

Doubly Robust Inference on Causal Derivative Effects for Continuous Treatments

Yikun Zhang

Joint work with *Professor Yen-Chi Chen*

*Department of Statistics,
University of Washington*

ENAR 2025 Spring Meeting
March 25, 2025

The Role of Derivatives in Causal Inference

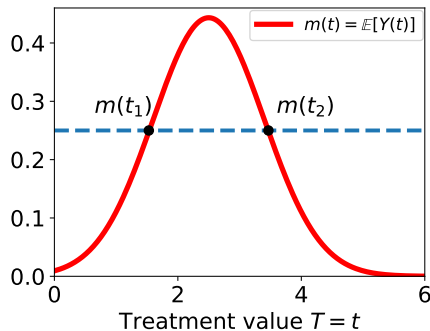
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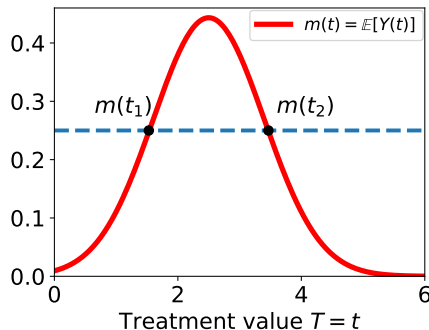
- $\mathbb{E}[Y(t)]$ = mean potential outcome under a static intervention $T = t$.
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- While $m(t_1) = m(t_2)$, the derivative effects $m'(t_1)$, $m'(t_2)$ are distinct!
- The **derivative effect curve** $\theta(t) = m'(t) = \frac{d}{dt}\mathbb{E}[Y(t)]$ is a continuous generalization to the average treatment effect $\mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$.

Our causal estimand of interest is the **derivative effect curve**

$$t \mapsto \theta(t) = m'(t) = \frac{d}{dt} \mathbb{E}[Y(t)] \quad \text{for } t \in \mathcal{T} \subset \mathbb{R}.$$

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There are some closely related but distinct estimands:

- *Incremental Causal/Treatment Effect* ([Kennedy, 2019](#); [Rothenhäusler and Yu, 2019](#)):

$$\mathbb{E}[Y(T + \delta)] - \mathbb{E}[Y(T)] \quad \text{for some deterministic } \delta > 0.$$

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$$\mathbb{E}[\theta(T)] = \mathbb{E} \left[\frac{\partial}{\partial t} \mathbb{E}(Y|T, S) \right], \quad \text{where } S \in \mathcal{S} \subset \mathbb{R}^d \text{ is a covariate vector.}$$

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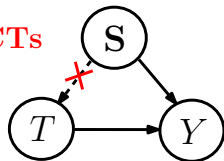
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Pros These new estimands may have more realistic interpretations in the actual context.

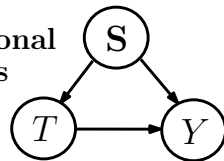
Cons They quantify only the overall causal effects, not those at a specific level of interest.

Identification Assumptions with Observational Data

RCTs

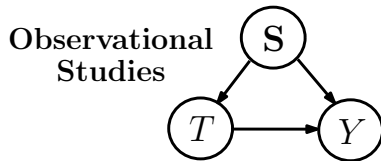
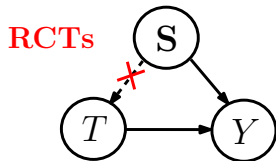


Observational
Studies



¹Some mild interchangeability assumptions are needed; see Theorem 1.1 in [Shao \(2003\)](#).

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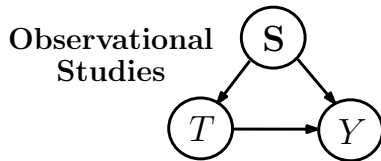
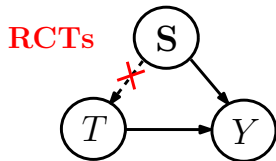


Assumption (Identification Conditions)

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- 3 (**Positivity**) The conditional density satisfies $p_{T|S}(t|s) \geq p_{\min} > 0$ for all $(t, s) \in \mathcal{T} \times S$.

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$$\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)] \stackrel{(*)^1}{=} \mathbb{E} \left[\frac{\partial}{\partial t} \mathbb{E}(Y|T = t, S) \right].$$

- The positivity condition is required for $\frac{\partial}{\partial t} \mu(t, s) = \frac{\partial}{\partial t} \mathbb{E}(Y|T = t, S = s)$ to be well-defined on $\mathcal{T} \times S$.

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An Example of the Positivity Violation

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There exists a constant $p_{\min} > 0$ such that $p_{T|S}(t|s) \geq p_{\min}$ for all $(t, s) \in \mathcal{T} \times \mathcal{S}$.

- Positivity is a very strong assumption with continuous treatments!

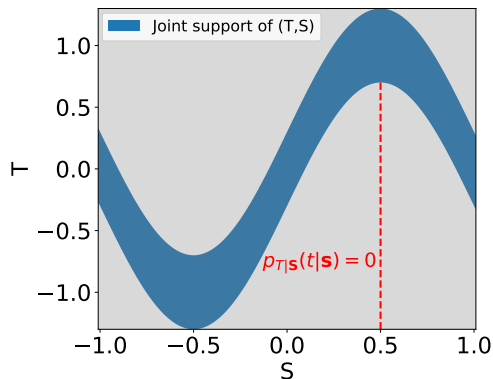
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$$T = \sin(\pi S) + E, \quad E \sim \text{Uniform}[-0.3, 0.3], \quad S \sim \text{Uniform}[-1, 1], \quad \text{and} \quad E \perp\!\!\!\perp S.$$



Note that $p_{T|S}(t|s) = 0$ in the gray regions, and the positivity condition fails.

Highlights of Today's Talk

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Under the positivity condition:

- 1 Propose a doubly robust (DR) estimator of $\theta(t)$ via kernel smoothing.

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- 2 $m(t)$ and $\theta(t)$ are identifiable with an additive structural assumption:

$$Y(t) = \bar{m}(t) + \eta(S) + \epsilon. \tag{1}$$

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- 4 Propose our bias-corrected IPW and DR estimators for $m(t)$ and $\theta(t)$.
 - Has a novel connection to nonparametric support and level set estimation problems.

Nonparametric Inference on $\theta(t)$ Under Positivity



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Given that $\mu(t, s) = \mathbb{E}(Y|T = t, S = s)$, we have

RA or G-computation:
$$\begin{cases} m(t) = \mathbb{E}[Y(t)] = \mathbb{E}[\mu(t, S)], \\ \theta(t) = \frac{d}{dt}\mathbb{E}[Y(t)] = \frac{d}{dt}\mathbb{E}[\mu(t, S)] = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t, S)\right]. \end{cases}$$

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IPW:
$$\begin{cases} m(t) = \mathbb{E}[Y(t)] = \lim_{h \rightarrow 0} \mathbb{E}\left[\frac{Y}{p_{T|S}(T|S)} \cdot \frac{1}{h}K\left(\frac{T-t}{h}\right)\right], \\ \theta(t) = \frac{d}{dt}\mathbb{E}[Y(t)] = ??? \end{cases}$$

- $K : \mathbb{R} \rightarrow [0, \infty)$ is a kernel function, e.g., $K(u) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) & \text{(Gaussian),} \\ \frac{3}{4}(1 - u^2) \cdot \mathbb{1}_{\{|u| \leq 1\}} & \text{(Parabolic).} \end{cases}$
- $h > 0$ is a smoothing bandwidth parameter.

Dose-Response Curve Estimation Under Positivity

Given the observed data $\{(Y_i, T_i, S_i)\}_{i=1}^n$, there are three main strategies for estimating

$$m(t) = \mathbb{E}[Y(t)] = \mathbb{E}[\mu(t, S)] = \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{Y \cdot K\left(\frac{T-t}{h}\right)}{h \cdot p_{T|S}(T|S)} \right].$$

① **RA Estimator** (Robins, 1986; Gill and Robins, 2001):

$$\hat{m}_{\text{RA}}(t) = \frac{1}{n} \sum_{i=1}^n \hat{\mu}(t, S_i).$$

② **IPW Estimator** (Hirano and Imbens, 2004; Imai and van Dyk, 2004):

$$\hat{m}_{\text{IPW}}(t) = \frac{1}{nh} \sum_{i=1}^n \frac{K\left(\frac{T_i-t}{h}\right)}{\hat{p}_{T|S}(T_i|S_i)} \cdot Y_i.$$

③ **DR Estimator** (Kallus and Zhou, 2018; Colangelo and Lee, 2020):

$$\hat{m}_{\text{DR}}(t) = \frac{1}{nh} \sum_{i=1}^n \left\{ \frac{K\left(\frac{T_i-t}{h}\right)}{\hat{p}_{T|S}(T_i|S_i)} \cdot [Y_i - \hat{\mu}(t, S_i)] + h \cdot \hat{\mu}(t, S_i) \right\}.$$

To estimate $\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)] = \mathbb{E}\left[\frac{\partial}{\partial t} \mu(t, S)\right]$ from $\{(Y_i, T_i, S_i)\}_{i=1}^n$, we could also have three strategies:

1 RA Estimator:

$$\hat{\theta}_{\text{RA}}(t) = \frac{1}{n} \sum_{i=1}^n \hat{\beta}(t, S_i) \quad \text{with} \quad \beta(t, s) = \frac{\partial}{\partial t} \mu(t, s).$$

Question: How can we generalize the IPW form $m(t) = \lim_{h \rightarrow 0} \mathbb{E}\left[\frac{Y \cdot K\left(\frac{T-t}{h}\right)}{h \cdot p_{T|S}(T|S)}\right]$ to identify and estimate $\theta(t)$?

RA and IPW Estimators of $\theta(t)$ Under Positivity

To estimate $\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)] = \mathbb{E}\left[\frac{\partial}{\partial t} \mu(t, S)\right]$ from $\{(Y_i, T_i, S_i)\}_{i=1}^n$, we could also have three strategies:

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2 IPW Estimator: Inspired by the derivative estimator in [Mack and Müller \(1989\)](#), we propose

$$\hat{\theta}_{\text{IPW}}(t) = \frac{1}{n} \sum_{i=1}^n \frac{Y_i \cdot \left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{h^2 \cdot \kappa_2 \cdot \hat{p}_{T|S}(T_i|S_i)} \quad \text{with} \quad \kappa_2 = \int u^2 \cdot K(u) du.$$

Doubly Robust Estimator for $\theta(t)$ Under Positivity

$$\hat{\theta}_{\text{RA}}(t) = \frac{1}{n} \sum_{i=1}^n \hat{\beta}(t, \mathbf{S}_i) \quad \text{"+"} \quad \hat{\theta}_{\text{IPW}}(t) = \frac{1}{nh^2} \sum_{i=1}^n \frac{\left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{\kappa_2 \cdot \hat{p}_{T|S}(T_i|\mathbf{S}_i)} \cdot Y_i \quad \Rightarrow$$

$$\hat{\theta}_{\text{DR}}(t) = \underbrace{\frac{1}{nh^2} \sum_{i=1}^n \frac{\left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{\kappa_2 \cdot \hat{p}_{T|S}(T_i|\mathbf{S}_i)} \left[Y_i - \hat{\mu}(t, \mathbf{S}_i) - (T_i - t) \cdot \hat{\beta}(t, \mathbf{S}_i) \right]}_{\text{New IPW component}} + \underbrace{\frac{1}{n} \sum_{i=1}^n \hat{\beta}(t, \mathbf{S}_i)}_{\text{RA component}}.$$

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The “New IPW component” leverages a local polynomial approximation to push the residual of the IPW component to (roughly) second order.

- Neyman orthogonality (Neyman, 1959; Chernozhukov et al., 2018) holds for this form of $\hat{\theta}_{\text{DR}}(t)$ as $h \rightarrow 0$.

Theorem (Theorem 1 in Zhang and Chen 2025)

Under some regularity assumptions and

- ① $\hat{\mu}, \hat{\beta}, \hat{p}_{T|S}$ are estimated on a dataset independent of $\{(Y_i, T_i, S_i)\}_{i=1}^n$;
- ② at least one of the model specification conditions hold:
 - $\hat{p}_{T|S}(t|s) \xrightarrow{P} \bar{p}_{T|S}(t|s) = p_{T|S}(t|s)$ (**conditional density model**),
 - $\hat{\mu}(t, s) \xrightarrow{P} \bar{\mu}(t, s) = \mu(t, s)$ and $\hat{\beta}(t, s) \xrightarrow{P} \bar{\beta}(t, s) = \beta(t, s)$ (**outcome model**);
- ③ $\sup_{|u-t| \leq h} \left\| \hat{p}_{T|S}(u|S) - p_{T|S}(u|S) \right\|_{L_2} \left[\left\| \hat{\mu}(t, S) - \mu(t, S) \right\|_{L_2} + h \left\| \hat{\beta}(t, S) - \beta(t, S) \right\|_{L_2} \right] = o_P \left(\frac{1}{\sqrt{nh}} \right),$

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we prove that

- $\sqrt{nh^3} \left[\hat{\theta}_{\text{DR}}(t) - \theta(t) \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_{h,t} \left(Y_i, T_i, S_i; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S} \right) + o_P(1).$
- $\sqrt{nh^3} \left[\hat{\theta}_{\text{DR}}(t) - \theta(t) - h^2 B_{\theta}(t) \right] \xrightarrow{d} \mathcal{N}(0, V_{\theta}(t)).$

An asymptotically valid inference on $\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)]$ can be conducted through

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① We estimate $V_{\theta}(t) = \mathbb{E} \left[\phi_{h,t}^2 \left(Y, T, S; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S} \right) \right]$ with

$$\phi_{h,t} \left(Y, T, S; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S} \right) = \frac{\left(\frac{T-t}{h} \right) K \left(\frac{T-t}{h} \right)}{\sqrt{h} \cdot \kappa_2 \cdot \bar{p}_{T|S}(T|S)} \cdot [Y - \bar{\mu}(t, S) - (T-t) \cdot \bar{\beta}(t, S)]$$

$$\text{by } \hat{V}_{\theta}(t) = \frac{1}{n} \sum_{i=1}^n \phi_{h,t}^2 \left(Y, T, S; \hat{\mu}, \hat{\beta}, \hat{p}_{T|S} \right).$$

Statistical Inference on $\theta(t)$

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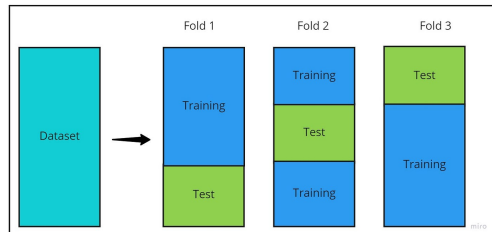
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- ① We estimate $V_{\theta}(t) = \mathbb{E} \left[\phi_{h,t}^2(Y, T, S; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S}) \right]$ with

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$$\text{by } \hat{V}_{\theta}(t) = \frac{1}{n} \sum_{i=1}^n \phi_{h,t}^2(Y, T, S; \hat{\mu}, \hat{\beta}, \hat{p}_{T|S}).$$

- ② $\hat{\mu}, \hat{\beta}, \hat{p}_{T|S}$ can be estimated via sample-splitting or cross-fitting.



An asymptotically valid inference on $\theta(t) = \frac{d}{dt} \mathbb{E} [Y(t)]$ can be conducted through

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- 2 $\hat{\mu}, \hat{\beta}, \hat{p}_{T|S}$ can be estimated via sample-splitting or cross-fitting.
- 3 The explicit form of $B_{\theta}(t)$ is complicated, but $h^2 B_{\theta}(t)$ is asymptotically negligible when $h = O \left(n^{-\frac{1}{5}} \right)$.
 - This order aligns with the outputs from usual bandwidth selection methods (Wand and Jones, 1994; Wasserman, 2006).

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- $t \mapsto \theta(t) := \Psi(P_0)(t)$ is *not* pathwise differentiable (Bickel et al., 1998; Hirano and Porter, 2012; Luedtke and van der Laan, 2016):

$$\forall t \in \mathcal{T}, \quad \exists \{P_\epsilon : \epsilon \in \mathbb{R}\} \quad \text{s.t.} \quad \lim_{\epsilon \rightarrow 0} \frac{\Psi(P_\epsilon)(t) - \Psi(P_0)(t)}{\epsilon} \quad \text{does not exist.}$$

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- Up to a shrinking bias $O(h^2)$, the efficient influence function for $\Phi(P_0)(t)$ leads to

$$\hat{\theta}_{\text{EIF}}(t) = \frac{1}{nh^2} \sum_{i=1}^n \frac{\left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{\kappa_2 \cdot \hat{p}_{T|S}(T_i|S_i)} [Y_i - \hat{\mu}(T_i, S_i)] + \frac{1}{n} \sum_{i=1}^n \hat{\beta}(t, S_i).$$

- The asymptotic variances of $\hat{\theta}_{\text{DR}}(t)$ and $\hat{\theta}_{\text{EIF}}(t)$ are the same (or differing by $O(h^2)$)!

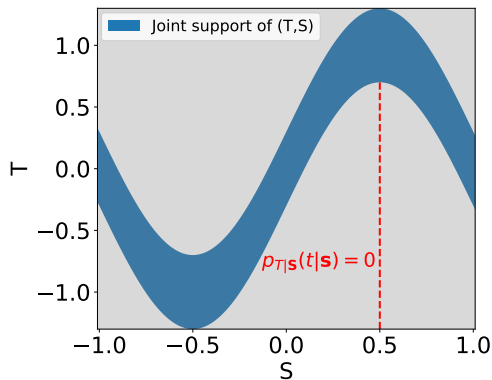
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Nonparametric Inference on $\theta(t)$ Without Positivity



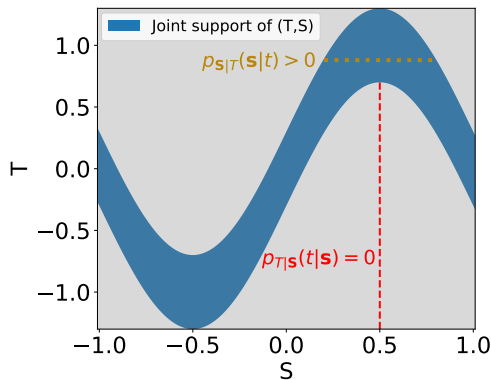
Assumption (Identification Conditions)

- 1 (Consistency) $Y = Y(t)$ whenever $T = t \in \mathcal{T}$.
- 2 (Ignorability) $Y(t)$ is conditionally independent of T given S for all $t \in \mathcal{T}$.
- 3 (Treatment Variation) $\text{Var}(T|S = s) > 0$ for all $s \in \mathcal{S}$.



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Assumption (Extrapolation; Zhang et al. 2024)

Assume $(t, s) \mapsto \mathbb{E}[Y(t)|S = s]$ to be differentiable w.r.to t for any $(t, s) \in \mathcal{T} \times \mathcal{S}$ with $p_{S|T}(s|t) > 0$ and

$$\begin{aligned} \theta(t) &= \frac{d}{dt} \mathbb{E}[Y(t)] = \mathbb{E} \left[\frac{\partial}{\partial t} \mathbb{E}[Y(t)|S] \right] \\ &\stackrel{*}{=} \mathbb{E} \left[\frac{\partial}{\partial t} \mathbb{E}[Y(t)|S] \mid T = t \right]. \end{aligned}$$

Additionally, it holds true that $\mathbb{E}(Y) = \mathbb{E}[m(T)]$.

Key Example: Additive Confounding Model

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Proposition 2 in [Zhang et al. \(2024\)](#) shows that the above equality holds under an additive structural assumption

$$Y(t) = \bar{m}(t) + \eta(S) + \epsilon.$$

- $\bar{m} : \mathcal{T} \rightarrow \mathbb{R}$ and $\eta : \mathcal{S} \rightarrow \mathbb{R}$ are deterministic functions.
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- **Identification:**

$$m(t) = \mathbb{E} \left[Y + \int_{u=T}^{u=t} \theta(u) du \right] \quad \text{and} \quad \theta(t) = \int \frac{\partial}{\partial t} \mu(t, s) dF_{S|T}(s|t).$$

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- **RA estimator without positivity:**

$$\hat{m}_{\text{C,RA}}(t) = \frac{1}{n} \sum_{i=1}^n \left[Y_i + \int_{\tilde{t}=T_i}^{\tilde{t}=t} \hat{\theta}_{\text{C,RA}}(\tilde{t}) d\tilde{t} \right] \quad \text{and} \quad \hat{\theta}_{\text{C,RA}}(t) = \int \hat{\beta}(t, s) d\hat{F}_{S|T}(s|t).$$

Question: How about IPW and DR estimators of $\theta(t)$ (and $m(t)$) without positivity?

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$$\tilde{m}_{\text{IPW}}(t) = \frac{1}{nh} \sum_{i=1}^n \frac{Y_i \cdot K\left(\frac{T_i - t}{h}\right)}{p_{T|S}(T_i | \mathbf{S}_i)} \quad \text{and} \quad \tilde{\theta}_{\text{IPW}}(t) = \frac{1}{nh^2} \sum_{i=1}^n \frac{Y_i \cdot \left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot p_{T|S}(T_i | \mathbf{S}_i)}.$$

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Proposition (Proposition 2 in [Zhang and Chen 2025](#))

$$\lim_{h \rightarrow 0} \mathbb{E} [\tilde{m}_{\text{IPW}}(t)] = \bar{m}(t) \cdot \rho(t) + \omega(t) \neq m(t), \quad \text{with} \quad \rho(t) = \mathbb{P}(S \in \mathcal{S}(t)),$$

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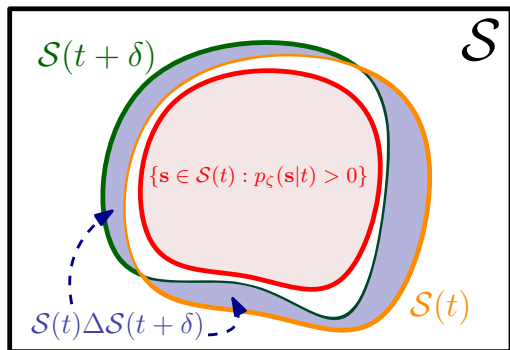
► **Key Issue:** The conditional support $\mathcal{S}(t)$ of $p_{S|T}(s|t)$ and the marginal support \mathcal{S} of $p_S(s)$ are different under the violations of positivity!!

$$\mathbb{E} \left[\frac{Y \cdot \left(\frac{T-t}{h} \right) K \left(\frac{T-t}{h} \right) p_{S|T}(S|t)}{h^2 \cdot \kappa_2 \cdot p_{T|S}(T|S) \cdot p_S(S)} \right] = \bar{m}'(t) + O(h^2) + \text{"Non-vanishing Bias"}.$$

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- 2 We replace $p_{S|T}(s|t)$ with a ζ -interior conditional density $p_\zeta(s|t)$ so that

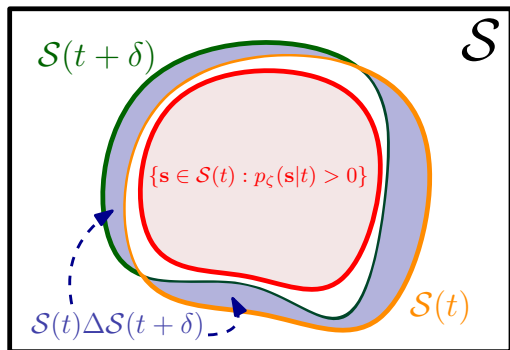
$$\{s \in \mathcal{S}(t) : p_\zeta(s|t) > 0\} \subset \mathcal{S}(t + \delta) \quad \text{for any } \delta \in [-h, h].$$



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Now, we have that

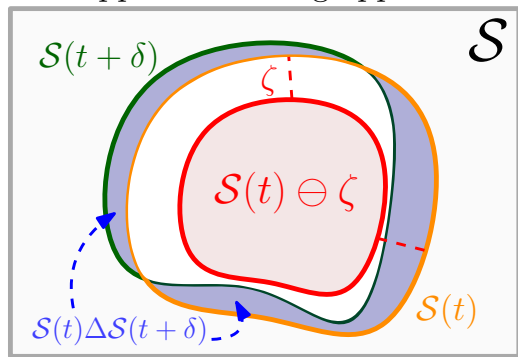
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ζ -Interior Conditional Density

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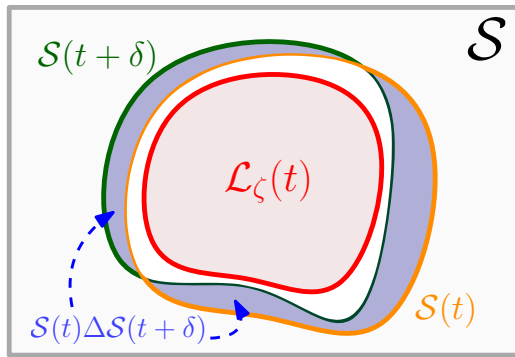
Support shrinking approach



$$\mathcal{S}(t) \ominus \zeta = \left\{ \mathbf{s} \in \mathcal{S}(t) : \inf_{\mathbf{x} \in \partial \mathcal{S}(t)} \|\mathbf{s} - \mathbf{x}\|_2 \geq \zeta \right\},$$

$$p_\zeta(\mathbf{s}|t) = \frac{p_{\mathcal{S}|T}(\mathbf{s}|t) \cdot \mathbb{1}_{\{\mathbf{s} \in \mathcal{S}(t) \ominus \zeta\}}}{\int_{\mathcal{S}(t) \ominus \zeta} p_{\mathcal{S}|T}(\mathbf{s}_1|t) d\mathbf{s}_1}.$$

Level set approach



$$\mathcal{L}_\zeta(t) = \left\{ \mathbf{s} \in \mathcal{S}(t) : p_{\mathcal{S}|T}(\mathbf{s}|t) \geq \zeta \right\},$$

$$p_\zeta(\mathbf{s}|t) = \frac{p_{\mathcal{S}|T}(\mathbf{s}|t) \cdot \mathbb{1}_{\{\mathbf{s} \in \mathcal{L}_\zeta(t)\}}}{\int_{\mathcal{L}_\zeta(t)} p_{\mathcal{S}|T}(\mathbf{s}_1|t) d\mathbf{s}_1}.$$

► Bias-Corrected IPW Estimator Without Positivity:

$$\hat{\theta}_{\text{C,IPW}}(t) = \frac{1}{nh^2} \sum_{i=1}^n \frac{Y_i \cdot \left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right) \hat{p}_{\zeta}(\mathbf{S}_i|t)}{\kappa_2 \cdot \hat{p}(T_i, \mathbf{S}_i)},$$

- $\hat{p}(t, \mathbf{s}), \hat{p}_{\zeta}(\mathbf{s}|t)$ are estimators of $p(t, \mathbf{s}), p_{\zeta}(\mathbf{s}|t)$ and $\zeta = 0.5 \cdot \max \{\hat{p}_{\mathbf{S}|T}(\mathbf{S}_i|t) : i = 1, \dots, n\}$.

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$$\begin{aligned} \hat{\theta}_{\text{C,DR}}(t) = & \underbrace{\frac{1}{nh^2} \sum_{i=1}^n \frac{\left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right) \hat{p}_{\zeta}(S_i|t)}{\kappa_2 \cdot \hat{p}(T_i, S_i)} \left[Y_i - \hat{\mu}(t, S_i) - (T_i - t) \cdot \hat{\beta}(t, S_i) \right]}_{\text{IPW component}} \\ & + \underbrace{\int \hat{\beta}(t, s) \cdot \hat{p}_{\zeta}(s|t) ds}_{\text{RA component}}. \end{aligned}$$

Theorem (Theorem 5 in Zhang and Chen 2025)

Under some regularity assumptions and

- ① $\hat{\mu}, \hat{\beta}, \hat{p}, \hat{p}_\zeta$ are estimated on a dataset independent of $\{(Y_i, T_i, S_i)\}_{i=1}^n$;
- ② $\sqrt{nh} \|\hat{p}_\zeta(S|t) - \bar{p}_\zeta(S|t)\|_{L_2} = o_P(1)$ with $\hat{p}_\zeta(s|t) \xrightarrow{P} \bar{p}_\zeta(s|t)$;
- ③ at least one of the model specification conditions hold:
 - $\hat{p}(t, s) \xrightarrow{P} \bar{p}(t, s) = p(t, s)$ (joint density model),
 - $\hat{\mu}(t, s) \xrightarrow{P} \bar{\mu}(t, s) = \mu(t, s)$ and $\hat{\beta}(t, s) \xrightarrow{P} \bar{\beta}(t, s) = \beta(t, s)$ (outcome model);
- ④ $\sup_{|u-t| \leq h} \|\hat{p}(u, S) - p(u, S)\|_{L_2} \left[\|\hat{\mu}(t, S) - \mu(t, S)\|_{L_2} + h \left\| \hat{\beta}(t, S) - \beta(t, S) \right\|_{L_2} \right] = o_P\left(\frac{1}{\sqrt{nh}}\right)$,

we prove that

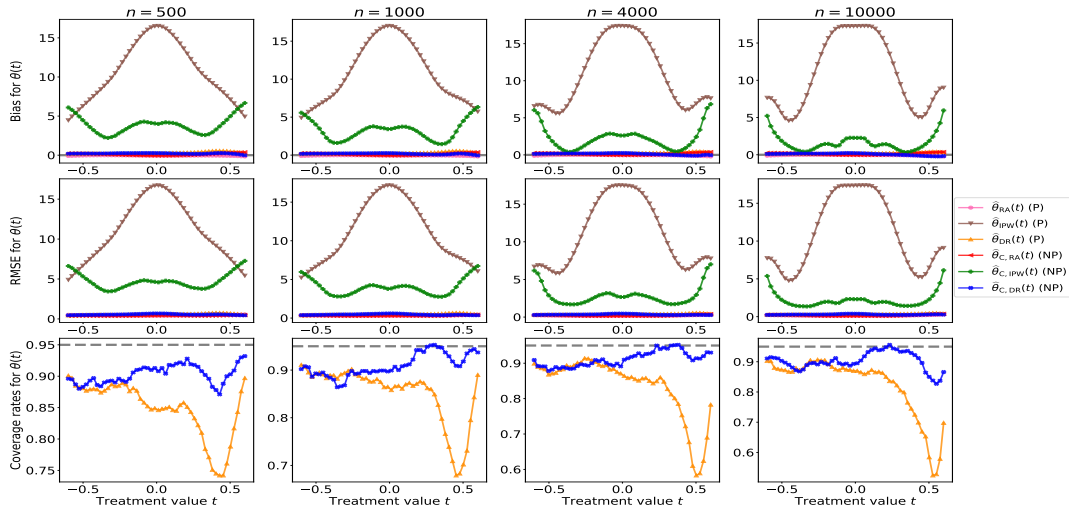
- $\sqrt{nh^3} [\hat{\theta}_{C,DR}(t) - \theta(t)] = \frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_{C,h,t}(Y_i, T_i, S_i; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S}) + o_P(1).$
- $\sqrt{nh^3} [\hat{\theta}_{C,DR}(t) - \theta(t) - h^2 \cdot B_{C,\theta}(t)] \xrightarrow{d} \mathcal{N}(0, V_{C,\theta}(t)).$

Simulations and Case Study



Simulations for $\hat{\theta}_{C,RA}(t)$, $\hat{\theta}_{C,IPW}(t)$, $\hat{\theta}_{C,DR}(t)$ Without Positivity

$$Y = T^3 + T^2 + 10S + \epsilon, \quad T = \sin(\pi S) + E, \quad S \sim \text{Unif}[-1, 1], \quad E \sim \text{Unif}[-0.3, 0.3].$$



Note: $\beta(t, s) = \frac{\partial}{\partial t} \mu(t, s)$ is estimated via automatic differentiation of a well-trained neural network (inspired by [Luedtke 2024](#)).

A Case Study Under Positivity

We compare our proposed DR estimator $\hat{\theta}_{\text{DR}}(t)$ under positivity with the finite-difference method ([Colangelo and Lee 2020](#); CL20) on the U.S. Job Corps program ([Schochet et al., 2001](#)).

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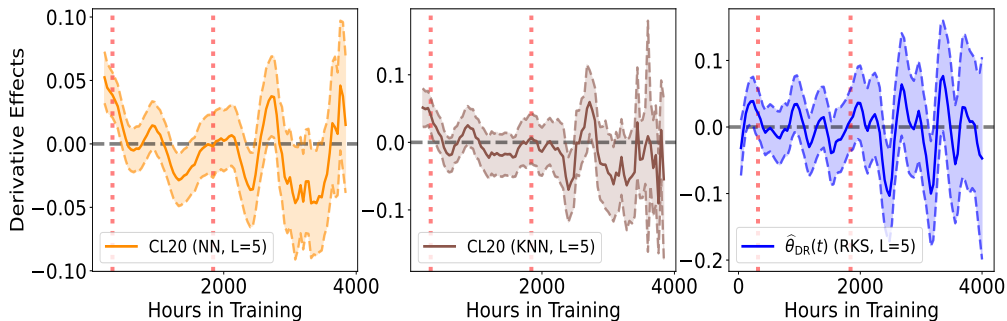
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Summary and Future Work

We study (nonparametric) doubly robust inference on $\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)]$, $t \in \mathcal{T} \subset \mathbb{R}$.

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① Under the positivity condition:

- We propose $\hat{\theta}_{\text{DR}}(t)$ with standard nonparametric consistency and efficiency guarantee:

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We study (nonparametric) doubly robust inference on $\theta(t) = \frac{d}{dt} \mathbb{E}[Y(t)]$, $t \in \mathcal{T} \subset \mathbb{R}$.

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- Our bias-corrected IPW and DR estimators $\hat{\theta}_{\text{C,IPW}}(t)$, $\hat{\theta}_{\text{C,DR}}(t)$ reveal interesting connections to nonparametric level set estimation problems (Bonvini et al., 2023):

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3 Future Works:

- Sensitivity analysis on unmeasured confounding (Chernozhukov et al., 2022).
- Generalize our derivative estimators to other causal estimands:
 - instantaneous causal effect $\frac{d}{dt} \mathbb{E}[Y(t)|S=s]$ (Stolzenberg, 1980);
 - direct and indirect effects in mediation analysis (Huber et al., 2020; Xu et al., 2021)?

Thank you!

More details can be found in

[1] Y. Zhang and Y.-C. Chen. Doubly Robust Inference on Causal Derivative Effects for Continuous Treatments. *arXiv preprint*, 2025. <https://arxiv.org/abs/2501.06969>.

All the code and data are available at
<https://github.com/zhangyk8/npDRDeriv>.

Python Package: [npDoseResponse](#).

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Detailed Regularity Assumptions

Assumption (Differentiability of the conditional mean outcome function)

For any $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $\mu(t, s) = \mathbb{E}(Y|T = t, S = s)$, it holds that

- ① $\mu(t, s)$ is at least four times continuously differentiable with respect to t .*
- ② $\mu(t, s)$ and all of its partial derivatives are uniformly bounded on $\mathcal{T} \times \mathcal{S}$.*

Let \mathcal{J} be the support of the joint density $p(t, s)$.

Assumption (Differentiability of the density functions)

For any $(t, s) \in \mathcal{J}$, it holds that

- ① The joint density $p(t, s)$ and the conditional density $p_{T|S}(t|s)$ are at least three times continuously differentiable with respect to t .*
- ② $p(t, s)$, $p_{T|S}(t|s)$, $p_{S|T}(s|t)$, as well as all of the partial derivatives of $p(t, s)$ and $p_{T|S}(t|s)$ are bounded and continuous up to the boundary $\partial\mathcal{J}$.*
- ③ The support \mathcal{T} of the marginal density $p_T(t)$ is compact and $p_T(t)$ is uniformly bounded away from 0 within \mathcal{T} .*

Assumption (Regular kernel conditions)

A kernel function $K : \mathbb{R} \rightarrow [0, \infty)$ is bounded and compactly supported on $[-1, 1]$ with $\int_{\mathbb{R}} K(t) dt = 1$ and $K(t) = K(-t)$. In addition, it holds that

- ① $\kappa_j := \int_{\mathbb{R}} u^j K(u) du < \infty$ and $\nu_j := \int_{\mathbb{R}} u^j K^2(u) du < \infty$ for all $j = 1, 2, \dots$
- ② K is a second-order kernel, i.e., $\kappa_1 = 0$ and $\kappa_2 > 0$.
- ③ $\mathcal{K} = \left\{ t' \mapsto \left(\frac{t' - t}{h} \right)^{k_1} K \left(\frac{t' - t}{h} \right) : t \in \mathcal{T}, h > 0, k_1 = 0, 1 \right\}$ is a bounded VC-type class of measurable functions on \mathbb{R} .

Assumption (Smoothness condition on $\mathcal{S}(t)$)

For any $\delta \in \mathbb{R}$ and $t \in \mathcal{T}$, there exists an absolute constant $A_0 > 0$ such that either (i) “ $\mathcal{S}(t) \ominus (A_0|\delta|) \subset \mathcal{S}(t + \delta)$ ” for the support shrinking approach or (ii) “ $\mathcal{L}_{A_0|\delta|}(t) \subset \mathcal{S}(t + \delta)$ ” for the level set approach.

The self-normalizing technique can reduce the instability of IPW and DR estimators (Kallus and Zhou, 2018):

1 Self-Normalized Estimators Under Positivity:

$$\hat{\theta}_{\text{IPW}}^{\text{norm}}(t) = \frac{\hat{\theta}_{\text{IPW}}(t)}{\frac{1}{nh} \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right)}{\hat{p}_{T|S}(T_j|S_j)}} = \frac{\sum_{i=1}^n \frac{Y_i\left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{\hat{p}_{T|S}(T_i|S_i)}}{\kappa_2 h \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right)}{\hat{p}_{T|S}(T_j|S_j)}},$$

and

$$\hat{\theta}_{\text{DR}}^{\text{norm}}(t) = \frac{\sum_{i=1}^n \frac{[Y_i - \hat{\mu}(t, S_i) - (T_i - t) \cdot \hat{\beta}(t, S_i)] \left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right)}{\hat{p}_{T|S}(T_i|S_i)}}{\kappa_2 h \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right)}{\hat{p}_{T|S}(T_j|S_j)}} + \frac{1}{n} \sum_{i=1}^n \hat{\beta}(t, S_i).$$

2 Self-Normalized Estimators Without Positivity:

$$\hat{\theta}_{C,IPW}^{\text{norm}}(t) = \frac{\hat{\theta}_{C,IPW}(t)}{\frac{1}{nh} \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right) \cdot \hat{p}_{\zeta}(S_j|t)}{\hat{p}(T_j, S_j)}} = \frac{\sum_{i=1}^n \frac{Y_i \left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right) \cdot \hat{p}_{\zeta}(S_i|t)}{\hat{p}(T_i, S_i)}}{\kappa_2 h \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right) \cdot \hat{p}_{\zeta}(S_j|t)}{\hat{p}(T_j, S_j)}},$$

and

$$\begin{aligned} \hat{\theta}_{C,DR}^{\text{norm}}(t) = & \frac{\sum_{i=1}^n \frac{[Y_i - \hat{\mu}(t, S_i) - (T_i - t) \cdot \hat{\beta}(t, S_i)] \left(\frac{T_i-t}{h}\right) K\left(\frac{T_i-t}{h}\right) \cdot \hat{p}_{\zeta}(S_i|t)}{\hat{p}(T_i, S_i)}}{\kappa_2 h \sum_{j=1}^n \frac{K\left(\frac{T_j-t}{h}\right) \cdot \hat{p}_{\zeta}(S_j|t)}{\hat{p}(T_j, S_j)}} \\ & + \int \hat{\beta}(t, s) \cdot \hat{p}_{\zeta}(s|t) ds. \end{aligned}$$

We generate i.i.d. observations $\{(Y_i, T_i, S_i)\}_{i=1}^n$ from the following data-generating model (Colangelo and Lee, 2020):

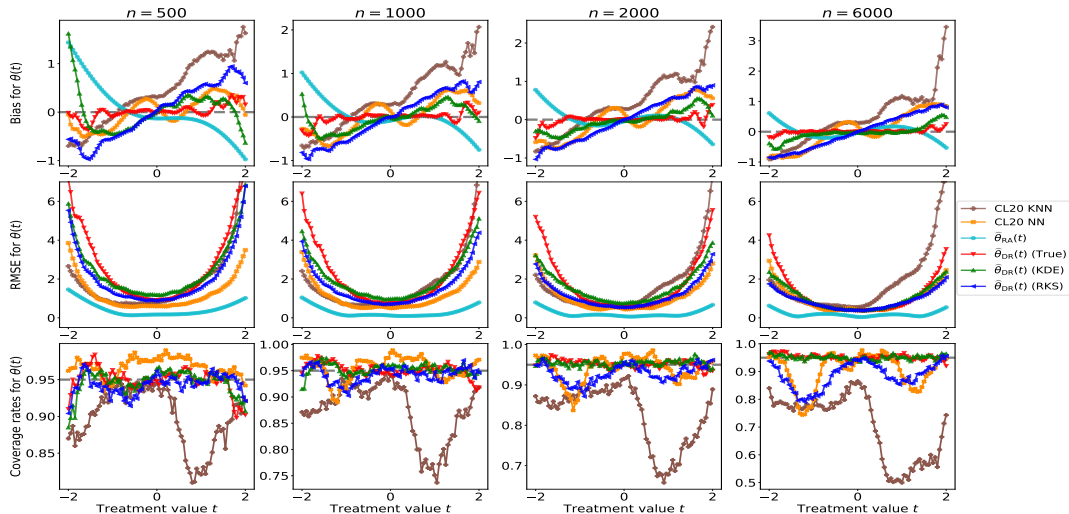
$$Y = 1.2T + T^2 + TS_1 + 1.2\boldsymbol{\xi}^T \mathbf{S} + \epsilon \sqrt{0.5 + F_{\mathcal{N}(0,1)}(S_1)}, \quad \epsilon \sim \mathcal{N}(0, 1),$$

$$T = F_{\mathcal{N}(0,1)}\left(3\boldsymbol{\xi}^T \mathbf{S}\right) - 0.5 + 0.75E, \quad \mathbf{S} = (S_1, \dots, S_d)^T \sim \mathcal{N}_d(\mathbf{0}, \Sigma), \quad E \sim \mathcal{N}(0, 1),$$

where

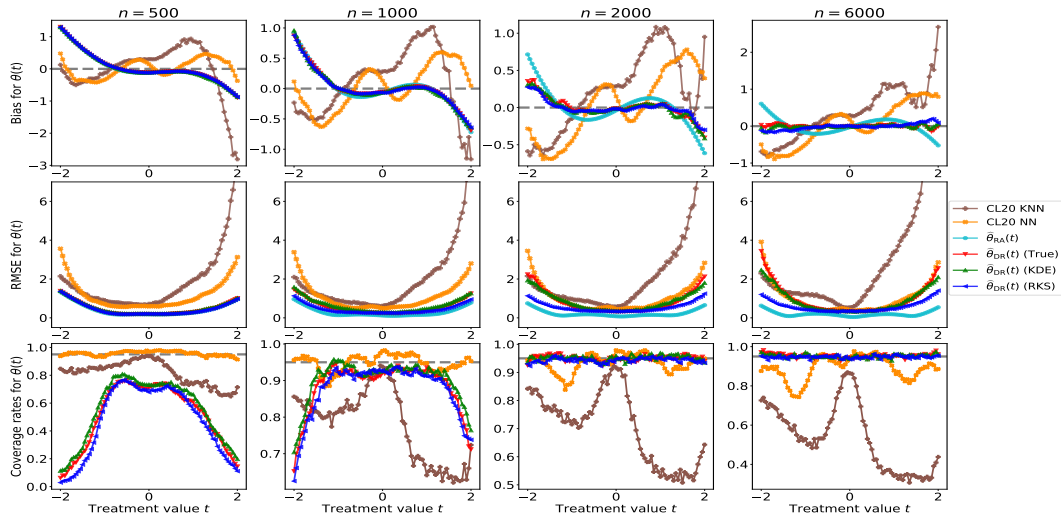
- $F_{\mathcal{N}(0,1)}$ is the CDF of $\mathcal{N}(0, 1)$ and $d = 20$.
- $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^T \in \mathbb{R}^d$ has its entry $\xi_j = \frac{1}{j^2}$ for $j = 1, \dots, d$ and $\Sigma_{ii} = 1$, $\Sigma_{ij} = 0.5$ when $|i - j| = 1$, and $\Sigma_{ij} = 0$ when $|i - j| > 1$ for $i, j = 1, \dots, d$.
- The dose-response curve is given by $m(t) = 1.2t + t^2$, and our parameter of interest is the derivative effect curve $\theta(t) = 1.2 + 2t$.

Simulations for Estimating $\theta(t)$ Under Positivity



Comparisons between our proposed estimators and the finite-difference approaches by Colangelo and Lee (2020) (“CL20”) under positivity and with 5-fold cross-fitting across various sample sizes.

Simulations for Estimating $\theta(t)$ Under Positivity



Comparisons between our proposed estimators and the finite-difference approaches by [Colangelo and Lee \(2020\)](#) (“CL20”) under positivity and **without cross-fitting** across various sample sizes.