Isolated cycles of critical random graphs

Marc Noy* Vonjy Rasendrahasina Vlady Ravelomanana† Juanjo Rué ‡

Abstract

Consider the Erdős-Rényi random graph $G(n, M)$ built with n vertices and M edges uniformly randomly chosen from the set of $\binom{n}{2}$ edges. Let L be a set of positive integers. For any number of edges $M \leqslant \frac{n}{2} + o(n^{3/4})$, we compute – via analytic combinatorics – the number of isolated cycles of $G(n, M)$ whose length is in L.

1 Introduction

Random graph theory (see [2, 9, 12]) is an active area of research that combines computer science, combinatorics, probability theory and graph theory. The uniform random graph model $G(n, M)$ studied in [5] consists of n vertices with M edges drawn uniformly at random from the set of $\binom{n}{2}$ possible edges. Erdős and Rényi showed in their seminal paper [5] how the structure of the connected components of $G(n, M)$ changes as M grows. More precisely, when $M = \frac{cn}{2}$ for constant c the largest component of $G(n, M)$ has asymptotically almost surely $\mathcal{O}(\log n)$, $\Theta(n^{2/3})$ or $\Theta(n)$ vertices according to whether $c < 1$, $c = 1$ or $c > 1$. This *double-jump* phenomenon in the structure of $G(n, M)$ is one of the most spectacular results in [5] and of the whole random graph theory. Due to this phase transition, researchers had worked around the critical value $\frac{n}{2}$ and one can distinguish three different phases: *subcritical* when $(M - n/2)n^{-2/3} \to -\infty$, *critical* $M = n/2 + \mathcal{O}(n^{2/3})$ and *supercritical* as $(M - n/2)n^{-2/3} \to \infty$. We refer to Bollobás [2] and Janson, Łuczak and Ruciński [12] for books devoted to the random graphs $G(n, M)$ and $G(n, p)$. If the $G(n, p)$ model is the one more commonly used today, partly due to the independence of the edges, the $G(n, M)$ model has more enumerative flavour allowing generating functions based approaches. By setting $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, the stated results of this paper can be transferred to the $G(n, p)$ model.

*Partiallay supported by the Spanish MICINN project MTM2014-54745-P

†Partially supported by grants ANR 2010 BLAN 0204 (Magnum) and the Project PHC Procope ID-57134837 "Analytic, probabilistic and geometric Methods for Random Constrained graphs"

‡Partially supported by the Project PHC Procope ID-57134837 "Analytic, probabilistic and geometric Methods for Random Constrained graphs", by the FP7-PEOPLE-2013-CIG project CountGraph (ref. 630749), and the Spanish MICINN projects MTM2014-54745-P, MTM2014-56350-P

2 Number of unicyclic components

The drastic structural change of $G(n, M)$ has fascinated researchers for years. This phenomena is partly due to the appearance of isolated cycles (or unicyclic component) in the evolving graph. Cycles have been the object of various theoretical studies as shown in Bollobás [2, Chapter V] and Kolchin [13, Chapter 1]. In this section, our goal is to quantify the distribution of unicyclic components (connected components with as many edges as vertices) of $G(n, M)$ using techniques from analytic combinatorics. We refer the reader to the masterful works of Flajolet, Knuth and Pittel [6] and of Janson, Knuth, Łuczak et Pittel [11] where generating functions and analytic combinatorics have been successfully used to study in depth the development of components of $G(n, M)$.

2.1 Previous works on $\mathbf{X_{n,M}}$ Let $X_{n,M}$ be the random variable associated to the number of isolated cycles of $G(n, M)$. Erdős and Rényi showed that $\lim_{n\to\infty} \mathbb{E}[X_{n,cn}] = -\frac{1}{2}\log(1 - 2c) - c - c^2$ for fixed $c < \frac{1}{2}$ [5, Theorem 5.a]. If $M = \frac{n}{2}(1 - \mu n^{-1/3})$, Kolchin obtained that $(X_{n,M} - r_0)/\sqrt{r_0}$ with $r_0 = \frac{1}{6}\log n - \frac{1}{2}\log\mu$ is $\mathcal{N}(0, 1)$ if $\mu \to \infty$ but $\mu = o(n^{1/3})$ (see [13, Theorem 1.1.15]). If $\mu = \mathcal{O}(1)$, Flajolet, Knuth and Pittel [6, Corollary 6] proved that $\mathbb{E}[X_{n,M}] \sim \frac{1}{6}\log n$. By symmetry, $X_{n,M}$ properly normalized should be Gaussian as $M = \frac{n}{2}(1 + \mu n^{-1/3})$ if $1 \ll \mu = o(n^{1/3})$ [12, Theorem 5.24]. For the supercritical case, i.e. as $c > \frac{1}{2}$, Fountoulakis has shown that $X_{n,cn}$ is Poisson [8, Theorem 1.1 : (2)] with parameter $-\frac{1}{2}\log(1 - 2ce^{-s}) - \frac{1}{2}(1 - 2ce^{-s}) - \frac{1}{4}(1 - 2ce^{-s})^2$ where s is the positive solution of $s/(1 - e^{-s}) = 2c$. Using methods from statistical physics, Ben-Naim and Krapivsky [1] studied also the number as well as the size of first cycles at the so called "gelation point".

2.2 Our results concerning isolated cycles In this subsection, we present the limiting distribution of $X_{n,M}$. In particular, we obtain full limiting distributions for the whole spectrum, improving the what was known before. It is important to note that in the following theorem there is *no discontinuity* between the equations (2.1)-(2.4).

THEOREM 2.1. *Let $X_{n,\,M}$ be the random variable counting the number of isolated cycles of $G(n,\,M)$. Then, the following limiting distributions hold: if $c := c(n)$ is such that $0 < \limsup_{n\to\infty} c < 1/2$ and $M = cn$ then*

$$\frac{X_{n,\,M}}{-\frac{1}{2}\log(1-2c)-c-c^2} \xrightarrow{\mathcal{D}} \text{Poisson}\,(1)\ . \tag{2.1}$$

If $M = \frac{n}{2}(1-\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/3}$, then

$$\frac{X_{n,\,M} - \frac{\log n}{6} + \frac{\log \mu}{2}}{\sqrt{\frac{\log n}{6} - \frac{\log \mu}{2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,. \tag{2.2}$$

If $M = \frac{n}{2}(1 \pm \mathcal{O}(1)n^{-1/3})$, then

$$\frac{X_{n,\,M} - \frac{\log n}{6}}{\sqrt{\frac{\log n}{6}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,. \tag{2.3}$$

If $M = \frac{n}{2}(1+\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/12}$ then

$$\frac{X_{n,\,M} - \frac{\log n}{6} + \frac{\log \mu}{2}}{\sqrt{\frac{\log n}{6} - \frac{\log \mu}{2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,. \tag{2.4}$$

These results are the respective reformulations of Theorems 4.1 and 4.2, Corollary 4.1 and Theorem 4.4 given in the paragraph 4. Finally, let us emphasize that the limiting distribution for the regimes $M = \frac{n}{2}(1-\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/3}$ and $M = \frac{n}{2}(1+\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/12}$ was already known (see [6, 8]). The proofs are also included in this paper, because the techniques used gives us an straightforward generalization of the general case to the study of the number of isolated cycles of length in a set of integers L (see Section 3).

3 The number of cycles of given length

The use of the analytic techniques developed to prove Theorem 2.1 generalizes in the following way: let L be an infinite set of positive integers greater or equal than 3. Following Flajolet, Knuth and Pittel [6], an L-cycle is defined as an isolated cycle whose length is in L. Let $X^{(L)}_{n,\,M}$ be the random variable counting the number of L-cycles of $G(n,\,M)$. [6, Corollary 7] states that if $\lim_{n\to\infty} \frac{M}{n} = c < \frac{1}{2}$, then the probability that a graph with n vertices and M edges has no cycle of length $l \in L$ is equal to

$$\sqrt{1-2c}\,\exp\left(\sum_{i>1,\,i\notin L} c^i\right) + \mathcal{O}\left(n^{-1/2}\right).$$

We can extend this result in the following way:

THEOREM 3.1. *Let L be a set of integers of unbounded cardinality and*

$$\ell(z) = \sum_{k\geqslant 3,\,k\in L} \frac{z^k}{2k}\,.$$

Let $L(n)$ be the counting function that gives the number of integers in L less than or equal to n. Let δ be a function such that

$$\delta(n) = \frac{n}{L(n)}\,(1+o(1))\,.$$

Define $\lambda = \delta\left(\frac{\log n}{6} - \frac{\log \mu}{2}\right)$ and let $X^{(L)}_{n,\,M}$ be the random variable counting the number of L-cycles of $G(n,\,M)$. Then the following holds:
If if $c := c(n)$ is such that $0 < \limsup_{n\to\infty} c < 1/2$ and $M = cn$ then

$$\frac{X^{(L)}_{n,\,M}}{\ell(2c)} \xrightarrow{\mathcal{D}} \text{Poisson}\,(1)\ .$$

If $M = \frac{n}{2}(1-\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/3}$ or $M = \frac{n}{2}(1 \pm \mathcal{O}(1)n^{-1/3})$ or $M = \frac{n}{2}(1+\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/12}$, then we have :

$$\frac{X^{(L)}_{n,\,M} - \lambda}{\sqrt{\lambda}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,.$$

4 Proof of Theorem 2.1

We present the key ingredients needed to prove Theorem 2.1. The full proofs of these statements can be found in the Appendix of this paper. The main idea of all proofs are based on encoding the structure random graphs in the regime under consideration using generating functions and estimating later larger powers by means of saddle point estimates. Specially in the critical phase, the required analysis is quite delicate.

4.1 Subcritical phase.

In this regime, the structure of the random graph is based on a set of acyclic graphs (a forest) plus a set of unicyclic graphs. We exploit this property in order to get the following results.

THEOREM 4.1. *Let $c := c(n)$ be a function of n such that $0 < \limsup_{n\to\infty} c < 1/2$, and $M = cn$. Then, for all fixed non-negative number $k \geqslant 0$ as $n \to \infty$, we have:*

$$\Pr\left[X_{n,\,M} = k\right] = e^{-\lambda(2c)}\frac{\lambda(2c)^k}{k!}\left(1+\mathcal{O}\left(n^{-1}\right)\right)\,,$$

with

$$\lambda(2c) = -\tfrac{1}{2}\log(1-2c) - c - c^2.$$

If $k \to \infty$ as $n \to \infty$ then there exists absolute constants $C > 0$ and $\varepsilon > 0$ such that

$$\Pr[X_{n,\,M} = k] \leqslant Ce^{-\varepsilon k}.$$

Proof. See the full proof in Appendix 6.1.

THEOREM 4.2. *Fix $y \in \mathbb{R}$. As n is large and $M = \frac{n}{2}(1 - \mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/3}$. Define $\alpha_n = \frac{1}{6}\log n + \frac{1}{2}\log\mu$. Then the number of isolated cycles of $G(n,\,M)$ satisfies*

$$\Pr\left[\frac{X_{n,\,M} - \alpha_n}{\sqrt{\alpha_n}} \leqslant y\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{u^2}{2}}du\,.$$

Proof. See the full proof in Appendix 6.2.

4.2 Critical phase. In the critical phase, we have to take into account the appearance of complex (multicyclic) components. Let $p_k(n, M, r)$ be the probability that $G(n,\,M)$ has a total excess [1] r with k unicyclic components.

THEOREM 4.3. *Let $M = \frac{n}{2}(1+\mu n^{-1/3})$ with $\mu = O(1)$, $k = \frac{1}{6}\log n + \rho\sqrt{\frac{1}{6}\log n}$ and $\rho \ll (\log(n))^{1/6}$.*

- *For each fixed $r \in \mathbb{N}$, the value of $p_k(n, M, r)$ is*

$$e^{-\left(\frac{1}{6}\log n\right)}\frac{\left(\frac{1}{6}\log n\right)^k}{k!}\sqrt{2\pi}e_r\; A(3r+1/2,\mu)$$
$$.\left(1 + \mathcal{O}\left((1+|\rho|)\frac{\log\log n}{\log(n)^{1/2}}\right) + \mathcal{O}\left(\frac{r^{3/2}}{n^{1/2}}\right)\right),$$

where

$$e_r = \frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!},$$

and

$$A(y,\mu) = \frac{e^{-\mu^3/6}}{3^{(y+1)/3}}\sum_{k\geqslant 0}\frac{(\frac{1}{2}3^{2/3}\mu)^k}{k!\Gamma((y+1-2k)/3)}\,.$$

- *As r is large, there exists absolute constants $C > 0$ and $\epsilon > 0$ such that*

$$p_k(n, M, r) \leqslant e^{-\left(\frac{1}{6}\log n\right)}\frac{\left(\frac{1}{6}\log n\right)^k}{k!} \times Ce^{-\epsilon r}. \tag{4.5}$$

Proof. See the full proof in Appendix 6.3.

As a consequence of this result we have the following corollary, which provides Equation (2.3) .

COROLLARY 4.1. *Let $M = \frac{n}{2}(1+\mu n^{-1/3})$ where μ is a real constant. As $n \to \infty$ for any $y \in \mathbb{R}$, we have :*

$$\Pr\left[\frac{X_{n,\,M} - \frac{1}{6}\log n}{\sqrt{\frac{1}{6}\log n}} \leqslant y\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du\,.$$

Proof. By the dominated convergence theorem, by splitting the summation into two parts and using (4.3), (4.5) we have that $\Pr[X_{n,\,M} = k] = \sum_{r\geqslant 0} p_k(n, M, r)$ is equal to

$$\sum_{r\geqslant 0} e^{-\left(\frac{1}{6}\log n\right)}\frac{\left(\frac{1}{6}\log n\right)^k}{k!}\sqrt{2\pi}e_r A(3r+1/2,\mu)$$
$$.\left(1 + \mathcal{O}\left((1+|\rho|)\frac{\log\log n}{(\log n)^{1/2}}\right) + \mathcal{O}\left(\frac{r^{3/2}}{n^{1/2}}\right)\right).$$

For any constant μ, Janson, Knuth, Łuczak and Pittel [11, Equation (13.17) and Corollary p. 61] have shown that the probability that $G(n,\,M)$ has total excess r is asymptotically $\sqrt{2\pi}e_r A(3r+1/2,\mu)$ with the s-th moment $\sum_{r\geqslant 0}\sqrt{2\pi}e_r r^s A(3r+1/2,\mu) = \mathcal{O}(\mu^{3s}) = \mathcal{O}(1)$ and $\sum_{r\geqslant 0}\sqrt{2\pi}e_r A(3r+1/2,\mu) = 1$. Moreover as $k = \left(\frac{1}{6}\log n\right) + \rho\sqrt{\left(\frac{1}{6}\log n\right)}$ where $\rho \ll (\log(n))^{1/6}$, by means of Theorem 6.1 we get that:

$$e^{-\left(\frac{1}{6}\log n\right)}\frac{\left(\frac{1}{6}\log n\right)^k}{k!}$$

is equal to

$$\frac{e^{-\rho^2/2}}{\sqrt{2\pi\left(\frac{1}{6}\log n\right)}}\left(1 + \mathcal{O}\left(\frac{1+|\rho|^3}{\sqrt{\log n}}\right)\right)\,.$$

4.2.1 Supercritical phase. Before stating the theorem, let us mention that Note that the condition $1 \ll \mu \ll n^{1/12}$ is needed to bound the error term in the calculation:

THEOREM 4.4. *Let $M = \frac{n}{2}(1+\mu n^{-1/3})$ with $1 \ll \mu \ll n^{1/12}$. Define $\alpha_n = \frac{1}{6}\log n - \frac{1}{2}\log\mu$. For any real number y as n is large, we have :*

$$\Pr\left[\frac{X_{n,\,M} - \alpha_n}{\sqrt{\alpha_n}} \leqslant y\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du\,.$$

Proof. Postponed in Appendix 6.4.

Let us finally mention about the proof of Theorem 3.1. When the critical parameter c is in the vicinity of $\frac{1}{2}$, the computations are more delicate but we rely on the proofs of Theorems 4.2, 4.4 and Corollary 4.1. Indeed, the proofs are similar to those of Theorem 2.1.

[1] The *total excess* of a graph is the number of edges plus the number of acyclic components, minus the number of vertices.

For instance for the subcritical case, the probability of interest is this time

$$\frac{n!}{\binom{\binom{n}{2}}{M}} [z^n] \frac{W_{-1}(z)^{n-M}}{(n-M)!} \exp\Big(W_0(z) - \ell(z)\Big) \frac{\ell(z)^k}{k!} .$$

Compare this equation with Equation (6.8) (where $W_{-1}(z)$, $W_0(z)$ are the generating functions of labelled trees and unicyclic connected graphs, see (6.6), (6.7)).

5 Conclusion

Although some of the results presented above have been suspected by many researchers in probability, combinatorics [6] or physics [1], this paper fixes rigorously results about the number of cycles of given length in random subcritical and critical graphs. If the results in [6, Corollary 7] (which extends the classical result in [5, Theorem 5b]) and in [14, Theorem 1] show how to capture various cycle parameters in the subcritical cases of $G(n, M = cn)$ (resp. $G(n, p = c/2n)$) with $\lim M/n < 1/2$ (resp. $\lim np < 1/2$) and introduced methods to deal with these objects, our paper shows that the critical case demands more scrutiny and involves technical details which can be dealt with methods from analytic combinatorics.

References

[1] E. Ben-Naim and P. L. Krapivsky. Kinetic theory of random graphs: From paths to cycles. *Phys. Rev. E*, 71(2):026129, 2005.

[2] B. Bollobás. *Random Graphs*. Cambridge Studies in Advanced Mathematics, 1985.

[3] A. Cayley. A theorem on trees. *Q. J. Pure Appl. Math.*, 23:376–378, 1889.

[4] H. Daudé and V. Ravelomanana. Random 2-XORSAT phase transition. *Algorithmica*, Special issue of LATIN 2008:1–18, 2009.

[5] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5:17–61, 1960.

[6] P. Flajolet, D. E. Knuth, and B. Pittel. The first cycles in an evolving graph. *Discrete Mathematics*, 75(1-3):167–215, 1989.

[7] P. Flajolet and B. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[8] N. Fountoulakis. On the structure of the core of sparse random graphs. *Preprint available at the webpage of the author*, 2000.

[9] A. Frieze and C. McDiarmid. Algorithmic theory of random graphs. *Random Struct. Algorithms*, 10:5–42, February 1997.

[10] F. Harary and E. Palmer. *Graphical Enumeration*. Academic Press, New-York and London, 1973.

[11] S. Janson, D. E. Knuth, T. Łuczak, and B. Pittel. The Birth of the Giant Component. *Random Structures and Algorithms*, 4(3):233–358, 1993.

[12] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs*. Wiley-Interscience, 2000.

[13] V. F. Kolchin. *Random Graphs*. Cambridge University Press, 1998.

[14] B. Pittel. A Random Graph With a Subcritical Number of Edges. *Transactions of the American Mathematical Society*, 309:51 – 75, 1988.

6 Appendix

6.1 Proof of Theorem 4.1 Let us recall briefly the main EGFs involved in our proofs. Let $W_{-1}(z)$ be the exponential generating function of labelled unrooted (unweighted) trees and $T(z)$ be the EGF of rooted labelled trees. The number of trees on n labelled vertices is given by Cayley's formula n^{n-2}. We know from [3] that:

(6.6)
$$W_{-1}(z) = T(z) - \frac{T(z)^2}{2},\ T(z) = ze^{T(z)} = \sum_{n=1}^{\infty} n^{n-1}\frac{z^n}{n!} .$$

The EGF $W_0(z)$ of unicyclic components (connected components with n vertices and n edges) is given by (see [11, Equation (3.5)]) :

$$W_0(z) = -\frac{1}{2}\log\left(1 - T(z)\right) - \frac{T(z)}{2} - \frac{T(z)^2}{4} . \tag{6.7}$$

In the range $0 < M < n/2$ with $n - 2M \gg n^{2/3}$, the probability that a random unweighted graph with n vertices and M edges ((n, M)-graph for short) contains only trees and unicycles is $1 - \mathcal{O}\left(\frac{n^2}{(n-2M)^3}\right)$ (see for instance [4, Theorem 3.2]). Thus, it suffices to consider graphs with trees and unicycles to prove the theorem.

For any $k \geqslant 0$, we denote by $p_k(n, M)$ the probability that a graph with M edges and n vertices has a set of acyclic components and exactly k unicyclic components. The number of (n, M)-graphs is $\binom{\binom{n}{2}}{M}$. Using the symbolic method[2], we obtain :

$$p_k(n, M) = \frac{n!}{\binom{\binom{n}{2}}{M}} [z^n] \frac{W_{-1}(z)^{n-M}}{(n-M)!} \frac{W_0(z)^k}{k!} . \tag{6.8}$$

Next, by using Cauchy integral's formula, we get :

$$[z^n]\left(T(z) - \tfrac{1}{2}T(z)^2\right)^{n-M} W_0(z)^k =$$
$$\frac{2^{M-n}}{2\pi i} \oint \left(2T(z) - T(z)^2\right)^{n-M} W_0(z)^k \frac{dz}{z^{n+1}} .$$

[2] we refer to Harary and Palmer [10] for graphical enumeration and to Flajolet and Sedgewick [7] for the symbolic method of generating functions.

After the substitution $u = T(z)$, it yields :

$$[z^n]\left(T(z) - \tfrac{1}{2}T(z)^2\right)^{n-M} W_0(z)^k = \frac{2^{M-n}}{2\pi i}\oint g(u)\lambda(u)^k e^{nh(u)}\frac{du}{u}, \tag{6.9}$$

where

$$g(u) = 1 - u, \tag{6.10}$$

$$\lambda(u) = -\tfrac{1}{2}\log(1-u) - \tfrac{u}{2} - \tfrac{u^2}{4}, \tag{6.11}$$

$$h(u) = u - \log u + \left(1 - \tfrac{M}{n}\right)\log\left(2u - u^2\right). \tag{6.12}$$

Note that the function h given by (6.12) is exactly the same as [4, Equation (30)], which satisfies $h'(2c) = h'(1) = 0$. In the range $M = cn$ with $0 < \limsup_n c < \frac{1}{2}$, we can apply saddle-point methods by choosing a circular path $\{2ce^{i\theta}, \theta \in [-\pi,\pi]\}$. As shown in [6], when splitting the integral in (6.9) into three parts, viz. $\int_{-\pi}^{-\theta_0} + \int_{-\theta_0}^{\theta_0} + \int_{\theta_0}^{\pi}$, we know that it suffices to integrate from $-\theta_0$ to θ_0, for a convenient value of θ_0, because the other integrals can be bounded by the magnitude of the central integrand. Then, by following the proof of [4, Theorem 3.2] and choosing $\theta_0 = n^{-2/5}$ (so $n\theta_0^2 \to \infty$ but $n\theta_0^3 \to 0$ as $n \to \infty$) we have that $\exp\left(nh(2ce^{i\theta})\right)$ is equal to

$$\exp\left(nh(2c) - \tfrac{nc(1-2c)}{2(1-c)}\theta^2\right)\left(1 + i\mathcal{O}(n\theta^3) + \mathcal{O}(n\theta^4)\right),$$

and for all $\theta \in [-\pi, -\theta_0] \cup [\theta_0, \pi]$, we have

$$|\exp\left(nh(2ce^{i\theta}) - nh(2c)\right)| = \mathcal{O}\left(\exp\left(-\mathcal{O}(n^{1/5})\right)\right). \tag{6.13}$$

Next, in the vicinity of θ_0, we have

$$g(2c\exp(i\theta)) = g(2c)\left(1 - i\mathcal{O}(\theta) + \mathcal{O}(\theta^2)\right), \tag{6.14}$$

and for fixed $k \geqslant 0$,

$$\lambda(2ce^{i\theta})^k = \lambda(2c)^k\left(1 + i\mathcal{O}(\theta) - \mathcal{O}(\theta^2)\right). \tag{6.15}$$

Then, using expansions (6.13), (6.14), (6.15) and then error bound (6.13) we have that the integral

$$\oint g(z)\lambda(z)^k e^{nh(z)}\frac{dz}{z}$$

is equal to

$$i\int_{-\theta_0}^{\theta_0} g(2ce^{i\theta})\lambda(2ce^{i\theta})^k e^{nh(2ce^{i\theta})}d\theta\left(1 + e^{-\mathcal{O}(n^{1/5})}\right).$$

This can be written as

$$ig(2c)\lambda(2c)^k e^{nh(2c)}\int_{-\theta_0}^{+\theta_0} e^{-n\sigma\frac{\theta^2}{2}}\left(1 + e^{-\mathcal{O}(n^{1/5})}\right) \cdot (1 + i\mathcal{O}(\theta) + \mathcal{O}(\theta^2) + i\mathcal{O}(n\theta^3) + \mathcal{O}(n\theta^4))d\theta,$$

where $\sigma = \frac{c(1-2c)}{1-c}$. If we set $x = \sqrt{n\sigma}\theta$, so $dx = \sqrt{n\sigma}d\theta$, the integral in the above equation leads to

$$\frac{1}{\sqrt{n\sigma}}\int_{-\sigma^{1/2}n^{1/10}}^{\sigma^{1/2}n^{1/10}} e^{-\frac{x^2}{2}} \cdot \left(1 + i\mathcal{O}\left(\tfrac{x}{\sqrt{n\sigma}}\right) + \mathcal{O}\left(\tfrac{x^2}{n\sigma}\right) + i\mathcal{O}\left(n\tfrac{x^3}{\sqrt{n\sigma^3}}\right) + \mathcal{O}\left(\tfrac{x^4}{n\sigma^2}\right)\right)dx. \tag{6.16}$$

By symmetry of the function, the integral of terms with odd exponents in x as ix and ix^3 vanish. Standard calculations show also that for M in the stated range, the multiplication of the factors of ix and ix^3 leads to a term of order of magnitude $\mathcal{O}\left(\frac{x^4}{n}\right)$. Therefore, (6.16) is equivalent to

$$\frac{1}{\sqrt{n\sigma}}\int_{-\sigma^{1/2}n^{1/10}}^{+\sigma^{1/2}n^{1/10}} \exp\left(-\tfrac{x^2}{2}\right)\left(1 + \mathcal{O}\left(\frac{x^2}{n}\right) + \mathcal{O}\left(\frac{x^4}{n}\right)\right)dx.$$

We deduce from the above that the integral

$$\int_{-\theta_0}^{\theta_0} g(2ce^{i\theta})\lambda(2ce^{i\theta})^k e^{nh(2ce^{i\theta})}d\theta$$

is equal to

$$\sqrt{\frac{2\pi}{\sigma n}}g(2c)\lambda(2c)^k e^{nh(2c)}\left(1 + \mathcal{O}\left(n^{-1}\right) + e^{-\mathcal{O}(n^{1/5})}\right).$$

That is,

$$[z^n]\left(T(z) - \frac{1}{2}T(z)^2\right)^{n-M} W_0(z)^k$$

is equal to

$$\frac{2^{M-n}}{\sqrt{2\pi\sigma n}}g(2c)\lambda(2c)^k e^{nh(2c)}\left(1 + \mathcal{O}\left(n^{-1}\right)\right). \tag{6.17}$$

Using Stirling's formula for the stated range of M, we have that $\frac{1}{\binom{\binom{n}{2}}{M}}\frac{n!}{(n-M)!}$ is equal to

$$\sqrt{\frac{2\pi nM}{n-M}}\frac{2^M n^n M^M}{n^{2M}(n-M)^{n-M}}e^{-2M+\frac{M}{n}+\frac{M^2}{n^2}}\left(1 + \mathcal{O}\left(n^{-1}\right)\right). \tag{6.18}$$

Multiplying (6.17) and (6.18), after cancellations, we get

$$p_k(n, M) = e^{-\lambda(2c)}\frac{\lambda(2c)^k}{k!}\left(1 + \mathcal{O}\left(n^{-1}\right)\right).$$

Now, suppose that $k \to \infty$ as $n \to \infty$. By choosing the same contour as above the circular path $\{z = 2ce^{i\theta}, \theta \in [-\pi, +\pi]\}$ we have that $p_k(n, M)$ is equal to

$$\frac{n!}{\binom{\binom{n}{2}}{M}}\frac{2^{n-M}}{(n-M)!}\frac{1}{2\pi i k!}\oint \lambda(z)^k e^{nh(z)}\frac{dz}{z} = \frac{n!}{\binom{\binom{n}{2}}{M}}\frac{2^{n-M}}{(n-M)!}\frac{1}{2\pi k!}\int_{-\pi}^{+\pi}\lambda(2ce^{i\theta})e^{nh(2ce^{i\theta})}d\theta,$$

where λ and h are given respectively by (6.11) and (6.12). Moreover, in the range $0 < \limsup_{n\to\infty} c < 1/2$, the real part of $h(2ce^{i\theta})$ verifies that

$$\frac{d}{d\theta}\Re\left(h(2ce^{i\theta})\right) = -c\sin\theta\left(2 - \frac{1-c}{1+c^2-2c\cos\theta}\right),$$

which satisfies that

$$\begin{aligned} &-c\sin\theta\left(2-\frac{1-c}{1+c^2-2c\cos\theta}\right)\\ &\leqslant -c\sin\theta\left(\frac{2(1-c)^2-(1-c)}{1+c^2-2c\cos\theta}\right)\\ &\leqslant -c\sin\theta\frac{(1-2c)(1-c)}{1+c^2+2c}\\ &\leqslant -\frac{4c}{9}\sin\theta(1-2c)(1-c). \end{aligned}$$

Next for $|\theta| < \pi$, $\cos\theta \leqslant 1 - \frac{2\theta^2}{\pi^2}$ and so

$$\begin{aligned} &\Re\left(h(2ce^{i\theta})\right)\\ &\leqslant h(2c) + \frac{4c}{9}\cos\theta(1-2c)(1-c)\\ &< h(2c) - \frac{8c}{9\pi^2}(1-2c)(1-c)\theta^2. \end{aligned}$$

By using (6.18), we get that $p_k(n,M)$ is less or equal than

$$\begin{aligned} &\frac{n!}{\binom{\binom{n}{2}}{M}}\frac{2^{M-n}}{(n-M)!}e^{nh(2c)}\frac{\lambda(2c)^k}{k!}\frac{1}{2\pi}\\ &\cdot\int_{-\pi}^{+\pi}\exp\left(-n\frac{8c}{9\pi^2}(1-2c)(1-c)\theta^2\right)d\theta\\ &=\sqrt{\frac{2\pi cn}{1-c}}\frac{\lambda(2c)^k}{k!}\frac{1}{2\pi}\left(1+\mathcal{O}\left(n^{-1}\right)\right)\\ &\cdot\int_{-\pi}^{+\pi}\exp\left(-n\frac{8c}{9\pi^2}(1-2c)(1-c)\theta^2\right)d\theta\\ &<\frac{3}{2}\frac{\pi}{(1-c)\sqrt{1-2c}}\frac{\lambda(2c)^k}{k!}. \end{aligned}$$

Using Stirling's formula for large k, we know that $\frac{1}{k!} < \frac{e^k}{k^k}$. Hence,

$$p_k(n,M) < \frac{3}{2}\frac{\pi}{(1-c)\sqrt{1-2c}}\left(\frac{e\lambda(2c)}{k}\right)^k.$$

Thus, there exist $C = \frac{3}{2}\frac{\pi}{(1-c)\sqrt{1-2c}} > 0$ and $\delta > 0$ (with $e\lambda(2c)/k < \delta < 1$) such that $p_k(n,M) \leqslant C\exp(\log(\delta)k)$. Finally, we set $\varepsilon = -\log(\delta) > 0$ to obtain desired results.

6.2 Proof of Theorem 4.2 In this regime we have that $M = \frac{n}{2}\left(1-\mu n^{-1/3}\right)$. As mentioned above, a random (n,M)-graph contains only trees and unicycles with probability $1-\mathcal{O}\left(\mu^{-3}\right)$ when $\mu\to\infty$ but $\mu n^{-1/3}\to 0$ as $n\to\infty$. Then, we can consider only graphs with acyclic and unicyclic components. So, we need to compute $p_k(n,M)$ given by (6.8) for this range.

As shown above, we use the same methods as in the proof of [4, Theorem 3.2] by choosing

$$\theta_0 = \sqrt{\frac{\tau}{n}}\omega(n), \tau = \frac{n(n-M)}{M(n-2M)},\ \omega(n) = \frac{(n-2M)^{1/4}}{n^{1/6}}.$$

The expansion of $h\left(\frac{2M}{n}e^{i\theta}\right)$ in the vicinity of θ_0 becomes

$$\begin{aligned} &h\left(\frac{2M}{n}\right) - \frac{M(n-2M)}{2n(n-M)}\theta^2 -\\ &i\frac{(n^2-5nM+2M^2)M}{6(n-M)^2}\theta^3 + \mathcal{O}(\theta^4), \end{aligned}\tag{6.19}$$

and for $\theta\in[-\theta_0,+\theta_0]$ and $k = \Theta\left(\lambda(\frac{2M}{n})\right) = \Theta(\log n)$, the expansion of

$$\frac{\lambda\left(\frac{2M}{n}e^{i\theta}\right)^k}{\lambda\left(\frac{2M}{n}\right)^k}$$

in the vicinity of θ_0 becomes

$$\begin{aligned} &1 + i\mathcal{O}\left(\frac{k}{\lambda\left(\frac{2M}{n}\right)}\frac{n}{(n-2M)}\theta\right) +\\ &\mathcal{O}\left(\frac{k^2}{\lambda\left(\frac{2M}{n}\right)^2}\frac{n^2}{(n-2M)^2}\theta^2\right)\\ &= 1 + i\mathcal{O}\left(\frac{n}{(n-2M)}\theta\right) + \mathcal{O}\left(\frac{n^2}{(n-2M)^2}\theta^2\right). \end{aligned}\tag{6.20}$$

The integrand can be bounded on $[-\pi,-\theta_0]\cup[\theta_0,\pi]$ because

$$\left|\exp\left(nh\left(\frac{2M}{n}e^{i\theta}\right) - nh\left(\frac{2M}{n}\right)\right)\right| = \mathcal{O}(e^{-\omega(n)^2/2}).\tag{6.21}$$

Then, combining (6.19),(6.20) and (6.21), we have $p_k(n,M)$ is equal to

$$\begin{aligned} &\frac{n!}{\binom{\binom{n}{2}}{M}(n-M)!}\frac{2^{M-n}}{2\pi}g\left(\frac{2M}{n}\right)\exp\left(nh\left(\frac{2M}{n}\right)\right)\\ &\cdot\frac{\lambda\left(\frac{2M}{n}\right)^k}{k!}\int_{-\theta_0}^{\theta_0}e^{-n\tau\frac{\theta^2}{2}}\\ &\cdot\left(1+i\mathcal{O}\left(\frac{n}{(n-2M)}\theta\right)+\mathcal{O}\left(\frac{n^2}{(n-2M)^2}\theta^2\right)\right)\\ &\cdot\left(1+in\frac{(n^2-5nM+2M^2)M}{6(n-M)^2}\theta^3+\mathcal{O}(n\theta^4)\right)d\theta\\ &\cdot\left(1+\mathcal{O}(e^{-\omega(n)^2/2})\right). \end{aligned}$$

Next, in substituting θ by $\sqrt{\frac{\tau}{n}}x$, terms in the above integral equal to

$$\sqrt{\frac{\tau}{n}}\int_{-\omega(n)}^{\omega(n)} \exp\left(-\frac{x^2}{2}\right)$$
$$\cdot\left(1+i\mathcal{O}\left(\frac{n}{(n-2M)^{3/2}}x\right)+\mathcal{O}\left(\frac{n^2}{(n-2M)^3}x^2\right)\right)$$
$$\cdot\left(1+i\mathcal{O}\left(\frac{n}{(n-2M)^{3/2}}x^3\right)+\mathcal{O}\left(\frac{n}{(n-2M)^3}x^4\right)\right)dx$$
$$=\sqrt{\frac{\tau}{n}}\int_{\omega(n)}^{\omega(n)} \exp\left(-\frac{x^2}{2}\right)\left(1+\mathcal{O}\left(\frac{n^2}{(n-2M)^3}x^4\right)\right)dx$$
$$=\sqrt{\frac{2\pi\tau}{n}}\left(1+\mathcal{O}\left(\frac{n^2}{(n-2M)^3}\right)\right).$$

After a bit of algebra we get

$$p_k(n,M)=e^{-\lambda\left(\frac{2M}{n}\right)}\frac{\lambda\left(\frac{2M}{n}\right)^k}{k!}\left(1+\mathcal{O}\left(\mu^{-3}\right)\right).$$

Then, by setting $k=\lambda\left(\frac{2M}{n}\right)+\rho\sqrt{\lambda\left(\frac{2M}{n}\right)}$ with $|\rho|\ll(\log(n))^{1/6}$, observing that $\lambda\left(\frac{2M}{n}\right)\sim\frac{1}{6}\log n-\frac{1}{2}\log\mu\to\infty$ as $n\to\infty$, and using Theorem 6.1, we obtain that $\Pr\left[X_{n,M}=k\right]$ is equal to:

$$\frac{1}{\sqrt{2\pi\lambda\left(\frac{2M}{n}\right)}}e^{-\rho^2/2}\left(1+\mathcal{O}\left(\mu^{-3}\right)+\mathcal{O}\left(\frac{1+|\rho|^3}{\sqrt{\log n}}\right)\right)$$
$$=\frac{1}{\sqrt{2\pi\lambda\left(\frac{2M}{n}\right)}}e^{-\rho^2/2}(1+o(1)).$$

In other words, the distribution of $X_{n,M}$ converges to the normal law with parameters $\left(\lambda\left(\frac{2M}{n}\right),\lambda\left(\frac{2M}{n}\right)\right)$. That is for any real y as $n\to\infty$, we have

$$\Pr\left[\frac{X_{n,M}-\left(\frac{1}{6}\log n-\frac{1}{2}\log\mu\right)}{\sqrt{\left(\frac{1}{6}\log n-\frac{1}{2}\log\mu\right)}}\leqslant y\right]\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{u^2}{2}}du.$$

6.3 Proof of Theorem 4.3 The proof below is based on techniques introduced in [11].

The probability $p_k(n,M,r)$ that a random (n,M)-graph have k unicyclic components and a total excess equal to r is exactly

(6.22)
$$\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{\left(T(z)-\frac{1}{2}T(z)^2\right)^{n-M+r}}{(n-M+r)!}E_r(z)\frac{W_0(z)^k}{k!},$$

where $E_r(z)$ the EGF of complex components of total excess r given by [11, Equation (6.8)]. As shown in [11] the EGF $E_r(z)$ can be approximated by $\frac{e_r}{(1-T(z))^{3r}}$ when $r=o(n^{1/3})$, where $e_r=\frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!}$ as in [11, Equation (6.8)] and the error term is of order $\mathcal{O}\left(\frac{r^{3/2}}{n^{1/2}}\right)$.

Next, using Cauchy's integral formula, we need to compute the expression below to evaluate (6.22) :

$$\frac{St(n,M,r)}{2\pi i}\oint(1-z)^{1-3r}e^{nh_1(z)}\frac{\lambda(z)^k}{k!}\frac{dz}{z}, \tag{6.23}$$

where λ is given by (6.11), $St(n,M,r)$ is equal to

$$\frac{n!}{\binom{\binom{n}{2}}{M}}\frac{2^{-n+M-r}e^n e_r}{(n-M+r)!}, \tag{6.24}$$

and $h_1(z)$ is equal to

$$z-1-\log z+\left(1-\tfrac{M}{n}\right)\log(2z-z^2). \tag{6.25}$$

Note that h_1 defined in (6.25) is exactly the same as in [11, Equation (10.12)] satisfying $h_1(1)=h'(1)=0$ and if $m=\frac{n}{2}$, $h''(1)=0$. Then, we can follow the proof of the one of [11, Lemma 3] to compute integral by choosing the path of integration $z=e^{-(\alpha+it)n^{-1/3}}$ where t runs from $-\pi n^{1/3}$ to $\pi n^{1/3}$ and α is the positive solution of $\mu=\frac{1}{\alpha}-\alpha$. The main difference is that in our case we have the factor $\lambda(z)^k$. In their case, it suffices to integrate from $t=-n^{1/12}$ to $t=n^{1/12}$, so that the error term becomes superpolynomially small. But in our case, we need $|t|\leqslant\log n$ to bound the error term due to the factor $\lambda(z)^k$.

We set $s=\alpha+it$ and $\nu=n^{-1/3}$, so that $z=e^{-s\nu}$. Furthermore, we suppose that $k=\frac{1}{6}\log n+\rho\sqrt{\frac{1}{6}\log n}$ with $\rho=o\left((\log n)^{1/6}\right)$. Then for $|t|\leqslant\log n$, in using the expansion of e^x and $\log(1+x)$ in vicinity of $x=0$, we obtain after some algebra:

$$\lambda(e^{-s\nu})^k=\left(-\tfrac{1}{2}\log(1-e^{-s\nu})-\tfrac{e^{-s\nu}}{2}-\tfrac{e^{-2s\nu}}{4}\right)^k$$
$$=e^{-3/4}\left(\tfrac{1}{6}\log n\right)^k s^{-1/2}\left(1+R\right)$$

where

$$|R|=\mathcal{O}\left((1+|\rho|)\frac{\log\log n}{(\log n)^{1/2}}\right).$$

Then, by following the rest of the proof of [11, Equation (10.1) of Lemma 3], and after using Stirling's formula for $St(n,M,r)$ defined in (6.24), we get the following estimate for $p_k(n,M,r)$:

$$n^{-1/6}\frac{\left(\frac{1}{6}\log n\right)^k}{k!}\sqrt{2\pi}e_r A(3r+1/2,\mu)$$
$$\cdot\left(1+\mathcal{O}\left(\frac{(1+|\rho|)\log\log n}{(\log n)^{1/2}}\right)+\mathcal{O}\left(\frac{r^{3/2}}{n^{1/2}}\right)\right).$$

Now, we suppose $r\to\infty$ as $n\to\infty$. The proof of (4.5) is the similar to [4, Equation (53)] (see also [11, Lemma 5]). We know that $E_r(z) \preceq \frac{e_r}{(1-T(z))^{3r}}$ [3](see for instance [11, Lemma 4]). First, we have from (6.22) that $p_k(n,M,r)$ is less or equal than:

$$\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{\left(T(z)-\frac{1}{2}T(z)^2\right)^{n-M+r}}{(n-M+r)!}\frac{W_0(z)^k}{k!}\frac{e_r}{(1-T(z))^{3r}}.$$

Then, we obtain that $p_k(n,M,r)$ is less or equal than

$$(6.26)\quad \frac{St(n,M,r)}{2\pi i}\oint\frac{z^r(2-z)^r}{(1-z)^{3r}}e^{nh_1(z)}\frac{(\lambda(z))^k}{k!}(1-z)dz$$

with h_1 is in (6.25) and λ in (6.11). Next, we choose a countour of integration a circle $\{\delta e^{i\theta}\}$ with $0<\delta<1$. On this circle, $|(2-z)/(1-z)|$ and $1/|1-z|$ attain their maxima at $z=\delta$. When $r\geqslant 1$, the countour including the factor $1/(2\pi i)$ is less than

$$\begin{aligned}&\frac{\lambda(\delta)^k}{k!}\frac{\delta}{2\pi}\left(\frac{\delta(2-\delta)}{(1-\delta)^3}\right)^r e^{nh_1(\delta)}(1-\delta)\\ &\cdot\int_{-\pi}^{\pi}\exp\left(\frac{-4n\delta(1-\delta)}{9\pi^2}\theta^2\right)d\theta\\ &<\frac{3}{4}\sqrt{\frac{\pi}{n}}\delta^{r+1/2}\left(2-\delta\right)^r\left(1-\delta\right)^{\frac{1}{2}-3r}e^{nh(\delta)}\frac{\lambda(\delta)^k}{k!},\end{aligned}$$

Note that $r\leqslant m=\frac{n}{2}(1+\mu n^{-1/3})$. Let $\delta=1-\frac{r^{1/3}}{n^{1/3}}$. We then have :

$$\frac{\delta^r(2-\delta)^r}{(1-\delta)^{3r}}=\frac{n^r}{r^r}\left(1-\frac{r^{2/3}}{n^{2/3}}\right)<\frac{n^r}{r^r},$$

$$(1-\delta)^{1/2}=\frac{r^{1/6}}{n^{1/6}}=r^{1/6}e^{-\frac{1}{6}\log n}.$$

We also have :

$$(6.27)\quad \lambda(\delta)\leqslant\frac{1}{6}\log n\quad\text{and}\quad nh_1(\delta)<\frac{13}{12}r+\frac{11}{6}\mu r^{2/3},$$

Then using Stirling's formula, we find :

$$\frac{n!}{\binom{\binom{n}{2}}{m}(n-m+r)!}e^n2^{-n+m-r}<\frac{n^{1/2}}{n^r}e^{-\mu^3/6+3/4}2^{-r},$$

and for $r\to\infty$, we have :

$$(6.28)\qquad e_r=\frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!}\leqslant\frac{1}{r^{1/2}}\left(\frac{3r}{2e}\right)^r.$$

Combining (6.27) and (6.28) with (6.26), we deduce that $p_k(n,M,r)$ is less than:

$$\begin{aligned}&e^{-\left(\frac{1}{6}\log n\right)}\frac{\left(\frac{1}{6}\log n\right)^k}{k!}\\ &\cdot\frac{c_0}{r^{1/3}}\exp\left(-\frac{\mu^3}{6}+\mu r^{2/3}+\left(\frac{13}{12}+\log\frac{3}{4}\right)r\right),\end{aligned}$$

for some constant $c_0>0$. Note that $\frac{13}{12}+\log\frac{3}{4e}\approx -0.2043$ We then have (4.5) as $r\to\infty$.

6.4 Proof of Theorem 4.4 Recall that the probability $p_k(n,M,r)$ that a graph with $M=\frac{n}{2}\left(1+\mu n^{-1/3}\right)$ edges and n vertices has total excess r with exactly k unicyclic components is given by (6.22). So, we have that $p_k(n,M,r)$ is equal to:

$$\frac{n!}{\binom{\binom{n}{2}}{M}}\frac{2^{-n+M-r}e_r}{(n-M+r)!}\frac{1}{2\pi i}\oint g(z)e^{H(z)}\frac{\lambda(z)^k}{k!}\frac{dz}{z},$$

where λ is defined in (6.10) and $H(z)$ is equal to

$$nz-n\log z+(n-M+r)\log(2z-z^2)-3r\log(1-z).$$

Let r and $\phi=o(u^{1/2})$ such that

$$(6.29)\quad r=\frac{2}{3}\mu^3+\phi\sqrt{\mu^3}=\frac{2}{3}x^3n+\phi\sqrt{x^3n}.$$

As n and $r\to\infty$, using Stirling formula gives :

$$e_r=\frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!}=\left(\frac{3}{2}\right)^r(r-1)!(1+\mathcal{O}(r^{-1})),$$

and the term

$$\log\frac{n!2^{-n+M-r}e_r}{\binom{\binom{n}{2}}{M}(n-M+r)!}$$

is equal to

(6.30)

$$\begin{aligned}&-n+\frac{1}{2}n\log n+\log n-\frac{1}{2}n\log\left(n-nx+2\,r\right)\\ &-r\log\left(n-nx+2\,r\right)-\frac{1}{2}nx\log n+\frac{1}{2}nx\log\left(1+x\right)\\ &+\frac{1}{2}n\log\left(1+x\right)-nx+\frac{1}{2}\,\log\left(1+x\right)-r\log 2+r\log r\\ &+r\log 3-\frac{1}{2}\,\log r+\frac{1}{2}\,nx\log\left(n-nx+2\,r\right)\\ &-\frac{1}{2}\,\log\left(n-nx+2\,r\right)\\ &+\log(1+\mathcal{O}(r^{-1}))+\frac{3}{4}+\mathcal{O}\left(n^{-1}\right).\end{aligned}$$

Next, using the expansion $\log(1-t)=-t-\frac{1}{2}t^2-\frac{1}{3}t^3+\mathcal{O}(t^4)$ as $t\to 0$, ans taking into account (6.29), we get that 6.30 can be written as:

(6.31)

$$\begin{aligned}&-n+\left(-\frac{5}{6}+2\,\ln\left(x\right)\right)nx^3+\frac{2}{3}nx^4+\frac{17}{60}nx^5\\ &+\mathcal{O}\left(nx^6\right)+3\,\phi\,\ln\left(x\right)n^{1/2}x^{3/2}+\mathcal{O}((1+|\phi|)\,n^{1/2}x^{5/2})\\ &-\frac{3}{2}\,\ln x+\frac{1}{2}\,\ln 3-\frac{1}{2}\,\ln 2+\frac{3}{4}\,\phi^2+\mathcal{O}\left(\frac{1+|\phi|^3}{n^{1/2}x^{3/2}}\right).\end{aligned}$$

[3]For any power series $A(z)=\sum a_nz^n$ and $B(z)=\sum b_nz^n$, we write $A(z)\preceq B(z)$ iff there exists $n_0\in\mathbb{N}$ such that for $n\geqslant n_0$, $a_n\leqslant b_n$.

Then with the same arguments as in [11, Lemma 7], the asymptotic value of the integral depends only on the behavior of the integrand near $z = 1$ since we are sufficiently far from the critical point $\mu = 0$. Note that when $r = \frac{2}{3}\mu^3$, we have a saddle point $z_0 < 1$, $z_0 \to 1$ as $x \to 0$, with :

$$
\begin{aligned}
z_0 = 1 - x + \frac{2}{15}x^3 - \frac{8}{375}x^5 + \frac{256}{84375}x^7 - \frac{416}{1265625}x^9 \\
+ \frac{2176}{158203125}x^{11} + \frac{167552}{35595703125}x^{13} \\
- \frac{4173824}{2669677734375}x^{15} + \mathcal{O}\left(x^{17}\right).
\end{aligned}
$$

Therefore, we choose as path of integration : $\left\{z = 1 - \left(x - \frac{2}{15}x^3\right) - \frac{3}{5}\frac{\phi}{\sqrt{xn}} + i\theta\right\} = \{z = a + i\theta\}$, with $a = 1 - \left(x - \frac{2}{15}x^3\right) - \frac{3}{5}\frac{\phi}{\sqrt{xn}}$ and ϕ given by (6.29). In the vicinity of $z = 1$, we have that $H(a + i\theta) - H(a)$ is equal to

$$
\begin{aligned}
& i\left(\mathcal{O}\left(\sqrt{n}x^{11/2}\right) + \mathcal{O}(x)\phi + \mathcal{O}\left(\frac{1+|\phi|^2}{\sqrt{nx^{3/2}}}\right)\right)\frac{\sqrt{nx}\theta}{1!} \\
& - \left(5 + \mathcal{O}(x) + \mathcal{O}\left(\frac{1+|\phi|}{\sqrt{nx^3}}\right)\right)\frac{nx\theta^2}{2!} + \\
& i\mathcal{O}\left(\frac{1}{\sqrt{nx^3}} + \frac{1+|\phi|}{nx^3}\right)\frac{(\sqrt{nx}\theta)^3}{3!} + \cdots .
\end{aligned}
$$

Let

$$
\theta_0 = \frac{\log n}{\sqrt{xn}} \quad \text{et} \quad \theta = \frac{t}{\sqrt{xn}} \quad \text{with} \quad t \in [-\log n, \log n].
$$

Since $\log n < \mu = x^3 n$ we have $nx\theta_0^2 \to \infty$ and $n\theta_0^3 \to 0$. Then, we obtain :

$$
H\left(a + i\frac{t}{\sqrt{xn}}\right) = H(a) - \frac{5}{2}t^2 + f(t),
$$

where $f(t)$ is equal to

$$
\begin{aligned}
& it\mathcal{O}\left(\frac{\mu^{11/2}}{n^{4/3}} + \frac{(1+|\phi|)\mu}{n^{1/3}} + \frac{1+\phi^2}{\sqrt{\mu^3}}\right) \\
& + t^2\mathcal{O}\left(\frac{1+|\phi|}{\sqrt{\mu^3}} + \frac{\mu}{n^{1/3}}\right),
\end{aligned}
$$

and $H(a)$ is equal to

$$
\begin{aligned}
& n + \left(\frac{5}{6} - 2\ln(x)\right)nx^3 + \mathcal{O}(nx^5) \\
& + \left(-3\sqrt{n}\ln(x)\,x^{3/2} + \mathcal{O}(\sqrt{n}x^{7/2})\right)\phi \\
& + \left(-\frac{9}{10} + \mathcal{O}(x)\right)\phi^2 + \mathcal{O}\left(\frac{1+|\phi|^3}{\sqrt{nx^3}}\right).
\end{aligned}
$$

Next, we have that $\lambda(z)$ is equal to:

$$
-\frac{1}{2}\log(1-z) - \frac{z}{2} - \frac{z^2}{4},
$$

which can be written as:

$$
\begin{aligned}
& -\frac{1}{2}\log\left(1 - \left(a + \frac{it}{\sqrt{xn}}\right)\right) - \frac{3}{4} + \mathcal{O}\left(\frac{\phi}{\sqrt{xn}}\right) \\
& + i\mathcal{O}\left(\frac{t}{\sqrt{xn}}\right) = -\frac{1}{2}\log x - \frac{3}{4} + \mathcal{O}(x) \\
& + \mathcal{O}\left(\frac{\phi}{\sqrt{x^3 n}}\right) + i\mathcal{O}\left(\frac{t}{\sqrt{x^3 n}}\right).
\end{aligned}
$$

Thus, for

$$
k = \alpha_n + \kappa\sqrt{\alpha_n}
$$

with $\alpha_n = -\frac{1}{2}\log x$ equal to $\frac{1}{6}\log n - \frac{1}{2}\log\mu$, we get that $\lambda(z)$ is equal to:

$$
\begin{aligned}
& \alpha_n\left(1 - \frac{3}{4\alpha_n} + \mathcal{O}\left(\frac{x}{\alpha_n}\right)\right) \\
& + \alpha_n\left(\mathcal{O}\left(\frac{\phi}{\alpha_n\sqrt{x^3 n}}\right) + i\mathcal{O}\left(\frac{t}{\alpha_n\sqrt{x^3 n}}\right)\right)
\end{aligned}
$$

hence, $\lambda(z)^k$ which can be written as

$$
\alpha_n^k \exp\left(-\frac{3}{4} + \mathcal{O}\left(\frac{\kappa}{\sqrt{\alpha_n}}\right) + \mathcal{O}\left(\frac{\phi}{\sqrt{\mu^3}}\right) + i\mathcal{O}\left(\frac{t}{\sqrt{\mu^3}}\right)\right),
$$

and

$$
\begin{aligned}
g(z)\frac{dz}{z} &= \left(1 - a - i\frac{t}{\sqrt{nx}}\right)\frac{in^{-1/2}x^{-1/2}dt}{1 - \left(1 - a - i\frac{t}{\sqrt{nx}}\right)} \\
&= x^{1/2}in^{-1/2}dt\sum_{k\geqslant 0}x^k\left(x^{-1}\left(1 - a - i\frac{t}{\sqrt{nx}}\right)\right)^{k+1} \\
&= x^{1/2}in^{-1/2}dt \\
&\cdot\sum_{k\geqslant 0}x^k\left(1 - \frac{2}{15}x^2 + \frac{3}{5}\frac{\phi}{\sqrt{x^3 n}} - i\frac{t}{\sqrt{x^3 n}}\right)^{k+1} \\
&= x^{1/2}in^{-1/2}dt\left(1 + S(t)\right),
\end{aligned}
$$

where $S(t)$ is an absolutely convergent series 0 for $t \in [-\log n, \log n]$. Since $|f(t)|$ converges to 0 for $t \in [-\log n, \log n]$ and $\phi = o(\sqrt{\mu})$ as $n \to \infty$, the error term can be bounded. Then, we have that

$$
\int_{-\log n}^{\log n} e^{-\frac{5}{2}t^2 + f(t)}(1 + S(t))dt
$$

is equal to

$$\int_{-\log n}^{\log n} e^{-\frac{5}{2}t^2}\left(1+\mathcal{O}(f(t))\right)\left(1+S(t)\right)dt$$
$$=\sqrt{\frac{2\pi}{5}}\left(1+\mathcal{O}\left(\sqrt{n}x^{11/2}+\frac{1+\phi^2}{\sqrt{x^3 n}}\right)\right)$$
$$=\sqrt{\frac{2\pi}{5}}\left(1+\mathcal{O}\left(\frac{\mu^{11/2}}{n^{4/3}}+\frac{1+\phi^2}{\sqrt{\mu^3}}\right)\right),$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} g(z)e^{H(z)}\lambda(z)^k\frac{dz}{z}$$

is equal to

$$\alpha_n^k e^{-3/4}e^{H(a)}x^{1/2}n^{-1/2}\sqrt{\frac{1}{10\pi}} \cdot\left(1+\mathcal{O}\left(\frac{\mu^{11/2}}{n^{4/3}}+\frac{1+|\kappa|}{\sqrt{\log n}}+\frac{1+|\phi|^2}{\sqrt{\mu^3}}\right)\right). \tag{6.32}$$

Combining (6.31) and (6.32), we have that $p_k(n,M,r)$ is equal to:

$$\frac{\alpha_n^k}{k!}\frac{1}{\mu n^{1/6}}\sqrt{\frac{3}{20\pi}}\exp\left(-\frac{3}{20}\phi^2\right) \cdot\left(1+\mathcal{O}\left(\frac{\mu^4}{n^{1/3}}+\frac{(1+|\phi|)\mu^{5/2}}{n^{1/3}}+\frac{1+|\phi|^3}{\sqrt{\mu^3}}+\frac{1+|\kappa|}{\sqrt{\log n}}\right)\right).$$

Then, converting the sum into an integral with $dr=\sqrt{\mu^3}d\phi$, for $\mu=o(n^{1/12})$, we have :

$$\sum_{r=0}^{+\infty}p_k(n,M,r)\sim\int_0^{+\infty}p_k(n,M,r)dr$$
$$=\int_{-\infty}^{+\infty}\frac{\alpha_n^k}{k!}\frac{1}{\mu n^{1/6}}\sqrt{\frac{3}{20\pi}}\exp\left(-\frac{3}{20}\phi^2\right)\sqrt{\mu^3}d\phi \cdot\left(1+\mathcal{O}\left(\frac{\mu^4}{n^{1/3}}+\frac{1}{\sqrt{\mu^3}}+\frac{1+|\kappa|}{\sqrt{\log n}}\right)\right)$$
$$=\frac{\mu^{1/2}}{n^{1/6}}\frac{\alpha_n^k}{k!}\left(1+\mathcal{O}\left(\frac{\mu^4}{n^{1/3}}+\frac{1}{\sqrt{\mu^3}}+\frac{1+|\kappa|}{\sqrt{\log n}}\right)\right)$$
$$=e^{-\alpha_n}\frac{\alpha_n^k}{k!}\left(1+o(1)\right).$$

Finally, for $k=\alpha_n+\kappa\sqrt{\alpha_n}$ where $\alpha_n=-\frac{1}{2}\log(\mu n^{-1/3})=\frac{1}{6}\log n-\frac{1}{2}\log\mu$ and $|\kappa|\ll\alpha_n^{1/6}$ and $\log n\leqslant\mu\leqslant n^{1/12}$, we have from 6.1 that $p_k(n,m)$ is equal to:

$$\frac{1}{\sqrt{2\pi\alpha_n}}e^{-\kappa^2/2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\mu^3}}+\frac{1+|\kappa|^3}{\sqrt{\log n}}+\frac{\mu^4}{n^{1/3}}\right)\right),$$

Note that the conditions $1\ll\mu\ll n^{1/12}$ and $\kappa=o((\log n)^{1/6})$ are needed to bound the error term.

To conclude, we include a full result by Kolchin needed in the proof of Corollary 4.1 :

Theorem 6.1. ([13, Theorem 1.1.15]) *If $(1+\rho_n)^6/\lambda_n\to 0$ as $n\to\infty$, where $k=\lambda_n+\rho_n\sqrt{\lambda_n}$, we then have :*

$$e^{-\lambda_n}\frac{\lambda_n^k}{k!}=\frac{1}{\sqrt{2\pi\lambda_n}}e^{-\rho_n^2/2}\left(1+\frac{\rho_n^3-\rho_n}{6\sqrt{\lambda_n}}+\mathcal{O}\left(\frac{1+\rho_n^6}{\lambda_n}\right)\right).$$