Degree distributions of generalized hooking networks *

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Abstract

A hooking network is grown from a set of graphs called blocks, each block with a labelled vertex called a hook. At each step in the growth of the network, a vertex called a latch is chosen from the hooking network, and a block is attached by joining the hook of the block with the latch. These graphs generalize trees, which are hooking networks grown from a single edge as the only block. Using Pólya urns, we show multivariate normal limit laws for the degree distributions of hooking networks. We extend previous results by allowing for more than one block in the growth of the network and by studying arbitrarily large degrees.

1 Introduction

Let $\mathcal{S} = \{G_1, G_2, \ldots, G_m\}$ be a finite set of connected graphs with at least 2 vertices, each with a labelled vertex h_i. We call G_i a *block* and h_i its *hook*. A sequence of *hooking networks* $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ is constructed recursively as follows: choose one of the blocks G_i in $\mathcal{S}$ and let $\mathcal{G}_0$ be a copy of G_i. The graph $\mathcal{G}_n$ is constructed from $\mathcal{G}_{n-1}$ by choosing a vertex v of $\mathcal{G}_{n-1}$, called a *latch*, choosing a block G_i, and attaching a copy of G_i to $\mathcal{G}_{n-1}$ by joining the hook h_i and the latch v. The vertex corresponding to the hook of the initial block used to make $\mathcal{G}_0$ is called the *root of the network*.

As an example, consider the set of blocks in Figure 1.

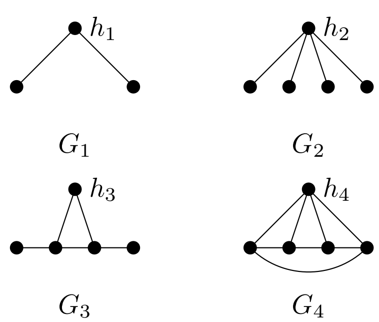

Figure 1: A set of graphs as blocks.

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The graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_3$ in Figure 2 are examples of hooking networks constructed by choosing a copy of G_3 as $\mathcal{G}_0$ and attaching a copy of G_4, then a copy of G_2, and finally a copy of G_1. The root of the network is labelled r and at each step, the vertex chosen to be the latch is denoted by $*$.

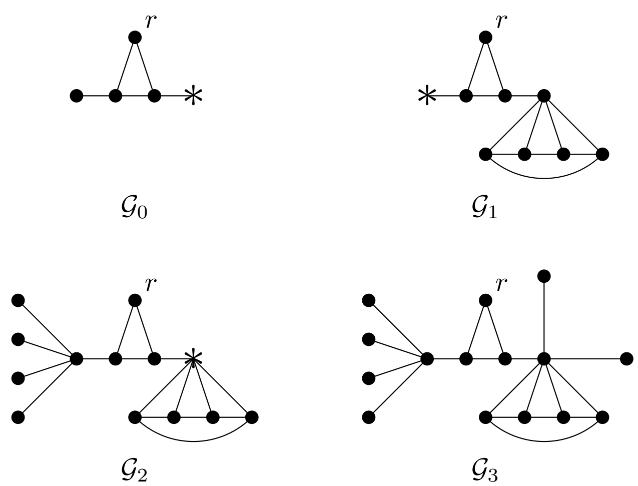

Figure 2: A sequence of hooking networks constructed from G_1, G_2, G_3 and G_4 in Figure 1.

Several well-known graphs can be considered as hooking networks. Any hooking network constructed from a single block consisting of an edge K_2 is simply a tree; the process of attaching K_2 to a latch v is equivalent to adding a child to v. The definition of hooking networks above generalizes *self-similar hooking networks* introduced by Mahmoud [11], which are hooking networks grown from a single block. A block graph is a hooking network whose blocks are complete graphs, and a cactus graph is a hooking network whose blocks are cycles and that may include an edge K_2 in the set of blocks. The study of these graphs has been very active recently [1, 2, 7], with applications in genome comparison [13] as well as in telecommunication networks and material handling networks (see [4]).

In this extended abstract we only consider hooking networks where at every step in the growth of the network the latch is chosen uniformly at random among all of the vertices of the network, and the block to be attached is chosen uniformly at random among all of the blocks. We prove that for any positive integer k, the number of vertices of degree k in such a hooking network converges to a normal distribution. In fact, we prove convergence to multivariate normal distributions for vectors of random variables counting the number of vertices with fixed degrees. We make use of the theory of generalized Pólya urns developed in [8] to prove our results. Our methods also work for more general hooking networks (see Remark 1.1).

In the example of hooking networks of Figure 2, all of the hooks of the blocks have even degrees, and every other vertex has an odd degree. As a result, during the growth of the hooking networks, only the root of the network r will have an even degree, while every other vertex will have an odd degree. In this case, we say the odd numbers are *admissible degrees*. More generally, we define admissible degrees as follows:

Definition 1.1. *Given a set of blocks $\mathcal{S}$, the positive integer k is called an admissible degree if there is a positive probability that for some n, the hooking network $\mathcal{G}_n$ grown from the blocks $\mathcal{S}$ has at least two vertices with degree k.*

Note that the definition of admissible degrees used here differs slightly from that of [3] and [11]. It is easy to verify that the root of the network is the only vertex that may have a degree that is not admissible.

For a set $\mathcal{S}$ of blocks, let N_d be the set of admissible degrees less than or equal to d, and let

$$k_1 < k_2 < \cdots < k_r$$

be the elements of N_d.

Theorem 1.1. *Let $\mathcal{X}_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r})$, where $X_{n,i}$ is the number of vertices with degree k_i in $\mathcal{G}_n$, and $\mathcal{G}_n$ is a hooking network constructed from the set of blocks $\mathcal{S}$ grown by choosing latches and blocks uniformly at random for n iterations. Let*

$$\mu_n := \mathbb{E}\mathcal{X}_n = (\mathbb{E}X_{n,1}, \ldots, \mathbb{E}X_{n,r}).$$

Then

$$n^{-1/2}(\mathcal{X}_n - \mu_n) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

for some covariance matrix Σ, where $\mathcal{N}(0,\Sigma)$ denotes a multivariate normal distribution.

Remark 1.1. *In the journal version of this paper (to appear), we prove similar results when the latch is chosen preferentially according to its degree, and when the blocks are chosen proportional to a fixed weight assigned to each block. Our methods also show similar results for blocks trees (see [5] for a definition) and bipolar networks (see [3] for a definition).*

The above theorem generalizes several previous results. A random recursive tree is equivalent to a hooking network grown from a single edge K_2. Theorem 1.1 therefore includes previously known results for the asymptotic degree distribution for random recursive trees [12, 9]. Blocks trees can be thought of as hooking networks with trees for blocks. Gopaladesikan, Mahmoud, and Ward [5] proved asymptotic results for the number of leaves in randomly grown blocks trees. Since leaves are vertices of degree 1, Theorem 1.1 extends their work to arbitrary degrees. Mahmoud [11] studied the distribution of vertices with the two smallest admissible degrees in self-similar hooking networks. Our theorem implies this result as well.

2 Pólya urns

A generalized Pólya urn process $(X_n)_{n=0}^{\infty}$ is defined as follows. There are q types (or colours) $1, 2 \ldots, q$ of balls and for each vector $X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,q})$, the entry $X_{n,i} \geq 0$ is the number of balls of type i in the urn at time n, starting with a given (random or not) vector X_0. Each type i is assigned an activity $a_i \in \mathbb{R}_{\geq 0}$ and a random vector $\xi_i = (\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,q})$ satisfying $\xi_{i,j} \geq 0$ for $i \neq j$ and $\xi_{i,i} \geq -1$. At each time $n \geq 1$, a ball is drawn at random so that the probability of choosing a ball of type i is

$$\frac{a_i X_{n-1,i}}{\sum_{j=1}^{q} a_j X_{n-1,j}}.$$

If the drawn ball is of type i, then it is replaced along with $\Delta X_{n,j}$ balls of type j for each $j = 1, \ldots, q$, where the vector $\Delta X_n = (\Delta X_{n,1}, \Delta X_{n,2}, \ldots, \Delta X_{n,q})$ has the same distribution as ξ_i and is independent of everything else that has happened so far. We allow for $\Delta X_{n,i} = -1$, in which case the drawn ball is not replaced.

The *intensity matrix* of the Pólya urn is the $q \times q$ matrix

$$A := (a_j \mathbb{E}\xi_{j,i})_{i,j=1}^{q}.$$

By the choice of $\xi_{i,j}$, the matrix $I + A$ has non-negative entries and so by the standard Perron-Frobenius theory, A has a real eigenvalue λ_1 such that all other eigenvalues $\lambda \neq \lambda_1$ satisfy $\mathrm{Re}\lambda < \lambda_1$.

We use basic assumptions on the Pólya urn, these are (A1)–(A7) gathered in [6] ((A1)–(A6) are also explicitly stated in [8]). A ball of type i is said to be *dominating* if with positive probability, every other ball of type j can be found at some time in an urn starting with a single ball of type i. The urn (and its matrix A) is *irreducible* if every type is dominating. In our applications, all of the urns will be irreducible. Using the Perron-Frobenius theorem, it is easy to verify that (A1)–(A6) are satisfied when the intensity matrix is irreducible. We will only use balls with positive activity and so (A7) is satisfied.

Denote column vectors as v with v' as its transpose. The transpose of a matrix A is also denoted as A'. Let $a = (a_1, \ldots, a_q)'$ denote the vector of activities, and let u_1' and v_1 be the left and right eigenvectors of A corresponding to the eigenvalue λ_1 normalized so that $a \cdot v_1 = a'v_1 = v_1'a = 1$ and $u_1 \cdot v_1 = u_1'v_1 = v_1'u_1 = 1$. Define $P_{\lambda_1} = v_1u_1'$ and $P_I = I_q - P_{\lambda_1}$. Define the matrices

$$B_i := \mathbb{E}(\xi_i\xi_i')$$

for every $i = 1, \ldots, q$, denote $v_1 = (v_{1,1}, v_{1,2}, \ldots, v_{1,q})'$, and define the matrix

$$B := \sum_{i=1}^{q} v_{1,i}a_iB_i.$$

In the case where $\mathrm{Re}\lambda < \lambda_1/2$ for every eigenvalue $\lambda \neq \lambda_1$, define

$$\Sigma_I := \int_0^\infty P_I e^{sA} B e^{sA'} P_I' e^{-\lambda_1 s} ds, \tag{2.1}$$

where $e^{tA} = \sum_{j=0}^\infty t^jA^j/j!$.

The result we use from [8] guarantees that if (A1)-(A7) hold and $\mathrm{Re}\lambda < \lambda_1/2$ for all eigenvalues $\lambda \neq \lambda_1$, then

$$n^{-1/2}(X_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

for some $\mu = (\mu_1, \ldots, \mu_q)$ and $\Sigma = (\sigma_{i,j})_{i,j=1}^q$. We use lemmas stated in [8] and gathered in [6, Theorem 4.1] to calculate μ and Σ.

Theorem 2.1. ([8, Thm. 3.22 and Lem. 5.4]) *Assume (A1)-(A7) and that the right and left eigenvectors corresponding to λ_1 are normalized as above. Assume that* $\mathrm{Re}\lambda < \lambda_1/2$ *for each eigenvalue $\lambda \neq \lambda_1$.*

(i) Then, as $n \to \infty$,

$$n^{-1/2}(X_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma) \tag{2.2}$$

with $\mu = \lambda_1 v_1$ and some covariance matrix Σ.

(ii) Suppose further that, there exists $c > 0$ so that for every $i = 1, \ldots, q$,

$$a \cdot \mathbb{E}(\xi_i) = c. \tag{2.3}$$

Then the covariance matrix is given by $\Sigma = c\Sigma_I$, with Σ_I as defined in (2.1).

Remark 2.1. *We will use Theorem 2.1 to show that the degree vectors of hooking networks converge in distribution to a multivariate normal distribution. A recent result by Janson and Pouyanne [10] guarantees convergence in moments for certain generalized Pólya urns. By [10, Remark 1.9], our urns satisfy the conditions needed for convergence in moments as well. In particular, from [10, Theorem 1.1], we get that*

$$\mathbb{E}X_n/n \to \mu.$$

3 Proof of main result

We start by describing the degrees of the vertices in the growth of a hooking network as balls in the evolution of a Pólya urn, and then show that the intensity matrix of our urn satisfies the conditions of Theorem 2.1.

3.1 Degrees in hooking networks as balls in Pólya urns We start by first looking at an urn with infinitely many types. We assign a type to each degree in the network so that a ball of type k represents a vertex of degree k. Each network starts as a copy of a graph from the list of blocks. This corresponds to starting a Pólya urn with a ball of the matching type for the degree of each vertex in the block. In the evolution of the network, the hook of a block being attached to a latch v corresponds to choosing a ball in the urn of type corresponding to the degree of v, and replacing it with a ball representing the new degree of v along with balls representing the degrees of the rest of the vertices of the block. Since a latch is chosen uniformly at random, each ball simply has activity 1.

The Pólya urn described above has infinitely many types, and so Theorem 2.1 does not apply. We would like to instead use an urn with finitely many types in a similar manner as was done in [6] and [9]. The urn is replaced with the following Pólya urn: let d be a positive integer corresponding to the largest degree we wish to study in this instance of the model. A new ball of special type $*$ with activity $a_* = 1$ is introduced, and for every $k > d$, each ball of type k is replaced with a ball of special type $*$. In this way, the probability of choosing a ball of special type in the new urn is equal to the probability of choosing a ball of type greater than d in the old urn. If a vertex v of degree $k \leq d$ is chosen as a latch, and a hook is attached so that v now has degree $k + j > d$, then the ball of type k is removed and a ball of special type is added. If a latch of degree $k > d$ is chosen and a hook is attached, then the ball of special type that was chosen is simply placed back in the urn. At each step we again also place balls representing the degrees of the vertices of the block that are not the hook.

Instead of adding a ball of type corresponding to the degree of the root of the network, a ball of special type is added. This guarantees that all types of balls in the urn that are not special types correspond only to degrees that are admissible; recall from Definition 1.1 that a degree is admissible if there is a positive probability that at some point in the growth of the network at least two vertices will have that degree. For a positive integer d, the possible types of balls present in the urn are exactly the elements of N_d, the set of admissible degrees less than or equal to d, together with a ball of special type $*$. In our intensity matrix, we can then omit the rows and columns corresponding to types that will never be present in the urn. By restricting to admissible degrees, it can be verified that now every ball in the urn is of dominating type, and so the urn (and its intensity matrix) is irreducible. As discussed in Section 2, it is easy to verify that the assumptions (A1)-(A7) are satisfied for irreducible urns. To avoid confusion, we will label the type of a ball with the degree of the vertex it represents.

To show how to calculate the intensity matrix for an urn corresponding to the growth of a hooking network, consider the example given in Figure 2 of the introduction. Suppose we look at all admissible degrees less than or equal to 9, so $N_9 = \{1, 3, 5, 7, 9\}$ (recall from the introduction that the admissible degrees for this example are all odd degrees). The images in Figure 3 illustrates the possibilities for replacing a ball of type k, corresponding to attaching a block to a latch v in the hooking network. Since the blocks are chosen uniformly at random, each has a probability of 1/4 of being attached to a latch.

The intensity matrix for this urn will have 6 rows and columns: one corresponding to the degrees 1,3,5,7,9, and the last one for balls of special type $*$. Let's consider what happens when a block is attached to a vertex with degree 3. This corresponds to choosing a ball of type 3. For example, with probability 1/4, the graph G_1 is attached to v. The hook has degree 2 and there are 2 vertices of degree 1. Then the ball of type 3 is removed and replaced with a ball of type 5 along with 2 balls of type 1. This explains the first column vector in the sum below. Using Figure 3 for the other blocks, we can calculate $\mathbb{E}\xi_3$ to be the following sum:

$$\frac{1}{4}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Remember that rows and columns for even types are removed, and so the first row represents balls of type 1, the second row for balls of type 3, etc, and the final row for balls of special type.

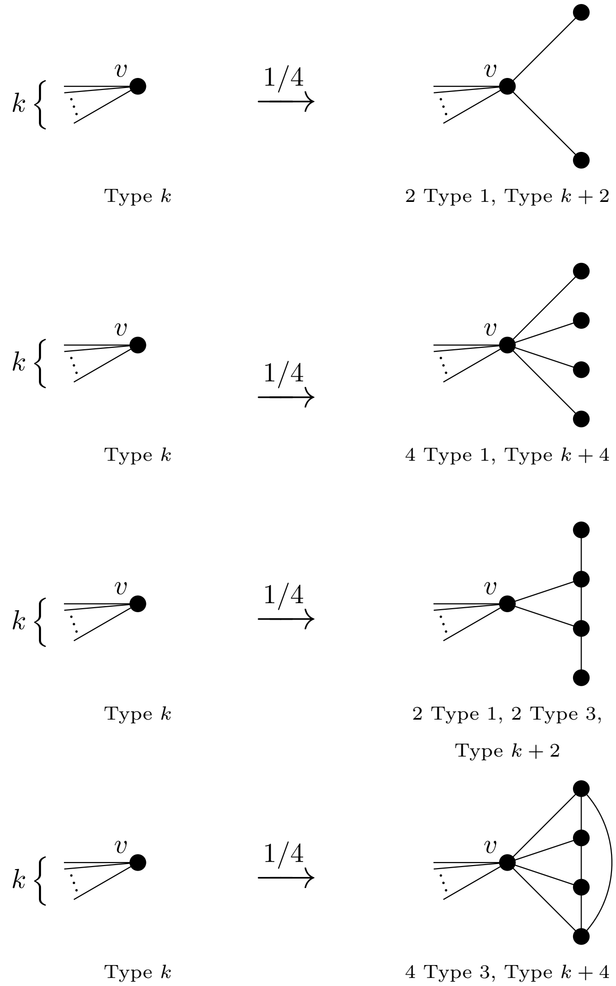

Figure 3: The replacements of a ball of type k in the hooking network from Figure 2.

Now consider what happens when a ball of type 7 is chosen. If a hook of degree 4 is attached to a vertex of degree 7, the resulting vertex will have degree 11. Recall that instead of adding a ball of type 11, we add a ball of special type. Then $\mathbb{E}\xi_7$ is the following sum:

$$\frac{1}{4}\begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 4 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Finally consider choosing a ball of special type; that is either a block was attached to a vertex of degree greater than 9, or a block was attached to the root of the network r. This time the ball of special type is simply replaced back in the urn. Thus, for the special type,

$\mathbb{E}\xi_*$ is the following sum:

$$\frac{1}{4}\begin{pmatrix}2\\0\\0\\0\\0\\0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}4\\0\\0\\0\\0\\0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}2\\2\\0\\0\\0\\0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0\\4\\0\\0\\0\\0\end{pmatrix}.$$

By finishing the calculations for the remaining degrees, we get the following intensity matrix:

$$A=\frac{1}{4}\begin{pmatrix}4&8&8&8&8&8\\8&2&6&6&6&6\\2&2&-4&0&0&0\\0&2&2&-4&0&0\\0&0&2&2&-4&0\\0&0&0&2&4&0\end{pmatrix}.$$

The matrix A has eigenvalues

$$7/2,-1,-1,-1,-1,-1$$

and so Theorem 2.1 applies. The right eigenvector of A corresponding to λ_1, normalized as is described in Section 2, is calculated (using MATHEMATICA) to be

$$v_1=\left(\frac{4}{9},\frac{31}{81},\frac{67}{729},\frac{346}{6561},\frac{949}{59049},\frac{716}{59049}\right)'.$$

Let $X_{n,i}$ be the number of vertices of degree $2i-1$ for $i=1,2,3,4,5$, and let $X_{n,6}$ be the number of vertices of degree greater than 9. Let $\mathcal{X}_n=(X_{n,1},\dots,X_{n,6})$. Then by Theorem 2.1,

$$n^{-1/2}(X_n-n\mu)\xrightarrow{d}\mathcal{N}(0,\Sigma),$$

where $\mu=\frac{7}{2}v_1$.

For Σ, every ball in the urn has activity $a_i=1$, and since all columns sum to $\frac{7}{2}$, then (2.3) holds, and we can calculate the covariance matrix Σ by following the steps laid out in Section 2. The left eigenvector of A associated with λ_1 normalized as is described in Section 2 is simply the vector of all 1's. Therefore P_{λ_1} is the matrix with copies of v_1 making up every column. We can use Figure 3 to calculate B_1,B_3,B_5,B_7,B_9 and B_*. For example,

$$B_3=\frac{1}{4}b_1b_1'+\frac{1}{4}b_2b_2'+\frac{1}{4}b_3b_3'+\frac{1}{4}b_4b_4',$$

where b_1, b_2, b_3, and b_4 are given below:

$$b_1=\begin{pmatrix}2\\-1\\1\\0\\0\\0\end{pmatrix}b_2=\begin{pmatrix}4\\-1\\0\\1\\0\\0\end{pmatrix}b_3=\begin{pmatrix}2\\1\\1\\0\\0\\0\end{pmatrix}b_4=\begin{pmatrix}0\\3\\0\\1\\0\\0\end{pmatrix}$$

The matrix Σ_I was calculated according to (2.1) using MATHEMATICA, giving the covariance matrix

$$\Sigma=\lambda_1\Sigma_I=\frac{7}{2}\Sigma_I$$

which is added in Appendix A.

3.2 Proof of Theorem 1.1

We start by introducing some useful notation. Let $\mathcal{S}$ be a set of blocks and let each block $G_i\in\mathcal{S}$ have vertex set $V(G_i)$ and hook h_i. For every positive integer j, define

$$f(j)=\sum_{G_i\in\mathcal{S}}|\{v\in V(G_i)\setminus\{h_i\}:\deg(v)=j\}|$$

and

$$g(j)=|\{h_i:\deg(h_i)=j,G_i\in\mathcal{S}\}|.$$

The number $f(j)$ is the number of vertices that are not hooks of degree j among all the blocks, and $g(j)$ is the number of hooks of degree j. Define $m=|\mathcal{S}|$ to be the number of blocks. Note also that $m=\sum_{j\geq 1}g(j)$. For the set $\mathcal{S}$ of blocks in Figure 1 as an example, $f(1)=8$, $f(3)=6$, $g(2)=2$, $g(4)=2$, and $m=4$.

It is useful to note for the proof below that if j is an admissible degree (recall from Definition 1.1) and k is not, then there are no hooks of degree $k-j$. Otherwise, there is a positive probability of attaching a hook of degree $k-j$ to a latch of degree j and increase the degree of the latch to k. As a result, $g(k-j)=0$ whenever j is admissible and k is not admissible. Also, if k is not an admissible degree, then $f(k)=0$.

Proof. We start by calculating the intensity matrix for our Pólya urn defined in Section 3.1. Suppose that we want to look at vertices of degree at most d. Recall that we let N_d be the set of admissible degrees less than or equal to d, and that $k_1<\cdots<k_r$ are the elements of N_d.

We will look at two cases: when a block is attached to a vertex that is not the root of the network with degree less than or equal to d, and when a block is attached to the root of the network or to a vertex of degree greater than d. Recall that the root of the network is the vertex corresponding to the hook of the initial block used to start the network, and is represented by a ball of special type $*$ in the urn.

CASE I: Let $j\in N_d$ and suppose that at some step in the growth of the network a vertex v is chosen as a latch where $\deg(v)=j$ and v is not the root of the network, and a block is attached to v. This corresponds to choosing a ball of type j. Let $i\in N_d$ be an arbitrary admissible degree less than or equal to d. The probability that the degree of v is increased to i

is equal to the number of hooks of degree $i-j$ divided by the total number of blocks, that is, $g(i-j)/m$. Other than the latch, the expected number of new vertices of degree i added to the network is equal to the number of vertices of degree i that are not a hook divided by the total number of blocks, that is, is equal to $f(i)/m$. For $i, j \in N_d$ and with $\mathbb{E}(\xi_{j,i})$ being the expected change in the number of balls of type i added when a ball of type j is chosen, then the arguments above show that

$$\mathbb{E}(\xi_{j,i}) = \begin{cases} f(i)/m & i < j, \\ f(i)/m - 1 & i = j, \\ f(i)/m + g(i-j)/m & i > j. \end{cases}$$

For every i that is an admissible degree greater than d, balls of special type are added instead of balls of type i. So by similar arguments as above and summing over all admissible degrees greater than d, the number of balls of special type added if a ball of type j is chosen is

$$\mathbb{E}(\xi_{j,*}) = \sum_{k>d} (f(k)/m + g(k-j)/m).$$

Case II: Look at what happens when a ball of special type is chosen. This corresponds to choosing a vertex v of degree greater than d or the the root of the network as a latch. Suppose that a block is attached to v. Again, for an arbitrary admissible degree $i \in N_d$, the expected number of new vertices of degree i added is $f(i)/m$. Therefore for $i \in N_d$,

$$\mathbb{E}(\xi_{*,i}) = f(i)/m.$$

The ball of special type that was chosen is simply placed back in the urn. We also add balls of special type for every new vertex other than the latch that is of degree greater than d. The expected number of balls of special type added is then

$$\mathbb{E}(\xi_{*,*}) = \sum_{k>d} f(k)/m.$$

The activity of every ball is simply $a_i = 1$, and so after removing rows and columns corresponding to degrees that are not admissible, we then get the following intensity matrix A:

$$\begin{pmatrix} \frac{f(k_1)}{m} - 1 & \cdots & \frac{f(k_1)}{m} & \frac{f(k_1)}{m} \\ \frac{f(k_2)+g(k_2-k_1)}{m} & \cdots & \frac{f(k_2)}{m} & \frac{f(k_2)}{m} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{f(k_r)+g(k_r-k_1)}{m} & \cdots & \frac{f(k_r)}{m} - 1 & \frac{f(k_r)}{m} \\ \sum\limits_{k>d} \frac{f(k)+g(k-k_1)}{m} & \cdots & \sum\limits_{k>d} \frac{f(k)+g(k-k_r)}{m} & \sum\limits_{k>d} \frac{f(k)}{m} \end{pmatrix}.$$

The first r rows and columns of A correspond to $k_1, k_2, \ldots, k_r$ and the last row and column corresponds to the ball of special type $*$. From here, the eigenvalues of A can be calculated directly. Look at $A - \lambda I$. Subtract column $r+1$ from all other columns. Then add rows $1, 2, \ldots, r$ to row $r+1$. Using the fact that $\sum_{j \geq 1} g(j) = m$, we get the matrix

$$A' = \begin{pmatrix} -1-\lambda & \cdots & 0 & \frac{f(k_1)}{m} \\ \frac{g(k_2-k_1)}{m} & \cdots & 0 & \frac{f(k_2)}{m} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{g(k_r-k_1)}{m} & \cdots & -1-\lambda & \frac{f(k_r)}{m} \\ 0 & \cdots & 0 & \sum_j \frac{f(j)}{m} - \lambda \end{pmatrix}.$$

Since the determinant of a matrix is unchanged by adding one row to another or by subtracting one column from another, both $A - \lambda I$ and A' have the same determinant. We can calculate the determinant of A' by expanding along the bottom row and see that A has characteristic polynomial

$$\pm \left(\lambda - \frac{1}{m} \sum_{j \geq 1} f(j) \right) (\lambda + 1)^r.$$

Therefore A has largest eigenvalue $\lambda_1 = \frac{1}{m} \sum_{j \geq 1} f(j)$, and all other eigenvalues are $\lambda = -1$. Theorem 2.1 applies, proving Theorem 1.1.

Remark 3.1. *By studying the matrix A in the proof above, we see that the entries of the last column sum to $\lambda_1 = \frac{1}{m} \sum_{j \geq 1} f(j)$. Furthermore, since $\sum_{j \geq 1} g(j) = m$, then the entries of any other column sum to*

$$\frac{1}{m} \sum_{j \geq 1} f(j) + \frac{1}{m} \sum_{j \geq 1} g(j) - 1 = \frac{1}{m} \sum_{j \geq 1} f(j) = \lambda_1.$$

Therefore, (2.3) is satisfied with $c = \lambda_1$, and the covariance matrix Σ in Theorem 1.1 is $\Sigma = \lambda_1 \Sigma_I$, where Σ_I is given by (2.1).

Recall the example of Figure 2 in the Introduction. We calculated at the end of Section 3.1 the intensity matrix A and its eigenvalues when we look at vertices with degrees less than or equal to 9. With $f(1) = 8$, $f(3) = 6$, $g(2) = 2$, $g(4) = 2$, and $m = 4$, then the proof above shows that A has eigenvalues

$$7/2, -1, -1, -1, -1, -1,$$

which is what was calculated at the end of Section 3.1.

4 More examples: cactus graphs

Recall from the introduction that a cactus graph is a hooking network grown from a set of cycles that may or may not include a single edge K_2. For a positive integer m, consider a hooking network whose blocks are the m smallest cycles: $C_3, C_4, \ldots, C_{m+2}$. The set of admissible degrees in such a hooking network consists of all even positive integers. For a positive integer r, suppose we want to count the vertices of degree $2, 4, \ldots, 2r$. We can do a similar analysis as was done in Section 3.1 to calculate the intensity matrix, or we can simply refer to the proof of Theorem 1.1 in Section 3.2. We calculate that $g(2) = m$ and that

$$f(2) = 2 + 3 + \ldots + m + 1 = \frac{m(m+3)}{2}.$$

Then the intensity matrix is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} \frac{m+3}{2} - 1 & \frac{m+3}{2} & \cdots & \frac{m+3}{2} & \frac{m+3}{2} \\ 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where for $i = 1, 2, \ldots, r$, the i-th row and column represent balls of type $2i$, and row and column $r+1$ represent balls of special type. By using the proof of Theorem 1.1 (or by direct calulation), we know that A has a largest real eigenvalue $\lambda_1 = \frac{m+3}{2}$. It is then elementary to verify that the right eigenvector v_1 associated with λ_1 whose entries sum to 1 is the vector

$$\left(\frac{m+3}{m+5}, \frac{2(m+3)}{(m+5)^2}, \ldots, \frac{2^{r-1}(m+3)}{(m+5)^r}, \frac{2^{r-1}}{(m+5)^r}\right)'.$$

For $i = 1, 2, \ldots, r$ let $X_{n,i}$ be the number of vertices of degree $2i$ in a hooking network at time n grown from the blocks $C_3, C_4, \ldots, C_{m+2}$, and let $X_{n,r+1}$ be the number of vertices of degree greater than $2r$ in the same hooking network at time n. Let $\mathcal{X}_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r}, X_{n,r+1})$. Then by Theorem 1.1,

$$n^{-1/2}\left(\mathcal{X}_n - \mathbb{E}\mathcal{X}_n\right) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

where

$$\mathbb{E}\mathcal{X}_n / n \to \frac{m+3}{2} v_1$$

by Remark 2.1 and $\Sigma = \frac{m+3}{2}\Sigma_I$, where Σ_I can be calculated by (2.1). For example, the matrix Σ when $m = 3$ and $r = 5$ is included in Appendix B.

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A The covariance matrix calculated for the example in Figure 2

$$\begin{pmatrix}
\frac{1358}{891} & \frac{-125300}{88209} & \frac{-226030}{8732691} & \frac{-76650812}{864536409} & \frac{525120302}{85589104491} & \frac{408156658}{85589104491} \\
\frac{-125300}{88209} & \frac{12903359}{8732691} & \frac{-86970905}{864536409} & \frac{5297948383}{171178208982} & \frac{1858133830}{8473321344609} & \frac{208893864151}{16946642689218} \\
\frac{-226030}{8732691} & \frac{-86970905}{864536409} & \frac{40236500857}{171178208982} & \frac{-529630593233}{8473321344609} & \frac{-41135904377813}{1677717626232582} & \frac{-18077457766130}{838858813116291} \\
\frac{-76650812}{864536409} & \frac{5297948383}{171178208982} & \frac{-529630593233}{8473321344609} & \frac{128853799246214}{838858813116291} & \frac{-1493303871893519}{83047022498512809} & \frac{-2559126802394561}{166094044997025618} \\
\frac{525120302}{85589104491} & \frac{1858133830}{8473321344609} & \frac{-41135904377813}{1677717626232582} & \frac{-1493303871893519}{83047022498512809} & \frac{763496328494992267}{16443310454705536182} & \frac{-84570373610799818}{8221655227352768091} \\
\frac{408156658}{85589104491} & \frac{208893864151}{16946642689218} & \frac{-18077457766130}{838858813116291} & \frac{-2559126802394561}{166094044997025618} & \frac{-84570373610799818}{8221655227352768091} & \frac{247872194436271685}{8221655227352768091}
\end{pmatrix}$$

B The covariance matrix calculated for a cactus graph example

$$\begin{pmatrix}
\frac{1}{8} & \frac{-61}{352} & \frac{337}{15488} & \frac{11771}{681472} & \frac{193993}{29984768} & \frac{83779}{29984768} \\
\frac{-61}{352} & \frac{4469}{15488} & \frac{-58473}{681472} & \frac{-673091}{29984768} & \frac{-7127009}{1319329792} & \frac{-2106699}{1319329792} \\
\frac{337}{15488} & \frac{-58473}{681472} & \frac{2598237}{29984768} & \frac{-21625921}{1319329792} & \frac{-266259915}{58050510848} & \frac{-94530713}{58050510848} \\
\frac{11771}{681472} & \frac{-673091}{29984768} & \frac{-21625921}{1319329792} & \frac{1473056309}{58050510848} & \frac{-7070936249}{2554222477312} & \frac{-2658012563}{2554222477312} \\
\frac{193993}{29984768} & \frac{-7127009}{1319329792} & \frac{-266259915}{58050510848} & \frac{-7070936249}{2554222477312} & \frac{780449246221}{112385789001728} & \frac{-73846108497}{112385789001728} \\
\frac{83779}{29984768} & \frac{-2106699}{1319329792} & \frac{-94530713}{58050510848} & \frac{-2658012563}{2554222477312} & \frac{-73846108497}{112385789001728} & \frac{239255434469}{112385789001728}
\end{pmatrix}$$