STEP Lectures

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1. **[Rudin 6.13 (6.14 is similar)]** Define

$$f(x) = \int_x^{x+1} \sin\left(t^2\right) dt$$

and prove that

a $|f(x)| < \frac{1}{x}$ if $x > 0$.

b

$$2xf(x) = \cos\left(x^2\right) - \cos\left[(x+1)^2\right] + r(x)$$

where $|r(x)| < \frac{c}{x}$ and c is constant.

c Find

$$\limsup_{x\to\infty} xf(x) \quad \text{and} \quad \liminf_{x\to\infty} xf(x).$$

d Does the improper integral

$$\int_0^\infty \sin(t^2)\, dt$$

converge?

2. **[Rudin 8.19]** A trigonometric polynomial is a finite formal sum

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

in a real formal variable x with all coefficients a_i and b_i in $\mathbb{C}$; it can be viewed as a complex-valued function. If $\frac{\alpha}{\pi}$ is irrational, prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt.$$

3. Define

$$F(a) = \int_0^1 |\sin x - a\cos x| dx.$$

Find the a that minimizes $F(a)$.

4. Let $f(x)$ be differentiable on $\mathbb{R}$ and $f(x) > 0$, $f'(0) = 0$. If f satisfies the functional equation $f(x+y) = f(x)f(y)e^{2xy}$, find $f(x)$.

5. Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that the improper integral

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges.

6. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

7. Fill an $n \times n$ table A^i_j with 0 and 1, prove that $n \geq 5$ is a sufficient condition for that there exists i_1, i_2, j_1, j_2 such that $A^{i_1}_{j_1} = A^{i_1}_{j_2} = A^{i_2}_{j_2} = A^{i_2}_{j_1}$.

8. For a function f such that $f(x) > 0$ on $\mathbb{R}$, define

$$I_a = \frac{1}{a} \int_0^a f(x)\,dx.$$

If $\lim_{a\to\infty} I_a = A$, then there exists a strictly monotonically increasing $\{x_n\}$ such that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} f(x_n) = A$.

9. Let f be continuous on $[1, 2]$, and $1 \leq x \leq 2$. Find A and B in

$$\int_{\frac{1}{x}}^{\frac{2}{x}} |\log y| f(xy) dy = 3x(\log x - 1) + A + \frac{B}{x}.$$

10. Let $a = \frac{2^8}{3^4}$. Consider the sequence

$$b_k = \frac{(k+1)^{k+1}}{a^k k!} \quad (k = 1, 2, 3, \cdots).$$

   a. Show that $f(x) = (x + 1) \log\left(1 + \frac{1}{x}\right)$ is monotonically increasing on $x > 1$.
   b. Find the maximum M of the sequence $\{b_k\}$, and nominate all k such that $M = b_k$.

11. Consider the Cauchy's functional equation

$$f(x + y) = f(x) + f(y).$$

It is clear that linear functions, i.e. functions of the form $f : x \mapsto cx$ where c is constant, is a family of solutions. Discover the structure of this and other families of solutions by following the steps below.

   a. Show that when f is a function with domain $\mathbb{Q}$, then linear functions are the only solutions to Cauchy's functional equation.

A subset A of $\mathbb{R}^2$ is called to be dense if any disk in $\mathbb{R}^2$, however small and wherever it is, contains a point from A.

   b. Show that on $\mathbb{R}$, a solution to the Cauchy's functional equation is either linear or having the pathological property that the graph set $G := \{(x, f(x)) \mid x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$.

The Hamel basis is a set $\{b_i\}_{i\in I} = \mathcal{B} \subset \mathbb{Q}$ such that for every $x \in \mathbb{R}$, it can be represented as a linear combination of the elements in $\mathcal{B}$, i.e. $x = \sum_{j=1}^n \lambda_j b_{i_j}$.

   c. Prove that there exists non-linear solutions to Cauchy's functional equation on $\mathbb{R}$, assuming the Hamel basis exists.

Two sets are said to be of the same cardinality if there exists a bijection between them. A is said to be smaller than B in the cardinality sense if an injection from A to B can be established but a surjection cannot.

   d. Show that there are more nonlinear solutions than linear ones to the Cauchy's functional equation over $\mathbb{R}$.