CS 70 Discrete Mathematics and Probability Theory
Fall 2025 Sabin, Hug

HW 01

1 Solving a System of Equations Review

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

Solution:

Letting a, b, and c be the dollar cost of an apple, beet, and carrot, respectively, we get the system of equations

$$\begin{aligned} a+b+c &= 16 \\ 2a+3b &= 23 \\ a+2b+3c &= 35. \end{aligned}$$

There are many approaches to solving this system (Gaussian Elimination, substitution, etc.). Here we show a solution via substitution.

Subtracting the third equation from three times the first equation gives

$$2a+b = 3(a+b+c)-(a+2b+3c) = 3\cdot 16-35 = 13.$$

Subtracting this equation from the second equation gives

$$2b = (2a+3b)-(2a+b) = 23-13 = 10,$$

so $b=5$. Backsolving gives $a=4$ and $c=7$.

2 Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t.$$

(b) Compute the double integral

$$\iint_R 2x + y \, \mathrm{d}A,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

Solution:

(a) Let $I = \int \sin(t)e^{-t} \, \mathrm{d}t$.
Use integration by parts, with $u = \sin(t)$ and $\mathrm{d}v = e^{-t}$.
This means $\mathrm{d}u = \cos(t)$ and $v = -e^{-t}$.

$$\begin{aligned} I = \int \sin(t)e^{-t} \, \mathrm{d}t &= uv - \int v \cdot \mathrm{d}u \\ &= -\sin(t)e^{-t} + \int e^{-t} \cos(t) \, \mathrm{d}t \end{aligned}$$

Use integration by parts again on $\int e^{-t} \cos(t) \, \mathrm{d}t$, with $u = \cos(t)$ and $\mathrm{d}v = e^{-t}$. This means $\mathrm{d}u = -\sin(t)$ and $\mathrm{d}v = -e^{-t}$.

$$\begin{aligned} \int e^{-t} \cos(t) \, \mathrm{d}t &= uv - \int v \cdot \mathrm{d}u \\ &= -\cos(t)e^{-t} - \int e^{-t} \cdot \sin(t) \, \mathrm{d}t \\ &= -\cos(t)e^{-t} - I \end{aligned}$$

Combining these results:

$$\begin{aligned} I &= -\sin(t)e^{-t} - \cos(t)e^{-t} - I \\ \Rightarrow 2I &= -\sin(t)e^{-t} - \cos(t)e^{-t} \\ \Rightarrow I &= \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2} \end{aligned}$$

Finally, we have:

$$I \Big|_0^\infty = \frac{0-0}{2} - \frac{0-1}{2} = \frac{1}{2}.$$

(b) We may set up the integral over the region R as follows:

$$\int_0^1 \int_0^x 2x + y \, \mathrm{d}y \, \mathrm{d}x.$$

Evaluating this integral gives

$$\int_0^1 \int_0^x 2x + y \,\mathrm{d}y \,\mathrm{d}x = \int_0^1 2xy + \frac{y^2}{2}\Big|_0^x \mathrm{d}x$$
$$= \int_0^1 \frac{5x^2}{2} \,\mathrm{d}x$$
$$= \frac{5x^3}{6}\Big|_0^1$$
$$= \frac{5}{6}.$$

3 Logical Equivalence?

Note 1 Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \vee \forall x\, Q(x)$

(c) $\exists x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \vee \exists x\, Q(x)$

(d) $\exists x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \wedge \exists x\, Q(x)$

Solution:

(a) **Correct**.

Assume that the left hand side is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) **Incorrect**.

Note that there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that x can take on) is $\{1,2\}$ and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false, $P(2)$ to be false and $Q(2)$ to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

Another more concrete example is if $P(x) = x < 0$ and $Q(x) = x \geq 0$, where the universe is the real numbers. For any $x \in \mathbb{R}$, exactly one of $P(x)$ or $Q(x)$ is true, but it is not the case that $P(x)$ holds for every x, and it is also not the case that $Q(x)$ holds for every x. Since the LHS and RHS have different values, the two sides are not equivalent.

(c) **Correct**

Assuming that the left hand side is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the right hand side is false.

(d) **Incorrect**.

Note, there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that x can take on) is the natural numbers $\mathbb{N}$, and that P and Q are truth functions defined on this universe. Here, suppose we set $P(1)$ to be true and $P(x)$ to be false for all other x, and $Q(2)$ to be true and $Q(x)$ to be false for all other x. (In other words, $P(x) = (x = 1)$ and $Q(x) = (x = 2)$.)

With these definitions, the right hand side would be true, since there exists some value of x that makes $P(x)$ true (namely, $x = 1$), and there exists some value of x that makes $Q(x)$ true (namely, $x = 2$). However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

4 If, Then

Note 2

Consider the conjecture: The sum of an even integer and an odd integer is odd.

(a) Write the conjecture as an if/then statement.

(b) An if/then proof always starts with an assumption. What can you assume here to begin the proof?

(c) Based on your assumption, what do you Want To Show that would complete the proof?

(d) Complete the proof of the conjecture.

Solution:

(a) For two integers e, o, if e is even and o is odd, then their sum $e + o$ is odd.

(b) Assume e is an even integer and o is an odd integer. Then $e = 2k$ for some integer k and $o = 2m + 1$ for some integer m.

(c) We want to show that the sum $e + o$ is odd; that is, $e + o$ can be written in the form $2n + 1$ for some integer n.

(d) $e + o = 2k + (2m + 1) = 2(k + m) + 1$, which is of the form $2n + 1$ for $n = k + m$. Therefore, $e + o$ is odd.

5 Prove

Note 2

Prove each of the following statements

(a) $\forall a, b, c \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

(b) $\forall n \in \mathbb{N}$, n is odd if and only if $3n+5$ is even

(c) $\forall n \in \mathbb{Z}$, n^2+n+6 is even.

Solution:

(a) Assume $a|b$ and $b|c$. Then there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 b$. Substituting for b gives $c = k_2(k_1 a) = (k_2 k_1)a$, so $a|c$.

(b) Assume n is odd. Then $n = 2k+1$ for some integer k. Thus, $3n+5 = 3(2k+1)+5 = 6k+3+5 = 6k+8 = 2(3k+4)$, which is even. Conversely, if $3n+5$ is even, then $3n+5 = 2m$ for some integer m, so $3n = 2m-5$. Since $2m$ is even, $2m-5$ is odd, and thus n must be odd.

(c) Assume $n \in \mathbb{Z}$. Then $n^2+n = n(n+1)$ is the product of two consecutive integers, which is always even. Thus, n^2+n+6 is even.

6 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then n^2+4n is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a+b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

(e) The product of a non-zero rational number and an irrational number is irrational.

Solution:

(a) **Answer**: True.

Proof. We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. This means that we have

$$\begin{aligned} n^2+4n &= (2k+1)^2+4(2k+1) \\ &= 4k^2+12k+5 \\ &= 2(2k^2+6k+2)+1 \end{aligned}$$

Since $2k^2+6k+2$ is a natural number, by the definition of odd numbers, n^2+4n is odd.

Alternatively, we could also factor the expression to get $n(n+4)$. Since n is odd, $n+4$ is also odd. The product of 2 odd numbers is also an odd number. Hence n^2+4n is odd. $\square$

(b) **Answer**: True.

Proof. We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg(a \leq 11 \vee b \leq 4)$). Since $a > 11$ and $b > 4$, $a+b > 15$ (note that $a+b > 15$ is equivalent to $\neg\big(a+b \leq 15\big)$). Thus, if $a+b \leq 15$, then $a \leq 11$ or $b \leq 4$. □

(c) **Answer**: True.

Proof. We will use a proof by contraposition. Assume that r is rational. Since r is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$. Then r^2 can be written as $\frac{a^2}{b^2}$. By the definition of rational numbers, r^2 is a rational number, since both a^2 and b^2 are integers, with $b \neq 0$. By contraposition, if r^2 is irrational, then r is irrational. □

(d) **Answer**: False.

Proof. We will show a counterexample. Let $n = 7$. Here, $5 \cdot 7^3 = 1715$, but $7! = 5040$. Since $5n^3 < n!$, the claim is false.

A counterexample that is easier to see without much calculation is for a much larger number like $n = 100$; here, $100!$ is clearly more than $5 \cdot 100^3 = 100 \cdot 50 \cdot 25 \cdot 5 \cdot 4 \cdot 2$, since the latter product contains only a subset of the terms in $100!$. □

(e) **Answer**: True.

Proof. We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where p, q, r, and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational. □

7 The Tell of the Digits

Note 2

Prove the following.

For all four-digit natural numbers, if the alternating sum (alternating subtracting and adding) of its digits is divisible by 11, then the number itself is divisible by 11.

For example, for the number 5291, the alternating sum of its digits is $5-2+9-1 = 11$ is divisible by 11 and, indeed, 5291 is itself divisible by 11. Yet 9718 has an alternating sum of $9-7+1-8 = -5$ is not divisible by 11 and so there is no guarantee on whether 9718 is divisible by 11 (and, in fact, it does not evenly divide it).

Solution: Let N be the four-digit number with decimal digits a, b, c, d, so

$$N = 1000a + 100b + 10c + d,$$

Write the powers of 10 as multiples of 11 plus a small remainder:

$$1000 = 1001 - 1 = 11 \cdot 91 - 1, \qquad 100 = 99 + 1 = 11 \cdot 9 + 1,$$

$$10 = 11 - 1 = 11 \cdot 1 - 1, \qquad 1 = 11 \cdot 0 + 1.$$

Substituting these into the expression for N gives

$$\begin{aligned} N &= (11 \cdot 91 - 1)a + (11 \cdot 9 + 1)b + (11 \cdot 1 - 1)c + (11 \cdot 0 + 1)d \\ &= 11(91a + 9b + c) + (-a + b - c + d). \end{aligned}$$

Thus N is equal to an integer multiple of 11 plus the quantity $-a+b-c+d$.

If the quantity $-a+b-c+d$ is divisible by 11, then the original alternating sum $a-b+c-d$is also divisible by 11, so the whole number N is a multiple of 11.

8 (Optional) Study Groups

This semester, we're facilitating the formation of study groups for students to collaborate on the homework and study together otherwise.

We especially want to encourage study groups to come to office hours together and receive support from our course staff. **Maybe: Study groups will given an opportunity to book a regular time at Wednesday and Thursday office hours to work on the HW together with staff support.**

If you're interested in being assigned a study group by us, please fill out this form: `https://docs.google.com/forms/d/e/1FAIpQLSdEblrc1r397szWCugMdwe7QRFYBgC2ru2aACmFHlel10Ey6Q/viewform?usp=header` by this homework's due date.

Solution: Filled out the form!