Advancement in Mathematical Analysis

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Contents

1 Limit and Continuity

1.1 Intermediate Value Theorem

Theorem. For function $f : [a, b] \subset D \to \mathbb{R}$, if f is continuous on $[a, b]$, then for any $y_0 \in [f(a), f(b)]$, there exists an x_0 such that $y = f(x_0)$.

Proof. Define the set
$$S = \{x \in [a, b] \,\big|\, f(v) \leq y \forall v \in [a, x]\},$$
then it is not empty because $a \in S$, and it is bounded above because b is an upper bound, then by the least upper bound property of real number, the set S has a least upper bound, denoted as c. It is claimed that $f(c) = y$.

Suppose $f(c) > y$, let $\epsilon = f(c) - y > 0$. Then by the continuity of f, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$, which suggests that $f(x) > y$. But these means that if $|x - c| < \delta$, $x \notin S$, then there would be other upper bound of S that is less than c, say $c - \frac{\delta}{2}$. This contradicts the fact that c is the **least** upper bound of S.

Suppose $f(c) < y$, let $\epsilon = y - f(c) > 0$. Then by the continuity of f, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$, which suggests that $f(x) < y$. Then there exists an $x \in S$ that is greater than c, say $c + \frac{\delta}{2}$. This contradicts the fact that c is an upper bound of S. □

1.2 Extreme Value Theorem

Lemma (Boundedness Lemma). For a function $f : [m, n] \to \mathbb{R}$, if f is continuous, then it is bounded.

Proof. Suppose f is unbounded, then it is possible to define a sequence $\{a_n\}$ such that $f(a_n) > n$. Since $\{a_n\} \in [m, n]$ which is bounded, by **Bolzano-Weierstraß Theorem**, there exists a convergent subsequence $\{a_{n_k}\}$, denoting its limit a, which is a limit point of $[m, n]$. $[m, n]$ being closed, $a \in [m, n]$. By the continuity of f, $\lim_{k\to\infty} f(a_{n_k}) = f(a)$, but $f(a_{n_k}) > n_k > k$ for $k \in \mathbb{N}$ thus $\lim_{k\to\infty} f(a_{n_k}) = \infty$, which contradicts the fact that it converges, as desired. □

Lemma (Subsequence Limit Lemma). For sequence $\{a_n\}$ and any one of its subsequence a_{n_k}, if $\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{k\to\infty} a_{n_k} = L$.

Proof. For any given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $n > N$, $|a_n - L| < \epsilon$. For $k > N$, it is obvious that $n_k > k > N$, thus $|a_{n_k} - L| < \epsilon$, as desired. □

Lemma. If a function f is continuous at a, then for any sequence $\{a_n\}$ that converges to a, $\lim\limits_{n\to\infty} f(a_n) = f(a)$.

Proof. Trivial. □

Theorem (Extreme Value Theorem). For a function $f : [m, n] \to \mathbb{R}$ that is continuous over the domain, f attains its maximum and minimum in $[m, n]$.

Proof. By the boundedness lemma, function f is bounded from above, thus by the least upper bound property of real number, there exists $M = \sup f(x)$. Since M is the least upper bound, it is possible to define a sequence $\{a_n\}$ such that $M - \frac{1}{n} < a_n$ for $n \in \mathbb{N}$. By definition, $M - \dfrac{1}{n} < f(a_n) \leq M$, apply the squeeze theorem, $M = \lim\limits_{n\to\infty} a_n$.

By **Bolzano-Weierstraß Theorem**, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges, say to a. The interval $[m, n]$ is closed, thus the limit point $a \in [m, n]$. Since $\{f(a_{n_k})\}$ is a subsequence of $\{f(a_n)\}$, and $\{f(a_n)\}$ converges to M, by the subsequence limit lemma, $M = \lim\limits_{k\to\infty} f(a_{n_k})$. And by the continuity of f and the previous lemma[1][2], $M = \lim_{k\to\infty} f(a_{n_k}) = f(a)$, which is the point at which f attains its maximum.

The case of minimum is similar. □

2 Sequence and Series

2.1 Cauchy Sequence

Theorem (Cauchy Convergence Criterion). A sequence $\{a_n\}$ is convergent if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $p, q > N$,

$$|a_p - a_q| < \epsilon.$$

Proof. The ($\implies$) direction is trivial, we only prove the ($\impliedby$) direction. The sequence $\{a_n\}$ is clearly bounded, thus by the **Bolzano-Weierstraß Theorem**, there exists a subsequence $\{a_{n_k}\}$ convergent to, say L. Choose N such that for all $p, q > N$, $|a_p - a_q| < \frac{\epsilon}{2}$, and choose K such that for $k > K$, $|a_{n_k} - L| < \frac{\epsilon}{2}$. For any $n > N$, pick k such that $k > K$ and $n_k > N$, then

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. □

2.2 Series Convergence Tests

3 Differentiation

3.1 Leibniz's Rules of Differentiation

Theorem (Leibniz's Product Rule). For two functions f and g differentiable at a,

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof.

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \\
&= \lim_{x\to a} \frac{f(x)g(x) - f(x)g(a) - f(a)g(a) + f(x)g(a)}{x - a} \\
&= \lim_{x\to a} \left(f(x)\frac{g(x) - g(a)}{x - a} - g(a)\frac{f(a) - f(x)}{x - a} \right) \\
&= f(a)g'(a) - g(a)(-f'(a)) = f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

□

Theorem (Leibniz's Quotient Rule). For two functions f and g differentiable at a, if $g(a) \neq 0$,

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

[1] f conti. at $a \iff \forall\{a_n\} \in \mathrm{Dom}(f).\ \lim_{n\to\infty} a_n = a \implies \lim_{n\to\infty} f(a_n) = f(a)$

[2] f lim exists at $a \iff \forall\{a_n\} \in \mathrm{Dom}(f).\ \lim_{n\to\infty} a_n = a \implies \lim_{n\to\infty} f(a_n) = \lim_{x\to a} f(x)$

Proof.

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{x\to a} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(a) - g(x)f(a)}{(x-a)g(x)g(a)} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) - g(x)f(a) + f(a)g(a)}{(x-a)g(x)g(a)} \\
&= \lim_{x\to a} \frac{1}{g(x)g(a)} \left(g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a} \right) \\
&= \frac{1}{g^2(a)}(g(a)f'(a) - f(a)g'(a)) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.
\end{aligned}$$

□

3.2 Derivative of Inverse Function

Theorem. For a bijection f differentiable at a, the derivative of its inverse function $g'(b) = \dfrac{1}{f'(g(b))}$ if $g(b) = a$.

Proof. By the definition of derivative and the property of inverse function,

$$g'(b) = \lim_{y\to b} \frac{g(y)-g(b)}{y-b} = \lim_{y\to f(a)} \frac{g(y)-g(f(a))}{y-f(a)} = \lim_{f(x)\to f(a)} \frac{g(f(x))-g(f(a))}{f(x)-f(a)}.$$

Since g is the inverse function of a continuous function f, it is continuous, i.e. $f(x) \to f(a)$ as $x \to a$, we substitute the variable of the limit to x,

$$g'(b) = \lim_{x\to a} \frac{x-a}{f(x)-f(a)} = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

□

3.3 Derivative of Composite Function

Lemma (Linear Decomposition Lemma). A function f is differentiable at a point a if and only if it can be decomposed near a to

$$f(x) = f(a) + A(x-a) + \eta(x-a)(x-a),$$

where A is a linear map (in this case, a real number) and $\lim\limits_{h\to 0} \eta(h) = 0 = \eta(0)$.

Proof. Trivial. □

Theorem (The Chain Rule). If $f : I \to J$ is differentiable at a, $g : J \to \mathbb{R}$ differentiable at $f(a)$, then $g \circ f$ is differentiable at a, and the derivative $(g \circ f)'(a) = g'(f(a))f'(a)$.

Proof. By the **Linear Decomposition Lemma**, f being differentiable at a implies that f can be decomposed by linear principal part near a as

$$f(x) = f(a) + f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a)$$

where $\lim\limits_{x\to a} \eta(x-a) = 0 = \eta(0)$. Similarly, g can be decomposed near $f(a)$ as

$$g(y) = g(f(a)) + g'(f(a)) \cdot (y - f(a)) + \xi(y - f(a)) \cdot (y - b)$$

where $\lim\limits_{y \to f(a)} \xi(y - f(a)) = 0 = \xi(0)$. Now we attempt to decompose the composition $g \circ f$ near x,

$$\begin{aligned} g(f(x)) =& g(f(a)) + g'(f(a)) \cdot (f(a) + f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a) - f(a)) + \\ & \xi(f(a) + f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a) - f(a)) \cdot \\ & (f(a) + f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a) - f(a)). \end{aligned}$$

Simplify, we get

$$\begin{aligned} g(f(x)) =& g(f(a)) + g'(f(x))f'(a) \cdot (x-a) + \big(g'(f(x))\eta(x-a) \cdot (x-a) + \\ & \xi(f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a)) \cdot (f'(a) \cdot (x-a) + \eta(x-a) \cdot (x-a))\big). \end{aligned} \tag{1}$$

Define

$$\chi(h) = g'(f(x))\eta(h) + \xi(f'(a) \cdot (h) + \eta(h) \cdot (h)) \cdot (f'(a) + \eta(h)),$$

then since $\eta(h), \xi(h) \to 0$ when $h \to 0$ and $\eta(0) = \xi(0) = 0$, when $h \to 0$, $\chi \to 0$ and $\chi(0) = 0$, which satisfies the condition for error term in the decomposition, thus (1) can be rewritten as

$$g(f(x)) = g(f(a)) + g'(f(x))f'(a) \cdot (x-a) + \chi(x-a) \cdot (x-a),$$

thus by the **Linear Decomposition Lemma**, $g \circ f$ is differentiable at a and the derivative is $g'(f(a))f'(a)$. □

3.4 Mean Value Theorems

3.4.1 Rolle's Theorem

Theorem (Rolle's Theorem). For a function f that is continuous on $[m, n]$, differentiable on (m, n), if $f(m) = f(n)$, then there exists a $c \in (m, n)$ such that $f'(c) = 0$.

Proof. By the **Extreme Value Theorem**, there exists a maximum and a minimum in $[m, n]$. If they are both on the border of the interval, namely m and n, then the function is a constant function, the derivative of which is constantly 0. If not, say the maximum x_0 is in (m, n) (if it's the minimum or both, the case is similar), then it is true that for all $x \in [m, n]$, $f(x) < f(x_0)$. We claim that $c = x_0$. The right derivative of f at point c

$$f'_+(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

is obvious smaller than 0. Similarly, $f'_-(c) \geq 0$. Since f is differentiable at c, $f'_+(c) = f'_-(c)$, then $f'(c) = 0$, as desired. □

3.4.2 Lagrange's Mean Value Theorem

Theorem (Lagrange's Mean Value Theorem). For a function f that is continuous on $[m, n]$, differentiable on (m, n), there exists a $c \in (m, n)$ such that $f'(c) = \dfrac{f(n) - f(m)}{n - m}$.

Proof. Construct function

$$F(x) = f(x) - \left(f(m) + \frac{f(n) - f(m)}{n - m}(x - m)\right).$$

It is true that $F(m) = F(n) = 0$, which brings f to satisfy the premises of the **Rolle's Theorem**. Applying it, the conclusion is that there exists a $c \in (m, n)$ such that $F'(c) = 0$. Then

$$F'(c) = f'(c) - \frac{f(n) - f(m)}{n - m} = 0,$$

as desired. □

3.4.3 Cauchy's Mean Value Theorem

Theorem (Cauchy's Mean Value Theorem). For two functions f and g that is continuous on $[m, n]$, differentiable on (m, n), there exists a $c \in (m, n)$ such that

$$f'(c)(g(n) - g(m)) = g'(c)(f(n) - f(m)).$$

Proof. Construct function

$$F(x) = f(x)(g(n) - g(m)) - g(x)(f(n) - f(m)).$$

F satisfies the premises of the **Rolle's Theorem**: $F(m) = F(n) = f(m)g(n) - g(m)f(n)$. Applying it, it concludes that there exists a $c \in (m, n)$ such that $F'(c) = 0$, which suggests

$$F'(c) = f'(x)(g(n) - g(m)) - g'(x)(f(n) - f(m)) = 0,$$

as desired. □

3.5 L'Hôpital's Rule

Theorem (L'Hôpital's Rule). For functions f and g that is differentiable on (a, b) with $g(x) \neq 0$ for all $x \in (a, b)$, if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, and $f(x), g(x) \to 0$ as $x \to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = A$.

Proof. Since $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$, for any $\epsilon > 0$, there exists $c = \min\{\delta, b\} \in (a, b)$ such that

$$A - \epsilon < \frac{f'(x)}{g'(x)} < A + \epsilon$$

when $a < x < c$. For $a < x < y < c$, by the **Cauchy's Mean Value Theorem**, there exists a point $t \in (x, y)$ such that

$$A - \epsilon < \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < A + \epsilon.$$

Since $f(x), g(x) \to 0$ as $x \to a$, take the limit $x \to a$ and $y \to a$ and apply the property that limit preserves non-strict order relation, we have

$$A - \epsilon \leq \lim_{y\to a} \frac{f(y)}{g(y)} \leq A + \epsilon.$$

Since ϵ is arbitrary, the theorem is proven. □

4 Riemann-Darboux Integral

4.1 Fundamental Theorem of Calculus

Lemma. For a Riemann-Darboux integrable function $f : [a, b] \to \mathbb{R}$ and partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$ for some ϵ, if t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\left| \sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1}) - \int_a^b f(x) \,\mathrm{d}\, x \right| < \epsilon.$$

Proof. Since t_i are arbitrary points in $[x_{i-1}, x_i]$,

$$L(P, f) = \sum_{i=1}^{n} \inf_{x_{i-1}\leq x\leq x_i} f(x)\cdot(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(t_i)\cdot(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \sup_{x_{i-1}\leq x\leq x_i} f(x)\cdot(x_i - x_{i-1}) = U(P, f).$$

And since f is Riemann-Darboux integrable,

$$L(P, f) \leq \int_a^b f(x) \,\mathrm{d}\, x \leq U(P, f).$$

Finally, since $U(P, f) - L(P, f) < \epsilon$, the result is proven. □

Theorem (Fundamental Theorem of Calculus). If f is Riemann-Darboux integrable and if there exists a function F differentiable on $[a, b]$ such that $F'(x) = f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}\,x = F(b) - F(a).$$

Proof. For given $\epsilon > 0$, it is possible to choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$, because f is Riemann-Darboux integrable. By the **Lagrange's Mean Value Theorem**, there exists $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i) \cdot (x_i - x_{i-1}).$$

Sum these up, we have

$$F(b) - (a) = \sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1}),$$

which satisfies the premises of previous lemma. Applying it, we have

$$\left|F(b) - (a) - \int_a^b f(x)\,\mathrm{d}\,x\right| = \left|\sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1}) - \int_a^b f(x)\,\mathrm{d}\,x\right| < \epsilon,$$

as desired. □

5 Exponential and Logarithmic Function

5.1 Equivalency of Usual Definition of $\exp x$ and that by Series

Lemma. For any function f, f is continuous at $a \iff$ for all sequence $\{a_n\} \in \mathrm{Dom}(f)$ with $\lim_{n\infty} a_n = a$, $\lim_{n\to\infty} f(a_n) = f(a)$.

From previous arguments, the following statements are assumed.

- $E(p) = e^p$ for $p \in \mathbb{Q}$;
- $E(x)$ is continuous on $\mathbb{R}$;
- $E(x)$ is strictly monotonically increasing on $\mathbb{R}$; and
- e^p is strictly monotonically increasing on $\mathbb{Q}$.

Theorem.

$$E(x) = e^x$$

where

$$e^x = \sup_{\substack{p\in\mathbb{Q}\\ p<x}} e^p.$$

Proof. Since $e^x = \sup_{\substack{p\in\mathbb{Q}\\ p<x}} e^p$, there exists a sequence $\{p_n\} \in \mathbb{Q}$ such that

1. $\lim_{n\to\infty} e^{p_n} = \sup_{\substack{p\in\mathbb{Q}\\ p<x}} e^{p_n}$;
2. $e^{p_n} < e^{p_{n+1}} \implies p_n < p_{n+1}$ (by monotonicity of e^p for $p \in \mathbb{Q}$); and
3. $p_n < x$.

This existence holds since there exists a rational number between any two real numbers. We claim that $\lim_{n\to\infty} p_n = x$, and this can be proven by contradiction. Suppose $L = \lim_{n\to\infty} p_n \neq x$, then either

- $L > x$, in which case there exists a $p_n > x$, which contradicts the hypotheses made on $\{p_n\}$, or
- $L < x$, in which case we can pick $k, r \in \mathbb{Q}$ such that $L < k < r < x$. Since e^p is monotonic, $e^{p_n} < e^k$ for all n; take limit on both sides, we have

$$\lim_{n\to\infty} p_n \le e^k < e^r \le \sup_{\substack{p\in\mathbb{Q}\\ p<x}} e^p \implies \lim_{n\to\infty} p_n < \sup_{\substack{p\in\mathbb{Q}\\ p<x}} e^p,$$

  which contradicts the hypotheses made on $\{p_n\}$.

By the statements assumed, $e^{p_n} = E(P_n)$. Take limit on both sides, $\lim_{n\to\infty} e^{p_n} = \lim_{n\to\infty} E(p_n)$. By continuity of $E(x)$, swap E and the limit sign, $\lim_{n\to\infty} E(p_n) = E(\lim_{n\to\infty} p_n) = E(x)$. $\square$