Notes on Lesson 1

Measurements and Operators - Monday Sep 9

- **1. Derivation of momentum operator**

Starting with the most basic measurable value: expectation value of position $<x>$

$$<x>=\int x\,\rho(x)\,dx=\int \psi^*\,x\,\psi\,dx$$

Then consider $<x>$ change with time:

$$\frac{d}{dt}<x>=\frac{d}{dt}\int \Psi^*\,x\,\Psi\,dx \tag{1}$$

$$=\int x\frac{\partial}{\partial t}|\Psi|^2\,dx \tag{2}$$

$$=\int x\left(\frac{\partial \Psi^*}{\partial t}\Psi+\frac{\partial \Psi}{\partial t}\Psi^*\right)\,dx \tag{3}$$

$$SchrdingerEq \Rightarrow i\hbar\frac{\partial}{\partial t}\Psi=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\Psi+V\Psi \tag{4}$$

$$-i\hbar\frac{\partial}{\partial t}\Psi^*=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\Psi^*+V\Psi^* \tag{5}$$

$$Conclude\ to \Rightarrow \frac{\partial \Psi}{\partial t}\cdot\Psi^*=+\frac{i\hbar}{2m}\frac{\partial}{\partial x^2}\Psi\cdot\Psi^*+\frac{V}{i\hbar}\Psi\cdot\Psi^* \tag{6}$$

$$\frac{\partial \Psi^*}{\partial t}\cdot\Psi=-\frac{i\hbar}{2m}\frac{\partial}{\partial x^2}\Psi^*\cdot\Psi-\frac{V}{i\hbar}\Psi^*\cdot\Psi \tag{7}$$

$$=\frac{i\hbar}{2m}\int x\cdot\left(\Psi^*\frac{\partial^2}{\partial x^2}\Psi-\Psi\frac{\partial^2}{\partial x^2}\Psi^*\right)\,dx \tag{8}$$

$$Math\ tricks \Rightarrow (u\,\ddot{v}-v\,\ddot{u})=\frac{d}{dx}(u\,\dot{v}-v\,\dot{u}) \tag{9}$$

$$=\frac{i\hbar}{2m}\int x\cdot\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)\,dx \tag{10}$$

$$Using\ integral\ by\ part \Rightarrow u\cdot dv=uv-v\cdot du \tag{11}$$

$$let \Rightarrow u=x \tag{12}$$

$$v=\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^* \tag{13}$$

$$We\ get \Rightarrow \tag{14}$$

$$x\cdot d\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)=u\cdot dv=\cancel{uv}-v\cdot du=-\left(\Psi^*\frac{\partial}{\partial x}\Psi-\Psi\frac{\partial}{\partial x}\Psi^*\right)\,dx \tag{15}$$

$$=-\frac{i\hbar}{2m}\int\left(\Psi^*\frac{\partial}{\cancel{\partial x}}\Psi-\Psi\frac{\partial}{\cancel{\partial x}}\Psi^*\right)\cancel{dx} \tag{16}$$

$$Using\ integral\ by\ part\ again \Rightarrow u \cdot dv - v \cdot du = -uv + 2u \cdot dv \tag{17}$$

$$let \Rightarrow u = \Psi^* \tag{18}$$

$$v = \Psi \tag{19}$$

$$We\ get \Rightarrow \Psi^* \partial \Psi - \Psi \partial \Psi^* = u \cdot dv - v \cdot du = \cancel{-uv} + 2v \cdot du = 2\,\Psi^* \partial \Psi = 2\,\Psi^* \frac{\partial}{\partial x} \Psi\, dx \tag{20}$$

$$= -\frac{i\hbar}{\cancel{2}m} \int \cancel{2} \cdot \Psi^* \frac{\partial}{\partial x} \Psi\, dx \tag{21}$$

$$= \int \Psi^* \left(-\frac{i\hbar}{m} \frac{\partial}{\partial x} \right) \Psi\, dx \tag{22}$$

Now we can find a new operator: momentum $\hat{p}$

$$< p > = m \frac{d < x >}{dt} = \int \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi\, dx = \int \Psi^* \left(\frac{\hbar}{i} \cdot \frac{\partial}{\partial x} \right) \Psi\, dx$$

Additional proof of $\frac{\partial}{\partial t} \int |\Psi|^2\, dx = 0$

$$\frac{d}{dt} \int |\Psi|^2\, dx = \int \frac{\partial}{\partial t} |\Psi|^2\, dx \tag{23}$$

$$= \int \frac{\partial}{\partial t} \left(\Psi^* \Psi \right) dx \tag{24}$$

$$= \int \left(\Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \right) dx \tag{25}$$

$$SchrdingerEq \Rightarrow \quad \frac{\partial}{\partial t} \Psi = \frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi - \frac{i}{\hbar} V \Psi \tag{26}$$

$$\frac{\partial}{\partial t} \Psi^* = -\frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi^* + \frac{i}{\hbar} V \Psi^* \tag{27}$$

$$= \int \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \Psi \frac{\partial^2 \Psi^*}{\partial x^2} \right) dx \tag{28}$$

$$= \frac{i\hbar}{2m} \int \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) dx \tag{29}$$

$$= \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) \Bigg|_{-\infty}^{+\infty} \tag{30}$$

$$Note\ that\ \Psi\ is\ normalizable \Rightarrow \ lim_{x \to \infty} \Psi = 0 \tag{31}$$

$$\Rightarrow \ lim_{x \to \infty} \frac{\partial \Psi}{\partial x} = 0 \tag{32}$$

$$= 0 \tag{33}$$

$$\Rightarrow \ Proved \tag{34}$$