Solutions to Optional Homework (Lecture 4)

Problem 4.15

(a) We solve this problem by direct substitution using orthogonality. First we have to show that F_m the DFT of f_n. Substituting Eq. (4-43) into (4-42) yields

$$
\begin{aligned}
F_m &= \sum_{n=0}^{M-1}\left[\frac{1}{M}\sum_{r=0}^{M-1} F_r e^{j2\pi rn/M}\right]e^{-j2\pi mn/M} \\
&= \frac{1}{M}\sum_{r=0}^{M-1} F_r\left[\sum_{n=0}^{M-1} e^{j2\pi rn/M}e^{-j2\pi mn/M}\right] \\
&= \frac{1}{M}F_m M \\
&= F_m
\end{aligned}
$$

where, because of orthogonality, the third step is 0 unless $r = m$.

Next, we have to show that f_n is the inverse DFT of F_m. Substituting Eq. (4-42) into (4-43) and using the same basic procedure yields as above,

$$
\begin{aligned}
f_n &= \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{r=0}^{M-1} f_r e^{-j2\pi rm/M}\right]e^{j2\pi mn/M} \\
&= \frac{1}{M}\sum_{r=0}^{M-1} f_r\left[\sum_{m=0}^{M-1} e^{-j2\pi rm/M}e^{j2\pi mn/M}\right] \\
&= \frac{1}{M}f_n M \\
&= f_n
\end{aligned}
$$

where, because of orthogonality, the third step is 0 unless $r = n$. By showing that F_m is the DFT of f_n and that f_n is the IDFT of F_m we have established that Eqs. (4-42) and (4-43) constitute a Fourier transform pair.

(b) We solve this problem as above, by direct substitution and using orthogonality. First we have to show that $F(u)$ the DFT of $f(x)$. Substituting Eq. (4-45) into (4-44) yields

$$
\begin{aligned}
F(u) &= \sum_{x=0}^{M-1}\left[\frac{1}{M}\sum_{r=0}^{M-1} F(u) e^{j2\pi rx/M}\right]e^{-j2\pi ux/M} \\
&= \frac{1}{M}\sum_{r=0}^{M-1} F(r)\left[\sum_{x=0}^{M-1} e^{j2\pi rx/M}e^{-j2\pi ux/M}\right] \\
&= \frac{1}{M}F(u)M \\
&= F(u)
\end{aligned}
$$

where, because of orthogonality, the third step is 0 unless $r = u$.

Next, we have to show that $f(x)$ is the inverse DFT of $F(u)$. Substituting Eq. (4-44) into (4-45) and using the same basic procedure yields as above,

$$\begin{aligned} f(x) &= \frac{1}{M}\sum_{u=0}^{M-1}\left[\sum_{r=0}^{M-1} f(r)e^{-j2\pi ur/M}\right]e^{j2\pi ux/M} \\ &= \frac{1}{M}\sum_{r=0}^{M-1} f(r)\left[\sum_{u=0}^{M-1} e^{-j2\pi ur/M}e^{j2\pi ux/M}\right] \\ &= \frac{1}{M}f(x)M \\ &= f(x) \end{aligned}$$

where, because of orthogonality, the third step is 0 unless $r = x$. By showing that $F(u)$ is the DFT of $f(x)$ and that $f(x)$ is the IDFT of $F(u)$, we have established that Eqs. (4-44) and (4-45) constitute a Fourier transform pair.

Problem 4.21

(a) Because all rows of the image are identical, we can focus attention on one row, which is a 1-D square wave with a period, P, of four pixels. Therefore, the frequency of this signal is $f = 1/4 = 0.25$ cycles/pixel. If the stripes are now four pixels wide, then the period is eight pixels, and the frequency of the signal is $f = 1/8 = 0.125$ cycles/pixel, which is one-half the frequency of the original signal. The center peak in the spectrum shown in the problem statement is the dc term, and the other two dominant peaks appear on the horizontal axis of the spectrum, exactly half-way between the center and ends of the horizontal axis of the spectrum. The corresponding peaks in the new spectrum have half the frequency, so they will appear midway between the original peaks and the center of the spectrum. That is, one-quarter of the axis length on either side of center. (The spectrum contains other harmonic frequency components that are of lower amplitude and are not shown.)

(b) The image is constant along each column, which means that variations are limited only along each row, with each row being identical. Therefore, the DFT will have frequency components only along the horizontal axis.

(c) We can arrive at the same solution in two different ways. The first is that one-pixel-wide stripes are the narrowest possible in a digital image. Therefore, the period will be the smallest possible period which implies the highest possible frequency. This means that the two spikes will appear at the furthest point on either side of the center. Another way to arrive at the same answer is to notice that, if the spikes of an image with two-pixel-wide stripes are midway between the center and ends of the horizontal axis of the spectrum, then the spikes of a signal with double the frequency must appear at twice the original distance. That is, the spikes of the signal with one-pixel-wide stripes must appear at the ends of the horizontal axis of the spectrum. The first answer is preferable because it shows a deeper understanding of the problem, but the second answer is acceptable in the context of how the problem statement was worded.

(d) The dc term is equal to the average value of the signal. Because the images are of the same size and also contain an integer number of complete periods, the total area occupied by black and white is the same in both images. Therefore, their dc terms are the same.

Problem 4.47

(a) The spatial average (excluding the center term) is

$$g(x,y) = \frac{1}{4}\left[f(x,y+1) + f(x+1,y) + f(x-1,y) + f(x,y-1)\right]$$

From property 3 in Table 4.4,

$$\begin{aligned} G(u,v) &= \frac{1}{4}\left[e^{j2\pi v/N} + e^{j2\pi u/M} + e^{-j2\pi u/M} + e^{-j2\pi v/N}\right]F(u,v) \\ &= H(u,v)F(u,v) \end{aligned}$$

where

$$H(u,v) = \frac{1}{2}\left[\cos(2\pi u / M) + \cos(2\pi v / N)\right]$$

is the filter transfer function in the frequency domain.

(b) To see that this is a lowpass filter transfer function, consider its values in the range $[-M/2, M/2]$. The function assumes its highest value at the origin and decreases on either side of it, so it passes the dc term and low frequencies, and suppresses higher frequencies. Thus, it acts as a lowpass filter transfer function.

Problem 4.53

The answer is no. The Fourier transform is a linear process, while the square and square roots involved in computing the gradient are nonlinear operations. The Fourier transform could be used to compute the derivatives as differences (as in Problem 4.50), but the squares and square root values, must be computed directly in the spatial domain.

Problem 4.57

(a) The ring in fact has a dark center area as a result of the highpass operation only (the following image shows the result of highpass filtering only). However, the dark center area is averaged out by the lowpass filter. The reason the final result looks so bright is that the discontinuity (edge) on boundaries of the ring are much higher than anywhere else in the image, thus dominating the display of the result.

(b) Filtering with the Fourier transform is a linear operation. The order does not matter.

Figure P4.57