随机响应计算

响应平均值

载荷向量$\boldsymbol{F}(t)$的均值为$\boldsymbol{m}_F$，再注意到平稳随机过程的平均值与时间无关，于是有

$$
\begin{aligned}
E[\boldsymbol{X}(t)] &= E\left[\int_{-\infty}^{+\infty} \boldsymbol{h}(\theta)\boldsymbol{F}(t-\theta)d\theta\right] \\
&= \int_{-\infty}^{+\infty} \boldsymbol{h}(\theta)E\left[\boldsymbol{F}(t-\theta)\right]d\theta \\
&= \boldsymbol{m}_F \int_{-\infty}^{+\infty} \boldsymbol{h}(\theta)d\theta
\end{aligned}
$$

或即为

$$
\boldsymbol{m}_X = \boldsymbol{H}(0)\boldsymbol{m}_F
$$

其中$\boldsymbol{H}(0)$是$\omega = 0$时的频响函数矩阵。上式表明响应均值可以很方便地由激励均值计算。

响应相关矩阵

$$
\begin{aligned}
\boldsymbol{R}_{XX}(\tau) &= E[\boldsymbol{X}(t)\boldsymbol{X}^{\mathrm{T}}(t+\tau)] \\
&= E\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{h}(\theta_1)\boldsymbol{F}(t-\theta_1)\boldsymbol{F}^{\mathrm{T}}(t+\tau-\theta_2)\boldsymbol{h}^{\mathrm{T}}(\theta_2)d\theta_1\theta_2\right] \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{h}(\theta_1)E\left[\boldsymbol{F}(t-\theta_1)\boldsymbol{F}^{\mathrm{T}}(t+\tau-\theta_2)\right]\boldsymbol{h}^{\mathrm{T}}(\theta_2)d\theta_1\theta_2
\end{aligned}
$$

即

$$
\boldsymbol{R}_{XX}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{h}(\theta_1)\boldsymbol{R}_{FF}(\tau+\theta_1-\theta_2)\boldsymbol{h}^{\mathrm{T}}(\theta_2)d\theta_1\theta_2
$$

这表明响应相关矩阵可由激励相关矩阵积分得到。

$$
\begin{aligned}
\boldsymbol{R}_{FX}(\tau) &= E\left[\boldsymbol{F}(t)\boldsymbol{X}^{\mathrm{T}}(t+\tau)\right] \\
&= E\left[\boldsymbol{F}(t)\int_{-\infty}^{+\infty} \boldsymbol{F}^{\mathrm{T}}(t+\tau-\theta)\boldsymbol{h}^{\mathrm{T}}(\theta)d\theta\right] \\
&= \int_{-\infty}^{+\infty} \boldsymbol{R}_{FF}(\tau-\theta)\boldsymbol{h}^{\mathrm{T}}(\theta)d\theta
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{R}_{XF}(\tau) &= E\left[\boldsymbol{X}(t)\boldsymbol{F}^{\mathrm{T}}(t+\tau)\right] \\
&= E\left[\int_{-\infty}^{+\infty}\boldsymbol{h}(\theta)\boldsymbol{F}(t-\theta)\boldsymbol{F}^{\mathrm{T}}(t+\tau)d\theta\right] \\
&= \int_{-\infty}^{+\infty}\boldsymbol{h}(\theta)\boldsymbol{R}_{FF}(\tau+\theta)d\theta
\end{aligned}
$$

响应功率谱矩阵

先考虑单点输入单点输出的情况

$$
\begin{aligned}
S_{XX}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{XX}(\tau)e^{-\mathrm{i}\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}R_{FF}(\tau+\theta_1-\theta_2)h(\theta_1)h(\theta_2)d\theta_1\theta_2\right]e^{-\mathrm{i}\omega\tau}d\tau \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{FF}(\tau+\theta_1-\theta_2)e^{-\mathrm{i}\omega(\tau+\theta_1-\theta_2)}d(\tau+\theta_1-\theta_2)h(\theta_1)e^{\mathrm{i}\omega\theta_1}d\theta_1h(\theta_2)e^{-\mathrm{i}\omega\theta_2}d\theta_2 \\
&= S_{FF}(\omega)\int_{-\infty}^{+\infty}h(\theta_1)e^{\mathrm{i}\omega\theta_1}d\theta_1\int_{-\infty}^{+\infty}h(\theta_2)e^{-\mathrm{i}\omega\theta_2}d\theta_2 \\
&= S_{FF}(\omega)H^*(\omega)H(\omega) \\
&= |H(\omega)|^2S_{FF}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
S_{FX}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{FX}(\tau)e^{-\mathrm{i}\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(\theta)R_{FF}(\tau-\theta)e^{-\mathrm{i}\omega\tau}d\theta d\tau \\
&= \int_{-\infty}^{+\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{FF}(\tau-\theta)e^{-\mathrm{i}\omega(\tau-\theta)}d(\tau-\theta)h(\theta)e^{-\mathrm{i}\omega\theta}d\theta \\
&= \int_{-\infty}^{+\infty}S_{FF}(\omega)h(\theta)e^{-\mathrm{i}\omega\theta}d\theta \\
&= H(\omega)S_{FF}(\omega)
\end{aligned}
$$

相似地

$$
S_{XF}(\omega) = H^*(\omega)S_{FF}(\omega)
$$

多点输入多点输出的谱矩阵为

$$
\boldsymbol{S}_{XX}(\omega) = \boldsymbol{H}^*\boldsymbol{S}_{FF}(\omega)\boldsymbol{H}^{\mathrm{T}}
$$

$$
\boldsymbol{S}_{FX}(\omega) = \boldsymbol{S}_{FF}(\omega)\boldsymbol{H}^{\mathrm{T}}
$$

$$
\boldsymbol{S}_{XF}(\omega) = \boldsymbol{H}^*\boldsymbol{S}_{FF}(\omega)
$$

CQC(complete quadratic combination，完全二次结合)法

SRSS(square root of the sum of squares，平方和开平方)法