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Combinatorial games modeling seki in GO

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ABSTRACT

The game SEKI is played on an $(m \times n)$-matrix A with non-negative integer entries. Two players R (for rows) and C (for columns) alternately reduce a positive entry of A by 1 or pass. If they pass successively, the game is a draw. Otherwise, the game ends when a row or column contains only zeros, in which case R or C wins, respectively. If a zero row and column appear simultaneously, then the player who made the last move is the winner. We will also study another version of the game, called D-SEKI, in which the above case is defined as a draw.

An integer non-negative matrix A is a *seki* or *d-seki* if the corresponding game results in a draw, regardless of whether R or C begins. Of particular interest are the matrices in which each player loses after every option except pass. Such a matrix is called a *complete seki* or a *complete d-seki*. For example, each matrix with entries in $\{0, 1\}$ that has the same sum (at least 2) in each row and column is a complete d-seki, and each such matrix with entries in $\{0, 1, 2\}$ is a complete seki. The game SEKI is closely related to the seki (shared life) positions in the classical game of GO.

1. Introduction

The games SEKI and D-SEKI. The game SEKI was introduced by the first two authors in 1981 in the manuscript [5].

Let $A : I \times J \rightarrow \mathbb{Z}_+$ be a non-negative integer $(m \times n)$-matrix having a positive entry in each row and column. The game *SEKI*(A) is defined as follows. Two players R and C alternate turns and it is specified who begins; this player is called the *first*, while the opponent is the *second*. On their turn, the players can either reduce any strictly positive entry of A by 1 (an *active move*) or pass. The game ends in a draw when two players pass successively. A row or column of A is *zero* if all its entries equal 0. Player R wins if a zero row appears before any zero column and player C wins when a zero column appears before any zero row. After some move, a zero row and column may appear simultaneously. In this case, the player who made this last move is claimed the winner. We will also study another version of the game, D-SEKI(A), in which the above case is defined as a draw. Frequently, when only one matrix is under consideration, we will omit the argument A and shorten SEKI(A) and D-SEKI(A) to just SEKI or D-SEKI, respectively. Thus, SEKI results in a draw only after two consecutive passes, while D-SEKI ends in a draw in this situation and also when a zero row and column appear simultaneously.

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Two matrices are isomorphic if one can be obtained from the other by permutations of its rows and columns.

A matrix A is a *seki* or *d-seki* if "perfect play" of both players in the games SEKI(A) or D-SEKI(A), respectively, results in a draw, regardless of whether R or C begins. The concept of perfect play is defined precisely as follows.

Solving SEKI and D-SEKI by backward induction. It is well known that any finite acyclic game can be standardly solved by backward induction [8,9,4], but SEKI and D-SEKI are not acyclic, since any pass is a loop. Nevertheless, some slightly modified backward induction is applicable to both of these games.

For each matrix A we have to solve the games SEKI(A) and D-SEKI(A) for two cases: when R or C begins. Thus, we assign to A one of the nine pairs $(R(A), C(A))$, where $R(A)$ and $C(A)$ take values in $\{W, L, D\}$ (which stand for "win", "lose", and "draw") and show the result for the first player, in other words, for the case when, R and C begin, respectively. For example, $(R(A), C(A)) = (W, W)$ means that the first player wins whether it is R or C, while (W, L) means that R wins whether or not (s)he begins.

We do not consider the matrices that contain more than one zero row or more than one zero column. To initialize the procedure, we assign

- (i) (W, L) to any matrix A that contains exactly one zero row but no zero column;
- (i′) (L, W) to any matrix A that contains exactly one zero column but no zero row;
- (ii) if A contains both one zero row i and one zero column j, then we set $(R(A), C(A)) = (D, D)$ in D-SEKI and $(R(A), C(A)) = (L, L)$ in SEKI.

The assignments (i) and (i′) are obvious: R wins in case (i) and C wins in case (i′), as the first or second player, but (ii) requires some comments. In this case the last move is uniquely defined: the entry $A(i, j)$ is reduced from 1 to 0. After this, the game is over and, according to the rules of SEKI, the player who made this last move wins; hence, we assign (L, L) to SEKI(A). In contrast, D-SEKI is a draw; hence, we assign (D, D) to D-SEKI(A).

After this initialization, the values $R(A)$ and $C(A)$ are defined recursively for both games, SEKI(A) and D-SEKI(A), as follows. Let $N(A)$ denote the set of matrices that can be reached from A by one move. If R begins, then

- $R(A) = W$ whenever $C(A') = L$ for some $A' \in N(A)$;
- $R(A) = D$ if $C(A') = L$ for no $A' \in N(A)$, but $C(A') = D$ for some $A' \in N(A)$.

In other words, R begins and wins (respectively, makes a draw) if (s)he has a move to a position in which C, as the first player, loses (respectively, makes a draw). One case remains:

- [R′] $C(A') = W$ for all $A' \in N(A)$, that is, C wins after any active move by R.

This case is slightly more difficult, since R can still pass, thus, making C the first player. It cannot be a winning move for R, because C can also pass making a draw. Hence, either $R(A) = L$ or $R(A) = D$ in case [R′]. The answer depends on $C(A)$. Let us temporarily set $R(A) = X$ and consider $C(A)$. We define it by symmetry (transposing A) in all cases but

- [C′] $R(A') = W$ for all $A' \in N(A)$, that is, R wins after any active move of C.

In this case, either $C(A) = L$ or $C(A) = D$ and the answer depends on $R(A)$. Let us temporarily set $C(A) = Y$. It looks like a vicious circle, yet, it can be easily broken.

Semi-complete and complete seki. If $R(A) = X$, in case [R′], we redefine it as follows:

- $R(A) = L$ when $C(A) = W$, that is, if C wins after pass of R.
- $R(A) = D$ when $C(A) = D$, that is, if R must pass but C has an active move that still results in a draw. In this case A is called a *semi-complete seki* or *d-seki*.
- $R(A) = D$ when $C(A) = Y$; in other words, both players, R and C, must pass. In this case A is a *complete seki* or *d-seki*.

We define $C(A)$ in case [C′] by symmetry. Notice that A is a complete seki if and only if

- $R(A) = X$ and $C(A) = Y$; in other words both players, R and C, must pass.

Each game, SEKI or D-SEKI, is over after two consecutive passes. Obviously, the optimal result would not change if we allow any number $k \geq 2$ or even an infinite sequence of passes.

The computer code for backward induction. To study the games SEKI and D-SEKI, we generate successively all matrices, up to permutations of their rows and columns, increasing the sum of all entries one by one. Isomorphism-free exhaustive generation (without checking the isomorphism) is a difficult computational problem in general [10]; in particular, no simple efficient procedure for the required matrix generation is known. Yet, for our purposes, we allow some, but not too frequent, repetitions. In this paper we use many examples of non-trivial seki and d-seki computed by this method. The corresponding computer codes were written for SEKI by Konrad Borys and Gabor Rudolf in 2005 and then a more powerful code, working for D-SEKI as well, was written by Diogo Andrade in 2006.

The importance of being first. In both games, SEKI and D-SEKI, for any player, R or C, to be the first is never worse than to be the second. More precisely, if the second player wins or makes a draw in SEKI or D-SEKI, then (s)he can do the same, as the first player. Indeed, it is enough just pass and then apply the same optimal strategy.

These arguments imply that $(R(A), C(A))$ never takes values (D, L) or (L, D), and also that (L, L) is taken only in SEKI, when the matrix A has exactly one zero row and column. Note that the remaining six values for $(R(A), C(A))$ are possible; see [1] for more details.

Another important corollary is as follows: if a matrix A is symmetric, then $R(A)$ and $C(A)$ can take only values W and D; in other words, the second player cannot win. Indeed, if s(he) wins, then the first player can steal the winning strategy by passing and then playing the transposed version of that strategy.

According to the terminology of John Milnor [11], GO, GO-MOKU, SEKI, and D-SEKI are games with *positive incentive*.

Simple relations between SEKI and D-SEKI. Let us begin with the (1×1)-matrices. In this case D-SEKI results in a draw but no player is ever forced to pass. Hence, no (1×1)-matrix is a complete d-seki. Now, let us consider SEKI. If $A(1, 1) = 1$ then the first player wins immediately. If $A(1, 1) = 2$ then A is a complete seki, since both players, R and C, must pass in this case. Finally, if $A(1, 1) > 2$ then A is a seki but not a complete seki.

Proposition 1.1. *Let S, DS, CS, and CDS stand for the sets of seki, d-seki, complete seki, and complete d-seki matrices, respectively. The following strict containments hold:*

$$CDS \subset CS \subset S \subset DS.$$

Proof. Given a matrix A, a winning strategy in the game SEKI(A) guarantees only a draw in D-SEKI(A), while a winning strategy in D-SEKI(A) is winning in SEKI(A) too. This observation implies all three containments. Furthermore, already the (1×1)-matrices enable us to demonstrate that all three are strict. If $A(1, 1) = 1$ then A is a d-seki but not a seki; if $A(1, 1) = 2$ then A is a complete seki but not a complete d-seki; if $A(1, 1) > 2$ then A is a seki but not a complete seki. □

As usual, the inequality $A' \le A$ for two $(m \times n)$-matrices $A : I \times J \to \mathbb{Z}_+$ and $A' : I \times J \to \mathbb{Z}_+$ means $m \times n$ entry-wise inequalities, $A'(i, j) \le A(i, j)$ for all $i \in I$ and $j \in J$.

Obviously, for each $m \times n$ seki A there is an $m \times n$ complete seki A' such that $A' \le A$, while for D-SEKI the similar claim fails. For example, each strictly positive (1×1)-matrix is a d-seki but none of these d-seki is complete.

Any matrix A is a seki, d-seki, complete seki, or complete d-seki whenever the transposed matrix A^T has the same property. Indeed, the transposition of A "swaps players": $R(A) = C(A^T)$ and $C(A) = R(A^T)$. It remains to notice that all versions of the concept of seki considered above are symmetric with respect to the players.

Structure of the paper.

- In Section 2 we explain how the game SEKI is related to seki (shared life) in GO.
- In Section 3, for both games D-SEKI and SEKI we provide simple conditions sufficient for the first and also for the second player to win. These conditions result in two infinite sets of complete seki and d-seki, each of which is a (square) matrix such that the sum of all entries in every its row or column takes the same value $s \ge 2$. We prove that all such matrices with entries 0, 1 and 0, 1, 2 are complete d-seki and complete seki, respectively.
- In Section 4 we restrict ourselves to the $(m \times n)$-matrices with $\min(m, n) \le 2$ and prove that the sufficient conditions of the previous section become also necessary in this case. From this result we derive that, unlike in D-SEKI, there are no $2 \times n$ seki with $n > 2$, there are only four 2×2 complete seki, and only one among them is a complete d-seki. Furthermore, we characterize all 2×2 seki and d-seki and construct a 2×3 d-seki.
- In Section 5 we study the games SEKI and D-SEKI on the direct sums of matrices.
- In Section 6 we collect experimental results and conjectures related to the games SEKI and D-SEKI. In particular, we conjecture that all (complete) seki are square matrices. In contrast, we give the example of a 2×3 d-seki. We construct also a 3×3 complete seki which rows and columns do not have the same sum of entries. We conjecture a characterization of complete d-seki but structure of complete seki remains unclear.

2. The game SEKI and the seki (shared life) positions in GO

The game SEKI is closely related to the concept of seki (shared life) positions in GO, which was the motivation to introduce SEKI in [5]; see also [2]. Here we explain this correspondence briefly, addressing the readers familiar with (the rules of) GO. Others can skip this section.

A seki position in GO is *complete* if both players must pass; more precisely, if any move of a player at a point of the board adjacent to an involved group results in losses for this player. Similarly, a seki position in GO is *semi-complete* if one player must pass, while the opponent has an active move such that the obtained position is still a seki.

Non-negative integer matrices and shared life of eyeless groups in GO. Given a $(m \times n)$-matrix $A : I \times J \to \mathbb{Z}_+$, let us consider a position of GO with m white and n black groups that are indexed by I and J, respectively, and let $A(i, j)$ be the number of common free points (also called liberties or dame) between the white group i and the black group j; see examples in Figs. 1–4. Two players R and C in SEKI correspond to Black and White in GO; R wants to create a zero row and C wants to create a zero column, while Black and White each want to capture (eliminate all liberties of) an opponent's group.

Remark 2.1. Only some special shared life positions are considered. We assume that:

- (i) The numbers m and n will not change, that is, no two groups of the same color can ever be united, even if the opponent always passes. 两个串不会连接起来变成一个串

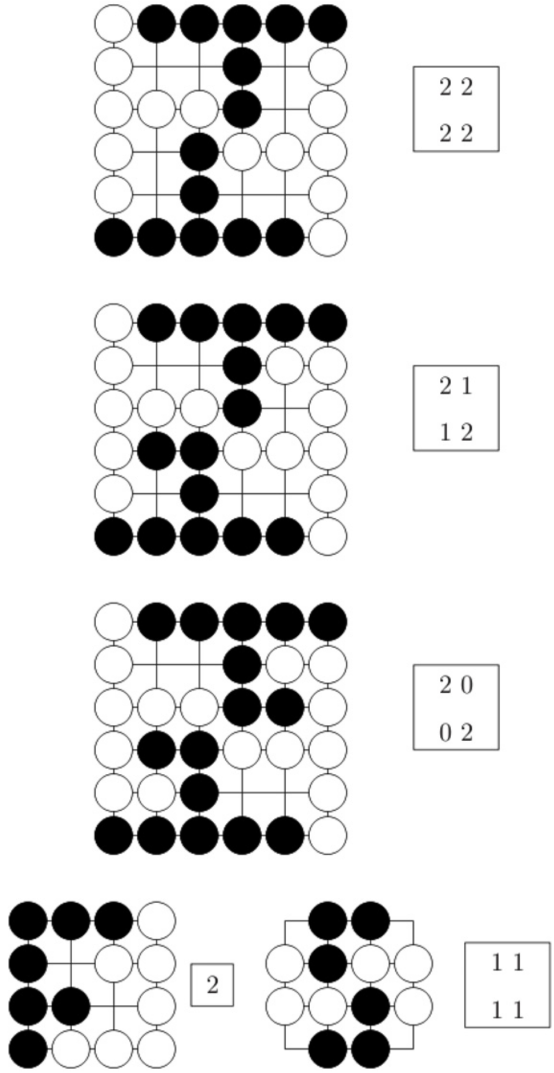


Fig. 1. Five standard complete seki positions in GO and the corresponding complete seki matrices.

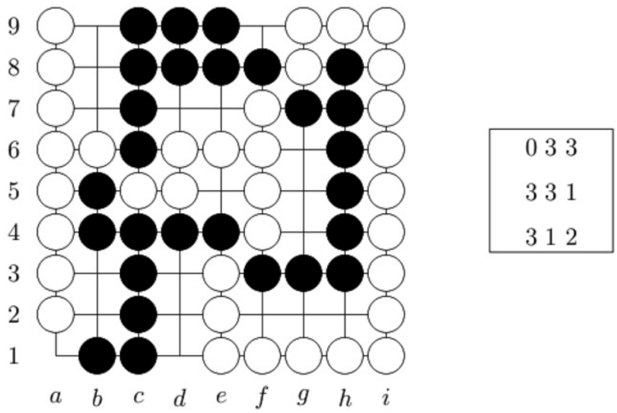
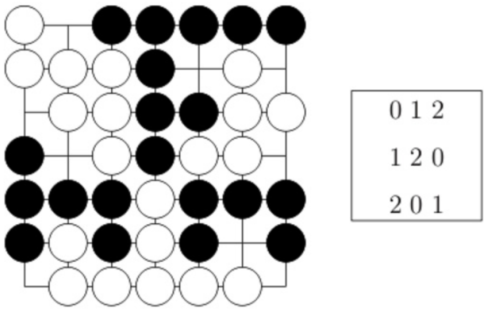
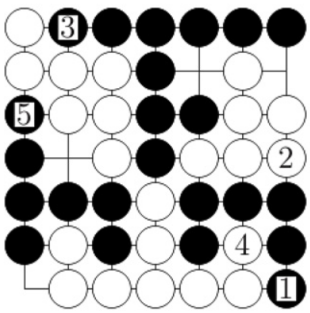
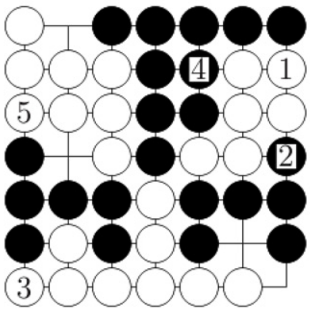


Fig. 2. A complete seki position in GO and the corresponding 3×3 complete seki.





seki上是双活，但围棋里面不是双活，围棋里面某方可以弃掉某个串后，再吃掉一个更大的串



无眼双活

Fig. 3. A non-seki position in GO which matrix is a complete seki.

- (ii) None of the $m+n$ involved groups has an eye and, moreover, no player can create an eye, even if the opponent always passes.

  In particular, both (i) and (ii) will hold if we require that

- (iii) Every liberty point is adjacent to exactly two groups, one black and one white.

  It is easy to verify that (iii) holds for all diagrams of Figs. 1–4 considered below.

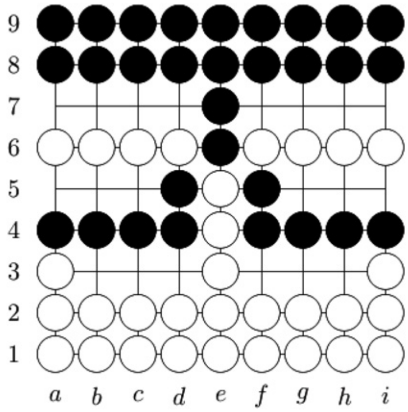
每个气都恰与一个白串和一个黑串相邻

Fig. 1 shows five examples of complete seki positions that frequently occur in GO and the corresponding five complete seki matrices. It is easy to verify that both players, Black and White, must pass in GO, similarly, R and C must pass in SEKI.

One can also check that only the last (2×2)-matrix is a complete d-seki. We prove that there are no other complete seki and d-seki among all $(1 \times n)$- and $(2 \times n)$-matrices.

Fig. 2 shows a non-trivial complete seki position in GO with three white and three black groups and the corresponding (3×3)-matrix A. Its rows 1, 2, 3 are assigned to the three white groups: the Left (LW), the Right (RW), and the Middle (MW) one, respectively; and its three columns, 1, 2, 3, are assigned to three black groups: the Middle (MB), the Lower (LB), and the Upper (UB) one, respectively. For example, $A(1, 1) = 0$, since LW and MB share no liberties; $A(1, 2) = 3$, since LW and LB share 3 liberties; $A(2, 3) = 1$, since RW and UB share 1 liberty; $A(3, 3) = 2$, since MW and UB share 2 liberties; etc.

Both players, R and C, must pass in SEKI(A). Here we analyze only one (but the most interesting) case pointed out by Thomas Wolf. Let C move at (2, 3), reducing $A(2, 3)$ from 1 to 0. The "natural" answer of R at (1, 2) fails to win. In response, C simply passes. In fact, the best response of R to the move (2, 3) of C is at (3, 1) rather than at (1, 2). Then, if C moves at (2, 1) or (1, 3), R answers at (2, 2) or (1, 2), respectively, and wins in both cases. These two variants are symmetric. All other cases are simpler and we leave their analysis to the careful reader. Thus, matrix A is a complete seki. Note that five entries



$$\begin{matrix} 0 & 3 & 3 \\ 4 & 0 & 3 \\ 4 & 3 & 0 \end{matrix}$$

Fig. 4. A semi-complete seki position in GO and the corresponding semi-complete seki matrix.

of A equal 3, the rows and columns of A have different sums of entries: 6 and 7, but all sums become equal 6 after the move at $(2, 2)$.

It is easy to check that if a player, Black or White, loses a group then (s)he will also lose the remaining two groups. In other words, no player can sacrifice a group to capture a (larger) opponent's group in return. However, in general this is possible, as the next example shows.

Fig. 3 demonstrates one more limitation of our approach. The (3×3)-matrix is a complete seki, both players must pass. In contrast, the corresponding play in GO is by far not over. If Black begins, he can exchange his group in the lower right corner for the larger white group in the upper left corner; if White begins, she can exchange her right group for Black's left group. Each of these active move is better than pass by two points. In other words, both Black and White can sacrifice a group and capture a larger opponent's group. It is important to emphasize the difference between the two games: SEKI is over as soon as a zero row or a zero column appears in the matrix; in contrast, GO is not necessarily finished when the first group is captured; sometimes a group may be sacrificed with profit.

反例：围棋里面是双活，seki上不是双活

Remark 2.2. In Diagram 12 of [5] and in Section 11.4 of [1], an "inverse example" was constructed: a complete seki position in GO such that the corresponding matrix A is not a seki. Each player, Black or White, can begin and capture an opponent's group, but then the opponent can retaliate by capturing a larger group in return. Thus, both Black or White must pass in GO, while the first player, R or C, wins in SEKI(A).

Fig. 4 presents a semi-complete seki position in GO and the corresponding semi-complete seki (3×3)-matrix. In this example, R (Black) passes. Indeed, due to symmetry, R has only three distinct options: to play at $(3, 2)$, $(2, 1)$, and $(1, 2)$. It is not difficult to verify that, in all three cases C wins, answering, respectively, at $(1, 2)$, $(1, 2)$, and $(3, 2)$ (or, less obviously, at $(2, 1)$). In contrast, C (White) can reduce $A(1, 2)$ from 3 to 2 and the result is still a draw. If R passes in response, C can even win playing at $(3, 2)$. Instead, R should reduce $A(3, 1)$ from 4 to 3 getting another (not complete) seki. Thus, playing GO by the Chinese rules, White can get extra points in this position, however, by the Japanese rules, these points do not count.

Which matrices correspond to positions of GO?

The standard GO board is the planar grid 19×19. However, the game can be naturally generalized to an arbitrary graph.

Let us consider the (3×3)-matrices in Figs. 2–4. Each of them contains at least one 0. This is not a coincidence. Indeed, the 3×3 complete bipartite graph $K_{3,3}$ is not planar. Hence, for any position of GO satisfying conditions (i) and (ii) of Remark 2.1 on a planar graph, any three rows and columns of the corresponding matrix must contain at least one zero among their nine intersections.

It is not difficult to answer the above question (in bold) in general. To a matrix A we assign a $(0, 1)$-matrix A' such that $A'(i, j) = 1$ if $A(i, j) > 0$ and $A'(i, j) = 0$ if $A(i, j) = 0$. Each $(0, 1)$-matrix A' is naturally associated with a bipartite graph $B(A')$. The following three properties are equivalent:

- the matrix A is associated with a position of GO in a planar graph;
- the matrix A is associated with a position of GO in a planar grid;
- the bipartite graph $B(A')$ is planar.

一个图是平面图的充要条件是图不包含同胚于完全图 K3 K5 的子图

We leave this topological exercise to the reader. A criterion of planarity is given by the famous Kuratowski theorem; see, e.g., [12].

SEKI and Whistette versus GO and Bridge. Following the terminology suggested by Aviezri Fraenkel in [3], SEKI can be viewed as a "math-game" corresponding to the "play-game" of GO. Similarly, a game *Whistette* (also called *Single-Suit*) introduced in 1929 by Emanuel Lasker can be viewed as a "math-game" corresponding to the "play-game" of Bridge; see [7, 13], and Section 4 of [6] for the definitions and more details. A polynomial algorithm solving Whistette was obtained in

2005 by Johan Wästlund [13]. Unlike the play-games, Bridge and GO, the math-games, Whistette and SEKI, seem too boring to practice. In return, they reveal deeper and nicer mathematical properties and are very complicated too. At least, some positions of SEKI and Whistette would be difficult to analyze even for advanced Bridge and GO players; see, for examples, Figs. 2 and 4.

3. Conditions sufficient to win in SEKI and D-SEKI

这两个条件其实是比较简单的

It seems difficult to characterize the winning positions of the games SEKI(A) and D-SEKI(A). However, there are simple cases, when the matrix A contains a "weak" row (respectively, column), which R (respectively, C) can eliminate making no moves elsewhere. The corresponding simple sufficient conditions imply important corollaries.

A matrix $A : I \times J \to \mathbb{Z}_+$ is a $(0, 1)$- or $(0, 1, 2)$-matrix if its entries take values in $\{0, 1\}$ or in $\{0, 1, 2\}$, respectively. Furthermore, let s_i^r and s_j^c denote the sum of all entries of row $i \in I$ and column $j \in J$, respectively. We call A an *integer doubly stochastic matrix (IDSM)* (with the sum $s = s(A)$) if the sum of the entries in every row and every column of A equal s. Obviously, any IDSM is a square matrix whenever $s > 0$.

When the first player wins in D-SEKI? A matrix $A : I \times J \to \mathbb{Z}_+$ is *R-winnable* if it has a row $i \in I$ such that $s_j^c - s_i^r \geq A(i, j)$ for every column $j \in J$. Similarly, A is *C-winnable* if it has a column $j \in J$ such that $s_i^r - s_j^c \geq A(i, j)$ for every row $i \in I$. Any row or column satisfying this condition is *winnable*.

Lemma 3.1. *If matrix A is R-winnable (respectively, C-winnable) then R (respectively, C), as the first player, wins in D-SEKI(A).*

Proof. It is enough to prove the statement only for R; the similar claim for C we derive by simply transposing A. Let R reduce any entry of any winnable row i and C replies arbitrarily. It is easy to check that these two moves maintain the inequality $s_j^c - s_i^r \geq A(i, j)$; in other words, row i remains winnable in the obtained (R-winnable) matrix. In particular, $s_i^r \leq s_j^c$ holds and $s_i^r = s_j^c = 1$ fails for any j. Hence, C cannot win or make a draw in D-SEKI. $\square$

This lemma implies that the following class of matrices contains only complete d-seki.

Proposition 3.2. *Any $(0, 1)$-IDSM A with $s(A) \geq 2$ is a complete d-seki.*

Proof. If a player reduces an entry of A, the opponent can win, since all conditions of Lemma 3.1 hold. This is also easy to prove from scratch: Let C (respectively, R) reduce an entry $A(i, j)$, then R (respectively, C) easily wins, reducing the entries of the same row i (respectively, of the same column j) in an arbitrary order. $\square$

The condition $s(A) \geq 2$ is essential. If $s(A) = 1$, the first player makes a draw in one move. Hence, in this case A is a d-seki but not a complete d-seki.

When the second player wins in SEKI? A matrix $A : I \times J \to \mathbb{Z}_+$ is *strictly R-winnable* (respectively, *strictly C-winnable*) if A has a row $i \in I$ such that $s_j^c - s_i^r \geq A(i, j)$ for every column $j \in J$ (respectively, A has a column $j \in J$ such that $s_i^r - s_j^c \geq A(i, j)$ for every row $i \in I$) and, moreover, this inequality is strict whenever $A(i, j) = 0$. Any row or column satisfying this condition is *strictly winnable*.

Lemma 3.3. *If A is strictly R-winnable (respectively, strictly C-winnable) then R (respectively, C) wins in SEKI(A), even as the second player.*

Proof. Again, due to symmetry, it is enough to prove the statement only for R. Let C reduce an arbitrary entry (i', j'). Then R reduces an entry (i, j) of a strictly winnable row i avoiding column j' whenever possible. The following simple case analysis shows that any such two moves either keep row i strictly winnable, or result in an immediate win by R.

Case $j \neq j'$. The sum s_i^r is reduced by at least 1, while s_j^c and $s_{j'}^c$ are reduced by exactly 1. The obtained matrix A' remains strictly R-winnable with the same strictly winnable row i. Even in case when R reduces $A(i, j) = 1$ to $A'(i, j) = 0$, still $s_j^c(A') > s_i^r(A')$ and row i remains strictly winnable; it can become zero, while column j cannot.

Case $j = j'$. This equality means that all entries of row i are zeros except $A(i, j)$; in particular, $s_j^c \geq 2A(i, j)$. By the considered two moves s_j^c is reduced by 2, while s_i^r and $A(i, j)$ are each reduced by 1. It is easily seen that row i remains strictly winnable in all cases, except $s_i^r = s_j^c - 1 = A(i', j) = A(i, j) = 1$. Then, the row i and column j become zero simultaneously, but still R wins in SEKI. $\square$

Theorem 3.4. *Any $(0, 1, 2)$-IDSM A with $s(A) \geq 2$ is a complete seki.*

Proof. Without loss of generality, assume that C begins by reducing an entry (i, j). In response R reduces another entry of the same row i. This entry can be chosen distinct from (i, j) unless $A(i, j) = s(A) = 2$. In the latter case the game is over and R won, otherwise it is not difficult to verify (using of the condition $A(i, j) \leq 2$) that the obtained matrix is strictly R-winnable. Since R becomes the second player again, by Lemma 3.3, (s)he wins. $\square$

All conditions of Theorem 3.4 are essential. If $s(A) = 1$ then the first (rather than the second) player wins by one move. If $A(i, j) > 2$ then the resulting row i is not strictly winnable. Finally, the requirement of the strict inequality for the case $A(i, j) = 0$ in the definition of the strictly R- and C-winnable matrices is also essential. Indeed, if $A(i, j) = 0$ and $s_i^r = s_j^c = s(A)$, then the first (rather than the second) player wins.

When the first player wins in SEKI? It would be good to have also some simple conditions sufficient for the *first* player to win in SEKI. Let us replace the strictly R-winnable matrices by a wider family satisfying the weaker inequality $s^c_j - s^r_i \geq A(i, j) - 1$ for a row i and every column j. This weaker condition fails to be sufficient for R to win. Moreover, it may hold even for a complete seki. For example, the following three matrices

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$

are complete seki by Theorem 3.4. Hence, R cannot win, even as the first player, but the inequality $s^c_j - s^r_i \geq A(i, j) - 1$ holds in all three cases for the (strictly positive) first row.

Of course, if a matrix A is strictly R-winnable, then player R wins in SEKI(A) not only as the second, but also as the first player. However, such sufficient condition is too strong and is not necessary even for the (2×2)-matrices.

In fact, the best what we can suggest is a triviality: As the first player, R wins in SEKI(A) whenever (s)he can reduce A by one move to a strictly R-winnable matrix. in which R wins, as the second player. This statement follows immediately from Lemma 3.3.

In spite of triviality, it allows us to analyze some examples. Consider the (2×2)-matrix A with entries $A(1, 1) = 1$, $A(1, 2) = A(2, 1) = A(2, 2) = k$. Clearly, it is a complete seki if $k = 1$, but if $k > 1$ then R begins and wins playing at $(1, 2)$. Indeed, in the obtained matrix A', with entries $A'(1, 1) = 1, A'(1, 2) = k - 1, A'(2, 1) = A'(2, 2) = k$, the first row becomes strictly R-winnable. More generally, the first row is strictly R-winnable in any matrix A'' with entries $A''(1, 1) = \ell, A''(1, 2) = k - \ell, A''(2, 1) = A''(2, 2) = k$ whenever $0 < \ell < k$. It is easily seen that $k > 1$ and that R wins, reducing the first row to zero, even if C begins.

Of course, in all above statements we can swap R and C, by simply transposing the matrices.

4. On matrices with at most two rows

On $(1 \times n)$-**matrices.** The trivial case $n = 1$ was already studied: Every (1×1)-matrix A is a d-seki; no (1×1)-matrix A is a complete d-seki; A is a seki if and only if $A(1, 1) > 1$; A is a complete seki if and only if $A(1, 1) = 2$. The case $n > 1$ is also trivial. Any active (never passing) strategy of C is winning in D-SEKI, and hence, in SEKI too. Indeed, if the unique row is zero then all columns are also zero. Hence, one of them became zero earlier.

In this section we will prove that there are no $m \times n$ complete seki with $2 = m < n$ and there exist only four 2×2 complete seki, shown in Fig. 1. Among them only the last one is a complete d-seki. However, there are many $(2 \times n)$-matrices with $n > 2$ that are (not complete) d-seki. For example, it is easy to check that neither R nor C, even as the first player, wins in D-SEKI(A) for the following (2×3)-matrix A:

$$\begin{matrix} 1 & 1 & 1 \\ 2 & 3 & 3. \end{matrix}$$

Note that A is neither C- nor R-winnable. Indeed, $s^r_1 = s^c_1 = 3$, while $A(1, 1) = 1$.

Some properties of the winnable and strictly winnable $(2 \times n)$-**matrices.** Obviously, for each matrix $A : I \times J \to \mathbb{Z}_+$ we have

$$\sum_{i \in I} s^r_i = \sum_{j \in J} s^c_j = \sum_{i \in I, j \in J} A(i, j). \tag{1}$$

Recall that A is R-winnable if there is a row $i \in I$ such that $s^c_j \geq s^r_i + A(i, j)$ for each column $j \in J$ and that A is strictly R-winnable if the above inequality is strict whenever $A(i, j) = 0$. In particular, if A is R-winnable and $m < n$, then there are $i \in I$ and $j \in J$ such that $s^r_i > s^c_j$. In case $m = 2$ a much stronger inequality holds: if A is R-winnable with the winnable row 1 then $s^c_j \geq s^r_1 + A(1, j)$ and $A(2, j) \geq s^r_1$ for each $j \in J$; hence,

$$s^r_2 = \sum_{j=1}^{n} A(2, j) \geq n s^r_1. \tag{2}$$

In other words, in any R-winnable $(2 \times n)$-matrix the sum of one row is at least n times the sum of the other. The same holds for the strictly R-winnable matrices.

The sufficient conditions of Lemmas 3.1 and 3.3 **become necessary for a** $(2 \times n)$-**matrix** A.

- (j) If R, as the first player, wins in D-SEKI(A), then A is R-winnable.
- (jj) If R, as the second player, wins in SEKI(A), then A is strictly R-winnable.

Proposition 4.1. *Let A be a $(2 \times n)$-matrix with $n \geq 2$ and let C begin in SEKI(A). Then R wins if and only if A is strictly R-winnable. Moreover, A can be a seki only if $n = 2$.*

Proof. By Lemma 3.3, if A is strictly R-winnable, then R, as the second player, wins in SEKI(A). To prove the inverse statement for the case $2 = m \leq n$, let us assume that A is not strictly R-winnable and show that C can always maintain this situation, that is, C has an active move such that, for any response of R, the obtain matrix remains not strictly R-winnable. Then, either A is a seki or C wins. Indeed, if in A' a zero row appears before a zero column, then A' is strictly R-winnable. Without loss of generality, let us assume that $s_1^r \le s_2^r$. The problem that C has to resolve is not difficult; for example, the following simple strategy works: C reduces $A(2, j)$ whenever possible. It is easy to check that, after any move of R, the obtained matrix is still not strictly R-winnable.

If $A(2, j) = 0$ then C can reduce $A(1, j)$, and if $A(1, j) = 1$ then (s)he wins in one move.

If $A(1, j) \ge 2$ then either C wins making column j zero before R makes row 1 zero, or the matrix is decomposed into the direct sum of two: a (1×1)-matrix at $(1, j)$ with $A(1, j) \ge 2$ and the "complementary" $(1 \times (n-1))$-matrix. If $n > 2$ then C wins, because (s)he wins in every $(1 \times (n-1))$-matrix. Thus, if $n > m = 2$ then R wins whenever A is strictly R-winnable and C wins otherwise. Yet, both obtained matrices may be seki if $n = 2$. □

Proposition 4.2. *Let A be a $(2 \times n)$-matrix with $n \ge 2$. Then R, as the first player, wins in D-SEKI(A) if and only if A is R-winnable.*

Proof. For any R-winnable $(m \times n)$-matrix A, by Lemma 3.1, R, as the first player, wins in D-SEKI(A). To prove the inverse statement for the case $n \ge m = 2$, let us assume that A is not R-winnable. A simple case analysis shows that for any move of R, there is a response of C that maintains this situation, that is, the matrix obtained after these two moves is still not R-winnable. Hence, either A is a d-seki or C wins. Indeed, any matrix that contains a zero row but no zero column is R-winnable. Hence, by preventing A to become R-winnable, C guarantees that no zero row will appear before a zero column. □

As we already know, d-seki, unlike seki, may exist for $n > m = 2$, but not for $n > m = 1$.

A standard form for (2×2)-**matrices**. Given a (2×2)-matrix $A : I \times J \to \mathbb{Z}_+$, where $I = J = \{1, 2\}$, without any loss of generality we can assume that

$$s_1^r \le s_2^r, \qquad s_1^c \le s_2^c, \qquad s_1^r \le s_1^c. \tag{3}$$

In terms of the entries of A, we can rewrite this system of inequalities as follows:

$$A(2, 2) - A(1, 1) \ge A(2, 1) - A(1, 2) \ge 0. \tag{4}$$

Clearly, (3) and (4) can be enforced by permutations of the rows and columns of A. From now on, we assume that these inequalities hold for all considered (2×2)-matrices.

All complete seki and d-seki of size 2×2**.** For a (2×2)-matrix A, by Proposition 4.1, R or C, as the second player, wins in SEKI(A) if and only if A is strictly R- or C-winnable, respectively. Using this result we will find all 2×2 complete seki explicitly.

Proposition 4.3. *There exist only four 2×2 complete seki given by the following matrices:*

$$\begin{array}{cccc} A_1^2 & A_2^2 & A_3^2 & A_4^2 \\ 1\,1 & 2\,0 & 2\,1 & 2\,2 \\ 1\,1 & 0\,2 & 1\,2 & 2\,2. \end{array}$$

Note that these four matrices are exactly the four 2×2 complete seki shown in Fig. 1.

Proof. It is easy to verify that each of these four matrices is indeed a complete seki.

To prove the inverse statement, assume that a (2×2)-matrix A is a complete seki. In particular, R wins after C moves at $(2, 2)$. By Proposition 4.1, R has a response resulting in a strictly R-winnable matrix. Consider two cases: row 1 or 2 becomes strictly winnable.

Case 1: row 1. It is easily seen that in this case the original matrix A is also strictly R-winnable with the winnable row 1, which contradicts the assumption that A is a seki.

Case 2: row 2. By two moves row 2 was reduced by at most 2. Hence, $s_1^r \ge 2(s_2^r - 2)$, by (2), and $s_2^r \ge s_1^r$, by (3). Thus, $s_2^r \ge 2s_2^r - 4$, implying, $s_2^r \le 4$. There are only a few (2×2)-matrices satisfying (3) with $s_2^r \le 4$ and it is not difficult to verify that among them only the considered four are complete seki. □

Corollary 4.4. *A_1^2 is a unique complete d-seki among all $(1 \times n)$- and $(2 \times n)$-matrices.*

Proof. Recall that each complete d-seki is a complete seki, by Proposition 1.1. Furthermore, A_ℓ^2 for $\ell = 1, 2, 3, 4$ are the only complete seki among all $(1 \times n)$- and $(2 \times n)$-matrices, and the only d-seki among them is A_1^2. □

All (2×2) **seki and d-seki.** The above results imply the following characterization of all (not only complete) seki and d-seki among the (2×2)-matrices.

Corollary 4.5. *Let A be a (2×2)-matrix that has no zero row or zero column. R, as the first player, wins in SEKI(A) if and only if (s)he has a move reducing A to a strictly R-winnable matrix. The similar (transposed) necessary and sufficient condition holds for C. If neither R nor C has such a move then the optimal result of SEKI is a draw. There exist only four 2×2 complete seki: A_ℓ^2, $\ell = 1, 2, 3, 4$.* □

Corollary 4.6. *Let A be a (2×2)-matrix that has no zero row or zero column. R, as the first player, wins in D-SEKI(A) if and only if A is R-winnable. The similar (transposed) necessary and sufficient condition holds for C. If A is neither R- nor C-winnable, then the optimal result of D-SEKI is a draw. There is a unique complete 2×2 d-seki: A_1^2.* □

Remark 4.7. The 2×2 seki and d-seki can be characterized explicitly in terms of disjunctive linear programming; see Section 6 of [1]. Although there are only four variables, the entries of A, the obtained disjunctive systems do not look elegant and will not be reproduced here.

使用矩阵直和的概念将矩阵分解为多个子阵，
原矩阵双活 iff 所有子阵双活

5. Direct sums of matrices for the games SEKI and D-SEKI

Given several non-negative integer matrices $A_\ell : I_\ell \times J_\ell \to \mathbb{Z}_+$, where $\ell \in [k] = \{1, \dots, k\}$, their direct sum $A = \oplus_{\ell\in[k]} A_\ell$ is standardly defined as follows: $A : I \times J \to \mathbb{Z}_+$, where $I = \cup_{\ell\in[k]} I_\ell$, $J = \cup_{\ell\in[k]} J_\ell$, and all $2k$ sets $I_\ell, J_\ell, \ell \in [k]$, are assumed pairwise disjoint. Then, $A(i,j) = A(i_\ell, j_\ell)$ if $i \in I_\ell, j \in J_\ell$ for some $\ell \in [k]$ and $A(i,j) = 0$ otherwise. Such a matrix A is frequently referred to as the block-diagonal matrix.

Proposition 5.1. *Let $A = \oplus_{\ell\in[k]} A_\ell$, then A is a seki, complete seki, or complete d-seki if and only if A_ℓ have the corresponding property for all $\ell \in [k]$. The game SEKI(A) is a draw if and only if SEKI(A_ℓ) is a draw for all $\ell \in [k]$. The "if" (but not "only if") part of the last claim holds for D-SEKI, as well.*

Proof. If in SEKI(A_ℓ) or D-SEKI(A_ℓ) for all $\ell \in [k]$, R or C, even as the first player, cannot win, then (s)he cannot win in SEKI(A) or D-SEKI(A), respectively. Indeed, the opponent guarantees at least a draw always replying optimally in the same A_ℓ, where the previous move was made and passing if it was pass.

For SEKI, the inverse also holds. If a player, say R, wins in SEKI(A_{ℓ_0}) then SEKI(A) cannot be a draw. Indeed, if R is the second, (s)he always responds optimally in the same subgame A_ℓ, where the previous move was made. If R is the first, (s)he starts optimally in A_{ℓ_0} and then applies the previous strategy. In both cases R plays optimally in A_{ℓ_0} whenever C passes. Obviously, R wins if A_{ℓ_0} is finished. Otherwise, if some other subgame A_ℓ is finished before A_{ℓ_0}, then either R or C wins, since SEKI(A_ℓ) (unlike D-SEKI(A_ℓ)) cannot result in a draw unless both players pass, while R always proceeds in A_{ℓ_0} rather than pass.

The last argument does not work for D-SEKI. For example, if A_ℓ is a (1×1)-matrix with an entry 1 for some $\ell \in [k]$ then any player can force a draw in one move in A.

For this reason, both statements of Proposition 5.1 are not extendable to D-SEKI, but both hold for the complete d-seki, which is defined it terms of winning rather than drawing: a matrix is a complete d-seki if after any active move of a player the opponent wins. Obviously, A has this property if and only if A_ℓ have it for all $\ell \in [k]$. $\square$

A matrix is *prime* if it is not a non-trivial ($k > 1$) direct sum. Every matrix A is a direct sum of prime ones and the corresponding representation $A = \oplus_{\ell\in[k]} A_\ell$ is unique.

In the endplay, GO is typically the sum of several independent games [11]. Proposition 5.1 can be interpreted in these terms: it shows that the sum of several independent seki is a seki; see, for example, the third diagram in Fig. 1.

6. Computations and conjectures related to the games SEKI and D-SEKI

Complete seki of size 3×3. For a $(m \times n)$-matrix $A : I \times J \to \mathbb{Z}_+$, its *height* $h = h(A)$ is defined as the maximum of its entries. By Theorem 3.4, each $(0, 1, 2)$-IDSM A with $s(A) \geq 2$ is a complete seki. This is a large family of complete seki of height at most two. Our computer analysis shows that there exist seven prime 3×3 complete seki of height 3:

A_1^3	A_2^3	A_3^3	A_4^3	A_5^3	A_6^3	A_7^3
033	133	301	320	320	320	033
303	313	022	212	203	213	331
330	331	121	023	032	033	312.

Even without a computer, one can check that each of these seven matrices is a complete seki. However, such verification requires a complicated case analysis; see Section 2.

All seven matrices are symmetric, yet, the last two are not IDSMs. In contrast, every known prime $(0, 1, 2)$ complete seki with $s \geq 2$ is an IDSM. Conversely, every $(0, 1, 2)$-IDSM with $s \geq 2$ is a complete seki, by Theorem 3.4.

Conjecture 6.1. *Any prime $(0, 1, 2)$ complete seki is an IDSM.*

This was proven in [1] for $(0, 1)$-matrices and verified by computer for $(n\times n)$-matrices with $n \leq 5$. If true, this conjecture would promote Theorem 3.4 to a characterization of the $(0, 1, 2)$ complete seki. However, even a prime complete seki of height greater than 2 may be not an IDSM; see, for example, A_6^3 and A_7^3. The first one results from an IDSM by one move; the last one results in an IDSM by one move. Hence, there is a symmetric IDSM in which the first player wins. Consider, for example, the matrix A obtained from A_7^3 by reducing its central entry by 1. The first player wins in SEKI(A), because A_7^3 is a complete seki. In contrast, the second player never wins in a symmetric matrix, as we know.

The height of a complete seki. Let H_n denote the maximum height of a $n \times n$ complete seki. We already know that $H_1 = H_2 = 2$, while $H_3 \geq 3$. Our computations show that either $H_3 = 3$ or $H_3 > 10$. In this section we prove that $H_n \geq 3$ for all $n \geq 3$.

Conjecture 6.2. *The function H_n is well defined (that is, $H_n < \infty$) for all integer $n \geq 1$; furthermore, this function is monotone, non-decreasing, and unbounded; $H_3 = 3$ but $H_4 > 3$.*

However, all known complete seki are of height at most 3. Also, they are all square.

Conjecture 6.3. *Every seki (and in particular, each complete seki) is a square matrix; in other words, either R or C wins in SEKI whenever $m \neq n$.*

It is sufficient to prove this conjecture for the prime complete seki. The conjecture follows from Proposition 4.1 if $\min(m, n) \leq 2$, but it is not yet known whether a 3×4 seki exist.

The semi-complete seki was defined as a seki in which exactly one player must pass, while the other one can make an active move such that the reduced matrix is still a seki. We found out only two such examples among the (3×3)-matrices.

A_8^3	A_9^3
0 3 3	0 3 5
4 0 3	3 0 5
4 3 0	4 4 0.

In A_8^3 player C can reduce $A(1, 2)$ from 3 to 2, yet, R must pass; see Section 2. Of course, two transposed matrices $(A_8^3)^T$ and $(A_9^3)^T$ and all matrices obtained from these four by permutations of their rows and columns, are semi-complete seki too. Our computations show that there are no others among the 3×3 matrices of height $h \leq 10$. In contrast, there are very many 4×4 semi-complete seki and some of them are IDSMs, e.g., there are ten (4×4)-IDSMs of height 3 that are semi-complete seki (C can move but R must pass).

0023	0023	0033	0123	0123	0123	0133	0312	0233	1133
2300	2201	3300	2301	2310	2121	3301	2112	3302	3311
1121	2210	1122	2121	2112	2211	2122	2121	3131	2222
2111	1121	2211	2121	2121	2211	2221	2121	2222	2222.

On (4×4)-**matrices of height** 3. There are thirteen (4×4)-IDSMs of height 3 in which R wins, even as the second player:

0023	0113	0113	0113	0113	0123	0123	0123	0123	0123	0123	0222	0223
1310	1130	3110	1220	2120	0321	0213	0213	0321	2103	1230	0222	3220
2012	2102	0221	2012	2210	3012	2220	3120	3012	2220	2202	3111	2113
2210	2210	2111	2210	1112	3210	2220	3210	3210	2220	3111	3111	2221.

There are (only) three (4×4)-IDSMs of height 3 in which R wins as the first player but can only make a draw as the second one.

0 1 3 3	0 2 2 3	0 2 3 3
3 1 0 3	2 2 3 0	3 3 1 1
2 3 2 0	2 1 1 3	3 1 2 2
2 2 2 1	3 2 1 1	2 2 2 2.

There are sixty (4×4)-IDSMs of height 3 in which the first player wins; see [6] for more details.

Infinite families of complete seki. Let us notice that the 3×3 complete seki A_i^3 for $i = 3, 4, 5, 6$, can be viewed as extensions of some 2×2 seki as follows:

	A_3^3		A_4^3		A_5^3		A_6^3
	3 0 1		3 2 0		3 2 0		3 2 0
2 2	0 2 2	3 2	2 1 2	2 3	2 0 3	3 3	2 1 3
2 2	1 2 1	2 3	0 2 3	3 2	0 3 2	3 3	0 3 3.

Let us define these two extensions explicitly. Given an $(m \times n)$-matrix $A : I \times J \to \mathbb{Z}_+$, fix arbitrary $i^* \in I, j^* \in J$, and define two new $(m + 1) \times (n + 1)$-matrices $E_1 = E_1(A, i^*, j^*)$ and $E_2 = E_2(A, i^*, j^*)$ by setting $E_1 : I' \times J' \to \mathbb{Z}_+$ and $E_2 : I' \times J' \to \mathbb{Z}_+$, where $I' = \{0, 1, \ldots, n\}$, $J' = \{0, 1, \ldots, m\}$, while $I = \{1, \ldots, n\}$, $J = \{1, \ldots, m\}$, as usual.

The added row and column of E_1 and E_2 are defined as follows: $E_1(0, 0) = E_2(0, 0) = 3$; $E_1(0, j^*) = E_1(i^*, 0) = 1$, $E_2(0, j^*) = E_2(i^*, 0) = 2$; $E_1(i, 0) = E_1(0, j) = E_2(i, 0) = E_2(0, j) = 0$ for all $i \in I \setminus \{i^*\}$ and $j \in J \setminus \{j^*\}$. We set $E_1(i^*, j^*) = A(i^*, j^*) - 1$ and $E_2(i^*, j^*) = A(i^*, j^*) - 2$, noticing that $E_1(A)$ is not defined when $A(i^*, j^*) = 0$ and $E_2(A)$ is not defined when $A(i^*, j^*) \leq 1$. Finally, all other entries of $E_1(A)$ and $E_2(A)$ are the same as in A, that is, $E_1(i, j) = E_2(i, j) = A(i, j)$ for all $i \in I \setminus \{i^*\}$ and $j \in J \setminus \{j^*\}$.

Two obtained matrices $E_1 = E_1(A, i^*, j^*)$ and $E_2 = E_2(A, i^*, j^*)$ is called, respectively, the $(3, 1, 1, -1)$- and the $(3, 2, 2, -2)$-*extension* of A at (i^*, j^*). For example, $E_2 = A_5^3$ is the $(3, 2, 2, -2)$-extension of a (2×2)-matrix at $(1, 1)$ and $E_1 = A_3^3$ is the $(3, 1, 1, -1)$-extension of a (2×2)-matrix at $(2, 2)$. Both obtained matrices are complete seki.

The $(3, 1, 1, -1)$-extension was suggested by Andrey Gol'berg in 1981. Being applied to a complete seki, it frequently, but not always, results in another complete seki. For example, our computations show that *all* $(3, 1, 1, -1)$-extension of the five 3×3 complete seki, $A_i^3,\ i = 1, 2, 3, 4, 5$ result in complete seki. Yet, the $(3, 1, 1, -1)$-extensions of A_6^3 at (i^*, j^*) are complete seki whenever $i^* + j^* \geq 4$, that is, at $(1, 3)$, $(1, 3)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, and $(2, 2)$, but at $(1, 1)$ it is not; the first player wins in the obtained matrix. Finally, the $(3, 1, 1, -1)$-extension of A_6^3 is not defined at $(1, 2)$ and at $(2, 1)$, since $A_6^3(1, 2) = A_6^3(2, 1) = 0$.

The $(3, 1, 1, -1)$-extensions of A_7^3 at (i^*, j^*) are complete seki whenever $\min(i^*, j^*) \geq 2$, that is, at $(2, 2)$, $(2, 3)$, $(3, 2)$, and $(3, 3)$, but at $(1, 2)$ and $(2, 1)$, it results in a semi-complete seki, and at $(1, 3)$, $(3, 1)$ it is not a complete seki either; see [6] for more details.

In [5] it was erroneously announced that the $(3, 1, 1, -1)$-extension of a complete seki is always a complete seki; more precisely, it was stated that if $E_1 = E_1(A, i^*, j^*)$ is the $(3, 1, 1, -1)$-extension of a complete seki A at (i^*, j^*) then E_1 is a complete seki whenever $s_{i^*}^r \geq 4$, $s_{j^*}^c \geq 4$. The above analysis of A_6^3 and A_7^3 shows that this claim was an overstatement: among all 2×2 and 3×3 complete seki there are five counterexamples: the $(3, 1, 1, -1)$-extensions of A_7^3 at $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 1)$ and of A_6^3 at $(1, 1)$. Note that conditions $s_{i^*}^r \geq 4$, $s_{j^*}^c \geq 4$ are necessary: if $s_{i^*}^r \leq 3$ (respectively, $s_{j^*}^c \leq 3$) then R (respectively, C) begins and wins in E_1 in at most 3 moves.

Let us apply the $(3, 1, 1, -1)$- and $(3, 2, 2, -2)$-extensions at $(1, 1)$ recursively beginning with the two (2×2)-matrices A^2 and B^2 given below (that is, $A^n = E_1(A^{n-1}, 1, 1)$ and $B^n = E_2(B^{n-1}, 1, 1)$ for $n \geq 3$). We obtain the following two infinite matrix sequences A^n, B^n for $n = 2, 3, \dots$

$$\begin{matrix}2&2\\2&2\end{matrix} \quad \begin{matrix}3&1&0\\1&1&2\\0&2&2\end{matrix} \quad \begin{matrix}3&1&0&0\\1&2&1&0\\0&1&1&2\\0&0&2&2\end{matrix} \quad \begin{matrix}3&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&1&2\\0&0&0&2&2\end{matrix} \quad \begin{matrix}3&1&0&0&0&0\\1&2&1&0&0&0\\0&1&2&1&0&0\\0&0&1&2&1&0\\0&0&0&1&1&2\\0&0&0&0&2&2\end{matrix} \quad \cdots \quad \begin{matrix}3&2\\2&3\end{matrix} \quad \begin{matrix}3&2&0\\2&1&2\\0&2&3\end{matrix} \quad \begin{matrix}3&2&0&0\\2&1&2&0\\0&2&1&2\\0&0&2&3\end{matrix} \quad \begin{matrix}3&2&0&0&0\\2&1&2&0&0\\0&2&1&2&0\\0&0&2&1&2\\0&0&0&2&3\end{matrix} \quad \cdots.$$

Proposition 6.4. *Except for B^2, both sequences contain only complete seki.*

Proof. By construction, these two sequences contain only IDSMs with the sums $s(A^n) = 4$ and $s(B^n) = 5$, for all $n \geq 2$. Moreover, except $A(1, 1) = 3$ and $B(1, 1) = B(n, n) = 3$, all other entries take only values 0, 1, 2. Thus, conditions of Lemma 3.3 and Theorem 3.4 "almost" hold. Moreover, by the same arguments we can show that after "almost" every active move of a player the opponent wins in 3 (respectively, in 4) moves. There are two important cases, yet, when the opponent cannot win immediately; instead (s)he can reduce A^n (respectively, B^n) to the $((n-1) \times (n-1))$-matrix from the same sequence, in which one entry is also reduced by 1. Thus, we can finish the proof by induction on n, since in both sequences the 3×3 matrices are known complete seki: $A^3 = A_3^3$ and $B^3 = A_4^3$, respectively.

Case 1. If a player, say R, begins at $(1, 1)$ then C replies at $(2, 1)$. It is easily seen that R must play at $(1, 2)$. (In case of the second sequence, C will repeat by playing at $(2, 1)$ once more and again R must answer at $(1, 2)$.) As a result of this exchange of moves, the original $(n \times n)$-matrix, A^n or B^n, is reduced to the direct sum of two matrices: a 1×1 complete seki with the entry 2 and the $(n-1) \times (n-1)$-matrix from the same sequence in which the first entry is 2 rather than 3. Since C has to move in the obtained game, (s)he can enforce the same reduction again, etc.

Case 2. If a player, say R, begins at $(2, 1)$ then C answers at $(1, 1)$ threatening to make the first column zero by the next 2 (respectively, 3) moves. The only defense of R is to play at $(1, 2)$. (In the second sequence, C will proceed by reducing $(2, 1)$ and R must reduce $(1, 2)$ in return). Such an exchange of moves results in the same reduction as in Case 1. Hence, C wins.

Note that the matrices of the second sequence are double-symmetric, that is, $B(i, j) = B(j, i)$ and $B(i, j) = B(n-i+1, n-j+1)$ for all $n \geq 3$ and $i, j \in [n] = \{1, \dots, n\}$. Hence, we can substitute $(1, 2)$, $(2, 1)$, and $(1, 1)$ by $(n-1, n)$, $(n, n-1)$, and (n, n), respectively.

It remains to verify that for any other move of a player, the opponent wins in 3 (respectively, in 4) moves by applying the strategy suggested in the proof of Lemma 3.3. $\square$

Corollary 6.5. *For each $n \geq 3$ there is a prime $n \times n$ complete seki of height 3; in other words, $H_n \geq 3$ for $n \geq 3$.*

Proof. For every $n \geq 2$, each of the above two sequences contains a unique $(n \times n)$-matrix. It is prime, it is of height 3, and by Proposition 6.4, it is a complete seki for $n \geq 3$. $\square$

D-SEKI. The family of all known complete d-seki is smaller and it looks "nicer" than the family of all known complete seki. By Proposition 3.2, a $(0, 1)$-IDSM A is a complete d-seki whenever $s(A) \geq 2$. We know few other examples. All of them are $(0, 2)$-matrices, that is, their entries take only values 0 and 2. In particular, the height is 2. The following $(0, 2)$-matrices are called 2-*cycles*:

$$\begin{matrix}2&2&0\\2&0&2\\0&2&2\end{matrix} \quad \begin{matrix}2&2&0&0\\2&0&2&0\\0&2&0&2\\0&0&2&2\end{matrix} \quad \begin{matrix}2&2&0&0&0\\2&0&2&0&0\\0&2&0&2&0\\0&0&2&0&2\\0&0&0&2&2\end{matrix}.$$

More precisely, for each $k \geq 3$, a (unique) 2-cycle C_k is defined as the $(k \times k)$-matrix in which the first and the last entries of the main diagonal and all entries of the two neighbor diagonals equal 2 (that is, $C_k(1, 1) = C_k(k, k) = C_k(i, i + 1) = C_k(i + 1, i) = 2$ for $i = 1, \ldots, k - 1$), while all remaining entries of C_k equal 0.

Lemma 6.6. *Every 2-cycle C_k is transitive. In other words, for any two entries $(i, j), (i', j')$ such that $i, i', j, j' \in [k] = \{1, \ldots, k\}$ and $C_k(i, j) = C_k(i', j') = 2$, there are permutations of the rows and columns of C_k that transform (i, j) to (i', j') and the matrix C_k to itself.*

Proof. First, let us consider the simple case $j = i' = i + 1$ and $j' = i$, that is, $(i, j) = (i, i + 1)$ and $(i', j') = (i + 1, i)$. In this case the matrix should be just transposed.

Since permutations form a group, it remains to transform $(i, i + 1)$ to $(1, 1)$ keeping C_k. The following (unique) permutations of the rows and columns solve the problem:

$$(f(i), f(i+2), f(i-2), f(i+4), f(i-4), \ldots); \qquad (f(i+1), f(i-1), f(i+3), f(i-3), f(i+5), \ldots),$$

where the first and second permutations correspond to the rows and columns, respectively, and the function $f : \mathbb{Z} \to [k]$ is defined by the formula:

$$f(\ell) = \begin{cases} \ell, & \text{if } \ell \in [k]; \\ 2k + 1 - \ell, & \text{if } \ell > k; \\ 1 - \ell, & \text{if } \ell < 1. \end{cases} \tag{5}$$

For example, the permutations $(3, 5, 1, 7, 2, 6, 4)$ of the rows and $(4, 2, 6, 1, 7, 3, 5)$ of the columns of C_7 transform $(3, 4)$ to $(1, 1)$ and keep C_7 itself. $\square$

Theorem 6.7. *Every 2-cycle is a prime complete d-seki.*

Proof. Obviously, any 2-cycle is prime matrix. Let us prove that it is a complete d-seki. To do so, we have to demonstrate that after any move of R or C, the opponent wins in D-SEKI. There are many possible moves in a 2-cycle that seem different, but by Lemma 6.6, they are all equivalent. Without loss of generality, we will assume that C plays at $(1, 1)$ and prove that R wins naturally replying at $(1, 2)$. By this she threatens to make zero the first row in two moves: $(1, 2)$ and $(1, 1)$. Against this, C has only one defense: $(2, 1)$. Now, against the move at $(1, 2)$ by R, he would simply repeat $(2, 1)$ forcing a draw. Yet, R has a stronger move, at $(2, 3)$, creating two threats simultaneously: $(1, 2), (1, 1)$ and $(2, 3), (2, 1)$. If C would pass then both plans work:

- (i) R plays at $(1, 2)$ and, against the response of C at $(2, 1)$, wins at $(2, 3)$.
- (ii) R plays at $(2, 3)$ and, against the response of C at $(1, 1)$, wins at $(1, 2)$.

Of course, C is not to pass. In fact, he has a defense against each of the above two threats, but not against both. The move of C at $(3, 2)$ destroys (i), but (ii) is still killing him. Also C can play at $(3, 3)$ if $k = 3$ or, respectively, at $(4, 3)$ if $k > 3$. This is a good defense against (ii), but (i) still works for R. It is easily seen that C has no other defense against (i) and (ii), except for the two above options. Yet, the corresponding two moves differ and C cannot avert both threats. $\square$

Yet, the above arguments do not hold for $k = 2$: the (2×2)-IDSM A_4^2 is a complete seki but not a complete d-seki. Respectively, A_4^2 is not a 2-cycle. Although, like a 2-cycle, A_4^2 is a $(0, 2)$-IDSM of sum 4, but its size is only 2×2, while the size of a 2-cycle is at least 3×3, by the definition.

Proposition 6.8. *Every $(0, 2)$-IDSM of sum 4 can be transformed, by permutations of its rows and columns, to the direct sum of 2-cycles and (2×2)-IDSMs A_4^2.*

Sketch of the Proof. This claim results from the following well-known statement of graph theory: each 2-regular finite multi-graph is the union of cycles. The $(0, 2)$-IDSMs of sum 4 correspond to the 2-regular bipartite multi-graphs. We leave details to careful readers. $\square$

It seems that Propositions 3.2, 6.8, and Theorem 6.7 can be reversed as follows.

Conjecture 6.9. *Each complete d-seki is the direct sum of $(0, 1)$-IDSMs with $s \geq 2$ and $(0, 2)$-IDSMs with $s = 4$ of size at least 3×3 (that is, 2-cycles). In other words, all prime complete d-seki are listed by Proposition 3.2 and Theorem 6.7.*

This conjecture was verified for all $m \times n$ matrices of height at most 3 and $\max(m, n) \leq 5$.

Each $(0, 1, 2)$-IDSM is a complete seki, by Theorem 3.4, but for each $k \geq 3$, only one among them (the 2-cycle C_k) is a $k \times k$ prime complete d-seki, if Conjecture 6.9 holds for the $(0, 1, 2)$-matrices.

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