_P(1)}.$$

Theorem (Theorem 7 in Zhang et al. 2023)

Under regularity conditions,

$$\sqrt{n} \left[\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) - m_0(x) \right] \xrightarrow{d} \mathcal{N} \left(0, \sigma_m^2(x) \right)$$

with $\sigma_m^2(x) = \lim_{n \to \infty} \sigma_\epsilon^2 \cdot x^T \left[\mathrm{E} \left(R X X^T \right) \right]^{-1} x.$

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1. For any fixed dimension $d > 0$, the asymptotic variance

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2. Under regularity conditions (Proposition 8 in Zhang et al. 2023),

$$\left| x^T \left[\text{E}\left(RXX^T\right)\right]^{-1} x - \sum_{i=1}^{n} \widehat{\pi}_i \widehat{w}_i^2 \right| = o_P(1).$$

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- Why don't we need sample splitting or cross fitting for estimating the propensity score by any machine learning method?

▶ **Answer:** Our asymptotic normality result depends on the *in-sample* estimation error r_π of the propensity score:

$$\max_{1 \leq i \leq n} |\widehat{\pi}_i - \pi_i| = O_P(r_\pi) \quad \text{with} \quad \pi_i = \pi(X_i), i = 1, ..., n.$$

- Our debiased estimator performs even better when we overfit the propensity scores!!

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- This coincides with *"benign overfitting"* in linear regression or neural networks (Bartlett et al., 2020; Li et al., 2021; Cao et al., 2022).

Simulation and Real-World Application

Lasso Pilot Estimate — Debiased Estimator (1SE) — Debiased Estimator (min-feas)

$\sqrt{n}|\hat{m}(x) - m_0(x)|/\hat{\sigma}_m(x)$
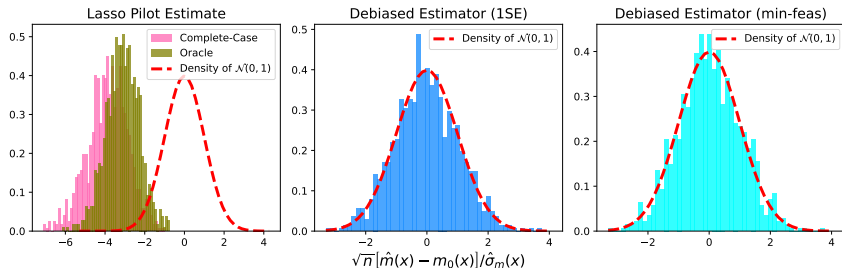
▶ **Effectiveness of Our Debiased Estimator:**

• Correct the bias of the Lasso pilot estimate.

• Asymptotically normal under a wide range of $\gamma > 0$.

▶ **Notes:** Our paper contains comprehensive comparisons with other existing methods.

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95% Confidence Intervals of Estimated Stellar Masses
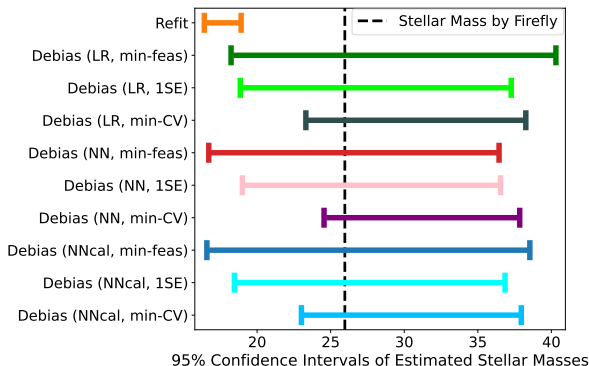
- The 95% confidence intervals by our debiasing methods cover the true stellar mass of a new galaxy.

Is it statistically significant that the stellar mass of a galaxy is negatively correlated with its distance to the nearby cosmic filament structures?

95% Confidence Intervals
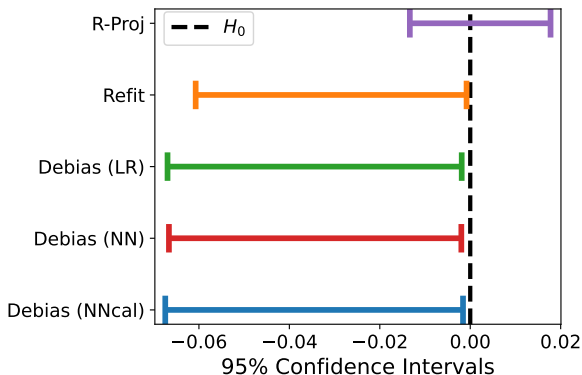
- 95% confidence intervals by our debiasing methods exclude 0 and are all negative.

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More details can be found in

[1] Y. Zhang, A. Giessing, and Y.-C. Chen. Efficient Inference on High-Dimensional Linear Models with Missing Outcomes. *arXiv preprint*, 2023. https://arxiv.org/abs/2309.06429.

Python Package: Debias-Infer and R Package: DebiasInfer.

We present an efficient debiasing method for conducting valid inference on high-dimensional linear models with MAR outcomes.

- The dual form explains its computational and statistical efficiencies.

- The nuisance propensity score can be nonparametrically estimated without sample splitting or cross fitting.

- A novel application to the inference on galactic stellar mass.

More details can be found in

[1] Y. Zhang, A. Giessing, and Y.-C. Chen. Efficient Inference on High-Dimensional Linear Models with Missing Outcomes. *arXiv preprint*, 2023. https://arxiv.org/abs/2309.06429.

Python Package: Debias-Infer and R Package: DebiasInfer.

Thank you!

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1. **Lasso pilot estimate:** We adopt the scaled Lasso (Sun and Zhang, 2012) with its universal regularization parameter $\lambda_0 = \sqrt{\frac{2 \log d}{n}}$ as the initialization. Specifically, it iteratively updates $\widehat{\beta}(\widetilde{\lambda}), \widehat{\sigma}_\epsilon(\widetilde{\lambda}), \widetilde{\lambda}$ via the jointly convex optimization program:

$$\left(\widehat{\beta}(\widetilde{\lambda}), \widehat{\sigma}_\epsilon(\widetilde{\lambda}) \right) = \underset{\beta \in \mathbb{R}^d, \sigma_\epsilon > 0}{\arg\min} \left[\frac{1}{2n\sigma_\epsilon} \sum_{i=1}^n R_i \left(Y_i - X_i^T \beta \right)^2 + \frac{\sigma_\epsilon}{2} + \widetilde{\lambda} \, ||\beta||_1 \right].$$

2. **Debiasing program:** We solve the primal program by Python package "CVXPY" (Diamond and Boyd, 2016; Agrawal et al., 2018) or R package "CVXR" (Fu et al., 2020). For the dual program, we formulate a coordinate descent algorithm (Wright, 2015) as:

$$\left[\widehat{\ell}(x) \right]_j \leftarrow \frac{\mathcal{S}_{\frac{\gamma}{n}} \left(-\frac{1}{2n} \sum_{i=1}^n \widehat{\pi}_i \left(\sum_{k \neq j} X_{ik} X_{jk} \left[\widehat{\ell}(x) \right]_k \right) - x_j \right)}{\frac{1}{2n} \sum_{i=1}^n \widehat{\pi}_i X_{ij}^2} \quad \text{for } j = 1, ..., d,$$

where $\mathcal{S}_{\frac{\gamma}{n}}(u) = \text{sign}(u) \cdot \left(u - \frac{\gamma}{n} \right)_+$ is the soft-thresholding operator.

- Suppose that we conduct a *K*-fold cross-validation on a candidate set $\Gamma = \{\gamma_1, ..., \gamma_m\}$ of the tuning parameter.

- For each $\gamma_i \in \Gamma$, we compute the cross-validated risk or error on each fold of the data as:
$$CV_k(\gamma_i), \quad k = 1, ..., K.$$

- For each $\gamma_i \in \Gamma$, we calculate the standard error of $CV_1(\gamma_i), ..., CV_K(\gamma_i)$ as:
$$SD(\gamma_i) = \sqrt{\text{Var}\left(CV_1(\gamma_i), ..., CV_K(\gamma_i)\right)}, \quad SE(\gamma_i) = SD(\gamma_i)/\sqrt{K}.$$

- Let
$$CV(\gamma) = \frac{1}{K}\sum_{k=1}^{K} CV_k(\gamma) \quad \text{and} \quad \widehat{\gamma} = \arg\min_{\gamma \in \Gamma} CV(\gamma).$$

The 1SE rule (Breiman et al., 1984; Chen and Yang, 2021) selects $\gamma_{1SE} \in \Gamma$ with as the one with the smallest $CV(\gamma)$ such that
$$CV(\gamma_{1SE}) \geq CV(\widehat{\gamma}) + SE(\widehat{\gamma}).$$
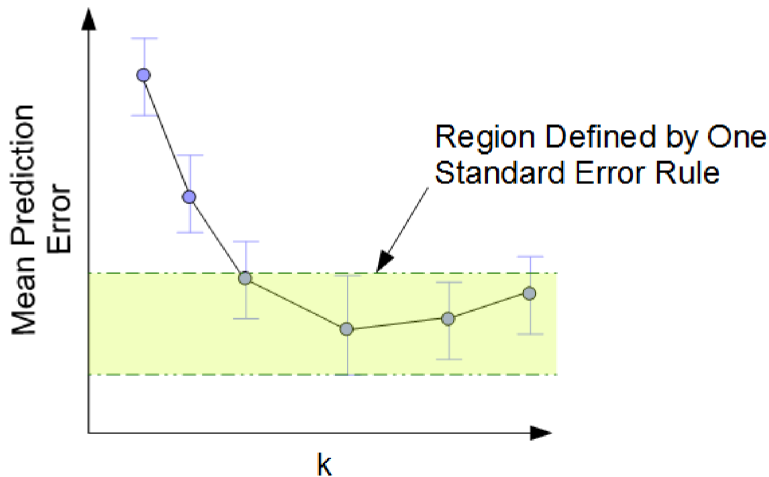
Figure: Illustration of the 1SE rule for selecting the model parameter.

- Consider the regression function $m \equiv m(x) \in \mathbb{R}$ as the main parameter to be inferred and $\beta \in \mathbb{R}^d$ as the high-dimensional nuisance parameter.

- Our generic debiased estimator $m^{\text{debias}}(x, \boldsymbol{w})$ solves the sample-based estimating equation

$$\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m^{\text{debias}}, \beta) = m^{\text{debias}}(x; \boldsymbol{w}) - x^T\beta - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i \cdot R_i \left(Y_i - X_i^T\beta\right) = 0.$$

- The Neyman near-orthogonalization condition (Chernozhukov et al., 2018) given $\boldsymbol{X} = (X_1, ..., X_n)^T \in \mathbb{R}^{n \times d}$ at $(m_0, \beta_0) = (x^T\beta_0, \beta_0)$ requires

$$\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m_0, \beta_0)\bigg|\boldsymbol{X}\right] = 0,$$

$$\sup_{\beta \in \mathcal{T}_n} \left|\left\{\frac{\partial}{\partial\beta}\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m, \beta)\bigg|\boldsymbol{X}\right]\bigg|_{(m_0, \beta_0)}\right\}^T (\beta - \beta_0)\right| \leq \frac{\delta_n}{\sqrt{n}}, \tag{2}$$

where $\mathcal{T}_n$ is a properly shrinking neighborhood of β_0 and $\delta_n = o(1)$.

- Both conditions in (2) hold true, because for any $\beta \in \mathcal{T}_n$ and some convex set $\mathcal{B}$ containing β_0, we have that

$$\left| \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \beta} \mathbb{E} \left[\Xi_x(Y_i, R_i, X_i; m, \beta) | X \right] \Big|_{(m_0, \beta_0)} \right\}^T (\beta - \beta_0) \right|$$

$$= \left| \left[x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i \cdot \pi(X_i) X_i \right]^T (\beta_0 - \beta) \right|$$

$$\text{``} \leq \text{''} \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i \cdot \widehat{\pi}_i \cdot X_i \right\|_{\infty} ||\beta - \beta_0||_1 \quad \text{by Hölder's inequality}$$

$$\leq \frac{\gamma}{n} ||\beta - \beta_0||_1 \quad \text{by the box constraint in our debiasing program}$$

$$\leq \frac{\delta_n}{\sqrt{n}} \quad \text{by setting } \mathcal{T}_n = \left\{ \beta \in \mathcal{B} \subset \mathbb{R}^d : ||\beta - \beta_0||_1 \leq \frac{\sqrt{n}\delta_n}{\gamma} \right\}.$$

- Our debiasing program optimizes the (estimated) variance among all the estimators satisfying Neyman near-orthogonalization (2).

- (2) also allows our debiasing program to *de-correlate* the Lasso pilot regression from propensity score estimation and weight optimization.

▶ **Goal:** Establish the asymptotic normality of our debiased estimator

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \widehat{w}_i R_i \left(Y_i - X_i^T \widehat{\beta} \right).$$

• Linearity assumption $Y_i = X_i^T \beta_0 + \epsilon_i$ for $i = 1, ..., n$ implies

$$\sqrt{n} \left[\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) - m_0(x) \right] = \underbrace{\sum_{i=1}^{n} \widehat{w}_i R_i \epsilon_i}_{\text{Not an i.i.d. sum!}} + \left[x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \widehat{w}_i R_i X_i \right]^T \sqrt{n} \left(\widehat{\beta} - \beta_0 \right),$$

• Dual relation $\widehat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \widehat{\ell}$ for $i = 1, ..., n$ and dual consistency $\widehat{\ell} \xrightarrow{P} \ell_0$ reveal that

$$\sqrt{n} \left[\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) - m_0(x) \right] = -\frac{1}{2\sqrt{n}} \sum_{i=1}^{n} R_i \epsilon_i X_i^T \widehat{\ell} + \left[x + \frac{1}{2n} \sum_{i=1}^{n} R_i X_i X_i^T \widehat{\ell} \right]^T \sqrt{n} \left(\beta_0 - \widehat{\beta} \right)$$

$$= \underbrace{-\frac{1}{2\sqrt{n}} \sum_{i=1}^{n} R_i \epsilon_i X_i^T \ell_0}_{\text{i.i.d. sum!}} + \underbrace{\text{``Bias terms''}}_{o_P(1)}.$$

1. The covariate vector $X \in \mathbb{R}^d$ and the noise $\epsilon \in \mathbb{R}$ are sub-Gaussian.

2. There exists a constant $\kappa_R > 0$ such that

$$\inf_{v \in \mathbb{S}^{d-1}} \mathrm{E}\left[R(X^T v)^2\right] \geq \kappa_R^2 \quad \text{with} \quad \mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d : ||x||_2 = 1\right\}.$$

3. Given any $n \geq 1$ and $\delta \in (0,1)$, there exists $r_\pi \equiv r_\pi(n,\delta) > 0$ such that

$$\mathrm{P}\left(\max_{1 \leq i \leq n} |\widehat{\pi}_i - \pi_i| > r_\pi\right) < \delta \quad \text{with} \quad \pi_i = \pi(X_i), i = 1,...,n.$$

4. Define the population dual program as:

$$\min_{\ell \in \mathbb{R}^d} \left\{\frac{1}{4} \mathrm{E}\left[R\left(X^T \ell\right)^2\right] + x^T \ell\right\},$$

whose exact solution is $\ell_0(x) = -2\left[\mathrm{E}\left(RXX^T\right)\right]^{-1} x$. We assume that the r_ℓ-approximation $\widetilde{\ell}(x)$ to $\ell_0(x)$ is sparse with $r_\ell \in \left[0, \frac{1}{2}\right]$, i.e.,

$$s_\ell(x) = \left|\left|\widetilde{\ell}(x)\right|\right|_0 \ll \min\{n,d\} \quad \text{with} \quad \widetilde{\ell}(x) = \underset{u \in \mathbb{R}^d}{\arg\min}\left\{||u||_0 : ||u - \ell_0(x)||_2 \leq r_\ell ||\ell_0(x)||_2\right\}.$$

Methods to be compared:

- **"DL-Jav":** The debiased Lasso by Javanmard and Montanari (2014).

- **"DL-vdG":** The debiased Lasso by van de Geer et al. (2014).

- **"Refit":** Run the regular least-square regression on the support set of the Lasso pilot estimate (Belloni and Chernozhukov, 2013).

Implementation settings of the above methods:

- Complete-case (CC) data $\{(X_i, Y_i, R_i = 1)\}_{i=1}^{n}$;

- Inverse probability weighted (IPW) data $\left\{ \left(\frac{X_i}{\sqrt{\hat{\pi}_i}}, \frac{Y_i}{\sqrt{\hat{\pi}_i}}, R_i = 1 \right) \right\}_{i=1}^{n}$;

- Oracle fully observed data (X_i, Y_i) for $i = 1, ..., n$.

Evaluation metrics over 1000 Monte Carlo experiments:

- Average absolute bias $\left| \hat{m}^{\text{debias}}(x) - m_0(x) \right|$;

- Average coverage and average length of the yielded 95% confidence intervals.

Figure: Sparse β_0^{sp} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{cs})$, $i = 1, ..., n$.

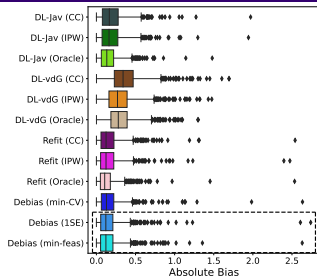Figure: Pseudo-dense β_0^{pd} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{ar}}), i = 1, ..., n$.

Simulation Results Under Laplace$(0, 1/\sqrt{2})$ Noises
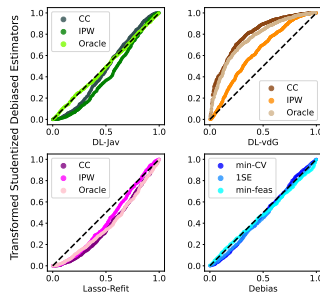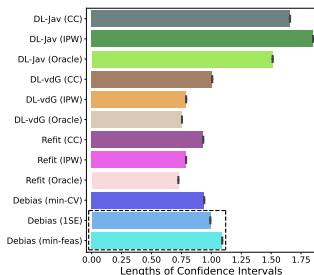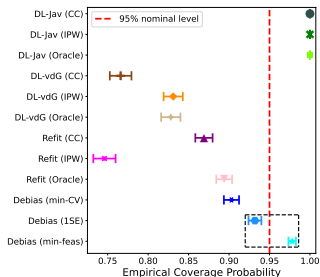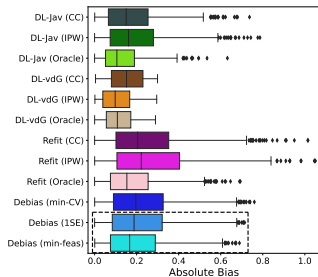
Figure: Dense β_0^{de} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{cs}), i = 1, ..., n$.
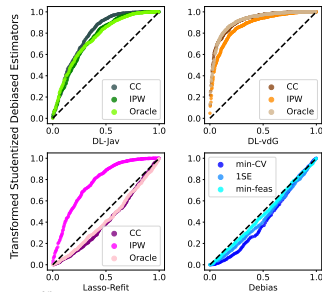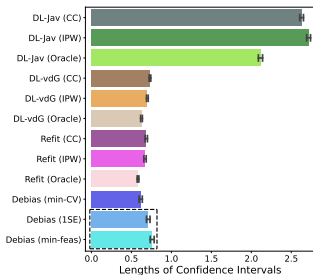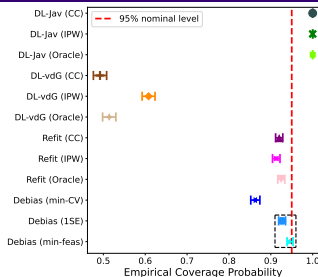
Figure: Pseudo-dense β_0^{pd} and dense $x^{(4)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{ar})$, $i = 1, ..., n$. Note that the mean-zero t_2 distribution has *infinite* variance.

1. True propensity score model: $P(R_i = 1|X_i) = \Phi\left(-4 + \sum_{k=1}^{K} Z_{ik}\right)$, where $(Z_{i1}, ..., Z_{iK})$ contains all polynomial combinations of the first eight components $X_{i1}, ..., X_{i8}$ of $X_i \in \mathbb{R}^{1000}$ with degrees ≤ 2.

2. Estimate the propensity scores $\pi(X_i), i = 1, ..., n$ by the following nonlinear/nonparametric machine learning methods:

   - **Gaussian Naive Bayes ("NB")**.

   - **Random Forest ("RF"):** 100 trees, bootstrapping samples, and the Gini impurity.

   - **Support Vector Machine ("SVM"):** Gaussian radial basis function.

   - **Neural Network ("NN"):** Two hidden layers of size 80×50 and ReLU $h(x) = \max\{x, 0\}$ as the activation function.

3. Include an extra evaluation metric as the average mean absolute error ("Avg-MAE") for the estimated propensity scores.

$$\sqrt{n}\left[\hat{m}^{debias}(x) - m_0(x)\right]/\hat{\sigma}_m(x)$$

Study Design For Stellar Mass Inference
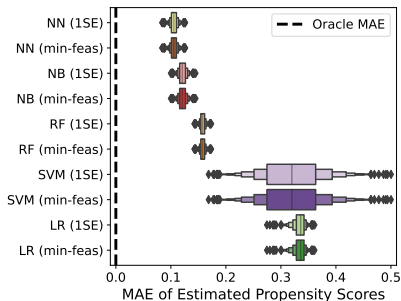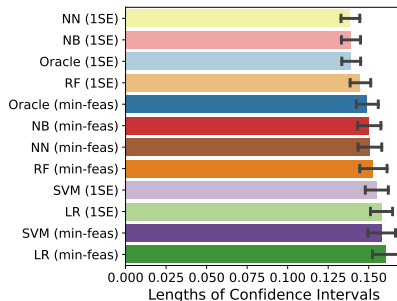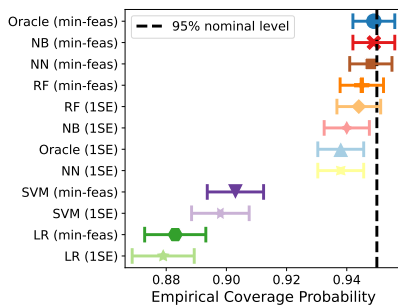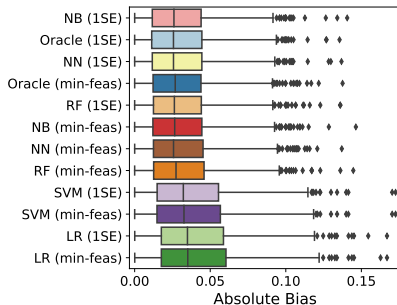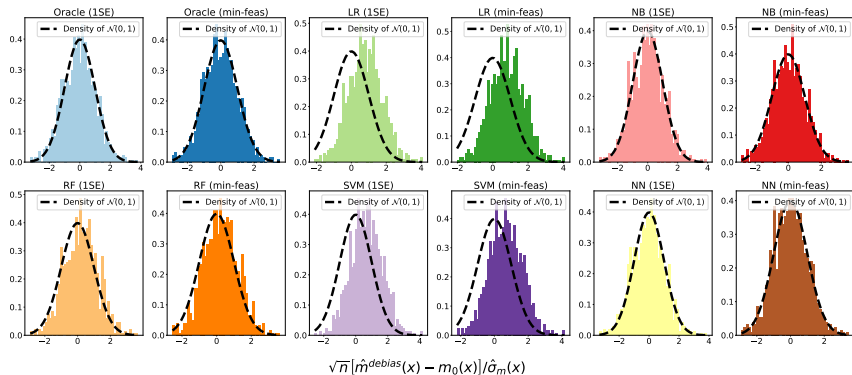
1. Consider all the observed galaxies by SDSS-IV within a thin redshift slice $0.4 \sim 0.4005$, among which 30.2% of their stellar masses are missing in the Firefly value-added catalog.

2. Fetch their spectroscopic and photometric properties from SDSS-IV DR16 database similar to the input catalog of Chang et al. (2015).

3. Apply feature transformation, remove highly linearly correlated covariates, and generate univariate B-spline base covariates of polynomial order 3 with 40 knots.

4. Incorporate RA, DEC, and the angular diameter distances from the galaxies to the two-dimensional spherical cosmic filaments by Zhang and Chen (2023); Zhang et al. (2022).

5. Control for the confounding effects by including the distances from galaxies to candidate galaxy clusters.

▶ **Final Dataset:** $n = 1185$ and $d = 1409$.

The observable data in causal inference are

$$\{(\mathbb{Y}_i, T_i, X_i)\}_{i=1}^{n} \subset \mathbb{R} \times \{0,1\} \times \mathbb{R}^d.$$

- $T_i \in \{0,1\}$ is a binary treatment assignment indicator;
- $\mathbb{Y}_i = T_i \cdot Y(1)_i + (1 - T_i) \cdot Y(0)_i$ with $Y(0), Y(1)$ as potential outcomes.

▶ **Objective:** Conduct valid inference on $\mathrm{E}\left[Y(1)|X, T = 1\right]$.

Treatment Group

X_1^T
⋮
$X_{\frac{n}{2}}^T$

$Y(1)_1$
⋮
$Y(1)_{\frac{n}{2}}$

Control Group

$X_{\frac{n}{2}+1}^T$
⋮
X_n^T

$Y(0)_{\frac{n}{2}+1}$
⋮
$Y(0)_n$

$\mathrm{E}\left[Y(1)|X, T = 1\right]$
based on
$\{(Y(1)_i, T_i, X_i)\}_{i=1}^{n}$

Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$E[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters $\gamma_1, \gamma_2 > 0$ is

$$\underset{\boldsymbol{w}_{(0)}, \boldsymbol{w}_{(1)} \in \mathbb{R}^n}{\arg\min} \sum_{i=1}^n \left[\widehat{\pi}_i w_{i(1)}^2 + (1 - \widehat{\pi}_i) w_{i(0)}^2 \right]$$

$$\text{s.t. } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(1)} \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \le \frac{\gamma_1}{n} \text{ and } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(0)} (1 - \widehat{\pi}_i) X_i \right\|_\infty \le \frac{\gamma_2}{n}.$$

- The extended debiased estimator becomes

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(0)})$$
$$= x^T \left(\widehat{\beta}_{(1)} - \widehat{\beta}_{(0)} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\widehat{w}_{i(1)} \cdot T_i \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(1)} \right) - \widehat{w}_{i(0)} \cdot (1 - T_i) \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(0)} \right) \right].$$

- The efficiency theory for this modified procedure is worth studying!

The observable data in causal inference are

$$\{(\mathbb{Y}_i, T_i, X_i)\}_{i=1}^n \subset \mathbb{R} \times \{0,1\} \times \mathbb{R}^d.$$

- $T_i \in \{0,1\}$ is a binary treatment assignment indicator;
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▶ **Objective:** Conduct valid inference on $\mathrm{E}\left[Y(1)|X, T = 1\right]$.

| Treatment Group | X_1^T $\vdots$ $X_{\frac{n}{2}}^T$ | $Y(1)_1$ $\vdots$ $Y(1)_{\frac{n}{2}}$ |
| Control Group | $X_{\frac{n}{2}+1}^T$ $\vdots$ X_n^T | $Y(0)_{\frac{n}{2}+1}$ $\vdots$ $Y(0)_n$ |

$\mathrm{E}\left[Y(1)|X, T = 1\right]$
based on
$\{(Y(1)_i, T_i, X_i)\}_{i=1}^n$

Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$\mathrm{E}[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters $\gamma_1, \gamma_2 > 0$ is

$$\underset{\boldsymbol{w}_{(0)}, \boldsymbol{w}_{(1)} \in \mathbb{R}^n}{\arg \min} \sum_{i=1}^{n} \left[\widehat{\pi}_i w_{i(1)}^2 + (1 - \widehat{\pi}_i) w_{i(0)}^2 \right]$$

$$\text{s.t. } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(1)} \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \le \frac{\gamma_1}{n} \text{ and } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(0)} (1 - \widehat{\pi}_i) X_i \right\|_\infty \le \frac{\gamma_2}{n}.$$

- The extended debiased estimator becomes

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(0)})$$

$$= x^T \left(\widehat{\beta}_{(1)} - \widehat{\beta}_{(0)} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\widehat{w}_{i(1)} \cdot T_i \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(1)} \right) - \widehat{w}_{i(0)} \cdot (1 - T_i) \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(0)} \right) \right].$$

- The efficiency theory for this modified procedure is worth studying!

The galaxy distribution is distorted along the line of sight due to the peculiar velocities of galaxies, *i.e.*, the so-called *finger-of-god* (Jackson, 1972) and *Kaiser* (Kaiser, 1987) effects.
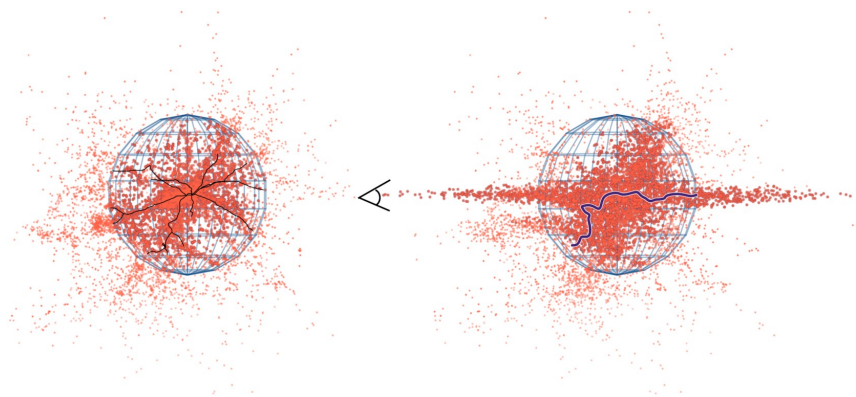

Figure: Redshift distortions along the line of sight (Kuchner et al., 2021).