

# Efficient Inference on High-Dimensional Linear Models With Missing Outcomes

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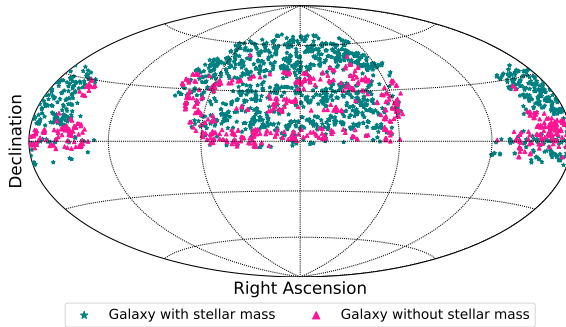
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Joint Work with *Alexander Giessing*<sup>2</sup> and *Yen-Chi Chen*<sup>1</sup>

<sup>1</sup>Department of Statistics, University of Washington

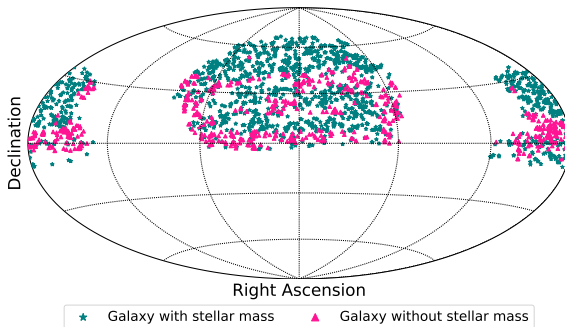
<sup>2</sup>Department of Statistics and Data Science, National University of Singapore

2024 ICSA Applied Statistics Symposium  
June 18, 2024



Observed galaxies on the high redshift slice  $0.4 \sim 0.401$ .

► **Notes:** Sloan Digital Sky Survey (SDSS) observes millions of galaxies, but some (estimated) galactic stellar masses are missing in the associated value-added catalog ([Comparat et al., 2017](#)).



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► **Scientific Question:**

*How can we quantify the uncertainty of the (estimated) stellar mass of a newly observed galaxy based on the spectroscopic and photometric properties?*

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- Potential data contamination;
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- Limiting usage of the observational run in SDSS for galaxy targets;
- Potential data contamination;
- Misclassification of galaxies as stars.

► **Statistical Problem:**

*How can we conduct valid and efficient inference on the regression function despite missing outcomes?*

- ① **Linearity:** The data  $\{(Y_i, R_i, X_i)\}_{i=1}^n$  are i.i.d. observations from a sparse linear model

$$Y = X^T \beta_0 + \epsilon \quad \text{with} \quad E(\epsilon|X) = 0 \quad \text{and} \quad E(\epsilon^2|X) = \sigma_\epsilon^2,$$

where  $\|\beta_0\|_0 = s_\beta \ll d$  and  $R \in \{0, 1\}$  when  $Y$  is missing or not.

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- ② **Missing At Random (MAR):**  $Y_i \perp\!\!\!\perp R_i | X_i$  for  $i = 1, \dots, n$ .

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- ① **Fully Observed Outcomes:** Debiased Lasso (Zhang and Zhang, 2014; van de Geer et al., 2014; Javanmard and Montanari, 2014):

$$\hat{\beta}^{\text{debias}} = \hat{\beta}_{\lambda} + \frac{1}{n} \hat{\Theta} \sum_{i=1}^n X_i (Y_i - X_i^T \hat{\beta}_{\lambda}),$$

- $\hat{\beta}_{\lambda}$  is a Lasso solution under the regularization parameter  $\lambda > 0$ ;
- $\hat{\Theta} \in \mathbb{R}^{d \times d}$  is an approximation to the matrix inverse  $(\frac{1}{n} \sum_{i=1}^n X_i X_i^T)^{-1}$ .

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- 2 **MAR Outcomes:** M-estimation framework with a Lasso-type debiased and doubly robust estimator ([Chakraborty et al., 2019](#)).

- **Drawbacks of Existing Approaches:** Inference on  $\beta_0 \in \mathbb{R}^d$ .
- ① Need to compute a  $d \times d$  debiasing matrix  $\hat{\Theta}$ .
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- ▶ **Our Focus:** Inference on  $m_0(x) = x^T \beta_0$ .
  - *Computational efficiency:* Our debiasing program is convex and only needs to solve for an  $n$ -dimensional weight vector.
  - *Statistical efficiency:* Our estimator is semi-parametrically efficient among all asymptotically linear estimators.

# Methodology and Asymptotic Theory

- The debiased Lasso estimator is given by

$$\hat{\beta}^{\text{debias}} = \hat{\beta}_{\lambda} + \frac{1}{n} \sum_{i=1}^n R_i \hat{\Theta} X_i \left( Y_i - X_i^T \hat{\beta}_{\lambda} \right).$$



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► **Issue:** This naive estimator may not be asymptotically normal in general ([van de Geer et al., 2014](#); [Javanmard and Montanari, 2014](#))!

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► **Idea:** Introduce a weight vector  $\mathbf{w} = (w_1, \dots, w_n)^T \in \mathbb{R}^n$  to replace

$$\frac{1}{\sqrt{n}} x^T \widehat{\Theta} \mathbf{X}_i \implies w_i \quad \text{for } i = 1, \dots, n$$

and formulate a generic debiased estimator

$$\widehat{m}^{\text{debias}}(x; \mathbf{w}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i R_i \left( Y_i - \mathbf{X}_i^T \widehat{\beta} \right). \quad (1)$$

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► **Question:** How do we estimate the weight vector  $\mathbf{w} = (w_1, \dots, w_n)^T$ ?

The conditional mean squared error of  $\sqrt{n} m^{\text{debias}}(x; \boldsymbol{w})$  is

$$\mathbb{E} \left[ \left( \sqrt{n} m^{\text{debias}}(x; \boldsymbol{w}) - \sqrt{n} m_0(x) \right)^2 \middle| X_1, \dots, X_n \right]$$

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 &= \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi_i}_{\text{Main Conditional Variance}} + \underbrace{\left[ \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x \right)^T \sqrt{n} (\beta_0 - \beta) \right]^2}_{\text{Conditional Bias}} \\
 &\quad + \underbrace{(\beta_0 - \beta)^T \left[ \sum_{i=1}^n w_i^2 \pi_i (1 - \pi_i) X_i X_i^T \right] (\beta_0 - \beta)}_{\text{Asymptotically Negligible Conditional Variance}}.
 \end{aligned}$$

► **Notes:**  $\pi_i := \mathbb{P}(R_i = 1 | X_i)$  is the propensity score under the MAR condition.

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- By Hölder's inequality,

$$\text{"Conditional Bias"} \leq \left[ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x \right\|_\infty \sqrt{n} \|\beta_0 - \beta\|_1 \right]^2.$$

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- We design our debiasing program as:

$$\min_{w \in \mathbb{R}^n} \sum_{i=1}^n w_i^2 \hat{\pi}_i \quad \text{subject to} \quad \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \hat{\pi}_i \cdot X_i \right\|_\infty \leq \frac{\gamma}{n}.$$



- 1 Compute the Lasso pilot estimate  $\hat{\beta}_\lambda$  on the complete-case data

$$\hat{\beta}_\lambda = \arg \min_{\beta \in \mathbb{R}^d} \left[ \frac{1}{2n} \sum_{i=1}^n R_i (Y_i - X_i^T \beta)^2 + \lambda \|\beta\|_1 \right].$$

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- ② Is our debiased estimator asymptotically normal?

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► **Answer:** The above two questions can be addressed by the *dual formulation* of our debiasing program!

## ► Primal Program:

$$\min_{w \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \hat{\pi}_i w_i^2 : \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \hat{\pi}_i \cdot X_i \right\|_{\infty} \leq \frac{\gamma}{n} \right\}.$$

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$$\min_{\ell \in \mathbb{R}^d} \left\{ \frac{1}{4n} \sum_{i=1}^n \hat{\pi}_i (X_i^T \ell)^2 + x^T \ell + \frac{\gamma}{n} \|\ell\|_1 \right\}.$$



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## ► Primal-Dual Relation: Under the strong duality,

$$\hat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \hat{\ell} \quad \text{for } i = 1, \dots, n.$$

► **Dual Program:**

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- It is an *unconstrained* optimization problem, and  $\gamma > 0$  can be fine-tuned via cross-validation.
- Primal-dual relation  $\hat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \hat{\ell}$ ,  $i = 1, \dots, n$  and dual consistency  $\hat{\ell} \xrightarrow{P} \ell_0$  reveal that

$$\sqrt{n} \left[ \hat{m}^{\text{debias}}(x; \hat{w}) - m_0(x) \right] = \underbrace{-\frac{1}{2\sqrt{n}} \sum_{i=1}^n R_i \epsilon_i X_i^T \ell_0}_{\text{i.i.d. sum!}} + \underbrace{\text{“Bias terms”}}_{o_P(1)}.$$

Theorem (Theorem 7 in [Zhang et al. 2023](#))

*Under regularity conditions,*

$$\sqrt{n} \left[ \widehat{m}^{\text{debias}}(x; \widehat{w}) - m_0(x) \right] \xrightarrow{d} \mathcal{N}(0, \sigma_m^2(x))$$

*with  $\sigma_m^2(x) = \lim_{n \rightarrow \infty} \sigma_\epsilon^2 \cdot x^T [\mathbb{E}(RXX^T)]^{-1} x$ .*

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- ② Under regularity conditions (Proposition 8 in [Zhang et al. 2023](#)),

$$\left| x^T [\mathbb{E}(RXX^T)]^{-1} x - \sum_{i=1}^n \widehat{\pi}_i \widehat{w}_i^2 \right| = o_P(1).$$

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- **Answer:** Our asymptotic normality result depends on the *in-sample* estimation error  $r_\pi$  of the propensity score:

$$\max_{1 \leq i \leq n} |\hat{\pi}_i - \pi_i| = O_P(r_\pi) \quad \text{with} \quad \pi_i = \pi(X_i), i = 1, \dots, n.$$

- Our debiased estimator performs even better when the estimated propensity scores on the training data are close to the true ones!!

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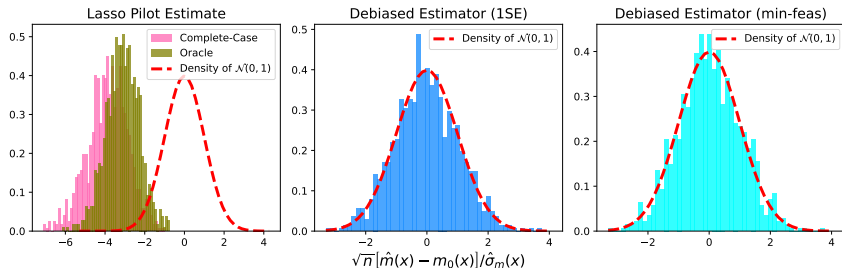
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- Our debiased estimator performs even better when the estimated propensity scores on the training data are close to the true ones!!
- This permits the use of complex machine learning methods with high learnability (Steinwart, 2001; Farrell et al., 2021; Gao et al., 2022).

# Simulation and Real-World Application



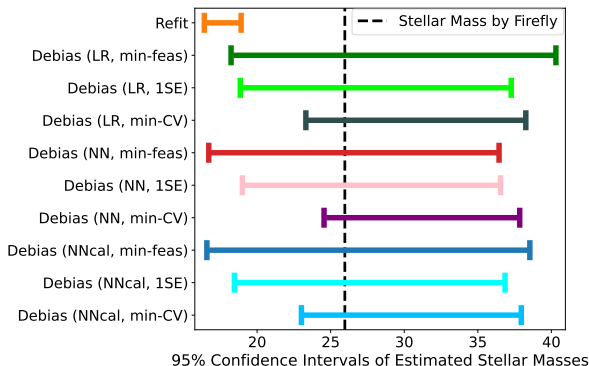
► **Effectiveness of Our Debiased Estimator:**

- Correct the bias of the Lasso pilot estimate.
- Asymptotically normal under a wide range of  $\gamma > 0$ .

► **Notes:** Our paper contains comprehensive comparisons with other existing methods.

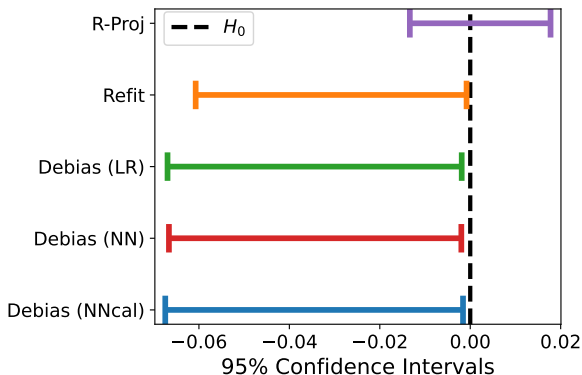
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- The 95% confidence intervals by our debiasing methods cover the true stellar mass of a new galaxy.

*Is it statistically significant that the stellar mass of a galaxy is negatively correlated with its distance to the nearby cosmic filament structures?*



- 95% confidence intervals by our debiasing methods exclude 0 and are all negative.

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- The dual form explains its computational and statistical efficiencies.
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- A novel application to the inference on galactic stellar mass.

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More details can be found in

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# Thank you!

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- ① **Lasso pilot estimate:** We adopt the scaled Lasso (Sun and Zhang, 2012) with its universal regularization parameter  $\lambda_0 = \sqrt{\frac{2 \log d}{n}}$  as the initialization. Specifically, it iteratively updates  $\hat{\beta}(\tilde{\lambda}), \hat{\sigma}_\epsilon(\tilde{\lambda}), \tilde{\lambda}$  via the jointly convex optimization program:

$$\left( \hat{\beta}(\tilde{\lambda}), \hat{\sigma}_\epsilon(\tilde{\lambda}) \right) = \arg \min_{\beta \in \mathbb{R}^d, \sigma_\epsilon > 0} \left[ \frac{1}{2n\sigma_\epsilon} \sum_{i=1}^n R_i(Y_i - X_i^T \beta)^2 + \frac{\sigma_\epsilon}{2} + \tilde{\lambda} \|\beta\|_1 \right].$$

- ② **Debiasing program:** We solve the primal program by Python package “CVXPY” (Diamond and Boyd, 2016; Agrawal et al., 2018) or R package “CVXR” (Fu et al., 2020). For the dual program, we formulate a coordinate descent algorithm (Wright, 2015) as:

$$\left[ \hat{\ell}(x) \right]_j \leftarrow \frac{\mathcal{S}_{\frac{\gamma}{n}} \left( -\frac{1}{2n} \sum_{i=1}^n \hat{\pi}_i \left( \sum_{k \neq j} X_{ik} X_{jk} \left[ \hat{\ell}(x) \right]_k \right) - x_j \right)}{\frac{1}{2n} \sum_{i=1}^n \hat{\pi}_i X_{ij}^2} \quad \text{for } j = 1, \dots, d,$$

where  $\mathcal{S}_{\frac{\gamma}{n}}(u) = \text{sign}(u) \cdot \left( u - \frac{\gamma}{n} \right)_+$  is the soft-thresholding operator.

- Suppose that we conduct a  $K$ -fold cross-validation on a candidate set  $\Gamma = \{\gamma_1, \dots, \gamma_m\}$  of the tuning parameter.
- For each  $\gamma_i \in \Gamma$ , we compute the cross-validated risk or error on each fold of the data as:

$$CV_k(\gamma_i), \quad k = 1, \dots, K.$$

- For each  $\gamma_i \in \Gamma$ , we calculate the standard error of  $CV_1(\gamma_i), \dots, CV_K(\gamma_i)$  as:

$$SD(\gamma_i) = \sqrt{\text{Var}(CV_1(\gamma_i), \dots, CV_K(\gamma_i))}, \quad SE(\gamma_i) = SD(\gamma_i)/\sqrt{K}.$$

- Let

$$CV(\gamma) = \frac{1}{K} \sum_{k=1}^K CV_k(\gamma) \quad \text{and} \quad \hat{\gamma} = \arg \min_{\gamma \in \Gamma} CV(\gamma).$$

The 1SE rule ([Breiman et al., 1984](#); [Chen and Yang, 2021](#)) selects  $\gamma_{1SE} \in \Gamma$  with as the one with the smallest  $CV(\gamma)$  such that

$$CV(\gamma_{1SE}) \geq CV(\hat{\gamma}) + SE(\hat{\gamma}).$$

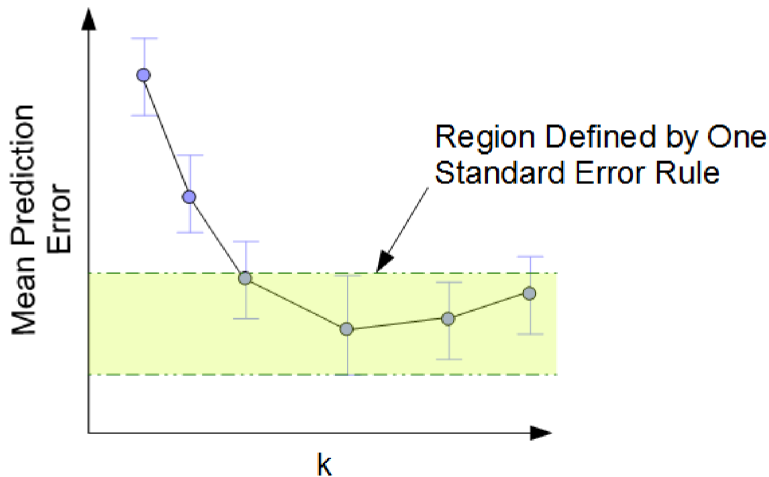


Figure: Illustration of the 1SE rule for selecting the model parameter.

- Consider the regression function  $m \equiv m(x) \in \mathbb{R}$  as the main parameter to be inferred and  $\beta \in \mathbb{R}^d$  as the high-dimensional nuisance parameter.
- Our generic debiased estimator  $m^{\text{debias}}(x, w)$  solves the sample-based estimating equation

$$\frac{1}{n} \sum_{i=1}^n \Xi_x(Y_i, R_i, X_i; m^{\text{debias}}, \beta) = m^{\text{debias}}(x; w) - x^T \beta - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot R_i \left( Y_i - X_i^T \beta \right) = 0.$$

- The Neyman near-orthogonalization condition ([Chernozhukov et al., 2018](#)) given  $\mathbf{X} = (X_1, \dots, X_n)^T \in \mathbb{R}^{n \times d}$  at  $(m_0, \beta_0) = (x^T \beta_0, \beta_0)$  requires

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \Xi_x(Y_i, R_i, X_i; m_0, \beta_0) \middle| \mathbf{X} \right] &= 0, \\ \sup_{\beta \in \mathcal{T}_n} \left| \left\{ \frac{\partial}{\partial \beta} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \Xi_x(Y_i, R_i, X_i; m, \beta) \middle| \mathbf{X} \right] \middle|_{(m_0, \beta_0)} \right\}^T (\beta - \beta_0) \right| &\leq \frac{\delta_n}{\sqrt{n}}, \end{aligned} \quad (2)$$

where  $\mathcal{T}_n$  is a properly shrinking neighborhood of  $\beta_0$  and  $\delta_n = o(1)$ .

- Both conditions in (2) hold true, because for any  $\beta \in \mathcal{T}_n$  and some convex set  $\mathcal{B}$  containing  $\beta_0$ , we have that

$$\begin{aligned}
 & \left| \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \beta} \mathbb{E} [\Xi_x(Y_i, R_i, X_i; m, \beta) | X] \Big|_{(m_0, \beta_0)} \right\}^T (\beta - \beta_0) \right| \\
 &= \left| \left[ x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \pi(X_i) X_i \right]^T (\beta_0 - \beta) \right| \\
 &\leq \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \hat{\pi}_i \cdot X_i \right\|_{\infty} \|\beta - \beta_0\|_1 \quad \text{by Hölder's inequality} \\
 &\leq \frac{\gamma}{n} \|\beta - \beta_0\|_1 \quad \text{by the box constraint in our debiasing program} \\
 &\leq \frac{\delta_n}{\sqrt{n}} \quad \text{by setting } \mathcal{T}_n = \left\{ \beta \in \mathcal{B} \subset \mathbb{R}^d : \|\beta - \beta_0\|_1 \leq \frac{\sqrt{n}\delta_n}{\gamma} \right\}.
 \end{aligned}$$

- Our debiasing program optimizes the (estimated) variance among all the estimators satisfying Neyman near-orthogonalization (2).
- (2) also allows our debiasing program to *de-correlate* the Lasso pilot regression from propensity score estimation and weight optimization.

- **Goal:** Establish the asymptotic normality of our debiased estimator

$$\widehat{m}^{\text{debias}}(x; \widehat{w}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{w}_i R_i \left( Y_i - X_i^T \widehat{\beta} \right).$$

- Linearity assumption  $Y_i = X_i^T \beta_0 + \epsilon_i$  for  $i = 1, \dots, n$  implies

$$\sqrt{n} \left[ \widehat{m}^{\text{debias}}(x; \widehat{w}) - m_0(x) \right] = \underbrace{\sum_{i=1}^n \widehat{w}_i R_i \epsilon_i}_{\text{Not an i.i.d. sum!}} + \left[ x - \frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{w}_i R_i X_i \right]^T \sqrt{n} \left( \widehat{\beta} - \beta_0 \right),$$

- Dual relation  $\widehat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \widehat{\ell}$  for  $i = 1, \dots, n$  and dual consistency  $\widehat{\ell} \xrightarrow{P} \ell_0$  reveal that

$$\begin{aligned} \sqrt{n} \left[ \widehat{m}^{\text{debias}}(x; \widehat{w}) - m_0(x) \right] &= -\frac{1}{2\sqrt{n}} \sum_{i=1}^n R_i \epsilon_i X_i^T \widehat{\ell} + \left[ x + \frac{1}{2n} \sum_{i=1}^n R_i X_i X_i^T \widehat{\ell} \right]^T \sqrt{n} \left( \beta_0 - \widehat{\beta} \right) \\ &= \underbrace{-\frac{1}{2\sqrt{n}} \sum_{i=1}^n R_i \epsilon_i X_i^T \ell_0}_{\text{i.i.d. sum!}} + \underbrace{\text{“Bias terms”}}_{op(1)}. \end{aligned}$$

- ① The covariate vector  $X \in \mathbb{R}^d$  and the noise  $\epsilon \in \mathbb{R}$  are sub-Gaussian.
- ② There exists a constant  $\kappa_R > 0$  such that

$$\inf_{v \in \mathbb{S}^{d-1}} \mathbb{E} [R(X^T v)^2] \geq \kappa_R^2 \quad \text{with} \quad \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}.$$

- ③ Given any  $n \geq 1$  and  $\delta \in (0, 1)$ , there exists  $r_\pi \equiv r_\pi(n, \delta) > 0$  such that

$$\mathbb{P} \left( \max_{1 \leq i \leq n} |\hat{\pi}_i - \pi_i| > r_\pi \right) < \delta \quad \text{with} \quad \pi_i = \pi(X_i), i = 1, \dots, n.$$

- ④ Define the population dual program as:

$$\min_{\ell \in \mathbb{R}^d} \left\{ \frac{1}{4} \mathbb{E} [R(X^T \ell)^2] + x^T \ell \right\},$$

whose exact solution is  $\ell_0(x) = -2 [\mathbb{E} (RXX^T)]^{-1} x$ . We assume that the  $r_\ell$ -approximation  $\tilde{\ell}(x)$  to  $\ell_0(x)$  is sparse with  $r_\ell \in [0, \frac{1}{2}]$ , i.e.,

$$s_\ell(x) = \left\| \tilde{\ell}(x) \right\|_0 \ll \min\{n, d\} \quad \text{with} \quad \tilde{\ell}(x) = \arg \min_{u \in \mathbb{R}^d} \{ \|u\|_0 : \|u - \ell_0(x)\|_2 \leq r_\ell \|\ell_0(x)\|_2 \}.$$

Methods to be compared:

- “DL-Jav”: The debiased Lasso by [Javanmard and Montanari \(2014\)](#).
- “DL-vdG”: The debiased Lasso by [van de Geer et al. \(2014\)](#).
- “Refit”: Run the regular least-square regression on the support set of the Lasso pilot estimate ([Belloni and Chernozhukov, 2013](#)).

Implementation settings of the above methods:

- Complete-case (CC) data  $\{(X_i, Y_i, R_i = 1)\}_{i=1}^n$ ;
- Inverse probability weighted (IPW) data  $\left\{ \left( \frac{X_i}{\sqrt{\hat{\pi}_i}}, \frac{Y_i}{\sqrt{\hat{\pi}_i}}, R_i = 1 \right) \right\}_{i=1}^n$ ;
- Oracle fully observed data  $(X_i, Y_i)$  for  $i = 1, \dots, n$ .

Evaluation metrics over 1000 Monte Carlo experiments:

- Average absolute bias  $|\hat{m}^{\text{debias}}(x) - m_0(x)|$ ;
- Average coverage and average length of the yielded 95% confidence intervals.



# W Simulation Results Under Gaussian Noises (I)

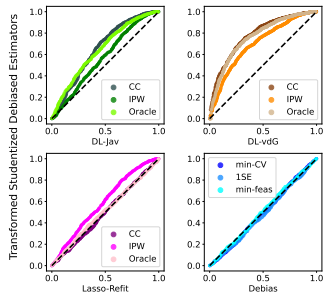
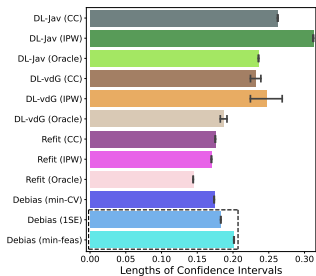
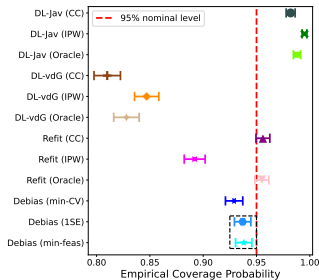
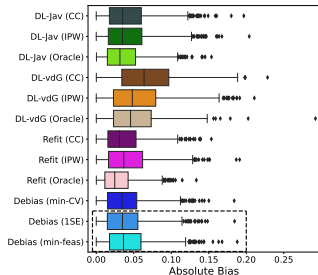


Figure: Sparse  $\beta_0^{sp}$  and sparse  $x^{(2)}$  with  $X_i \sim \mathcal{N}(\mathbf{0}, \Sigma^{\text{cs}})$ ,  $i = 1, \dots, n$ .

# W Simulation Results Under Gaussian Noises (II)

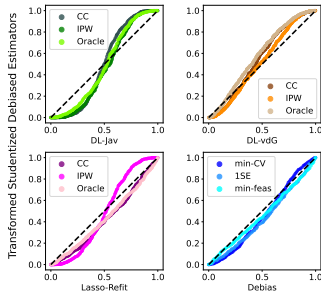
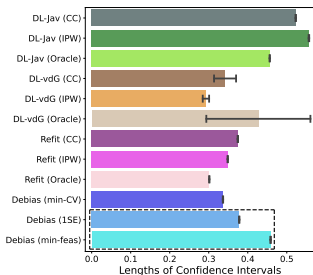
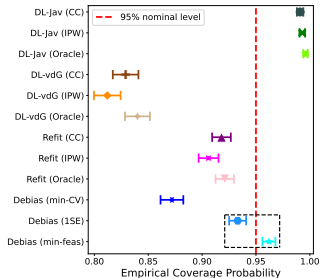
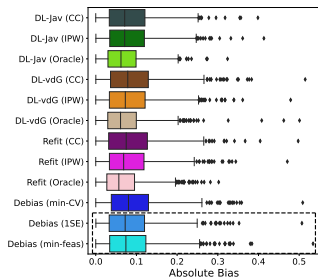


Figure: Pseudo-dense  $\beta_0^{pd}$  and sparse  $x^{(2)}$  with  $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{ar}})$ ,  $i = 1, \dots, n$ .

# Simulation Results Under Laplace(0, 1/√2) Noises

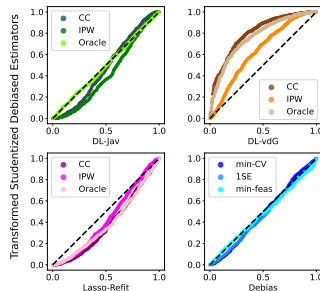
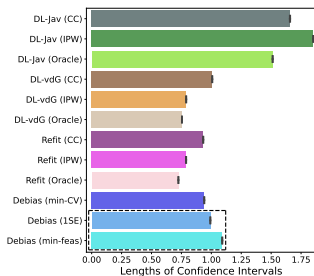
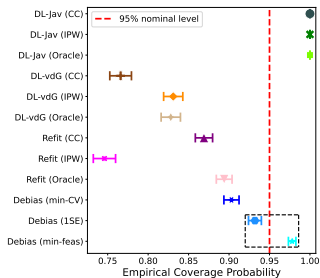
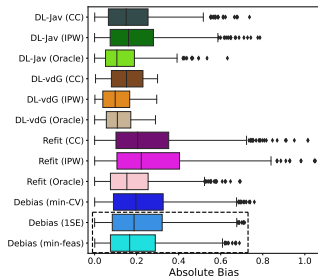


Figure: Dense  $\beta_0^{de}$  and sparse  $x^{(2)}$  with  $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{cs}})$ ,  $i = 1, \dots, n$ .

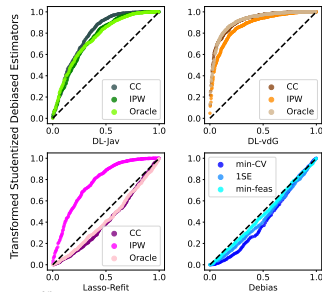
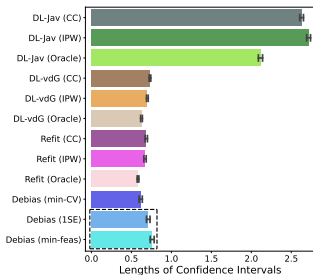
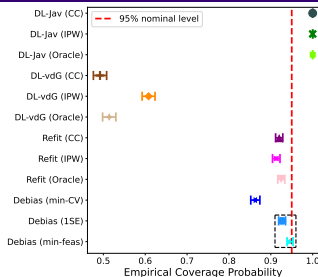
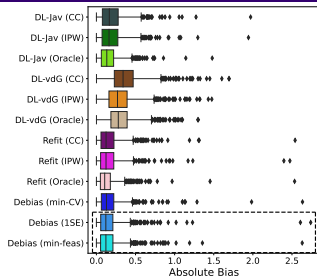
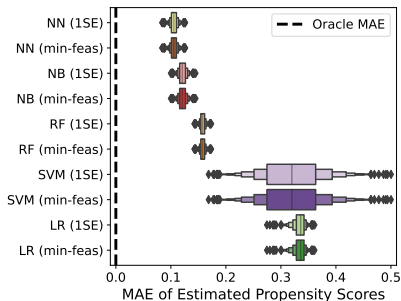
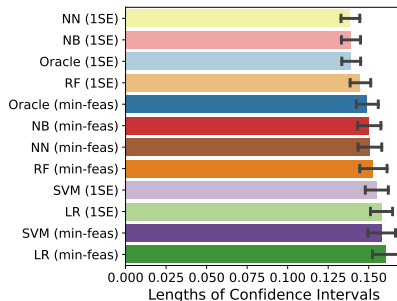
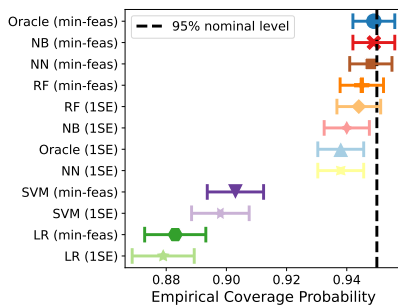
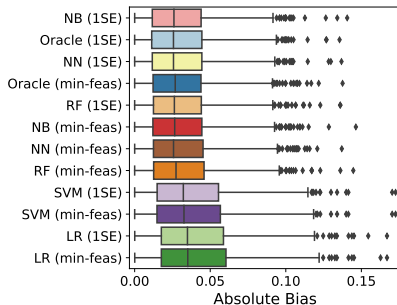
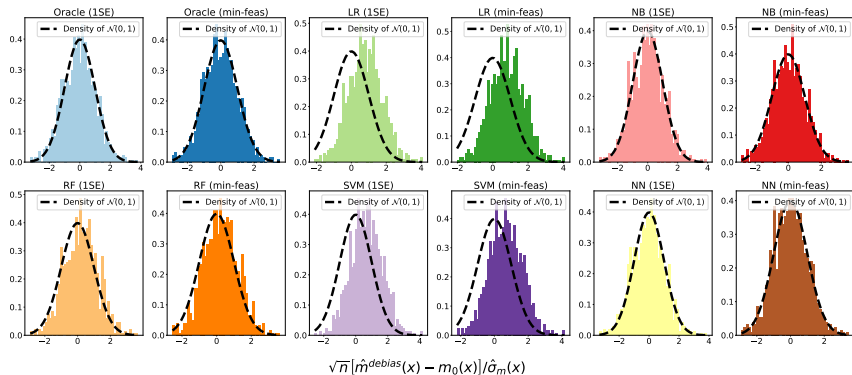


Figure: Pseudo-dense  $\beta_0^{pd}$  and dense  $x^{(4)}$  with  $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{ar}})$ ,  $i = 1, \dots, n$ . Note that the mean-zero  $t_2$  distribution has *infinite* variance.

- ① True propensity score model:  $P(R_i = 1|X_i) = \Phi\left(-4 + \sum_{k=1}^K Z_{ik}\right)$ , where  $(Z_{i1}, \dots, Z_{iK})$  contains all polynomial combinations of the first eight components  $X_{i1}, \dots, X_{i8}$  of  $X_i \in \mathbb{R}^{1000}$  with degrees  $\leq 2$ .
- ② Estimate the propensity scores  $\pi(X_i), i = 1, \dots, n$  by the following nonlinear/nonparametric machine learning methods:
  - **Gaussian Naive Bayes ("NB")**.
  - **Random Forest ("RF")**: 100 trees, bootstrapping samples, and the Gini impurity.
  - **Support Vector Machine ("SVM")**: Gaussian radial basis function.
  - **Neural Network ("NN")**: Two hidden layers of size  $80 \times 50$  and ReLU  $h(x) = \max\{x, 0\}$  as the activation function.
- ③ Include an extra evaluation metric as the average mean absolute error ("Avg-MAE") for the estimated propensity scores.





- ① Consider all the observed galaxies by SDSS-IV within a thin redshift slice  $0.4 \sim 0.4005$ , among which 30.2% of their stellar masses are missing in the Firefly value-added catalog.
  - ② Fetch their spectroscopic and photometric properties from SDSS-IV DR16 database similar to the input catalog of [Chang et al. \(2015\)](#).
  - ③ Apply feature transformation, remove highly linearly correlated covariates, and generate univariate B-spline base covariates of polynomial order 3 with 40 knots.
  - ④ Incorporate RA, DEC, and the angular diameter distances from the galaxies to the two-dimensional spherical cosmic filaments by [Zhang and Chen \(2023\)](#); [Zhang et al. \(2022\)](#).
  - ⑤ Control for the confounding effects by including the distances from galaxies to candidate galaxy clusters.
- **Final Dataset:**  $n = 1185$  and  $d = 1409$ .




The observable data in causal inference are

$$\{(\mathbb{Y}_i, T_i, X_i)\}_{i=1}^n \subset \mathbb{R} \times \{0, 1\} \times \mathbb{R}^d.$$

- $T_i \in \{0, 1\}$  is a binary treatment assignment indicator;
- $\mathbb{Y}_i = T_i \cdot Y(1)_i + (1 - T_i) \cdot Y(0)_i$  with  $Y(0), Y(1)$  as potential outcomes.

► **Objective:** Conduct valid inference on  $E[Y(1)|X, T = 1]$ .

Treatment Group	$X_1^T$	$Y(1)_1$
	$\vdots$	$\vdots$
	$X_{\frac{n}{2}}^T$	$Y(1)_{\frac{n}{2}}$
Control Group	$X_{\frac{n}{2}+1}^T$	$Y(0)_{\frac{n}{2}+1}$
	$\vdots$	$\vdots$
	$X_n^T$	$Y(0)_n$


 $E[Y(1)|X, T = 1]$   
 based on  
 $\{(Y(1)_i, T_i, X_i)\}_{i=1}^n$

Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$E[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters  $\gamma_1, \gamma_2 > 0$  is

$$\begin{aligned} & \arg \min_{\mathbf{w}_{(0)}, \mathbf{w}_{(1)} \in \mathbb{R}^n} \sum_{i=1}^n \left[ \hat{\pi}_i w_{i(1)}^2 + (1 - \hat{\pi}_i) w_{i(0)}^2 \right] \\ \text{s.t. } & \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(1)} \cdot \hat{\pi}_i \cdot X_i \right\|_{\infty} \leq \frac{\gamma_1}{n} \quad \text{and} \quad \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(0)} (1 - \hat{\pi}_i) X_i \right\|_{\infty} \leq \frac{\gamma_2}{n}. \end{aligned}$$

- The extended debiased estimator becomes

$$\begin{aligned} & \hat{m}^{\text{debias}}(x; \hat{\mathbf{w}}_{(1)}, \hat{\mathbf{w}}_{(0)}) \\ &= x^T \left( \hat{\beta}_{(1)} - \hat{\beta}_{(0)} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \hat{w}_{i(1)} \cdot T_i \left( \mathbb{Y}_i - X_i^T \hat{\beta}_{(1)} \right) - \hat{w}_{i(0)} \cdot (1 - T_i) \left( \mathbb{Y}_i - X_i^T \hat{\beta}_{(0)} \right) \right]. \end{aligned}$$

- The efficiency theory for this modified procedure is worth studying!


The observable data in causal inference are

$$\{(\mathbb{Y}_i, T_i, X_i)\}_{i=1}^n \subset \mathbb{R} \times \{0, 1\} \times \mathbb{R}^d.$$

- $T_i \in \{0, 1\}$  is a binary treatment assignment indicator;
- $\mathbb{Y}_i = T_i \cdot Y(1)_i + (1 - T_i) \cdot Y(0)_i$  with  $Y(0), Y(1)$  as potential outcomes.

► **Objective:** Conduct valid inference on  $E[Y(1)|X, T = 1]$ .

Treatment Group	$X_1^T$	$Y(1)_1$
	$\vdots$	$\vdots$
	$X_{\frac{n}{2}}^T$	$Y(1)_{\frac{n}{2}}$
Control Group	$X_{\frac{n}{2}+1}^T$	$Y(0)_{\frac{n}{2}+1}$
	$\vdots$	$\vdots$
	$X_n^T$	$Y(0)_n$


 $E[Y(1)|X, T = 1]$   
 based on  
 $\{(Y(1)_i, T_i, X_i)\}_{i=1}^n$

Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$E[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters  $\gamma_1, \gamma_2 > 0$  is

$$\begin{aligned} & \arg \min_{\mathbf{w}_{(0)}, \mathbf{w}_{(1)} \in \mathbb{R}^n} \sum_{i=1}^n \left[ \hat{\pi}_i w_{i(1)}^2 + (1 - \hat{\pi}_i) w_{i(0)}^2 \right] \\ \text{s.t. } & \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(1)} \cdot \hat{\pi}_i \cdot X_i \right\|_{\infty} \leq \frac{\gamma_1}{n} \quad \text{and} \quad \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_{i(0)} (1 - \hat{\pi}_i) X_i \right\|_{\infty} \leq \frac{\gamma_2}{n}. \end{aligned}$$

- The extended debiased estimator becomes

$$\begin{aligned} & \hat{m}^{\text{debias}}(x; \hat{\mathbf{w}}_{(1)}, \hat{\mathbf{w}}_{(0)}) \\ &= x^T (\hat{\beta}_{(1)} - \hat{\beta}_{(0)}) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \hat{w}_{i(1)} \cdot T_i \left( \mathbb{Y}_i - X_i^T \hat{\beta}_{(1)} \right) - \hat{w}_{i(0)} \cdot (1 - T_i) \left( \mathbb{Y}_i - X_i^T \hat{\beta}_{(0)} \right) \right]. \end{aligned}$$

- The efficiency theory for this modified procedure is worth studying!

The galaxy distribution is distorted along the line of sight due to the peculiar velocities of galaxies, *i.e.*, the so-called *finger-of-god* (Jackson, 1972) and *Kaiser* (Kaiser, 1987) effects.

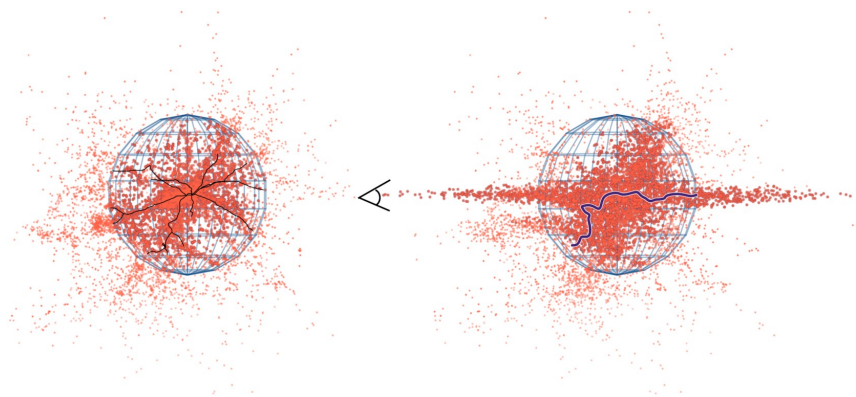


Figure: Redshift distortions along the line of sight (Kuchner et al., 2021).