On the exponential decay of the characteristic function of the quicksort distribution

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Abstract. We prove that the characteristic function of the quicksort distribution is exponentially decreasing at infinity. As a consequence it follows that the density of the quicksort distribution can be analytically extended to the vicinity of the real line.

Keywords: Quicksort, characteristic function, density, Laplace transform, analytic continuation

1 Introduction

Let X_n be the number of steps required by Quicksort algorithm to sort the list of values $\sigma(1), \sigma(2), \ldots, \sigma(n)$ where σ is a random permutation chosen with uniform probability from the set of all permutations S_n of order n. It has been proven by Régnier (1989) and Rösler (1991) that the appropriately scaled distribution of X_n converges to some limit law

$$\frac{X_n - \mathbb{E}X_n}{n} \to^d Y$$

as $n \to \infty$. Let us denote as f(t) the characteristic function of the limiting distribution

$$f(t) = \mathbb{E}e^{itY}$$

Tan and Hadjicostas (1995) proved that the characteristic function f(t) has a density p(x). Knessl and Szpankowski (1999) using heuristic approach established a number of very precise estimates for the behavior of p(x) at infinity. Later Fill and Janson (2000) showed that the characteristic function f(t) of the limit quicksort distribution together with its all derivatives decrease faster than any polynomial at infinity. More precisely they showed that for all real p > 0 there is such a constant c_p that

$$|f(t)| \leqslant \frac{c_p}{|t|^p}, \quad \text{for all} \quad t \in \mathbb{R}.$$

They also proved that

$$c_p \leqslant 2^{p^2 + 6p}.$$

Hence

$$|f(t)| \le \inf_{p>0} \frac{2^{p^2+6p}}{|t|^p}.$$

The infimum in the above inequality can be evaluated as

$$|f(t)| \le \inf_{p>0} \frac{2^{p^2+6p}}{|t|^p} \le |t|^3 e^{-\frac{\log^2|t|}{4\log 2}}.$$

The main result of this paper is the following theorem stating that the characteristic function f(t) of limiting Quicksort distribution decreases exponentially at infinity.

Theorem 1 There is a constant $\eta > 0$ such that

$$f(t) = O(e^{-\eta|t|})$$

as
$$|t| \to \infty$$
.

Corollary 2 *Quicksort distribution has a bounded density that can be extended analytically to the vicinity of the real line* $|\Im(s)| < \eta$. Where η is the same positive number as in the formulation of Theorem 1.

2 Proofs

It has been shown in Rösler (1991) that the characteristic function f(t) satisfies the functional equation

$$f(t) = e^{it} \int_0^1 f(tx) f(t(1-x)) e^{2itx \log x + 2it(1-x) \log(1-x)} dx$$

which after a change of variables $x \to y/t$ becomes

$$tf(t)e^{2it\log t} = e^{it} \int_0^t f(y)f(t-y)e^{2iy\log y + 2i(t-y)\log(t-y)} dy$$

It follows hence by taking Laplace transform of the both sides that function

$$\psi(s) = \int_0^\infty f(t)e^{2it\log t}e^{-st} dt$$

satisfies an equation

$$-\psi'(s) = \psi^2(s-i). \tag{1}$$

The Laplace transform $\psi(s)$ together with the above differential equation will be the main tool of proving the result stated in the introduction.

It is well known that the quicksort distribution has finite moments of all orders. In the following analysis we will only need the fact that it has finite first moment, which implies that |f'(t)| is bounded. Thus integrating by parts we conclude that

$$\psi(s) = \int_0^\infty f(t)e^{2it\log t}e^{-st} dt$$

$$= \frac{1}{s} + \frac{1}{s} \int_0^\infty \left(f'(t)e^{2it\log t} + f(t)e^{2it\log t} (2i\log t + 2i) \right)e^{-st} dt$$

$$\leqslant \frac{A}{|s|} \left(1 + \frac{|\log \Re s|}{\Re s} \right), \tag{2}$$

for all s lying in the right half-plane $\Re s > 0$ and A > 0 being some positive absolute constant.

Lemma 3 For all s lying in the right half-plane $\Re s > 0$ and all integer $n \geqslant 0$ holds the inequality

$$|\psi^{(n)}(s)| \le n! \left(\max_{r \in \{0,1,\dots,n\}} |\psi(s-ir)| \right)^{n+1}$$

Proof: The proof is done by applying mathematical induction on n and using the fact that the differential equation for $\psi(s)$ allows us to express the derivatives $\psi^{(n)}(s)$ as a polynomial function of $\psi(s-ik)$ with $0 \le k \le n$.

Indeed, for n=0 the above inequality becomes an identity. Suppose this identity holds for all n not exceeding m. Let us consider now n=m+1. Replacing the first derivative of $\psi(s)$ by $-\psi^2(s-i)$ we obtain

$$\begin{split} \psi^{(m+1)}(s) &= \left(\psi'(s)\right)^{(m)} = -\left(\psi^2(s-i)\right)^{(m)} \\ &= -\sum_{k=0}^m \binom{m}{k} \psi^{(k)}(s-i) \psi^{(m-k)}(s-i). \end{split}$$

Thus applying the inductive hypothesis to the derivatives of $\psi(s-i)$ we get

$$\begin{split} \left| \psi^{(m+1)}(s) \right| & \leq \sum_{k=0}^{m} \binom{m}{k} k! \left(\max_{r \in \{0,1,\dots,k\}} \left| \psi(s-i-ir) \right| \right)^{k+1} (m-k)! \left(\max_{r \in \{0,1,\dots,m-k\}} \left| \psi(s-i-ir) \right| \right)^{m-k+1} \\ & \leq (m+1)! \left(\max_{r \in \{0,1,\dots,n\}} \left| \psi(s-ir) \right| \right)^{m+2}. \end{split}$$

The last inequality is the same as stated in the lemma with n=m+1. This completes the proof of the lemma

Lemma 4 For all s lying in the lower part of the right half-plane $\Re s > 0$ and $\Im s < 0$ holds the inequality

$$\left|\psi^{(n)}(s)\right| \leqslant n! \left(\frac{C(\sigma)}{|s|}\right)^{n+1}$$

Where $\sigma = \Re s$ and

$$C(\sigma) = A\left(1 + \frac{|\log \sigma|}{\sigma}\right)$$

with some absolute constant A > 0.

Proof: Our upper bound (2) for $\psi(s)$ implies that for $\Re s > 0$ and $\Im s < 0$ we have

$$\max_{r \in \{0,1,\dots,n\}} \left| \psi(s-ir) \right| \leqslant \max_{r \in \{0,1,\dots,n\}} \frac{C(\sigma)}{|s-ir|} \leqslant \frac{C(\sigma)}{|s|}.$$

Since imaginary part of s is negative so $|s - ir| \ge |s|$. Using this inequality to evaluate the right hand side of the inequality of Lemma 3 we complete the proof of the lemma.

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Proposition 5 The function $\psi(s)$ can be continued analytically to the whole complex plane. Moreover, for all s belonging to the lower half-plane $\Im(s) < 0$ and $\Re s \geqslant -B$ with any fixed B > 0 holds the estimate

$$\psi(s) = O_B(1/|s|)$$

Proof: For $\Re(s) \ge 1$ the estimate of the proposition already follows from (2). By this estimate of Lemma 4 we have that the Taylor series

$$\psi(s) = \sum_{i=0}^{\infty} \frac{\psi^{(j)}(1 - iK)}{j!} (s - (1 - iK))^{j}$$

converges in the circle |1 - iK - s| < |1 - iK|/C(1) and moreover in this circle holds the estimate

$$|\psi(s)| \leqslant \sum_{j=0}^{\infty} \left(\frac{C(1)}{|1-iK|}\right)^{j+1} \left|s - (1-iK)\right|^{j} = \frac{C(1)}{|1-iK|} \frac{1}{1 - \frac{C(1)}{|1-iK|} |1 - iK - s|}.$$

This means that $\psi(s)$ can be analytically continued to the region of complex plane that consists of such s that are contained in any of the circles of radius |1-iK|/C(1) with center at 1-iK with some K>0. Note that all complex number s with negative imaginary part such that $1+\frac{\Im(s)}{C(1)}\leqslant \Re(s)$ satisfy such condition. See the figure 1.

Note that $\psi(s)$ satisfies a shift-differential equation (1) which is by integrating its both sides yields the identity

$$\psi(s) = \psi(s-i) + i \int_0^1 \psi(s-i-it)^2 dt.$$

The repeated application of the above identity allows us to continue $\psi(s)$ analytically to the whole complex plane.

We have already proven that for $\Im(s) \leqslant 0$ we have

$$\psi(s) = O\left(\frac{1}{|s|}\right)$$

when $\Re(s) \geqslant -H$ with an arbitrary fixed H > 0. Let us now try to obtain a similar estimate for the values of s lying in the upper half-plane.

Lemma 6 For all $\sigma > 0$ we have

$$\sup_{y \in \mathbb{R}} |\psi(\sigma + iy)| < \frac{1}{\sigma}.$$

Proof: The proof of the lemma relies on a standard trick that is used to prove that if a modulus of a characteristic function of a random variable reaches 1 at some point other than 0 then the random variable has a lattice distribution. We have

$$|\psi(\sigma+iy)| = \left| \int_0^\infty f(t) e^{2it\log t} e^{-(\sigma+it)t} \, dt \right| \leqslant \left| \int_0^\infty e^{-\sigma t} \, dt \right| \leqslant \frac{1}{\sigma},$$

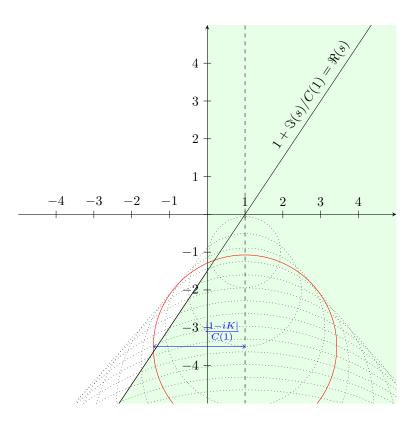


Fig. 1: The continuation of $\psi(s)$ to the left half-plane

for $\sigma > 0$. Note that the estimate 2 for fixed $\sigma > 0$ implies that $\psi(\sigma + iy) = O(1/|y|)$ as $|y| \to \infty$ which means that the supremum of $|\psi(\sigma + iy)|$ will be reached on some finite point $y_0 = y_0(\sigma)$. It remains to prove that this supremum cannot be equal to $1/\sigma$. Indeed if

$$|\psi(\sigma + iy_0)| = \frac{1}{\sigma},$$

then recalling the definition of ψ we can rewrite this identity as

$$\left| \int_0^\infty f(t)e^{2it\log t}e^{-\sigma t}e^{-iy_0t}\,dt \right| = \int_0^\infty e^{-\sigma t}\,dt$$

or equivalently

$$e^{i\theta} \int_0^\infty f(t)e^{2it\log t}e^{-\sigma t}e^{-iy_0t} dt = \int_0^\infty e^{-\sigma t} dt$$

for some real θ . Since $|f(t)| \leq 1$ taking the real part of the above equation we have

$$\Re(e^{i\theta}f(t)e^{2it\log t}e^{-iy_0t}) \equiv 1.$$

The above identity together with the fact that $\left|e^{i\theta}f(t)e^{2it\log t}e^{-iy_0t}\right|\leqslant 1$ implies that $\Im\left(e^{i\theta}f(t)e^{2it\log t}e^{-iy_0t}\right)\equiv 0$ and thus

$$e^{i\theta} f(t)e^{it\log t}e^{-iy_0t} \equiv 1.$$

Which means that

$$\psi(s) = \int_0^\infty f(t)e^{2it\log t}e^{-st} dt = e^{-i\theta} \int_0^\infty e^{-st}e^{iy_0t} dt = \frac{e^{-i\theta}}{s - iy_0}.$$

However such function does not satisfy the equation $-\psi'(s) = \psi^2(s-i)$. \square With the help of the just proven lemma we can obtain an upper bound for $\psi(s)$ in the vicinity of the imaginary line $\Im(s) = 0$.

Lemma 7 We have

$$|\psi(s)| \leqslant \frac{1-\varepsilon}{1-|\Re(s)-1|(1-\varepsilon)},$$

for s belonging to the vertical strip $-\frac{\varepsilon}{1-\varepsilon} < \Re(s) < \frac{2-\varepsilon}{1-\varepsilon}$, where ε is such that $\sup_{y \in \mathbb{R}} |\psi(1+iy)| = 1-\varepsilon$.

Proof: Applying the inequality of Lemma 6 with $\sigma = 1$ we have

$$\psi(1+iy) \leq 1-\varepsilon$$

for all $y \in \mathbb{R}$ and some fixed $\varepsilon > 0$. Hence inequality of Lemma 3 yields that

$$\psi^{(k)}(1+iy) \leqslant k!(1-\varepsilon)^{k+1} \tag{3}$$

uniformly for $y \in \mathbb{R}$. This implies that $\psi(s)$ is bounded in the vicinity of the imaginary line $\Re(s) \geqslant -\varepsilon'$ where $\varepsilon' < \varepsilon$. Indeed by Taylor expansion

$$\psi(s) = \sum_{k=0}^{\infty} \frac{\psi^{(k)}(1+iy)}{k!} (s-1-iy)^k$$

Thus

$$|\psi(s)| \leqslant \sum_{k=0}^{\infty} (1-\varepsilon)^{k+1} |s-1-iy|^k = \frac{1-\varepsilon}{1-|s-1-iy|(1-\varepsilon)}$$

for $|s-1-iy|<\frac{1}{1-\varepsilon}$. Suppose $|\Re(s)-1|<\frac{1}{1-\varepsilon}$ then taking $y=\Im(s)$ we get

$$|\psi(s)| \le \frac{1-\varepsilon}{1-|\Re(s)-1|(1-\varepsilon)},$$

for all s lying in the strip $|\Re(s) - 1| < \frac{1}{1-\epsilon}$.

A more precise estimate can be obtained combining the obtained two upper bounds for derivatives of $\psi(s)$.

Lemma 8 We have an upper bound

$$|\psi(s)| = O\left(\frac{1}{|s|}\right)$$

in the region $\Re(s) > -\frac{\varepsilon'}{1-\varepsilon'}$. Where ε' is a fixed number that $0 < \varepsilon' < \varepsilon = 1 - \sup_{y \in \mathbb{R}} |\psi(1+iy)|$, the constant in the symbol depends on ε' only.

Proof: Putting $\sigma = 1$ in our non-uniform bound (2) for $\psi(s)$ we have

$$|\psi(1+iy)| \leqslant D/|y|$$

for some fixed D > 0. Again by induction for $k \leq |y|/2$ we have

$$|\psi^{(k)}(1+iy)| \le k! \left(\frac{2D}{|y|}\right)^{k+1}.$$

Suppose $|\Re(s) - 1| < \frac{1}{1-\varepsilon'}$. Let us take $y = \Im(s)$. Combining the above upper bound with our previous uniform estimate (3) for the derivatives of $\psi^{(j)}(1+iy)$ we get

$$\begin{split} |\psi(s)| &\leqslant \sum_{k \leqslant |y|/2} \left(\frac{2D}{|y|}\right)^{k+1} |s - 1 - iy|^k + \sum_{k > |y|/2} |s - 1 - iy|^{k+1} (1 - \varepsilon)^k \\ &\leqslant \frac{2D}{|y|} \frac{1}{1 - \frac{2D}{|y|} |s - 1 - iy|} + \frac{|s - 1 - iy| \left(|s - 1 - iy|(1 - \varepsilon)\right)^{|y|/2}}{1 - |s - 1 - iy|(1 - \varepsilon)} \\ &\leqslant \frac{2D}{|y| - \frac{2D}{(1 - \varepsilon')}} + \frac{\left(\frac{1 - \varepsilon}{1 - \varepsilon'}\right)^{|y|/2}}{(1 - \varepsilon') \left(1 - \frac{1 - \varepsilon'}{1 - \varepsilon}\right)}, \end{split}$$

for $|\Re(s)-1|<\frac{1}{1-\varepsilon'}$ and $|y|>\frac{2D}{1-\varepsilon'}$. Since $\frac{1-\varepsilon}{1-\varepsilon'}<1$ we have

$$|\psi(s)| = O\left(\frac{1}{|s|}\right).$$

☐ A number of conclusions can be drawn from the estimate of the just proven lemma.

Proof of Theorem 1: The Laplace transform of $tf(t)e^{2it\log t}$ is $-\psi'(s)$ so, by inversion formula we have

$$-f(t)e^{2it\log t} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi'(s)e^{ts} ds = \frac{-1}{2\pi it} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi^2(s-i)e^{ts} ds$$

and taking into account that $|\psi(s-i)| \ll 1/|s|$ in the region $\Re(s) \geqslant -2\eta$ for some fixed $\eta > 0$ we can shift the integration line to the left and obtain

$$f(t)e^{2it\log t} = \frac{1}{2\pi it} \int_{-\eta - i\infty}^{-\eta + i\infty} \psi^2(s-i)e^{ts} \, ds \ll e^{-\eta t}.$$

Proof of Corollary 2: The density is given by formula

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-ixt} dt.$$

The fact that f(t) is exponentially decreasing $|f(t)| \ll e^{-\eta |t|}$ at infinity $|t| \to \infty$ immediately implies that the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-ist} dt.$$

is absolutely convergent in the vicinity of the real line $|\Im(s)| < \eta$ where it defines an analytic function that coincides with the density of the quicksort distribution p(x) on the real line $s = x \in \mathbb{R}$.

Corollary 9 The density function p(x) of the quicksort distribution can have only finite number of zeros in any finite interval. The same is true for the derivatives of p(x) of all orders.

Proof: Since an analytic function that is not identically equal to zero can have only finite number of zeros in any closed circle $|s-x| \le r/2$ for any $x \in \mathbb{R}$, so the density p(x) can have only finite number of zeros in any finite interval [x-r/2, x+r/2] with all $x \in \mathbb{R}$.

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