Quantum Mechanics Fall 2019

Yuxuan Zhang, XJTU, 2160909016

Important equations and constants

Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V\Psi$$

$$\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

Hamiltonian operator

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

Week 1

Get into the quntum world

Notes on Lesson 1

Monday Sep 2

Notes on Lesson 2

Tuesday Sep 3

Bohr-Sommerfeld Quantization (Old quantum theory)

$$\oint p\,dq = nh$$

While p is the momentum of a particle, which is a function of position of particle q. This equation constrains the movement of a particle in a specific potential field V(q).

Notes on Lesson 3

Friday Sep 6

Born Statistical Interpretation of Wave Function

The wave function $\Psi(x,t)$ discribes the "Amplitude" of the probability finding the paticle at coordinate(x,t). While the probability density equals amplitude squared.

That is, the probability finding the particle among $x \in [a,b]$ at time t_0 is discribed as:

$$P(x \in [a,b], t = t_0) = \int_a^b |\psi|^2 dx = \int_a^b \psi^* \psi dx$$

Or, to say, the *Probability Density* can be written as:

$$p(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$$

Homework

[week1 hw1]

- **Problem 1 - [Thinking][Optional] A Brief History of Clues**

  PENDING: This part will be done later this week.

- **Problem 2 - Caculate wave length of an electron after a 1000V potential field**

  Utilizing ***de Broglie wave*** equation (ignoring relativity effect):

$$
\begin{aligned}
Energy\ obtained\ by\ this\ electron: &\quad E = q \cdot U = 1000\, eV \\
Corresponding\ momentum\ of\ this\ electron: &\quad p = \sqrt{2m \cdot E} \\
Wave\ length\ of\ this\ electorn: &\quad \lambda = \frac{h}{p} \\
&\quad = \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1000 \times 1.6 \times 10^{-19}}} \\
&\quad = 3.88 \times 10^{-9}\, m
\end{aligned}
$$

- **Problem 3 - Derivation of Klein-Gordon equation**

  PENDING: This part will be done later this week.

[week1 hw2]

- **Problem 4 - Griffiths 1.2**

(a) Find the standard deviation of the distribution in Example 1.1.

First caculate the probability of finding this object at position x, represented by $\rho(x)$:

$$total\ travel\ time\ T = \sqrt{\frac{2\,h}{g}}$$

$$dt = \frac{dx}{v} = \frac{1}{\sqrt{2gx}}\, dx$$

$$\rho(x) = \frac{dt}{T\, dx} = \frac{1}{2\sqrt{hx}}$$

Accoring to statistics:

$$\langle x\rangle^2 = \left(\int_0^h x\cdot\rho(x)\,dx\right)^2$$
$$= \left(\int_0^h \frac{x}{2\sqrt{hx}}\,dx\right)^2$$
$$= \frac{h^2}{9}$$

$$\langle x^2\rangle = \int_0^h x^2\cdot\rho(x)\,dx$$
$$= \int_0^h \frac{x^2}{2\sqrt{hx}}\,dx$$
$$= \frac{h^2}{5}$$

Hence:

$$\delta = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$
$$= \frac{2}{3\sqrt{5}}h$$

(b) What is the probability that a photograph, selected at ramdom, would show a distance x more than one standard deviation away from the average?

We have known that the average is $\langle x\rangle = \frac{h}{3}$, and the standard deviation is $\delta = \frac{2}{3\sqrt{5}}h$. We now want to know the probability of finding x among $0 < x < \langle x\rangle - \delta$ or $\langle x\rangle + \delta < x < h$:

$$P = \int_0^{\langle x\rangle-\delta} \rho(x)\,dx + \int_{\langle x\rangle+\delta}^h \rho(x)\,dx$$
$$= 1 - \int_{\langle x\rangle-\delta}^{\langle x\rangle+\delta} \rho(x)\,dx$$
$$= 1 - \frac{1}{\sqrt{h}}\left(\sqrt{x}\Big|_{\frac{5-2\sqrt{5}}{15}h}^{\frac{5+2\sqrt{5}}{15}h}\right)$$
$$= 1 - \left(\sqrt{\frac{5+2\sqrt{5}}{15}} - \sqrt{\frac{5-2\sqrt{5}}{15}}\right)$$
$$\approx 0.39$$

- **Problem 5 - Griffiths 1.4**

At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\dfrac{x}{a}, & if\ \ 0 \le x \le a, \\[2ex] A\dfrac{(b-x)}{(b-a)}, & if\ \ a \le x \le b, \\[2ex] 0, & otherwise, \end{cases}$$

where A, a and b are constants.

(a) Normalize Ψ (that is, find A in terms of a and b)

The integral of $|\Psi|^2$ in all space is:

$$\int_{-\infty}^{+\infty} \Psi^*\Psi\, dx = \int_0^a A^2\,\frac{x^2}{a^2}\,dx + \int_a^b A^2\,\frac{(b-x)^2}{(b-a)^2}\,dx$$

$$= A^2\left(\frac{a}{3}+\frac{b-a}{3}\right)$$

$$= \frac{1}{3}A^2 b$$

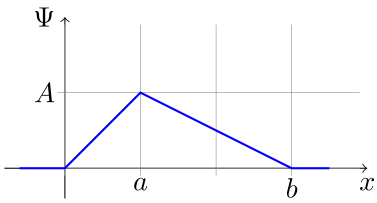
$$= 1$$

Hence:

$$A = \sqrt{\frac{3}{b}}$$

(b) Sketch $\Psi(x,0)$, as a function of x.

Let $a = 1,\ b = 3\ \Rightarrow A = 1$:



(c) Where is the particle most likely to be found, at $t = 0$?.

Obviously, at $x = a$.

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

In case of $b = a$:

$$A = \sqrt{\frac{3}{b}} = \sqrt{\frac{3}{a}}$$

$$P = \int_0^a \frac{3}{a}\left(\frac{x}{a}\right)^2 dx = 1$$

In case of $b = 2a$:

$$A = \sqrt{\frac{3}{b}} = \sqrt{\frac{3}{2a}}$$

$$P = \int_0^a \frac{3}{2a}\left(\frac{x}{a}\right)^2 dx = \frac{1}{2}$$

(e) What is the expectation value of x?

$$\langle x\rangle = \frac{3}{b}\left(\int_0^a x\cdot\frac{x^2}{a^2}\,dx + \int_a^b x\cdot\frac{(b-x)^2}{(b-a)^2}\,dx\right) = \frac{3}{b}\left(\frac{a^2}{4} + \frac{b^4-3a^4+8a^3b-6a^2b^2}{12\,(b-a)^2}\right)$$

$$= \frac{3a^2(b-a)^2+b^4-3a^4+8a^3b-6a^2b^2}{4b\,(b-a)^2} = \frac{\cancel{3a^4}+3a^2b^2-6a^3b+b^4-\cancel{3a^4}+8a^3b-6a^2b^2}{4b\,(b-a)^2}$$

$$= \frac{\cancel{b}(b^3+2a^3-3a^2b)}{4\cancel{b}\,(b-a)^2} = \frac{(b-a)^{\cancel{3}}+3a\cancel{(b-a)^2}}{4\cancel{(b-a)^2}} = \frac{2a+b}{4}$$

Notes on Lesson 1

Measurements and Operators - Monday Sep 9

Starting with the most basic measurable value: ***expectation value of position*** $<x>$

$$<x>=\int x\,\rho(x)\,dx=\int \psi^*\,x\,\psi\,dx$$

Then consider $<x>$ **change with time:**

$$\frac{d}{dt}<x>=\frac{d}{dt}\int \Psi^*\,x\,\Psi\,dx \tag{1}$$

$$=\int x\frac{\partial}{\partial t}|\Psi|^2\,dx \tag{2}$$

$$=\int x\left(\frac{\partial \Psi^*}{\partial t}\Psi+\frac{\partial \Psi}{\partial t}\Psi^*\right)dx \tag{3}$$

$$Schrdinger Eq \Rightarrow i\hbar\frac{\partial}{\partial t}\Psi=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\Psi+V\Psi \tag{4}$$

$$-i\hbar\frac{\partial}{\partial t}\Psi^*=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\Psi^*+V\Psi^* \tag{5}$$

$$Conclude\ to \Rightarrow \frac{\partial \Psi}{\partial t}\cdot\Psi^*=+\frac{i\hbar}{2m}\frac{\partial}{\partial x^2}\Psi\cdot\Psi^*+\frac{V}{i\hbar}\Psi\cdot\Psi^* \tag{6}$$

$$\frac{\partial \Psi^*}{\partial t}\cdot\Psi=-\frac{i\hbar}{2m}\frac{\partial}{\partial x^2}\Psi^*\cdot\Psi-\frac{V}{i\hbar}\Psi^*\cdot\Psi \tag{7}$$

$$=\frac{i\hbar}{2m}\int x\cdot\left(\Psi^*\frac{\partial^2}{\partial x^2}\Psi-\Psi\frac{\partial^2}{\partial x^2}\Psi^*\right)dx \tag{8}$$

$$Math\ tricks \Rightarrow (u\,\ddot{v}-v\,\ddot{u})=\frac{d}{dx}(u\,\dot{v}-v\,\dot{u}) \tag{9}$$

$$=\frac{i\hbar}{2m}\int x\cdot\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)dx \tag{10}$$

$$Using\ integral\ by\ part \Rightarrow u\cdot dv=uv-v\cdot du \tag{11}$$

$$let \Rightarrow u=x \tag{12}$$

$$v=\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^* \tag{13}$$

$$We\ get \Rightarrow \tag{14}$$

$$x\cdot d\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)=u\cdot dv=\cancel{uv}-v\cdot du=-\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)dx \tag{15}$$

$$=-\frac{i\hbar}{2m}\int\left(\Psi^*\frac{\partial}{\cancel{\partial x}}\Psi-\Psi\frac{\partial}{\cancel{\partial x}}\Psi^*\right)\cancel{dx} \tag{16}$$

$$Using\ integral\ by\ part\ again \Rightarrow u \cdot dv - v \cdot du = -uv + 2u \cdot dv \tag{17}$$

$$let \Rightarrow u = \Psi^* \tag{18}$$

$$v = \Psi \tag{19}$$

$$We\ get \Rightarrow \Psi^* \partial \Psi - \Psi \partial \Psi^* = u \cdot dv - v \cdot du = \cancel{-uv} + 2v \cdot du = 2\,\Psi^* \partial \Psi = 2\,\Psi^* \frac{\partial}{\partial x} \Psi\, dx \tag{20}$$

$$= -\frac{i\hbar}{\cancel{2}m} \int \cancel{2} \cdot \Psi^* \frac{\partial}{\partial x} \Psi\, dx \tag{21}$$

$$= \int \Psi^* \left(-\frac{i\hbar}{m} \frac{\partial}{\partial x} \right) \Psi\, dx \tag{22}$$

Now we can find a new operator: momentum $\hat{p}$

$$<p> = m \frac{d <x>}{dt} = \int \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi\, dx = \int \Psi^* \left(\frac{\hbar}{i} \cdot \frac{\partial}{\partial x} \right) \Psi\, dx$$

Additional proof of $\frac{\partial}{\partial t} \int |\Psi|^2\, dx = 0$

$$\frac{d}{dt} \int |\Psi|^2\, dx = \int \frac{\partial}{\partial t} |\Psi|^2\, dx \tag{23}$$

$$= \int \frac{\partial}{\partial t} \left(\Psi^* \Psi \right) dx \tag{24}$$

$$= \int \left(\Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \right) dx \tag{25}$$

$$Schrdinger Eq \Rightarrow \quad \frac{\partial}{\partial t} \Psi = \frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi - \frac{i}{\hbar} V \Psi \tag{26}$$

$$\frac{\partial}{\partial t} \Psi^* = -\frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi^* + \frac{i}{\hbar} V \Psi^* \tag{27}$$

$$= \int \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \Psi \frac{\partial^2 \Psi^*}{\partial x^2} \right) dx \tag{28}$$

$$= \frac{i\hbar}{2m} \int \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) dx \tag{29}$$

$$= \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) \Bigg|_{-\infty}^{+\infty} \tag{30}$$

$$Note\ that\ \Psi\ is\ normalizable \Rightarrow \quad lim_{x \to \infty} \Psi = 0 \tag{31}$$

$$\Rightarrow \quad lim_{x \to \infty} \frac{\partial \Psi}{\partial x} = 0 \tag{32}$$

$$= 0 \tag{33}$$

$$\Rightarrow \quad Proved \tag{34}$$

No Homework For Week 2

Week 3

To be edited

Notes on Lesson 1

Measurements and Operators - Monday Sep 16

Homework

[week3 hw1]

• **Problem 1 - Griffiths 2.1**

Prove the following three theorems:
(a) For normalizable solutions, the separation constant E muust be $real$.

Proof:

(b) The time-independent wave function $\psi(x)$ can always be taken to be $real$ (unlike $\Psi(x,t)$, which is necessarily complex). This doesm't mean that every solution to the time-independent Schrödinger equation is real; what it says is that if you've got one that is not, it can always be expressed as a linear combination of solutions (with the same energy) that are. So you $might\ as\ well$ stick to ψ's that are real.

Proof:

(c) If $V(x)$ is an **even function** (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd.

Proof:

• **Problem 2 - Griffiths 2.2**

Show that E must exceed the minimum value of $V(x)$, fot every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint:* Rewrite Equation 2.5 in the form

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar}\left[V(x) - E\right]\psi$$

if $E \leq V_min$, then ψ and its second derivative always have the *same sign* – argue that such a function cannot be normalized.