Solutions to Foundations of Algebraic Geometry

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Abstract

These notes contain my solutions to Ravi Vakil's Foundations of Algebraic Geometry. For more information I refer you to the official website/blog (http://math216.wordpress.com/) and the page with the actual notes (http://math.stanford.edu/~vakil/216blog/).

The solutions are provided as is. I don't claim these to be correct or well written (although I certainly intend them to be for my own benefit). If you encounter any flagrant mistakes, you can contact me by e-mail (pieterbelmans@gmail.com) or add a patch to the GitHub repository at https://github.com/pbelmans/math216. Please do so by the way, preferably by submitting patches.

As the notes are still being written when I started writing up these solutions and there may be changes to the numbering system upcoming: I am using the June 27 version. If I am still interested in these exercises when a final version is posted, I might edit in possible changes.

Contents

Todo list

Chapter 1

Introduction

There are no exercises in this chapter.

Part I

Preliminaries

Chapter 2

Some category theory

2.1 Motivation

There are no exercises in this section.

2.2 Categories and functors

2.2.A Exercise. (a) The elements of the group(oid) correspond to the category's morphisms. As every morphism is an isomorphism, we can only compose morphisms on the same object. Now in case of a single object, all axioms for a group are satisfied, as isomorphisms lead to inverse elements.

(b) Take the category of a group and copy the unique object, together with all its morphisms. Voilà, a groupoid.

A natural example of groupoids is the fundamental groupoid of a topological space. You cannot combine loops that have different base points.

□

2.2.B Exercise. By definition of *invertible element of* $\mathrm{Mor}(A, A)$ the automorphisms form a group: we get composition and associativity from the category and the identity and inverse from our choice of elements.

In case of Sets the automorphisms are the permutations of the set, *i.e.*, bijections. In case of Vec_k the automorphisms are bijective linear self-maps.

By conjugation, isomorphic objects have isomorphic automorphism groups. □

2.2.C Exercise. Linear algebra exercise. I will do this one if I feel inspired. □

??: isomorphism of double dual

2.2.D Exercise. A basis for a finite-dimensional vectorspace has a well-defined cardinality, defining its dimension. So the inverse functor $\mathrm{f.d.}\ \mathsf{Vec}_k \to \mathcal{V}$ maps an n-dimensional vectorspace V to k^n, while every linear map between finite-dimensional vectorspaces can be written against a choice of bases. As we were allowed to pick a basis for each vectorspace simultaneously, every linear map admits by linear algebra magic with tons of indices a representation as a matrix. □

2.3 Universal properties determine an object up to unique isomorphism

2.3.A Exercise. Take A and B initial objects. By definition of an initial object have (unique) morphisms $A \to B$ and $B \to A$, we can compose them, obtaining morphisms $A \to A$ and $B \to B$. But the identity is another candidate for this morphisms, so by uniqueness of the morphisms A and B are isomorphic.

The proof for final objects is completely the same. □

2.3.B Exercise. Sets The initial object is the empty set $\emptyset$, the singleton $\{x\}$ is the final object (all singletons are isomorphic as stated before in **??**).

Rings As the image of $1 \in \mathbb{Z}$ determines the entire ring morphism, the ring of integers $\mathbb{Z}$ is the initial object. The final object is the trivial ring (in which $0 = 1$).

Top The initial object is the empty set $\emptyset$ equipped with the topology consisting of 1 open set, the final object is the singleton equipped with the topology consisting of the empty set and the entire space.

The category subsets of a set and the category of open sets in a topological space are both *bounded lattices*. Or, as there is either no morphism (if two sets are incomparable) or one morphism (if one set is contained in the other), we need to find objects that are either smaller or greater than all other objects. These are the empty set and the set X. □

2.3.C Exercise. Take $s \in S$ a zerodisivor and let $b \in A$ such that $bs = 0$. Now the image of b is equal to zero as

$$\frac{b}{1} = \frac{0}{1} \iff s\,(b-0) = 0. \tag{2.1}$$

Conversely, take $a, b \in A$ and $a \neq b$ such that their images are equal in the localization. That means there exists an $s \in S$ such that $s(a-b) = 0$, so $a-b \neq 0$ is a zerodivisor as $0 \notin S$. □

2.3.D Exercise. The A-algebra $S^{-1}A$ is a member of this category: an element of the multiplicative subset $s \in S$ is a unit as it is inverted by $1/s$. It is furthermore initial among these algebras because the unique morphism $\overline{\varphi}\colon S^{-1}A \to B$ is given by $\overline{\varphi}(r/s) = \varphi(r)\varphi(s)^{-1}$ where $\varphi\colon A \to B$ is the structure map.

Now this morphism $\overline{\varphi}$ is unique because if ψ would be another morphism extending $i\colon A \to S^{-1}A$ we'd find

$$\psi\left(\frac{r}{s}\right) = \psi\left(\frac{r}{1}\right)\psi\left(\frac{1}{s}\right) = \varphi(r)\varphi(s)^{-1} \tag{2.2}$$

as we split the fraction into parts on which i works. □

2.3.E Exercise. The definition of $S^{-1}M$ is already given thoroughly in the hint, we obtain the map $\phi\colon m \mapsto (m/1)$ because $1 \in S$ is required. The checks for the $S^{-1}A$-module structure are trivial and the universal property is satisfied by the proof from **??**. □

2.3.F Exercise. The isomorphism is given by

$$\frac{1}{s}(m_1, \dots, m_n) \mapsto \left(\frac{m_1}{s}, \frac{m_2}{s}, \dots, \frac{m_n}{s}\right) \tag{2.3}$$

with the inverse map being

$$\left(\frac{m_1}{s_1}, \dots, \frac{m_n}{s_n}\right) \mapsto \frac{1}{\prod_{i=1}^n s_i} \left(m_1 \prod_{i \neq 1} s_i, \dots, m_n \prod_{i \neq n} s_i\right). \tag{2.4}$$

In the infinite case the product of the nominators is not defined. In the scenario of the hint that is given, the image under the inverse map has both a division by zero and a multiplication by infinity. □

2.3.G Exercise. It is possible to proof that

$$\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(d) \tag{2.5}$$

with $d := \gcd(m, n)$. In order to do so: observe that

$$x \otimes y = (xy) \otimes 1 = xy\,(1 \otimes 1) \tag{2.6}$$

hence $1 \otimes 1$ is the generator of a cyclic group that represents the tensor product. Now $d(1 \otimes 1) = 0$ because both $m(1 \otimes 1)$ and $n(1 \otimes 1)$ are zero by bringing it into the correct factor and using Bézout's identity. So the order of the cyclic group divides d.

Now we can map the direct product into $\mathbb{Z}/(d)$ in an obvious way, which induces a map from the tensor product to $\mathbb{Z}/(d)$. The element $1 \otimes 1$ is mapped to 1 and consequently has order d, so there is an element of order *at least* d. Hence we have obtained the desired isomorphism.

In this special case: $\gcd(10, 12) = 2$, so $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$. □

2.3.H Exercise. That $f \otimes \mathrm{id}\colon M \otimes N \to M'' \otimes N$ is still surjective is obvious.

For the surjection of $M' \otimes N$ onto the kernel of $f \otimes \mathrm{id}$ we have to prove that in

$$M \otimes N \to (M \otimes N)/\operatorname{im}((M' \to M) \otimes \mathrm{id}) \to M'' \otimes N, \tag{2.7}$$

the last morphism being the induced morphism from $f \otimes \mathrm{id}$ is invertible. Construct the induced $\overline{\varphi}$ from

$$\varphi\colon \begin{array}{ccc} M'' \otimes N & \longrightarrow & (M \otimes N)/\operatorname{im}((M' \to M) \otimes \mathrm{id}) \\ m'' \otimes n & \longmapsto & m \otimes n + \operatorname{im}((M' \to M) \otimes \mathrm{id}) \end{array} \tag{2.8}$$

where $m \in f^{-1}(m'')$ by surjectivity of f. Now φ is well defined, bilinear and therefore extends to $\overline{\varphi}$. Now the composition $\overline{\varphi} \circ \overline{f \otimes \mathrm{id}}$ is the identity, hence the induced morphism is invertible and we have obtained exactness. □

2.3.I Exercise. Unique up to unique isomorphism means that the object is not necessarily unique, but all objects satisfying the universal property are isomorphic using a unique (iso)morphism (as f is said to be so). Hence these objects have trivial automorphism groups. By the same proof as for the universal property of categorical products this holds for tensor products. □

2.3.J Exercise. The construction exactly quotients those objects that correspond to the bilinearity of the morphisms that are considered. Therefore all morphisms $M \times N \to T'$ factor uniquely through T. □

2.3.K Exercise. (a) As stated in **??**: giving a ring map $A \to B$ is the same as giving B an A-algebra structure. So we can consider both B and M as A-modules.

We now use the fact that B is more than a module: it is an algebra. So we take by definition that $B \otimes_A M$ interacts with scalars such that elements of $B \setminus A$ are absorbed in the first factor, while scalar elements of A can (as by definition of the tensor product over A) move around as we like.

Let's tediously check all module axioms now. Take $b_1, b_2 \in B$ as scalars for the B-structure and $b \otimes m, b' \otimes m' \in B \otimes_A M$ as B-module elements. We have

$r(x+y) = rx + ry$: This one is immediate.

$(r+s)x = rx + sx$: We manipulate:

$$\begin{aligned}(b_1 + b_2)(b \otimes m) &= (b_1 + b_2)b \otimes m \\ &= (b_1 b + b_2 b) \otimes m \\ &= b_1 b \otimes m + b_2 b \otimes m \\ &= b_1(b \otimes m) + b_2(b \otimes m)\end{aligned}. \tag{2.9}$$

$(rs)x = r(sx)$: By definition of the B-module structure we obtain

$$(b_1 b_2)(b \otimes m) = b_1 b_2 b \otimes m = b_1(b_2 b \otimes m). \tag{2.10}$$

$1_R x = x$: We easily obtain

$$1_B(b \otimes m) = 1_B b \otimes m = b \otimes m. \tag{2.11}$$

The A-linearity (we're taking the tensor product over A) is induced by the construction and requires no explicit checking.

The functorality of the mapping $\mathsf{Mod}_A \to \mathsf{Mod}_B$ follows from assigning to each morphism $f\colon M_1 \to M_2$ in the first category the map $\mathrm{id} \otimes f$ in the second.

(b) Now both B and C carry the desired A-algebra structure necessary for this multiplication to hold and the result follows from the previous point.

□

2.3.L Exercise. The natural isomorphism is given by

$$\frac{a}{s} \otimes m \mapsto \frac{am}{s} \tag{2.12}$$

which is compatible with both the $S^{-1}A$- and A-module structure by definition of $S^{-1}M$. The inverse map is obviously defined as

$$\frac{m}{s} \mapsto \frac{1}{s} \otimes m \tag{2.13}$$

and this is the correct inverse because we're tensoring over the ring A. □

2.3.M Exercise. The condition imposed on the elements of the Cartesian product that will act as the fibered product in Sets are necessary by the commutativity of the square. The maps π_X and π_Y are the obvious projections on the first or second factor.

A map $W \to X \times_Z Y$ is by the agreement of compositions to W given by using the maps $W \to X$ and $W \to Y$ for each component. Now this must be unique: if we'd try to fit in another map the commutativity of the diagram is broken. □

2.3.N Exercise. It is the *intersection* of the open sets: the fibered product must be an element of the category (which in this case only contains open sets). As for morphisms: these depict inclusion ($U \subseteq V$ implies $U \to V$) and are either unique or non-existing. So if W is a map to both X and Y which map to Z, we have a chain of inclusions in which we can fit $W \subseteq X \cap Y$. □

2.3.O Exercise. The definition of fibered product depends on the choice of f and g. But as Z is the final object, these morphisms are unique. Now place the product in the position reserved for the fibered product in the definition. As the compositions from W to Z agree (Z being final implies uniqueness of these maps, hence equality!) we have a unique isomorphism between these two products. □

2.3.P Exercise. The projection $U \to V$ is given by the definition of the fibered product in the second square. The projection $U \to Y$ is given by composing the projections $U \to W$ and $W \to Y$. The compositions of the obtained projections with both $V \to Z$ and $X \to Z$ agree by commutativity of the diagram.

Now take an object A with maps $A \to V$ and $A \to Y$ such that the compositions with $V \to Z$ and $Y \to Z$ agree. We wish to consider a unique map $A \to U$. Consider the map $A \to W$ (which exists by chasing the diagram), by definition of the first fibered product, this map is unique.

Now consider $A \to V$, we have defined $V \to Z$ by the composition of projections. By commutativity of the diagram $A \to W \to X$ and $A \to V \to X$ agree so we can use the second fibered product. So just construct a unique map $A \to U$. We have obtained a tower of fibered products.

Essentially, it boils down to lifting $A \to Y$ to $A \to W$ in a unique way. □

2.3.Q Exercise. Let’s draw a picture.

$$
\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_2 \arrow[dd] \\
X_1 \arrow[r] \arrow[d] & Y \arrow[dr] & \\
X_1 \arrow[rr] & & Z
\end{tikzcd}
\tag{2.14}
$$

By definition of $X_1 \times_Y X_2$ the maps to Y through X_1 and X_2 agree. So the composition with $Y \to Z$ agrees as well. We can now put the fibered product $X_1 \times_Y X_2$ in the position of W for the definition of the fibered product $X_1 \times_Z X_2$, obtaining a unique or natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$. □

2.3.R Exercise. I’m not sure to what extent the map should (or even *can*) be described, but it is induced by the diagram

$$
\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] \arrow[dd, dashed, bend right, "\exists!"'] & X_2 \arrow[d] \arrow[dd, bend left, "\mathrm{id}_{X_2}"] \\
X_1 \arrow[r] \arrow[dd, bend right, "\mathrm{id}_{X_1}"'] & Y \arrow[dd, bend left, "Y \to Z"] \\
X_1 \times_Z X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_1 \arrow[r] & Z
\end{tikzcd}
\tag{2.15}
$$

as all maps agree by taking the compositions and we can use the definition of the fibered diagram to obtain the unique map $X_1 \times_Y X_2 \to X_1 \times_Z X_2$. □

2.3.S Exercise. Let's draw another picture.

$$\begin{tikzcd}
P' & & \\
& P \arrow[ul, dashed, "\exists!"] & N \arrow[l, "\nu"] \arrow[ull, "\nu'"'] \\
& M \arrow[u, "\mu"'] \arrow[uul, "\mu'"] &
\end{tikzcd} \tag{2.16}$$

When $P = M \sqcup N$ we have obvious morphisms μ and ν which are the canonical injections. But μ' and ν' are uniquely determining $P \to P'$ as exactly the right information is stored in P. ☐

2.3.T Exercise. We check the axioms for the ring morphism $b \mapsto b \otimes 1$:

additivity By definition of the tensor product of two A-modules we obtain

$$(b_1 + b_2) \otimes c = b_1 \otimes c + b_2 \otimes c, \tag{2.17}$$

hence the map is additive.

multiplicativity By the definition of multiplication induced on $B \otimes_A C$ in **??** we obtain $(b_1 b_2) \otimes 1 = (b_1 \otimes 1)(b_2 \otimes 1)$.

preservation of unity By definition we have $1 \mapsto 1 \otimes 1$ which acts as unity for the multiplication.

We have maps $B \to B \otimes_A C$ and $C \to B \otimes_A C$ by the previous observations. Now these maps agree by definition of $\otimes_A$: the images of elements of A in either B or C can be moved back and forth after the map to $B \otimes_A C$.

For any other ring A' equipped with maps $f\colon B \to A'$ and $g\colon C \to A'$ such that they agree when composed with $A \to B$ and $A \to C$ we have an obvious map $B \otimes_A C \to A'$ by taking $b \otimes c \mapsto f(b)g(c)$. As unity is preserved under ring morphisms, this makes everything commute. By the axioms of ring morphisms, this is the unique map making everything commute as this multiplicative structure of the map is imposed. ☐

2.3.U Exercise. Consider two maps $g_1, g_2\colon Z \to X$ and two consecutive monomorphisms $f_1\colon X \to Y_1$, $f_2\colon Y_1 \to Y_2$ such that $f_2 \circ f_1 \circ g_1 = f_2 \circ f_1 \circ g_2$, by associativity and f_2 monomorphic we obtain $f_1 \circ g_1 = f_1 \circ g_2$ which by f_1 monomorphic yields $g_1 = g_2$. ☐

2.3.V Exercise. If $f\colon X \to Y$ is a monomorphism taking X as the fibered product proves its existence. The unique map $Z \to X \times_Y X = X$ is given by $g_1 = g_2$, where $g_1, g_2\colon Z \to X$.

Conversely, if the fibered product exists we have that $f \circ g_1 = f \circ g_2$ as these maps must agree. Now there is a *unique* map $Z \to X \times_Y X$ so we obtain $g_1 = g_2$ as g_1 and g_2 both equal the composition of this unique map with the equal projections $X \times_Y X \to X$.

By putting X in the position for a map $X \to X \times_Y X$ we obtain an induced isomorphism as we can use the identity map $X \to X$ at both sides in the diagram. $\square$

2.3.W Exercise. Using the magic diagram we obtain

$$
\begin{tikzcd}
X_1 \times_Z X_2 \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\
& Y \arrow[r] & Y \times_Z Y \cong Y
\end{tikzcd}
\tag{2.18}
$$

where we have $Y \times_Z Y \cong Y$ by **??**. The isomorphism is immediate by the uniqueness of $X_1 \times_Z X_2 \to X_1 \times_Y X_2$. $\square$

2.3.X Exercise. By plugging in $C = A$ we get the candidate $g := i_A(\mathrm{id}_A)$. $\square$

2.4 Limits and colimits

2.4.A Exercise. As stated in the exercise the morphisms f_j are the obvious projection maps. The maps $F(m)$ for all $m\colon j \to k$ in $\mathcal{I}$ commute with these projections by the identification $F(m)(a_j) = a_k$.

Now take an object W such that the $g_i\colon W \to A_i$ commute with all the $F(m)$. Construct $g\colon W \to \varprojlim_{\mathcal{I}} A_i$ by using he value $g_i(w)$ for the ith position of the direct limit. By demanding commutativity this map is necessarily unique. $\square$

2.4.B Exercise. (a) The rational numbers $\mathbb{Q}$ is the object that captures all of the information contained in the $\frac{1}{n}\mathbb{Z}$. We define a morphism $\frac{1}{n}\mathbb{Z} \to \frac{1}{m}\mathbb{Z}$ is $n \mid m$, by $\frac{z}{n} \mapsto \frac{(m/n)z}{m}$. The remark in the notes after **??** is helpful in this respect.

(b) Take subsets A_j and A_k of A such that $m\colon A_j \hookrightarrow A_k$ is the obvious inclusion map (if it exists). The diagram becomes

$$
\begin{tikzcd}
& \varinjlim_{\mathcal{I}} A_i \\
A_j \arrow[ur, "f_j"] \arrow[r, hook] & A_k \arrow[u, "f_k"']
\end{tikzcd}
\tag{2.19}
$$

where $\varinjlim_{\mathcal{I}} A_i$ should capture enough but no more of the information contained in the A_i in order to make maps out of it to compatible objects unique.

Therefore two different elements are mapped to different elements of the colimit, but they are identified for the inclusion maps. The union captures all of this information: we have the obvious inclusions f_i, the $F(m)$ are still the inclusion maps from the previous paragraph and as to maps out of $\varinjlim_{\mathcal{I}} A_i$ to compatible objects: these are unique because the g_j define them uniquely.

$\square$

2.4.C Exercise. Extending the answer to **????** we see that the maps out of $\varinjlim_{\mathcal{I}} A_i$ are defined uniquely by this quotient of the disjoint union. $\square$

2.4.D Exercise. First the well-definedness:

addition the pointwise addition is compatible with the construction because the maps are all A-module maps, hence the identification is preserved;

scalar multiplication the same reasoning holds.

And this construction serves as a colimit because two elements that get identified will have an equal image out of the colimit (*i.e.*, just one choice) while the compatibility of the maps gives us the construction of the map $\varinjlim_{\mathcal{I}} \to W$[1]. $\square$

2.4.E Exercise. The maps $F(m)\colon \frac{1}{s_1}A \to \frac{1}{s_2}A$ where $s_2 = s_1 s'$ are defined by $\frac{1}{s_1}a \mapsto \frac{1}{s_2}s'a$. So an element in $S^{-1}A$ which can be regarded as a fraction appears somewhere in the direct system and stays there by the integrality. The direct limit essentially captures all the information of the localization, for which integral domains are the most intuitive case.

In this more general case, the torsion will disappear. $\square$

2.4.F Exercise. The diagram defining the colimit is now broken into several disjoint parts by lack of the filtered condition. The construction works for these parts and they are independent, hence these are not affected by the quotient of the direct sum. $\square$

2.5 Adjoints

2.5.A Exercise. Extending the diagram given in (2.5.0.2) we obtain

$$\begin{array}{ccccc} \mathrm{Mor}_{\mathcal{B}}(F(A'), B) & \xrightarrow{Ff^*} & \mathrm{Mor}_{\mathcal{B}}(F(A), B) & \xrightarrow{g^*} & \mathrm{Mor}_{\mathcal{B}}(F(A), B') \\ \downarrow{\scriptstyle \tau_{A',B}} & & \downarrow{\scriptstyle \tau_{A,B}} & & \downarrow{\scriptstyle \tau_{A,B'}} \\ \mathrm{Mor}_{\mathcal{A}}(A', G(B)) & \xrightarrow{f^*} & \mathrm{Mor}_{\mathcal{A}}(A, G(B)) & \xrightarrow{Gg^*} & \mathrm{Mor}_{\mathcal{A}}(A, G(B')) \end{array} \tag{2.20}$$

in an obvious way, where g^* is *appending* the morphism to $f\colon F(A) \to B$. $\square$

[1] You might have noticed it: I am not god at this kind of arguments. Please consult a good algebra textbook.

2.5.B Exercise. The map η_A should behave somewhat like an identity. If we take

$$\eta_A := \tau^{-1}_{A,F(A)}\left(\mathrm{id}_{F(A)}\right) \tag{2.21}$$

with $\mathrm{id}_{F(A)} \in \mathrm{Mor}_{\mathcal{B}}(F(A), F(A))$ we find the correct definition.
Analogously we take

$$\epsilon_B := \tau^{-1}_{G(B),B}\left(\mathrm{id}_{G(B)}\right) \tag{2.22}$$

in $\mathrm{Mor}_{\mathcal{A}}(G \circ F(A), A)$ where $\mathrm{id}_{G(B)} \in \mathrm{Mor}_{\mathcal{A}}(G(B), G(B))$.
Now these maps are the ones we were looking by the functoriality of F and G. □

2.5.C Exercise. Taking $f\colon M \otimes_A N \to P$, transform it to $g\colon M \to \mathrm{Hom}_A(N, P)$ by defining $g(m)(-) := f(m \otimes -)$. This is a bijection:

injective Take f_1, f_2 such that $f_1 \neq f_2$, *i.e.*, there exist $m \in M$ and $n \in N$ such that $f_1(m \otimes n) \neq f_2(m \otimes n)$. Taking the corresponding g_i we obtain $g_1(m)(-) \neq g_2(m)(-)$ as their values in the point n differ.

surjective By the universal property of the tensor product.

□

2.5.D Exercise. It just boils down to unwinding the diagram from **??**. Set $F := -\otimes_A N$ and $G := \mathrm{Hom}_A(N, -)$, $A := M$, $A' := M'$, $B := P$ and $B' := P'$. Now taking $h\colon M'\otimes_A N \to P$, chasing the diagram using $\tau_{A,B} \circ Ff^*$ we obtain $h\circ(f\otimes \mathrm{id})(-_M \otimes -_N)$ first and then $h(f(-_M))(-_N)$.
Chasing the diagram in the other direction, through $f^* \circ \tau_{A',B}$ we first obtain $h(-_M)(-_N)$ and then end up with $h(f(-_M))(-_N)$, hence equality. The rest is analogous, now you have to compose with the map $g\colon B \to B'$. □

2.5.E Exercise. Let's take the equivalence relation described in the notes with *pointwise addition*. Now we check the group axioms:

closure taking (a, b) and (c, d) in $S \times S$, the pointwise sum is $(a+c, d+e)$ and both components are defined by the binary operation in the semigroup;

associativity again, the pointwise sum is associative by the associativity of the binary operation in the semigroup;

identity the element (s, s) for $s \in S$ is the identity: all of them are identified by the equivalence relation and we have $(a, b) \sim (a + s, b + s)$ because $a + b + s + e = b + a + s + e$ as S is an *abelian* semigroup;

inverse element the inverse element of (a, b) is given by (b, a): their sum is $(a + b, b + a)$ which is a representative of the equivalence class of the identity.

The map $S \to H(S)$ is given by choosing a fixed element e and mapping s to $\overline{(s,e)}$ in $S \times S/\sim$. By the equivalence relation, any choice will do and induce the same abelian group.

Reproducing the diagram for **??** with the corresponding functors in place we obtain

(2.23)
$$\begin{tikzcd} \mathrm{Mor}_{\mathsf{Ab}}(H(S'),G) \arrow[r, "Hf^*"] \arrow[d, "\tau_{S',G}"] & \mathrm{Mor}_{\mathsf{Ab}}(H(S),G) \arrow[r, "g^*"] \arrow[d, "\tau_{S,G}"] & \mathrm{Mor}_{\mathsf{Ab}}(H(S),G') \arrow[d, "\tau_{S,G'}"] \\ \mathrm{Mor}_{\mathsf{AbS}}(S',F(G)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathsf{AbS}}(S,F(G)) \arrow[r, "Fg^*"] & \mathrm{Mor}_{\mathsf{AbS}}(S,F(G')) \end{tikzcd}$$

where $f\colon S \to S'$ in AbS and $g\colon G \to G'$ in Ab. The maps $\tau_{S,G}$ are obtained by forgetting the group structure on $H(S)$, *i.e.*, given $m\colon H(S) \to G$ we let $\tau_{S,G}m$ be the map sending $s \in S$ to the image of $\overline{(s,e)}$ under m which is $\overline{(m(s),m(e))}$.

Now the commutativity follows from the following observation: taking a morphism $n\colon H(S') \to G$, applying Hf^* yields a map

$$Hf^*(g)\colon \overline{(s,e)} \mapsto g\left(\overline{(f(s),f(e))}\right). \tag{2.24}$$

We immediately see that going the other direction is equal by definition of f^*. The second part is completely analogous. $\square$

2.5.F Exercise. By filling in $\pi = \mathrm{id}_S$ in the universal property defining groupification we easily see the unique morphism $G \to G'$ is given by $S \to G$. $\square$

2.5.G Exercise. Because by definition $1 \in S$ we have the desired inclusion of categories $\mathsf{Mod}_{S^{-1}A} \hookrightarrow \mathsf{Mod}_A$ as every S^{-1}-module is an A-module using $a \mapsto a/1$. The converse doesn't hold obviously.

Now this embedding is fully faithful: every A-module morphism $M \to M'$ is an $S^{-1}A$-module morphism: the localization can occur either before or after the morphism by the corresponding linearity.

The adjointness is still to come. $\square$

??: prove adjointness of inclusion functor and forgetful functor in Mod_A and $\mathsf{Mod}_{S^{-1}A}$

2.6 (Co-)kernels and exact sequences (an introduction to abelian categories)

2.6.A Exercise. By the Freyd-Mitchell embedding theorem, we can diagram-chase elements. By the element-wise definition of $\operatorname{im} f^i$ there is an injection $\operatorname{im} f^i \hookrightarrow A^{i+1}$ and likewise the cokernel is *defined as* $A^{i+1}/\operatorname{im} f^i$, hence we immediately obtain the surjection in the second part of the diagram.

The ith cohomology $H^i(A^\bullet)$ is defined as $\ker f^i/\operatorname{im} f^{i-1}$ which gives us the injection into $\operatorname{coker} f^{i-1}$, defined as $A^i/\operatorname{im} f^{i-1}$, by the injection $\ker f^i \hookrightarrow A^i$. For the surjectivity of $\operatorname{coker} f^{i-1} \to \operatorname{im} f^i$ we use the fact that it is a complex: $f^i \circ f^{i-1} = 0$. $\square$

2.6.B Exercise. We first use the fact from linear algebra that for the exact sequence

$$0 \to A^1 \to A^2 \to A^3 \to 0 \tag{2.25}$$

where $\dim A^2 = \dim A^1 + \dim A^3$.

Now by taking the long sequence apart and setting $A^1 := \operatorname{im} d^i = \ker d^{i+1}$ and likewise $A^3 := \operatorname{im} d^i = \ker d^{i+1}$ we can chain all these sums together and obtain the desired equality. □

2.6.C Exercise. The key insight is: *positionwise*. The category $\mathsf{Com}_{\mathsf{Mod}_A}$ is additive by imposing the positionwise (as in: each position in a complex) structures. The kernels and cokernels of maps of complexes are defined by putting the appropriate (co)kernels at each position. The additional axioms of an abelian category follow likewise. □

2.6.D Exercise. The truth lies in the commutativity of diagram (2.6.4.5), but I have to come up with a good formulation. □

??: prove induced map of homology

2.6.E Exercise. This is by breaking apart the short exact sequences and putting them back together equivalent to the definition. □

2.6.F Exercise. (a) We consider an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{2.26}$$

in Mod_A that is mapped by S^{-1} to a sequence

$$0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0 \tag{2.27}$$

in $\mathsf{Mod}_{S^{-1}A}$ and we'll study its exactness.

This is almost by definition: if we take a morphism $f\colon M \to M'$ in Mod_A we define $S^{-1}f\colon S^{-1}M \to S^{-1}M'$ to be $S^{-1}f\colon (m/s) \mapsto (f(m)/s)$. The surjectivity is obviously preserved and if $(m_1'/s), (m_2'/s) \in S^{-1}M'$ are two different element such that $S^{-1}f(m_1'/s) = S^{-1}f(m_2'/s)$ we have that $ss'(f(m_1') - f(m_2')) = 0$ but this implies $f(m_1') = f(m_2')$, a contradiction.

(b) See **??**.

(c) Now considering the exact sequence

$$0 \to M_1 \to M_2 \to M_3 \tag{2.28}$$

with $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ in Mod_A, which is transformed to

$$0 \to \operatorname{Hom}(M, M_1) \to \operatorname{Hom}(M, M_2) \to \operatorname{Hom}(M, M_3), \tag{2.29}$$

in Mod_A. The preservation of injectivity is satisfied by definition of a monomorphism: take $g_1, g_2\colon m \to M_1$ such that the induced $\operatorname{Hom}(M, f)$ maps these to $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.

Now for the exactness at $\mathrm{Hom}(M, g)$, I was heavily inspired by this question at `math.stackexchange.com` as my original proof looked quite like the asker's proof. We have $\mathrm{im}\, f = \ker g$, so $g \circ f = 0$ and f has the universal property

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists! h'"'] \arrow[d, "h"] \arrow[dr, "0"] & \\ M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] & M_3 \end{tikzcd} \tag{2.30}$$

where $h\colon M \to M_2$ such that $g \circ h = 0$ and we obtain a factorization $f \circ h' = h$. So if $\mathrm{Hom}(M, g)(h) = 0$ we have $\mathrm{Hom}(M, f)(h') = 0$ hence we obtain the first inclusion $\ker \mathrm{Hom}(M, g) \subseteq \mathrm{im}\, \mathrm{Hom}(M, f)$ and $\mathrm{Hom}(M, i)$ is injective.

Now we draw the second diagram

$$\begin{tikzcd} & M \arrow[dl, "h'"'] \arrow[d, "h"] \arrow[dr, dashed, "0"] & \\ M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] & M_3 \end{tikzcd} \tag{2.31}$$

where $h\colon M \to M_1$ is such that $h = f \circ h' = \mathrm{Hom}(M, f)h'$ we easily obtain $\mathrm{Hom}(M, g)h = 0$ because

$$\mathrm{Hom}(M, g)h = \mathrm{Hom}(M, g)\,\mathrm{Hom}(M, f)h' = \mathrm{Hom}(M, g \circ f)h' = 0 \circ f \tag{2.32}$$

so $\mathrm{im}\, \mathrm{Hom}(M, f) \subseteq \ker \mathrm{Hom}(M, g)$. We have obtained equality.

The covariance is by the position of M: a morphism $M \to M_1$ gives rise to a morphism $M \to M_2$ by composing it on the right with $M_1 \to M_2$ (where composition on the right means "on the right in the diagram", it's on the left when notated as functions).

And finally: the proof holds in general abelian categories, we have not assumed anything about a module structure on our objects.

(d) The proof of the left-exactness of $\mathrm{Hom}(M, \cdot)$ can be adapted to proof right-exactness of $\mathrm{Hom}(\cdot, M)$, noting that we have contravariance: given $M_2 \to M$ we obtain a morphism $M_1 \to M$ by composing it on the left.

$\square$

There are more exercises, but I haven't done them yet.

finish exercises in Section 2.6

2.7 Spectral sequences

I decided to leave this section as-is, doing it when the need arises as suggested in the introduction.

finish exercises in Section 2.7

Chapter 3

Sheaves

3.1 Motivating example: the sheaf of differentiable functions

3.1.A Exercise. As every element of $\mathcal{O}_p \setminus \mathfrak{m}$ is nonzero in a neighbourhood of p we can restrict an element such that it is invertible there, a property which is preserved when taking the stalk. Hence the germ of a non-vanishing function is invertible and $\mathfrak{m}$ is invertible. □

3.1.B Exercise. I don't really have a differential geometry background and I fail to see what should be proved. But I should revisit this exercise later. □

??: $\mathfrak{m}/\mathfrak{m}^2$ is real vector space

3.2 Definition of sheaf and presheaf

3.2.A Exercise. A functor assigns an object in Sets to every object in the category of open sets of the topological space X. As an inclusion of sets is reflected as a morphism of the two open sets involved, this is translated to a morphism of sets in the codomain category[1]. Identities are preserved by functors, so $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ by definition. For the commutativity of restriction maps, this is by the *contravariance* of the functor. □

3.2.B Exercise. (a) The presheaf axioms are (trivially) true by restriction of functions. Yet it is not a sheaf, take $x \mapsto |x|$. On $B(0,n)$ the function is bounded by n, we can take $\mathbb{C} = \bigcup_{n\in\mathbb{N}} B(0,n)$ and glue together a function on all of $\mathbb{C}$ yet it is not bounded hence not a section over $\mathcal{C}$ of this sheaf.

(b) Inspired by `math.stackexcange.com` (and my complex analysis course) the idea is to circle a zero, ending up with a multiple-valued function, which is clearly not a section of the sheaf.

[1] I have never seen this terminology, don't shoot me if I missed some more obvious wording.

Take $f\colon z \mapsto z$ on the annulus $U = \{1-\epsilon < |z| < 1+\epsilon\}$. By the Cauchy integral formula we have

$$\oint_{|z|=1} \frac{g'(z)}{g(z)}\,dz \in 2\pi i\mathbb{Z} \tag{3.1}$$

where g is holomorphic, without zeroes on the unit circle. Hence for f we obtain $2\pi i$.

But if $f = g^2$ we would obtain

$$\oint_{|z|=1} \frac{f'(z)}{f(z)}\,dz = 2\oint_{|z|=1} \frac{g'(z)}{g(z)}\,dz \in 4\pi i\mathbb{Z} \tag{3.2}$$

which is a contradiction on the value we previously obtained. Now f can be patched together from holomorphic functions that admit a square root, as long as the open set doesn't wind around zero: take open balls to cover U and we're done.

□

3.2.C Exercise. As we have maps $\mathcal{F}(\bigcup U_i) \to \mathcal{F}(U_i)$ (or arbitrary unions of U_i) this is a limit: the arrows from our desired object map *to* all the objects as described in 2.4.4. □

3.2.D Exercise. (a) Such functions are defined in their points, if all the restrictions agree for a covering, the functions are obviously equal. The identity axiom is satisfied. But given a covering on which all restrictions agree we can just paste together a global function, hence the gluability axiom is satisfied too.

Glueing functions preserves their extra properties: these are all defined in a neighbourhood of a point, hence valid in an open set and therefore lift to the global function.

(b) Analogous.

□

3.2.E Exercise. The presheaf axioms are trivially satisfied, local functions restrict easily. The identity axiom is obvious too. For the gluability: observe that the compatibility boils down to defining a function on the *connected components* of the covering because sections are constant if there is a non-empty intersection and therefore this axiom is satisfied too. □

3.2.F Exercise. The key idea is "a function is continuous" if and only if "locally a function is continuous". Hence restriction of continuous functions is well defined and the commutativity of the restriction triangle is immediate. Now for the identity axiom: equality of functions is checked in the points and as all restrictions agree the global sections must agree. Likewise we can check the gluability axiom. □

3.2.G Exercise. (a) The sections of f over U form a subset of the sections of the sheaf from **??** over U. The restrictions are trivially fine, the identity axiom follows from the previous exercise too and gluability is obviously true as well as $f \circ s = \mathrm{id}|_U$ holds for the pointwise glueing of a section.

(b) There is nothing to add to the arguments of **??**. The set of sections carries a natural pointwise abelian group structure.

□

3.2.H Exercise. The presheaf axioms are satisfied because f is continuous hence the inverse f^{-1} maps open sets to open sets. The presheaf axioms are satisfied on the open sets in the domain of the map and the presheaf axioms for $f_*\mathcal{F}$ are a subset (sketchy wording) of those for $\mathcal{F}$.

The sheaf axioms are satisfied because inverse maps are compatible with unions and intersections. As for the presheaf axioms: everything is transferred. □

3.2.I Exercise. Using the definition of the direct limit and the construction as described in **??** we see that *less* relations have to be quotiented, as the direct system defining $(f_*\mathcal{F})_y$ is contained in the direct system defining $\mathcal{F}_p$. □

3.2.J Exercise. As germs in the stalk $\mathcal{F}_x$ are equivalence classes of functions defined in the neighbourhood modulo equality on a (smaller) neighbourhood, we can define the $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$ by defining it on representatives of these classes. By commutativity of (3.2.12.1) this is well defined. The actual checking of the structure is straightforward and familiar. □

3.3 Morphisms of presheaves and sheaves

3.3.A Exercise. The induced morphism is given by applying the morphism to representatives of equivalence classes of the stalks. By commutativity of the diagram in the definition of (pre)sheaf morphisms this is correct. □

3.3.B Exercise. To every sheaf $\mathcal{F}$ in Sets_X there is a (unique) associated sheaf $f_*\mathcal{F}$ in Sets_Y. Obviously $f_*(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{f_*(\mathcal{F})}$ and we have $f_*(h \circ g)$ with $g\colon \mathcal{F} \to \mathcal{G}$ and $h\colon \mathcal{G} \to \mathcal{H}$ by applying this to every open set U. □

3.3.C Exercise. The presheaf axioms are easy: the associated restriction is just the obvious restriction of a function $f\colon \mathcal{F}(U) \to \mathcal{G}(U)$ to $f|_V\colon \mathcal{F}(V) \to \mathcal{G}(V)$ in $\mathcal{H}om(\mathcal{F},\mathcal{G})(V)$ for $V \subseteq U$.

The identity axiom for $\mathcal{H}om$ follows from the separatedness of $\mathcal{G}$. Take a cover $(U_i)_i$ of U, any open subset V of U and a section $s \in \mathcal{F}(V)$. Now we have $f(V)(s)|_{U_i \cap V} = g(V)(s)|_{U_i \cap V}$ (which are elements of the sheaf $\mathcal{G}$) for the covering $(U_i \cap V)_i$ of V. As $\mathcal{G}$ is assumed to be separated we obtain $f(V)(s) = g(V)(s)$. As this holds for any V and all sections s, we have that $\mathcal{H}om$ is a separated presheaf.

Now consider a family $(f_i)_i$ of compatible sections (which are morphisms of sets of sections) on a cover $(U_i)_i$ of U. Like the case of the identity axiom, we consider $V \subseteq U$ open and $s \in \mathcal{F}(V)$. This gives

$$f_i(U_i \cap V)(s|_{U_i \cap V}) \in \mathcal{G}(U_i \cap V). \tag{3.3}$$

Now glue together $f(V)(s)$ in $\mathcal{G}(V)$. This defines a morphism of presheaves, by separatedness of $\mathcal{G}$ (the compatibility of restrictions is induced in a unique way).

If $\mathcal{G}$ is a sheaf of abelian groups, morphisms $\mathcal{F} \to \mathcal{G}$ carry a natural group structure, by pointwise addition. The neutral element is the zero map. □

3.3.D Exercise. (a) By definition

$$\mathcal{H}\mathrm{om}\left(\underline{\{p\}}, \mathcal{F}\right) = \mathrm{Mor}\left(\underline{\{p\}}|_U = \{p\}, \mathcal{F}|_U\right). \tag{3.4}$$

As there is a unique map $f_x \colon \{p\} \to \mathcal{F}|_U : p \mapsto x$ for every element of $x \in \mathcal{F}|_U$ (hence a bijection of sets) we have obtained an isomorphism of sheaves.

(b) Analogously we now have a map $f_g \colon \underline{\mathbb{Z}}(U) = \mathbb{Z} \to \mathcal{F}(U)$ such that $1 \mapsto g$, defined for every $g \in \mathcal{F}(U)$. Now take sections g and $g' \in \mathcal{F}(U)$. We obtain $f_{g+g'} = f_g + f_{g'}$ because the image of 1 defines everything (*i.e.*, $\mathbb{Z}$ is the free group generated by $\{1\}$).

(c) Analogous to **??**.

□

3.3.E Exercise. In the situation of the diagram as given in the notes, we have injective maps $\ker_{\mathrm{pre}} \phi(V) \to \mathcal{F}(V)$ for every V open. Define $\mathrm{res}^{\ker}_{V,U}$ by mapping $g \in \ker_{\mathrm{pre}} \phi(V)$ to the preimage of $\mathrm{res}_{V,U}$ under the injection maps. As both maps are injective, this preimage is a well-defined element. We have found our induced restriction map. It is unique by commutativity of the square and the injectivity: any two maps such that $f \circ g_1 = f \circ g_2$ imply $g_1 = g_2$.

We now check the conditions required for a presheaf. Obviously $\mathrm{res}^{\ker}_{U,U}$ is the identity map and the commutativity of the restriction triangle follows from the uniqueness of the restriction maps. □

3.3.F Exercise. Given a presheaf morphism $\phi \colon \mathcal{F} \to \mathcal{G}$, we would like to characterize $\mathrm{coker}_{\mathrm{pre}} f$ by the universal property

$$\begin{tikzcd} & & \mathcal{H} \\ \mathcal{F} \arrow[r, "\phi"'] \arrow[urr, "0"] & \mathcal{G} \arrow[r, "p"'] \arrow[ur] & \mathrm{coker}_{\mathrm{pre}} f \arrow[u, dashed, "\exists!"'] \end{tikzcd}. \tag{3.5}$$

Defining $(\mathrm{coker}_{\mathrm{pre}} \phi)(U) := \mathrm{coker}\, \phi(U)$, we can conclude that the universal property characterizes this definition because categorical properties of presheaves and presheaf morphisms are verified on open sets. The map p is induced by the definition and the map $\mathrm{coker}_{\mathrm{pre}} \phi$ exists by the "check on open sets" mantra and is unique for the same reason. □

3.3.G Exercise. A morphism between sheaves is defined as a morphism of the category over which we are considering the sheaf for *every* open set. We now restrict ourselves to a specific open set, hence the morphisms (the only nontrivial part of a functor's definition) are preserved.

This functor is exact because the exactness of a sequence of sheaves if checked over open sets by the fact that all abelian-categorical notions are verified "open set by open set". And we only consider a specific open set, over which the exactness is given. □

3.3.H Exercise. The abelian-categorial notions are verified "open set by open set", hence we consider a family of functors as in **??** indexed by all open sets in the topology on X. The equivalence follows. □

3.3.I Exercise. The uniqueness of the induced restriction map and the injectivity for every open set give us the identity axiom: if all restrictions are equal we can just move everything one morphism to the right and identify things there. The same holds for the gluability axiom: given local f_i, we move everything using the injectivity to the sheaf's object $\mathcal{F}(U_i)$ and lift things to $\mathcal{F}(U)$. Now we don't know yet that this section lives in $\ker_{\mathrm{pre}} \phi$, but if it wouldn't, there needs to be an open set for which the image under ϕ is nonzero. □

??: check that presheaf kernel is a sheaf

3.3.J Exercise. The inclusion $\underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_X$ is obvious: constant functions are holomorphic. Under the mapping $f \mapsto \exp(2\pi i f$ the kernel are exactly those functions such that $\exp(2\pi i f)$ is 1 (the neutral element in this multiplicative group). This is true for all constant integer functions: $2\pi i n$ evaluates under exponentiation to 1. As $\mathcal{F}$ is the presheaf of functions admitting a holomorphic logarithm, they must come from the exponentiation of a holomorphic function contained in $\mathcal{O}_X$ and we have the surjectivity of $\mathcal{O}_X \twoheadrightarrow \mathcal{F}$.

The failure of the sheaf axioms is for tomorrow □

??: show that holomorphic functions admitting a logarithm do not form a sheaf

3.4 Properties determined at the level of stalks, and sheafification

3.4.A Exercise. Assume the map is not injective, *i.e.*, there are sections f and g in $\mathcal{F}(U)$ such that the product of their values in the stalks in U is equal. Every value arises by construction from a section on a neighbourhood, hence for the intersection of these neighbourhoods the restrictions of f and g have to agree. Now use the identity axiom because $\mathcal{F}$ is a separated presheaf, which leads to $f = g$, a contradiction. □

3.4.B Exercise. The notion of compatible germ corresponds to the notion of compatible sections in the gluability axiom: a germ is an equivalence class represented by an open neighbourhood with a section on it, which induces equality for the restrictions $U_i \cap U_j$ under the equivalence relation on the stalks. Now glue together a section over U, this serves as the preimage of a product of compatible germs. □

3.4.C Exercise. Given a germ $f \in \mathcal{F}_p$, represented by (f_U, U) where $p \in U$ open, we define

$$\phi_p \colon \overline{(f_U, U)} \mapsto \overline{(\phi(f_U), U)}. \tag{3.6}$$

By compatibility of restrictions, this is well-defined. □

3.4.D Exercise. It is actually enough for $\mathcal{G}$ to be a monopresheaf.

Take any section $s \in \mathcal{F}(U)$. We have $f(s)$ and $g(s)$ in $\mathcal{G}(U)$. Because the stalk morphisms are equal, *i.e.*, $f_x(s_x) = g_x(s_x)$, we can take a representative of these, obtaining an open set U_x for each $x \in U$ where the restrictions of the sheaf morphisms agree. Using the identity axiom we have $f(s) = g(s)$. □

3.4.E Exercise. Using the diagram in (3.4.3.1) we have injectivity for both vertical arrows by **??**. Now let $f\colon \mathcal{F} \to \mathcal{G}$ be an isomorphism of sheaves. The induced stalk morphisms are by construction isomorphisms.

For the other direction, assume all stalk morphisms are isomorphisms. The map $\mathcal{F}(U) \to \mathcal{G}(U)$ must be injective: if there are two sections of $\mathcal{F}$ over U mapped by f to the same section of $\mathcal{G}$ (over U) we get a contradiction on the commutativity of the square. Chasing the diagram the other way around is through an injection and a product of isomorphisms, which is injective, a contradiction. Now for the surjectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$: take a section $g \in \mathcal{G}(U)$, consider the corresponding compatible germs, take the inverse isomorphism to the product of stalks of $\mathcal{F}$ and use gluability to construct a section of $\mathcal{F}$ over U which is mapped to g. □

3.4.F Exercise. (a) Consider $X = \{x, y\}$ equipped with the discrete topology and let $\mathcal{F}(\{x\} = 0$, $\mathcal{F}(\{y\}) = 0$ and $\mathcal{F}(X) = \mathbb{Z}$. The stalks are the trivial group, the product is still the trivial group, so for $U = X$ we get a contradiction on the injectivity.

(b) After a little detour to (pre)sheaves of abelian groups, back to (pre)sheaves of sets. Take $\mathcal{F} := \underline{\{p\}}$ and $\mathcal{G}$ such that

$$\begin{aligned} \mathcal{G}(\{x\}) = \mathcal{G}(\{y\}) = \{p\} \\ \mathcal{G}(\{x, y\}) = \left\{p, p'\right\} \end{aligned} . \tag{3.7}$$

We have $\phi_1(X)(p) = p$ and $\phi_2(X)(p) = p'$ as possible candidates, yet the induced stalk morphisms are equal.

(c) Using the same setup: we have trivial stalk isomorphisms, yet by construction the presheaves are not isomorphic.

□

3.4.G Exercise. The standard argument: take $\mathcal{F}_1^{\mathrm{sh}}$ and $\mathcal{F}_2^{\mathrm{sh}}$ both sheafifications of $\mathcal{F}$. There are unique morphisms $\mathcal{F}_1^{\mathrm{sh}} \to \mathcal{F}_2^{\mathrm{sh}}$ and $\mathcal{F}_2^{\mathrm{sh}} \to \mathcal{F}_1^{\mathrm{sh}}$. Now we can compose these. But if I give you the identity map between both compositions, you have to agree we've obtained isomorphisms, right?

If $\mathcal{F}$ already is a sheaf, setting $\mathcal{F}^{\mathrm{sh}} := \mathcal{F}$ satisfies the conditions of sheafification if $\mathrm{sh} = \mathrm{id}$: the map f equals g and is therefore uniquely defined. □

3.4.H Exercise. The definition of $\mathcal{F}^{\mathrm{sh}}(U)$ takes the compatible germs as sections. The identity axiom follows by definition of the sections: sections are defined by pointwise products. The gluability axiom is by definition of the compatibility: if the restrictions agree just taking the product gives us a global section. □

3.4.I Exercise. Define

$$\mathrm{sh}\colon \begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^{\mathrm{sh}} \\ s & \longmapsto & \prod_{x\in U} s_x \end{array} . \tag{3.8}$$

□

3.4.J Exercise. First of all, by **??** $\mathcal{F}^{\mathrm{sh}}$ is a sheaf.

Using **??** we get the unique induced morphism: take the induced stalk morphisms as the morphisms for g. In case it's not clear how this induces a sheaf morphism (instead of a bunch of stalk morphisms), use **??**. □

3.4.K Exercise. Take $\mathcal{G}$ in the definition of the sheafification to be $\mathcal{G}^{\mathrm{sh}}$, the map g being $\mathrm{sh}\circ\phi$. Now we get the induced $f = \phi^{\mathrm{sh}}$ on the sheafifications.

The sheafification preserves the identity, the induced ϕ^{sh} is unique by definition, but $\mathrm{id}_{\mathcal{F}^{\mathrm{sh}}}$ does the trick. And it agrees with compositions by the uniqueness: the composition of the sheafified morphisms serves as the candidate for the sheafification of the composition. □

3.4.L Exercise. I'm not a category-lover, but I'll do this exercise later □

??: prove adjointness of sheafification functor and forgetful functor

3.4.M Exercise. Stalks are defined as equivalence classes of sections on open sets. The definition of the sheafification requires the existence of a section inducing stalks. The stalks of the sheafification are the canonical projections. So we get a morphism from the stalks of the presheaf to the stalks of the sheaf by mapping $\overline{(s_U, U)}_x$ to $s_{U,x}$ which is by construction a stalk of the sheafification. This mapping is bijective: surjectivity follows from construction of sh where a product of stalks arises from a section over a certain open set, take this section and set as representative of the equivalence class of a stalk in the presheaf as preimage. For the injectivity: consider two germs in x such that their image is equal in the stalk of $\mathcal{F}^{\mathrm{sh}}$, by the previous observation we could've picked the inducing compatible germs with their corresponding open set as the representative of the stalk of the presheaf, which makes the two germs equal.

And I should add the answer using Remark 3.6.3. □

??: sheafification is isomorphism of stalks, category theory style

3.4.N Exercise. Following the order of implications as indicated by the hints:

(a) ⇒ (c) A morphism of sheaves is given by a morphism of sets for each set of sections over an open set U. If ϕ is a monomorphism we have $g_1 = g_2$, but this condition can be applied for each open set (using the indicator sheaf if you want), obtaining the injectivity of $\phi(U)$.

(c) ⇒ (b) Take germs t_1 and t_2 in $\mathcal{F}_x$ such that $\phi_x(t_1) = \phi_x(t_2)$. Consider their representatives, there must be an open set contained in the intersection of the two open sets representing the germs such that their image is equal. But the injectivity on the level of open sets gives us that these representatives must be equal, hence the germs are too.

(b) ⇒ (a) Morphisms are determined by their stalks using **??**, if all the stalk morphisms are injective we can use the pointwise monomorphisms to determine $g_1 = g_2$ in the definition of a monomorphism.

These proofs hold for monopresheaves as well.

Observe that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) is possible as well using the definition of compatible germs and their correspondence with sheaves. For this chain of implications to be easy to proof we do need the sheaf condition though. □

3.4.O EXERCISE. By the observations from **??** and the construction of the sheafification, all the properties are defined on the level of stalks. If ϕ is an epimorphism, this holds for the products of stalks, implying epimorphisms on each stalk and the other direction is analogous. □

3.4.P EXERCISE. The map $\exp\colon \mathcal{O}_X \to \mathcal{O}_X^*$ is an epimorphism: at the level of the stalks we have for a nowhere zero g that g is nowhere zero in a neighbourhood of x, which can be taken as a representative and which admits a local logarithm.

But if we take a open neighbourhood of zero and puncture it there (or more general: any open set that is not contractible) we can construct a function that is nowhere zero, but is not in the image of exp: the winding number causes the function $z \mapsto z$ to be an section of $\mathcal{O}_X^*$ but there is no section in $\mathcal{O}_X$ mapping to it. □

3.5 Sheaves of abelian groups, and $\mathcal{O}_X$-modules, form abelian categories

3.5.A EXERCISE. We have $g \in (\ker(\mathcal{F} \to \mathcal{G}))_x$ if and only if there is an open set U and a section $s \in \ker(\mathcal{F} \to \mathcal{G})(U)$ representing the germ g. But this is equivalent to $s \in \mathcal{F}(U)$ being mapped to $(\mathcal{F} \to \mathcal{G})(U)(s) = 0$, or $(\mathcal{F}_x \to \mathcal{G}_x)(g) = 0$. □

3.5.B EXERCISE. Analogous to **??**. □

3.5.C EXERCISE. The image presheaf is defined by $\operatorname{im}^{\mathrm{pre}} \phi(U) = \phi(U)(\mathcal{F}(U))$. □

??: definition of image presheaf

3.5.D EXERCISE. Take $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ an exact sequence of sheaves. By **??** and **??** and the relation of the image to the cokernel (is it already a fact at this moment or am I using facts from Tennison which are unknown as of now?) the sequence of the stalks is exact too. □

3.5.E EXERCISE. By **??** the left-exactness is checked on open sets, hence we obtain the result.

Using **??** we easily obtain an example, patching together a function on an open set that is not simply connected. □

3.5.F EXERCISE. Take $U = f^{-1}(V)$, then we have by **??** the exactness of

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \tag{3.9}$$

which implies by definition of the pushforward the exactness of

$$0 \to f_*\mathcal{F}(V) \to f_*\mathcal{G}(V) \to f_*\mathcal{H}(V). \tag{3.10}$$

□

3.5.G Exercise. By **??** we can take Hom for each abelian group of sections and using the exactness of that sequence we obtain by left-exactness of Hom the result. □

3.5.H Exercise. This is boring but important. I should do this. □

??: $\mathcal{O}_X$-modules abelian category

3.5.I Exercise. (a) We want it to be defined using the tensor-Hom adjunction we saw in **??**. The presheaf tensor product of two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{3.11}$$

and the tensor product of two $\mathcal{O}_X$-modules is the sheafification of this construction.

This construction satisfies the adjointness by the adjunction of modules.

??: expand on tensor product of $\mathcal{O}_X$-modules

(b) Because stalks are preserved by sheafification, we have to show that the tensor product of stalks is the stalk of the presheaf tensor product. Using the fact that the tensor product commutes with direct limits (can we?), we obtain

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \varinjlim_{U \ni x} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x. \tag{3.12}$$

□

3.6 The inverse image sheaf

3.6.A Exercise. The induced restriction morphism follows from the construction using the direct limit: if $U_1 \subseteq U_2 \subseteq X$ are open sets the direct system $V \supseteq f(U_2)$ is contained in the direct system $V \supseteq f(U_1)$ as $f(U_1) \subseteq f(U_2)$. Using the construction of **??** we get the restriction by the canonical injection of coproducts.

For the commutativity of the restriction triangle the same idea holds. □

3.6.B Exercise. Let $f\colon f^{-1}\mathcal{G} \to \mathcal{F} \in \mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}$. We can apply the pushforward to both sheaves, resulting in $f_* f^{-1}\mathcal{G} \to f_*\mathcal{F}$. But by categorical definition in 3.6.1 we get a canonical map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ and we can compose these to get $\mathcal{G} \to f_*\mathcal{F}$.

The inverse map is given analogously by taking $g\colon \mathcal{G} \to f_*\mathcal{F}$, applying the inverse image f^{-1} and composing it with the natural $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

As for the bijectivity of the map, I don't have a clue (yet). □

??: adjointness of f_* and f^{-1}

3.6.C Exercise. We're interested in the stalks of the presheaf as they are preserved by sheafification. We obtain

$$
\begin{aligned}
(f^{-1}\mathcal{G})_p &= \varinjlim_{U \ni p} f^{-1}\mathcal{G}(U) \\
&= \varinjlim_{U \ni p} \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \\
&= \varinjlim_{f(U) \ni f(p)} \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \\
&= \lim_{V \ni f(p)} \mathcal{G}(V) = \mathcal{G}_q.
\end{aligned} \tag{3.13}
$$

□

3.6.D Exercise. In the construction of sections over V of the inverse image presheaf we have a direct system of open sets containing $i(V) =$. But the open set V appears in this system, hence we have $i^{-1}\mathcal{G}(V) = \mathcal{G}(V)$ where $V \subseteq U$. □

3.6.E Exercise. An exact sequence of sheaves of abelian groups on Y is transformed in an exact sequence of abelian groups by considering the stalks using **??**. Now consider the stalks at $q = f(p)$, these are by **??** isomorphic to the stalk of the inverse images in p. Now the inverse image is sheafified using these stalks, hence the inverse image sequence is exact. □

3.6.F Exercise. (a) If $y \in Z$, then $i^{-1}(U) = Z \cap U$ where $y \in U$ open. Now the stalk $(i_*\mathcal{F})_y$ is defined as the direct system of the $U \cap Z$ because $U \cap Z$ is open in the closed subspace Z. Therefore $(i_*\mathcal{F})_y = \mathcal{F}_y$ in this case. For $y \notin Z$ we have by definition the one element-set as the stalk.

(b) As $\operatorname{Supp}\mathcal{G} \subset Z$ the inverse image functor preserves all stalks by **??**. Now we use the previous observation to obtain the isomorphism.

□

3.6.G Exercise. (a) If $y \in U$ then the stalk at y is determined by the direct system of open sets containing y. Because we are working with sheaves we can cut off a part of the system on the left: compatibility ensures that the smaller direct system still defines the same stalk. Now if $y \notin U$ we have that all sections over open sets containing y are zero by definition.

(b) Exactness is checked at the stalks. These stalks are either all zero, or correspond to the stalks of an exact sequence of sheaves on the open subspace, which are exact by **??**.

(c) The inverse image functor i^{-1} defines the sheaf on U with stalks $\mathcal{G}_p$ for $p \in U$. Now the extension by zero sends this to $\mathcal{G}|_U$, hence the inclusion.

(d) This follows from the previous observations and I should write a more decent answer.

□

3.7 Recovering sheaves from a "sheaf on a base"

3.7.A Exercise. We have enough information to determine the stalk at p: we can drop the unions as the only interesting information is present in the open sets containing the point. $\square$

3.7.B Exercise. An element of $\mathcal{F}(B)$ is given by $(f_p \in F_p)_{p \in B}$ where we can take $p \in B = B$ such that $s_q = f_q$ for $q \in B$. But $F_p = \varinjlim F(B_i)$, such that the germ $s_q = f_q$ for all $q \in B$. $\square$

??: prove isomorphism of sheaves on a base

3.7.C Exercise. (a) Use the fact that sheaves on a base correspond uniquely to their corresponding sheaves up to unique isomorphisms: what happens in the stalks didn't change, so the induced morphisms of sheaves on a base still completely determines the morphism of sheaves.

(b) The morphism of sheaves on a base induces morphisms of stalks, which induce a morphism of the induced sheaves.

$\square$

3.7.D Exercise. By the hint: take a base of open sets consisting of open sets completely contained in an element of the cover. Arbitrary open sets are obviously unions of these.

We can now define the stalks of $\mathcal{F}$: the sheaves on the $(U_i)_i$ determine the sheaf on a base F by the cocycle condition which provides the compatibility of the base. Now we have obtained our induced sheaf. $\square$

3.7.E Exercise. It might be unimportant, but I would like to answer it. Unfortunately I fail to do so. $\square$

??: surjective on base implies surjective

Part II

Schemes

Chapter 4

Toward affine schemes

4.1 Toward schemes

4.1.A Exercise. Take U and V open in respectively X and Y such that for the corresponding charts ϕ and ψ the images $\phi(U)$ and $\psi(V)$ are homeomorphic to an open subset of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$. Let x_i and y_j be respectively the ith and the jth coordinate function for these charts. Now f is locally described by the maps $y_j \circ f \circ \phi^{-1}$ which is differentiable, so $\psi \circ f \circ \phi^{-1}$ is differentiable too and this is all compatible, describing the map f as a differentiable map of differentiable manifolds.

Or shorter: the pullback of $g\colon Y \to \mathbb{R}$ a differentiable function on Y is defined as $f_* g\colon X \to \mathbb{R} : x \mapsto g(f(x))$. $\square$

4.1.B Exercise. We have that $f_*\mathcal{O}_X$ has stalk isomorphisms for every $f(p) = q$. So by **??** we get induced stalk morphisms $\mathcal{O}_{Y,q} \to f_*\mathcal{O}_{X,p}$, inducing the desired morphism of stalks $f^\#$.

By the definition of $f^\#$ we have that a function $g\colon Y \to \mathbb{R}$ such that $g(q) = 0$ is mapped by **??** to $f^\#(g)(p) = g(f(p) = g(q) = 0$, so we have the inclusion $f^\#(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$. $\square$

4.2 The underlying set of affine schemes

4.2.A Exercise. (a) So we're looking for the prime ideals of $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. But these correspond to the prime ideals of $\operatorname{Spec} k[\epsilon]$ containing (ϵ^2). Now the only prime ideal of this form is (ϵ). This corresponds to the polynomials in ϵ with no constant term. If there would be a constant term, *i.e.*, something of the form $a + b\epsilon$ it would be invertible modulo ϵ^2 using a geometric series. There is only one point.

Notice that $\operatorname{Spec} k[\epsilon]/(\epsilon)$ is not an integral domain: ϵ^2 is contained in (0) yet $\epsilon \notin (0)$.

(b) By commutative algebra the prime ideals of the localization correspond to the prime ideals of Spec $k[x]$ not containing (x). So the set Spec $k[x]_{(x)}$ corresponds to Spec $k[x] \setminus \{(x)\}$ because there is only one prime ideal containing x namely (x): if there would be another one we could reduce it to a constant ending up the whole ring, a contradiction.

□

4.2.B Exercise. Using the discriminant we obtain the two roots of the quadratic which look like

$$x_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{4.1}$$

where $a^2 - 4b < 0$. Now using operations of $\mathbb{R}$ we can reduce this to i. □

4.2.C Exercise. This set corresponds to all polynomials that are irreducible over $\mathbb{Q}$. There are the obvious points $(x - a)$ where $a \in \mathbb{Q}$, but all roots of polynomials are present too but they are glued together by the corresponding Galois actions. It corresponds to the identification of roots in the algebraic closure $\mathbb{Q}^{\text{alg}}$. □

4.2.D Exercise. Suppose $\mathfrak{p}$ is a prime ideal that is not a principal ideal. Take two essential generators $f(x, y)$ and $g(x, y)$ (*i.e.*, with not all factors of one contained in the other). This must be possible because otherwise we wouldn't have a principal ideal: one can be written as a product of the other with a polynomial containing the missing factors. Now because $\mathfrak{p}$ is prime we can remove all common factors.

By applying the Euclidean algorithm in $\mathcal{C}(x)[y]$ we can find a polynomial in the variable x contained in $(f(x, y), g(x, y)) \subseteq \mathfrak{p}$, which by the algebraic closedness of $\mathcal{C}$ reduces to a linear factor $(x - a)$ contained in $\mathfrak{p}$ and analogously $(y - b) \in \mathfrak{p}$.

Obviously any principal ideal must be generated by an irreducible polynomial. So having reduced all non-principal ideals to ideals of the form $(x - a, y - b)$ we have finished the proof. □

4.2.E Exercise. The first maximal ideal is $(x^2 + y^2 - 4, x - y)$ while the second is $(x^2 + y^2 - 4, x + y)$. The residue fields are $\mathbb{Q}(\sqrt{2})$ in both cases: substituting the second generator in the first yields this result. □

4.2.F Exercise. (a) Let $m_\pi(x)$ be the minimal polynomial of $\pi \in \mathbb{C}$ if $\pi \in \mathbb{Q}^{\text{alg}}$ and zero otherwise. Now the image of (π, π^2) under ϕ is given by the irreducible polynomial $m_\pi(x) + m_{\pi^2}(y)$.

(b) I don't know. **??**: prove surjectivity of $\mathbb{C}^2 \to \mathbb{A}^2_{\mathbb{Q}}$

□

4.2.G Exercise. I have used this fact in **????**. It boils down to

$$A/J \cong (A/I)/(J/I) \tag{4.2}$$

where $I \subseteq J$ are prime ideals of A, and this is equivalent to $\bar{J}$ being prime in A/I. □

4.2.H Exercise. I have used this fact in **????**. A prime ideal $\mathfrak{p}$ of A that contains an element of S will become $S^{-1}A$ under localization: take an element $s \in \mathfrak{p} \cap S$, this is mapped to $(s/1)$ by the canonical $A \to S^{-1}A$, but $(1/s)$ is an element of $S^{-1}A$, so $1 \in S^{-1}\mathfrak{p}$ and it is not a prime ideal.

We now have by the preservation of primes under an inverse ring morphism that $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ associates prime ideals to prime ideals. The fact that this is a bijection comes from taking the forward map: a prime $\mathfrak{p}$ of $S^{-1}A$ not meeting S is mapped to itself (as mentioned before, primes that do meet S are mapped to the whole of $S^{-1}A$).

Conversely, suppose $\mathfrak{q}$ prime in A, then $S^{-1}(A \setminus \mathfrak{q})$ is a multiplicative subset of $S^{-1}A$ and $S^{-1}A = S^{-1}(A \setminus \mathfrak{q}) \coprod S^{-1}\mathfrak{q}$ so $S^{-1}\mathfrak{q}$ is prime. This was inspired by Atiyah-Macdonald. □

4.2.I Exercise. Elements of $(\mathbb{C}[x,y]/(xy))_x$ look like the sum of a polynomial in x with a polynomial in y (mixed terms are sent to zero under the quotient with (xy)) divided by a power of x.

The obvious ring morphism $(\mathbb{C}[x,y]/(xy))_x \to \mathbb{C}[x]_x$ is given by forgetting the polynomial in y. This is easily seen to be an isomorphism: a general element (containing terms in y) is in the localization of $\mathbb{C}[x,y]/(xy)$ already equal to the part without the terms in y because $(a_1/s_1) = (a_2/s_2)$ if and only if $s'(a_1 s_2 - a_2 s_1)$ and we can always multiply by $s' = x$ in order to drop all terms containing y. □

4.2.J Exercise. Assume $b_1 b_2 \in \phi^{-1}(\mathfrak{p})$, we have

$$\phi(b_1 b_2) = \phi(b_1)\phi(b_2) \in \phi\left(\phi^{-1}(\mathfrak{p})\right) = \mathfrak{p}, \tag{4.3}$$

hence $\phi(b_1)$ or $\phi(b_2)$ as $\phi(\phi^{-1}(\mathfrak{p})) = \mathfrak{p}$ is a prime ideal in A. We obtain that $\phi^{-1}(\phi(b_1)) = b_1$ or $\phi^{-1}(\phi(b_2)) = b_2$ must be an element of $\phi^{-1}(\mathfrak{p})$. □

4.2.K Exercise. (a) Using **??** everything is already clear: the primes of A containing I form a subset of $\operatorname{Spec} A$ and ϕ^{-1} is an inclusion-preserving bijection, giving us the suggested idea.

(b) Using **??** everything is analogous.

□

4.2.L Exercise. The fiber of $a \in \mathbb{C}$ corresponds to the preimage of the prime ideal (in this case: maximal ideal) defining a, *i.e.*, $(x - a)$. This obviously gives us $y^2 - a = (y - \sqrt{a})(y + \sqrt{a})$, hence the result. □

4.2.M Exercise. (a) This is a restatement of **????**.

(b) The Nullstellensatz gives us that all maximal ideals (*i.e.*, points) of $\mathbb{C}^n$ are exactly the ideals of the form $(x_1 - a_1, \dots, x_m - a_m)$, which by ϕ are mapped to the corresponding points of $\mathbb{C}^n$.

□

4.2.N Exercise. We have the situation of **??**, where $A = \mathbb{Z}[x_1, \dots, x_n]$ and the ideal $I = (p)$. So primes of $\mathbb{F}_p[x_1, \dots, x_n]$ correspond to primes of A that contain (p), which are exactly the points lying in the fiber $f^{-1}([(p)])$.

As for the field that appears in the figure, I guess it must be $\mathbb{Q}$. □

??: find out what field lies over $[(0)]$

4.2.O Exercise. (a) We have that every prime ideal contains all nilpotents: if c is a nilpotent such that $c^n = 0$, we immediately find that c is an element of the prime ideal. The bijection is between primes of A/I and primes of A containing I, but this latter set contains all primes, hence there is a bijection of the underlying sets.

(b) Let's check the axioms. The sum of two nilpotents is again a nilpotent: take x and y nilpotents such that $x^n = y^m = 0$, we easily obtain

$$(x+y)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} x^i y^{n+m-i} \tag{4.4}$$

such that there always is a vanishing factor present in the expansion. Closed under multiplication is obviously true too: we have $(bx)^n = b^n x^n = 0$ for $b \in B$.

□

4.2.P Exercise. Let $N(A)$ be the intersection of all prime ideals of A, let $\mathfrak{p}$ be a prime ideal of A and $x \in \mathfrak{N}(A)$. We have $x^{k-1}x = 0 \in P$ for some positive integer k and by primality of $\mathfrak{p}$ we have $x \in \mathfrak{p}$ or $x^{k-1} \in \mathfrak{p}$, so by induction $x \in \mathfrak{p}$, implying $N(A) \subseteq \mathfrak{p}$ and $\mathfrak{N}(A) \subseteq N(A)$.

Now take x in the complement, *i.e.*, in $A \setminus \mathfrak{N}(A)$. Define

$$S = \left\{ I \lhd A \,|\, x^k \notin I \forall k \in \mathbb{N}_0 \right\} \tag{4.5}$$

and there is the obvious partial order, induced by inclusion. Furthermore we have $(0) \in S$ so it is non-empty. By Zorn's lemma we get a maximal element $I \in S$.

Suppose $a, b \in A \setminus I$, the ideals (I, a) and (I, b) are strictly bigger, hence they are not in S. Therefore we have the existence of n and m such that $x^n \in (I, a)$ and $x^m \in (I, b)$, so $x^{n+m} \in (I, a)(I, b)$ which is a subset of (I, ab). We obtain $ab \notin I$ so I is a prime ideal. But in that case we would have $x \in I$, a contradiction because $I \in S$. □

4.2.Q Exercise. I fail to find a decent argument. □

??: prove that open subsets in an reduced affine algebraic set contains a closed points

4.2.R Exercise. A polynomial $f \in k[x]$ corresponds to $\sum_{k=0}^n a_k x^k$. Now considering it over $k[x, \epsilon]/(\epsilon^2)$ and "evaluating" it at $x + \epsilon$ we find

$$f(x+\epsilon) = \sum_{k=0}^{n} a_k (x+\epsilon)^k = \sum_{k=0}^{n} a_k \left(x^n + n x^{n-1} \epsilon\right) \tag{4.6}$$

because every term containing ϵ^2 is gone. If we move the first term of the inner sum to the left-hand side and dividing both sides by ϵ (which isn't really possible, but for the sake of argument we can assume this), we see the fact $(x^n)' = n x^{n-1}$. □

4.3 Visualing schemes I: generic points

There are no exercises in this section.

4.4 The underlying topological space of an affine scheme

4.4.A Exercise. The x-axis corresponds to the ideal (y, z). This ideal contains the ideal (xy, yz), hence we have the inclusion of the axis in the vanishing set. $\square$

4.4.B Exercise. We have the obvious inclusion $S \subseteq (S)$, so for the vanishing sets we have the opposite inclusion:

$$V\,V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\} \supseteq \{[\mathfrak{p}] \in \operatorname{Spec} A \mid (S) \subseteq \mathfrak{p}\} = V\left((S)\right). \tag{4.7}$$

But if we take an element of the left-hand side, *i.e.*, a prime ideal such that $S \subseteq \mathfrak{p}$ we can take an element of the generated ideal (S) and see that it is contained in $\mathfrak{p}$ by the axioms of an ideal. I might have wasted too much words on this exercise. $\square$

4.4.C Exercise. (a) We need to find vanishing sets such that their complements are $\varnothing$ and $\operatorname{Spec} A$. The vanishing set $V(A)$ contains all of A, hence its complement is empty. On the other hand, $V(\{0\})$ contains all prime ideals of A, hence equals $\operatorname{Spec} A$.

(b) This equality is obvious: for a point to be contained in the intersection, it must be contained in all ideals $(I_i)_i$, but that means it is contained in the vanishing set of the sum of the ideals because this contains exactly all those elements.

(c) We obviously have $V(I_1) \cup V(I_2) \subseteq V(I_1 I_2)$ because $I_1 I_2 \subseteq I_1, I_2$.

For the other direction, consider $\mathfrak{p} \in \operatorname{Spec} A \setminus (V(I_1) \cup V(I_2))$. That means there exist $f \in I_1 \setminus \mathfrak{p}$ and $g \in I_2 \setminus \mathfrak{p}$ such that $fg \notin \mathfrak{p}$. But $fg \in I_1 I_2$, so $\mathfrak{p} \not\supseteq I_1 I_2$ so the other inclusion is proved.

$\square$

4.4.D Exercise. The fact that $\sqrt{I}$ is an ideal is a restatement of **????**.

Because $I \subseteq \sqrt{I}$ we obviously have $V(\sqrt{I}) \subseteq V(I)$, and assume the inclusion is strict, so let's take $[\mathfrak{p}] \in V(I) \setminus V(\sqrt{I})$. That means there exists an element $a \in \sqrt{I} \setminus \mathfrak{p}$, but there exists an $n \in \mathbb{N}_0$ such that $a^n \in I \subseteq \mathfrak{p}$, which because $\mathfrak{p}$ is prime gives $a \in \mathfrak{p}$, a contradiction.

The fact that $\sqrt{\sqrt{I}} = \sqrt{I}$ follows from consecutive exponentiation. And if $f^m \in \mathfrak{p}$ for some m we have $f \in \mathfrak{p}$ by primality. $\square$

4.4.E Exercise. Consider the scenario of two ideals, hence we wish to prove

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}. \tag{4.8}$$

Because taking the radical preserves inclusion we have the inclusion from left to right as $I \cap J \subseteq I, J$ and therefore $\sqrt{I \cap J} \subseteq \sqrt{I}, \sqrt{J}$ so $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$.

Now take $x \in \sqrt{I} \cap \sqrt{J}$, there must exist an n such that $x^n \in \sqrt{I}$ and m such that $x^m \in \sqrt{J}$. But now $x^{n+m} \in I \cap J$, so $x \in \sqrt{I \cap J}$.

The general result follows by induction. $\square$

4.4.F Exercise. Perform it on A/I. The ideal I is sent to 0 under the quotient map, hence the nilradical $\mathfrak{N}(A/I)$ is the intersection of all prime ideals containing I, but this corresponds to $\sqrt{I}$ in A. □

4.4.G Exercise. By **??** closed points pull back to closed points. Now a closed set is the finite union of closed points, which under an inverse map is preserved. □

4.4.H Exercise. (a) By **??** we have that $\operatorname{Spec} B/I$ corresponds to the vanishing set $V(I)$, hence it is closed by definition of the Zariski topology.

Analogously, the vanishing set $V((f))$ described the complement of $\operatorname{Spec} S^{-1}B$ in this case.

This construction doesn't work for every localization as is clear from Figure 4.5 and **??**. The complement of the "shred of $\mathbb{A}^2_{\mathbb{C}}$" doesn't fulfill the conditions for a closed set described there and in Example 7 of that section.

(b) By **??** the primes in B/I correspond bijectively to primes in B containing I. So a closed subset in $\operatorname{Spec} B/I$ corresponds to a vanishing set that looks like $V(S) = \{\mathfrak{p} \in \operatorname{Spec} B/I \mid S \subset \mathfrak{p}\}$ which is associated to a vanishing set $V(S+I) = \{\mathfrak{p} \in \operatorname{Spec} B \mid I \subset S + I \subset \mathfrak{p}\}$. This set contains only points of $\operatorname{Spec} B/I$, so we have the subspace topology on $\operatorname{Spec} B/I$.

For $\operatorname{Spec} S^{-1}B$ we use **??**, which states that closed sets in $\operatorname{Spec} S^{-1}B$ correspond to sets of points in $\operatorname{Spec} B$ that don't meet S. But if $\mathfrak{p}$ doesn't meet S in $\operatorname{Spec} B$ it is contained in $D(f)$ (terminology from Section 4.5).

□

4.4.I Exercise. By definition we have

$$V(I) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid I \subseteq [\mathfrak{p}]\} \tag{4.9}$$

and f vanishes on this set if and only if it is contained in the prime ideals that constitute the points of $V(I)$. But using **??** this is exactly equivalent to $f \in \sqrt{I}$. □

4.4.J Exercise. By **??** we have the subspace topology on $\operatorname{Spec} k[x]_{(x)}$ so we need to state what the subspace is. By using **??** we obtain the affine line with the origin (corresponding to (x)) removed. □

4.5 A base of the Zariski topology on Spec A: distinguished open sets

4.5.A Exercise. Using the hint we find

$$X \setminus V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid S \nsubseteq [\mathfrak{p}]\} \tag{4.10}$$

and $I \nsubseteq \mathfrak{p}$ if there is an $f \in S$ such that $f \notin \mathfrak{p}$, so $D(f) \subseteq X \setminus V(S)$ in this case. We obtain $X \setminus V(S) = \bigcup_{f \in S} D(f)$ as desired. □

4.5.B Exercise. I have to write down a nice answer[1]. □

[1] **??**: $\bigcup_{i \in J} D(f_i) = A$ if and only if $(f_i) = A$

4.5.C Exercise. If $\bigcup_{j\in J} D(f_j) = \operatorname{Spec} A$ we have $(f_j)_{j\in J} = A$ by **??**, which means there is a finite subset J' of the index set J such that $\sum_{j\in J'} a_j f_j = 1$, so $(f_j)_{j\in J'} = A$, such that $\bigcup_{j\in J'} D(f_j) = \operatorname{Spec} A$. $\square$

4.5.D Exercise. We have

$$\begin{aligned} D(f) \cap D(g) &= \{[\mathfrak{p}] \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \wedge g \notin \mathfrak{p}\} \\ &= \{[\mathfrak{p}] \in \operatorname{Spec} A \mid fg \notin \mathfrak{p}\} \\ &= D(fg) \end{aligned} \tag{4.11}$$

because $\mathfrak{p}$ is prime, so fg cannot be an element of it unless at least one of the factors is. $\square$

4.5.E Exercise. By taking complements we can reduce this to **??**.

Now if $f^n \in (g)$ we have $f^n = ag$ with $a \in A$, which after multiplication with $(1/f^n)$ in A_f is sent to a unit. Conversely, if g is a unit in A_f there exists an $(a/f^n) \in A_f$ such that $(ag/f^n) = 1$, or $ag = f^n$, implying $f^n \in (g)$. $\square$

4.5.F Exercise. If $f \in \mathfrak{N}$ we have $f^n = 0$ for some n, which by **????** (this is more background information than the actual reason) and **??** gives $D(f) \subseteq D(0) = \varnothing$. $\square$

4.6 Topological definitions

4.6.A Exercise. If there were a nonempty non-dense open subset, take its complement as Y and its closure as Z in the definition of irreducibility. We have a contradiction. $\square$

4.6.B Exercise. Take $X = Y \cup Z$ as in the definition. The generic point (0) (because A is integral) should be contained in at least one of these. But by definition of a vanishing set we have that *all* points of Spec A will be contained in this vanishing set, hence Y or Z equals X. $\square$

4.6.C Exercise. If $\{x\}$ is a closed subset, it corresponds to a vanishing set. But this vanishing set equals $\{x\}$ if and only if the only point it contains is the prime ideal (corresponding to) x, which happens if and only if it is a maximal ideal. If it wasn't maximal, we'd have by application of Zorn's lemma a maximal ideal containing it and maximal ideals are prime, contradiction the fact that the set is a singleton. $\square$

4.6.D Exercise. (a) By **??** the distinguished opens form a basis for the Zariski topology and by **??** we can reduce a covering using distinguished opens to a finite covering. Replace the general covering $\bigcup_{i\in I} U_i$ by the covering where every U_i is replaced by the (arbitrary) union of distinguished opens. Take the finite subcover of these distinguished opens and take the original open sets that corresponded to the distinguished opens, these are bigger, hence constitute a finite subcover themselves.

(b) The ideal $\mathfrak{m} = (x_1, x_2, \ldots)$ is a maximal ideal as $A/\mathfrak{m} = k$, a field. The complement of $V(\mathfrak{m})$ can be covered using distinguished sets $D((x_i))_{i\in\mathbb{N}}$, this covering doesn't admit a finite subcovering.

□

4.6.E EXERCISE. (a) Given a cover of X, consider the induced coverings of the subspaces, take finite subcovers there and take the finite union of these covers.

(b) Given a cover of a closed subset of a quasicompact space, we can add the complement to every open set used in the covering, which gives us by definition of the subspace topology (all open sets in the subspace arise by intersecting with the subspace) a covering of the big space using open sets. Apply the quasicompactness condition here to obtain a finite covering and remove the complement of the closed subset, resulting in a finite subcover .

□

4.6.F EXERCISE. By definition $V(\mathfrak{p}) = \{[\mathfrak{q}] \in \operatorname{Spec} A \mid \mathfrak{p} \subseteq \mathfrak{q}\}$, while the closure of $\{[\mathfrak{p}]\}$ contains all points in $\operatorname{Spec} A$ that are contained in the point $[\mathfrak{p}]$, we see that these definitions are equivalent. □

4.6.G EXERCISE. This is obvious from **??** and the definition: the point x is represented by $[\mathfrak{p}] = [(y - x^2)]$ which is a prime ideal and we obtain

$$\overline{\{x\}} = \overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p}) = V(y - x^2) = K. \tag{4.12}$$

□

4.6.H EXERCISE. (a) This is too tricky for the moment, I'll come back to this later.

(b) This is too tricky for the moment, I'll come back to this later.

□

??: irreducibility of the cone over the twisted cubic

??: generalization to degree n rational normal curves

4.6.I EXERCISE. Assume for the sake of contradiction that we have an infinite descending chain. Using 4.4.3 the first closed subset that is not the entire space[2] is built using a finite number of curves and closed points. The next set (ignoring equalities, of which there are only finitely many between each real step) contains one curve or one point less (that cannot lie on one of the curves!). This process can only be repeated a finite number of times, ending in $\varnothing$, a contradiction on the infiniteness assumption. □

4.6.J EXERCISE. Assume A is Noetherian and take an ideal I that is not finitely generated, *i.e.*, we have $I = (x_1, x_2, \ldots)$. Now construct the chain

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \ldots \tag{4.13}$$

and this would constitute an infinitely ascending chain, contradiction. At some point the ideal I should pop up and we have a finite set of generators.

Assume every ideal of A is finitely generated, but A is not Noetherian. Take an infinite ascending chain of ideals, the union is again an ideal. This ideal is finitely generated, so take the index for which each of the generators is contained in the ideal at that position of the chain. We have equality for all subsequent ideals, contradicting the lack of Noetherianness. □

[2] We've assumed an infinite descending chain. Just repeating the whole space isn't quite infinite.

4.6.K Exercise. Suppose $I \subset A[[x]]$ is an ideal, let $I_n \subset A$ be the set of coefficients of the term x^n in elements of I. Because I is an ideal, the sum of two elements in I_n corresponds to the sum of the inducing power series and by the pointwise definition of the sum we have that the sum of the coefficient of x^n is contained in I_n. Likewise for the pointwise scalar multiplication, as A is embedded in $A[[x]]$ as the constant power series.

By multiplication with x of an $f \in I$ we have $I_n \subseteq I_{n+1}$, which by the Noetherianness of A stabilizes.

The last part of the hint baffles me though. □

??: Noetherianness of power series ring over Noetherian ring

4.6.L Exercise. Assume there exists an infinitely descending chain of closed subsets in the topological space $\operatorname{Spec} A$. Each of these subsets corresponds to a vanishing set $V(S_i)$ such that $S_i \nsubseteq S_{i+1}$. Consider the ideal generated by choosing $x_i \in S_{i+1} \setminus S_i$, this gives a sequence of ascending ideals that must stabilize as A is Noetherian. We've obtained a contradiction.

For a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space, consider the scenario from **????** gives the canonical example. □

4.6.M Exercise. Take a covering $(U_i)_i$ of an open subset using open sets and define Z_1 to be a random open set. Now inductively define a chain by picking an element not yet contained in the chain and taking the union of the previous open sets together with an open set containing this previously uncovered element. By taking the complement we obtain an descending chain of closed subsets. This must stabilize, but if it stabilizes to a non-empty set, these elements are not considered in our process, a contradiction. □

4.6.N Exercise. A topological space is irreducible if and only if it is nonempty and it cannot be written as the union of two proper closed subsets. A closed subset is an irreducible component if and only if it is a maximal irreducible subspace. Now take a minimal prime $[p]$ of A, this corresponds to a point $[\mathfrak{p}]$ that we can take the closure of: now all primes lying over this minimal prime are considered.

This is an irreducible subset: one of the subsets used in the union should contain the (generic) point and the closure is the *smallest* subset such that it contains that point, so we've got irreducibility. This construction is maximal for the inclusion ordering: take a closed subset containing this irreducible subset, by the minimality of the prime ideal, there are other prime ideals necessary for constructing the vanishing set, when considering these it is obvious the subset is reducible. □

??: refine argument about minimal primes in bijection with irreducible components

4.6.O Exercise. This follows from **??** but I should add some explanation maybe. □

4.6.P Exercise. The geometric picture of this are the two axes of $\mathbb{A}^2_k$, the irreducible components are easily seen to be the two axes, their vanishing sets given by $V(\overline{x})$ and $V(\overline{y})$. □

4.6.Q Exercise. If X is not connected we have two disjoint open sets such that their union is X. Now take the complement, we have two proper closed subsets

??: prove irreducibility if and only if one minimal prime ideal (that by acciden are disjoint, but that's not necessary) such that their union is X, so X is not irreducible. □

4.6.R Exercise. In **??** we have discussed something that looks a lot like $\times$: two axes of the affine plane yield the hint. Set $A = \mathbb{R}[x,y]/(xy)$, we've proved that $\operatorname{Spec} A$ is reducible. Yet it is connected: the topology on $\operatorname{Spec} A$ is given by open sets that do not vanish, but a polynomial expression in the plane $\mathbb{A}^2_{\mathbb{R}}$ has only a finite number of intersections with the axes, so the topology corresponds to the cofinite topology, in which there are no disjoint open sets! □

4.6.S Exercise. I still have to do this exercise. □

??: prove connected components equal unions irreducible components in Noetherian rings

4.6.T Exercise. A prime ideal of $\prod_{i=1}^n A_i$ is given by $\mathfrak{p}_j \times \prod_{i\neq j} A_i$ where $\mathfrak{p}_j$ is a prime ideal in A_j: this is easily seen to be a prime ideal (all factors except A_j go to zero, $A_j/\mathfrak{p}_j$ becomes a domain). Likewise, these are the *only* prime ideals: if there are two nontrivial factors, we'd obtain zerodivisors $(0,1)$ and $(1,0)$ (neglecting other factors) so it can never be a domain.

Now define a map that looks like an injection:

$$f\colon \begin{array}{ccc} \coprod_{i=1}^n \operatorname{Spec} A_i & \to & \operatorname{Spec} \prod_{i=1}^n A_i \\ (\mathfrak{p}_j, j) & \mapsto & (A_1,\dots,A_{j-1},\mathfrak{p}_j,A_{j+1},\dots,A_n) \end{array} \tag{4.14}$$

with the inverse being something that looks like a projection:

$$f^{-1}\colon \begin{array}{ccc} \operatorname{Spec} \prod_{i=1}^n A_i & \to & \coprod_{i=1}^n \operatorname{Spec} A_i \\ (A_1,\dots,A_{j-1},\mathfrak{p}_j,A_{j+1},\dots,A_n) & \mapsto & (\mathfrak{p}_j, j) \end{array} \tag{4.15}$$

but this definition is rather redundant. These maps are continuous: take $V(S_1\times\ldots\times S_n)$ a closed subset in the codomain of f, and this is when my computer crashed and I'll finish then when TEX Live is installed but now I'm gonna type up my handwritten solutions to the following section. □

??: finish $\operatorname{Spec} \prod_{i=1}^n A_i = \coprod_{i=1}^n \operatorname{Spec} A_i$

4.7 The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

4.7.A Exercise. By simply filling in the definition we obtain

$$I(S) = \bigcap_{[\mathfrak{p}] S} \mathfrak{p} = (x) \cap (x-1,y) = (x^2-x, xy) \tag{4.16}$$

and we observe that this ideal describes the function vanishing on the set S. □

4.7.B Exercise. Hoping I have not been tricked, I say

$$I(S) = (xy, yz, xz). \tag{4.17}$$

If we take a point on one of the axes (*i.e.*, $(x,0,0)$, $(0,y,0)$ or $(0,0,z)$) we immediately see that these vanishing on $I(S)$. On the other hand, any point not on these axes has 2 nonzero coordinates, hence, it does not vanish. □

4.7.C Exercise. I don't have a satisfying answer. □

??: prove $V(I(S)) = \overline{S}$

4.7.D Exercise. Using **??** we have that f vanishes on $V(J)$ if and only if $f \in \sqrt{J}$, so by definition of $I(\cdot)$ we have $f \in \sqrt{J}$, as $I(V(J))$ is the set of functions vanishing on $V(J)$. It's just jotting down the definitions of the functions in words. $\square$

4.7.E Exercise. Assume S is reducible, so we can write $S = V_1 \cup V_2$ where V_i is closed for $i = 1, 2$. By Theorem 4.7.1 we have a bijection between closed subsets and radical ideals, so $I(S) = I(V_1 \cup V_2) \subsetneqq I(V_i)$ where $i = 1, 2$ by the inclusion-reversing property of $I(\cdot)$ and this inclusion is strict by the preceding theorem.

Now take $f_i \in I(V_i) \setminus I(S)$, we have $f_1 f_2 \in I(S)$ because it vanishes on both components, hence $I(S)$ is not a prime ideal.

Conversely, take an ideal I that is not prime and f, g in $A \setminus I$ such that $fg \in I$. Now $V(I)$ is a closed subset in the Zariski topology on $\operatorname{Spec} A$ and so are $V(I, f_1)$ and $V(I, f_2)$. We wish to prove that $V(I) = V(I, f_1) \cup V(I, f_2)$ where $V(I, f_i) \subsetneqq V(I)$.

Resorting to the manipulation of symbols we have

$$\begin{aligned} V(I, f_1) \cup V(I, f_2) &= \{[\mathfrak{p}] \in \operatorname{Spec} A \mid (I, f_1) \subseteq \mathfrak{p} \vee (I, f_2) \subseteq \mathfrak{p}\} \\ &= \{[\mathfrak{p}] \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p} \vee f_1 f_2 \in \mathfrak{p}\} \\ &= V(I) \end{aligned} . \tag{4.18}$$

$\square$

CHAPTER 5

The structure sheaf and the definition of schemes in general

5.1 The structure sheaf of an affine scheme

5.1.A EXERCISE. We have $V(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}$, functions not vanishing outside $V(f)$ are by **??** correspond to g such that $f^n \in (g)$. Therefore the localisation $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ consists of fractions such that the denominator is an element of the principal ideal (g) such that $f^n \in (g)$. Remark that this is obviously a multiplicative set and the isomorphism is given by the result of **??**. □

5.1.B EXERCISE. This really boils down to judiciously replacing A by A_f. We obtain:

> We check identity on the base. Suppose that $\operatorname{Spec} A_f = \bigcup_{i \in I} D(f_i)$ where i runs over some index set I. Then there is some finite subset of I, which we name $\{1, \dots, n\}$, such that $\operatorname{Spec} A_f = \bigcup_{i=1}^n D(f_i)$, *i.e.*, $(f_1, \dots, f_n) = A_f$ (quasicompactness of $\operatorname{Spec} A_f$, **??**).
>
> Suppose we are given $s \in A_f$ such that $\operatorname{res}_{\operatorname{Spec} A_f, D(f_i)} s = 0$ in A_{f_i} for all i. We wish to show that $s = 0$. The fact that $\operatorname{res}_{\operatorname{Spec} A_f, D(f_i)} s = 0$ in A_{f_i} implies that there is some m such that for each $i \in \{1, \dots, n\}$, $f_i^m s = 0$. Now $(f_1^m, \dots, f_n^m) = A_f$, for example, from
>
> $$\operatorname{Spec} A_f = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n D(f_i^m), \tag{5.1}$$
>
> so there are $r_i \in A_f$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A_f, from which
>
> $$s = \left(\sum_{i=1}^n r_i f_i^m \right) s = \sum_{i=1}^n r_i (f_i^m s) = 0. \tag{5.2}$$
>
> Thus we have checked the "base identity" axiom for $\operatorname{Spec} A_f$.

□

5.1.C Exercise. Again, replacing A with A_f, open covers of $\operatorname{Spec} A$ with open covers of the open subspace $\operatorname{Spec} A_f$ and copying the entire proof suffices. Only three occurences of A should be replaced with A_f □

5.1.D Exercise. We have to redo Theorem 5.1.2, but now for this more general construction. □

??: prove that $\tilde{M}$ is a sheaf

5.2 Visualing schemes II: nilpotents

There are no exercises in this section.

5.3 Definition of schemes

5.3.A Exercise. If $f\colon A' \to A$ is a isomorphism of rings, the induced affine schemes are obviously isomorphic too: the underlying spaces are homeomorphic (the vanishing sets are "equal" by the isomorphism) and we can put the induced isomorphism for every localization that occurs in the corresponding sheaves.

If $f\colon \operatorname{Spec} A \to \operatorname{Spec} A'$ is an isomorphism of affine schemes, the rings of global sections are isomorphic. Now this is a bijection because the rings A and A' determine everything there is to know about these ringed spaces and their isomorphisms. □

5.3.B Exercise. We have the homeomorphism between $D(f)$ and $\operatorname{Spec} A_f$ by Section 4.5. Now the sheaves are isomorphic too: the restriction $\mathcal{O}_{\operatorname{Spec} A}|_{D(f)}$ has $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$ as global ring of sections which induces the desired isomorphism. □

5.3.C Exercise. Consider $p \in U \subseteq X$. Take V a neighbourhood of p such that $(V, \mathcal{O}_X|_V)$ is an affine scheme, *i.e.*, is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. Now consider the restriction of $\mathcal{O}_{\operatorname{Spec} A}$ to $U \cap V$ which is again an affine scheme. □

??: an open subscheme of an affine scheme is again affine, but formalize this please

5.3.D Exercise. By definition of a scheme we have for $U \subseteq X$ open that $(U, \mathcal{O}_X|_U)$ is an affine scheme, which induces the Zariski topology on U. Now all the restrictions are given from that point, so we just have to take arbitrary unions of open sets regarding every open U to obtain all open sets in X. □

5.3.E Exercise. (a) Everything follows from **??**: we have the homeomorphism $\coprod_{i=1}^n \operatorname{Spec} A_i \to \operatorname{Spec} \prod_{i=1}^n A_i$ which gives us that the finite disjoint union is isomorphic to the spectrum of a finite product of rings, which is an affine scheme, just take $A = \prod_{i=1}^n A_i$ in the definition of the affine scheme $\coprod_{i=1}^n \operatorname{Spec} A_i$.

(b) By **????** we have that affine schemes are quasicompact, but the infinite disjoint union can be covered by taking the (affine) opens $\operatorname{Spec} A_i$, which are by definition disjoint but open. We've obtained an infinite cover that cannot be reduced. □

5.3.F Exercise. The stalk in $[\mathfrak{p}]$ is defined as the direct limit of the sections over all open subsets containing $[\mathfrak{p}]$. These open sets are generated by the $D(f)$ for $f \notin \mathfrak{p}$. The sections over these sets are the localizations A_f. In the direct system these will be identified accordingly, but because $\mathfrak{p}$ is prime, one will never obtain a localization at an element of $\mathfrak{p}$. Yet all elements in $A \setminus \mathfrak{p}$ *will* be inverted.

The localization at a prime ideal on the other hand is the localization at the complement $A \setminus \mathfrak{p}$. We obtain the desired result. □

5.4 Three examples

5.4.A Exercise. The set X is given by $\coprod_i X_i/\sim$ where $\sim$ identifies homeomorphic open sets by the isomorphisms of ringed spaces $f_{i,j} X_{i,j} \to X_{j,i}$ and X_i is the underlying topological space of the ringed space[1]. This immediately induces a topology on the obtained set X.

Now for the sections of the sheaf: open sets are arbitrary unions of open sets in each X_i, where identification can take place. A section is nothing more but pasting sections together, which by the isomorphisms $f_{i,j}$ and the cocycle condition is completely valid. We have defined a scheme X such that every scheme X_i is an open subscheme, the isomorphic parts are glued together in the canonical way, everything is unique up to unique isomorphism as there hasn't been any possible choice. □

5.4.B Exercise. The only point in which sections over U and V can disagree are the origins. Sections over U and V lie in the rings $k[t, 1/t]$ and $k[u, 1/u]$ which are identified by $t \mapsto u$. So for functions to agree they have to be equal under this identification and be defined on the part that is not identified. But an element of $k[t, 1/t]$ extends to $\mathbb{A}_k^1$ if and only if it is defined in the origin, so we only have polynomials to choose from! Therefore $\Gamma(X, \mathcal{O}_X) \cong k[t]$. But if the scheme were affine, we'd have $X = \operatorname{Spec} k[t]$, a contradiction as we obviously have two origins. □

5.4.C Exercise. Is there something special about the definition?

??: define affine plane with doubled origin

Two affine open sets with a non-affine open intersection are given by taking U and V and adding the respective origins. We have that these open subsets are affine, being isomorphic to $\operatorname{Spec} k[x, y]$ yet their intersection is $\mathbb{A}_k^2 \setminus \{(0,0)\}$ which is not affine by the previous remarks. □

5.4.D Exercise. Because we're considering the restrictions to the intersections of the isomorphisms, we have $x_{i/j} x_{j/k} x_{k/i} = 1$ which gives us the desired cocycle condition. I might have to phrase this a little bit more concretely though. □

5.4.E Exercise. Like in Proposition 5.4.8 where we've proved that $\mathbb{P}_k^1$ is not affine we can take two affine open sets and we have that a section on their overlap are polynomials in the x_i, adding $1/x_i$ in one open set and $1/x_j$ in the other. Now we have $f(x_1, \ldots, x_n, 1/x_i) = g(x_1, \ldots, x_n, 1/x_j)$ (so x_j isn't inverted in f

[1] Just to be completey unambiguous.

and x_i not in g), so we only have the constant functions as sections over two affine open subsets. This extends to the global sections. $\square$

5.4.F Exercise. A point $[a_0; \ldots; a_n]$ in $\mathbb{P}^n_k$ such that not all the a_i are zero is contained in every affine open subset U_j where $a_j \neq 0$, by considering it to be the point $(a_0/a_j, \ldots, a_{j-1}/a_j, a_{j+1}/a_j, \ldots, a_n/a_j)$ in $\mathbb{A}^n_k$ which is compatible under the isomorphisms. Now the $\lambda \in k^\times$ all vanish under these fractions, we never divide by zero, so these points are identified. $\square$

Chapter 6

Some properties of schemes