On the cycle structure of the product of random maximal cycles*

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Abstract

The subject of this paper is the cycle structure of the random permutation σ of $[N]$, which is the product of k independent random cycles of maximal length N. We use the character-based Fourier transform to study the number of cycles of σ and also the distribution of the elements of the subset $[\ell]$ among the cycles of σ.

1 Introduction

Enumeration of permutations of a set $[N] = \{1, 2, \ldots, N\}$ according to the numbers of cycles of various lengths has a long and glorious history. The plentiful results are not infrequently cast in the probabilistic light, if the assumption is made that a permutation is chosen *uniformly at random* among all $N!$ permutations. The techniques vary widely, from bijective methods to multivariate generating functions to functional limit theorems, allowing to find solutions, exact or asymptotic, of rather delicate, enumerative-probabilistic, problems. More recently there has been a growing interest in the probabilities regarding distribution of the elements of a subset $S \subseteq [N]$ among the cycles of the random permutation. For instance, we can determine the probability that each of the entries in S will be in a different cycle, or that all entries of S will be in the same cycle, or that each cycle of p will contain at least one entry of S. See Lovász [22] for results of this kind.

The classic, and more recent, problems become much more difficult if instead of the uniformly random permutation, we consider a random permutation which is a *product* of random *maximal* cycles. That is, our sample space is now that of all ordered k-tuples $(p_1, p_2, \cdots, p_k)$, where all p_i are maximal cycles of length N. One can investigate the random permutation $\sigma := p_1 \cdots p_k$ under the assumption that $p_1, \ldots, p_k$ are maximal cycles, chosen uniformly at random, and independently of each other, from all $(N-1)!$ such cycles.

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1.1 Motivation and recent results

Among the sources of our inspiration are Zagier's formula for the distribution of the number of cycles in σ for $k = 2$, and the more recent results by Stanley [27] and Bernardi et al. [2], again for $k = 2$. For instance, in [2] a formula is proved for the probability that σ, the product of two maximal cycles, separates the *given disjoint* subsets of $[N]$, i.e. no two of those subsets are represented in the same cycle of σ. In particular, the probability that σ separates the entries $1, \ldots, \ell$ is equal to $1/\ell!$ if $N - \ell$ is odd. In other words, in this aspect, the product of two independent maximal cycles behaves as the uniformly random permutation!

Beside their intrinsic interest, solutions of the mentioned problems may lead to surprising applications. In [4], Bóna and Flynn used a result of Stanley [27] concerning the special case $S = \{1, 2\}$ and $k = 2$ to prove an exact formula for the average number of block interchanges needed to sort a permutation, a problem motivated by genome sorting. We will discuss sorting algorithms in more detail in Section 7. Equally interesting are the methods that can be used, as they come from a wide array of areas in mathematics, such as character theory, multivariate Gaussian integration, bijective combinatorics and the summation techniques for hypergeometric sums.

1.2 Overview: methods and results

In 1986 Harer and Zagier [16] discovered a remarkable formula for the bivariate generating function of the number of cycles in the product of a maximal cycle and the random, fixed-point free, involution of $[2n]$, thus solving a difficult problem of enumerating the chord diagrams by the genus of an associated surface. The proof was based on evaluation of the multidimensional Gaussian integrals. Soon after Jackson [17] and later Zagier [32] found alternative proofs that used characters of the symmetric group S_{2n}. Recently the second author [23] found a different, character-based proof. Its core is computing and marginally inverting the Fourier transform of the underlying probability measure on S_{2n}. In the present paper, we use the techniques in [23], see also an earlier paper by Chmutov and Pittel [7], to investigate the product of k maximal cycles in S_N. To make the discussion reasonably self-contained we will introduce the necessary definitions and facts from [23] in Section 2.

We begin Section 3 with Lemma 3.1 that states an explicit formula for the probability distribution of the number of cycles in σ, the product of k random, independent, maximal cycles in S_N. Not surprisingly, the distribution is expressed through the Stirling numbers of first kind. In particular, this formula yields the known results, Stanley [26], for the probabilities that σ is the identity permutation, or that σ is a maximal cycle. Our analysis also delivers a well-known formula found by Zagier for the case $k = 2$. See Corollary 3.4 for this special case; see the Appendix by Zagier in Lando and Zvonkin [18] for the original result of Zagier. In Corollary 3.5, we also obtain a bivariate generating function for the distribution of the number of cycles for the product of three cycles. We conclude this section with a relatively compact, integral formula for the probability that the product of two cycles belongs to a given conjugacy class.

Then, in Section 4, we turn to the following general question. Let $p_A(N, \ell; k)$ be the probability that the number of elements of $[\ell] = \{1, 2, \cdots, \ell\}$ in each cycle of σ comes from the set $A \subseteq \mathbb{Z}_{\geq 0}$. What can we say about $p_A(N, \ell; k)$?

To this end, for a general A, we first enumerate the admissible permutations by the cycle counts and then evaluate the sum of character values over all admissible permutations for irreducible representations labeled by one-hook Young diagrams. Then we consider the special case when $A = \mathbb{Z}_{>0}$, i.e. when each cycle of σ contains at least one element of $[\ell]$. Using the inverse Fourier transform, we find an alternating sum expression for this probability with $N-\ell+1$ binomial-type summands. This result is proved in Theorem 4.2. For $k = 2$, this sum reduces to two notably simpler expressions, that can be efficiently computed for moderate ℓ and moderate $N - \ell$ respectively.

Next we investigate the case of $A = \{0, \ell\}$, that is, when all elements of ℓ are in the same cycle of σ. This computation is longer than its counterpart in the previous case, and it leads to a general formula for $p_A(N, \ell; k)$, given in Theorem 4.3, that is analogous to that for $A = \mathbb{Z}_{>0}$. Again, if $k = 2$, then the formula shrinks to a pair of computationally efficient sums for moderate ℓ and moderate $N - \ell$ respectively. For $\ell = 2$ and $\ell = 3$, we recover the results obtained by Stanley [27].

Having experimented with Maple, we feel confident that the residual sums for $k = 2$ in either of the two cases do not have a more compact presentation.

After this, in Section 5, we turn to our most general problem. We consider disjoint subsets $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_\sqcup$ of $[N]$ so that $|\mathcal{S}_j| = \ell_j$; define $\ell = \sum_j \ell_j$. Let $p(N, \vec{\ell}; k)$ denote the probability that no cycle of σ contains elements from more than one $\mathcal{S}_j$, a property to which we refer by saying that σ *separates* the sets $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_t$. Bernardi et al. [2] found a striking formula for $p(N, \vec{\ell}; 2)$ that contained an alternating sum of $\ell - t + 1$ terms. Remarkably, the factor $\prod_j \ell_j!$ aside, the rest of the formula depends on ℓ and t only. In Lemma 5.1, we show that the separation probability continues to have this latter property for all $k \geq 2$, and find an alternating sum formula with $N-\ell+t+1$ terms for this probability, which is computationally efficient if t and $N - \ell$ are both bounded as N grows. Then, for $k = 2$, we are able to simplify this formula to one that is close in appearance, but is significantly different from the formula in [2]. This formula is given in Theorem 5.2, and it still contains a sum of $\ell - t + 1$ summands, but the signs are no longer alternating.

Finally, in Section 6, we consider the following question. Let us say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if no two elements of $[\ell]$ are neighbors, *and* each element of $[\ell]$ has a neighbor from $[N] \setminus [\ell]$. Then, for a general $k \geq 2$, we find a *two-term* formula for the probability that σ blocks the elements of $[\ell]$. This formula is proved in Theorem 6.1.

While on occasion our proofs deliver the already known results, we hope that the employed techniques can be used for a broader variety of problems on cyclic structure of the products of random permutations.

2 Preliminaries

A key observation is that the set of all maximal cycles forms a *conjugacy class* in the symmetric group S_N, a class with particularly simple character values. We mention that permutations generated by a given conjugated class were studied for instance by Diaconis [8, 9], Lulov and Pak [19], and, from a more algebraic point of view, by Liebeck, Nikolov, and Shalev [21].

Let us start with the Fourier inversion formula for a general probability measure P on S_N:

$$(2.1) \quad P(s) = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda \operatorname{tr}\big(\rho^\lambda(s^{-1}) \hat{P}(\rho^\lambda)\big); \quad s \in S_N;$$

see Diaconis and Shahshahani [10] and Diaconis [11]. Here λ is a generic partition of the integer N, ρ^λ is the irreducible representation of S_N associated with λ, $f^\lambda = \dim(\rho^\lambda)$, and $\hat{P}(\rho^\lambda)$ is the $f^\lambda \times f^\lambda$ matrix-valued Fourier transform of $P(\cdot)$ evaluated at ρ^λ, $\hat{P}(\rho^\lambda) = \sum_{s \in S_N} \rho^\lambda(s) P(s)$. Let us evaluate the right-hand side of (2.1) for $P = P_\sigma$, the probability measure on S_N induced by $\sigma = \prod_{j=1}^k \sigma_j$, where σ_j is uniform on a conjugacy class C_j. As the σ_j are independent, we have that $P_\sigma(s) = \sum_{s_1, \dots, s_k} \prod_j P_{\sigma_j}(s_j)$, $(s_1 \cdots s_k = s)$, that is, P_σ is the convolution of $P_{\sigma_1}, \dots, P_{\sigma_k}$.

So, by multiplicativity of the Fourier transform for convolutions, $\hat{P}_\sigma(\rho^\lambda) = \prod_j \hat{P}_{\sigma_j}(\rho^\lambda)$. Since each P_{σ_j} is supported by the single conjugacy class C_j, we have $\hat{P}_{\sigma_j}(\rho^\lambda) = \frac{\chi^\lambda(C_j)}{f^\lambda} I_{f^\lambda}$, I_{f^λ} being the $f^\lambda \times f^\lambda$ identity matrix, see [11]. So

$$\hat{P}_\sigma(\rho^\lambda) = \prod_{j=1}^{k} \hat{P}_{\sigma_j}(\rho^\lambda) = (f^\lambda)^{-k} \prod_{j=1}^{k} \chi^\lambda(C_j)\, I_{f^\lambda},$$

and (2.1) becomes

(2.2)

$$P_\sigma(s) = \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \left(\prod_{j=1}^{k} \chi^\lambda(C_j) \right) \operatorname{tr}\!\left(\rho^\lambda(s^{-1}) I_{f^\lambda}\right)$$
$$= \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \chi^\lambda(s) \prod_{j=1}^{k} \chi^\lambda(C_j);$$

see Stanley [26], Exercise 7. 67.

Note. For the special case $s = \text{id}$, the identity (2.2) becomes

$$\mathrm{P}_\sigma(\text{id}) = \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^{k} \chi^\lambda(C_j).$$

Since the left-hand side is just $\mathcal{N}(C_1, \ldots, C_k)$, the number of ways to write the identity permutation as the product of elements of $C_1, \ldots, C_k$, divided by $\prod_{j=1}^{k} |C_j|$, we obtain the well-known S_N-version of Frobenius's identity

(2.3)

$$\mathcal{N}(C_1, \ldots, C_k) = \frac{\prod_{j=1}^{k} |C_j|}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^{k} \chi^\lambda(C_j).$$

We will use (2.2) for $C_j \equiv \mathcal{C}_\mathcal{N}$, where $\mathcal{C}_\mathcal{N}$ is the conjugacy class of all maximal cycles. By the Murnaghan-Nakayama rule, Sagan [24] (Lemma 4.10.2) or Stanley [26] (Section 7.17, Equation (7.75)), $\chi^\lambda(\mathcal{C}_\mathcal{N}) = 0$ unless the diagram λ is a single hook λ^*, with one row of length λ_1 and one column of height λ^1, so $\lambda_1 + \lambda^1 = N + 1$. In that case

$$\chi^\lambda(\mathcal{C}_N) = (-1)^{\lambda^1 - 1}. \tag{2.4}$$

As for f^{λ^*}, the number of Standard Young Tableaux of shape λ^*, applying the hook length formula (or simply selecting the entries that go in the first column), we obtain

$$f^{\lambda^*} = \frac{N!}{N \prod_{r=1}^{\lambda_1 - 1} r \prod_{s=1}^{\lambda^1 - 1} s} = \binom{N-1}{\lambda_1 - 1}. \tag{2.5}$$

The equations (2.2), (2.4) and (2.5) imply

(2.6)

$$P_\sigma(s) = \frac{1}{N!} \sum_{\lambda^*} (-1)^{k(\lambda^1 - 1)} \binom{N-1}{\lambda_1 - 1}^{-k+1} \chi^{\lambda^*}(s).$$

By the Murnaghan-Nakayama rule, given a hook diagram λ^*, the value of $\chi^{\lambda^*}(s)$ depends on s only through $\vec{\nu} = \vec{\nu}(s) := \{\nu_r\}_{r \geq 1}$, where $\nu_r = \nu_r(s)$ is the total number of r-long cycles in the permutation s. It was proved in [23] that

$$\chi^{\lambda^*}(s) = (-1)^{\lambda^1 + \nu} [\xi^{\lambda_1}] \frac{\xi}{1 - \xi} \prod_{r \geq 1} (1 - \xi^r)^{\nu_r}, \tag{2.7}$$

$\nu(s) := \sum_r \nu_r(s)$ being the total number of cycles of s. From (2.7) it follows that

(2.8)

$$\sum_{s:\, \vec{\nu}(s) = \vec{\nu}} \chi^{\lambda^*}(s) = (-1)^N N!\, \mathcal{A}(N, \nu, \lambda_1),$$
$$\mathcal{A}(N, \nu, \lambda_1) := \binom{N-1}{N-\lambda_1} \sum_{\ell \geq 1} (-1)^\ell \frac{s(\ell, \nu)}{\ell!} \binom{N - \lambda_1}{N - \ell},$$

where $s(\ell, \nu)$ is the signless, first-kind, Stirling number of permutations of $[\ell] = \{1, 2, \cdots, \ell\}$ with ν cycles; see the proof of Theorem 2.1 and the equation (2.20) in [23]. The formulas (2.2), (2.7) and (2.8) are the basis of the proofs that follow.

3 Distribution of the number of cycles in σ

To stress dependence of σ on k, in this section we will write $\sigma^{(k)}$ instead of σ. Combining (2.8) and (2.6), and using $\lambda^1 + \lambda_1 = N + 1$, we obtain the following formula, which will be useful in our computations.

LEMMA 3.1. *We have*

(3.9)

$$P(\nu(\sigma^{(k)}) = \nu) = (-1)^N \sum_{\lambda_1 = 1}^{N} (-1)^{k(N - \lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2}$$
$$\times \sum_{\ell \geq 1} (-1)^\ell \frac{s(\ell, \nu)}{\ell!} \binom{N - \lambda_1}{N - \ell}.$$

COROLLARY 3.1. *For $k \geq 2$, we have*

$$P(\sigma^{(k)} = id) = P(\nu(\sigma) = N) \tag{3.10}$$
$$= \frac{1}{N!} \sum_{r=0}^{N-1} (-1)^{kr} \binom{N-1}{r}^{-k+2}.$$

Proof. Use formula (3.9) and the fact that $s(\ell, \nu) = 0$ for $\ell < \nu$.

Note that formula (3.10) appears as equation (7.181) in [26]. In the special case of $k = 2$ Corollary 3.1 yields

$$\mathrm{P}(\sigma^{(2)} = \text{id}) = \tfrac{1}{(N-1)!}. \tag{3.11}$$

This is an obvious result, since the inverse of the uniformly random cycle is again the uniformly random cycle.

The special case of $k = 3$ is not so obvious. However, combining (3.10) and the identity

$$\sum_{r=a}^{n} \frac{(-1)^r}{\binom{n}{r}} = \frac{n+1}{n+2}\left[\frac{(-1)^a}{\binom{n+1}{a}} + (-1)^n\right] \tag{3.12}$$

(Sury [28], Stanley [26], equation (7.211), Sury et al. [29]), we get the non-obvious formula

$$\mathrm{P}(\sigma^{(3)} = \mathrm{id}) = \frac{1+(-1)^{N-1}}{(N-1)!(N+1)}, \tag{3.13}$$

see [26], Exercise 7.67 (d).

The remarkable identity (3.12) followed from the elementary, yet surprisingly powerful, formula

$$\binom{n}{r}^{-1} = (n+1)\int_0^1 t^r(1-t)^{n-r}\,dt. \tag{3.14}$$

Note that for the even N, equation (3.13) returns zero probability, and that is how it should be, since the product of three even cycles is an odd permutation, and therefore, cannot be the identity. Furthermore, since $\sigma^{(k)} = \sigma^{(k-1)}\sigma_k$, $\sigma^{(k)}$ is the identity iff $\sigma^{(k-1)} = (\sigma_k)^{-1}$, which is a maximal cycle. As $(\sigma_k)^{-1}$ is uniform on the set of all $(N-1)!$ maximal cycles, *and* independent of $\sigma^{(k-1)}$, we see then that

$$\mathrm{P}(\sigma^{(k-1)} \text{ is a cycle}) = (N-1)!\mathrm{P}(\sigma^{(k)} = \mathrm{id}). \tag{3.15}$$

In the special case of $k = 2$, we rediscover a result that has been proved several times, with different methods.

Corollary 3.2. *We have*

$$P(\sigma^{(2)} \text{ is a cycle}) = \frac{1+(-1)^{N-1}}{N+1}. \tag{3.16}$$

Proof. Immediate from equations (3.13) and (3.15).

For even N, the statement of Corollary 3.2 is obvious, since the product of two maximal cycles is an even permutation, and hence, it cannot be an N-cycle for even N. For odd N, the result is equivalent to a well-known, but not at all obvious, fact that there are $\frac{2(N-1)!}{N+1}$ ways to factor a given maximal cycle into a product of two maximal cycles; see for instance [6] and the references therein. In general, the equations (3.10), (3.15) imply the following.

Corollary 3.3. *For all positive integers k, we have*

$$P(\sigma^{(k)} \text{ is a cycle}) = \frac{1}{N}\sum_{r=0}^{N-1}(-1)^{(k+1)r}\binom{N-1}{r}^{-k+1}. \tag{3.17}$$

Further, it follows from (3.9) that for every real number x, we have

$$\begin{aligned} \mathrm{E}\big[x^{\nu(\sigma)}\big] &= (-1)^N \sum_{\lambda_1=1}^{N}(-1)^{k(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+2} \\ &\quad\times \sum_{\ell\ge 1}\frac{(-1)^\ell}{\ell!}\binom{N-\lambda_1}{N-\ell}\sum_{\nu\ge 1}x^\nu s(\ell,\nu) \\ &= (-1)^N\sum_{r=0}^{N-1}(-1)^{kr}\binom{N-1}{r}^{-k+2}\binom{r-x}{N}. \end{aligned} \tag{3.18}$$

For a positive integer x, the non-zero contributions to the sum come from $r < \min\{N, x\}$. So, for instance, for $N > 1$, we have

$$\mathrm{E}\big[2^{\nu(\sigma^{(k)})}\big] = N + 1 + \frac{(-1)^k}{(N-1)^{k-2}},$$

and for $N > 2$, we have

$$\mathrm{E}\big[3^{\nu(\sigma^{(k)})}\big] = 2(N+2)_2 - \frac{N+1}{(N-1)^{k-2}} + \binom{N-1}{2}^{-k+2}.$$

where we use the notation $(a)_b = a(a-1)\cdots(a-b+1)$, for integers $a \ge b \ge 0$.

For $k = 2$ and $x > N$, equation (3.18) implies the following formula.

Corollary 3.4. *We have*

$$\begin{aligned} E\big[x^{\nu(\sigma^{(2)})}\big] &= (-1)^N\sum_{\lambda_1=1}^{N}\binom{N-\lambda_1-x}{N} \\ &= \sum_{\lambda_1=1}^{N}\binom{\lambda_1+x-1}{N} \\ &= \sum_{j=N}^{N+x-1}\binom{j}{N} - \sum_{j=N}^{x-1}\binom{j}{N} \\ &= \binom{N+x}{N+1} - \binom{x}{N+1} \\ &= \binom{N+x}{N+1} + (-1)^N\binom{N-x}{N+1}. \end{aligned} \tag{3.19}$$

Of course, the identity (3.19) holds for all x. It is equivalent to Zagier's result, (see the Appendix by Zagier in Lando and Zvonkin [18]), stating that

$$\mathrm{P}(\nu(\sigma^{(2)}) = \nu) = (1 + (-1)^{N-\nu})\,[x^\nu]\binom{N+x}{N+1}.$$

For $k = 3$, we can prove the following analogue of Corollary 3.4.

COROLLARY 3.5. *We have*

$$\sum_{N\geq 1}\frac{y^N}{N}E\big[x^{\nu(\sigma^{(3)}(N))}\big] \tag{3.20}$$

$$=\int_0^1 \frac{(1-y(1-t))^{-x}-(1-y(1-t))^{x}}{1-yt(1-t)}\,dt,$$

where $\sigma^{(3)}(N)$ is the product of 3 random cycles of length N, and $|x|\leq 1$, $|y|<1$.

The right-hand side of (3.20) is an odd function of x, which should be expected, since –regardless of the parity of N– the number of cycles in $\sigma^{(3)}(N)$ is odd. Differentiating both sides at $x=1$, we obtain that for $y\in[0,1)$,

$$\begin{aligned}\sum_{N\geq 1}\frac{y^N}{N}\mathrm{P}(\sigma^{(3)}(N)\text{ is a cycle}) &=\\ 2\int_0^1\frac{\log\big(1-y(1-t)\big)^{-1}}{1-yt(1-t)}\,dt &=\\ 2\sum_{j>0}\frac{1}{j}\int_0^1\frac{\big(y(1-t)\big)^j}{1-yt(1-t)}\,dt &=\\ 2\sum_{j>0,h\geq 0}\frac{y^{j+h}}{j}\int_0^1(1-t)^{j+h}t^h\,dt &=\\ 2\sum_{j>0,h\geq 0}\frac{y^{j+h}}{j}(j+2h+1)^{-1}\binom{j+2h}{h}^{-1}&.\end{aligned}$$

This implies that

$$\begin{aligned}\mathrm{P}(\sigma^{(3)}(N)\text{ is a cycle}) &=\\ 2N\sum_{h<N}(N-h)^{-1}(N+h+1)^{-1}\binom{N+h}{h}^{-1}; &=\end{aligned}$$

compare with the equation (3.17) for $k=3$.

Our final result in this section is a relatively compact, integral, formula for $P_n(\boldsymbol{\nu})$, the probability that $\sigma^{(2)}$ has ν_ℓ cycles of length ℓ, $1\leq\ell\leq n$, for the *arbitrary* $\boldsymbol{\nu}$, i.e. satisfying the only constraint $\sum_\ell \ell\nu_\ell=N$. Since $\sigma^{(2)}$ is even, $P_n(\boldsymbol{\nu})=0$ if $\sum_{\ell\text{ even}}\nu_\ell$ is odd.

THEOREM 3.1. *The equality*

$$P_n(\boldsymbol{\nu})=\frac{N}{\prod_\ell \ell^{\nu_\ell}\nu_\ell!}\int_0^1\prod_{\ell\geq 1}\big[t^\ell+(-1)^{\ell+1}(1-t)^\ell\big]^{\nu_\ell}\,dt,$$

holds.

Proof. First of all the number of permutations s with cycle parameter $\boldsymbol{\nu}$ is $N!/\prod_\ell \ell^{\nu_\ell}\nu_\ell!$. Furthermore, for every such permutation s, by (2.7), $\lambda^1+\lambda_1=N+1$ and $\nu=\sum_\ell\nu_\ell$, we obtain: setting $r=N-\lambda_1$, and choosing a positive ρ,

$$\chi^{\lambda^*}(s)=(-1)^{N+r}\frac{1}{2\pi i}\oint_{|\xi|=\rho}\frac{1}{\xi^{r+1}(1-\xi)}\prod_{\ell\geq 1}(\xi^\ell-1)^{\nu_\ell}\,d\xi.$$

here the circular contour is traversed counter-clockwise, and ρ is arbitrary, as the integrand is singular at $\xi=0$ only. Substituting $\xi=1/\eta$, we have

$$\chi^{\lambda^*}(s)=(-1)^{N+r}\frac{1}{2\pi i}\oint_{|\eta|=1/\rho}\frac{1}{\eta^{N-r}(\eta-1)}\prod_{\ell\geq 1}(1-\eta^\ell)^{\nu_\ell}\,d\eta,$$

with the contour traversed counter-clockwise again. Plugging this formula into the equation (2.6), and using (3.14), we have

$$\begin{aligned}P_{\sigma^{(2)}}(s)&=\frac{(-1)^N}{N!}\sum_{r=0}^{N-1}\binom{N-1}{r}^{-1}\chi^{\lambda^*(s)}(s)\\ &=\frac{(-1)^NN}{N!}\sum_{r=0}^{N-1}\chi^{\lambda^*}(s)\int_0^1 t^r(1-t)^{N-1-r}\,dt\\ &=\frac{(-1)^N}{(N-1)!}\,\frac{1}{2\pi i}\times\\ &=\oint_{|\eta|=1/\rho}\left(\int_0^1(1-t)^{N-1}\sum_{r=0}^{N-1}\left(-\frac{t\eta}{1-t}\right)^r dt\right)\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta\\ &=l\frac{(-1)^N}{(N-1)!}\times\\ &\int_0^1\left(\frac{1}{2\pi i}\oint_{|\eta|=1/\rho}\frac{(1-t)^N-(-t\eta)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta\right)dt\end{aligned}$$

Pick $\varepsilon\in(0,1)$ and consider $t\leq 1-\varepsilon$. Choose $\rho>(1-\varepsilon)/\varepsilon$. For this ρ, the internal integrand has two singular points, $\eta=0$ and $\eta=-(1-t)/t$, respectively within and without the integration contour. Crucially,

$$\frac{-(-t\eta)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}=\frac{-(-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta-1}$$

has no singularity at $\eta=0$. Furthermore, for $t>0$,

$$\frac{(1-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}=O(|\eta|^{-2}),\quad |\eta|\to\infty,$$

as $\sum_\ell \ell\nu_\ell=N$. So, by the residue theorem, the internal integral equals

$$\frac{1}{2\pi i}\oint\limits_{|\eta|=1/\rho}\frac{(1-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta$$
$$=-t^{-1}\,(1-t)^N\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\Bigg|_{\eta=-\frac{1-t}{t}}$$
$$=(-1)^N\prod_{\ell\ge1}\big[t^\ell+(-1)^{\ell+1}(1-t)^\ell\big]^{\nu_\ell},$$

for all $0<t\le1-\varepsilon$. Letting $\varepsilon\to0$, we obtain

$$P_{\sigma^{(2)}}(s)=\frac{1}{(N-1)!}\int\limits_0^1\prod_{\ell\ge1}\big[t^\ell+(-1)^{\ell+1}(1-t)^\ell\big]^{\nu_\ell}.$$

Multiplying the result by $N!/\prod_\ell\ell^{\nu_\ell}\nu_\ell!$ we complete the proof.

Corollary 3.6. *Let $P_{N,r}$ denote the probability that all cycles of $\sigma^{(2)}$ are of the same length $r\ge2$, i.e. $N\equiv0\,(mod\,r)$ and $\nu_r=N/r$. (So $P_{N,r}=0$ if r is even and $N\not\equiv0\,(mod\,2r)$.) Then*

(3.21)
$$P_{N,r}=\frac{N}{r^{N/r}(N/r)!}\int_0^1\big[t^r+(-1)^{r+1}(1-t)^r\big]^{N/r}dt$$
$$=\frac{N}{(N+1)r^{N/r}(N/r)!}\sum_{\substack{0\le j\le N\\ j\equiv0(mod\ r)}}(-1)^{j(r+1)/r}\frac{\binom{N/r}{j/r}}{\binom{N}{j}}$$

In particular,

$$P_{N,2}=\frac{N}{2^{N/2}(N/2+1)!},\tag{3.22}$$

and

$$P_{N,3}=\frac{N}{(N/3)!\,(12)^{N/3}}\sum_{j=0}^{N/3}\binom{N/3}{j}\frac{3^j}{2j+1}.\tag{3.23}$$

The derivation of (3.22) and (3.23) seems to indicate that the second line identity in (3.21) is the preferred expression for the probability $P_{N,r}$ when $r>3$.

Proof. The second identity in (3.21) follows from Theorem 3.1 using the binomial formula for the integrand $\big[t^r+(-1)^{r+1}(1-t)^r\big]^{N/r}$ and term-by-term integration. The formulas (3.22) and (3.23) follow immediately by integration from the first identity in (3.21), as $t^2-(1-t)^2=2t-1$, $t^3+(1-t)^3=1/4+3u^2$, and $u=t-1/2$.

Corollary 3.7. *For all positive integers N, we have*

$$P(\sigma^{(2)}\ \textit{is an involution})=N\sum_{\substack{\nu_1+2\nu_2=N\\ \nu_2\ even}}\frac{1}{\nu_1!\,2^{\nu_2}(\nu_2+1)!}.$$

The identities equivalent to (3.22) and(3.23) were proved in Doignon and Labarre [13] by using the sum-type formulas for the total number of ways to represent a maximal cycle as a product of a maximal cycle and a permutation from a given conjugacy class, see Goupil [14], Stanley [25], Goupil and Schaeffer [15]. The sequence $(N-1)!P_{N,2}$ is listed by Sloane as A035319, and known as the counts of certain rooted maps, see Walsh and Lehman [30]. The sequence $(N-1)!P_{N,3}$ is listed in Sloane as A178217.

4 Probability that the occupancy numbers of σ by the elements of $[\ell]$ belong to a given set

In the section title and elsewhere below σ is $\sigma^{(k)}$, the product of k random maximal cycles. Let $A\subseteq\mathbb{Z}_{\ge0}$ be given. Introduce $p_A(N,\ell;k)$, the probability that the number of elements of $[\ell]$ in each cycle of σ belongs to the set A.

The examples include: (1) $A_1=\mathbb{Z}_{>0}$; each cycle must contain at least one element of $[\ell]$; (2) $A_2=\{0,\ell\}$; one of the cycles of σ contains the whole set $[\ell]$; (3) $A_3=\{0,1\}$; each element of $[\ell]$ belongs to a distinct cycle of σ. The case of $k=2$, $\ell=2$ and $A=\{0,2\}$ or $A=\{0,1\}$ was solved by Stanley [27]. Very recently Bernardi et al. [2] solved the case $k=2$, $A=\{0,1\}$ for $\ell\ge2$. In fact they solved a general problem of separation probability for t disjoint sets $\mathcal{S}_1,\dots,\mathcal{S}_t$.

To evaluate $p_A(N,\ell;k)$, consider first $Q_A(\vec\nu,\ell)$, the total number of permutations s of $[N]$, with $\vec\nu(s)=\{\nu_r(s)\}=\{\nu_r\}=\vec\nu$, such that the number of elements of $[\ell]$ in every cycle is an element of A. The reason we need $Q_A(\vec\nu,\ell)$ is that the key formula (2.7) expresses $\chi^{\lambda*}(s)$ through the cycle counts $\nu_r(s)$, $r\ge1$.

To evaluate $Q_A(\vec\nu,\ell)$, introduce the non-negative integers $a_{r,j}$, $b_{r,j}$ that stand for the generic numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ in the j-th cycle of length r, $(j\le\nu_r)$.

Theorem 4.1. *For all $\ell\ge2$, we have*

(4.24)
$$Q_A(\vec\nu,\ell)=(N-\ell)!\,\ell!\,[w^\ell]\prod_r\frac{1}{\nu_r!}\left(\frac{\sum_{a\in A}\binom{r}{a}w^a}{r}\right)^{\nu_r}.$$

Proof. For $\mathbf{a}$, $\mathbf{b}$ to be admissible we must have

$$a_{r,j}+b_{r,j}=r,\tag{4.25}$$
$$a_{r,j}\in A,\tag{4.26}$$
$$\sum_{r,j}a_{r,j}=\ell,\quad\sum_{r,j}b_{r,j}=N-\ell.\tag{4.27}$$

Therefore $Q_A(\vec\nu,\ell)$ is equal to

$$(N-\ell)!\,\ell!\sum_{\substack{\mathbf{a},\,\mathbf{b}\text{ meet}\\ (4.25),(4.26),(4.27)}}\prod_r\frac{((r-1)!)^{\nu_r}}{\nu_r!}\prod_{j\le r}\frac{1}{a_{r,j}!\,b_{r,j}!}$$

$$= (N-\ell)!\,\ell!\,[w^\ell]\prod_r \frac{1}{r^{\nu_r}\nu_r!}\prod_{j\le\nu_r}\sum_{a_{r,j}\in A}\binom{r}{a_{r,j}}w^{a_{r,j}}$$

$$= (N-\ell)!\,\ell!\;[w^\ell]\prod_r \frac{1}{\nu_r!}\left(\frac{\sum_{a\in A}\binom{r}{a}w^a}{r}\right)^{\nu_r}.$$

So, using (2.7) and $\nu = \sum_r \nu_r$, we conclude that

$$\sum_{s\colon \vec\nu(s)=\vec\nu}\chi^{\lambda^*}(s) = (-1)^{\lambda^1}(N-\ell)!\,\ell!$$

$$\times[\xi^{\lambda_1}w^\ell]\;\frac{\xi}{1-\xi}\prod_r\frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}.$$

Call a permutation s of $[N]$ admissible if the numbers of elements from $[\ell]$ in each cycle of s meet the constraint (4.26). The above identity implies

$$\text{(4.28)}\qquad \sum_{s\text{ admissible}}\chi^{\lambda^*}(s) = (-1)^{\lambda^1}(N-\ell)!\,\ell!$$

$$\times[\xi^{\lambda_1}w^\ell]\;\frac{\xi}{1-\xi}\sum_{\vec\nu}\prod_r\frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}.$$

The expression in the second line of (4.28) equals the coefficient of $\xi^{\lambda_1}w^\ell x^N$ in

$$\frac{\xi}{1-\xi}\sum_{\vec\nu\ge\mathbf{0}}\prod_r\frac{(x^r)^{\nu_r}}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r},$$

or, equivalently, in

$$\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\ge0}\frac{1}{\nu_r!}\left(-\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}.$$

This is further equal to

$$[\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\;\prod_r\exp\left(-\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)$$

$$\text{(4.29)}\qquad =[\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\;\exp\left(-\sum_{r\ge1}\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right).$$

4.1 Probability that each cycle of σ contains at least one element of $[\ell]$

In this case $A = A_1 = \mathbb{Z}_{>0}$. Therefore

$$\sum_{a\in A}\binom{r}{a}w^a = (1+w)^r - 1.$$

We are going to prove the following result and discuss some of its special cases.

THEOREM 4.2. *For all positive integers ℓ and k, the probability $p_{A_1}(N,\ell;k)$ equals*

$$\text{(4.30)}\qquad \binom{N}{\ell}^{-1}\sum_{\lambda_1=\ell}^{N}(-1)^{(k-1)(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+1}\binom{\lambda_1-1}{\ell-1}.$$

Proof. Using (4.28), (4.29) and the identity $\sum_{j\ge1}z^j/j = -\log(1-z)$, $|z|<1$, we obtain the formula

$$\text{(4.31)}\qquad \sum_{s\text{ admissible}}\chi^{\lambda^*}(s)$$

$$= (-1)^{\lambda^1}(N-\ell)!\,\ell!$$

$$\times[\xi^{\lambda_1}w^\ell x^N]\;\frac{\xi}{1-\xi}\;\frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)}.$$

Let us simplify this formula. Write

$$[w^\ell]\frac{1-x(1+w)}{1-\xi x(1+w)} = \frac{\xi-1}{\xi}[w^\ell]\frac{1}{1-\xi x(1+w)}$$
$$=\frac{1-\xi}{\xi^2x}\,[w^\ell]\left(w-\frac{1-\xi x}{\xi x}\right)^{-1}$$
$$=-\,\frac{1-\xi}{\xi^2 x}\left(\frac{\xi x}{1-\xi x}\right)^{1+\ell}.$$

Therefore

$$[\xi^{\lambda_1}w^\ell x^N]\;\frac{\xi}{1-\xi}\;\frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)}$$
$$= -[\xi^{\lambda_1}x^N]\,(1-x)^{-1}\left(\frac{\xi x}{1-\xi x}\right)^{\ell}$$
$$= -[x^N]\,x^{\lambda_1}(1-x)^{-1}\cdot[y^{\lambda_1}]\left(\frac{y}{1-y}\right)^{\ell}$$
$$= -[y^{\lambda_1-k}](1-y)^{-\ell} = -\binom{\lambda_1-1}{\lambda_1-\ell},$$

where $\binom{a}{b}=0$ for $b<0$. So (4.31) becomes

$$\text{(4.32)}\qquad \sum_{s\text{ admissible}}\chi^{\lambda^*}(s) = (-1)^{\lambda^1-1}(N-\ell)!\,\ell!\binom{\lambda_1-1}{\lambda_1-\ell}.$$

Combining (4.32) and (2.6) we conclude that $p_{A_1}(N,\ell;k)$ equals

$$= \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)$$
$$= \binom{N}{\ell}^{-1}\sum_{\lambda_1=\ell}^{N}(-1)^{(k-1)(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+1}\binom{\lambda_1-1}{\ell-1},$$

which was to be proved.

Note that as $N \to \infty$, the dominant contribution to the right-hand side in (4.33) comes from $\lambda_1 = \ell$ and $\lambda_1 = N$, so that $p_{A_1}(N,\ell;k) = \ell/N + O(N^{-2\ell+1})$; the formula is useful for $\ell > 1$. We remark that ℓ/N is the probability that every cycle of the uniformly random permutation of $[N]$ contains at least one element of $[\ell]$; see Lovász [22], Section 3, Exercise 6.

COROLLARY 4.1. *For all positive integers ℓ, the identity*

(4.33)
$$p_{A_1}(N,\ell;2) = (-1)^{N+\ell} N \binom{N}{\ell}^{-1} \sum_{i=0}^{N-\ell} (-1)^i \binom{N}{i} \frac{1}{i+\ell}$$

holds.

We will not give a full proof here, but we mention that a key element of the proof is the sum

$$S_{n,a,b} = \sum_{r=a+b}^{n} (-1)^r \frac{\binom{r-a}{b}}{\binom{n}{r}}. \tag{4.34}$$

This function is relevant since (4.33) is equivalent to

$$p_{A_1}(N,\ell;2) = (-1)^{N-1} \binom{N}{\ell}^{-1} S_{N-1,0,\ell-1}. \tag{4.35}$$

Two relevant formulas can be proved for $S_{n,a,b}$. The first one is

(4.36)
$$S_{n,a,b} = \sum_{r=a+b}^{n} (-1)^r \frac{\binom{r-a}{b}}{\binom{n}{r}} = (n+1)\Bigg[\frac{(-1)^{a+b}}{(n+2+b)\binom{n+b+1}{a+b}} + \sum_{j=0}^{b} (-1)^{n+b-j} \binom{n-a+1}{j} \frac{1}{n+2+b-j}\Bigg].$$

For large n, this formula is a significant improvement of the initial definition of $S_{n,a,b}$ if b remains moderately valued. Using yet another identity

$$\sum_{j=0}^{u} (-1)^j \binom{u}{j} \frac{1}{v+j+1} = \frac{1}{(u+v+1)\binom{u+v}{v}},$$

from Sury et al. [29], equation (4.36) is easily transformed into

$$S_{n,a,b} = (-1)^{a+b}(n+1) \sum_{i=0}^{n-a-b} (-1)^i \frac{\binom{n-a+1}{i}}{i+a+b+1}. \tag{4.37}$$

This alternative formula is efficient for the extreme case, when $n-a-b$ is moderately valued as n grows.

Example. As a special case of the formula in Corollary 4.1, we obtain

$$p_{A_1}(N,1;2) = \begin{cases} \frac{2}{N+1} & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

This is equivalent to the result already mentioned in Section 3, since $p_{A_1}(N,1;2)$ is indeed equal to the probability that σ is a maximal cycle. For the next few values of ℓ, we have

$$p_{A_1}(N,2;2) = \begin{cases} \frac{2}{N+1} & \text{if } N \text{ is odd,} \\ \frac{2N}{(N+2)(N-1)} & \text{if } N \text{ is even.} \end{cases}$$

$$p_{A_1}(N,3;2) = \begin{cases} \frac{3(N^2+N-4)}{(N+1)(N+3)(N-2)} & \text{if } N \text{ is odd,} \\ \frac{3N}{(N+2)(N-1)} & \text{if } N \text{ is even.} \end{cases}$$

$$p_{A_1}(N,4;2) = \begin{cases} \frac{4(N^2+N-3)}{(N+1)(N+3)(N-2)} & \text{if } N \text{ is odd,} \\ \frac{4N(N^2+N-11)}{(N-3)(N-1)(N+2)(N+4)} & \text{if } N \text{ is even.} \end{cases}$$

4.2 Probability that the elements $1,\dots,\ell$ are in the same cycle of σ This time $A = A_2 = \{0,\ell\}$, so that

$$\sum_{a\in A_2} \binom{r}{a} w^a = 1 + \binom{r}{\ell} w^\ell. \tag{4.38}$$

Our goal in this section is to prove the following theorem and its special case of $k=2$.

THEOREM 4.3. *For all integers $\ell \ge 2$, the probability $p_{A_2}(N,\ell;k)$ equals*

$$\frac{1}{N!} \sum_{\lambda_1=1}^{N} (-1)^{k(\lambda^1-1)} \binom{N-1}{\lambda_1-1}^{-k+1} \sum_{s \text{ admissible}} \chi^{\lambda^*}(s)$$
$$= \frac{1}{\ell}\binom{N}{\ell}^{-1} \sum_{\lambda_1} (-1)^{(k+1)(\lambda^1-1)} \binom{N-1}{\lambda_1-1}^{-k+1}$$
$$\times \left\{ 1_{\{\lambda_1<N\}} \left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1} \right] + 1_{\{\lambda_1=N\}} \binom{N}{\ell} \right\}.$$

Proof. In this case, the computation is more involved than it was for A_1. Formula (4.38) implies

(4.39)
$$Q_{A_2}(\vec{\nu},\ell) = (N-\ell)!\,\ell!\,[w^\ell] \prod_r \frac{1}{\nu_r!} \left(\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r}.$$

So, using (2.7) and $\nu = \sum_r \nu_r$, we conclude that

$$\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1}(N-\ell)!\,\ell! \tag{4.40}$$
$$\times [\xi^{\lambda_1} w^\ell]\, \frac{\xi}{1-\xi} \sum_{\substack{\vec{\nu}: \\ 1\nu_1+2\nu_2+\cdots=N}} \prod_r \frac{1}{\nu_r!} \left(-(1-\xi^r) \frac{1+\binom{r}{\ell}w^\ell}{r} \right)^{\nu_r}.$$

Since $\sum_r r\nu_r = N$, the identity

$$\sum_r z^r/r = -\log(1-z),$$

($|z| < 1$), implies that the second line expression in (4.40) equals

$$\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\sum_{\vec\nu\geq\mathbf{0}}\prod_r \frac{(x^r)^{\nu_r}}{\nu_r!}\left(-(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r}\\
&=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\geq 0}\frac{1}{\nu_r!}\left(-x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r}\\
&=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r \exp\left(-x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)\\
&=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\exp\left(-\sum_{r\geq 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right).
\end{aligned}$$

Here, using $\sum_{b\geq a}\binom{b}{a}z^b = \frac{z^a}{(1-z)^{a+1}}$,

$$\begin{aligned}
&\sum_{r\geq 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r} =\\
&-\log(1-x)+\log(1-x\xi)+\frac{w^\ell}{\ell}\sum_{r\geq 1}\binom{r-1}{\ell-1}\left(x^r-(x\xi)^r\right)\\
&=\log\frac{1-x\xi}{1-x}+\frac{w^\ell}{\ell}\left(\frac{x^\ell}{(1-x)^\ell}-\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[w^\ell]\exp\left(-\sum_{r\geq 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)\\
&=\frac{1-x}{1-x\xi}[w^\ell]\exp\left[-\frac{w^\ell}{\ell}\left(\frac{x^\ell}{(1-x)^\ell}-\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\right)\right]\\
&=\frac{1}{\ell}\frac{1-x}{1-x\xi}\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right).
\end{aligned}$$

Therefore the expression in the second line of (4.40) is equal to

$$\begin{aligned}
&\frac{1}{\ell}[\xi^{\lambda_1}x^N]\frac{\xi}{1-\xi}\cdot\frac{1-x}{1-x\xi}\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)\\
&=\frac{1}{\ell}[\xi^{\lambda_1}x^N]\left(\frac{1}{1-\xi}-\frac{1}{1-x\xi}\right)\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)\\
&=:\frac{1}{\ell}(T_1+T_2+T_3+T_4).
\end{aligned}$$

Here

$$\begin{aligned}
T_1 &= [\xi^{\lambda_1}x^N]\frac{1}{1-\xi}\cdot\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\\
&=[\xi^{\lambda_1}]\frac{\xi^N}{1-\xi}[sy^N]\frac{y^\ell}{(1-y)^\ell} = 1_{\{\lambda_1=N\}}\binom{N-1}{\ell-1};
\end{aligned}\tag{4.41}$$

next

$$\begin{aligned}
T_2 &= -[\xi^{\lambda_1}x^N]\frac{1}{1-\xi}\cdot\frac{x^\ell}{(1-x)^\ell}\\
&=-[x^{N-\ell}]\frac{1}{(1-x)^\ell} = -\binom{N-1}{\ell-1};
\end{aligned}\tag{4.42}$$

next

$$\begin{aligned}
T_3 &= -[\xi^{\lambda_1}x^N]\frac{(x\xi)^\ell}{(1-x\xi)^{\ell+1}}\\
&=-1_{\{\lambda_1=N\}}[y^{N-\ell}]\frac{1}{(1-y)^{\ell+1}} = -1_{\{\lambda_1=N\}}\binom{N}{\ell};
\end{aligned}\tag{4.43}$$

and finally

$$\begin{aligned}
T_4 &= [\xi^{\lambda_1}x^N]\frac{1}{1-x\xi}\frac{x^\ell}{(1-x)^\ell}\\
&=[x^N]\frac{x^{\lambda_1+\ell}}{(1-x)^\ell} = [x^{N-\lambda_1-\ell}]\frac{1}{(1-x)^\ell}\\
&=1_{\{\lambda_1<N\}}\binom{N-\lambda_1-1}{\ell-1}.
\end{aligned}\tag{4.44}$$

It follows from (4.41), (4.42), (4.43) and (4.44) that $\frac{1}{\ell}(T_1+T_2+T_3+T_4)$ is equal to

$$-\frac{1}{\ell}\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1}-\binom{N-\lambda_1-1}{\ell-1}\right]+1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.\tag{4.45}$$

So (4.40) becomes

$$\begin{aligned}
&\sum_{s\text{ admissible}}\chi^{\lambda^*}(s) = (-1)^{\lambda^1-1}(N-\ell)!\,(\ell-1)!\\
&\times\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1}-\binom{N-\lambda_1-1}{\ell-1}\right]+1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.
\end{aligned}\tag{4.46}$$

Combining (4.46) and (2.6) we obtain the statement that was to be proved.

Corollary 4.2. *For all integers $\ell\geq 2$, we have*

$$\begin{aligned}
&p_{A_2}(N,\ell;2) = \frac{1}{\ell}-\frac{1}{(N+1)_2}\\
&+(-1)^{\ell+1}\binom{N-1}{\ell-1}^{-1}\sum_{i=0}^{N-\ell}(-1)^i\binom{N-1}{i}\frac{1}{i+\ell+1}.
\end{aligned}\tag{4.47}$$

Proof. For $k=2$, introducing $r=N-\lambda_1$, we have

$$\begin{aligned}
&p_{A_2}(N,\ell;2) = \frac{1}{\ell}+\frac{1}{\ell}\binom{N}{\ell}^{-1}\\
&\times\sum_{r=1}^{N-1}(-1)^r\binom{N-1}{r}^{-1}\left[\binom{N-1}{\ell-1}-\binom{r-1}{\ell-1}\right].
\end{aligned}\tag{4.48}$$

By (4.34), the last sum is the linear combination of $S_{N-1,0,0}-1$ and $S_{N-1,1,\ell-1}$. According to (4.36) and (4.37), we have

$$S_{N-1,0,0} = \left[1+(-1)^{N-1}\right]\frac{N}{N+1},$$

$$S_{N-1,1,\ell-1} = (-1)^\ell\left[\binom{N+\ell}{\ell}^{-1} + N\sum_{j=0}^{\ell-1}(-1)^{N-j}\frac{\binom{N-1}{j}}{N+\ell-j}\right]$$
$$= (-1)^\ell N\sum_{i=0}^{N-1-\ell}(-1)^i\binom{N-1}{i}\frac{1}{i+\ell+1}.$$

Plugging these expressions into (4.48), we obtain after simple algebra

(4.49)
$$p_{A_2}(N,\ell;2) = \frac{1}{\ell} + \left[\frac{1+(-1)^{N-1}}{N+1} - \frac{1}{N}\right] + \frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}}\left[\binom{N+\ell}{\ell}^{-1} + N\sum_{j=0}^{\ell-1}(-1)^{N-j}\frac{\binom{N-1}{j}}{N+\ell-j}\right]$$

(4.50)
$$= \frac{1}{\ell} - \frac{1}{(N+1)_2} + \frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}}\left[\binom{N+\ell}{\ell}^{-1} + N\sum_{j=0}^{\ell-2}(-1)^{N-j}\frac{\binom{N-1}{j}}{N+\ell-j}\right]$$

(4.51)
$$= \frac{1}{\ell} - \frac{1}{(N+1)_2} + (-1)^{\ell+1}\binom{N-1}{\ell-1}^{-1}\sum_{i=0}^{N-\ell}(-1)^i\binom{N-1}{i}\frac{1}{i+\ell+1},$$

as claimed.

The equivalent formulas (4.49) and (4.47) are computationally efficient for moderate ℓ and moderate $N-\ell$ respectively. In particular, plugging $\ell = 2,3$ into (4.49) and simplifying, we recover Stanley's results, [27].

$$p_{A_2}(N,2;2) = \begin{cases} \frac{1}{2} & \text{if } N \text{ is odd,}\\ \frac{1}{2} - \frac{2}{(N-1)(N+2)} & \text{if } N \text{ is even,}\end{cases}$$

and

$$p_{A_2}(N,3;2) = \begin{cases} \frac{1}{3} + \frac{1}{(N-2)(N+3)} & \text{if } N \text{ is odd,}\\ \frac{1}{3} - \frac{1}{(N-1)(N+2)} & \text{if } N \text{ is even.}\end{cases}$$

Here are the new formulas for $\ell = 4,5$ obtained from (4.49): denoting $(a)^{(b)} = a(a+1)\cdots(a+b-1)$,

$$p_{A_2}(N,4;2) = \begin{cases} \frac{1}{4} + \frac{2}{(N)^{(2)}-(2)^{(2)}} & N \text{ odd,}\\ \frac{1}{4} - \frac{-4(N)^{(2)}+44}{[(N)^{(2)}-(1)^{(2)}]\,[(N)_2-(3)_2]} & N \text{ even.}\end{cases}$$

and

$$p_{A_2}(N,5;2) = \begin{cases} \frac{1}{5} + \frac{5(N)^{(2)}-50}{[(N)^{(2)}-(2)^{(2)}]\cdot[(N)^{(2)}-(4)^{(2)}]} & N \text{ odd,}\\ \frac{1}{5} + \frac{-5(N)^{(2)}+50}{[(N)^{(2)}-(1)^{(2)}]\,[(N)^{(2)}-(3)^{(2)}]} & N \text{ even.}\end{cases}$$

While the denominators certainly follow a simple pattern, the numerators do not exhibit a discernibly regular behavior, except that they are also polynomials of $(N)^{(2)}$.

5 The probability that σ separates the disjoint sets $\mathcal{S}_1,\dots,\mathcal{S}_t$

Let $\ell_j = |\mathcal{S}_j|$, $1\le j\le t$, $\ell = \sum_j \ell_j$. Introduce $p(N,\vec{\ell};k)$, the probability that the permutation σ separates the sets $\mathcal{S}_1,\dots,\mathcal{S}_t$, meaning that no cycle of σ contains a pair of elements from two distinct sets $\mathcal{S}_i$ and $\mathcal{S}_j$. Bernardi et al. [2] were able to derive a striking formula for $p(N,\vec{\ell};2)$:

(5.52)
$$p(N,\vec{\ell};2) = \frac{(N-\ell)!\prod_j \ell_j!}{(N+t)(N-1)!} \times\left[\frac{(-1)^{N+\ell}\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}} + \sum_{j=0}^{\ell-t}\frac{(-1)^j\binom{\ell-t}{j}\binom{N+j+1}{\ell}}{\binom{N+t+j}{j}}\right],$$

which is a sum of $\ell-t+2$ terms. Remarkably, $\prod_j \ell_j!$ aside, the rest of this expression does not depend on the individual ℓ_j. The equation (5.52) is very efficient for values of ℓ, t relatively small compared to N.

In this section first we apply our approach to obtain a formula for this probability for a general $k\ge 2$. Similarly to $p(N,\vec{\ell};2)$, it is of a form $\prod_j \ell_j!$ times an expression that depends on ℓ, but not on individual ℓ_j.

Lemma 5.1. *Introduce*

$$K(N,\ell,t;r) = \left[\xi^{r-\ell+t}\eta^{N-\ell}\right]\left(\frac{1-\xi}{1-\eta}\right)^{t-1}(1-\xi\eta)^{-\ell-1},$$

and define $\alpha_k(N,t) = t-1$ if k is odd, and $\alpha_k(N,t) = N+t$ if k is even. Then

(5.53)
$$p(N,\vec{\ell};k)\frac{(-1)^{\alpha_k(N,t)}\prod_j \ell_j!}{(N)_\ell} \times\sum_{r=\ell-t}^{N-1}(-1)^{(k+1)r}\binom{N-1}{r}^{-k+1}K(N,\ell,t;r).$$

While we do not prove the lemma here, we point out that an important step of the proof is to show the identities

(5.54)
$$K(N,\ell,t;r) = [\xi^{r-\ell+t}z^{N-\ell}]\,(1-\xi)^{t-1}(1-z)^{-t+1}(1-\xi z)^{-\ell-1} = \sum_j(-1)^{r-\Delta-j}\binom{\ell+j}{j}\binom{t-1}{r-\Delta-j}\binom{N-\Delta-j-2}{t-2},$$

where we set $\Delta = \ell - t$.

The sum in (5.53) depends only on ℓ and t, rather than the individual $\ell_1, \ldots, \ell_t$, and $K(N, \ell, t, r)$ is given by each of two lines in (5.54). In particular,

$$K(N, N, t; r) = [\xi^{r-N+t}](1-\xi)^{t-1} = (-1)^{r-N+t}\binom{t-1}{r-N+t}.$$

Let $\ell = \sum_j \ell_j = N$. Introducing $\beta_k(N) = N-1$ for k odd, $\beta_k(N) = 0$ for k even, equation (5.53) becomes

$$p(N, \vec{\ell}; k) = \frac{(-1)^{\beta_k(N)} \prod_j \ell_j!}{(N)_\ell} \times \sum_{r=N-t}^{N-1} (-1)^{kr} \binom{N-1}{r}^{-k+1} \binom{t-1}{r-N+t},$$

an alternating sum of t terms. For $t = N$, $p(N, \vec{\ell}; k) = \mathrm{P}(\sigma = \mathrm{id})$; the resulting formula agrees with (3.10), since for k odd and N even the sum over $r \in [0, N-1]$ is zero.

5.1 When $k = 2$.

From now on we focus on $k = 2$, and general $\vec{\ell}$. We begin with a relatively compact formula that represents $p(N, \vec{\ell}; 2)$ as a composition of integration operation and coefficient extraction operation.

THEOREM 5.1. *We have*

$$p(N, \vec{\ell}; 2) = \frac{(-1)^{N+\ell} N \prod_j \ell_j!}{(N)_\ell} \times [z^{N-\ell}](1-z)^{-t+1} \int_0^1 \frac{(1-u)^{N+1} u^{\ell-t}}{(1-u+zu)^{\ell+1}}\, du. \tag{5.55}$$

COROLLARY 5.1. *For $\ell = N$ the formula* (5.55) *yields*

$$p(N, \vec{\ell}; 2) = \frac{N \prod_j \ell_j!}{N!} \int_0^1 u^{N-t}\, du = \frac{\prod_j \ell_j!}{(N-1)!(N-t+1)}. \tag{5.56}$$

To compare, the separation probability for the uniformly random permutation of $[N]$ is $\prod_j \ell_j!/N!$.

More generally,

$$p(N, \vec{\ell}; 2) = \frac{N \prod_j \ell_j!}{(N)_\ell} \sum_{k \le N-\ell} (-1)^k \frac{\binom{t+k-2}{t-2}\binom{N-k}{\ell}}{(N-t+1)\binom{N-t}{k}}, \tag{5.57}$$

an equation computationally efficient for moderate $N - \ell$, but progressively less useful for larger values of $N - \ell$.

5.2 An alternative formula deduced by the WZ-method.

In this section, we will show that equation (5.55) can be transformed so that extraction of the coefficient of $z^{N-\ell}$ will lead to a sum with $\ell - t + 2$ number of terms, close in appearance to the formula (5.52) by Bernardi et al.

LEMMA 5.2. *The identity*

$$\Sigma(N, \ell, t) = \frac{(N-1)_{t-2}\,(\ell-t)!}{(t-2)!(N+t)^{(\ell-t+1)}}. \tag{5.58}$$

holds.

Proof. We confirmed this conjecture via the powerful Wilf-Zeilberger algorithm, see Nemes et al. [20], Wilf and Zeilberger [31]. Given $\Delta \ge 0$, introduce a function of $t \ge 2$, defined by

$$S(t) = \sum_{j=1}^{t-1+\Delta} \frac{(t+1)^{(j-1)}(j-1)_\Delta}{(t+1)^{(\Delta)}(N+2)^{(j)}} \binom{N-j-1}{t+\Delta-j-1}.$$

The summands are nonzero if $j \in [\Delta+1, t-1+\Delta]$. We can extend summation to $j \in [1, \infty)$, since the last binomial is zero for $j \ge t + \Delta$. We need to show that

$$S(t) = S^*(t) := \frac{(N-1)_{t-2}\Delta!}{(t-2)!(N+t)^{(\Delta+1)}}. \tag{5.59}$$

To do so, first we compute

$$\frac{S^*(t)}{S^*(t-1)} = \frac{\beta(t)}{\alpha(t)},$$

$$\alpha(t) := (t-2)(N+t+\Delta), \quad \beta(t) := (N-t+2)(N+t-1).$$

Next, let $F(t, j)$ stand for the j-term in the series $S(t)$. Introduce the "partner" sequence $G(t, j)$ (which again for each t is 0 for all but finitely many j) such that

$$G(t,j) - G(t,j-1) = \alpha(t)F(t,j) - \beta(t)F(t-1,j), \quad j \ge \Delta+1, \tag{5.60}$$

and $G(t, \Delta) = 0$.

The equation (5.59) will be proved if we demonstrate that $G(t, j) = 0$ for j large enough.

Computation by Maple shows that

$$G(t, \Delta+1) = -\frac{(\Delta+1)!(\Delta+2t-2)}{(N+2)^{(\Delta+1)}}\binom{N-\Delta-2}{t-3},$$

$$G(t, \Delta+2) = -\frac{(\Delta+2)!(\Delta+2t-2)(t+\Delta+1)}{(N+2)^{(\Delta+2)}}\binom{N-\Delta-3}{t-4},$$

$$G(t, \Delta+3) = -\frac{(\Delta+3)!(\Delta+2t-2)(t+\Delta+2)_2}{2(N+2)^{(\Delta+3)}}\binom{N-\Delta-4}{t-5}.$$

The evidence is unmistakable: it must be true that for all $u \ge 1$

$$G(t, \Delta+u) = -\frac{(\Delta+u)!(\Delta+2t-2)\binom{t+\Delta+u-1}{u-1}}{(N+2)^{(\Delta+u)}}\binom{N-\Delta-u-1}{t-u-2}. \tag{5.61}$$

Sure enough, the inductive step based on the recurrence (5.60) is easily carried out with a guided assistance of Maple. It remains to notice that the last binomial coefficient is zero for $u > t - 2$.

Now we are in a position to announce the main result of this section.

THEOREM 5.2. *We have*

$$p(N,\vec{\ell};2)=\frac{(N-\ell)!\prod_j \ell_j!}{(N-1)!\,(N+t)}\left[(-1)^{N+\ell}\frac{\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}}\right.$$
$$\left.+\frac{(N+t)(N+1)_{\ell+1}}{(N-t+1)(N+\ell)!\,(\ell)_t}\sum_{\nu=0}^{\ell-t}\frac{(N+\ell-\nu-1)!(N-1)_\nu}{(\ell-t-\nu)!(N-t)_\nu}\right]. \tag{5.62}$$

6 Probability that σ blocks the elements of $[\ell]$

We say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if in every cycle of s (1) no two elements of $[\ell]$ are neighbors, and (2) each element from $[\ell]$ has a neighbor from $[N]\setminus[\ell]$.

Let $p(N,\ell;k)$ denote the probability of the event that σ blocks the elements of $[\ell]$. In this final section, we are going to prove the following theorem.

THEOREM 6.1. *For all positive integers ℓ and k, we have*

$$p(N,\ell;k)=\frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}}+(-1)^{k+1}\frac{\binom{N-\ell-1}{\ell-1}}{(N-1)^{k-1}\binom{N}{\ell}}. \tag{6.63}$$

Proof. Start again with $Q(\vec{\nu},\ell)$, the total number of permutations with cycle counts $\vec{\nu}$ such that the elements of $[\ell]$ are blocked. To evaluate $Q(\vec{\nu},\ell)$, introduce the non-negative integers $a_{r,j}$, $b_{r,j}$ that stand for the generic numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ in the j-th cycle of length r, $(j\le\nu_r)$. Then

$$a_{r,j}+b_{r,j}=r, \tag{6.64}$$
$$b_{r,j}>0, \tag{6.65}$$
$$\sum_{r,\,j\le\nu_r}a_{r,j}=\ell,\qquad \sum_{r,\,j\le\nu_r}b_{r,j}=N-\ell. \tag{6.66}$$

For $a_{r,j}>0$, the number of admissible cycles with parameters $a_{r,j}$, $b_{r,j}$ is

$$\begin{aligned}c(a_{r,j},b_{r,j})&:=(a_{r,j}-1)!\,b_{r,j}!\binom{b_{r,j}-1}{a_{r,j}-1}\\&=(b_{r,j}-1)!a_{r,j}!\binom{b_{r,j}}{a_{r,j}}.\end{aligned} \tag{6.67}$$

The last expression works for $a_{r,j}=0$ as well.

Indeed $(a_{r,j}-1)!$ is the total number of directed cycles formed by $a_{r,j}$ elements from $[\ell]$; $b_{r,j}!$ is the total number of ways to order, linearly, $b_{r,j}$ elements from $[N]\setminus\ell$, and $\binom{b_{r,j}-1}{a_{r,j}-1}$ is the total number of ways to break any such $b_{r,j}$-long sequence into $a_{r,j}$ blocks of positive lengths to be fitted between $a_{r,j}$ cyclically arranged elements from $[\ell]$, starting with the smallest element among them and moving in the cycle's direction, say.

Therefore

$$\begin{aligned}Q(\vec{\nu},\ell)&=(N-\ell)!\,\ell!\sum_{\substack{\mathbf{a},\,\mathbf{b}\text{ meet}\\(6.64)-(6.66)}}\prod_{r\ge1}\frac{1}{\nu_r!}\prod_{j\le\nu_r}\frac{c(a_{r,j},b_{r,j})}{a_{r,j}!\,b_{r,j}!}\\&=(N-\ell)!\,\ell!\,[w^\ell]\prod_{r\ge1}\frac{1}{\nu_r!}\left(\sum_{b>0,\,a+b=r}\frac{1}{b}\binom{b}{a}w^a\right)^{\nu_r}.\end{aligned} \tag{6.68}$$

Having found $Q(\vec{\nu},\ell)$, we turn to $p(N,\ell,k)$, the probability that σ blocks the elements of $[\ell]$. Using (2.7), the equality $\nu=\sum_r\nu_r$, and and (6.68), we obtain

$$\sum_{s:\,\vec{\nu}(s)=\vec{\nu}}\chi^{\lambda^*}(s)=(-1)^{\lambda^1}(N-\ell)!\,\ell!$$
$$\times[\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}\prod_r\frac{1}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]^{\nu_r}.$$

Call a permutation s of $[N]$ admissible if it blocks the elements of $[\ell]$. The above identity implies

$$\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)=(-1)^{\lambda^1}(N-\ell)!\,\ell!\times \tag{6.69}$$
$$[\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}\sum_{\substack{\vec{\nu}:\\\sum i\nu_i=N}}\prod_r\frac{1}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]^{\nu_r}.$$

The expression in the second line of (6.69) equals the coefficient of $[\xi^{\lambda_1}w^\ell x^N]$ in

$$\begin{aligned}&\frac{\xi}{1-\xi}\sum_{\vec{\nu}\ge\mathbf{0}}\prod_r\frac{(x^r)^{\nu_r}}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]^{\nu_r}\\=\;&\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\ge0}\frac{1}{\nu_r!}\left[-x^r(1-\xi^r)\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]^{\nu_r}\\=\;&\frac{\xi}{1-\xi}\prod_r\exp\left[-x^r(1-\xi^r)\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]\\=\;&\frac{\xi}{1-\xi}\exp\left[-\sum_{r\ge1}[x^r-(x\xi)^r]\left(\sum_{\substack{b>0,\\a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right].\end{aligned}$$

Since

$$\begin{aligned}&\sum_{r\ge1}y^r\sum_{b>0,\,a+b=r}\frac{1}{b}\binom{b}{a}w^a=\sum_{b>0}\frac{y^b}{b}\sum_a\binom{b}{a}(yw)^a\\&=\sum_{b>0}\frac{y^b}{b}(1+yw)^b=\sum_{b>0}\frac{[y(1+yw)]^b}{b}\\&=\log\frac{1}{1-y(1+yw)},\end{aligned}$$

the bottom part (6.70) becomes

$$
\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \\
&\quad \times \exp\left(-\log\frac{1}{1-x(1+xw)} + \log\frac{1}{1-x\xi(1+x\xi w)}\right) \\
=&[\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \frac{1-x(1+xw)}{1-x\xi(1+x\xi w)} \\
=&[\xi^{\lambda_1} x^N] \frac{\xi(1-x)}{(1-\xi)(1-x\xi)} [w^\ell] \frac{1-\frac{x^2}{1-x}w}{1-\frac{(x\xi)^2}{1-x\xi}w} \\
=&[\xi^{\lambda_1} x^N] \frac{\xi(1-x)}{(1-\xi)(1-x\xi)} \\
&\times \left[\left(\frac{(x\xi)^2}{1-x\xi}\right)^\ell - \frac{x^2}{1-x}\left(\frac{(x\xi)^2}{1-x\xi}\right)^{\ell-1}\right] \\
=&[\xi^{\lambda_1} x^N] \frac{\xi}{1-x\xi}\left(\frac{(x\xi)^2}{1-x\xi)}\right)^{\ell-1} \frac{x^2}{1-x\xi}(x\xi-1-\xi) \\
=&-[\xi^{\lambda_1} x^N] \left(\frac{x^{2\ell}\xi^{2\ell-1}}{(1-x\xi)^\ell} + \frac{x^{2\ell}\xi^{2\ell}}{(1-x\xi)^{\ell+1}}\right) \\
=&-[\xi^{\lambda_1-2\ell+1} x^{N-2\ell}](1-x\xi)^{-\ell} \\
&-[\xi^{\lambda_1-2\ell} x^{N-2\ell}](1-x\xi)^{-\ell-1} \\
=&-\binom{N-\ell-1}{\ell-1} 1_{\{\lambda_1=N-1\}} - \binom{N-\ell}{\ell} 1_{\{\lambda_1=N\}}.
\end{aligned}
$$

So (6.69) simplifies, greatly, to

$$
\begin{aligned}
(6.70)\quad &\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1-1}(N-\ell)!\,\ell! \\
&\times\left[\binom{N-\ell-1}{\ell-1} 1_{\{\lambda_1=N-1\}} + \binom{N-\ell}{\ell} 1_{\{\lambda_1=N\}}\right].
\end{aligned}
$$

The rest is easy. By (2.6), we know that $p(N,\ell;k)$ is equal to

$$
(6.71)\quad \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1} \sum_{s \text{ adm.}} \chi^{\lambda^*}(s).
$$

Combining this with (6.70) we conclude that

$$
(6.72)\quad p(N,\ell;k) = \frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}} + (-1)^{k+1}\frac{\binom{N-\ell-1}{\ell-1}}{(N-1)^{k-1}\binom{N}{\ell}}.
$$

Note. The equation (6.72) shows that $\lim_{k\to\infty} p(N,\ell;k) = \binom{N-\ell}{\ell}/\binom{N}{\ell}$, the probability that the *uniformly* random permutation blocks $[\ell]$.

7 Sorting algorithms

Our original motivation to study products of random maximal cycles came from a problem concerning block interchange sorting of permutations, a problem rooted in evolutionary biology.

A *block interchange* in a permutation (written in the one-line notation) is the interchange of two consecutive strings of entries. These two strings do not have to have the same length, and they do not have to be adjacent. For instance, a block interchange of the permutation 2513746 can result in the permutation 7461325, if we interchange the string of the first two entries and the string of the last three entries In [4], the average number of block interchanges needed to get from a random permutation of length N to the identity permutation was studied. It turned out that the crucial step in answering that question was to find the probability that the entries i and j are in the same cycle of the product of two random maximal cycles.

However, block interchange sorting is not the only biologically motivated sorting defined on permutations, *block transpositon sorting*, in which the two interchanged blocks have to be adjacent, is another one. Interestingly, sorting by block transpositions is not nearly as well understood as sorting by block interchanges. The connection between block transposition sorting and block interchange sorting was first established by Bafna and Pevzner in [3]. This leads to the following question.

QUESTION 1. *Can we use the techniques described in this section to compute the average number of block transpositions needed to sort a permutation?*

Block interchange sorting and block transposition sorting are special cases of biologically motivated sorting algorithms. In the mathematical model of these algorithms, we need to get from a given permutation p to the identity permutation using as few steps as possible, but each step must come from a finite list of allowed steps. This leads to a notion of *distance*. Let us say that the distance of the permutation p from the permutation q is k if k is the minimum number of allowed steps that are needed to turn p into q. Let us denote this fact by $d(p,q)=k$. Most of the frequently studied notions of distance are *left-invariant*, meaning that for all permutations r of length n, the identity $d(rp,rq)=d(p,q)$ holds. Setting $r=q^{-1}$, this simplifies to $d(q^{-1}p,\mathrm{id})=d(p,q)$. That is, if we can find the distance between any permutation and the identity, then we can find the distance between any two permutations. This reduces the problem of finding distances to a *sorting* problem.

Fix a set of allowed steps (such as block transpositions, or block interchanges, or block reversals). Fix n. Let $a(n,k)$ be the number of permutations of length n whose distance from the identity is k.

QUESTION 2. *Is the sequence $a(n,0), a(n,1)\cdots$ log-concave for all fixed n?*

It follows, for example, from the results in [4] that the answer to this question is positive for the block interchange distance. In the book [12], elements of the sequence $d(n,k)$ are computed for $n \leq 10$ for ten different notions of distance, and they always show the log-concave property, though the proof of that property is not known for general n for all these ten notions of distance. However, it is tempting to think that log-concavity always holds, and for one general reason. The first author studied this question with co-authors Bruce Sagan and Marie-Louise Lackner in [5] for another natural notion of distance, the *Ulam distance*.

7.1 Other conjugacy classes We obtained the results in this paper were using the fact that maximal cycles form a *conjugacy class* in the symmetric group, and for this conjugacy class, values of the character table are easy to compute. This suggests the following general family of questions.

QUESTION 3. *What can be said about the product of k random, independently selected element of a conjugacy class $\mathcal{C}$ of S_n?*

QUESTION 4. *For which conjugacy classes will we be able to prove results similar to those discussed in this paper?*

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