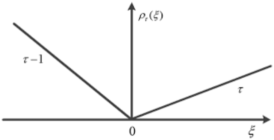


*Yikun Zhang*
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$*$ Part of the slides were made when I was an
Advanced Algorithmic Engineer at Trip.com

# Conditional Quantile Regression

## With Applications to User-Preferred Price Prediction

December 23, 2021

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# Introduction to Our Hotel Ranking Task

A group of candidate hotels (in a searched city).


*Logistic Regression, XGBoost, Deep Neural Networks,...

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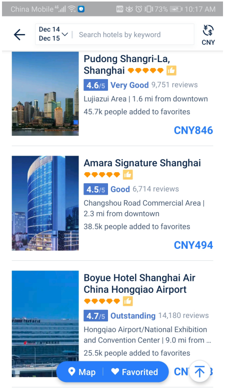
Ranking Algorithms* $\Longrightarrow$

A well-sorted list of hotels.




*Logistic Regression, XGBoost, Deep Neural Networks,...

# Objective of the Hotel Ranking Task

Return a list of hotels with user-preferred ones placed on the top.
$\Rightarrow$ Optimizing the *conversion rate* (on hotels with high commissions).

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Return a list of hotels with user-preferred ones placed on the top.
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Features/Predictors: $\boldsymbol{X}_i = \Big[\underbrace{V_1^{(i)}, ..., V_q^{(i)}}_{\text{Hotel Features}}, \underbrace{U_1^{(i)}, ..., U_p^{(i)}}_{\text{User Features}}\Big]$ for $i = 1, ..., n$.

Responses: $Y_i \in \{0 : \text{Not Booked}, 1 : \text{Booked}\}$ for $i = 1, ..., n$.

# How to Identify User-Preferred Hotels?

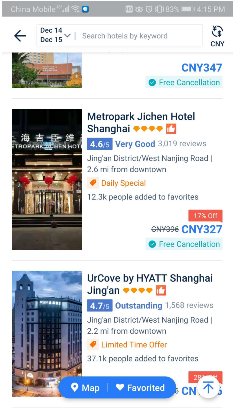

Figure 1: Six-group factors (Image source: the confluence page written by Jone Zhang).

## How to Identify User-Preferred Hotels?

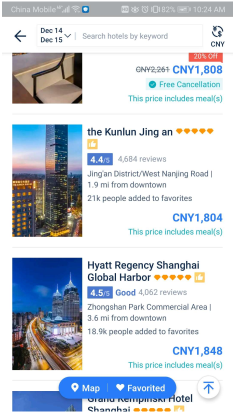

Figure 1: Six-group factors (Image source: the confluence page written by Jone Zhang).

- The prices of hotels clicked/booked by a user quantify his/her affordability.
- The price preferences of users on our platform are diverse.

# Variety of User Price Preferences




(a) Users that prefer low-priced hotels




(b) Users that prefer high-priced hotels

# Multimodal Nature of User Price Preferences

The price preferences varies between different groups of users on our platform.

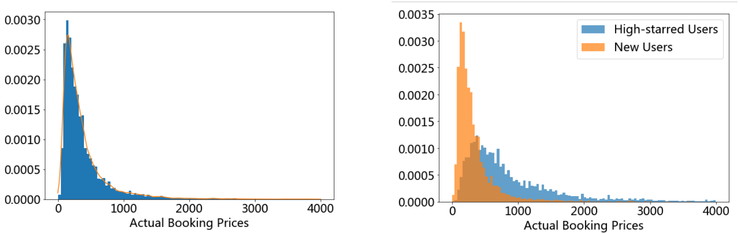



Figure 3: Overall and group-specific distributions of actual booking prices on December 6, 2021.

## Main Objective: User-Preferred Hotel Price Prediction

Correctly predicting the preferred hotel prices or *price intervals* is of great significance to our hotel ranking task!

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Mathematically, given a user $\boldsymbol{X}_i = \boldsymbol{x}_i = \left[u_1^{(i)}, ..., u_p^{(i)}\right]$, we intend to predict his/her preferred price interval

$$\left[\widehat{Q}_l(\boldsymbol{x}_i), \widehat{Q}_u(\boldsymbol{x}_i)\right].$$

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$$\left[\widehat{Q}_l(\boldsymbol{x}_i), \widehat{Q}_u(\boldsymbol{x}_i)\right].$$

Here, the features $u_j^{(i)}, j = 1, ..., p$ range from

- user behaviors (such as historical clicked/booked hotels, user IDs, etc.)
- location information (such as city IDs, average GMV in that city, etc.)

# Drawback of the Current Online Model (Baseline)

**Current online model:** It is a weighted sum of historical booked prices, real-time clicked prices, and the specific quantile price in the searched city.

$$\text{Predicted Price} = \frac{\sum_i \omega_{\text{time}} \cdot \omega_{\text{type}} \cdot \omega_{\text{abnormal}} \cdot \omega_{\text{city}} \cdot \text{Price}_i}{\sum_i \omega_{\text{time}} \cdot \omega_{\text{type}} \cdot \omega_{\text{abnormal}} \cdot \omega_{\text{city}}}.$$

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- The choices weights $\omega_{\text{time}}, \omega_{\text{type}}, \omega_{\text{abnormal}}, \omega_{\text{city}}$ are heuristic and outdated.
- The preferred price interval is symmetrically extended from the above point estimate.
- The accuracy of the current predicted prices (or price intervals) is also limited.
- ...

# Our Proposed Method: Conditional Quantile Regression

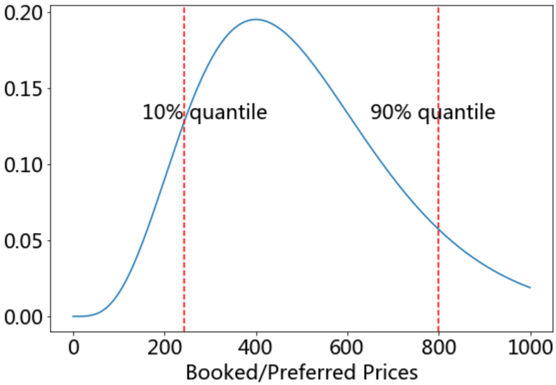



Figure 4: (Smoothed) conditional distribution of historical booked/preferred prices for a user with feature $\boldsymbol{X}_i = \boldsymbol{x}_i$. The synthetic density function (blue curve) is given by $f(u|\boldsymbol{x}_i) = \frac{1}{\Gamma(5)\cdot 100^5} \cdot u^4 \exp\left(-\frac{u}{100}\right)$.

# Our Proposed Method: Conditional Quantile Regression

Given the conditional cumulative distribution function $F(y|\boldsymbol{X} = \boldsymbol{x})$ of booked prices, we pursue an interval

$$\Big[Q_\tau(\boldsymbol{x}),\, Q_{1-\tau}(\boldsymbol{x})\Big],$$

where $Q_\tau(\boldsymbol{x}) = \inf\{y : F(y|\boldsymbol{X} = \boldsymbol{x}) \geq \tau\}$ and $\tau \in (0, 1/2]$.†

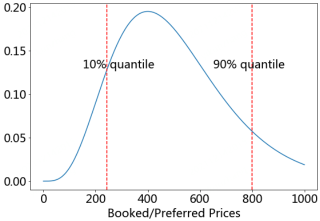



Figure 5: $\tau$ and $(1-\tau)$ quantile of $F(y|\boldsymbol{X} = \boldsymbol{x})$ with $\tau = 0.1$.

† Koenker, R., & Bassett Jr, G. (1978). Regression quantiles. *Econometrica: Journal of the Econometric Society*, 33-50.

# How to Fit the Conditional Quantile?

## How to Fit the Conditional Quantile?

The conditional quantile $Q_\tau(\boldsymbol{x})$ is the solution to the following optimization problem:

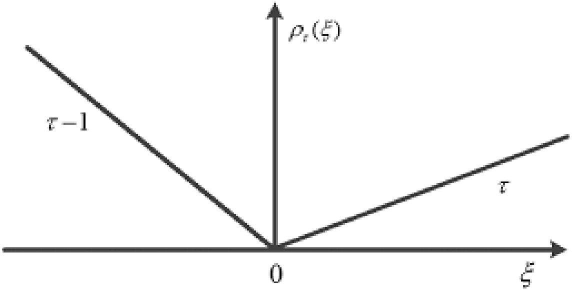
$$Q_\tau(\boldsymbol{x}) = \arg\min_q \mathbb{E}\left[\rho_\tau(Y-q)|\boldsymbol{X}=\boldsymbol{x}\right], \tag{1}$$

where

$$\rho_\tau(\xi) = \xi\left[\tau - \mathbb{1}_{\{\xi<0\}}\right] = \begin{cases} \tau\xi, & \xi \geq 0, \\ -(1-\tau)\xi, & \xi < 0 \end{cases} \tag{2}$$

is the so-called "pinball" loss (Koenker and Bassett, 1978; Firpo et al., 2009; Steinwart and Christmann, 2011).

## "Pinball Loss"




**Remark:**

- When $\tau = 0.5$, the aforementioned optimization problem (1) recovers the absolute deviation problem.
- The loss is robust to outliers (Hampel, 1971; John, 2015).

## Correctness of the (Conditional) Quantile Regression

### Proposition

*Given the conditional distribution function $F(y|\boldsymbol{X} = \boldsymbol{x})$,*

$$Q_\tau(\boldsymbol{x}) = \inf\left\{y : F(y|\boldsymbol{X} = \boldsymbol{x}) \geq \tau\right\}$$

*is the solution to (1).*
*More generally, given any càdlàg function $F(y)$,*

$$q_\tau = \inf\left\{y : F(y) \geq \tau\right\}$$

*is the solution to the unconditional quantile regression problem $\arg\min_q \mathbb{E}\left[\rho_\tau(Y - q)\right]$.*

## Quantile Regression in Practice

Theoretically, $Q_\tau(\boldsymbol{x}) = \arg\min_q \mathbb{E}\left[\rho_\tau(Y-q)|\boldsymbol{X}=\boldsymbol{x}\right]$.

## Quantile Regression in Practice

Theoretically, $Q_\tau(\boldsymbol{x}) = \arg\min_q \mathbb{E}\left[\rho_\tau(Y-q)|\boldsymbol{X}=\boldsymbol{x}\right]$.

Practically, given the training set with clicked/booked hotel entries

$$\{(\boldsymbol{X}_i, Y_i)\} = \left\{\left(\left[U_1^{(i)},...,U_p^{(i)}\right], Y_i\right)\right\},$$

we solve the following empirical risk minimization (ERM) problem:

$$\widehat{Q}_\tau = \arg\min_{Q\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n} \rho_\tau\left(Y_i - Q(\boldsymbol{X}_i)\right),$$

where $\mathcal{F}$ is the function class spanned by our (neural network) models.

# Fitting the Empirical Quantiles $\widehat{Q}_\tau$ and $\widehat{Q}_{1-\tau}$

**Input:** $\{(\boldsymbol{X}_i, Y_i)\} = \left\{\left(\left[U_1^{(i)}, ..., U_p^{(i)}\right], Y_i\right)\right\}$, where the continuous features are standardized and discrete ones are converted to embedding vectors.

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**Architecture:** One shared hidden layer $512 \times 200$ with additional separate $200 \times 100 \times 1$ full-connected Relu layers.

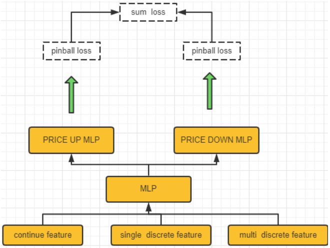


Figure 6: Double-tower architecture (image credit: Dr. Xianzhang Xiang)

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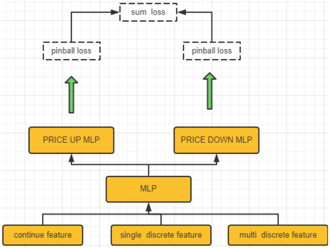


Figure 6: Double-tower architecture (image credit: Dr. Xianzhang Xiang)

**Objective:** $\left\{\widehat{Q}_\tau, \widehat{Q}_{1-\tau}\right\} = \underset{\{f,g\}\subset\mathcal{F}}{\arg\min} \frac{1}{n}\sum_{i=1}^{n}\left[\rho_\tau\left(Y_i - f(\boldsymbol{X}_i)\right) + \rho_{1-\tau}\left(Y_i - g(\boldsymbol{X}_i)\right)\right]$

with $\tau = 0.1$.

# Why do we use Relu Neural Network? (Minimax Theory)

Assume that

- the true quantile function $Q_\tau$ belongs to the Hölder class $\mathcal{H}$ or Besov space $\mathcal{B}$.
- the number of layers $L$ satisfies $\log_2(n) \lesssim L \lesssim n^{\frac{p}{2s+p}}$.
- the maximum norm of network coefficients $\|\boldsymbol{\beta}\|_{\max} \lesssim n^{\frac{p}{2s+p}} \log n$.

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Then,

$$\|\widehat{Q}_\tau - Q_\tau\|_{\ell_2}^2 \leq C \cdot (\log n)^2 n^{-\frac{2s}{2s+p}},$$

where $s$ is the smoothness parameter, $p$ is the dimension of the feature space, and $n$ is the sample size (Schmidt-Hieber, 2020; Padilla et al., 2020).

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Based on the nonparametric theory (Wasserman, 2006; Tsybakov, 2008), this rate of convergence is indeed *minimax* up to a log factor!

$\widehat{f}^*$ is minimax

$$\iff \sup_{f \in \mathcal{H}} \mathbb{E}\left[\left(\widehat{f}^*(\boldsymbol{x}_0) - f(\boldsymbol{x}_0)\right)^2\right] = \inf_{\widehat{f}_n} \sup_{f \in \mathcal{H}} \mathbb{E}\left[\left(\widehat{f}_n(\boldsymbol{x}_0) - f(\boldsymbol{x}_0)\right)^2\right],$$

where the infimum is taken among all the estimators.

## Summary of Our Proposed Model

- **Goal:** Preferred Price Interval $\left[Q_\tau(\boldsymbol{x}),\ Q_{1-\tau}(\boldsymbol{x})\right]$ with $Q_\tau(\boldsymbol{x}) = \inf\left\{y : F(y|\boldsymbol{X} = \boldsymbol{x}) \geq \tau\right\}$ and $\tau \in (0, 1/2]$.

- **Theoretical Solution:** Conditional Quantile Regression,

$$Q_\tau(\boldsymbol{x}) = \arg\min_q \mathbb{E}\left[\rho_\tau(Y - q)|\boldsymbol{X} = \boldsymbol{x}\right].$$

- **Practical Model:** Empirical Risk Minimization with Relu Networks,

$$\widehat{Q}_\tau = \arg\min_{Q \in \mathcal{F}} \frac{1}{n}\sum_{i=1}^{n} \rho_\tau\left(Y_i - Q(\boldsymbol{X}_i)\right).$$

- **Minimax Guarantee:** $\|\widehat{Q}_\tau - Q_\tau\|_{\ell_2}^2 \to 0$ as $n \to \infty$.

## Other Potential Choices of Quantile Regression Models

- **Quantile Regression Forests** (Meinshausen, 2006): The random forests method has the uniform consistency in estimating the cumulative distribution function (CDF) of $Y|\boldsymbol{X} = x$.

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- **Quadratic Programming and Reproducing Kernel Hilbert Space (RKHS) Methods** (Takeuchi et al., 2006), **Nadaraya-Watson Nonparametric Regression Estimator** (Huang and Nguyen, 2018), etc.

## Evaluation Metrics

- **Coverage Accuracy**:

$$ACC(\mathcal{Y}, \widehat{\mathcal{I}}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\left\{Y_i \in \left[\widehat{I}(\boldsymbol{x}_i)\right]\right\}},$$

where $\mathcal{Y} = \{Y_i\}_{i=1}^{n}$ is a collection of booked hotel prices and $\widehat{I}(\boldsymbol{x}_i) = \left[\widehat{Q}_\tau(\boldsymbol{x}_i), \widehat{Q}_{1-\tau}(\boldsymbol{x}_i)\right]$ is the predicted preferred price interval for the user with feature $\boldsymbol{x}_i$.

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- **Average Interval Length**:

$$\text{Average Length}(\widehat{\mathcal{I}}) = \frac{1}{n}\sum_{i=1}^{n} \left|\widehat{Q}_\tau(\boldsymbol{x}_i) - \widehat{Q}_{1-\tau}(\boldsymbol{x}_i)\right|.$$

# Neural Network Quantile Regression on the “My Location” Scenario

| | *Cov. Acc. (fh_prices)* | *Cov. Acc. (order prices)* | *Average Interval Length* |
|---|---|---|---|
| **Baseline Model** | 0.7521 | 0.8019 | 283.8624 |
| **Our NN QR** | **0.8718** | **0.8593** | **233.5811** |

Table 1: Comparison between our neural network quantile regression model and the current online model (baseline) on the “My Location” scenario.

## Neural Network Quantile Regression on the “Main Ranking” Scenario

| | *Cov. Acc. (fh_prices)* | *Cov. Acc. (order prices)* | *Average Interval Length* |
|---|---|---|---|
| **Baseline Interval I** | 0.5590 | 0.5804 | 213.1197 |
| **Baseline Interval II** | 0.8593 | 0.8534 | 597.1224 |
| **Our NN QR (Before calibration)** | 0.7883 | 0.7351 | 317.5999 |
| **Our NN QR (After calibration)** | **0.9268** | **0.8954** | 482.3805 |

Table 2: Comparison between our neural network quantile regression model and the current online model (baseline) on the “Main Ranking” scenario.

- Notes: The calibration means that we extend our predicted interval as:

$$\left[\widehat{Q}_\tau(\boldsymbol{x}_i) - \alpha \cdot \left|\widehat{Q}_\tau(\boldsymbol{x}_i) - \widehat{Q}_{1-\tau}(\boldsymbol{x}_i)\right|, \widehat{Q}_{1-\tau}(\boldsymbol{x}_i) + \alpha \cdot \left|\widehat{Q}_\tau(\boldsymbol{x}_i) - \widehat{Q}_{1-\tau}(\boldsymbol{x}_i)\right|\right],$$

where $\alpha = 0.3 \sim 0.5$.

# Discussion: Non-Crossing Property of Quantile Regression

Recall that our current optimization framework is

$$\left\{\widehat{Q}_\tau, \widehat{Q}_{1-\tau}\right\} = \mathop{\arg\min}_{\{Q_\tau, Q_{1-\tau}\}\subset\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n}\left[\rho_\tau\left(Y_i - Q_\tau(\boldsymbol{X}_i)\right) + \rho_{1-\tau}\left(Y_i - Q_{1-\tau}(\boldsymbol{X}_i)\right)\right].$$

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However, a constraint is required for the monotonicity of quantiles, *i.e.*, for any $\tau \in (0, 1/2]$, we should solve the constrained optimization problem:

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subject to $Q_\tau(\boldsymbol{X}_i) \leq Q_{1-\tau}(\boldsymbol{X}_i)$ for all $i = 1, ..., n$.

**Challenges:** Solving the constrained optimization problem is difficult due to the nature of stochastic gradient descent (Padilla et al., 2020).

# Discussion: Solution to the Non-Crossing Constrained Quantile Regression

**Feasible Approaches**:

- Penalized Method: With a large $\lambda > 0$, we optimize the following problem:

$$\left\{\widehat{Q}_\tau, \widehat{Q}_{1-\tau}\right\} = \operatorname*{arg\,min}_{\{Q_\tau, Q_{1-\tau}\} \subset \mathcal{F}} \sum_{i=1}^n \Big[\rho_\tau\left(Y_i - Q_\tau(\boldsymbol{X}_i)\right) + \rho_{1-\tau}\left(Y_i - Q_{1-\tau}(\boldsymbol{X}_i)\right)\Big]$$
$$+ \lambda \cdot \sum_{i=1}^n \mathbb{1}_{\left\{\rho_\tau\left(Y_i - Q_\tau(\boldsymbol{X}_i)\right) > \rho_{1-\tau}\left(Y_i - Q_{1-\tau}(\boldsymbol{X}_i)\right)\right\}}.$$

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- Redefined Objective (Padilla et al., 2020):

$$\left\{\widehat{h}_1, \widehat{h}_2\right\} = \underset{\{h_1, h_2\}\subset\mathcal{F}}{\arg\min} \sum_{i=1}^n \rho_\tau\left(Y_i - h_1(\boldsymbol{X}_i)\right) + \sum_{i=1}^n \rho_{1-\tau}\left\{Y_i - h_1(\boldsymbol{X}_i) - \log\left[1 + e^{h_2(\boldsymbol{X}_i)}\right]\right\}$$

and set $\widehat{Q}_\tau(\boldsymbol{x}) = \widehat{h}_1(\boldsymbol{x})$ and $\widehat{Q}_{1-\tau}(\boldsymbol{x}) = \widehat{h}_1(\boldsymbol{x}) + \log\left[1 + e^{\widehat{h}_2(\boldsymbol{x})}\right]$.

# Motivation of Our Proposed Method: Conformal Inference

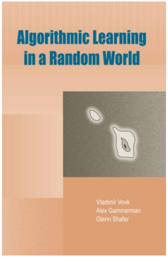
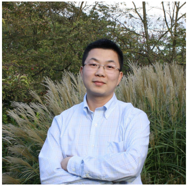
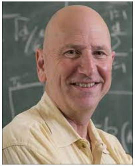



Figure 7: Algorithmic Learning in a Random World (Vovk et al., 2005).



(a) Jing Lei



(b) Larry Wasserman

(c) Emmanuel Candès

# Motivation of Our Proposed Method: Conformal Inference

What is conformal prediction/inference (Vovk et al., 1999, 2005; Lei et al., 2018)?

- Given a training set $\{(\boldsymbol{X}_i, Y_i)\} \subset \mathbb{R}^p \times \mathbb{R}$ and the unknown value $Y_{n+1}$ at a test point $\boldsymbol{X}_{n+1}$, it aims to construct a *marginal distribution-free prediction interval* $\mathcal{C}(\boldsymbol{X}_{n+1}) \subset \mathbb{R}$ such that

$$\mathbb{P}\left(Y_{n+1} \in \mathcal{C}(\boldsymbol{X}_{n+1})\right) \geq 1 - \alpha$$

for some nominal miscoverage level $\alpha \in (0, 1)$.

- Notes: The $(1-\alpha)$-*confidence interval* is defined as:

$$\mathbb{P}\left(\mathbb{E}[Y|\boldsymbol{X}] \in \mathcal{C}(\boldsymbol{X})\right) \geq 1 - \alpha.$$

# Classical (Split) Conformal Prediction: A Preview

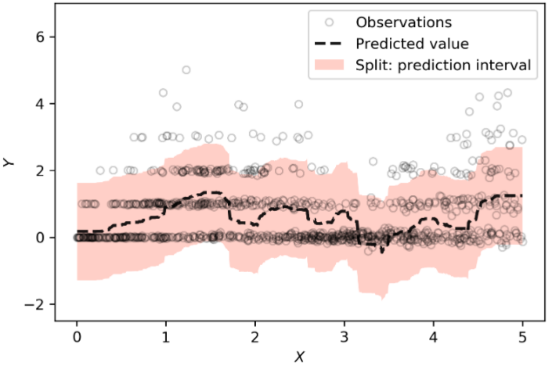



Figure 9: Classical (Split) Conformal Prediction (Average coverage: 91.4%; Average interval length: 2.91.)

# Classical (Split) Conformal Prediction: Detailed Procedures

1. Split the training set $\mathcal{D} = \{(\boldsymbol{X}_i, Y_i)\} \subset \mathbb{R}^p \times \mathbb{R}$ into $\mathcal{D} = \mathcal{D}_T \cup \mathcal{D}_C$:
   - A proper training set $\mathcal{D}_T = \{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_1\}$,
   - A calibration set $\mathcal{D}_C = \{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_2\}$.

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2. Fit $\widehat{\mu}(\boldsymbol{x}) \leftarrow \mathcal{A}\left(\{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_1\}\right)$ via any regression algorithm $\mathcal{A}$ on $\mathcal{D}_T$.

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$$R_i = |Y_i - \widehat{\mu}(\boldsymbol{X}_i)| \quad \text{with} \quad i \in \mathcal{I}_2.$$

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$$R_i = |Y_i - \widehat{\mu}(\boldsymbol{X}_i)| \quad \text{with} \quad i \in \mathcal{I}_2.$$
4. Compute the $(1-\alpha)$ empirical quantile of the absolute residuals,
$$Q_{1-\alpha}(R, \mathcal{I}_2) := (1-\alpha)\left(1 + \frac{1}{|\mathcal{I}_2|}\right)\text{-th empirical quantile of } \{R_i : i \in \mathcal{I}_2\}.$$

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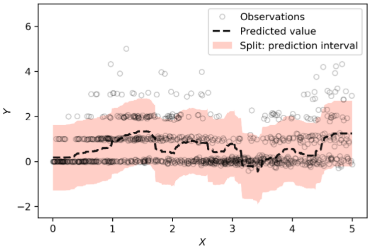
$$Q_{1-\alpha}(R, \mathcal{I}_2) := (1-\alpha)\left(1 + \frac{1}{|\mathcal{I}_2|}\right)\text{-th empirical quantile of } \{R_i : i \in \mathcal{I}_2\}.$$
5. The prediction interval at a new point $\boldsymbol{X}_{n+1}$ is given by
$$\mathcal{C}(\boldsymbol{X}_{n+1}) = \left[\widehat{\mu}(\boldsymbol{X}_{n+1}) - Q_{1-\alpha}(R, \mathcal{I}_2),\, \widehat{\mu}(\boldsymbol{X}_{n+1}) + Q_{1-\alpha}(R, \mathcal{I}_2)\right].$$

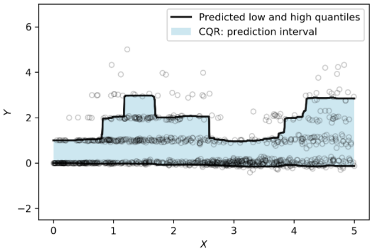
# Conformalized Quantile Regression (Romano et al., 2019)

1. Split the training set $\mathcal{D} = \{(\boldsymbol{X}_i, Y_i)\} \subset \mathbb{R}^p \times \mathbb{R}$ into $\mathcal{D} = \mathcal{D}_T \cup \mathcal{D}_C$:
   - A proper training set $\mathcal{D}_T = \{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_1\}$,
   - A calibration set $\mathcal{D}_C = \{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_2\}$.
2. Fit $\left\{\widehat{Q}_{\alpha_{\text{low}}}, \widehat{Q}_{\alpha_{\text{high}}}\right\} \leftarrow \mathcal{A}_q\left(\{(\boldsymbol{X}_i, Y_i) : i \in \mathcal{I}_1\}\right)$ via any **quantile regression** algorithm $\mathcal{A}_q$ on $\mathcal{D}_T$.
3. Compute the **conformity scores** of $\widehat{C}(\boldsymbol{x}) = \left[\widehat{Q}_{\alpha_{\text{low}}}(\boldsymbol{x}), \widehat{Q}_{\alpha_{\text{high}}}(\boldsymbol{x})\right]$ on $\mathcal{D}_C$ as:
$$E_i := \max\left\{\widehat{Q}_{\alpha_{\text{low}}}(\boldsymbol{X}_i) - Y_i, Y_i - \widehat{Q}_{\alpha_{\text{high}}}(\boldsymbol{X}_i)\right\} \quad \text{with} \quad i \in \mathcal{I}_2.$$
4. Compute the $(1-\alpha)$ empirical quantile of the conformity scores,
$$Q_{1-\alpha}(E, \mathcal{I}_2) := (1-\alpha)\left(1 + \frac{1}{|\mathcal{I}_2|}\right)\text{-th empirical quantile of } \{E_i : i \in \mathcal{I}_2\}.$$
5. The prediction interval at a new point $\boldsymbol{X}_{n+1}$ is given by
$$\mathcal{C}(\boldsymbol{X}_{n+1}) = \left[\widehat{Q}_{\alpha_{\text{low}}}(\boldsymbol{X}_{n+1}) - Q_{1-\alpha}(R, \mathcal{I}_2),\ \widehat{Q}_{\alpha_{\text{high}}}(\boldsymbol{X}_{n+1}) + Q_{1-\alpha}(R, \mathcal{I}_2)\right].$$

## Comparisons Between Split Conformal Prediction and Conformalized Quantile Regression




(a) Classical (Split) Conformal Prediction
(Average coverage: 91.4%; Average interval length: 2.91).




(b) Conformalized Quantile Regression
(Average coverage: 91.06%; Average interval length: 1.99).

## Conclusion and Future Works

What we have done:

- We proposed a user-preferred price prediction model via (conditional) quantile regression with a Relu neural network.
- The model is well-performed based on offline evaluations.

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- We proposed a user-preferred price prediction model via (conditional) quantile regression with a Relu neural network.
- The model is well-performed based on offline evaluations.

Ongoing works:

- Handle the non-crossing properties/constraints of our model.
- Extend the user-preferred price prediction model to other scenarios and develop an unified modeling framework.

# Thank You

Comments or Questions?

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## Proof of Proposition 1.

Let $g(u) = \mathbb{E}\left[\rho_\tau(Y-u)\right]$. Some simple algebra show that

$$\begin{aligned} g(u) &= \int_{-\infty}^{\infty} \rho_\tau(y-u)dF(y) \\ &= \int_u^{\infty} \tau(y-u)dF(y) - \int_{-\infty}^{u} (1-\tau)(y-u)dF(y). \end{aligned}$$

Applying the Leibniz integral rule shows that

$$g'(u) = 0 \iff -\tau\int_u^{\infty} dF(y) + (1-\tau)\int_{-\infty}^{u} dF(y) = F(u) - \tau = 0.$$

Therefore, $u = q_\tau$ is the smallest point satisfying $F(u) - \tau = 0$ and will be unique when $F$ is strictly monotonic on $q_\tau$. □