The Maximum Block Size of Critical Random Graphs[†]

Vonjy Rasendrahasina[1], Andry Rasoanaivo[2], Vlady Ravelomanana[3]

[1]*ENS – Université d'Antananarivo, Madagascar.* `rasendrahasina@gmail.com`
[2]*LIMA – Université d'Antananarivo, Madagascar.* `r.andry.rasoanaivo@gmail.com`
[3]*IRIF UMR CNRS 8243 – Université Denis Diderot, France.* `vlad@liafa.univ-paris-diderot.fr`

Abstract. Let $G(n, M)$ be the uniform random graph with n vertices and M edges. Let $\wp_{n, M}$ be the maximum block-size of $G(n, M)$ or the maximum size of its maximal 2-connected induced subgraphs. We determine the expectation of $\wp_{n, M}$ near the critical point $M = n/2$. As $n - 2M \gg n^{2/3}$, we find a constant c_1 such that

$$c_1 = \lim_{n \to \infty} \left(1 - \frac{2M}{n} \right) \mathbb{E}(\wp_{n, M}).$$

Inside the window of transition of $G(n, M)$ with $M = \frac{n}{2}(1 + \lambda n^{-1/3})$, where λ is any real number, we find an exact analytic expression for

$$c_2(\lambda) = \lim_{n \to \infty} \frac{\mathbb{E}\left(\wp_{n, \frac{n}{2}(1+\lambda n^{-1/3})} \right)}{n^{1/3}}.$$

This study relies on the symbolic method and analytic tools coming from generating function theory which enable us to describe the evolution of $n^{-1/3} \mathbb{E}\left(\wp_{n, \frac{n}{2}(1+\lambda n^{-1/3})} \right)$ as a function of λ.

Keywords: Random graph, Analytic Combinatorics, Maximum block-size

1 Introduction

Random graph theory Frieze and C. (1997); Bollobás (2001); Janson et al. (2000) is an active area of research that combines algorithmics, combinatorics, probability theory and graph theory. The uniform random graph model $G(n, M)$ studied in Erdos and Renyi (1960) consists in n vertices with M edges drawn uniformly at random from the set of $\binom{n}{2}$ possible edges. Erdős and Rényi showed that for many properties of random graphs, graphs with a number of edges slightly less than a given threshold are unlikely to have a certain property, whereas graphs with slightly more edges are almost guaranteed to satisfy the same property, showing paramount changes inside their structures (refer to as *phase transition*). As shown in their seminal paper Erdos and Renyi (1960), when $M = \frac{cn}{2}$ for constant c the largest component of $G(n, M)$ has a.a.s. $O(\log n)$, $\Theta(n^{2/3})$ or $\Theta(n)$ vertices according to whether $c < 1$, $c = 1$ or $c > 1$.

This *double-jump* phenomenon about the structures of $G(n, M)$ was one of the most spectacular results in Erdos and Renyi (1960) which later became a cornerstone of the random graph theory. Due to such a dramatic change, researchers worked around the critical value $\frac{n}{2}$ and one can distinguish three different phases: *sub-critical* when $(M - n/2)n^{-2/3} \to -\infty$, *critical* $M = n/2 + O(n^{2/3})$ and *supercritical* as $(M - n/2)n^{-2/3} \to \infty$. We refer to Bollobás Bollobás (2001) and Janson, Łuczak and Ruciński Janson et al. (2000) for books devoted to the random graphs $G(n, M)$ and $G(n, p)$. If the $G(n, p)$ model is the one more commonly used today, partly due to the independence of the edges, the $G(n, M)$ model has more enumerative flavors allowing generating functions based approaches. By setting $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, the stated results of this paper can be extended to the $G(n, p)$ model.

Previous works. In graph theory, a block is a maximal 2-connected subgraph (formal definitions are given in Section 2). The problem of estimating the maximum block size has been well studied for some class of graphs. For a graph drawn uniformly from the class of simple labeled planar graphs with n vertices, the expectation of the number of vertices in the largest block is αn asymptotically almost surely (a.a.s) where $\alpha \approx 0.95982$ Panagiotou and Steger (2010); Giménez et al. (2013). They found that the largest block in random planar graphs is related to a distribution of the exponential-cubic type, corresponding to distributions that involve the Airy function Banderier et al. (2001).

For the labeled connected class, these authors proved also independently that a connected random planar graph has a unique block of linear size.

When we restrict to sub-critical graph (graph that the block-decomposition looks tree-like), Drmota and Noy Drmota and Noy (2013) proved that the maximum block size of a random connected graph in an aperiodic[(i)] sub-critical graph class is $O(\log n)$.

For random maps (a map is a planar graph embedded in the plane), Gao and Wormald Gao and Wormald (1999) proved that a random map with n edges has almost surely $n/3$ edges. That is, the probability that the size of the largest block is about $n/3$ tends to 1 as n goes to infinity. This result is improved by Banderier *et al.* Banderier et al. (2001) by finding the density Airy distribution of the map type.

Panagiotou Panagiotou (2009) obtained more general results for any graph class $\mathcal{C}$. He showed that the size of largest block of a random graph from $\mathcal{C}$ with n vertices and m edges belongs to one of the two previous categories ($\Theta(n)$ and $O(\log n)$). In particular, the author pointed out that random planar graphs with cn edges belong to the first category, while random outerplanar and series-parallel graphs with fixed average degree belong to the second category.

For the Erdős-Rényi $G(n, M)$ model, the maximum block-size is implicitly a well-studied graph property when $M = \frac{cn}{2}$ for fixed $c < 1$. For this range, $G(n, M)$ contains only trees and unicyclic components a.a.s. Erdos and Renyi (1960). So, studying maximum block-size and the largest cycle are the same in this case. Denote by $\wp_{n, M}$ the maximum block-size of $G(n, M)$. It is shown in (Bollobás, 2001, Corollary 5.8) that as $M = \frac{cn}{2}$ for fixed $c < 1$ then $\wp_{n, M}$ is a.a.s at most ω for any function $\omega = \omega(n) \to \infty$. Pittel Pittel (1988) then obtained the limiting distribution (amongst other results) for $\wp_{n, M}$ for $c < 1$. Note that the results of Pittel are extremely precise and include other parameters of random graphs with c satisfying $c < 1 - \varepsilon$ for fixed $\varepsilon > 0$.

Our results. In this paper, we study the fine nature of the Erdős and Rényi phase transition, with emphasis on what happens as the number of edges is close to $\frac{n}{2}$: within the window of the phase transition and near to it, we quantify the maximum block-size of $G(n, M)$.

[(i)] In the periodic case, $n \equiv 1 \mod d$ for some $d > 1$ (see Drmota and Noy (2013) for more details)

For sub-critical random graphs, our finding can be stated precisely as follows :

Theorem 1 *If $n - 2M \gg n^{2/3}$, the maximum block-size $\wp_{n,M}$ of $G(n, M)$ satisfies*

$$\mathbb{E}(\wp_{n,M}) \sim c_1 \left(\frac{n}{n - 2M} \right), \tag{1}$$

where $c_1 \approx 0.378\,911$ is the constant given by

$$c_1 = \int_0^\infty \left(1 - e^{-E_1(v)} \right) dv \text{ with } E_1(x) = \frac{1}{2} \int_x^\infty e^{-t} \frac{dt}{t}. \tag{2}$$

For critical random graphs, we have the following :

Theorem 2 *Let λ be any real constant and $M = \frac{n}{2}(1 + \lambda n^{-1/3})$. The maximum block-size $\wp_{n,M}$ of $G(n, M)$ verifies :*

$$\mathbb{E}(\wp_{n,M}) \sim c_2(\lambda)\, n^{1/3}, \tag{3}$$

where

$$c_2(\lambda) = \frac{1}{\alpha} \int_0^\infty \left(1 - \sqrt{2\pi} \sum_{r \geq 0} \sum_{d \geq 0} A\left(3r + \frac{1}{2}, \lambda \right) e^{-E_1(u)}\, e_{r,d}\left(e^{-u} \right) \right) du \tag{4}$$

$E_1(x)$ *is defined in (2), α is the positive solution of*

$$\lambda = \alpha^{-1} - \alpha, \tag{5}$$

the function A is defined by

$$A(y, \lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}} \sum_{k \geq 0} \frac{\left(\frac{1}{2} 3^{2/3} \lambda \right)^k}{k!\, \Gamma\big((y + 1 - 2k)/3\big)}, \tag{6}$$

and the $(e_{r,d}(z))$ are polynomials with rational coefficients defined recursively by (22).

The accuracy of our results is of the same vein as the one on the probability of planarity of the Erdős-Rényi critical random graphs Noy et al. (2015) or on the finite size scaling for the core of large random hypergraphs Dembo and Montanari (2008) which have been also expressed in terms of the Airy function. This function has been encountered in the physics of random graphs Janson et al. (1993) and is shown in Flajolet et al. (1989) related to $A(y, \lambda)$ defined by (6) and appearing in our formula (4).

It is important to note that there is *no discontinuity* between Theorems 1 and 2. First, observe that as $M = \frac{n}{2} - \frac{\lambda(n)n^{2/3}}{2}$ with $1 \ll \lambda(n) \ll n^{1/3}$, equation (1) states that $\mathbb{E}(\wp_{n,M})$ is about $c_1 \frac{n^{1/3}}{\lambda(n)}$. Next, to see that this value matches the one from (3), we argue briefly as follows. In (5), as $\lambda(n) \to -\infty$ we have $\alpha \sim |\lambda(n)|$ and (see (Janson et al., 1993, equation (10.3)))

$$A\left(3r + \frac{1}{2}, \lambda \right) \sim \frac{1}{\sqrt{2\pi} |\lambda(n)|^{3r}}.$$

Thus, all the terms in the inner double summation 'vanish' except the one corresponding to $r = 0$ and $d = 0$ (this term is the coefficient for graphs without multicyclic components $e_{0,0}^{[k]} = 1$). It is then remarkable that as $\lambda(n) \to -\infty$, $c_2(\lambda(n))$ behaves as $\frac{c_1}{|\lambda(n)|}$.

Outline of the proofs and organization of the paper. In (Flajolet and A., 1990, Section 4), Flajolet and Odlyzko described generating functions based methods to study extremal statistics on random mappings. Random graphs are obviously harder structures but as shown in the masterful work of Janson *et al.* Janson et al. (1993), analytic combinatorics can be used to study in depth the development of the connected components of $G(n, M)$. As in Flajolet and A. (1990), we will characterize the expectation of $\wp_{n, M}$ by means of truncated generating functions.

Given a family $\mathcal{F}$ of graphs, denote by (F_n) the number of graphs of $\mathcal{F}$ with n vertices. The *exponential generating function* (EGF for short) associated to the sequence (F_n) (or family $\mathcal{F}$) is $F(z) = \sum_{n \geq 0} F_n \frac{z^n}{n!}$. Let $F^{[k]}(z)$ be the EGF of the graphs in $\mathcal{F}$ but *with all blocks of size at most k*. From the formula for the mean value of a discrete random variable X,

$$\mathbb{E}(X) = \sum_{k \geq 0} k \mathbb{P}\left[X = k\right] = \sum_{k \geq 0} \left(1 - \mathbb{P}\left[X \leq k\right]\right),$$

we get a generating function version to obtain

$$\Xi(z) = \sum_{k \geq 0} \left[F(z) - F^{[k]}(z)\right]$$

and the expectation of the maximum block-size of graphs of $\mathcal{F}$ is[ii] $\frac{n![z^n]\Xi(z)}{F_n}$. Turning back to $G(n, M)$, realizations of random graphs when M is close to $\frac{n}{2}$ contain a set of trees, some components with one cycle and complex components with 3-regular 3-cores a.a.s. In this paper, our plan is to apply this scheme above by counting realizations of $G(n, M)$ with all blocks of size less than a certain value. Once we get the forms of their generating functions, we will use complex analysis techniques to get our results.

This extended abstract is organized as follows. Section 2 starts with the enumeration of trees of given degree specification. We then show how to enumerate 2-connected graphs with 3-regular 3-cores. Combining the trees and the blocks graphs lead to the forms of the generating functions of connected graphs under certain conditions. Section 2 ends with the enumeration of complex connected components with all blocks of size less than a parameter k. Based on the previous results and by means of analytic methods, Section 3 (resp. 4) offers the proof of Theorem 1 (resp. 2).

2   Enumerative tools

Trees of given degree specification. Let $U(z)$ be the exponential generating function of labelled unrooted trees and $T(z)$ be the EGF of rooted labelled trees, it is well-known that[iii]:

$$U(z) = \sum_{n=1}^{\infty} n^{n-2} \frac{z^n}{n!} = T(z) - \frac{T(z)^2}{2} \quad \text{and} \quad T(z) = \sum_{n=1}^{\infty} n^{n-1} \frac{z^n}{n!} = z e^{T(z)}. \tag{7}$$

[ii] For any power series $A(z) = \sum a_n z^n$, $[z^n]A(z)$ denotes the n-th coefficient of $A(z)$, viz. $[z^n]A(z) = a_n$.

[iii] We refer for instance to Goulden and Jackson Goulden and Jackson (1983) for combinatorial operators, to Harary and Palmer Harary and Palmer (1973) for graphical enumeration and to Flajolet and Sedgewick Flajolet and Sedgewick (2009) for the symbolic method of generating functions.

For a tree with exactly m_i vertices of degree i, define its *degree specification* as the $(n-1)$-tuple $(m_1, m_2, \cdots, m_{n-1})$. We have the following.

Lemma 1 *The number of labeled trees with n vertices and degree specification $(m_1, m_2, \cdots m_{n-1})$ with $\sum_{i=1}^{n} m_i = n$ and $\sum_{i=1}^{n} im_i = 2n - 2$ is*

$$a_n(m_1, m_2, \ldots, m_{n-1}) = \frac{(n-2)!}{\prod_{i=1}^{n-1}((i-1)!)^{m_i}} \binom{n}{m_1, m_2, \cdots, m_{n-1}}.$$

Proof (sketched). Using Prüfer code, the number of trees with degree sequence $d_1, d_2, \cdots, d_n$ (that is with node numbered i of degree d_i) is $\frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!}$. The result is obtained by regrouping nodes of the same degree. ∎

Define the associated EGF to $a_n(m_1, m_2, \ldots, m_{n-1})$ with

$$U(\delta_1, \delta_2, \cdots; z) = \sum_{n=2}^{\infty} \sum a_n(m_1, m_2, \ldots, m_{n-1})\delta_1^{m_1} \delta_2^{m_2} \cdots \delta_{n-1}^{m_{n-1}} \frac{z^n}{n!} \qquad (8)$$

where the inner summation is taken other all i such that $\sum im_i = 2n - 2$ and $\sum m_i = n$. Define $U_n(\delta_1, \delta_2, \ldots, \delta_{n-1})$ as

$$U_n(\delta_1, \delta_2, \ldots, \delta_{n-1}) = [z^n]U(\delta_1, \delta_2, \ldots, \delta_{n-1}; z). \qquad (9)$$

The following result allows us to compute recursively $U_n(\delta_1, \cdots, \delta_{n-1})$.

Lemma 2 *The generating functions U_n defined in (9) satisfy $U_2(\delta_1) = \frac{\delta_1^2}{2}$ and for any $n \geq 3$,*

$$\begin{aligned} U_n(\delta_1, \ldots, \delta_{n-1}) &= \delta_2 U_{n-1}(\delta_1, \ldots, \delta_{n-2}) \\ &+ \sum_{i=2}^{n-2} \delta_{i+1} \int_0^{\delta_1} \frac{\partial}{\partial \delta_i} U_{n-1}(x, \delta_2, \ldots, \delta_{n-2}) dx. \end{aligned}$$

Proof. Postponed in the Appendix – 6.1.

Enumerating 2-connected graphs whose kernels are 3-regular. A *bridge* or *cut-edge* of a graph is an edge whose removal increases its number of connected components. Especially, the deletion of such an edge disconnects a connected graph. Similarly an *articulation point* or *cut-vertex* of a connected graph is a vertex whose removal disconnects a graph. A connected graph without an articulation point is called a *block* or a 2-*connected* graph.

Following the terminology of Janson et al. (1993), a connected graph has *excess* r if it has r edges more than vertices. Trees (resp. *unicycles* or *unicyclic components*) are connected components with excess $r = -1$ (resp. $r = 0$). Connected components with excess $r > 0$ are called *complex connected components*. A graph (not necessarily connected) is called *complex* when all its components are complex. The *total excess* of a graph is the number of edges plus the number of acyclic components, minus the number of vertices.

Given a graph, its 2-*core* is obtained by deleting recursively all nodes of degree 1. A *smooth* graph is a graph without vertices of degree one.

The 3-*core* (also called *kernel*) of a complex graph is the graph obtained from its 2-core by repeating the following process on any vertex of degree two : for a vertex of degree two, we can remove it and splice

together the two edges that it formerly touched. A graph is said *cubic* or 3-*regular* if all of its vertices are of degree 3. Denote by $\mathcal{B}_r$ the family of 2-connected smooth graphs of excess r with 3-regular 3-cores and let

$$\mathcal{B} = \bigcup_{r=1}^{\infty} \mathcal{B}_r .\tag{10}$$

In this paragraph, we aim to enumerate asymptotically the graphs of $\mathcal{B}_r$. In Chae et al. (2007), the authors established recurrence relations for the numbers of labeled cubic multigraphs with given connectivity, number of double edges and number of loops. For instance, they were able to rederive Wormald's result about the numbers of labeled connected simple cubic graphs with $3n$ simple edges and $2n$ vertices (Chae et al., 2007, equation (24)). They proved that the number of such objects is given by

$$\frac{(2n)!}{3n2^n} (t_n - 2t_{n-1}), \, n \geq 2\tag{11}$$

with

$$t_1 = 0, t_2 = 1 \text{ and } t_n = 3nt_{n-1} + 2t_{n-2} + (3n-1) \sum_{i=2}^{n-3} t_i t_{n-1-i}, \, n \geq 2 \,.\tag{12}$$

From the sequence (t_n), they found the number of 2-connected multigraphs.

Lemma 3 (Chae, Palmer, Robinson) *Let $g(s,d)$ be the number of cubic block (2-connected labelled) multigraphs with s single edges and d double-edges. Then, the numbers $g(s,d)$ satisfy*

$$g(s, \, d) = 0 \text{ if } s < 2 \,, g(s, \, s) = (2s-1)! \text{ and } g(3s, 0) = \frac{(2s)!}{3s2^s} (t_s - 2t_{s-1})$$

with t_s defined as in (12). In all other cases,

$$g(s, \, d) = 2n(2n-1) \left(\frac{(s-1)}{d} g(s-1, d-1) + g(s-3, d) \right) \,.$$

We are now ready to enumerate asymptotically the family $\mathcal{B}_r$. Throughout the rest of this paper if $A(z)$ and $B(z)$ are two EGFs we write

$$A(z) \asymp B(z) \text{ if and only if } [z^n]A(z) \sim [z^n]B(z) \text{ as } n \to +\infty \,.$$

Lemma 4 *For $r \geq 1$, let $B_r(z)$ be the EGF of smooth graphs of excess r whose kernels are 3-regular and 2-connected. $B_r(z)$ satisfies $B_r(z) \asymp \frac{b_r}{(1-z)^{3r}}$ where $b_1 = \frac{1}{12}$ and for $r \geq 2$*

$$b_r = \sum_{s+2d=3r} \frac{g(s, \, d)}{2^d (2r)!}\tag{13}$$

with the $g(s, \, d)$ defined as in lemma 3.

Proof. Postponed in the Appendix – 6.2.

We need to count graphs of excess r with at most k vertices so that all the blocks of such structures are of size at most k. We begin our task with the graphs with cubic and 2-connected kernels.

Lemma 5 *Let $\mathcal{B}_r^{[k]}$ be the family of 2-connected graphs of excess r, with at most $k - 2r$ vertices of degree two in their 2-cores and whose 3-cores are cubic. For any fixed $r \geq 1$, we have*

$$B_r^{[k]}(z) \asymp b_r \, \frac{1 - z^k}{(1 - z)^{3r}} \,.$$

Proof. Postponed in the Appendix – 6.3.

Let $\mathcal{B}_r^{\bullet s}$ be the set of graphs of $\mathcal{B}_r$ such that s vertices of degree two of their 2-cores are distinguished amongst the others. In other words, an element of $\mathcal{B}_r^{\bullet s}$ can be obtained from an element of $\mathcal{B}_r$ by marking (or pointing) s unordered vertices of its 2-core. In terms of generating functions, we simply get (see Harary and Palmer (1973); Goulden and Jackson (1983); Flajolet and Sedgewick (2009)) :

$$B_r^{\bullet s}(z) = \frac{z^s}{s!} \frac{\partial^s}{\partial z^s} B_r(z, t)\bigg|_{t=z} = \frac{z^s}{s!} \frac{\partial^s}{\partial z^s} \left(b_r \frac{t^{2r}}{(1 - z)^{3r}} \right)\bigg|_{t=z}, \tag{14}$$

where $B_r(z, t)$ is the bivariate EGF of $\mathcal{B}_r$ with t the variable for the vertices of degree 3. (The substitution $t = z$ is made after the derivations.)

Define

$$b_r^{\bullet s} = \frac{1}{s!} b_r \prod_{i=1}^{s} [3r + (s - i)]$$

so that $B_r^{\bullet s}(z) \asymp \frac{b_r^{\bullet s}}{(1-z)^{3r+s}}$. Now if we switch to the class of graphs with blocks of size at most k then by similar arguments, the asymptotic number of graphs of $\mathcal{B}_r^{\bullet s}$ with s distinguished vertices and at most k vertices on their 2-cores behaves as

$$B_r^{\bullet s,\,[k]}(z) \asymp b_r^{\bullet s} \frac{1 - z^k}{(1 - z)^{3r+s}} \,.$$

Counting 2-cores with cubic kernels by number of bridges. In this paragraph, we aim to enumerate connected smooth graphs whose 3-cores are 3-regular according to their number of bridges (or cut-edges) and their excess. To that purpose, let $\mathcal{C}_r$ be the family of such graphs with excess $r \geq 0$, and for any $d \geq 0$ let

$$\mathcal{C}_{r,d} \overset{\text{def}}{=} \{G \in \mathcal{C}_r : G \text{ is a cycle or its } 3\text{-core is } 3\text{-regular and has } d \text{ bridges}\} \,.$$

Clearly, we have $\mathcal{C}_{r,0} = \mathcal{B}_r$. If we want to mark the excess of these graphs by the variable w, we simply have

$$C_{r,d}(w, z) = w^r C_{r,d}(z) \,.$$

Lemma 6 *For any $r \geq 1$ and $d \geq 1$,*

$$
\begin{aligned}
C_{r,d}(z) &= [w^r] U_{d+1}\bigg(B^{\bullet 1}(w, z),\, 2! B^{\bullet 2}(w, z),\, 3! B^{\bullet 3}(w, z) + w^{-1} z, \\
&\qquad\qquad 4! B^{\bullet 4}(w, z),\, \ldots,\, d! B^{\bullet d}(w, z) \bigg) \frac{w^d}{(1 - z)^d},
\end{aligned}
$$

where U_{d+1} are the EGF given by lemma 2, $B_0(w,z) = -\frac{1}{2}\log(1-z) - z/2 - z^2/4$, $B_0^{\bullet s}(w,z) = \frac{1}{s!}\frac{\partial^s}{\partial z^s} B_0(w,z)$ and $B^{\bullet s}(w,z) = \sum_{r\geq 0} w^r B_r^{\bullet s}(z)$.

Proof. Postponed in the Appendix – 6.4

Lemma 7 *For $r \geq 1$ and $d \geq 1$, we have*

$$C_{r,d}(z) \asymp \frac{c_{r,d}}{(1-z)^{3r}}$$

where the coefficients $c_{r,d}$ are defined by

$$c_{r,d} = [w^r]U_{d+1}\left(\beta_1(w), \beta_2(w), \beta_3(w) + w^{-1}, \beta_4(w), \ldots, \beta_d(w)\right)w^d,$$

with b_ℓ given by (13) and

$$\beta_s(w) = \frac{(s-1)!}{2} + \sum_{\ell=1}^{r-1} w^\ell b_\ell \prod_{i=1}^{s}[3\ell + (s-i)] \quad \text{with } s \geq 1.$$

Proof. Postponed in the Appendix – 6.5.

Let us restrict our attention to elements of $\mathcal{C}_{r,d}$ with blocks of size at most k. Denote by $\mathcal{C}_{r,d}^{[k]}$ this set of graphs. Since they can be obtained from a tree with $d+1$ vertices by replacing each vertex of degree s by a s-marked block (block with a distinguished degree of degree two) of the family $\bigcup_{r=0}^{\infty} \mathcal{B}^{\bullet s,[k]}$, we infer the following :

Lemma 8 *For fixed values of r, the EGF of graphs of $\mathcal{C}_{r,d}^{[k]}$ verifies*

$$C_{r,d}^{[k]} \asymp c_{r,d}\frac{(1-z^k)^{d+1}}{(1-z)^{3r}}.$$

From connected components to complex components. Denote by $\mathcal{E}_r^{[k]}$ the family of complex graphs (not necessarily connected) of total excess r with all blocks of sized $\leq k$. Let $E_r^{[k]}$ be the EGF of $\mathcal{E}_r^{[k]}$. Using the symbolic method and sprouting the rooted trees from the smooth graphs counted by $C_{r,d}^{[k]}(z)$, we get

$$\sum_{r=0}^{\infty} w^r E_r^{[k]}(z) = \exp\left(\sum_{r=1}^{\infty} w^r \sum_{d\geq 0}^{2r-1} C_{r,d}^{[k]}\left(T(z)\right)\right).$$

We now use a general scheme which relates behavior of connected components and complex components (see for instance (Janson et al., 1993, Section 8)). If $E(w,z) = 1 + \sum_{r\geq 1} w^r E_r(z)$ with $E_r(z) \asymp \frac{e_r}{(1-T(z))^{3r}}$ and $C_r(z) \asymp \frac{c_r}{(1-T(z))^{3r}}$ are EGFs satisfying

$$1 + \sum_{r\geq 1} w^r E_r(z) = \exp\left(\sum_{r=1}^{\infty} w^r C_r(z)\right).$$

then the coefficients (e_r) and (c_r) are related by

$$e_0 = 1 \text{ and } e_r = c_r + \frac{1}{r}\sum_{j=1}^{r-1} jc_j e_{r-j} \text{ as } r \geq 1.$$

Similarly, after some algebra we get

Lemma 9 *For fixed $r \geq 1$,*

$$E_r^{[k]}(z) \asymp \sum_{d=0}^{2r-1} \frac{e_{r,d}^{[k]}(T(z))}{(1-T(z))^{3r}}$$

where the functions $(e_{r,d}^{[k]})$ are defined recursively by $e_{0,0}^{[k]}(z) = 1$, $e_{r,d}^{[k]}(z) = 0$ if $d > 2r - 1$ and

$$e_{r,d}^{[k]}(z) = c_{r,d} \left(1 - z^k\right)^{d+1} + \frac{1}{r} \sum_{j=1}^{r-1} jc_{j,d} \, e_{r-j,d}^{[k]}(z) \left(1 - z^k\right)^{d+1} . \tag{15}$$

Remark. Note that the maximal range $2r - 1$ of d appears when the 2-core is a cacti graph (each edge lies on a path or on a unique cycle), each cycle have exactly one vertex of degree three and its 3-core is 3-regular.

3   Proof of Theorem 1

Following the work of Flajolet and Odlyzko Flajolet and A. (1990) on extremal statistics of random mappings, let us introduce the relevant EGF for the expectation of the maximum block-size in $G(n, M)$.

On the one hand, if there are n vertices, M edges and with a total excess r there must be exactly $n - M + r$ acyclic components. Thus, the number of (n, M)-graphs[iv] of total excess r without blocks of size larger than k is

$$n![z^n] \frac{U(z)^{n-M+r}}{(n-M+r)!} \left(e^{W_0(z) - \sum_{i=k+1}^{\infty} \frac{T(z)^i}{2i}}\right) E_r^{[k]}(z) .$$

where $W_0(z) = -\frac{1}{2}\log(1 - T(z)) - \frac{T(z)}{2} - \frac{T(z)^2}{4}$ is the EGF of connected graphs of excess $r = 0$ (see (Janson et al., 1993, equation (3.5))).

On the other hand, the EGF of all (n, M)-graphs is

$$G_{n,M}(z) = \sum_{n \geq 0} \binom{\binom{n}{2}}{M} \frac{z^n}{n!} .$$

Define

$$\Xi(z) = \sum_{k \geq 0} \left[G_{n,M}(z) - \sum_{n \geq 0} \left(n![z^n] \frac{U(z)^{n-M+r}}{(n-M+r)!} \left(e^{W_0(z) - \sum_{i=k+1}^{\infty} \frac{T(z)^i}{2i}}\right) E_r^{[k]}(z) \right) \frac{z^n}{n!} \right] , \tag{16}$$

so that

$$\frac{n![z^n]}{\binom{\binom{n}{2}}{M}} \Xi(z) = \sum_{k \geq 0} \left[1 - \frac{n!}{\binom{\binom{n}{2}}{M}} [z^n] \frac{U(z)^{n-M+r}}{(n-M+r)!} \left(e^{W_0(z) - \sum_{i=k+1}^{\infty} \frac{T(z)^i}{2i}}\right) E_r^{[k]}(z) \right] , \tag{17}$$

[iv] Graph with n vertices and M edges

is the expectation of $\wp_{n,M}$.

We know from the theory of random graphs that in the sub-critical phase when $n - 2M \gg n^{2/3}$ $G(n, M)$ has no complex components with probability $1 - O\left(\frac{n^2}{(n-2M)^3}\right)$ (cf (Daudé and Ravelomanana, 2009, Theorem 3.2)). In this abstract, we restrict our attention to the typical random graphs. Otherwise, we will obtain the same result as stated by bounds on the $E_r^{[k]}(z)$ in (16) since

$$1 \leq E_r^{[k]}(z) \leq E_r(z) \leq \frac{e_r T(z)}{(1 - T(z))^{3r}}$$

(where inequality between the EGFs means that the coefficients of every power of z obeys the same relation and the last inequality is (Janson et al., 1993, equation (15.2)) with $e_r = \frac{(6r)!}{2^{5r} 3^{2r} (3r)! (2r)!}$). Assuming that the graphs are typical (i.e. without complex components), $\Xi(z)$ behaves as

$$\Xi(z) \asymp \sum_{k \geq 0} \left[G_{n,M}(z) - \sum_{n \geq 0} \left(n![z^n] \frac{U(z)^{n-M}}{(n-M)!} \frac{e^{-\frac{T(z)}{2} - \frac{T(z)^2}{4}}}{(1-T(z))^{1/2}} \exp\left(-\sum_{j \geq k+1} \frac{T(z)^j}{2j} \right) \right) \frac{z^n}{n!} \right] \quad (18)$$

We need the following Lemma to quantify large coefficients of (18).

Lemma 10 *Let a and b be any fixed rational numbers. For any sequence of integers $M(n)$ such that $\delta n < M$ for some $\delta \in \left[0, \frac{1}{2}\right]$ but $n - 2M \gg n^{2/3}$, define*

$$f_{a,b}(n, M) = \frac{n!}{\binom{\binom{n}{2}}{M}} [z^n] \frac{U(z)^{n-M}}{(n-M)!} \frac{U(z)^b e^{-T(z)/2 - T(z)^2/4}}{(1-T(z))^a} .$$

We have

$$f_{a,b}(n, M) \sim 2^b \left(\frac{M}{n} \right)^b \left(1 - \frac{M}{n} \right)^b \left(1 - \frac{2M}{n} \right)^{1/2-a} .$$

Proof. Postponed in the Appendix – 6.6.

Using Lemma 10 with $a = 1/2$ and $b = 0$, after a bit of algebra (change of variable $u = T(z)$ and approximating the sum by an integral), we first obtain

$$\mathbb{E}(\wp_{n,M}) \sim \sum_{k \geq 0} \left(1 - \exp\left(-\frac{1}{2} \int_{(k+1)(1-\frac{2M}{n})}^{\infty} e^{-v} \frac{dv}{v} \right) \right) .$$

Then by Euler-Maclaurin summation formula and after a change of variable $((k+1)(1 - \frac{2M}{n}) = u$ so $dk = (1 - \frac{2M}{n})^{-1} du)$, we get the result.

4 Proof of Theorem 2

The following technical result is essentially (Janson et al., 1993, Lemma 3). We give it here in a modified version tailored to our needs (namely involving truncated series). We refer also to the proof of (Flajolet et al., 1989, Theorem 5) and Banderier et al. (2001) for integrals related to the Airy function.

Lemma 11 *Let $M = \frac{n}{2}\left(1 + \lambda n^{-1/3}\right)$. Then for any natural integers a, k and r we have*

$$
\frac{n!}{\binom{\binom{n}{2}}{M}} [z^n] \frac{U(z)^{n-M+r}}{(n-M+r)!} \frac{T(z)^a \left(1 - T(z)^k\right)}{(1 - T(z))^{3r}} \exp\left(W_0(z) - \sum_{i=k}^{\infty} \frac{T(z)^i}{2i}\right)
$$

$$
= \sqrt{2\pi} \exp\left(-\sum_{j=k}^{\infty} \frac{e^{-j\alpha n^{-1/3}}}{2j}\right) \left(1 - e^{-k\alpha n^{-1/3}}\right) A\left(3r + \frac{1}{2}, \lambda\right) \left(1 + O\left(\frac{\lambda^4}{n^{1/3}}\right)\right),
\tag{19}
$$

uniformly for $|\lambda| \leq n^{1/12}$ where $A(y, \mu)$ is defined by (6) and α is given by (5).

Proof. Postponed in the Appendix – 6.7.

Using this lemma, equation (17) and next approximating a sum by an integral using Euler-Maclaurin summation, the expectation of $\wp_{n, M}$ is about

$$
\sum_{k=0}^{n} \left(1 - \sum_r \sum_d \sqrt{2\pi} \exp\left(-\sum_{j=k}^{\infty} \frac{e^{-j\alpha n^{-1/3}}}{2j}\right) e_{r,d}^{[k]}\left(e^{-k\alpha n^{-1/3}}\right) A\left(3r + \frac{1}{2}, \lambda\right)\right)
\tag{20}
$$

$$
= \alpha^{-1} n^{1/3} \int_0^{\alpha n^{2/3}} \left(1 - \sum_r \sum_d \sqrt{2\pi} \exp\left(-\int_u^{\infty} \frac{e^{-v}}{2v} dv\right) e_{r,d}\left(e^{-u}\right) A\left(3r + \frac{1}{2}, \lambda\right)\right) du
\tag{21}
$$

where

$$
e_{r,d}(z) = c_{r,d}(1 - z)^{d+1} + \frac{1}{r} \sum_{j=1}^{r-1} j c_{j,d}\, e_{r-j,d}(z)\, (1 - z)^{d+1} .
\tag{22}
$$

5 Conclusion

We have shown that the generating function approach is well suited to make precise the expectation of maximum block-size of random graphs. Our analysis is a first step towards a fine description of the various graph parameters inside the window of transition of random graphs.

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6 Appendix

6.1 Proof of Lemma 2

The case $n = 2$ is immediate. Let $\mathcal{U}_n$ be the family of trees of size n and $\mathcal{U}_n^\bullet$ be the family of rooted trees of size n whose roots are of degree 1. Deleting the root of the latter trees gives unrooted trees of size $n - 1$. Conversely, an element of $\mathcal{U}_n^\bullet$ can be obtained from an element of $\mathcal{U}_{n-1}$, by choosing any vertex and by attaching to this vertex a new vertex which is the root of the newly obtained tree. In terms of EGF, we have :

$$U_n^\bullet(\delta_1, \ldots, \delta_{n-1}) = \sum_{i=1}^{n-2} \delta_1 \delta_{i+1} \frac{\partial}{\partial \delta_i} U_{n-1}(\delta_1, \ldots, \delta_{n-2}).$$

The combinatorial operator that consists to choose a vertex of degree i and add the root is $\delta_1 \delta_i \frac{\partial}{\partial \delta_i}$. The multiplication by the terms $\delta_{i+1}\delta_i^{-1}$ reflects the fact that we have a vertex of degree i that becomes a vertex of degree $i + 1$ after the addition of the new vertex of degree 1 (thus the term δ_1). Next, we have to unmark the root which is by construction of degree 1. After a bit of algebra, we obtain the result. ∎

6.2 Proof of Lemma 4

The numbers $g(s, d)$ count labeled cubic multigraphs. If $s + 2d = 3r$, these multigraphs are exactly the 3-cores of the graphs of the family $\mathcal{B}_r$. Starting from the EGF $g(s, d)\frac{w^{3r}z^{2r}}{(2r)!}$ – with the variable w (resp. z) marking the edges (resp. vertices) – if we want to reconstruct from these multigraphs the graphs of the family $\mathcal{B}_r$ each edge w of these multigraphs is substituted by a sequence of vertices of degree 2 introducing the term $\frac{1}{(1-z)}$ (for each of the $3r$ edges of the multigraphs). Next, we have to compensate the symmetry of each double-edge introducing d times the factor $\frac{1}{2!}$. ∎

6.3 Proof of Lemma 5

The 3-cores of the graphs of $\mathcal{B}_r$ have as bivariate EGF $b_r w^{3r} t^{2r}$ (with w the variable for the edges and t for the vertices of degree 3). In order to reconstruct the 2-cores of $\mathcal{B}_r^{[k]}$, we insert at most $k - 2r$ vertices on each of the $3r$ paths between the vertices of degree 3. Hence,

$$
\begin{aligned}
b_r \sum_{i=0}^{k-2r} \binom{3r + i - 1}{i} z^i t^{2r} &= b_r \sum_{i=0}^{k-2r} \frac{(3r + i - 1)(3r + i - 2)\cdots(i+1)}{(3r-1)!} z^i t^{2r} \\
&\asymp b_r \frac{1 - z^{k+1-2r}}{(1-z)^{3r}} t^{2r} \asymp b_r \frac{1 - z^k}{(1-z)^{3r}} t^{2r}
\end{aligned}
$$

∎

6.4 Proof of Lemma 6

Any element of the family $\mathcal{C}_{rd}$ can be obtained from a tree with $d+1$ vertices as follows. Consider a tree $\mathcal{T}$ of size $d+1$. For each vertex v of $\mathcal{T}$ of degree s, we can substitute v by an element of $\mathcal{B}^{\bullet s}$ in $s!$ manners. We distinguish two cases according to the degree of v : vertices of degree 3 can be left unchanged or substituted by elements of $\mathcal{B}^{\bullet 3}$. Thus, the term $3!B^{\bullet s}(w, z) + w^{-1}z$ in (15). Next, each edge of $\mathcal{T}$ can be substituted by a path of length at least 1 with a factor w which parametrizes the excess of the obtained graph. Thus, the factor $\frac{w^d}{(1-z)^d}$. ■

6.5 Proof of Lemma 7

Applying the operator of $\frac{z^s}{s!}\frac{\partial^s}{\partial z^s}$ on unicyclic components gives $b_0^{\bullet s} = \frac{1}{s!}\frac{(s-1)!}{2}$. Define the ordinary generating function of $(b_\ell^{\bullet s})_{\ell \geq 0}$ as

$$b^{\bullet s}(w) = \sum_{\ell=0}^{\infty} b_\ell^{\bullet s} w^\ell = \frac{1}{s!}\left(\frac{(s-1)!}{2} + \sum_{\ell=1}^{\infty} b_\ell \prod_{i=1}^{s}[3\ell + (s-i)]\, w^\ell\right). \tag{23}$$

After a bit of algebra, we get

$$c_{r,d} = [w^r]U_{d+1}\left(b^{\bullet 1}(w), 2!b^{\bullet 2}(w), 3!b^{\bullet 3}(w) + w^{-1}, 4!b^{\bullet 4}(w), \ldots, d!b^{\bullet d}(w)\right)w^d. \tag{24}$$

Observe that for any $d \geq 1$, each involved block to obtain an element of $\mathcal{C}_{r,d}$ is necessarily of excess at most $r - 1$. So, the summation in (23) can be truncated to $r - 1$. ■

6.6 Proof of Lemma 10

We split the formula in two parts : $f_{a,b}(m, n) = St(m, n) \cdot Ca(m, n)$ with

$$St(m, n) = \frac{n!}{\binom{\binom{n}{2}}{m}(n-m)!} \quad \text{and} \quad Ca(m, n) = [z^n]\frac{U(z)^{n-m}}{(n-m)!}\frac{U(z)^b\, e^{-t(z)/2 - t(z)^2/4}}{(1-T(z))^a}.$$

Using Stirling's formula, we have for the stated range of m

$$\frac{n!m!}{(n-m)!} = \sqrt{2\pi}\frac{n^{n+1/2}m^{m+1/2}}{(n-m)^{n-m+1/2}}e^{-2m}\left(1 + O\left(\frac{1}{n}\right)\right).$$

We also have

$$\binom{\binom{n}{2}}{m} = \frac{n^{2m}}{2^m m!}\exp\left(-\frac{m}{n} - \frac{m^2}{n^2} + O\left(\frac{m}{n^2}\right) + O\left(\frac{m^3}{n^4}\right)\right).$$

Next, we get

$$St(m, n) = \left(\frac{2\pi nm}{n-m}\right)^{1/2}\frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}}\exp\left(-2m + \frac{m}{n} + \frac{m^2}{n^2}\right)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{25}$$

For $Ca(m, n)$, in using Cauchy integral's formula and substituting z by ze^{-z}, we obtain :

$$Ca(m, n) = \frac{2^{m-n}}{2\pi i} \oint \left(2T(z) - T(z)^2\right)^{n-m} \frac{U(z)^b \, e^{-T(z)/2 - T(z)^2/4}}{(1 - T(z))^a} \frac{dz}{z^{n+1}} \tag{26}$$

$$= \frac{2^{m-n}}{2\pi i} \oint g(z) e^{nh(z)} \frac{dz}{z} \tag{27}$$

where

$$g(z) = \frac{(z - z^2/2)^b \, e^{-z/2 - z^2/4}}{(1 - z)^{a-1}},$$

$$h(z) = z - \frac{m}{n} \log z + \left(1 - \frac{m}{n}\right) \log(2 - z).$$

$h'(z) = 0$ for $z = 1$ or $z = 2m/n$. $h''(1) = 2m/n - 1 < 0$ and $h''(2m/n) = \frac{n(n-2m)}{4m(n-m)} > 0$. As in Flajolet et al. (1989), we can apply the saddle-point method integrating around a circular path $|z| = 2m/n$. Let $\Phi(\theta)$ be the real part of $h(2m/ne^{i\theta})$. We have

$$\Phi(\theta) = 2\frac{m}{n} \cos\theta + \left(1 - 2\frac{m}{n}\right) \log 2 - \frac{m}{n} \log\left(\frac{m}{n}\right) + \frac{\left(1 - \frac{m}{n}\right)}{2} \log\left(1 + \frac{m^2}{n^2} - 2\frac{m}{n} \cos\theta\right)$$

and

$$\Phi'(\theta) = -2\frac{m}{n} \sin\theta + \frac{(1 - m/n)m}{n(1 + m^2/n^2 - 2m/n\cos\theta)} \sin\theta.$$

We note that $\Phi(\theta)$ is a symmetric function of θ. Fix sufficiently small positive constant θ_0. Then, $\Phi(\theta)$ takes its maximum value at $\theta = \theta_0$ as $\theta \in [-\pi, -\theta_0] \cup [\theta_0, \pi]$. In fact,

$$\Phi(0) - \Phi(\pi) = 4\frac{m}{n} + \left(1 - \frac{m}{n}\right) \log\left(\frac{n-m}{n+m}\right) + O(\theta^2).$$

Therefore, if $\theta \to 0$ $\Phi(0) > \Phi(\pi)$. Also, $\Phi'(\theta) = 0$ for $\theta = 0$ and $\theta = \theta_1$ (for some $\theta_1 > 0$). Standard calculus show that $\Phi(\theta)$ is decreasing from 0 to θ_1 and then increasing from θ_1 to π. We also have

$$h^{(p)}(z) = (p-1)! \left((-1)^p \frac{m}{n \, z^p} - \frac{(n-m)}{n(2-z)^p}\right), \quad p \geq 2.$$

Hence,

$$h(2me^{i\theta}/n) = h(2m/n) + \sum_{p \geq 2} \xi_p(e^{i\theta} - 1)^p,$$

where $\xi_p = \frac{(2m/n)^p}{p!} h^{(p)}(2m/n)$ and $|\xi_p| \leq \frac{m}{np}\left(\frac{2m}{n}\right)^p + \frac{n-m}{np}$. We then have

$$|\sum_{p \geq 4} \xi_p(e^{i\theta} - 1)^p| = O(\theta^4).$$

This allows us to write

$$h(2m/ne^{i\theta}) = h(2m/n) - \frac{m(n-2m)}{2\,n(n-m)}\theta^2 - i\frac{(n^2-5nm+2m^2)m}{6n(n-m)^2}\theta^3 + O(\theta^4).$$

Let $\tau = n(n-m)/\left(m(n-2m)\right)$ and

$$\theta_0 = \left(\frac{(n-m)}{(n-2m)m}\right)^{1/2} \cdot \omega(n) = \sqrt{\frac{\tau}{n}} \cdot \omega(n)$$

where we need a function $\omega(n)$ satisfying $n\theta_0^2 \gg 1$ but $n\theta_0^3 \ll 1$ as $n \to \infty$. We choose

$$\omega(n) = \frac{(n-2m)^{1/4}}{n^{1/6}}. \tag{28}$$

We can now use the magnitude of the integrand at θ_0 to bound the error and our choice of θ_0 verifies

$$|g(2m/ne^{i\theta_0})\left(\exp\left(nh(2m/ne^{i\theta_0})\right) - \exp\left(nh(2m/n)\right)\right)| = O\left(e^{-\omega(n)^2/2}\right).$$

Thus, we obtain

$$Ca(m,n) = \frac{2^{m-n}}{2\pi} \int_{-\theta_0}^{\theta_0} g\left(2\frac{m}{n}e^{i\theta}\right)\exp\left(nh(2m/ne^{i\theta})\right)d\theta \times \left(1 + O\left(e^{-\omega(n)^2/2}\right)\right).$$

We replace θ by $\frac{\tau^{1/2}}{n^{1/2}}t$. The integral in the above equation leads to

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(\frac{2m}{n}\exp\left(it\sqrt{\tau/n}\right)\right)\exp\left(nh\left(\frac{2m}{n}\exp\left(it\sqrt{\tau/n}\right)\right)\right)dt.$$

Expanding $g(2m/ne^{it\sqrt{\tau/n}})$, we obtain

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(2m/n\right)\left(1 - i\frac{2m\tau^{1/2}(n^2-2m^2)}{n^{5/2}(n-2m)}t + O\left(\frac{n^2}{(n-2m)^3}t^2\right)\right)$$
$$\times \quad \exp\left(nh\left(\frac{2m}{n}\exp\left(it\sqrt{\tau/n}\right)\right)\right)dt.$$

Observe that our choice of $\omega(n)$ in (28) and the hypothesis $n - 2m \gg n^{2/3}$ justify such an expansion. Similarly, using the expansion of $h(2m/ne^{it\sqrt{\tau/n}})$ yields

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(2m/n\right)\left(1 - i\frac{2m\tau^{1/2}(n^2-2m^2)}{n^{5/2}(n-2m)}t + O\left(\frac{n^2}{(n-2m)^3}t^2\right)\right)$$
$$\times \quad \exp\left(nh\left(\frac{2m}{n}\right) - \frac{1}{2}t^2\right)$$
$$\times \quad \left(1 - i\frac{(n^2-5nm+2m^2)}{6(n-m)^{1/2}m^{1/2}(n-2m)^{3/2}}t^3 + O\left(\frac{n}{(n-2m)^2}t^4\right)\right)dt.$$

Using the symmetry of the function, we can cancel terms such as it and it^3 (in fact all odd powers of t). Standard calculations show also that for m in the stated ranges, the multiplication of the factors of it and it^3 leads to a term of order of magnitude $O(n^2/(n-2m)^3 t^4)$. Therefore we obtain,

$$
Ca(m,n) = \frac{2^{m-n}}{2\pi} \left(\frac{\tau}{n}\right)^{1/2} g(2m/n) \, e^{nh(2m/n)} \int_{-\omega(n)}^{\omega(n)} e^{-t^2/2} \left(1 - O\left(\frac{n^2}{(n-2m)^3} t^4\right)\right) dt
$$

$$
Ca(m,n) = 2^{m-n} \left(\frac{\tau}{2\pi n}\right)^{1/2} g(2m/n) \, e^{nh(2m/n)} \left(1 - e^{-O(\omega(n)^2)} - O\left(\frac{n^2}{(n-2m)^3}\right)\right).
$$

$$(29)$$

Multiplying (25) and (29) leads to the result after nice cancellations. (Note that the error terms $e^{-O(\omega(n)^2)}$ and $O(1/n)$ can be regrouped with the $O(n^2(n-2m)^{-3})$.) ∎

6.7 Proof of Lemma 11

Proof. Using Stirling's formula, we get

$$
\begin{aligned}
St(M,n) &= \frac{n!}{\binom{\binom{n}{2}}{M}} \frac{1}{(n-M+r)!} \\
&= \sqrt{2\pi n} \, \frac{2^{n-M+r}}{n^r} \exp\left(-\frac{\lambda^3}{6} + \frac{3}{4} - n\right) \\
&\quad \times \left(1 + O\left(\frac{\lambda^4}{n^{1/3}}\right)\right).
\end{aligned}
$$

$$(30)$$

Using Cauchy integral's formula and substituting z by ze^{-z}, we obtain :

$$
\begin{aligned}
Ca(M,n) &= [z^n] \, U(z)^{n-M+r} \frac{T(z)^a (1-T(z)^k)}{(1-T(z))^{3r}} e^{(V(z)-\sum_{j=k}^{\infty} \frac{T(z)^j}{2j})} \\
&= \frac{1}{2\pi i} \oint \left(T(z) - \frac{T(z)^2}{2}\right)^{n-M+r} \frac{T(z)^a \, e^{-T(z)/2 - T(z)^2/4 - \sum_{j=k}^{\infty} T(z)^j/2j}}{(1-T(z))^{3r+1/2}} \frac{dz}{z^{n+1}} \\
&= \frac{2^{M-n-r} e^n}{2\pi i} \oint g(u) \exp(nh(u)) \frac{du}{u},
\end{aligned}
$$

$$(31)$$

where the integrand has been splitted into

$$
g(u) = \frac{u^a (2u - u^2)^r \, e^{-u/2 - u^2/4 - \sum_{j=k}^{\infty} u^j/2j} (1-u^k)}{(1-u)^{3r-1/2}}
$$

and

$$
h(u) = u - 1 - \log u - \left(1 - \frac{M}{n}\right) \log \frac{1}{1-(u-1)^2}.
$$

The contour in (31) should keep $|u| < 1$. Precisely at the critical value $M = \frac{n}{2}$ we also have $h(1) = h'(1) = h''(1) = 0$. This triple zero accounts in the procedure Janson, Knuth, Łuczak and Pittel used

when investigating the value of the integral for large n. Let $\nu = n^{-1/3}$, and let α be the positive solution of (5). Following the proof of (Janson et al., 1993, Lemma 3), we will evaluate (31) on the path $z = e^{-(\alpha+it)\nu}$, where t runs from $-\pi n^{1/3}$ to $\pi n^{1/3}$:

$$\oint f(z)\,\frac{dz}{z} = i\nu \int_{-\pi n^{1/3}}^{\pi n^{1/3}} f(e^{-(\alpha+it)\nu})\,dt\,.$$

The main contribution to the value of this integral comes from the vicinity of $t = 0$. The magnitude of $e^{h(z)}$ depends on the real part of $h(z)$, viz. $\Re h(z)$. $\Re h(e^{-(\alpha+it)\nu})$ decreases as $|t|$ increases and $|e^{nh(z)}|$ has its maximum on the circle $z = e^{-(\alpha+it)\nu}$ when $t = 0$.

We have $nh(e^{-s\nu})$

$$n\,h(e^{-s\nu}) = \tfrac{1}{3}s^3 + \tfrac{1}{2}\lambda s^2 + O\big((\lambda^2 s^2 + s^4)\nu\big)\,,$$

uniformly in any region such that $|s\nu| < \log 2$. In (Janson et al., 1993, equation (10.7)), the authors define

$$A(y,\mu) = \frac{1}{2\pi i}\int_{\Pi(1)} s^{1-y} e^{K(\mu,s)}\,ds\,,$$

where $K(\mu, s)$ is the polynomial

$$K(\mu, s) = \frac{(s+\mu)^2(2s-\mu)}{6} = \frac{s^3}{3} + \frac{\mu s^2}{2} - \frac{\mu^3}{6}$$

and $\Pi(\alpha)$ is a path in the complex plane that consists of the following three straight line segments:

$$s(t) = \begin{cases} -e^{-\pi i/3}t, & \text{for} -\infty < t \le -2\alpha; \\ \alpha + it\sin\pi/3, & \text{for} -2\alpha \le t \le +2\alpha; \\ e^{+\pi i/3}t, & \text{for} +2\alpha \le t < +\infty\,. \end{cases}$$

In particular, they proved that $A(y, \mu)$ can be expressed as (6).
For the function $g(u)$, we have

$$\begin{aligned} g(e^{-s\nu}) &= \frac{\big(2e^{-s\nu} - e^{-2s\nu}\big)^r}{(1 - e^{-s\nu})^{3r-1/2}}\, e^{-as\nu - e^{-s\nu}/2 - e^{-2s\nu}/4 - \sum_{j=k}^{\infty} e^{\frac{-js\nu}{2j}}}\,(1 - e^{-ks\nu}) \\ &= (s\nu)^{1/2-3r} e^{-3/4 - \sum_{j=k}^{\infty} e^{-js\nu}/2j}\,(1 - e^{-ks\nu})\,(1 + O(s\nu))\,. \end{aligned}$$

For $g(u)e^{nh(u)}$ in the integrand of (31), we have

$$\begin{aligned} e^{-\lambda^3/6} f(e^{-s\nu}) &= e^{-3/4 - \sum_{j=k}^{\infty} e^{-js\nu}/2j} \nu^{1/2-3r}\,(1 - e^{-ks\nu})\,, s^{1-(3r+1/2)} e^{K(\lambda,\,s)} \\ &\times\ (1 + O(s\nu) + O(\lambda^2 s^2 \nu) + O(s^4 \nu)) \end{aligned}$$

when $s = O(n^{1/12})$. Finally,

$$\frac{e^{-\lambda^3/6}}{2\pi i}\oint g(u)e^{nh(u)}\,\frac{du}{u} = \exp\left(-3/4 - \sum_{j=k}^{\infty} e^{-j\alpha\nu}/2j\right)(1 - e^{-k\alpha\nu})$$

$$\times \quad \nu^{3/2-3r} A(3r + \frac{1}{2}, \lambda) + O\left(\nu^{5/2-3r} e^{-\lambda^3/6} \lambda^{3r/2+1/4}\right)$$

where the error term has been derived from those already in Janson et al. (1993). The proof of the lemma is completed by multiplying (30) and (31). ∎