Definition S2 (Orientation Bundle of a Vector Bundle). Assume we have a vector bundle $\eta : E \xrightarrow{p} B$ and a choice of trivialization $\{(U_a, \phi_a)\}_{a\in A}$. We define the **orientation bundle** (a fiber bundle, not a vector bundle) of η, $\hat{\eta} = \langle \hat{E}, B, q, \pm 1\rangle$ as follows. First, the underlying set of the total space is defined to be

$$\hat{E} = \bigsqcup_{b\in B} \left({}^{F_{\mathrm{GL}}(E_b)}\Big/_{\mathrm{GL}^+} \right);$$

and the base space $\hat{B} = B$. Given an open set $B' \subseteq U_a$ for some a, define

$$\mu_{B'} = \left(\bigsqcup_{b'\in B'} F_{\mathrm{GL}}(E_{b'}) \right) \Big/ \sim$$

where $e \sim e'$ if and only if there exists $g^+ \in \mathrm{GL}^+$ such that $(\varphi_a(e))_2 \cdot g^+ = (\varphi_a(e'))_2$; for $b' \in B'$, define a pseudo-inclusion map

$$\begin{aligned} \psi_{b'}^{B'} : \mu_B &\to (b', -) \in \hat{\eta} \\ \nu &\mapsto [(\beta, \cdots) \in \mu_{B'} \mid \beta = b'] \end{aligned}$$

and define $U(\mu_{B'})$ to be the set of all $\mu_{b'} \in \hat{\eta}$ such that $b' \in B'$ and $\mu_{b'} = \psi_{b'}^{B'}(\mu_{B'})$. Topologizing $\hat{E}$ with the basis of the topology being the sets $U(\mu_{B'})$, indexed over all possible B', a projection $q : \hat{\eta} \to B$ is just a projection into the first factor.

Lemma S3. The orientation bundle of any vector bundle is a two-sheeted covering space thereof.

Proof. Trivial since

$$\left| {}^{F_{\mathrm{GL}}(E_b)}\Big/_{\mathrm{GL}^+} \right| = |\mathrm{GL} : \mathrm{GL}^+| = 2.$$

□

Lemma S4 (Orientability of Orientation Bundle). The orientation bundle of any vector bundle with connected and compact base space is orientable.

Proof. Choose a finite open cover U_i and its corresponding family of trivialization maps $\varphi_i : p^{-1}(U_i) \to B \times F^n$. Consider the intersection graph of U_i, which is clearly connected since the base space itself is connected, thus if we have a procedure to glue $\mu_{B'_k}$ and $\mu_{B'_l}$ together, we will have a method to create two global sections of the orientation bundle, which results in an assignment of the ± 1 thereto that is continuous, as required by the orientability condition for fiber bundle, i.e. fiberwise orientation-preserving trivialization maps. And indeed we have: On the intersection of two sets U_i, U_j in the open cover, two equivalence classes $[e] \in \mu_{U_i}, [e'] \in \mu_{U_j}$ are equivalent and to be merged if and only if $(\phi_i(e))_2 \cdot g^+ = (\phi_j(e'))_2$ for some $g^+ \in \mathrm{GL}^+$, and we are done. □

Proposition S5 (Criterion for Orientability of Vector Bundle). Let $\eta : E \xrightarrow{p} B$ a vector bundle with B connected, then η is orientable if and only if the orientation bundle $\hat{\eta}$ has two connected components.

Proof. If B is connected, $\hat{\eta}$ has either one or two component(s) since it's a two-sheeted covering space of B, If it has two, then they are each mapped homeomorphically to B by the covering projection defined above, splitting the fibers into 2 classes: voilà, une section d'orientation par l'axiome du choix! Conversely, if η is orientable, it has two orientations since it is connected, and each of these orientations corresponds to one of the global section of the orientation bundle, de facto et de jure! □

Problem 2. Disprove that $\xi \simeq \eta$.

Proof. Referring to Proposition S5, the former is orientable; the latter's orientation bundle is path-connected, thus it's unorientable. Since orientability of vector bundle is a vector bundle isomorphism invariant, they are not isomorphic. □