The recurrence function of a random Sturmian word. *

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Abstract

This paper describes the probabilistic behaviour of a random Sturmian word. It performs the probabilistic analysis of the recurrence function which can be viewed as a waiting time to discover all the factors of length n of the Sturmian word. This parameter is central to combinatorics of words. Having fixed a possible length n for the factors, we let α to be drawn uniformly from the unit interval $[0, 1]$, thus defining a random Sturmian word of slope α. Thus the waiting time for these factors becomes a random variable, for which we study the limit distribution and the limit density.

1 Introduction

Recurrence and Sturmian words. The recurrence function measures the "complexity" of an infinite word and describes the possible occurrences of finite factors inside it together with the maximal gaps between successive occurrences. This recurrence function is thus widely studied, notably in the case of Sturmian words. Sturmian words are central in combinatorics of words, as they are in a precise sense the simplest infinite words which are not eventually periodic [7]. Each Sturmian word is associated with an irrational number α, which is called the slope of the Sturmian word, and many of its characteristics depend on the continued fraction expansion of α. This is in particular the case for the recurrence function $n \mapsto R(\alpha, n)$, where the integer $R(\alpha, n)$ is the length of the smallest "window" which is needed for discovering the set $\mathcal{L}_\alpha(n)$ of all the finite factors of length n inside α. As this set $\mathcal{L}_\alpha(n)$ is widely used in many applications of Sturmian words (for instance quasicrystals, or digital geometry), the function $n \mapsto R(\alpha, n)$ thus intervenes very often as a pre-computation cost.

From a result due to Morse and Hedlund [8], it is known that the recurrence function $R(\alpha, n)$ depends on α via its continued fraction expansion, and, notably, its *continuants*. The continuant $q_k(\alpha)$ is the denominator of the k-th convergent of α, and, for an irrational α, the sequence $k \mapsto q_k(\alpha)$ is strictly increasing. The result of Morse and Hedlund expresses $R(\alpha, n)$ in terms of the integer n together with the two ends of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ which contains n. More precisely, for any $n \in [q_{k-1}(\alpha), q_k(\alpha)[$, one has

$$(1.1) \qquad R(\alpha, n) = n - 1 + q_k(\alpha) + q_{k-1}(\alpha)\,.$$

It is thus natural to study the recurrence function via

$$(1.2) \quad S(\alpha, n) := \frac{R(\alpha, n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha)}{n} + \frac{q_k(\alpha)}{n}\,,$$

called the recurrence quotient. Most of the classical studies deal with a *fixed* α, and the usual focus is put on *extremal* behaviours of the recurrence function. The following result exhibits the large variability of the function $n \mapsto S(\alpha, n)$.

PROPOSITION 1.1. *The following holds for the recurrence quotient defined in* (1.2)*:*

(*i*) *For any irrational real α, one has*

$$\liminf_{n\to\infty} S(\alpha, n) \le 3.$$

(*ii*) [Morse and Hedlund] [8] *For almost any irrational α, and any positive sequence $(\gamma(n))$, one has*

$$\limsup_{n\to\infty} \frac{S(\alpha, n)}{\gamma(n)} = +\infty \qquad or \qquad \limsup_{n\to\infty} \frac{S(\alpha, n)}{\gamma(n)} = 0$$

depending if the series of general term $(n\gamma(n))^{-1}$ is divergent or convergent.

This result also shows that the quotient recurrence is "small" for integers n which are close to the right end of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$, whereas it is "large" when n is close to $q_{k-1}(\alpha)$ (see Figure 1).

Two different probabilistic settings. Here, we adopt a probablistic approach, and consider a random Sturmian word, associated with a random irrational slope α of the unit interval. There are now two possibilities:

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(i) fix the integer n (corresponding to the length of the factors, which will further tend to ∞); now the index k of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ which contains n is a random variable $k = k(\alpha, n)$. This model may be called the model "with a large fixed n". The sequence $n \mapsto S(\alpha, n)$ is now a sequence of random variables.

(ii) fix a depth k (that further tends to ∞), and a fixed $\mu \in [0, 1]$. For any slope α, we consider the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ delimited by the two successive continuants with indices $k-1$ and k, and we choose there the integer $n := n_\mu(\alpha, k)$ at a barycentric position, which now becomes a random variable. This model may be called the model "with a large fixed k". The sequence $k \mapsto S_k^{\langle\mu\rangle}(\alpha) := S(\alpha, n_\mu(\alpha, k))$ is now a sequence of random variables.

In both cases, we are interested by the same type of questions about the sequence of random variables: does there exist a limit for the expectations? a limit distribution? a limit density?

Main results. We have already performed the probabilistic study of type (ii) (the model with "a large fixed k") in [1], and we return to it in Section 5.1.
Here we deal with the recurrence quotient within model (i) (the model with "a large fixed n"). We obtain three results for the recurrence quotient; we consider the random variables $\alpha \mapsto S(\alpha, n)$ and study them for large n. We exhibit a limit for their distribution, and prove that there exists a limit density, as $n \to \infty$. We also study the conditional expectation of the recurrence quotient, when we exclude the possibility for n to be too close of the left end of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$. More generally, we exhibit a class of events for which the order of this conditional mean value is exactly of order $\log n$. This can be viewed as a probabilistic extension of the Morse and Hedlund result given in Proposition 1.1.

Our proofs use elementary methods: they are based on a precise comparison between an integral and its Riemann sum ; however, the integral is improper (but convergent) and the Riemann sum is constrained by a coprimality condition, what we call a "coprime Riemann sum".

We also introduce a general family of functions, called continuant-functions or $\mathcal{Q}$-functions, which are defined via the sequence of continuants $k \mapsto q_k(\alpha)$. The recurrence quotient is an instance of such a function, but the other "geometric" parameters of interest provide other natural examples of such a notion. The framework of the paper is naturally adapted to the study of a general $\mathcal{Q}$-function.

Plan of the paper. Section 2 gives a precise definition of the parameters under study, introduces the class of $\mathcal{Q}$-functions and states our three results: limit distributions in Theorem 2.1, limit densities in Theorem 2.2, and conditional expectations in Theorem 2.3. Section 3 is devoted to the proof of the first two results, whereas Section 4 focuses on the study of conditional expectations. Section 5 compares the results obtained in the two models, the present model (with large fixed n), and the model (with large fixed k) previously studied in [1].

2 General framework and main results.

This section starts off by making precise the notions that were informally defined in the introduction, notably Sturmian words and recurrence. Then, it introduces parameters which describe the geometry of the "continuant intervals" or the position of the integer n inside the continuant interval. Section 2.3 defines the class of $\mathcal{Q}$- functions that provides a convenient framework for our study. Then, we state Theorems 2.1 2.2 in Sections 2.5 and 2.6, for general $\mathcal{Q}$-functions. We return to our specific parameters of interest, notably the recurrence function in Section 2.7, with two figures (Figures 2 and 3). Finally, Section 2.8 concludes with a study of the conditional expectations.

2.1 More on Sturmian words and recurrence function.

We consider a finite set $\mathcal{A}$ of *symbols*, called *alphabet*. Let $\boldsymbol{u} = (u_n)_{n\in\mathbb{N}}$ be an infinite word in $\mathcal{A}^{\mathbb{N}}$. A finite word w of length n is a factor of $\boldsymbol{u}$ if there exists an index m for which $w = u_m \dots u_{m+n-1}$. Let $\mathcal{L}_{\boldsymbol{u}}(n)$ stand for the set of factors of length n of $\boldsymbol{u}$. Two functions describe the set $\mathcal{L}_{\boldsymbol{u}}(n)$ inside the word $\boldsymbol{u}$, namely the complexity and the recurrence function.

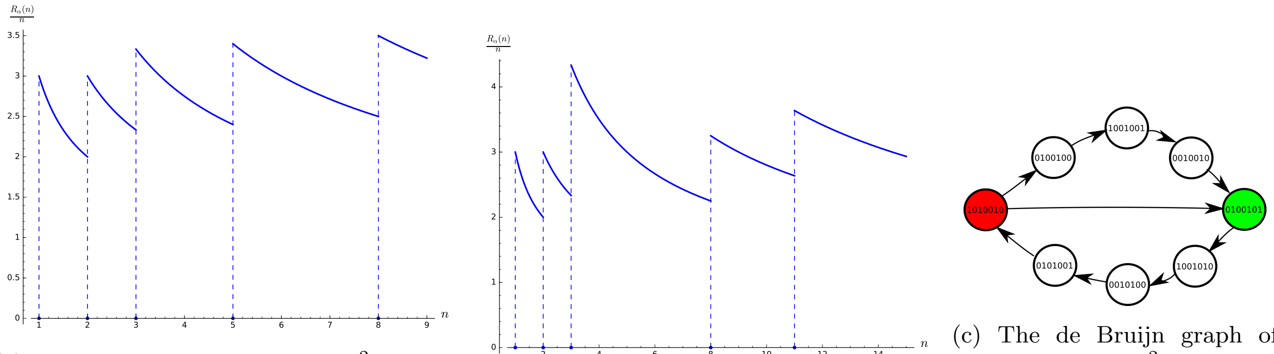
Complexity. The (*factor*) *complexity function* of the infinite word $\boldsymbol{u}$ is defined as the sequence $n \mapsto p_u(n) := |\mathcal{L}_{\boldsymbol{u}}(n)|$. The eventually periodic words are the simplest ones, in terms of the complexity function, and satisfy $p_{\boldsymbol{u}}(n) \le n$ for some n.

The simplest words that are not eventually periodic satisfy the equality $p_{\boldsymbol{u}}(n) = n + 1$ for each $n \ge 0$. As $p_{\boldsymbol{u}}(1) = 2$, the alphabet is necessarily binary, we consider $\mathcal{A} = \{0, 1\}$. Such words do exist, they are called *Sturmian words*. Moreover, Morse and Hedlund provided a powerful arithmetic description of Sturmian words (see also [7] for more on Sturmian words).

PROPOSITION 2.1. [Morse and Hedlund] [8] *Associate with a pair $(\alpha, \beta) \in [0, 1]^2$ the two infinite words $\underline{\mathfrak{S}}(\alpha, \beta)$ and $\overline{\mathfrak{S}}(\alpha, \beta)$ whose n-th symbols are respectively*

$$\underline{u}_n = \lfloor \alpha(n + 1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor,$$

$$\overline{u}_n = \lceil \alpha(n + 1) + \beta \rceil - \lceil \alpha n + \beta \rceil.$$



(a) Recurrence quotient for $\alpha = \varphi^2$, with $\varphi = (\sqrt{5}-1)/2$.

(b) Recurrence quotient for $\alpha = 1/e$.

(c) The de Bruijn graph of order 7 for $\alpha = \varphi^2$, with $\varphi = (\sqrt{5}-1)/2$.

Figure 1: Some examples about Sturmian words of slope α.

Then a word $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian if and only if it equals $\underline{\mathfrak{S}}(\alpha,\beta)$ or $\overline{\mathfrak{S}}(\alpha,\beta)$ for a pair (α,β) formed with an irrational $\alpha \in]0,1[$ and a real $\beta \in [0,1[$.

Remark. The most important properties of a Sturmian word only depend on α, which is called its slope. A Sturmian word of slope α is denoted by $\mathfrak{S}(\alpha)$.

The complexity of an infinite word $\boldsymbol{u}$ can be seen from its *de Bruijn* graphs $\mathcal{G}_{\boldsymbol{u}}(n)$: the de Bruijn graph $\mathcal{G}_{\boldsymbol{u}}(n)$ associated with an infinite word $\boldsymbol{u}$ is a finite automaton with state space $\mathcal{L}_{\boldsymbol{u}}(n)$ which describes a moving "sliding window" of length n along the word $\boldsymbol{u}$; From a state $b \in \mathcal{L}_{\boldsymbol{u}}(n)$ upon scanning letter u, the next word is $b \cdot u \in \mathcal{L}_{\boldsymbol{u}}(n+1)$: we label the edge with u and the next state is $\tau(b \cdot u)$ where $\tau(f)$ for a word f just erases the leftmost symbol of f. A Sturmian de Bruijn graph of order n is thus particularly sparse: it has $n+1$ vertices and $n+2$ edges (see Figure 1).

Recurrence. It is also important to study where finite factors occur inside the infinite word u. An infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is *uniformly recurrent* if every factor of $\boldsymbol{u}$ appears infinitely often and with bounded gaps. More precisely, denote by $w_{\boldsymbol{u}}(q,n)$ the minimal number of symbols u_k with $k \geq q$ which have to be inspected for discovering the whole set $\mathcal{L}_{\boldsymbol{u}}(n)$ from the index q. Then, the integer $w_{\boldsymbol{u}}(q,n)$ is a sort of "waiting time" and $\boldsymbol{u}$ is uniformly recurrent if each set $\{w_{\boldsymbol{u}}(q,n) \mid q \in \mathbb{N}\}$ is bounded, and the *recurrence function* $n \mapsto R_{\boldsymbol{u}}(n)$ is defined as

$$R_{\boldsymbol{u}}(n) := \max\{w_u(q,n) \mid q \in \mathbb{N}\}\,. \tag{2.3}$$

We then recover the usual definition: Any factor of length $R_{\boldsymbol{u}}(n)$ of u contains all the factors of length n of $\boldsymbol{u}$, and the length $R_{\boldsymbol{u}}(n)$ is the smallest integer which satisfies this property.
The inequality $R_{\boldsymbol{u}}(n) \geq p_{\boldsymbol{u}}(n) + n - 1$ thus holds.

Any Sturmian word is uniformly recurrent. Its recurrence function only depends on the slope α and is thus denoted by $n \mapsto R(\alpha,n)$. As we already said, it only depends on α via the sequence of its *continuants* $k \mapsto q_k(\alpha)$, and satisfies (1.1).

As a Sturmian word $\boldsymbol{u}$ is uniformly recurrent, its de Bruijn graph $\mathcal{G}_{\boldsymbol{u}}(n)$ is strongly-connected. The structure of $\mathcal{G}_{\boldsymbol{u}}(n)$ is related to the Three-Distance Theorem (described in [10] or [2]) which states that the words of $\mathcal{L}_{\boldsymbol{u}}(n)$ have only three possible frequencies, giving concrete expressions for them, these frequencies correspond to the three oriented paths, that form two cycles, in $\mathcal{G}_{\boldsymbol{u}}(n)$. There are two possible choices at the unique branching point (marked in red in Figure 1), and the two cycles are needed to discover all its nodes. The recurrence $R_{\boldsymbol{u}}(n)$ is the worst-case time needed to discover all these, starting from any position in $\boldsymbol{u}$.

2.2 Position parameters. Besides the recurrence quotient, there are also three other parameters ν, μ, ρ which describe the geometry of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ which contains n (this is the case for ρ) or the position of n inside this interval (the case for μ and ν)

$$\rho(\alpha,n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)}\,, \tag{2.4}$$

$$\mu(\alpha,n) := \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)}, \quad \nu(\alpha,n) = \frac{n}{q_k(\alpha)}\,. \tag{2.5}$$

When n belongs to the interval $[q_{k-1}(\alpha), q_k(\alpha)[$, the recurrence quotient is expressed with ρ and ν as

$$S(\alpha,n) = 1 + \frac{1+\rho(\alpha,n)}{\nu(\alpha,n)}. \tag{2.6}$$

As $\nu(\alpha,n)$ belongs to the interval $[\rho(\alpha,n),1]$, the fol-

Parameter	Function $f(x,y)$	Density $\frac{12}{\pi^2} J_f(\lambda)$
S	$1+x+y$	$\begin{cases} \frac{12}{\pi^2}\frac{1}{\lambda-1}\log(\lambda-1) & \text{if } 2\le\lambda\le 3 \\ \frac{12}{\pi^2}\frac{1}{\lambda-1}\log(1+\frac{1}{\lambda-2}) & \text{if } \lambda\ge 3\,. \end{cases}$
ρ	$\frac{x}{y}$	$\frac{12}{\pi^2}\frac{1}{1+\lambda}\lvert\log\lambda\rvert \quad \text{for } 0\le\lambda\le 1$
μ	$\frac{1-x}{y-x}$	$\begin{cases} \frac{12}{\pi^2}\frac{1}{2\lambda-1}\left(2\log 2-\frac{\log\lambda}{\lambda-1}\right) & \text{if } \lambda\ne 1/2 \\ \frac{24}{\pi^2}(1-\log 2) & \text{if } \lambda=1/2\,. \end{cases}$
ν	$\frac{1}{y}$	$\frac{12}{\pi^2}\frac{1}{\lambda}\log(1+\lambda) \quad \text{for } 0\le\lambda\le 1$

Figure 2: Limit densities for the main parameters.

lowing bounds hold

$$2+\rho(\alpha,n)\le S(\alpha,n)\le 2+\frac{1}{\rho(\alpha,n)} \tag{2.7}$$

(the lower bound holds for n close to $q_k(\alpha)$ whereas the upper bound is attained for $n=q_{k-1}(\alpha)$).

The ratio $\rho(\alpha,n)$ belongs to $]0,1]$, and the Borel-Bernstein Theorem proves that $\liminf_{n\to\infty}\rho(\alpha,n)=0$ for almost any irrational α. This is the main step for proving Proposition 1.1.

2.3 $\mathcal{Q}$-functions. More generally, we are interested in functions whose definition strongly depends on the partition defined by the continuants, and consider the functions $(\alpha,n)\mapsto\Lambda(\alpha,n)$ that are associated with some function f and are written as,

$$\Lambda(\alpha,n)=f\left(\frac{q_{k-1}(\alpha)}{n},\frac{q_k(\alpha)}{n}\right), \tag{2.8}$$

as soon as $n\in[q_{k-1}(\alpha),q_k(\alpha)[$.
In the following, we restrict ourselves to a function f that satisfes the following three properties

(i) it is written as the non trivial quotient of two linear functions

$$f(x,y)=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\,; \tag{2.9}$$

(ii) it is defined on the unbounded rectangle

$$\mathcal{R}:=\{(x,y)\mid 0<x\le 1<y\},$$

(iii) it is non negative on $\mathcal{R}$.

A function Λ which is written as in (2.8) in terms of such a function f is called a continuant-function, or a $\mathcal{Q}$-function.

Our four parameters of interest, namely the recurrence quotient, the ratio ρ and the two parameters which describe the position of integer n with respect to the interval $[q_{k-1}(\alpha),q_k(\alpha)[$ are $\mathcal{Q}$-functions, associated to the following functions f

$$f_S(x,y)=1+x+y,$$

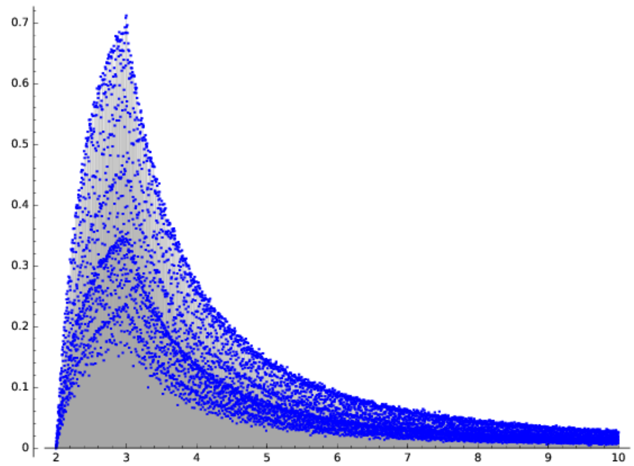
$$f_\rho(x,y)=\frac{x}{y},\quad f_\mu(x,y)=\frac{1-x}{y-x},\quad f_\nu(x,y)=\frac{1}{y}\,.$$

Ustinov has already considered in [11] a similar framework, with a definition of continuant functions which is slightly different from ours. We return to his model in the conclusion.

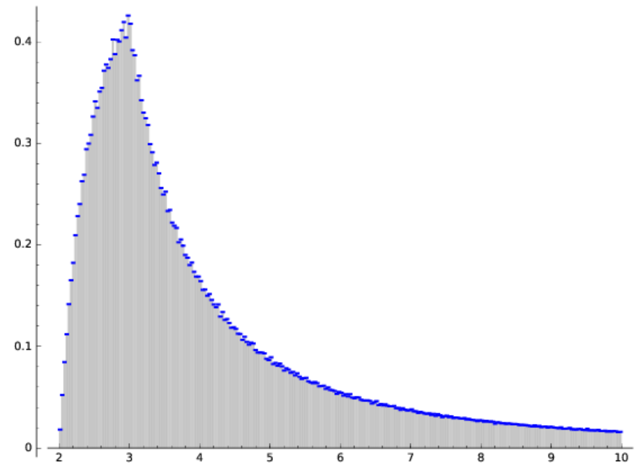
2.4 Probabilistic setting. We recall the present setting, already described in the introduction. We consider a fixed integer n, and a random real α in the unit interval $[0,1]$. The sequence $\Lambda_n(\alpha):=\Lambda(\alpha,n)$ is now a sequence of random variables. We are interested in the limit distribution of the sequence when $n\to\infty$. Does there exist a limit distribution? a limit density? Ustinov also adopts the same probabilistic model. His main motivation was mainly answering a question asked by Sinai and Ulcigrai [9], about the distributions of truncated continued fraction expansions. He does not study limit densities.

2.5 General results - distributions. In the distribution study, we associate with a real $\lambda\ge 0$ the subdomain of $\mathcal{R}$,

$$\Delta_f(\lambda):=\{(x,y)\mid 0\le x\le 1\le y;\ f(x,y)\le\lambda\} \tag{2.10}$$



(a) Experimental histogram for $S(\alpha, n)$ with step $\epsilon(n) = 1/n$.



(b) Experimental histogram for $S(\alpha, n)$ with step $\epsilon(n) = 1/\lceil\sqrt{n}\rceil$.

Figure 3: Limiting densities for the sequence $n \mapsto S(\alpha, n)$ as estimated by the scaled histograms. The number of experiments is $M = 10^7$, while $n = 1000$. The histograms have been scaled so that they integrate to 1.

(which is a convex domain due to the particular form of the function f), and associate the integral

$$I_f(\lambda) = \iint_{\Delta_f(\lambda)} \omega(x, y) dx dy = I[\omega, \Delta_f(\lambda)] , \tag{2.11}$$

which involves the function ω defined on $\mathcal{R}$ as

$$\omega(x, y) = \frac{1}{y(x + y)} , \tag{2.12}$$

whose integral on $\mathcal{R}$ satisfies $I(\omega, \mathcal{R}) = \pi^2/12$. The associated density

$$\psi(x, y) = \frac{12}{\pi^2} \frac{1}{y(x + y)} \tag{2.13}$$

plays a fundamental role in the sequel, as our originally discrete distribution smooths out (converges weakly) to the distribution associated with the density ψ, as the following result shows:

THEOREM 2.1. *Consider a $\mathcal{Q}$-function associated with a function f. Then the sequence $n \mapsto \Lambda_n(\alpha)$ as $n \to \infty$ admits a limit distribution, and the sequence*

$$F_n(\lambda) := \mathbb{P}[\Lambda_n \le \lambda] = \frac{12}{\pi^2} I_f(\lambda) + O\left(\frac{1}{n}\right) , \tag{2.14}$$

involves the integral $I_f(\lambda)$ defined in (2.11). Moreover, the constant does not depend on the pair (f, λ)

2.6 General results - densities. For the densities, we deal with boundary curves $\{(x, y) \mid f(x, y) = \lambda\}$ and their intersection with $\mathcal{R}$. We prove the following:

THEOREM 2.2. *Consider a $\mathcal{Q}$-function associated with a function f which is written as in (2.9). Then,*

(a) The function $\lambda \mapsto I_f(\lambda)$ and its derivative J_f exist for any λ. The derivative J'_f exists except perhaps on a finite set, which contains the point b_1/b_2 and two possible other values λ_0 and λ_1. The following holds:

(i) At each of the points $\lambda = \lambda_i$, the function J_f admits a left and a right derivative, each of them being finite.

(ii) When the determinant $r(a, b) := a_1 b_2 - a_2 b_1$ is zero, the derivative J'_f exists at $\lambda = b_1/b_2$.

(iii) When the determinant $r(a, b) := a_1 b_2 - a_2 b_1$ is not zero, the derivative J'_f does not exist at b_1/b_2 and is $O(|b_2\lambda - b_1|^{-1})$ for $\lambda \to b_1/b_2$.

(b) For any stricly positive sequence $n \mapsto \epsilon(n)$ which tends to 0 with $n\epsilon(n) \to \infty$, the secants of the distribution F_n with step $\epsilon(n)$ converge to $J_f(\lambda)$ and the following holds

$$\frac{F_n(\lambda + \epsilon(n)) - F_n(\lambda)}{\epsilon(n)} = \frac{12}{\pi^2} J_f(\lambda) + E(\lambda, \epsilon(n)) , \tag{2.15}$$

(c) The error term satisfies

$$E(\lambda, \epsilon(n)) = O\left(\frac{1}{\epsilon(n) n}\right) + O\Big(|J'_f(\lambda)| \epsilon(n)\Big) ,$$

and the constants in the O-term do not depend on the pair (f, λ).

2.7 Return to the parameters under study. We now apply the previous two results to the quotient recurrence S together with the three geometric parameters. Figure 2 exhibits the limit densities, whereas Figure 3 focuses on the recurrence quotient and compares scaled experimental histograms to the limit density.

Our theorems entail the following estimates for the recurrence quotient

$$\lim_{n\to\infty} \mathbb{P}\left[S_n \in [2,3]\right] = \frac{6}{\pi^2}(\log 2)^2\,,$$

$$\text{for } b \geq 2 \quad \lim_{n\to\infty} \mathbb{P}\left[S_n \geq b+1\right] = \frac{12}{\pi^2}\mathrm{Li}_2\left(\frac{1}{b}\right)\,,$$

which involve the dilogarithm $\mathrm{Li}_2(x) := \sum_{k\geq 1} \frac{x^k}{k^2}$.

This means (for instance) that for large enough n the probability $\mathbb{P}[S_n \leq 14]$ exceeds 0.9. We also derive a tail-inequality valid for all n, and any $b \geq 3$,

$$\mathbb{P}[S_n \geq b] \leq \frac{2}{b-1}\,.$$

2.8 Conditional expectations. We now focus on the position parameters ρ, ν and μ defined in (2.4) and (2.5), and consider the three sequences

$$\rho_n(\alpha) := \rho(\alpha, n), \quad \nu_n(\alpha) := \nu(\alpha, n), \quad \mu_n(\alpha) := \mu(\alpha, n)\,.$$

We have explained that the largest values of the recurrence quotient arise when ν or μ are small. In particular, the event $[\nu_n \geq \epsilon(n)]$ gathers the reals α for which the integer n is not too close of the left end of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$, and, at the same time, the length of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ is of the same order as the right end $q_k(\alpha)$. We then consider a sequence $\epsilon(n) \to 0$, and condition with one of the events

$$[\rho_n \geq \epsilon(n)], \quad [\nu_n \geq \epsilon(n)], \quad [\mu_n \geq \epsilon(n)]\,.$$

Theorem 2.3. *Consider a parameter $\Gamma \in \{\rho, \mu, \nu\}$ defined in (2.4) and (2.5) and a sequence $\epsilon(n)$ which is $\Omega(1/(n \log n))$. Then the conditional expectation of the recurrence quotient S_n with respect to the event $[\Gamma_n \geq \epsilon(n)]$ satisfies*

$$\mathbb{E}\left[S_n \middle| \Gamma_n \geq \epsilon(n)\right] \sim_{n\to\infty} \frac{12}{\pi^2}\left|\log \epsilon(n)\right|\,.$$

This result exhibits a sequence of events, for which the integer n is not too close to the left end of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$. When we are sure not to be too close to this left end, the recurrence quotient is (on average) roughly of logarithmic order.

This can be viewed as a probabilistic counterpart of Proposition 1.1, in the case of particular sequences of the form $\epsilon(n) = 1/(n\gamma(n))$, for which the series of general term $\epsilon(n)$ is divergent. The logarithm which appears in Theorem 2.3 arises naturally from the expressions of the recurrence quotient $f_S(x,y)$ and the smoothed density $\psi(x,y)$ given in Figure 2 and (2.13). The central quantity is indeed

$$f_S(x,y)\psi(x,y) = \psi(x,y) + 1/y,$$

so any condition $[\Gamma_n \geq \epsilon(n)]$ that makes f_S bounded, must bound y, and integrating $1/y$ produces the logarithm. We return to this study in Section 4.

3 Proofs of Theorems 2.1 and 2.2.

We first introduce the main objects of interest: continued fraction expansions and coprime Riemann sums. Then, we prove the existence of limit distribution and limit densities for a general $\mathcal{Q}$-function. The proof of Theorem 2.1 has three main steps, described in Sections 3.3, 3.4, and 3.5, and we conclude the proof of Theorem 2.1 in Section 3.6. Section 3.7 and 3.8 are devoted to the proof of Theorem 2.2.

3.1 Continued fractions, fundamental intervals and continuants. (See here [4] for more details). The continued fraction of an irrational number α of the unit interval $[0,1]$ is

$$\alpha = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_k + \cfrac{1}{\ddots}}}}}.$$

Truncated at depth k, it gives rise to a rational number p_k/q_k associated with a coprime integer pair (p_k, q_k). The numerator $p_k = p_k(\alpha)$ and the denominator $q_k = q_k(\alpha)$ are uniquely defined by the irrational number α. All the irrational numbers α which begin with the same sequence $\boldsymbol{m} = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k$ belong to an interval, called a fundamental interval of depth k and denoted by $I_k(\boldsymbol{m})$. As the irrational numbers of $I_k(\boldsymbol{m})$ have the same convergents of order $\ell \leq k$, we denote their numerator and denominator by $p_\ell(\boldsymbol{m}), q_\ell(\boldsymbol{m})$. The ends of the interval $I_k(\boldsymbol{m})$ are

$$\frac{p_k(\boldsymbol{m})}{q_k(\boldsymbol{m})}, \quad \frac{p_k(\boldsymbol{m}) + p_{k-1}(\boldsymbol{m})}{q_k(\boldsymbol{m}) + q_{k-1}(\boldsymbol{m})}.$$

As the equality $|p_k(\boldsymbol{m})q_{k-1}(\boldsymbol{m}) - p_{k-1}(\boldsymbol{m})q_k(\boldsymbol{m})| = 1$ holds, the length of the fundamental interval involves the function ω defined in (2.12) under the form

$$|I_k(\boldsymbol{m})| = \omega(q_{k-1}(\boldsymbol{m}), q_k(\boldsymbol{m}))\,. \tag{3.16}$$

This explains why the function ω defined in (2.12) and the associated density ψ are ubiquitous in the study of the $\mathcal{Q}$-functions.

3.2 Distributions. Strategy of the proof. There are two main steps in the proofs of Theorem 2.1.

(*i*) *Discrete step.* We express in Proposition 3.1 the distribution of a $\mathcal{Q}$-function in terms of a variant of a Riemann sum, that is called in the following a "coprime" Riemann sum.

(*ii*) *Continuous step.* We compare the "coprime" Riemann sum to the associated integral. We begin by the comparison of the "plain" Riemann sum to the integral in Proposition 3.2, then, we take into account the coprimality condition in Proposition 3.3.

This general framework, in particular the coprime Riemann sums, was already introduced and used in [3]. Even if Ustinov does not explicitely deal in [11] with this type of Riemann sums, we guess that he uses them in an implicit way. Here, we make precise and extend the framework of the two previous papers [3] or [11] which only deal with finite domains.

3.3 Distributions and Riemann sums. We begin with the alternative expression of a $\mathcal{Q}$-function Λ, associated with f, (already defined in (2.8)), which is written with the Iverson bracket[1] under the form

$$\Lambda(\alpha, n)$$
$$= \sum_{k\geq 0} f\left(\frac{q_{k-1}(\alpha)}{n}, \frac{q_k(\alpha)}{n}\right) \left[\!\left[n \in [q_{k-1}(\alpha), q_k(\alpha)[\right]\!\right].$$

The distribution of a $\mathcal{Q}$-function associated with f is

$$\mathbb{P}[\Lambda_n \leq \lambda] = \int_0^1 d\alpha \sum_{k\geq 0} \left[\!\left[\left(\frac{q_{k-1}(\alpha)}{n}, \frac{q_k(\alpha)}{n}\right) \in \Delta_f(\lambda) \right]\!\right].$$

For each k, the family of fundamental intervals $I_k(\boldsymbol{m})$ defines a pseudo-partition when $\boldsymbol{m}$ goes through $\mathbb{N}^k$, and, for any $\alpha \in I_k(\boldsymbol{m})$, the equality $q_k(\alpha) = q_k(\boldsymbol{m})$ holds. We deduce

$$\mathbb{P}[\Lambda_n \leq \lambda]$$
$$= \sum_{k=0}^{\infty} \sum_{\boldsymbol{m}\in\mathbb{N}^k} |I_k(\boldsymbol{m})| \left[\!\left[\left(\frac{q_{k-1}(\boldsymbol{m})}{n}, \frac{q_k(\boldsymbol{m})}{n}\right) \in \Delta_f(\lambda) \right]\!\right].$$

[1] The Iverson bracket is a Boolean function defined by $[\mathcal{P}] = 1$ as soon as Property $\mathcal{P}$ is true

Then, with the expression of the length $|I_k(\boldsymbol{m})|$ in terms of the function ω given in (3.16) and the fact that ω is homogeneous of degree -2, we obtain

$$|I_k(\boldsymbol{m})| = \frac{1}{n^2}\, \omega\left(\frac{q_{k-1}(\boldsymbol{m})}{n}, \frac{q_k(\boldsymbol{m})}{n}\right).$$

Now, as we go through all the sequences $\boldsymbol{m} \in \mathbb{N}^\star$, the coprime pairs $(q_{k-1}(\boldsymbol{m}), q_k(\boldsymbol{m}))$ give rise to all the coprime pairs (a, b). Moreover, each coprime pair (a, b), except the pair $(1, 1)$, appears exactly twice, due to the existence of two continued fraction expansions, the proper one (in which the last digits strictly greater than 1), and the improper one (in which the last digit is equal to 1). Then, the equality holds

$$\mathbb{P}[\Lambda_n \leq \lambda] = \frac{2}{n^2} \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)=1}} \omega\left(\frac{a}{n}, \frac{b}{n}\right) \left[\!\left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Delta_f(\lambda) \right]\!\right].$$

The right member is the Riemann sum of the function 2ω on the domain $\Delta_f(\lambda)$ with step $1/n$, with an extra condition of coprimality. More generally, for a function g integrable on a subset Ω, we are led to the following two Riemann sums with step $1/n$: the first one $R_n(g, \Omega)$ is the usual one,

$$R_n(g, \Omega) = \frac{1}{n^2} \sum_{(a,b)\in\mathbb{Z}^2} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\!\left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega \right]\!\right],$$

whereas the second one $\widehat{R}_n(g, \Omega)$ takes into account the coprimality of (a, b), and is called the "coprime" Riemann sum,

$$\widehat{R}_n(g, \Omega) = \frac{1}{n^2} \sum_{\substack{(a,b)\in\mathbb{Z}^2\\ \gcd(a,b)=1}} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\!\left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega \right]\!\right],$$

We summarize:

PROPOSITION 3.1. *Consider a $\mathcal{Q}$-function Λ associated with a function f. Then the distribution $F_n(\lambda) := \mathbb{P}[\Lambda_n \leq \lambda]$ is expressed with a coprime Riemann sum,*

$$\mathbb{P}[\Lambda_n \leq \lambda] = \widehat{R}_n(2\omega, \Delta_f(\lambda)). \tag{3.17}$$

which involves the density ω defined in (2.12) *and the domain $\Delta_f(\lambda)$ defined in* (2.10).

The previous result extends if we replace $\Delta_f(\lambda)$ by any other domain $\Omega \subset \mathcal{R}$. In particular, in Section 4, we will deal with two $\mathcal{Q}$-functions Λ and Γ associated respectively to f and g, together with the domain

$$\underline{\Delta}_{f,g}(\lambda, \epsilon) := \{(x, y) \in \mathcal{R} \mid f(x, y) \geq \lambda, g(x, y) \geq \epsilon\}, \tag{3.18}$$

and use the equality

$$\mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \widehat{R}_n\left(2\omega, \underline{\Delta}_{f,g}(\lambda, \epsilon)\right). \tag{3.19}$$

3.4 Usual Riemann sums and integrals. We first deal with the usual Riemann sum, and compare it to its associated integral $I(g,\Omega)$. This is a classical proof, but we consider improper integrals and we wish to have precise error terms.
We now deal (only within this subsection) with

$$\mathcal{S} := [0,1]\times(0,\infty)\,, \tag{3.20}$$

consider a subset $\Omega\subset\mathcal{S}$ and associate with it the family of subsets

$$\Omega(k) := \Omega\cap([0,1]\times[k,k+1])\,,$$

for any $k\geq 1$, which form a pseudo-partition of Ω. We also consider a positive function g defined on Ω of class $\mathcal{C}^1$, bounded on any bounded subset on Ω for which the following two finite bounds[2]

$$C_g(\Omega,k) := \sup\{g(x,y)\mid (x,y)\in\Omega(k)\},$$

$$D_g(\Omega,k) := \sup\left\{\left|\frac{\partial g}{\partial y}(x,y)\right| \mid (x,y)\in\Omega(k)\right\}\,,$$

define sequences whose associated series are convergent,

$$\sum_{k\geq 0} C_g(\Omega,k)<\infty,\qquad \sum_{k\geq 0} D_g(\Omega,k)<\infty\,.$$

Their sums are denoted by $C_g(\Omega)$ and $D_g(\Omega)$, and we denote by $M_g(\Omega)$ their maximum.

Such a function g is called strongly decreasing on Ω with bound $M_g(\Omega)$. Such a function is integrable on Ω and the inequality $I(g,\Omega)\leq M_g(\Omega)$ holds.

PROPOSITION 3.2. *Consider the domain $\mathcal{S}$ defined in* (3.20) *and a function g which is strongly decreasing on a convex $\Omega\subset\mathcal{S}$ with bound $M_g(\Omega)$. Then, the Riemann sum of the function g on Ω compares to the integral,*

$$|R_n(g,\Omega)-I(g,\Omega)|\leq \frac{5}{n}M_g(\Omega)\,. \tag{3.21}$$

Proof. We will prove the estimate, for each $k\geq 0$,

$$|R_n(g,\Omega(k))-I(g,\Omega(k)|\leq\frac{4}{n}\left(C_g(\Omega,k)+D_g(\Omega,k)\right).$$

This will entail the result by taking the sum over $k\geq 0$.

We consider the elementary squares of side $1/n$, namely

$$\mathcal{R}_{a,b}=\left[\frac{a}{n},\frac{a+1}{n}\right]\times\left[\frac{b}{n},\frac{b+1}{n}\right],$$

and we concentrate on those which meet $\Omega(k)$. There are two cases for such rectangles $\mathcal{R}_{a,b}$, namely

$$(i)\ \ \mathcal{R}_{a,b}\subset\Omega(k),\qquad \text{or}\quad (ii)\ \ \mathcal{R}_{a,b}\cap\Omega(k)^c\neq\emptyset\,.$$

In the first case (i), the definition of the bound D_g entails the estimate

$$\left|\frac{1}{n^2}g\left(\frac{a}{n},\frac{b}{n}\right)-I(g,\mathcal{R}_{a,b})\right|$$

$$\leq I\left(\left|g\left(\frac{a}{n},\frac{b}{n}\right)-g\right|,\mathcal{R}_{a,b}\right)\leq\frac{1}{n^3}\,D_g(\Omega,k)\,.$$

As the number of such squares is at most n^2, the contribution from case (i) is at most $(1/n)\,D_g(\Omega,k)$.

In the second case (ii), the positivity of g and the definition of the bound C_g entails the estimate

$$\left|\frac{1}{n^2}g\left(\frac{a}{n},\frac{b}{n}\right)-I(g,\Omega\cap\mathcal{R}_{a,b})\right|\leq\frac{1}{n^2}\,C_g(\Omega,k)\,.$$

But, the convexity of Ω entails that there are at most $4n$ such squares, and the contribution of the second case is at most $(4/n)\,C_g(\Omega,k)$.
To see where the constant 4 comes from, we first replace $\Omega(k)$ by a closed convex polygon $\mathcal{C}_n\subset\Omega$, without affecting the bound : in each square $\mathcal{R}_{a,b}$ of the second case, pick a point in $\Omega(k)$ and then take the convex hull. If $\Omega(k)$ is a closed convex polygon, we go through the border in clockwise order and look at the grid rectangles we encounter as explained in Figure 4. ■

3.5 Coprime Riemann sums and integrals. The following result is an extension of the results obtained in [3], that are only proven for finite domains.

PROPOSITION 3.3. *Consider a positive function g defined on $\mathcal{R}$, homogeneous of degree $-\beta$ there with $\beta>1$. Such a function is strictly decreasing on $\mathcal{R}$. Consider also a convex subset $\Omega\subset\mathcal{R}$. Then, the coprime Riemann sum of the function g on Ω compares to the integral of g on Ω, namely*

$$\left|\widehat{R}_n(g,\Omega)-\frac{6}{\pi^2}I(g,\Omega)\right|\leq\frac{1}{n}\left(1+5\zeta(\beta)\right)M_g(\mathcal{R})\,.$$

Proof. To filter the cases in which $\gcd(a,b)>1$, we use the Mobius function μ which performs "inclusion-exclusion". The Mobius function $\mu:\mathbb{N}\to\{-1,0,+1\}$ satisfies

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\,.\end{cases} \tag{3.22}$$

[2] By convention, we consider that $C_g(\Omega,k)$ and $D_g(\Omega,k)$ are 0 if the set $\Omega(k)$ is empty.

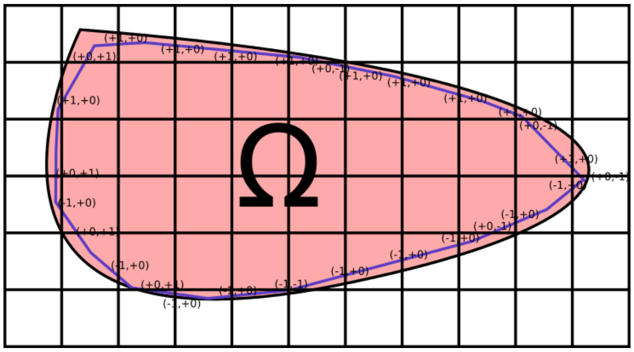


Figure 4: The convex domain Ω, and, in blue, a convex polytope $\mathcal{P}$. These two convex sets have the same grid squares that intersect both themselves and their complement. We traverse the polygon clockwise from the lowest vertex. Each time we intersect a horizontal line we move ± 1 square horizontally in the grid, similarly for the vertical lines, and diagonals. Being the polygon convex, once we stop moving upwards vertically (at most n steps), we can only move downwards (at most n steps) when moving vertically. A similar observation for the horizontal case tells us that there can be at most $2n$ horizontal steps.

We consider the restricted "coprime" Riemann sum, where the sum is taken over the pairs (a, b) with $\gcd(a, b) = 1$, namely

$$n^2\, \widehat{R}_n(g, \Omega) = \sum_{\substack{(a,b)\in\mathbb{Z}^2\\ \gcd(a,b)=1}} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\!\left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega \right]\!\right].$$

We then "insert" the μ-function inside this restricted "coprime" Riemann sum,

$$n^2\, \widehat{R}_n(g, \Omega)$$
$$= \sum_{(a,b)\in\mathbb{Z}^2} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\!\left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega \right]\!\right] \left(\sum_{d \mid \gcd(a,b)} \mu(d) \right).$$

As the point $(a/n, b/n)$ belongs to $\mathcal{R}$ with $a > 0$, the inequality $\gcd(a, b) \le n$ holds. Then, interverting the summations entails the equality

$$n^2\, \widehat{R}_n(g, \Omega)$$
$$= \sum_{d\le n} \mu(d) \sum_{(a,b)\in\mathbb{Z}^2} g\left(\frac{ad}{n}, \frac{bd}{n}\right) \left[\!\left[\left(\frac{ad}{n}, \frac{bd}{n}\right) \in \Omega \right]\!\right].$$

Finally, the following equality holds

$$\widehat{R}_n(g, \Omega) = \sum_{d\le n} \mu(d)\, R_n(g_d, \Omega_d)\,, \tag{3.23}$$

and involves the function g_d and the subset Ω_d defined as

$$g_d(x, y) := g(dx, dy), \qquad \Omega_d = \frac{1}{d}\Omega\,.$$

As the inclusion $\Omega_d \subset \mathcal{S}$ holds, we now apply the previous Proposition 3.2 to each (plain) Riemann sum $R_n(g_d, \Omega_d)$ and obtain

$$|R_n(g_d, \Omega_d) - I(g_d, \Omega_d)| \le \frac{5}{n} M_{g_d}(\Omega_d)\,. \tag{3.24}$$

We now use three properties. We first remark the equality

$$I(g_d, \Omega_d) = \frac{1}{d^2} I(g, \Omega)\,,$$

due to the change of variables $(x', y') = (dx, dy)$. Second, the series of general term $\mu(d)/d^2$ is convergent, and, with the Mobius inversion, its sum equal $1/\zeta(2)$ and

$$\left| \sum_{d\le n} \frac{\mu(d)}{d^2} - \frac{6}{\pi^2} \right| \le \frac{1}{n}\,.$$

Third, we relate the bound $M_{g_d}(\Omega_d)$ to its analogous. As g is homogeneous of degree $-\beta$, its derivative is homogeneous of degree $(-\beta - 1)$ and the two relations

$$g_d(x, y) = g(dx, dy) = \frac{1}{d^\beta} g(x, y)\,,$$

$$\frac{\partial g_d}{\partial y}(x, y) = d \frac{\partial g}{\partial y}(dx, dy) = \frac{1}{d^\beta} \frac{\partial g}{\partial y}(x, y)\,,$$

hold for $(x, y) \in \mathcal{R}$. As g and its derivative are 0 outside $\mathcal{R}$, the same holds for g_d and its derivative, and

$$M_{g_d}(\Omega_d) = M_{g_d}(\Omega_d \cap \mathcal{R}) = \frac{1}{d^\beta} M_g(\Omega_d \cap \mathcal{R}) \le \frac{1}{d^\beta} M_g(\mathcal{R})\,.$$

Then, as $\beta > 1$, one has

$$\sum_{d\le n} M_{g_d}(\Omega_d) \le \zeta(\beta) M_g(\mathcal{R})\,.$$

With the three previous properties, together with Eq. (3.24), we obtain the final result. ■

3.6 Distributions. Proof of Theorem 2.1. Theorem 2.1 is a particular case of the previous Proposition 3.3, when it applies to ω and $\Delta_f(\lambda)$ defined in (2.12) and (2.10). The function ω is homogeneous of degree -2 and the domain $\Delta_f(\lambda)$ is convex, as it is the intersection of the unbounded rectangle $\mathcal{R}$ with the halfplane $\{f(x, y) \le \lambda\}$. Applying Proposition 3.3 then proves Theorem 2.1.

3.7 Proof of Theorem 2.2. First step. We first prove Assertion (b) assuming Assertion (a). We let

$$F_n(\lambda) := \mathbb{P}[\Lambda_n \le \lambda], \qquad F_\infty(\lambda) = \frac{12}{\pi^2} I_f(\lambda)\,.$$

We know from Assertion (a) that the derivative $J_f(\lambda)$ of $\lambda \mapsto I_f(\lambda)$ exists. This is the same for the function F_∞ and we wish to estimate the difference

$$\left|\frac{F_n(\lambda+\epsilon(n)) - F_n(\lambda)}{\epsilon(n)} - F'_\infty(\lambda)\right| .$$

We begin with the triangle inequality

$$(3.25)\qquad \left|\frac{F_n(\lambda+\epsilon(n)) - F_n(\lambda)}{\epsilon(n)} - F'_\infty(\lambda)\right|$$

$$\leq \left|\frac{F_n(\lambda+\epsilon(n)) - F_\infty(\lambda+\epsilon(n))}{\epsilon(n)}\right| + \left|\frac{F_\infty(\lambda) - F_n(\lambda)}{\epsilon(n)}\right|$$

$$+ \left|\frac{F_\infty(\lambda+\epsilon(n)) - F_\infty(\lambda)}{\epsilon(n)} - F'_\infty(\lambda)\right| .$$

With the special form of function f, the domain $\Delta_f(\lambda)$ is convex, and Theorem 2.1 provides the estimates

$$|F_n(\lambda) - F_\infty(\lambda)| = O(1/n) ,$$
$$|F_n(\lambda+\epsilon(n)) - F_\infty(\lambda+\epsilon(n))| = O(1/n) ,$$

where the constant in the O-terms does not depend on λ and $\epsilon(n)$. Then, the first two terms in Inequality (3.25) are $O(1/(n\epsilon(n))$ and tend to 0 because $n\epsilon(n) \to \infty$. For the last term in (3.25), we use Taylor expansion of order 2 of the function F_∞ together Assertion (a). This ends the proof of Assertion (b) assuming Assertion (a).■

3.8 Proof of Theorem 2.2. Second step.

We now prove Assertion (a). There will be four sub-steps.

The set of lines $\mathcal{F}$. In the set $\mathcal{F}$ of lines, defined as

$$\mathcal{F} := \{f(x,y) = \lambda \mid \lambda \in \mathbb{R}\},$$

the equation of the line $f(x,y) = \lambda$ is written in terms of coefficients described in (2.9) as

$$(3.26)\qquad (a_1x + b_1y + c_1) - \lambda\,(a_2x + b_2y + c_2) = 0 .$$

The case where the two vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are colinear is excluded, as in this case $f(x,y)$ is constant. The case where $b_1 = b_2 = 0$ is also excluded as we wish that f depend on y. Then, there is at most one vertical line in $\mathcal{F}$.

There are two cases for the set $\mathcal{F}$ defined in (3.26).

(i) the case when the determinant $r(a,b) := a_1b_2 - a_2b_1$ is zero and in this case the determinant $r(a,c) := a_1c_2 - a_2c_1$ is not zero. The set $\mathcal{F}$ is formed with parallel lines of slope $-a_1/b_1$. This is for instance the case of the recurrence quotient with slope -1 or the case of ν with slope 0.

(ii) the case when the determinant $r(a,b) := a_1b_2 - a_2b_1$ is not zero. In this case, we can choose $r(a,b) = 1$ due to the homogeneity of the problem. Then, the set $\mathcal{F}$ is formed with all the lines which contain the point (x_0, y_0) uniquely defined by the relations

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix} \quad\text{or}\quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} r(b,c) \\ -r(a,c) \end{pmatrix} .$$

The point (x_0, y_0) is called the basic point of $\mathcal{F}$. Remark that case (i) can be seen as the limit of the case (ii) when (x_0, y_0) tends to ∞ in the direction a_1/b_1. The basic points attached to our parameters ρ, μ are $(0,0)$ for ρ and $(1,1)$ for μ.

In the set $\mathcal{F}$ of basic point (x_0, y_0), the value of λ and the inverse τ of the slope $1/\tau$ of the line $f(x,y) = \lambda$, are related via linear fractional transformations with determinant equal to 1, namely

$$(3.27)\quad \lambda = F(\tau) = \frac{a_1\tau + b_1}{a_2\tau + b_2},\ \tau = G(\lambda) = -\frac{b_2\lambda - b_1}{a_2\lambda - a_1}.$$

In the set $\mathcal{F}$ of basic point (x_0, y_0), the parametrization of the line $f(x,y) = \lambda$ of slope $1/\tau$ is thus

$$x = x_0 + \tau(y - y_0), \qquad \tau = G(\lambda) .$$

Expressions of I_f and its derivative. Consider a function f as in (2.9); denote by $\delta_f(\tau)$ the segment (possibly empty or unbounded) which is the intersection of the line $f(x,y) = \lambda = F(\tau)$ of slope $1/\tau$ with the rectangle $\mathcal{R}$. Now, the function f is fixed, the point (x_0, y_0) is fixed, and all the indices which involve f are removed. There is an open interval D which gathers the values of τ for which the segment $\delta(\tau)$ is not empty, and we denote by $A(\tau), B(\tau)$ the ordinates of the two ends of the segment $\delta(\tau)$.

As soon as the line $f(x,y) = F(\tau)$ is not horizontal, we consider the natural parametrization h_τ of the segment $\delta(\tau)$, namely a map $h_\tau\ :]A(\tau), B(\tau)[\to\ \delta(\tau)$ which associates to y the point

$$h_\tau(y) = h(\tau, y) = (x_0 + \tau(y - y_0), y)$$

of the segment $\delta(\tau)$. The map $\tau \mapsto h_\tau$ is of class $\mathcal{C}^\infty$ on the interval D.

We deal with the integral $L(\tau) := I_f(F(\tau))$, where I_f and F are defined in (2.11) and (3.27). As F is monotonic, we assume without loss of generality that it is increasing (otherwise take $\lambda \mapsto -\lambda$), the domain $\Delta_f(F(\tau))$ defined in (2.10) is written as an union of segments, namely

$$\Delta_f(F(\tau)) = \bigcup_{\theta \leq \tau} \delta(\theta) .$$

Using the change of variables $(\theta, y) \mapsto (h(\theta, y), y)$, and its Jacobian $|(\partial h)/(\partial\theta)(\theta, y)| = |y - y_0|$, the integral is written as

$$L(\tau) = \int_{-\infty}^{\tau} d\theta \int_{A(\theta)}^{B(\theta)} Q(\theta, y) dy\,,$$

$$\text{with}\quad Q(\theta, y) = |y - y_0|\, \omega(x_0 + \theta(y - y_0), y)\,.$$

Then the derivative of L admits the expression

$$L'(\tau) = \int_{A(\tau)}^{B(\tau)} Q(\tau, y) dy\,. \tag{3.28}$$

The function L' is itself differentiable on the set D, except perhaps on a finite set (as we will see in the next paragraph) and involves the previous functions under the form

$$L''(\tau) = \int_{A(\tau)}^{B(\tau)} \frac{\partial Q}{\partial \tau}(\tau, y) dy \tag{3.29}$$

$$+B'(\tau)\, Q(\tau, B(\tau)) - A'(\tau)\, Q(\tau, A(\tau))\,, \tag{3.30}$$

$$\text{with}\quad \frac{\partial Q}{\partial \tau}(\tau, y) = \frac{\partial \omega}{\partial x}(x_0 + \tau(y - y_0), y)\, |y - y_0|^2\,.$$

We prefer to deal with the function L, as it is easier to "see the geometry". We will return to the function I and its two derivatives with the relations

$$I'(\lambda) = \frac{L'(\tau)}{F'(\tau)}, \quad I''(\lambda) F'(\tau)^2 = L''(\tau) - L'(\tau)\frac{F''(\tau)}{F'(\tau)}\,, \tag{3.31}$$

and use the special form of F defined in (3.27).

The role of the corners. The values of τ in D for which I' is *a priori* not differentiable are those for which the line of slope $1/\tau$ is vertical (namely $\tau = 0$) or meets one of the two "corners" of $\mathcal{R}$, namely the slope $1/\tau_0$ for which it meets the point $(0, 1)$, and the slope $1/\tau_1$ for which it meets the point $(0, 1)$.
There are now two different geometric cases, the generic case $(\mathcal{G})$ or the exceptional case $(\mathcal{E})$:

($\mathcal{G}$) If the point (x_0, y_0) does not belong to the line $y = 1$, there are exactly two lines in $\mathcal{F}$, each of them containing one corner of $\mathcal{R}$, associated with two distinct values τ_0 and τ_1.

($\mathcal{E}$) If the point (x_0, y_0) belongs to the line $y = 1$, there is only one value $\tau_0 = \tau_1 = \infty$.

Finally, there are at most three values of τ in the set $\{0, \tau_0, \tau_1\}$ where L' is possibly not differentiable. But, L' possesses at each finite τ_i a left and a right derivative, each of them being finite. This is thus the same for the derivative I' of the function I. At $\tau = 0$, the derivatives $F'(0)$ and $F''(0)$ are finite as soon as $b_2 \neq 0$.

Behaviour of $L''(\tau)$ ***for*** $\lambda \to 0$**.** The ratio $R(\tau) := B(\tau)/A(\tau)$ is important, as the estimates

$$Q(\tau, y) = \Theta(y^{-1}), \qquad \frac{\partial Q}{\partial \tau}(\tau, y) = \Theta(y^{-1})$$

entail that $L'(\tau)$ and the first term of $L''(\tau)$ in (3.29) are both $\Theta(\log R(\tau))$.
The bound $B(\tau)$ always tends to $+\infty$ but there are two cases for $A(\tau)$: it remains bounded or not.

(i) The case when $A(\tau)$ remains bounded occurs if and only if the basic point belongs to one of the two vertical lines $x_0 = 1$ or $x_0 = 1$. Then the estimates $R(\tau) = \Theta(\tau^{-1})$ and $L'(\tau) = \Theta(\log \tau)$, directly entail that $L''(\tau)$ is $\Theta(\tau^{-1})$.

(ii) If $A(\tau)$ tends also to ∞, then the ratio $R(\tau)$ tends to $|x_0 - 1|/|x_0|$, and this limit may be only finite non zero. Then, the derivatives $A'(\tau)$ et $B'(\tau)$ are $\Theta(\tau^{-2})$ whereas $A(\tau)$ and $B(\tau)$ are $\Theta(\tau^{-1})$ and thus $Q(\tau, B(\tau))$ and $Q(\tau, B(\tau))$ are $\Theta(\tau)$ and each term of (3.30) is $\Theta(\tau^{-1})$, whereas the first term in (3.29) tends to a finite limit. More precisely, the estimate holds,

$$B'(\tau) Q(\tau, B(\tau)) - A'(\tau) Q(\tau, A(\tau)) \sim \frac{1}{\tau}\,.$$

In the two cases, with (3.31), this ends the proof of Theorem 2.2 (a) and, together with Section 3.7, the proof of Theorem 2.2. ∎

4 Conditional expectations. Proof of Thm 2.3.

We now focus on conditional expectations. Our final purpose is to prove Theorem 2.3 which is devoted to the recurrence quotient. However, we begin by a more general study and we obtain in Section 4.3 a general result on conditional expectations (Theorem 4.2). We then apply it in Section 4.4 to the particular case of the recurrence quotient, and this provides Theorem 4.3, which can be viewed itself as an extension of Theorem 2.3.

4.1 Limit expectation of bounded $\mathcal{Q}$-functions.

Thus far, we dealt with distributions of $\mathcal{Q}$-functions. Now, we consider expected values of a $\mathcal{Q}$-function, and use the equality

$$\mathbb{E}[\Lambda_n] = \int_0^{\infty} \mathbb{P}[\Lambda_n \geq \lambda]\, d\lambda\,,$$

valid when $\Lambda \geq 0$, as in our case. We consider here the case of a $\mathcal{Q}$-function Λ associated with a bounded function f (which is the case when b_2 is not zero). It is then possible to interchange the limit and the integral and use Theorem 2.1.

We thus first integrate with respect to λ, and we are led to the integral

$$\mathbb{E}_\psi[f] := \frac{6}{\pi^2} I(f \cdot 2\omega, \mathcal{R}) \tag{4.32}$$

which is exactly the expectation $\mathbb{E}_\psi[f]$ of the function f on the rectangle $\mathcal{R}$ with respect to the density $\psi := (12/\pi^2)\,\omega$. We thus obtain the following result which provides an extension of Theorem 2.1:

THEOREM 4.1. *Consider a $\mathcal{Q}$-function Λ associated with a function f bounded by B_f. Then the sequence $n \mapsto \Lambda_n$ admits a limit expected value as $n \to \infty$ equal to the expectation $\mathbb{E}_\psi[f]$ of the function f on the rectangle $\mathcal{R}$ with respect to the density $\psi := (12/\pi^2)\,\omega$, and*

$$\mathbb{E}\left[\Lambda_n\right] = \mathbb{E}_\psi[f] + B_f\, O\left(\frac{1}{n}\right), \tag{4.33}$$

where the constant in the O-term does not depend on f and λ.

4.2 Case of the recurrence quotient.

The function f associated with the recurrence quotient $S(\alpha, n)$ is $f_S(x, y) = 1 + x + y$. It is unbounded on $\mathcal{R}$, and the function f_S is not integrable with respect to ψ. In fact, by the argument of Proposition 3.1 the expected value can be worked out to be

$$\mathbb{E}[S_n] = \widehat{R}_n(2\omega f_S), \mathcal{R}),$$

and here $\widehat{R}_n(2\omega f_S, \mathcal{R})$ is infinite for each n.

This is why we consider the conditional expectations for the sequence S_n with respect to an event $[\Gamma_n \geq \epsilon(n)]$ associated with another $\mathcal{Q}$-function Γ, namely

$$\mathbb{E}[S_n | \Gamma_n \geq \epsilon(n)]\,.$$

We will choose in the sequel the $\mathcal{Q}$-function Γ from the set $\{\mu, \nu, \rho\}$ and a positive sequence $\epsilon(n)$ tending to 0 not all too quickly.

4.3 General conditional expectations.

We first consider more general conditional expectations,

$$\mathbb{E}[\Lambda_n | \Gamma_n \geq \epsilon] \qquad (\epsilon > 0)\,,$$

when Γ is a $\mathcal{Q}$-function associated with a function g which tends to 0 for $y \to \infty$. (This means that the pair (b_1, b_2) in (2.9) satisfies $b_1/b_2 = 0$). The subset

$$\{(x, y) \in \mathcal{R} \mid g(x, y) \geq \epsilon\}$$

is bounded for $\epsilon > 0$, and we denote, for $\epsilon > 0$,

$$B_{f|g}(\epsilon) := \sup\{f(x, y)\,|(x, y) \in \mathcal{R},\ g(x, y) \geq \epsilon\}\,. \tag{4.34}$$

In this case, the expectation of f with respect to ψ conditioned to the event $[g \geq \epsilon]$ is well defined, and denoted as

$$\mathbb{E}_\psi[f | g \geq \epsilon]\,.$$

The following holds:

THEOREM 4.2. *Consider two $\mathcal{Q}$-functions Λ and Γ with respective associated functions f and g. Assume that g tends to 0 for $y \to \infty$. Then the conditional expectation of Λ_n with respect to the event $[\Gamma_n \geq \epsilon]$ satisfies*

$$\mathbb{E}[\Lambda_n | \Gamma_n \geq \epsilon] \cdot \mathbb{P}[\Gamma_n \geq \epsilon] = \mathbb{E}_\psi[f | g \geq \epsilon] \cdot \mathbb{P}_\psi[g \geq \epsilon]$$

$$+ B_{f|g}(\epsilon)\, O\left(\frac{1}{n}\right)$$

where $B_{f|g}(\epsilon)$ is defined in (4.34) and the constant in the O-term does not depend on either f, g or ϵ.

Proof. The proof is very similar to the proof of Theorem 2.1. The conditional expectation is a ratio; the denominator is $\mathbb{P}[\Gamma_n \geq \epsilon]$ whereas the numerator

$$\int_0^\infty \mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon]\, d\lambda\,.$$

Associate with the pair (Λ, Γ) its function pair (f, g) and, for any pair (λ, ϵ) of positive real numbers, consider the bounded convex subset already described in (3.18)

$$\underline{\Delta}_{f,g}(\lambda, \epsilon) := \{(x, y) \in \mathcal{R} \mid f(x, y) \geq \lambda, g(x, y) \geq \epsilon\}\,.$$

We have remarked in Section 3 that a slight extension of Proposition 3.1 entails the equality

$$\mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \widehat{R}_n\left(2\omega, \underline{\Delta}_{f,g}(\lambda, \epsilon)\right)\,.$$

Moreover, with the convexity of the domain $\underline{\Delta}_{f,g}(\lambda, \epsilon) \subset \mathcal{R}$, Proposition 3.3 applies, yielding

$$\mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \frac{12}{\pi^2} I[\omega, \underline{\Delta}_{f,g}(\lambda, \epsilon)] + O\left(\frac{1}{n}\right)\,.$$

Now we integrate on λ, noticing that we need only integrate from 0 to $B_{f|g}(\epsilon)$

$$\int_0^\infty \mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] d\lambda$$

$$= \frac{12}{\pi^2} \int_0^\infty I[\omega, \underline{\Delta}_{f,g}(\lambda, \epsilon)] d\lambda + B_{f|g}(\epsilon) O\left(\frac{1}{n}\right)\,.$$

We are led to the integral of ω on the domain of $\mathbb{R}^3$ defined by

$$\{(x,y,\lambda) \in \mathcal{R} \times \mathbb{R}_{\geq 0} \mid f(x,y) \geq \lambda, g(x,y) \geq \epsilon\}$$

We interchange the summation, and first integrate with respect to λ (which provides the term $f(x,y)$); we obtain

$$\int_0^\infty I[\omega, \underline{\Delta}_{f,g}(\lambda,\epsilon)]d\lambda = \iint_{\substack{(x,y)\in\mathcal{R},\\ g(x,y)\geq\epsilon}} \omega(x,y)\cdot f(x,y)dxdy\,,$$

$$= \frac{\pi^2}{12}\mathbb{E}_\psi[f|g \geq \epsilon]\cdot \mathbb{P}_\psi[g\geq\epsilon]\,. \qquad \blacksquare$$

4.4 Return to the conditional expectation of the recurrence quotient. Proof of Theorem 2.3. We will prove here a stronger version of Theorem 2.3, where the remainder terms are more precise.

THEOREM 4.3. *Consider a parameter $\Gamma \in \{\rho,\mu,\nu\}$ defined in (2.4) and (2.5), and a sequence $n \mapsto \epsilon(n)$ which tends to 0. Then the conditional expectation of the recurrence quotient S_n with respect to the event $[\Gamma_n \geq \epsilon(n)]$ satisfies*

$$\mathbb{E}\left[S_n \middle| \Gamma_n \geq \epsilon(n)\right] = \frac{12}{\pi^2}|\log\epsilon(n)| + C(\Gamma)$$

$$+O\left(\frac{1}{\epsilon(n)n} + \epsilon(n)|\log\epsilon(n)|^2\right)\,. \tag{4.35}$$

Moreover, the constants $C(\Gamma)$ satisfy

$$C(\nu) = +1, \quad C(\mu) = 0, \quad C(\rho) = +1\,.$$

Proof. The proof is an application of Theorem 4.2. First, a direct computation with Theorem 2.1 shows that if Γ is one of the $\mathcal{Q}$-functions ρ, μ or ν, the following estimates hold

$$\mathbb{P}[\Gamma_n \geq \epsilon(n)] = 1 + O(\epsilon(n) + 1/n)\,.$$

Along with the bounds and the integrals provided in Figure 5, this implies the result. $\blacksquare$

Now, in the case when $\epsilon(n)$ is $\Omega(1/(n\log n))$, the remainder term in (4.35) is $o(|\log\epsilon(n)|)$. Then, Theorem 2.3 is an immediate application of Theorem 4.3.

5 Comparison between the two models.

We now compare the two models, the present model (with a large fixed n) and the model which was previously studied in [1], namely, the model with a large fixed k. We first recall our result of [1], we then study the number of continuants $q_k(\alpha)$ in an interval of the form $[n, cn]$ for $c > 1$.

5.1 Results in the previous model. When the integer n of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ is at a position μ there, the recurrence quotient admits the expression

$$S_k^{\langle\mu\rangle}(\alpha) = f_\mu\left(\frac{q_{k-1}(\alpha)}{q_k(\alpha)}\right) \tag{5.36}$$

which involves the function

$$f_\mu(x) := 1 + \frac{1+x}{x+\mu(1-x)}. \tag{5.37}$$

The main idea of the study "with a fixed k" relates the recurrence quotient and the k-th iterate of the Euclidean transfer operator $\mathbf{H}$ via the equality

$$\mathbb{E}[S_k^{\langle\mu\rangle}] = \mathbf{H}^k\left[x \mapsto \frac{f_\mu(x)}{1+x}\right](0)\,. \tag{5.38}$$

The operator $\mathbf{H}$ admits nice dominant spectral properties, and, notably, the celebrated Gauss density

$$x \mapsto \left(\frac{1}{\log 2}\right)\frac{1}{1+x} \tag{5.39}$$

as its fixed density. This leads to the estimate

$$\lim_{k\to\infty}\mathbb{E}[S_k^{\langle\mu\rangle}] = 1 + \frac{1}{\log 2}\int_0^1 \frac{1}{t+\mu(1-t)}dt\,, \tag{5.40}$$

More precisely, we have shown the following in [1]: for the sequence $\mu_k = \tau^k$, with $\tau \in [\varphi^2, 1[$, (where φ is the inverse of the Golden ratio), the following holds

$$\mathbb{E}[S_k^{\langle\tau^k\rangle}] \sim \frac{1}{\log 2}\,k|\log\tau| \quad (k\to\infty)\,. \tag{5.41}$$

5.2 Relation between the two models. We now wish to relate the two (asymptotic) models: the present model "with fixed large n" and the previous model "with fixed large k"? Of course, these two models should be close if the behaviour of the sequence $k \mapsto q_k(\alpha)$ does not depend too strongly on α, and we know that it is not the case. However, the behaviour of the sequence $k \mapsto \log q_k(\alpha)$ is much more regular, as it is well known (see for instance [6]) that

$$\lim_{k\to\infty}\frac{1}{k}\log q_k(\alpha) = L = \frac{\pi^2}{12\log 2} \quad \text{for almost all } \alpha\,. \tag{5.42}$$

Consider first the present model "with n fixed", and a sequence $\ell \mapsto n(\ell) = \tau^\ell$. Then Theorem 2.3 reads

$$\mathbb{E}[S_{n(\ell)} \mid \mu_{n(\ell)} \geq \tau^{-\ell}] \sim \left[\frac{12}{\pi^2}\log\tau\right]\ell\,. \tag{5.43}$$

Furhermore, as $n(\ell)$ belongs to the interval $[q_{k-1}(\alpha), q_k(\alpha)[$, the existence of the limit for the

Parameter Γ	Bound for S	$\mathbb{E}_\psi[f_S \mid f_\Gamma \geq \epsilon(n)]\,\mathbb{P}_\psi[f_\Gamma \geq \epsilon(n)]$
ρ	$S \leq 2 + 1/\rho \Longrightarrow B_{f_S \mid f_\rho}(\epsilon) = O(1/\epsilon)$	$A\lvert\log(\epsilon(n)\rvert + 1 - A\epsilon(n)\lvert\log\epsilon(n)\rvert$
μ	$S \leq 1 + 1/\mu \Longrightarrow B_{f_S \mid f_\mu}(\epsilon) = O(1/\epsilon)$	$A\lvert\log\epsilon(n)\rvert + \dfrac{A}{1-\epsilon(n)}\epsilon(n)\lvert\log\epsilon(n)\rvert$
ν	$S \leq 1 + 2/\nu \Longrightarrow B_{f_S \mid f_\nu}(\epsilon) = O(1/\epsilon)$	$A\lvert\log\epsilon(n)\rvert + 1$

Figure 5: In the second column, the bounds for S for each parameter $\Gamma \in \{\rho, \mu, \nu\}$. In the third column, the values of the product $\mathbb{E}_\psi[f_S|f_\Gamma \geq \epsilon(n)]\,\mathbb{P}_\psi[f_\Gamma \geq \epsilon(n)]$ needed to apply Theorem 4.2. The constant A is $12/\pi^2$.

quotient $q_k(\alpha)/k$, that holds for almost any α, and is recalled in (5.42) entails the relation between the index ℓ and the index $k := k(\alpha, n(\ell))$, that holds for almost any α, namely

$$\log n(\ell) = \ell \log \tau \sim \frac{\pi^2}{12\log 2}\, k(\alpha, n(\ell))\,. \tag{5.44}$$

Now, we deal with the model "with k fixed", and we consider that the index $k(\alpha, n(\ell))$ satisfies (5.44) *everywhere*. Then, the application of the result in the model "with k fixed", described in (5.41) should entail

$$\mathbb{E}[S_k^{\langle\tau^k\rangle}] \sim \left[\frac{1}{\log 2}\log\tau\right] k \sim \left[\frac{12}{\pi^2}\log^2\tau\right]\ell\,. \tag{5.45}$$

Remark that the conditional events are not the same in the two equations (5.43) and (5.45):
– in (5.43), the event is $\{\alpha \mid \mu(\alpha, n(\ell)) \geq \tau^{-\ell}\}$,
– in (5.45) the event is $\{\alpha \mid \mu(\alpha, n(\ell)) \sim \tau^{-\ell}\}$.
This (heuristic) comparison exhibits in both cases a linear growth with respect to ℓ. However, the events of interest are not the same, and we have considered that the index $k(\alpha, n(\ell))$ satisfies (5.44) everywhere.

5.3 Number of continuants in an interval.

There is also an interesting connection between the two models, that counts the number of terms of the sequence $k \mapsto q_k(\alpha)$ that belongs to the interval $[n, cn[$, for some fixed $c > 1$. We thus study the function

$$(\alpha, n) \mapsto T(\alpha, n) := \sum_{k\geq 0} [\![q_k(\alpha) \in [n, cn[\,]\!]\,.$$

PROPOSITION 5.1. *Consider the Lévy constant $\kappa := \exp\left(\pi^2/(12\log 2)\right)$. Then the mean number of continuants in the interval $[n, \kappa n]$ tends to 1 as $n \to \infty$*

Proof. Even if T is not a $\mathcal{Q}$-function, its expectation $\mathbb{E}[T_n]$ is expressed as a Riemann sum of the function 2ω, in a domain $\mathcal{T}_c$. However the domain $\mathcal{T}_c$ is not a subset of the rectangle $\mathcal{R}$. We have indeed

$$\mathbb{E}[T_n] = \int_0^1 T(\alpha, n)d\alpha = \int_0^1 d\alpha \sum_k [\![q_k(\alpha) \in [n, cn[\,]\!]$$

$$= \sum_k \sum_{\boldsymbol{m}\in\mathbb{N}^k} |I_k(\boldsymbol{m})|\, [\![q_k(m) \in [n, cn[\,]\!]$$

$$= \frac{1}{n^2}\sum_k \sum_{\boldsymbol{m}\in\mathbb{N}^k} \omega\left(\frac{q_{k-1}(\boldsymbol{m})}{n}, \frac{q_k(\boldsymbol{m})}{n}\right)\left[\!\!\left[\frac{q_k(\boldsymbol{m})}{n} \in [1, c[\right]\!\!\right]$$

$$= 2\sum_{\substack{(a,b)\in\mathbb{Z}^2\\ \gcd(a,b)=1}} \omega\left(\frac{a}{n}, \frac{b}{n}\right)\left[\!\!\left[\frac{a}{n} \leq \frac{b}{n}, 1 \leq \frac{b}{n} \leq c\right]\!\!\right]$$

$$= \widehat{R}_n(2\omega, \mathcal{T}_c), \quad \text{with} \quad \mathcal{T}_c = \{(x,y) \mid x \leq y, 1 \leq y \leq c\}\,.$$

Even if $\mathcal{T}_c$ is not a subset of $\mathcal{R}$, Proposition 3.3 applies, and the coprime Riemann series admits a limit equal to the integral

$$\frac{6}{\pi^2} I(2\omega, \mathcal{T}_c) = \frac{12\log 2}{\pi^2}\log c\,. \qquad \blacksquare$$

6 Conclusion and further studies

Conclusions. Beginning from the question "what does the recurrence function of a random Sturmian word look like?", we define and work within a model that is natural at least from an algorithmic standpoint: pick a large integer n and let the slope of the word be drawn at random from $[0, 1]$. We are led to the notion of the so-called $\mathcal{Q}$-functions: functions that, given n and a slope α, place n within the sequence of continuants $k \mapsto q_k(\alpha)$ of α, namely consider the index k for which $n \in [q_{k-1}(\alpha), q_k(\alpha)[$, and then return a value depending only on the two ratios $(1/n)q_{k-1}(\alpha)$ and $(1/n)q_k(\alpha)$. The recurrence quotient of Sturmian words defines such a $\mathcal{Q}$-function, via a Theorem of Morse and Hedlund, where n is the length of the factors and α the slope of the word.

Then, we study the distribution of a general $\mathcal{Q}$-function. It defines in fact a sequence of distributions, and we prove that the limit distribution and the limit densities exist. They all involve, as a sort of reference density, the density ψ defined in (2.13), which plays a similar role to that of the Gauss density (defined in (5.39)) when one studies functions that depend on the ratio $q_{k-1}(\alpha)/q_k(\alpha)$, and appears in our study [1].

Our results apply in particular to the recurrence quotient of Sturmian words; we exhibit the limit distribution (and the limit density) of such a quotient. We compare our probabilistic study to the results of Morse and Hedlund, which exhibit extreme behaviours, attained when n is close to the left border $q_{k-1}(\alpha)$ of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ containing the integer n. That is why we also consider conditional expectations, where the conditional events are related to the various parameters which describe the position of the integer n inside $[q_{k-1}(\alpha), q_k(\alpha)[$. We then compare this "constrained probabilistic" behaviours to the extreme behaviours, in a precise manner.

We had previously performed a similar study in [1] under another probabilistic model, where it is rather the index k of the interval $[q_{k-1}(\alpha), q_k(\alpha)[$ the integer n belongs to that is fixed. Then for $k \to \infty$, we exhibited limit distribution and limit densities all of which involve, as a sort of reference density, the Gauss density. The two models are clearly different, but the two types of results show certain similarities.

Further studies. We first plan to extend the present probabilistic study in three directions.

Reals with Bounded Partial Quotients. This type of slope α gives rise to Sturmian words whose recurrence function is proven to be linear. For a bound M, we restrict α to the set of numbers whose partial quotients are at most M, endowed with the Hausdorff measure, and we wish to observe the transition when $M \to \infty$.

Rational Numbers. This type of slope gives rise to periodic words, and occurs for Christoffel words. For a bound N, we restrict α to the set of rationals with denominator at most N, endowed with the uniform distribution, and we wish to observe the transition when $N \to \infty$. This will explain how a periodic word "becomes" Sturmian. In this study, it would be useful to work inside the general framework introduced by Ustinov in [11], where he deals with tails of continued fractions extensions.

Quadratic Irrationals. This type of slope α occurs for substitutive Sturmian words. There is a natural notion of size associated with such numbers, closely related to the period of their continued fraction expansion, and we wish to observe the transition when the size becomes large.

We also plan to study the "form" of a random de Bruijn Sturmian graph. What are the mean values of the lengths of its two cycles, for instance?

Finally, we are interested in the "mean-recurrence", where we replace in (2.3) the maximum by a mean-value (which has to be well-defined). The "mean-recurrence" of a random Sturmian word would be very interesting to study (and probably difficult...)

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