

# Quantum Mechanics Fall 2019

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## Important equations and constants

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### Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V\Psi$$
$$\hbar \frac{\partial}{\partial t} \Psi = H\Psi$$

### Hamiltonian operator

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V$$

## Week1 Notes

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*Get into the quantum world*

### Notes on Lesson 1

### Notes on Lesson 2

- 1. Bohr-Sommerfeld Quantization (Old quantum theory)

$$\oint p dq = nh$$

While  $p$  is the momentum of a particle, which is a function of position of particle  $q$ . This equation constrains the movement of a particle in a specific potential field  $V(q)$ .

### Notes on Lesson 3

- 1. Born Statistical Interpretation of Wave Function

The wave function  $\Psi(x, t)$  describes the "Amplitude" of the probability finding the particle at coordinate  $(x, t)$ . While the probability density equals amplitude squared.

That is, the probability finding the particle among  $x \in [a, b]$  at time  $t_0$  is described as:

$$P(x \in [a, b], t = t_0) = \int_a^b |\psi|^2 dx = \int_a^b \psi^* \psi dx$$

Or, to say, the *Probability Density* can be written as:

$$p(x, t) = |\psi(x, t)|^2 = \psi^*(x, t)\psi(x, t)$$

## Week1 Homework

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## [week1 hw1]

### • Problem 1 - [Thinking][Optional] A Brief History of Clues

**PENDING: This part will be done later this week.**

### • Problem 2 - Caculate wave length of an electron after a 1000V potential field

Utilizing *de Broglie wave* equation (ignoring relativity effect):

$$\text{Energy obtained by this electron : } E = q \cdot U = 1000 \text{ eV}$$

$$\text{Corresponding momentum of this electron : } p = \sqrt{2m \cdot E}$$

$$\text{Wave length of this electorn : } \lambda = \frac{h}{p}$$

$$\begin{aligned} &= \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1000 \times 1.6 \times 10^{-19}}} \\ &= 3.88 \times 10^{-9} \text{ m} \end{aligned}$$

### • Problem 3 - Derivation of Klein-Gordon equation

**PENDING: This part will be done later this week.**

## [week1 hw2]

### • Problem 4 - Griffiths 1.2

(a) Find the standard deviation of the distribution in Example 1.1.

First caculate the probability of finding this object at position  $x$ , represented by  $\rho(x)$ :

$$\text{total travel time } T = \sqrt{\frac{2h}{g}}$$

$$dt = \frac{dx}{v} = \frac{1}{\sqrt{2gx}} dx$$

$$\rho(x) = \frac{dt}{T dx} = \frac{1}{2\sqrt{hx}}$$

Accoring to statistics:

$$\begin{aligned}
\langle x \rangle^2 &= \left( \int_0^h x \cdot \rho(x) dx \right)^2 \\
&= \left( \int_0^h \frac{x}{2\sqrt{hx}} dx \right)^2 \\
&= \frac{h^2}{9}
\end{aligned}$$

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^h x^2 \cdot \rho(x) dx \\
&= \int_0^h \frac{x^2}{2\sqrt{hx}} dx \\
&= \frac{h^2}{5}
\end{aligned}$$

Hence:

$$\begin{aligned}
\delta &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \frac{2}{3\sqrt{5}} h
\end{aligned}$$

(b) What is the probability that a photograph, selected at random, would show a distance  $x$  more than one standard deviation away from the average?

We have known that the average is  $\langle x \rangle = \frac{h}{3}$ , and the standard deviation is  $\delta = \frac{2}{3\sqrt{5}} h$ . We now want to know the probability of finding  $x$  among  $0 < x < \langle x \rangle - \delta$  or  $\langle x \rangle + \delta < x < h$ :

$$\begin{aligned}
P &= \int_0^{\langle x \rangle - \delta} \rho(x) dx + \int_{\langle x \rangle + \delta}^h \rho(x) dx \\
&= 1 - \int_{\langle x \rangle - \delta}^{\langle x \rangle + \delta} \rho(x) dx \\
&= 1 - \frac{1}{\sqrt{h}} \left( \sqrt{x} \Big|_{\frac{5-2\sqrt{5}}{15} h}^{\frac{5+2\sqrt{5}}{15} h} \right) \\
&= 1 - \left( \sqrt{\frac{5+2\sqrt{5}}{15}} - \sqrt{\frac{5-2\sqrt{5}}{15}} \right) \\
&\approx 0.39
\end{aligned}$$

#### • Problem 5 - Griffiths 1.4

At time  $t = 0$  a particle is represented by the wave function

$$\Psi(x, 0) = \begin{cases} A \frac{x}{a}, & \text{if } 0 \leq x \leq a, \\ A \frac{(b-x)}{(b-a)}, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where  $A$ ,  $a$  and  $b$  are constants.

(a) Normalize  $\Psi$  (that is, find  $A$  in terms of  $a$  and  $b$ )

The integral of  $|\Psi|^2$  in all space is:

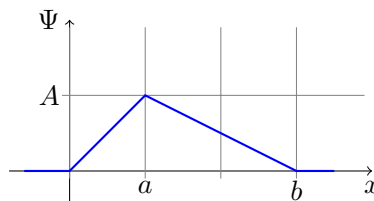
$$\begin{aligned}\int_{-\infty}^{+\infty} \Psi^* \Psi dx &= \int_0^a A^2 \frac{x^2}{a^2} dx + \int_a^b A^2 \frac{(b-x)^2}{(b-a)^2} dx \\ &= A^2 \left( \frac{a}{3} + \frac{b-a}{3} \right) \\ &= \frac{1}{3} A^2 b \\ &= 1\end{aligned}$$

Hence:

$$A = \sqrt{\frac{3}{b}}$$

(b) Sketch  $\Psi(x, 0)$ , as a function of  $x$ .

Let  $a = 1$ ,  $b = 3 \Rightarrow A = 1$ :



(c) Where is the particle most likely to be found, at  $t = 0$ ?

Obviously, at  $x = a$ .

(d) What is the probability of finding the particle to the left of  $a$ ? Check your result in the limiting cases  $b = a$  and  $b = 2a$ .

In case of  $b = a$ :

$$\begin{aligned}A &= \sqrt{\frac{3}{b}} = \sqrt{\frac{3}{a}} \\ P &= \int_0^a \frac{3}{a} \left( \frac{x}{a} \right)^2 dx = 1\end{aligned}$$

In case of  $b = 2a$ :

$$\begin{aligned}A &= \sqrt{\frac{3}{b}} = \sqrt{\frac{3}{2a}} \\ P &= \int_0^a \frac{3}{2a} \left( \frac{x}{a} \right)^2 dx = \frac{1}{2}\end{aligned}$$

(e) What is the expectation value of  $x$ ?

$$\begin{aligned}\langle x \rangle &= \frac{3}{b} \left( \int_0^a x \cdot \frac{x^2}{a^2} dx + \int_a^b x \cdot \frac{(b-x)^2}{(b-a)^2} dx \right) = \frac{3}{b} \left( \frac{a^2}{4} + \frac{b^4 - 3a^4 + 8a^3b - 6a^2b^2}{12(b-a)^2} \right) \\ &= \frac{3a^2(b-a)^2 + b^4 - 3a^4 + 8a^3b - 6a^2b^2}{4b(b-a)^2} = \frac{\cancel{3a^4} + 3a^2b^2 - 6a^3b + b^4 - \cancel{3a^4} + 8a^3b - 6a^2b^2}{4b(b-a)^2} \\ &= \frac{\cancel{b}(b^3 + 2a^3 - 3a^2b)}{4\cancel{b}(b-a)^2} = \frac{(b-a)^3 + 3a(b-a)^2}{4(b-a)^2} = \frac{2a+b}{4}\end{aligned}$$

Notes on Lesson 1

Measurements and Operators - Monday Sep 9

• 1. Derivation of momentum operator

Starting with the most basic measurable value: expectation value of position  $\langle x \rangle$

$$\langle x \rangle = \int x \rho(x) dx = \int \psi^* x \psi dx$$

Then consider  $\langle x \rangle$  change with time:

$$\frac{d}{dt} \langle x \rangle = \frac{d}{dt} \int \Psi^* x \Psi dx \quad (1)$$

$$= \int x \frac{\partial}{\partial t} |\Psi|^2 dx \quad (2)$$

$$= \int x \left( \frac{\partial \Psi^*}{\partial t} \Psi + \frac{\partial \Psi}{\partial t} \Psi^* \right) dx \quad (3)$$

$$\text{Schrödinger Eq} \Rightarrow i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + V \Psi \quad (4)$$

$$-i\hbar \frac{\partial}{\partial t} \Psi^* = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi^* + V \Psi^* \quad (5)$$

$$\text{Conclude to} \Rightarrow \frac{\partial \Psi}{\partial t} \cdot \Psi^* = +\frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi \cdot \Psi^* + \frac{V}{i\hbar} \Psi \cdot \Psi^* \quad (6)$$

$$\frac{\partial \Psi^*}{\partial t} \cdot \Psi = -\frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi^* \cdot \Psi - \frac{V}{i\hbar} \Psi^* \cdot \Psi \quad (7)$$

$$= \frac{i\hbar}{2m} \int x \cdot \left( \Psi^* \frac{\partial^2}{\partial x^2} \Psi - \Psi \frac{\partial^2}{\partial x^2} \Psi^* \right) dx \quad (8)$$

$$\text{Math tricks} \Rightarrow (u \ddot{v} - v \ddot{u}) = \frac{d}{dx} (u \dot{v} - v \dot{u}) \quad (9)$$

$$= \frac{i\hbar}{2m} \int x \cdot \frac{\partial}{\partial x} \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) dx \quad (10)$$

$$\text{Using integral by part} \Rightarrow u \cdot dv = uv - v \cdot du \quad (11)$$

$$\text{let} \Rightarrow u = x \quad (12)$$

$$v = \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \quad (13)$$

$$\text{We get} \Rightarrow \quad (14)$$

$$x \cdot d \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) = u \cdot dv = \cancel{uv} - v \cdot du = - \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) dx \quad (15)$$

$$= -\frac{i\hbar}{2m} \int \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) dx \quad (16)$$

$$Using\ integral\ by\ part\ again \Rightarrow u \cdot dv - v \cdot du = -uv + 2u \cdot dv \quad (17)$$

$$let \Rightarrow u = \Psi^* \quad (18)$$

$$v = \Psi \quad (19)$$

$$We\ get \Rightarrow \Psi^* \partial \Psi - \Psi \partial \Psi^* = u \cdot dv - v \cdot du = \cancel{u \cdot v} + 2v \cdot du = 2 \Psi^* \partial \Psi = 2 \Psi^* \frac{\partial}{\partial x} \Psi dx \quad (20)$$

$$= -\frac{i\hbar}{2m} \int \Psi^* \cdot \Psi^* \frac{\partial}{\partial x} \Psi dx \quad (21)$$

$$= \int \Psi^* \left( -\frac{i\hbar}{m} \frac{\partial}{\partial x} \right) \Psi dx \quad (22)$$

Now we can find a new operator: momentum  $\hat{p}$

$$\langle p \rangle = m \frac{d \langle x \rangle}{dt} = \int \Psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \Psi dx = \int \Psi^* \left( \frac{\hbar}{i} \cdot \frac{\partial}{\partial x} \right) \Psi dx$$

$$Additional\ proof\ of\ \frac{\partial}{\partial t} \int |\Psi|^2 dx = 0$$

$$\frac{d}{dt} \int |\Psi|^2 dx = \int \frac{\partial}{\partial t} |\Psi|^2 dx \quad (23)$$

$$= \int \frac{\partial}{\partial t} (\Psi^* \Psi) dx \quad (24)$$

$$= \int \left( \Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \right) dx \quad (25)$$

$$SchrödingerEq \Rightarrow \frac{\partial}{\partial t} \Psi = \frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi - \frac{i}{\hbar} V \Psi \quad (26)$$

$$\frac{\partial}{\partial t} \Psi^* = -\frac{i\hbar}{2m} \frac{\partial}{\partial x^2} \Psi^* + \frac{i}{\hbar} V \Psi^* \quad (27)$$

$$= \int \frac{i\hbar}{2m} \left( \Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \Psi \frac{\partial^2 \Psi^*}{\partial x^2} \right) dx \quad (28)$$

$$= \frac{i\hbar}{2m} \int \frac{\partial}{\partial x} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) dx \quad (29)$$

$$= \frac{i\hbar}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) \Big|_{-\infty}^{+\infty} \quad (30)$$

$$Note\ that\ \Psi\ is\ normalizable \Rightarrow \lim_{x \rightarrow \infty} \Psi = 0 \quad (31)$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\partial \Psi}{\partial x} = 0 \quad (32)$$

$$= 0 \quad (33)$$

$$\Rightarrow Proved \quad (34)$$

## Week3 Notes

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## Week3 Homework

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### [week1 hw1]

#### • Problem 1 - Griffiths 2.1

Prove the following three theorems:

(a) For normalizable solutions, the separation constant  $E$  must be *real*.

*Proof:*

(b) The time-independent wave function  $\psi(x)$  can always be taken to be *real* (unlike  $\Psi(x, t)$ , which is necessarily complex). This doesn't mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to  $\psi$ 's that are real.

*Proof:*

(c) If  $V(x)$  is an **even function** (that is,  $V(-x) = V(x)$ ) then  $\psi(x)$  can always be taken to be either even or odd.

*Proof:*

#### • Problem 2 - Griffiths 2.2

Show that  $E$  must exceed the minimum value of  $V(x)$ , for every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint:* Rewrite Equation 2.5 in the form

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2} [V(x) - E] \psi$$

if  $E \leq V_{\min}$ , then  $\psi$  and its second derivative always have the *same sign* – argue that such a function cannot be normalized.

#### • Problem 3 - Rewrite the entanglement of XXX when $c_n$ and $\psi_n$ are not necessarily *real*.