

Bootstrap Percolation on Periodic Trees

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Abstract

We study bootstrap percolation with the threshold parameter $\theta \geq 2$ and the initial probability p on infinite periodic trees that are defined as follows. Each node of a tree has degree selected from a finite predefined set of non-negative integers, and starting from a given node, called root, all nodes at the same graph distance from the root have the same degree. We show the existence of the critical threshold $p_f(\theta) \in (0, 1)$ such that with high probability, (i) if $p > p_f(\theta)$ then the periodic tree becomes fully active, while (ii) if $p < p_f(\theta)$ then a periodic tree does not become fully active. We also derive a system of recurrence equations for the critical threshold $p_f(\theta)$ and compute these numerically for a collection of periodic trees and various values of θ , thus extending previous results for regular (homogeneous) trees.

1 Introduction

Bootstrap percolation is a dynamic growth model generalizing cellular automata from square grids to arbitrary graphs. Starting from a random distribution of some feature over the nodes of a network (often infinite), new nodes iteratively may acquire the feature based on the density of nodes possessing it in their immediate neighborhoods. The goal is to determine under what conditions the feature spreads over almost all the nodes. In particular, there may exist a probability p_f , the percolation threshold, which characterizes the initial distribution of the said feature necessary to cause this contagious effect. Clearly such a threshold will depend on the structure of the network and the local activation rule characterized by a parameter $\theta \geq 2$ which determines when a node that does not possess the feature (an inactive node) acquires it (becomes active). Bootstrap percolation is therefore a useful model to study spread of viruses between communities, diffusion of attacks on the web or growth of the so-called “viral content” in social networks. Bootstrap percolation is therefore a

useful model to study spread of viruses between communities, diffusion of attacks on the web or growth of the so-called “viral content” in social networks. There are a number of analytical results on the percolation threshold on different graph structures such as regular trees, Euclidean lattices, some random graph models, hypercubes [11, 1, 12, 6, 8, 9, 14, 13, 5, 3, 7, 2, 15, 10, 16, 17, 4], to mention a few.

In this work, we study both analytically and numerically bootstrap percolation on periodic trees. Periodic trees are useful for estimating upper bounds on the percolation thresholds of various types of semi-regular Euclidean and non-Euclidean lattices. In general, trees can play an important role in estimating or bounding percolation threshold for more complicated graphs. For example, the percolation threshold of a spanning tree of a graph is an upper bound on the percolation threshold of the graph. Note that depending on the graph itself, the percolation thresholds of the graph and its spanning tree may be far apart. When the spanning tree is regular, as in Figure 1, existing results can be used [11, 12]. Our goal is to extend those results further through derivation of exact thresholds for *periodic trees* in which: (i) nodal degrees form a finite set of non-negative integers, and (ii) nodes at the same graph distance from a given node, called root, have the same degree, see Figure 2 (left). We make these definitions more precise in Section 2. To this end, we derive explicit equations for the percolation threshold for periodic trees as function of the degree sequence and θ , the threshold parameter. To illustrate, we compute the percolation threshold for several periodic trees.

Prior work on bootstrap percolation on trees includes the original paper of Chalupa *et al* [11] which introduced bootstrap percolation (on regular trees) and obtained a fundamental recursion for computation of the critical threshold. More recently Balogh *et al* [6] obtained new results for non-regular (infinite) trees, and Bollobás *et al* on Galton-Watson trees. Our work uses techniques introduced by Fontes and Schonmann [12] on the percolation threshold for almost sure activation of bootstrap percolation on regular trees. For the same process, the authors additionally showed the percolation threshold for the existence of infinite cluster [12].

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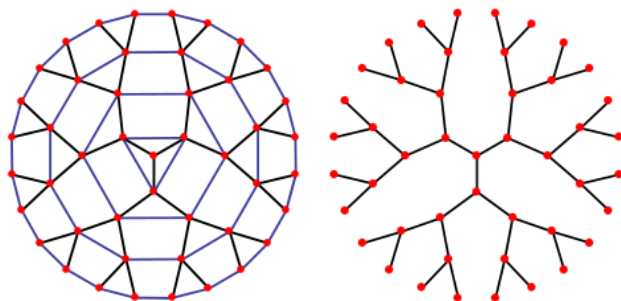


Figure 1: A regular spanning tree to approximate the percolation threshold of a graph.

2 Bootstrap percolation process on periodic trees

Bootstrap percolation (BP) is a cellular automaton defined on an underlying graph $G = (V, E)$ with state space $\{0, 1\}^V$ whose initial configuration is chosen by a Bernoulli product measure. In other words, every node is in one of two different states 0 or 1, *inactive* or *active* respectively, and a node is active with probability p , independently of other nodes, within the initial configuration.

After drawing an initial configuration at time $t = 0$, a discrete time deterministic process updates the configuration according to a local rule: an inactive node becomes active at time $t + 1$ if the number of its active neighbors at t is greater than or equal to some specified *threshold parameter* θ . Once an inactive node becomes active it remains active forever. A configuration that does not change at the next time step is a *stable* configuration. A configuration is *fully active* if all its nodes are active.

An interesting phenomenon to study is metastability near a first-order phase transition: Does there exist $0 < p_c < 1$ such that:

$$(\forall p < p_c) \lim_{t \rightarrow \infty} \mathbb{P}_p(V \text{ becomes fully active}) = 0,$$

and

$$(\forall p > p_c) \lim_{t \rightarrow \infty} \mathbb{P}_p(V \text{ becomes fully active}) = 1?$$

In this work we study bootstrap percolation processes and associated p_c 's on periodic trees defined as follows.

DEFINITION 2.1. (Periodic Tree) Let $\ell, m_0, m_1, \dots, m_{\ell-1} \in \mathbb{N}$. An ℓ -periodic tree $\mathbb{T}_{m_0, m_1, \dots, m_{\ell-1}}$ is recursively defined as follows.

There exists a node \emptyset , called root. The nodes at the distance $k \bmod \ell$ from \emptyset have degree $m_k + 1$ for $k \in \mathbb{N}$.

A regular (ordinary) tree T_d is a 1-periodic tree where each node has degree $d + 1$.

The schematic presentation of a finite restriction of $T_{3,2}$ is given in Figure 2. Notice that nodes in this tree have degrees 4 and 3; those at even distance from the node in the center have degree 4 and those at odd distance have degree 3.

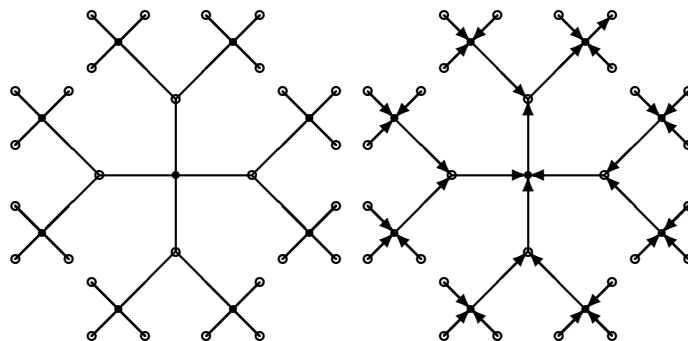


Figure 2: The periodic tree $T_{3,2}$ (left), and its oriented version $\vec{T}_{3,2}$ (right): their finite restrictions.

DEFINITION 2.2. (Oriented ℓ -Periodic Tree) Let $\ell, m_0, m_1, \dots, m_{\ell-1} \in \mathbb{N}$. An oriented ℓ -periodic tree $\vec{\mathbb{T}}_{m_0, m_1, \dots, m_{\ell-1}}$ is recursively defined as follows. There exists a node \emptyset , called root. The nodes at the distance $k \bmod \ell$ from \emptyset have in-degree m_k and out-degree 1 for $k \in \mathbb{N}$.

The adjacency relation among the nodes, and bootstrap percolation itself, on an oriented tree will follow the orientation of the edges. For details see Section 3.1.

The schematic presentation of a finite restriction of $T_{3,2}$ and its oriented version $\vec{T}_{3,2}$ are given in Figure 2. The following Lemma 2.1 is an important ingredient for our main result given by Theorem 3.1, which we prove directly. This result has appeared in different forms in [12, 8].

LEMMA 2.1. Given $n, \theta \in \mathbb{N}$ such that $2 \leq \theta \leq n - 1$ and $x \in [0, 1]$ let

$$(2.1) \quad \phi_{n,p,\theta}(x) := p + (1-p) \sum_{k=\theta}^n \binom{n}{k} x^k (1-x)^{n-k}.$$

There exists the smallest $p_c \in (0, 1)$ such that for any $p > p_c$ we have $\phi_{n,p,\theta}(x) > x$ for every $x \in (0, 1)$, and 1 is the only solution of $\phi_{n,p,\theta}(x) = x$ in $[0, 1]$.

Proof. From (2.1) it is immediately clear that $x = 1$ is a solution of $\phi_{n,p,\theta}(x) = x$ in $[0, 1]$. Given n and θ , let us define the function $\Phi_p(x) := \phi_{n,p,\theta}(x) - x$. The proof follows from analyzing $\Phi_p(x)$ as a function of x and its continuity and monotonicity as a function of p . The first and second derivatives, as functions of x , are given by:

$$\begin{aligned}\Phi'_p(x) &= (1-p)n \binom{n-1}{\theta-1} x^{\theta-1} (1-x)^{n-\theta} - 1, \\ \Phi''_p(x) &= (1-p)n \binom{n-1}{\theta-1} x^{\theta-2} (1-x)^{n-\theta-1} \\ &\quad \times (\theta-1 - (n-1)x) .\end{aligned}$$

By considering $\Phi''_p(x)$, we see that $x^* := (\theta-1)/(n-1)$ is a unique stationary point of the first derivative $\Phi'_p(x)$ in the open interval $(0, 1)$. Therefore $\Phi'_p(x)$ is strictly increasing on $[0, x^*)$, strictly decreasing on $(x^*, 1]$, and attains its maximum value at x^* given by (2.2)

$$\Phi'_p(x^*) = (1-p)n \binom{n-1}{\theta-1} \left(\frac{\theta-1}{n-1} \right)^{\theta-1} \left(\frac{n-\theta}{n-1} \right)^{n-\theta} - 1 .$$

For a given n , it is evident from (2.2) that there exists $p^* \in (0, 1)$ such that $\Phi'_p(x^*) < 0$. Hence, for every $p > p^*$, the first derivative is less than zero $\Phi'_p(x) < 0$, and the function $\Phi_p(x)$ is strictly decreasing. We have $\Phi_p(0) = p$ and $\Phi_p(1) = 0$. Therefore $\Phi_p(x)$ is strictly positive in $(0, 1)$ and thus $\phi_{n,p,\theta}(x) > x$ in $(0, 1)$.

This does not conclude the proof yet, as p^* is not the p_c in the assertion of the lemma. To identify p_c , we analyze $\Phi_0(x) = \sum_{k=\theta}^n \binom{n}{k} x^k (1-x)^{n-k} - x$. The idea is to show that $\Phi_0(x) = 0$ has a real root in $(0, 1)$. Then the monotonicity and continuity of $\Phi_p(x)$ in p (a linear function of p) will lead to the existence of the critical p_c in $(0, 1)$ for which $\Phi_{p_c}(x) = 0$ has a unique solution in $(0, 1)$.

By simple substitution, we have $\Phi_0(0) = \Phi_0(1) = 0$ and $\Phi'_0(0) = \Phi'_0(1) = -1$, so there exists a root $r \in (0, 1)$ such that $\Phi_0(r) = 0$. We have already shown that for any $p > p^*$, $\Phi_p(x) > 0$ for every x in $(0, 1)$. Observe that $\Phi_p(x)$ is strictly increasing and continuous in $p \in [0, 1]$. Hence there exists $0 < p_c < p^*$ such that the equation $\Phi_p(x) = 0$ has real root(s) in $(0, 1)$ for every $p \leq p_c$, which is given by

$$(2.3) \quad p_c = \inf \{p \in (0, 1) : \phi_{n,p,\theta}(x) > x \text{ for every } x \text{ in } (0, 1)\} .$$

This concludes the proof.

REMARK 2.1. The critical value p_c can be computed as the solution for p of the system of two equations $\phi_{n,p,\theta}(x) = x$ and $\phi'_{n,p,\theta}(x) = 1$ (respectively, $\Phi_p(x) = 0$

and $\Phi'_p(x) = 0$) in p and x in $(0, 1)^2$. Concretely, this system is given by:

$$\begin{aligned}p + (1-p) \sum_{k=\theta}^n \binom{n}{k} x^k (1-x)^{n-k} &= x, \\ (1-p)n \binom{n-1}{\theta-1} x^{\theta-1} (1-x)^{n-\theta} &= 1 .\end{aligned}$$

REMARK 2.2. It is not hard to show, by analyzing $\Phi'_p(x)$ and $\Phi''_p(x)$, that for $p < p_c$ the equation $\phi_{n,p,\theta}(x) = x$ has exactly two real solutions in $(0, 1)$, see Figure 3, and no roots when $p > p_c$, see Figure 4. (The fact that $0.3 < p_c < 0.4$ may be found in Figure 6, top, in the fourth curve from the bottom which corresponds to $a = b = 8$ and $\theta = 5$.)

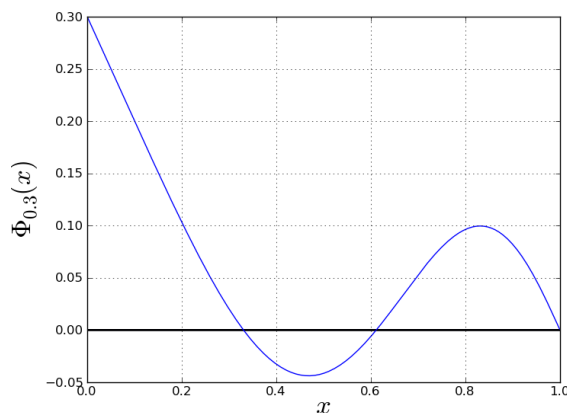


Figure 3: $\Phi_{0.3}(x)$ for $n = 7, \theta = 5, p = 0.3$.

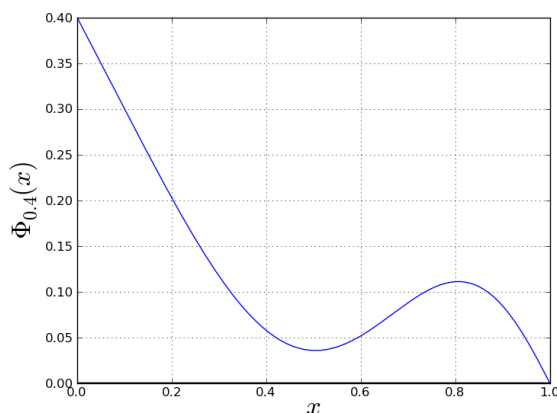


Figure 4: $\Phi_{0.4}(x)$ for $n = 7, \theta = 5, p = 0.4$.

3 Main result

We provide a proof for the case of a tree of periodicity two and then indicate how the result may be proved for larger periodicity.

THEOREM 3.1. *Given $a, b \in \mathbb{N}$ and $2 \leq \theta < a, b$ consider a bootstrap percolation on $\mathbb{T}_{a,b}$ with the initial probability p . There exists $p_f \in (0, 1)$ such that for all $p \geq p_f$, the tree $\mathbb{T}_{a,b}$ is fully active a.a.s.¹, and $\mathbb{T}_{a,b}$ is not fully active a.a.s. for $p < p_f$.*

To prove Theorem 3.1, we adopt the methodology of [12]. That is, we first derive the percolation threshold, denoted by \bar{p}_f , for the oriented periodic tree $\vec{\mathbb{T}}_{a,b}$ (see Definition 2.2), using a system of recurrence equations (Theorem 3.2). Next, we show that the percolation threshold, denoted by p_f , for the unoriented periodic tree $\mathbb{T}_{a,b}$ has to be the same as the threshold for $\vec{\mathbb{T}}_{a,b}$, that is $p_f = \bar{p}_f$ (Theorem 3.3). These two theorems complete the proof.

3.1 BP on an oriented tree $\vec{\mathbb{T}}_{a,b}$

THEOREM 3.2. *Given $a, b \in \mathbb{N}$ and $2 \leq \theta < a, b$ consider a bootstrap percolation on $\mathbb{T}_{a,b}$ with the initial probability p . There exists $\bar{p}_f \in (0, 1)$ such that for all $p > \bar{p}_f$, the tree $\vec{\mathbb{T}}_{a,b}$ is fully active a.a.s., and $\vec{\mathbb{T}}_{a,b}$ is not fully active a.a.s. for $p < \bar{p}_f$.*

Proof. Let V_a and V_b be the two sets of nodes of degrees $a + 1$ and $b + 1$ respectively in $\mathbb{T}_{a,b}$, that is, the sets of nodes of in-degrees a and b in $\vec{\mathbb{T}}_{a,b}$. The dynamics of bootstrap percolation process on $\vec{\mathbb{T}}_{a,b}$ is captured by knowing the states of all nodes, denoted by $\vec{\eta}_t(v) \in \{0, 1\}$ and $\vec{\zeta}_t(u) \in \{0, 1\}$, for every $v \in V_a$ and $u \in V_b$ at $t \in \mathbb{N}_0$.

Choose any node $v \in V_a$. Conditioning upon whether this node v was active at time 0 or not (i.e., $\vec{\eta}_0(v) = 0$ or $\vec{\eta}_0(v) = 1$), the probability that the node v is active at time t is given by

$$\begin{aligned} \mathbb{P}(\vec{\eta}_t(v) = 1) &= \mathbb{P}(\vec{\eta}_0(v) = 1) + \mathbb{P}(\vec{\eta}_0(v) = 0) \\ &\quad \times \mathbb{P}\left(\sum_{u \rightsquigarrow v} \vec{\zeta}_t(u) \geq \theta \mid \vec{\eta}_0(v) = 0\right), \end{aligned}$$

where the symbol “ \rightsquigarrow ” indicates that u is a neighbor of v in the oriented tree $\vec{\mathbb{T}}_{a,b}$ and the edge orientation is from u to v .

Also, choose any node $u \in V_b$, independently of v . Analogously, the probability that node u is active at

time t is given by

$$\begin{aligned} \mathbb{P}(\vec{\zeta}_t(u) = 1) &= \mathbb{P}(\vec{\zeta}_0(u) = 1) + \mathbb{P}(\vec{\zeta}_0(u) = 0) \\ &\quad \times \mathbb{P}\left(\sum_{v \rightsquigarrow u} \vec{\eta}_t(v) \geq \theta \mid \vec{\zeta}_0(u) = 0\right). \end{aligned}$$

Given symmetry and dynamical rules of the BP process, $\vec{\zeta}_t(x)$ are independent Bernoulli random variables and moreover independent of $\vec{\eta}_0(v)$. Hence letting $\vec{x}_t := \mathbb{P}(\vec{\eta}_t(v) = 1)$ and $\vec{y}_t := \mathbb{P}(\vec{\zeta}_t(u) = 1)$, we obtain

$$(3.4) \quad \vec{x}_t = p + (1 - p) \sum_{k=\theta}^a \binom{a}{k} \vec{y}_{t-1}^k (1 - \vec{y}_{t-1})^{a-k},$$

and

$$(3.5) \quad \vec{y}_t = p + (1 - p) \sum_{k=\theta}^b \binom{b}{k} \vec{x}_{t-1}^k (1 - \vec{x}_{t-1})^{b-k},$$

where $\vec{x}_0 = p$ and $\vec{y}_0 = p$.

In order to simplify the notation, we consider the auxiliary function $\phi_{n,p,\theta}$, defined in (2.1), for $x \in [0, 1]$, where $n, \theta \in \mathbb{N}$ are given, such that $2 \leq \theta \leq n - 1$. The function $\phi_{n,p,\theta}(x)$ is strictly increasing in x (the first derivative in x is positive in $(0, 1)$). Moreover, given $x \in [0, 1]$, the mapping $p \rightarrow \phi_{n,p,\theta}(x)$ is strictly increasing in p in $(0, 1)$, (the first derivative in p is positive in $(0, 1)$).

From the definition of $\phi_{a,p,\theta}$ and $\phi_{b,p,\theta}$, the recurrence equations (3.4) and (3.5) can be rewritten in a more compact form

$$(3.6) \quad \vec{x}_t = \phi_{a,p,\theta}(\vec{y}_{t-1}),$$

$$(3.7) \quad \vec{y}_t = \phi_{b,p,\theta}(\vec{x}_{t-1}).$$

We now show that the limits $\vec{x}_\infty := \lim_{t \rightarrow \infty} \vec{x}_t$ and $\vec{y}_\infty := \lim_{t \rightarrow \infty} \vec{y}_t$ exist. First, we show that the sequences $\{\vec{x}_t\}_{t=0}^\infty$ and $\{\vec{y}_t\}_{t=0}^\infty$ are increasing in t . By definition $\vec{x}_0 = p$ and $\vec{y}_0 = p$. The monotonicity of $\phi_{a,p,\theta}$ and $\phi_{b,p,\theta}$, and (3.6) and (3.7) yield $\vec{x}_1 = \phi_{a,p,\theta}(\vec{y}_0) \geq \vec{y}_0 = p$ and similarly $\vec{y}_1 = \phi_{b,p,\theta}(\vec{x}_0) \geq \vec{x}_0 = p$. Hence $\vec{x}_1 \geq \vec{x}_0$ and $\vec{y}_1 \geq \vec{y}_0$. Assume that $\vec{x}_t \geq \vec{x}_{t-1}$ and $\vec{y}_t \geq \vec{y}_{t-1}$ for some $t \geq 1$. Then it follows $\vec{x}_{t+1} = \phi_{a,p,\theta}(\vec{y}_t) \geq \phi_{a,p,\theta}(\vec{y}_{t-1}) = \vec{x}_t$ and similarly $\vec{y}_{t+1} = \phi_{b,p,\theta}(\vec{x}_t) \geq \phi_{b,p,\theta}(\vec{x}_{t-1}) = \vec{y}_t$. Hence by mathematical induction the sequences $\{\vec{x}_t\}_{t=0}^\infty$ and $\{\vec{y}_t\}_{t=0}^\infty$ are increasing, and upper bounded by 1. By the monotone convergence theorem the (unique) limits \vec{x}_∞ and \vec{y}_∞ exist in $[0, 1]$, and from (3.6) and (3.7), satisfy

$$(3.8) \quad \vec{x}_\infty = \phi_{a,p,\theta}(\vec{y}_\infty),$$

$$(3.9) \quad \vec{y}_\infty = \phi_{b,p,\theta}(\vec{x}_\infty).$$

¹a.a.s. stands for asymptotically almost surely.

Concretely,

$$(3.10) \quad \vec{x}_\infty = \phi_{a,p,\theta}(\phi_{b,p,\theta}(\vec{x}_\infty)) ,$$

$$(3.11) \quad \vec{y}_\infty = \phi_{b,p,\theta}(\phi_{a,p,\theta}(\vec{y}_\infty)) .$$

We also note that \vec{x}_∞ and \vec{y}_∞ are non-decreasing in $p \in [0, 1]$. This follows from the fact that \vec{x}_t and \vec{y}_t are non-decreasing in p for every $t \geq 0$.

We now show that there exists $\vec{p}_f \in (0, 1)$ such that $\vec{x}_\infty < 1$ and $\vec{y}_\infty < 1$ for all $p < \vec{p}_f$, and $\vec{x}_\infty = 1$ and $\vec{y}_\infty = 1$ for all $p > \vec{p}_f$. Let us first consider \vec{x}_∞ and \vec{y}_∞ as p varies in $[0, 1]$. From (3.8) and (3.9), $\vec{x}_\infty = 1$ is equivalent to $\vec{y}_\infty = 1$ (as well as $\vec{x}_\infty < 1$ is equivalent to $\vec{y}_\infty < 1$). Trivially, when $p = 0$, the initial probabilities $\vec{x}_0 = \vec{y}_0 = 0$, yielding $\vec{x}_t = \vec{y}_t = 0$ for every $t \geq 0$. Similarly, $\vec{x}_t = \vec{y}_t = 1$ for every $t \geq 0$, when $p = 1$. Thus there exists a value \vec{p}_f in $[0, 1]$ such that \vec{x}_∞ and \vec{y}_∞ are less than 1 for every $p < \vec{p}_f$, and equal to 1 for every $p > \vec{p}_f$.

We still have to show that the critical value \vec{p}_f is indeed in $(0, 1)$. Without loss of generality let us assume $a = \min(a, b)$ and $b = \max(a, b)$. Consider the sequences $\{\vec{s}_t\}_{t=0}^\infty$ and $\{\vec{S}_t\}_{t=0}^\infty$ defined by

$$\vec{s}_t = \phi_{a,p,\theta}(\vec{s}_{t-1}) ,$$

$$\vec{S}_t = \phi_{b,p,\theta}(\vec{S}_{t-1}) ,$$

for $t \geq 1$, where $\vec{s}_0 = p$, $\vec{S}_0 = p$. From the stochastic dominance on the Binomial random variable

$$(3.12) \quad \mathbb{P}(\text{Bin}(a, p) \geq \theta) \leq \mathbb{P}(\text{Bin}(b, p) \geq \theta) .$$

Now, it easily follows by mathematical induction that the sequences $\{\vec{s}_t\}_{t=0}^\infty$ and $\{\vec{S}_t\}_{t=0}^\infty$ represent, respectively, a lower and an upper bound on both $\{\vec{x}_t\}_{t=0}^\infty$ and $\{\vec{y}_t\}_{t=0}^\infty$, that is, $\vec{s}_t \leq \vec{x}_t, \vec{y}_t \leq \vec{S}_t$, for every $t \geq 0$. Analogously to the proof of the existence of \vec{x}_∞ and \vec{y}_∞ , one can show that the limits $\vec{s}_\infty := \lim_{t \rightarrow \infty} \vec{s}_t$ and $\vec{S}_\infty := \lim_{t \rightarrow \infty} \vec{S}_t$ exist, and satisfy

$$(3.13) \quad \vec{s}_\infty \leq \vec{x}_\infty, \vec{y}_\infty \leq \vec{S}_\infty ,$$

and moreover

$$\vec{s}_\infty = \phi_{a,p,\theta}(\vec{s}_\infty) ,$$

$$\vec{S}_\infty = \phi_{b,p,\theta}(\vec{S}_\infty) .$$

By Lemma 2.1, given a , there exists the critical value $p_a \in (0, 1)$ such that $\vec{s}_\infty < 1$ for every $p < p_a$, and $\vec{s}_\infty = 1$ for every $p > p_a$. Similarly, given b , there exists the critical value $p_b \in (0, 1)$ such that $\vec{S}_\infty < 1$ for every $p < p_b$, and $\vec{S}_\infty = 1$ for every $p > p_b$. From the fact that $\vec{x}_\infty, \vec{y}_\infty, \vec{s}_\infty, \vec{S}_\infty$ are non-decreasing in p and their relation given in (3.13), it follows that the critical value \vec{p}_f satisfies $p_b \leq \vec{p}_f \leq p_a$ and indeed belongs to $(0, 1)$.

REMARK 3.1. One can prove Theorem 3.2 for an oriented tree with periodicity greater than two. The steps of the proof are analogous to those presented above, but instead of two sequences $\{\vec{x}_t\}_{t=0}^\infty$ and $\{\vec{y}_t\}_{t=0}^\infty$ we have ℓ sequences, where ℓ is equal to the periodicity of the tree.

3.2 BP on an unoriented tree $\mathbb{T}_{a,b}$ To determine the critical threshold for BP on $\mathbb{T}_{a,b}$, we use the result of Section 3.1 on oriented trees. The dynamics of bootstrap percolation process on $\mathbb{T}_{a,b}$ is captured by knowing the states of nodes in a graph, that is, $\zeta_t(v) \in \{0, 1\}$ and $\eta_t(u) \in \{0, 1\}$ for every $v \in V_a$ and $u \in V_b$ at $t \in \mathbb{N}_0$. Denote by x_t the probability that a node of degree $a+1$ is active at time t , and similarly by y_t the probability that a node of degree $b+1$ is active at time t , where $x_0 = p$ and $y_0 = p$.

THEOREM 3.3. The probabilities $x_\infty, \vec{x}_\infty, y_\infty, \vec{y}_\infty$ satisfy

$$(3.14) \quad x_\infty = p + (1-p) \sum_{k=\theta}^{a+1} \binom{a+1}{k} \vec{y}_\infty^k (1-\vec{y}_\infty)^{b+1-k} ,$$

and

$$(3.15) \quad y_\infty = p + (1-p) \sum_{k=\theta}^{b+1} \binom{b+1}{k} \vec{x}_\infty^k (1-\vec{x}_\infty)^{a+1-k} .$$

Proof. The proof consists of two parts. In Part 1, we derive the equation for the probability that a randomly selected node (w.l.o.g. of degree $b+1$) in the unoriented BP becomes active as a function of the probabilities of activation of root nodes in truncated unoriented subtrees. In Part 2, we relate the probability of activation of the root node in truncated unoriented subtrees to that of the truncated oriented subtrees. Combining Parts 1 and 2 establishes (3.14) and (3.15).

Part 1. As previously, choose a node $v_0 \in \mathbb{T}_{a,b}$ of degree $b+1$. Denote by v_1, v_2, \dots, v_{b+1} the neighbors of v_0 . Let $T_i = \mathbb{T}_{a,b} - (v_0, v_i)$ be a tree incident to v_i obtained by removing the edge (v_0, v_i) from $\mathbb{T}_{a,b}$, see Figure 5. In T_i , node v_i has degree a , while all other nodes have degree either $b+1$ or $a+1$.

Consider BP denoted by $\zeta_t^{(i)}$ that (starts and) runs only on T_i , instead of the entire tree $\mathbb{T}_{a,b}$. Given $t \geq 0$, for $i = 1, 2, \dots, b+1$, the dynamics $\zeta_t^{(i)}(v_i)$ of the nodes $v_i \in T_i$ at time t , are i.i.d. random variables.

Now consider BP denoted by Ξ that (starts and) runs on $\mathbb{T}_{a,b}$. By symmetry and dynamics of BP, the process $\Xi(v_0)$ is the same in distribution for any choice of v_0 . The node v_0 becomes active, $\Xi(v_0) = 1$, if and only if: either (i) $\Xi_0(v_0) = 1$, or (ii) $\sum_{i=1}^{b+1} \zeta_\infty^{(i)}(v_i) \geq \theta$, given $\Xi_0(v_0) = 0$. But given $\Xi_0(v_0) = 0$, the event

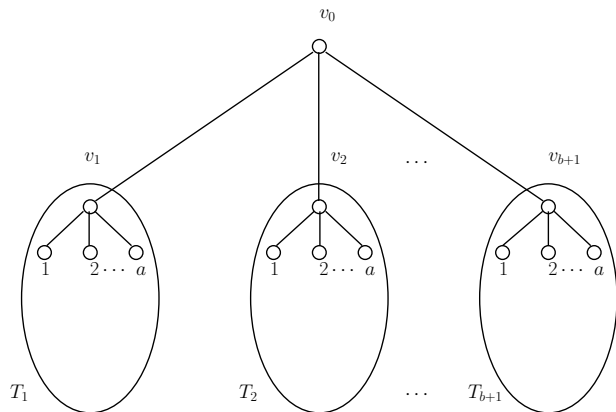


Figure 5: Periodic tree with the root node v_0 and subtrees T_1, T_2, \dots, T_{b+1} incident to the root.

$\Xi_\infty(v_0) = 0$ is equivalent to having at most $\theta - 1$ eventually active neighbors, i.e., $\zeta_\infty^{(i)}(v_i) = 1$. Moreover, the two BP processes: (1) Ξ restricted to the tree T_i given $\Xi_\infty(v_0) = 0$, and (2) $\zeta_t^{(i)}$ (which runs on T_i only) are equivalent. By this equivalence

$$(3.16) \quad \mathbb{P} \left(\sum_{i=1}^{b+1} \Xi_\infty(v_i) < \theta \mid \Xi_0(v_0) = 0 \right) = \sum_{k=0}^{\theta-1} \binom{b+1}{k} \times \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right)^k \left(1 - \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right) \right)^{b+1-k}.$$

Now, the probability that v_0 becomes active is given by

$$(3.17) \quad \mathbb{P}(\Xi_\infty(v_0) = 1) = \mathbb{P}(\Xi_0(v_0) = 1) + \mathbb{P}(\Xi_0(v_0) = 0) \times \left(1 - \mathbb{P} \left(\sum_{i=1}^{b+1} \Xi_\infty(v_i) < \theta \mid \Xi_0(v_0) = 0 \right) \right),$$

therefore from (3.16),

$$(3.18) \quad \mathbb{P}(\Xi_\infty(v_0) = 1) = p + (1-p) \sum_{k=\theta}^{b+1} \binom{b+1}{k} \times \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right)^k \left(1 - \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right) \right)^{b+1-k}.$$

Equation (3.18) expresses the probability that a randomly selected node v_0 in $\mathbb{T}_{a,b}$ becomes active as a function of the probability of activation of the root node in a truncated unoriented subtree.

Part 2. First, given the oriented edges in \vec{T}_1 and unoriented edges in T_1 , it follows by stochastic dominance that

$$(3.19) \quad \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right) \leq \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 1 \right).$$

Next, we show that $\zeta_\infty^{(1)}(v_1) = 0$ implies $\zeta_\infty^{(1)}(v_1) = 0$, which will yield

$$(3.20) \quad \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 0 \right) \leq \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 0 \right).$$

The equivalence of activation in the directed and undirected trees will follow from (3.19) and (3.20).

To show (3.20), we call a node v in \vec{T}_1 *eventually-inactive* if $\zeta_\infty^{(1)}(v) = 0$ and *eventually-active* if $\zeta_\infty^{(1)}(v) = 1$. Let us consider the root v_1 of \vec{T}_1 . The node v_1 is eventually-inactive, $\zeta_\infty^{(1)}(v_1) = 0$, if and only if v_1 is initially inactive and has at least $a - (\theta - 1) = a + 1 - \theta$ eventually-inactive neighbors. For $j \geq 0$, denote by L_j the set of nodes at the level j in \vec{T}_1 . In other words, $L_0 = \{v_1\}$, L_1 is the set of neighbors in \vec{T}_1 of the nodes in L_0 , similarly L_2 is the set of neighbors in \vec{T}_1 of nodes in L_1 , etc. Every eventually-inactive node in L_1 has at most $\theta - 1$ eventually-active neighbors in \vec{T}_1 . In other words, it has at least $b - (\theta - 1) = b + 1 - \theta$ eventually-inactive neighbors from L_2 . Given that $v_1 \in L_0$ is eventually-inactive, it follows that every eventually-inactive node in L_1 has at least $b + 2 - \theta$ eventually-inactive neighbors in \vec{T}_1 . Similarly, every eventually-inactive node in L_2 has at least $a + 2 - \theta$ eventually-inactive neighbors in \vec{T}_1 . Then, by mathematical induction on j , every eventually-inactive node in L_j has at least $b + 2 - \theta$ (respectively $a + 2 - \theta$) eventually-inactive neighbors in \vec{T}_1 , for odd j (respectively even j).

Hence v_1 is eventually-inactive in \vec{T}_1 if there exists an eventually-inactive subtree $\vec{T} \subseteq \vec{T}_1$, which consists of the root v_1 , and previously recursively defined eventually-inactive nodes from \vec{T}_1 . (Specifically, every node in the eventually-inactive tree \vec{T} is inactive at time $t = 0$.)

Now consider the unoriented BP $\zeta_t^{(1)}$ (on the tree T_1). Let \mathcal{T} be unoriented copy of \vec{T} . By construction of \vec{T} , at time $t = 0$, every node of \mathcal{T} is inactive, and moreover has at least $b + 2 - \theta$ (respectively $a + 2 - \theta$) inactive neighbors in T_1 , for odd j (respectively even j). That is, at time $t = 0$, every node of \mathcal{T} is inactive and has at most $\theta - 1$ active neighbors. Therefore \mathcal{T} is eventually-inactive under the unoriented BP $\zeta_t^{(1)}$, and specifically the root v_1 is eventually-inactive, $\zeta_\infty^{(1)}(v_1) = 0$. This yields (3.20), and thus

$$(3.21) \quad \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 0 \right) = \mathbb{P} \left(\zeta_\infty^{(1)}(v_1) = 0 \right).$$

Introducing $x_\infty := \mathbb{P}(\Xi_\infty(v_0) = 1)$ in (3.18) and using (3.21) gives (3.14).

Analogously, one can prove the result given in (3.15) for the choice of a node u_0 of degree $a + 1$, which concludes the proof.

PROPOSITION 3.1. *The percolation threshold on oriented and unoriented trees are the same:*

$$(3.22) \quad \vec{p}_f = p_f.$$

Proof. From (3.14) and (3.15) it follows that $(x_\infty, y_\infty) = (1, 1)$ if and only if $(\vec{x}_\infty, \vec{y}_\infty) = (1, 1)$. Now, by using the definition of \vec{p}_f and p_f , we obtain that $\vec{p}_f = p_f$.

REMARK 3.2. *The proof readily generalizes for trees of periodicity greater than 2 using analogous arguments.*

4 Numerical evaluation of the critical probability p_f

In this section we present numerical values of p_f for threes $T_{a,b}$, where degree $a = 3, \dots, 10$, for a non-trivial range of the threshold parameter $2 \leq \theta \leq 9$, when degree $b = a$, $b = a + 1$, $b = a + 2$, and $b = 2a$. Concretely, we numerically find the smallest \vec{p}_f such that the only solution of the recurrence system (3.10) and (3.11) is $(1, 1)$ as justified by Theorem 3.2.

Figure 6 and 7 show numerical evaluations of the critical threshold p_f for said values of a and b in the two-periodic tree $T_{a,b}$. Each curve represents p_f for a given $2 \leq \theta \leq 9$ and higher curves correspond to higher values of θ . The value of p_f for $a = b = 8$ and $\theta = 5$ corresponds to Figures 3 and 4 and is highlighted as a large dot in the top Figure 6.

We observe that for a fixed θ , the critical threshold p_f monotonically decreases for a fixed value of a for increasing values of b , as expected.

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References

- [1] AIZENMAN, M., AND LEBOWITZ, J. L. Metastability effects in bootstrap percolation. *Journal of Physics A: Mathematical and General* 21, 19 (1988), 3801–3813.
- [2] AMINI, H. Bootstrap percolation and diffusion in random graphs with given vertex degrees. *Electronic Journal of Combinatorics* 17, #R25 (2010).
- [3] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H., AND MORRIS, R. The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* 364, 5 (2012), 2667–2701.
- [4] BALOGH, J., BOLLOBÁS, B., AND MORRIS, R. Majority bootstrap percolation on the hypercube. *Combinatorics, Probability and Computing* 18, 1-2 (2009), 17–51.

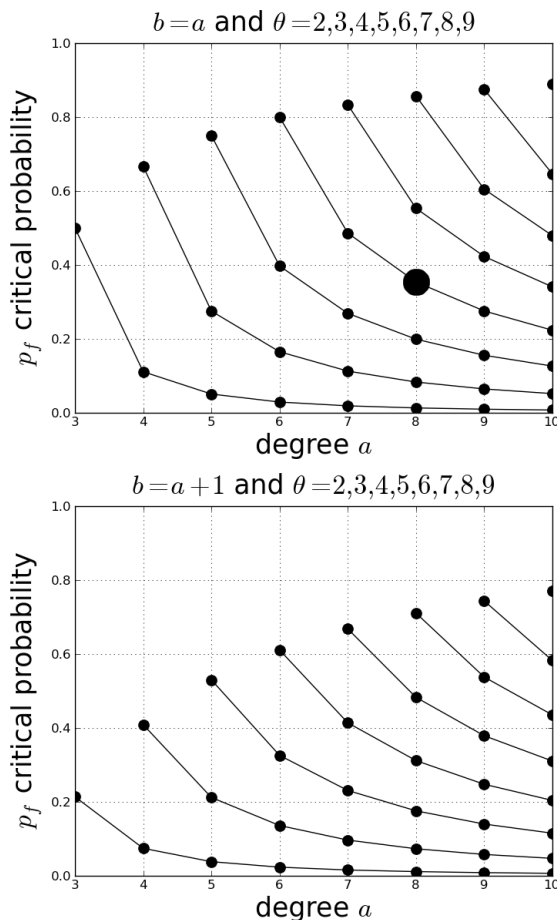


Figure 6: Numerical evaluation of the critical threshold p_f for different values of a and b in the two-periodic tree $T_{a,b}$, for $b = a$ and $b = a + 1$, where $2 \leq \theta \leq 9$. The large dot in the top figure corresponds to the value of p_f for $a = b = 8$ and $\theta = 5$ in Figures 3 and 4.

- [5] BALOGH, J., BOLLOBÁS, B., AND MORRIS, R. Bootstrap percolation in high dimensions. *Combinatorics, Probability and Computing* 19, 5-6 (2010), 643–692.
- [6] BALOGH, J., PERES, Y., AND PETE, G. Bootstrap percolation on infinite trees and non-amenable groups. *Combinatorics, Probability & Computing* 15, 5 (2006), 715–730.
- [7] BALOGH, J., AND PITTEL, B. Bootstrap percolation on the random regular graph. *Random Structures & Algorithms* 30, 1-2 (2007), 257–286.
- [8] BISKUP, M., AND SCHONMANN, R. H. Metastable behavior for bootstrap percolation on regular trees. *Journal of Statistical Physics* 136, 4 (2009), 667–676.
- [9] BOLLOBÁS, B., GUNDERSON, K., HOLMGREN, C., JANSON, S., AND PRZYKUCKI, M. Bootstrap percolation on Galton-Watson trees. *Electron. J. Probab.* 19 (2014), no. 13, 1–27.
- [10] BRADONJIĆ, M., AND SANIEE, I. Bootstrap percola-

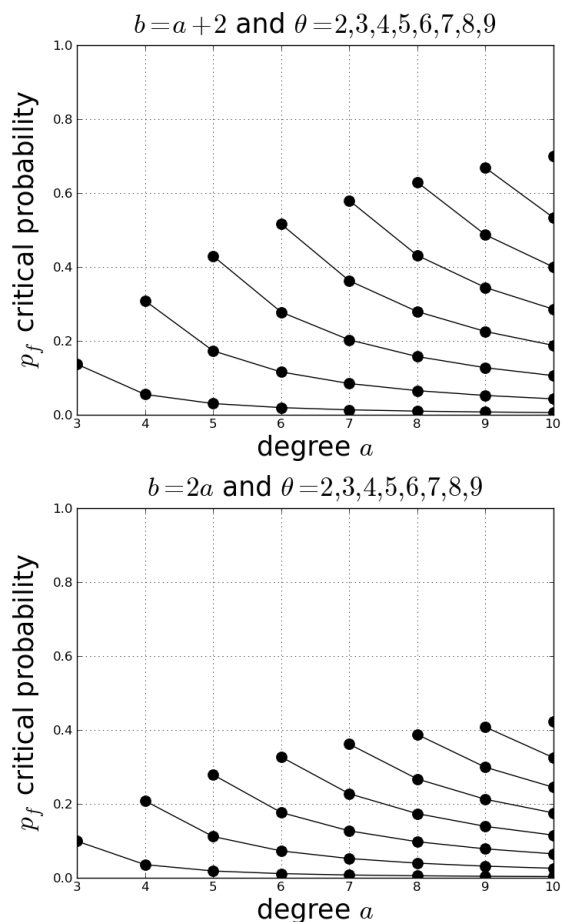


Figure 7: Numerical evaluation of the critical threshold p_f for different values of a and b in the two-periodic tree $T_{a,b}$, for $b = a + 2$ and $b = 2a$, where $2 \leq \theta \leq 9$.

- tion on random geometric graphs. *Probability in the Engineering and Informational Sciences* 28, 2 (2014), 169–181.
- [11] CHALUPA, J., LEATH, P. L., AND REICH, G. R. Bootstrap percolation on a Bethe lattice. *J. Phys. C* 12, L31 (1979).
- [12] FONTES, L., AND SCHONMANN, R. Bootstrap percolation on homogeneous trees has 2 phase transitions. *Journal of Statistical Physics* 132, 5 (2008), 839–861.
- [13] GRAVNER, J., HOLROYD, A. E., AND MORRIS, R. A sharper threshold for bootstrap percolation in two dimensions. *Probability Theory and Related Fields* 153, 1-2 (2012), 1–23.
- [14] HOLROYD, A. E. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probability Theory and Related Fields* 125 (2003), 195–224.
- [15] JANSON, S., LUCZAK, T., TUROVA, T., AND VALLIER, T. Bootstrap percolation on the random graph $G_{n,p}$. *Ann. of Appl. Probab* 22, 5 (2012), 1989–2047.
- [16] SCHONMANN, R. Critical points of two-dimensional

bootstrap percolation-like cellular automata. *Journal of Statistical Physics* 58, 5-6 (1990), 1239–1244.

[17] SCHONMANN, R. H. On the behavior of some cellular automata related to bootstrap percolation. *The Annals of Probability* 20, 1 (1992), 174–193.