QuickSort: Improved right-tail asymptotics for the limiting distribution, and large deviations (Extended Abstract)*

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Abstract

We refine asymptotic logarithmic upper bounds—extending from one term to three—produced by Svante Janson (2015) on the right tail of the limiting `QuickSort` distribution function F and by Fill and Hung (2018) on the right tails of the corresponding density f and of the absolute derivatives of f of each order. All our results match two-term lower bounds for the functions in question to two terms and match conjectured asymptotic expansions to three terms. Using the refined asymptotic bounds on F, we derive right-tail large deviation (LD) results for the distribution of the number of comparisons required by `QuickSort` that sharpen somewhat the two-sided LD results of McDiarmid and Hayward (1996).

1 Introduction

To set the stage, and for the reader's convenience, we repeat here relevant portions of Section 1 of Fill and Hung [2]. Let X_n denote the (random) number of comparisons when sorting n distinct numbers using the algorithm `QuickSort`. Clearly $X_0 = 0$, and for $n \geq 1$ we have the recurrence relation

$$X_n \stackrel{\mathcal{L}}{=} X_{U_n-1} + X^*_{n-U_n} + n - 1,$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law (i.e., in distribution); $X_k \stackrel{\mathcal{L}}{=} X^*_k$; the random variable U_n is uniformly distributed on $\{1, \ldots, n\}$; and $U_n, X_0, \ldots, X_{n-1}$, $X^*_0, \ldots, X^*_{n-1}$ are all independent. It is well known that

$$\mu_n := \mathbb{E} X_n = 2\,(n+1)\,H_n - 4n,$$

where H_n is the nth harmonic number $H_n := \sum_{k=1}^n k^{-1}$ and (from a simple exact expression) that

$$\operatorname{Var} X_n = (1 + o(1)) \left(7 - \frac{2\pi^2}{3}\right) n^2.$$

To study distributional asymptotics, we first center and scale X_n as follows:

$$(1.1) \qquad Z_n = \frac{X_n - \mu_n}{n}.$$

Using the Wasserstein d_2-metric, Rösler [11] proved that Z_n converges to Z weakly as $n \to \infty$. Using a martingale argument, Régnier [10] proved that the slightly renormalized $\frac{n}{n+1} Z_n$ converges to Z in L^p for every finite p, and thus in distribution; equivalently, the same conclusions hold for Z_n. The random variable Z has everywhere finite moment generating function with $\mathbb{E}Z = 0$ and $\operatorname{Var} Z = 7 - \left(2\pi^2/3\right)$. Moreover, Z satisfies the distributional identity

$$(1.2) \qquad Z \stackrel{\mathcal{L}}{=} UZ + (1-U)Z^* + g(U).$$

On the right, $Z^* \stackrel{\mathcal{L}}{=} Z$; U is uniformly distributed on $(0, 1)$; U, Z, Z^* are independent; and

$$g(u) := 2u \ln u + 2(1-u)\ln(1-u) + 1.$$

Further, the distributional identity together with the condition that $\mathbb{E}Z$ (exists and) vanishes characterizes the limiting `Quicksort` distribution; this was first shown by Rösler [11] under the additional condition that $\operatorname{Var} Z < \infty$, and later in full by Fill and Janson [3].

Fill and Janson [4] derived basic properties of the limiting `QuickSort` distribution $\mathcal{L}(Z)$. In particular, they proved that $\mathcal{L}(Z)$ has a (unique) continuous density f which is everywhere positive and infinitely differentiable.

Janson [6] studied logarithmic asymptotics in both tails for the corresponding distribution function F, and Fill and Hung [2] did the same for f and each of its derivatives. For right tails, all these results can be summarized in the following theorem. We let $\overline{F}(x) := 1 - F(x)$, and for a function $h : \mathbb{R} \to \mathbb{R}$ we write

$$(1.3) \qquad \|h\|_x := \sup_{t \geq x} |h(t)|.$$

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THEOREM 1.1. (a) *As $x \to \infty$, the limiting* `QuickSort` *density function f satisfies*

$$\exp[-x \ln x - x \ln\ln x + O(x)] \le f(x)$$
$$\le \exp[-x \ln x + O(x)]. \tag{1.4}$$

(b) *Given an integer $k \ge 0$, as $x \to \infty$ the k^{th} derivative of the limiting* `QuickSort` *distribution function F satisfies*

$$\exp[-x \ln x - (k \vee 1) x \ln\ln x + O(x)]$$
$$\le \|\overline{F}^{(k)}\|_x \le \exp[-x \ln x + O(x)]. \tag{1.5}$$

As discussed in [6, Section 1] and in [2, Remark 1.3(b)], non-rigorous arguments of Knessl and Szpankowski [7] suggest very refined asymptotics, which to three logarithmic terms assert that for each $k \ge 0$ we have

$$\overline{F}^{(k)}(x)$$
$$= \exp[-x \ln x - x \ln\ln x + (1 + \ln 2)x + o(x)] \tag{1.6}$$

as $x \to \infty$ (and hence that the same asymptotics hold for $\|\overline{F}^{(k)}\|_x$). Note that for $k = 0, 1$ these expansions match the lower bounds on f and $\overline{F}$ in Theorem 1.1 to two logarithmic terms.

In this note, we refine the upper bounds of Theorem 1.1 to match (1.6), and we are also able to improve the lower bound in (1.5) to match (1.6) to two terms. Here is our main theorem:

THEOREM 1.2. (a) *As $x \to \infty$, the limiting* `QuickSort` *density function f satisfies*

$$\exp[-x \ln x - x \ln\ln x + O(x)]$$
$$\le f(x) \tag{1.7}$$
$$\le \exp[-x \ln x - x \ln\ln x + (1 + \ln 2)x + o(x)]. \tag{1.8}$$

(b) *Given an integer $k \ge 0$, as $x \to \infty$ the k^{th} derivative of the limiting* `QuickSort` *distribution function F satisfies*

$$\exp[-x \ln x - x \ln\ln x + O(x)]$$
$$\le \|\overline{F}^{(k)}\|_x \tag{1.9}$$
$$\le \exp[-x \ln x - x \ln\ln x + (1 + \ln 2)x + o(x)]. \tag{1.10}$$

We next argue that to prove our main Theorem 1.2 we need only establish the following equivalent version of the upper bound (1.10) in the case $k = 0$:

PROPOSITION 1.1. *As $x \to \infty$, the limiting* `QuickSort` *distribution function F satisfies*

$$\overline{F}(x) \le \exp[-x \ln x - x \ln\ln x + (1 + \ln 2)x + o(x)].$$

Here is the argument proving Theorem 1.2 from Proposition 1.1. We already know (1.7) and, for $k = 0, 1$, the lower bounds (1.9) from Theorem 1.1. Next, the upper bounds in (1.10) for general values of k follow inductively from Proposition 1.1 using Proposition 6.1 of [2] with $r(x) \equiv x$, according to which

$$\limsup_{x \to \infty} x^{-1} \left(\ln \|\overline{F}^{(k+1)}\|_x - \ln \|\overline{F}^{(k)}\|_x \right) \le 0.$$

The upper bound (1.8) follows immediately from (1.10) with $k = 1$. Finally, the lower bounds (1.9) for fixed $k \ge 2$ follow as in Section 6.1 of [2], using the improved upper bound of Proposition 1.1. To spell this out, as in [2, Section 6.1] the Landau–Kolmogorov inequality (see [2, Lemma 2.1]) implies

$$\|\overline{F}^{(k)}\|_x \ge c_{k,1}^{-k} \, \|\overline{F}\|_x^{-(k-1)} \, [f(x)]^k,$$

where it is already known from [2] [consult Lemma 2.1 therein and (1.4) above] that

$$c_{k,1} \le e^2 k / 4$$

and

$$f(x) \ge \exp\left[-x \ln x - x \ln\ln x + O(x)\right].$$

Moreover, now we have from Proposition 1.1 the improved upper bound (1.10) on $\|\overline{F}\|_x$. Plugging in these bounds, we obtain (1.9).

REMARK 1.1. It follows immediately from our improved upper bound (1.10) that the first conjecture in [2, Remark 6.2] that

$$\rho_k := \lim_{x \to \infty} x^{-1} \left(x \ln x + \ln \|\overline{F}^{(k)}\|_x \right) = -\infty$$

is true. However, because the third term in (1.10) is not matched in (1.9), the second conjecture in that remark, namely, that

$$\lim_{x \to \infty} x^{-1} \left(\ln \|\overline{F}^{(k+1)}\|_x - \ln \|\overline{F}^{(k)}\|_x \right) = 0,$$

remains unproved.

We prove Proposition 1.1 in Section 2. In Section 3 we use our refined asymptotic bounds on F to derive right-tail large deviation results for the distribution of the number of comparisons required by `QuickSort` that sharpen somewhat the two-sided large-deviation results of McDiarmid and Hayward [8].

We conclude this section with an open problem concerning *left*-tail behavior.

Open Problem. With $\underline{F}(x) := F(-x)$ can the *lower* bounds as $x \to \infty$ in the left-tail results

(1.11)
$$\exp\left[-e^{\gamma x + \ln\ln x + O(1)}\right] \le f(-x) \le \exp\left[-e^{\gamma x + O(1)}\right],$$
(1.12)
$$\exp\left[-e^{\gamma x + \ln\ln x + O(1)}\right] \le \|\underline{F}^{(k)}\|_x \le \exp\left[-e^{\gamma x + O(1)}\right]$$

of [6] and [2] be improved to match the asymptotics

$$\underline{F}^{(k)}(x) = \exp\left[-e^{\gamma x + O(1)}\right]$$

suggested by Knessl and Szpankowski [7], where $\gamma := (2 - \frac{1}{\ln 2})^{-1}$?

2 Proof of the main Proposition 1.1

Let ψ denote the moment generating function of Z. It was shown by Rösler [11] that ψ is everywhere finite. As we next show, Proposition 1.1 follows easily by (i) combining the Chernoff bound

$$\overline{F}(x) = \mathbb{P}(Z \ge x) \le e^{-tx}\psi(t),$$

choosing $t = \ln\left(\frac{1}{2+\epsilon} x \ln x\right)$, with the following lemma; and (ii) letting $\epsilon \downarrow 0$.

Lemma 2.1. *For every $\epsilon > 0$ there exists $a \equiv a(\epsilon) \ge 0$ such that the moment generating function ψ of Z satisfies*

(2.13)
$$\psi(t) \le \exp[(2+\epsilon)t^{-1}e^t + at]$$

for every $t > 0$.

Granting Lemma 2.1 for the moment, let us follow the outline above to establish Proposition 1.1.

Proof of Proposition 1.1. Choosing $t \equiv t(x) = \ln\left(\frac{1}{2+\epsilon} x \ln x\right)$, for each $\epsilon > 0$ we find

$$\overline{F}(x) \le e^{-tx}\psi(t) \le \exp[-tx + (2+\epsilon)t^{-1}e^t + at].$$

But for fixed $\epsilon > 0$ we have

$$\begin{aligned} e^t &= (2+\epsilon)^{-1} x \ln x, \\ t &= \ln x + \ln\ln x - \ln(2+\epsilon), \\ t^{-1} &= (1 + o(1))/\ln x, \end{aligned}$$

so

$$\begin{aligned} &-tx + (2+\epsilon)t^{-1}e^t + at \\ &\quad = -x\ln x - x\ln\ln x + [\ln(2+\epsilon)]x + (1+o(1))x. \end{aligned}$$

We conclude that

$$\limsup_{x\to\infty} x^{-1}[\ln \overline{F}(x) - (-x\ln x - x\ln\ln x)] \le 1 + \ln(2+\epsilon).$$

Let $\epsilon \downarrow 0$ to complete the proof.

Inspired by the proof of Lemma 6.1 in [6], which states that there exists $a \ge 0$ such that

$$\psi(t) \le \exp(e^t + at) \quad \text{for every } t \ge 0,$$

we now establish the right-tail upper bound for ψ given by Lemma 2.1.

Proof of Lemma 2.1. Since $\psi(0) = 1$, it follows by continuity that there exists $t_1 > 0$ such that (2.13) holds [for every $\epsilon > 0$ and any choice of $a(\epsilon) \ge 0$] for $t \in (0, t_1]$. Also, we can choose $\alpha \ge 0$ such that $\psi(t) \le \exp(\alpha t)$ for $t \in [0, 1]$.

As the proof unfolds, we will see how to choose three parameters

$$\begin{aligned} a \equiv a(\epsilon) &\ge \alpha \text{ sufficiently large,} \\ t_2 \equiv t_2(\epsilon) &\ge t_1 \text{ sufficiently large,} \\ \delta \equiv \delta(\epsilon) &> 0 \text{ sufficiently small,} \end{aligned}$$

to effect a proof of (2.13).

However t_2 is chosen, we can certainly choose $a \ge \alpha$ so that (2.13) holds for all $t \in [t_1, t_2]$ and hence for $t \in (0, t_2]$. Assume (for the sake of contradiction) that (2.13) fails for some $t > 0$, and let

$$T \equiv T(\epsilon) := \inf\{t > 0 : (2.13) \text{ fails}\}.$$

Then $T > t_2$, and, by continuity,

(2.14)
$$\psi(T) = \exp[(2+\epsilon)T^{-1}e^T + aT].$$

We will make frequent use of (2.14) and also use the identity

(2.15)
$$\psi(t) = 2\int_{u=0}^{1/2} \psi(ut)\psi((1-u)t)e^{tg(u)}\,du, \quad t \in \mathbb{R},$$

[which follows from (1.2) and symmetry] with $t = T$, together with the simple bound

$$g(u) = 2u\ln u + 2(1-u)\ln(1-u) + 1 \le 1,$$

to obtain a contradiction by showing that RHS(2.15) for $t = T$ is bounded above by $(1 - \frac{1}{6}\epsilon)\psi(T)$.

For this, break the integral $2\int_{u=0}^{1/2}$ in (2.15) into "small", "medium", and "large" ranges of u:

$$\text{RHS}(2.15) = S + M + L = 2\int_{u=0}^{\delta/T} + 2\int_{u=\delta/T}^{1/T} + 2\int_{u=1/T}^{1/2}$$

with $\delta \equiv \delta(\epsilon) < 1$. To complete the proof, we will show

$$L \le \tfrac{1}{12}\epsilon\,\psi(T), \quad S \le (1-\tfrac{1}{3}\epsilon)\psi(T), \quad M \le \tfrac{1}{12}\epsilon\,\psi(T)$$

when a, t_2, δ are suitably chosen.

We start with L:

$$\begin{aligned}L &\le 2e^T \int_{u=1/T}^{1/2} \exp\left[(2+\epsilon)\frac{e^{uT}}{uT} + auT \right.\\ &\qquad\qquad \left. +(2+\epsilon)\frac{e^{(1-u)T}}{(1-u)T} + a(1-u)T\right] du\\ &\le 2e^{T+aT}\int_{u=1/T}^{1/2} \exp\left[(2+\epsilon)\left(e^{T/2} + \frac{e^{(1-u)T}}{(1-u)T}\right)\right] du.\end{aligned}$$

We now apply the bound

$$\frac{e^{-uT}}{1-u} \le 2e^{-1} < 0.8$$

to obtain

$$\begin{aligned}L &\le 2\exp[T + aT + (2+\epsilon)e^{T/2}] \times \tfrac{1}{2}\\ &\qquad \times \exp[(2+\epsilon)(0.8)T^{-1}e^T]\\ &= \exp[T + (2+\epsilon)e^{T/2}]\exp[-(0.2)(2+\epsilon)T^{-1}e^T]\\ &\qquad \times \exp[(2+\epsilon)T^{-1}e^T + aT]\\ &= \exp[T + (2+\epsilon)e^{T/2}]\exp[-(0.2)(2+\epsilon)T^{-1}e^T]\psi(T)\\ &\le \exp[(T^2e^{-T} + (2+\epsilon)Te^{-T/2} - 0.4)T^{-1}e^T]\psi(T).\end{aligned}$$

Recall that $T > t_2$. Provided t_2 is chosen sufficiently large, we clearly have

$$L \le \exp[-(0.3)T^{-1}e^T]\psi(T) \le \tfrac{1}{12}\epsilon\,\psi(T).$$

For the contributions S and M, we begin by observing that the first factor $\psi(uT)$ in the integrand can be bounded above by $\exp(\alpha uT) \le \exp(auT)$ and the second factor by (2.13):

$$\psi((1-u)T) \le \exp\left[(2+\epsilon)\frac{e^{(1-u)T}}{(1-u)T} + a(1-u)T\right].$$

For S, this gives the bound

$$S \le 2e^{T+aT}\int_{u=0}^{\delta/T} \exp\left[(2+\epsilon)\frac{e^{(1-u)T}}{(1-u)T}\right] du. \tag{2.16}$$

Observe that the function

$$h(v) := v^{-1}e^v$$

satisfies

$$\begin{aligned}h'(v) &= v^{-2}(v-1)e^v,\\ h''(v) &= v^{-3}(v^2 - 2v + 2)e^v,\\ h'''(v) &= v^{-4}(v^3 - 3v^2 + 6v - 6)e^v \ge 0,\\ h(v) &\le h(T) - (T-v)h'(T) + \tfrac{1}{2}(T-v)^2h''(T)\end{aligned}$$

where the inequalities hold for $T \ge v \ge 1.6$. Provided $t_2 - \delta \ge 1.6$, we may then conclude

$$\begin{aligned}S &\le 2e^{T+aT}\\ &\quad\times \int_{u=0}^{\delta/T} \exp\left\{(2+\epsilon)[h(T) - uTh'(T) + \tfrac{1}{2}u^2T^2h''(T)]\right\} du\\ &\le 2e^{T+aT}\\ &\quad\times \int_{u=0}^{\delta/T} \exp\left\{(2+\epsilon)[h(T) - uTh'(T) + \tfrac{1}{2}\delta uTh''(T)]\right\} du\\ &= 2e^{T+aT}\\ &\qquad\times \int_{u=0}^{\delta/T} \exp\left\{(2+\epsilon)\frac{e^T}{T}[1 - (T-1)u\right.\\ &\qquad\qquad\qquad \left. + \tfrac{1}{2}\delta T^{-1}((T-1)^2+1)u]\right\} du\\ &= \psi(T)\\ &\qquad\times 2e^T\int_{u=0}^{\delta/T} \exp\left\{-(2+\epsilon)\frac{e^T}{T}\right.\\ &\qquad\qquad\qquad \left. \times [(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2+1)]u\right\} du\\ &\le \psi(T)\\ &\qquad\times 2e^T\int_{u=0}^{\infty} \exp\left\{-(2+\epsilon)\frac{e^T}{T}\right.\\ &\qquad\qquad\qquad \left. \times [(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2+1)]u\right\} du\\ &= \psi(T)\\ &\quad\times 2e^T\left\{(2+\epsilon)\frac{e^T}{T}[(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2+1)]\right\}^{-1}\\ &= \psi(T) \times \frac{2}{2+\epsilon}[(1-T^{-1}) - \tfrac{1}{2}\delta T^{-2}((T-1)^2+1)]^{-1}.\end{aligned}$$

Without loss of generality $\epsilon < 1$. Provided we choose (independently!) t_2 sufficiently large and δ sufficiently small (in relation to ϵ), we can ensure

$$S \le (1 - \tfrac{1}{3}\epsilon)\psi(T).$$

The inequality analogous to (2.16) for M is

$$M \le 2e^{T+aT}\int_{u=\delta/T}^{1/T} \exp\left[(2+\epsilon)\frac{e^{(1-u)T}}{(1-u)T}\right] du.$$

For M, our bound on $e^{(1-u)T}/[(1-u)T]$ is simpler than for S:

$$\frac{e^{(1-u)T}}{(1-u)T} \le e^{-\delta}\frac{e^T}{T-1}.$$

The resulting bound on the integrand is then constant in u, and the length of the interval of integration is bounded above by $1/T$. This leads easily to

$$\begin{aligned} M &\le 2T^{-1} \times \exp\Big[-\Big\{1 - e^{-\delta}\frac{T}{T-1} - \frac{1}{2}T^2 e^{-T}\Big\} \\ &\qquad\qquad \times (2+\epsilon)\frac{e^T}{T}\Big]\,\psi(T) \\ &\le 2T^{-1}\psi(T) \\ &\le \tfrac{1}{12}\epsilon\,\psi(T), \end{aligned}$$

where the second inequality holds provided t_2 is chosen sufficiently large relative to δ that the expression in $\{\cdot\}$ is nonnegative, and the third inequality holds provided $t_2 \ge 24/\epsilon$.

This completes the proof of Lemma 2.1.

3 Large deviations for QuickSort

McDiarmid and Hayward [8] study large deviations for the variant of `QuickSort` in which the pivot (that is, the initial partitioning key) is chosen as the median of $2t+1$ keys chosen uniformly at random without replacement from among all the keys. The case $t = 0$ is the classical `QuickSort` algorithm of our ongoing focus. Restated equivalently in terms of the random variable Z_n in (1.1) (as straightforward calculation reveals), the following is their main theorem for classical `QuickSort`.

THEOREM 3.1. ([8]) *Let x_n satisfy*

$$\frac{\mu_n}{n \ln n} < x_n \le \frac{\mu_n}{n}. \tag{3.17}$$

Then as $n \to \infty$ we have

$$\mathbb{P}(|Z_n| > x_n) = \exp\{-x_n[\ln x_n + O(\log\log\log n)]\}. \tag{3.18}$$

Observe that (3.17) is roughly equivalent to the condition that x_n lie between 2 and $2\ln n$, and rather trivially the range can be extended to $1 < x_n \le \mu_n/n$. But notice also that if $x_n = (\ln\ln n)^{c_n}$ with c_n nondecreasing (say), then (3.18) provides a nontrivial upper bound on $\mathbb{P}(|Z_n| > x_n)$ if and only if $c_n \to \infty$.

McDiarmid and Hayward require a fairly involved proof utilizing primarily the method of bounded differences pioneered by McDiarmid [9] to establish the $\le$ half of (3.18). The $\ge$ half is proven by establishing (by means of another substantial argument) the right-tail lower bound

$$\mathbb{P}(Z_n > x_n) \ge \exp\{-x_n[\ln x_n + O(\log\log\log n)]\}, \tag{3.19}$$

again assuming (3.17) (see [8, Lemma 2.9]). It follows from (3.18)–(3.19) that we have the right-tail large deviation result that

$$\mathbb{P}(Z_n > x_n) = \exp\{-x_n[\ln x_n + O(\log\log\log n)]\}. \tag{3.20}$$

The main point of this section [see Theorem 3.2(b)–(d)] is to note that (3.20) can be refined, for deviations not allowed to be quite as large as those permitted by Theorem 3.1, rather effortlessly by combining the case $k = 0$ of Theorem 1.2(b) with the following bound on Kolmogorov–Smirnov distance between the distributions of Z_n and Z (see [5, Section 5]):

LEMMA 3.1. ([5]) *We have*

$$\sup_x |\mathbb{P}(Z_n > x) - \mathbb{P}(Z > x)| \le \exp\left[-\tfrac{1}{2}\ln n + O\left((\log n)^{1/2}\right)\right].$$

We state next our right-tail large-deviations theorem for `QuickSort`. With the additional indicated restriction on the growth of x_n (which allows for x_n nearly as large as $\frac{1}{2}\frac{\ln n}{\ln\ln n}$), parts (b)–(c) strictly refine (3.19) and the asymptotic upper bound on $\mathbb{P}(Z_n > x_n)$ implied by (3.20). The left-hand endpoint of the interval I_n in Theorem 3.2 is chosen as $c > 1$ simply to ensure that $\sup\{-\ln\ln x : x \in I_n\} < \infty$.

THEOREM 3.2. *Let (ω_n) be any sequence diverging to $+\infty$ as $n \to \infty$ and let $c > 1$. For integer $n \ge 3$, consider the interval $I_n := \left[c, \frac{1}{2}\frac{\ln n}{\ln\ln n}\left(1 - \frac{\omega_n}{\ln\ln n}\right)\right]$.*

(a) *Uniformly for $x \in I_n$ we have*

$$\mathbb{P}(Z_n > x) = (1 + o(1))\mathbb{P}(Z > x) \quad \textit{as } n \to \infty.$$

(b) *If $x_n \in I_n$ for all large n, then*

$$\mathbb{P}(Z_n > x_n) \ge \exp[-x_n \ln x_n - x_n \ln\ln x_n + O(x_n)].$$

(c) *If $x_n \in I_n$ for all large n and $x_n \to \infty$, then*

$$\mathbb{P}(Z_n > x_n) \le \exp[-x_n \ln x_n - x_n \ln\ln x_n + (1 + \ln 2)x_n + o(x_n)].$$

(d) *If $x_n \in I_n$ for all large n, then*

$$\mathbb{P}(Z_n > x_n) = \exp[-x_n \ln x_n - x_n \ln\ln x_n + O(x_n)].$$

Proof. Parts (b)–(c) follow immediately from part (a) and Theorem 1.2(b), and part (d) by combining parts (b)–(c). So we need only prove part (a), for which by Lemma 3.1 it is sufficient to prove that

$$\exp\left[-\tfrac{1}{2}\ln n + O\left((\log n)^{1/2}\right)\right] \le o(\mathbb{P}(Z > x_n))$$

with $x_n \equiv \frac{1}{2}\frac{\ln n}{\ln\ln n}\left(1 - \frac{\omega_n}{\ln\ln n}\right)$; this assertion decreases in strength as the choice of ω_n is increased, so we may assume that $\omega_n = o(\log\log n)$. Since, by Theorem 1.2(b), we have

$$\mathbb{P}(Z > x_n) \ge \exp[-x_n \ln x_n - x_n \ln\ln x_n + O(x_n)],$$

it suffices to show that for any constant $C < \infty$ we have

$$-\tfrac{1}{2}\ln n + C(\ln n)^{1/2} + x_n \ln x_n + x_n \ln\ln x_n + Cx_n \to -\infty.$$

But, writing L for ln and L_k for the kth iterate of L, and abbreviating $\alpha_n := 1 - \frac{\omega_n}{L_2 n}$, this follows from the observation that, for n large,

$$\begin{aligned}
&x_n(Lx_n + L_2x_n + C) \\
&= \frac{1}{2}\frac{Ln}{L_2n}\alpha_n[(L_2n - L_3n - L2 + L\alpha_n) \\
&\qquad + L(L_2n - L_3n - L2 + L\alpha_n) + C] \\
&= \frac{1}{2}\frac{Ln}{L_2n}\alpha_n\Big[L_2n + C - L2 + L\alpha_n \\
&\qquad + L\Big(1 - \frac{L_3n + L2 - L\alpha_n}{L_2n}\Big)\Big] \\
&= \frac{1}{2}\frac{Ln}{L_2n}\alpha_n\left[L_2n + C - L2 + L\alpha_n - (1+o(1))\frac{L_3n}{L_2n}\right] \\
&= \frac{1}{2}\frac{Ln}{L_2n}\alpha_n\left[L_2n + C - L2 + o(1)\right] \\
&= \left(\frac{1}{2}Ln\right)\alpha_n\left[1 + \frac{C - L2 + o(1)}{L_2n}\right] \\
&= \frac{1}{2}Ln - (1+o(1))\omega_n\frac{Ln}{2L_2n}.
\end{aligned}$$

For completeness we next present a left-tail analogue of Theorem 3.2 [but, for brevity, only parts (b)–(c) thereof]. Theorem 3.3 follows in similar fashion using the case $k = 0$ of (1.12) in place of Theorem 1.2(b). No such left-tail large-deviation result is found in [8]. Recall $\gamma := (2 - \frac{1}{\ln 2})^{-1}$ and the notation L_k used in the proof of Theorem 3.2.

THEOREM 3.3. *If* $1 < x_n \le \gamma^{-1}(L_2n - L_4n - \omega_n)$ *with* $\omega_n \to \infty$, *then*

$$\begin{aligned}
\exp\left[-e^{\gamma x_n + L_2x_n + O(1)}\right] &\le \mathbb{P}(Z_n \le -x_n) \\
&\le \left[-e^{\gamma x_n + O(1)}\right].
\end{aligned}$$

REMARK 3.1. The upper bound in Theorem 3.3 requires only the weaker restriction

$$-M \le x_n \le \gamma^{-1}(L_2n - \omega_n)$$

with $M < \infty$ and $\omega_n \to \infty$.

REMARK 3.2. If we let $N := n+1$ and study the slight modification $\widehat{Z}_n := (X_n - \mu_n)/N = [n/(n+1)]Z_n$ instead of (1.1), then large deviation upper bounds based on tail estimates of the limiting F have broader applicability and are easier to derive, too. The reason is that (i) both Proposition 1.1 and the upper bound for $k = 0$ in (1.12) have been derived by establishing an upper bound on the limiting moment generating function ψ and using a Chernoff bound, and (ii) according to [5, Theorem 7.1], ψ majorizes the moment generating function $\widehat{\psi}_n$ of $\widehat{Z}_n$ for every n. It follows immediately that $\mathbb{P}(\widehat{Z}_n > x)$ (respectively, $\mathbb{P}(\widehat{Z}_n \le -x)$) is bounded above uniformly in n by

$$\exp[-x\ln x - x\ln\ln x + (1+\ln 2)x + o(x)]$$

(resp., by $\exp\left[-e^{\gamma x + O(1)}\right]$) as $x \to \infty$; there is *no restriction at all* on how large x can be in terms of n.

Here are examples of *very* large values of x for which the tail probabilities are nonzero and the aforementioned bounds still match logarithmic asymptotics to lead order of magnitude, albeit not to lead-order term. Let lg denote binary log. The largest possible value of X_n is $\binom{n}{2}$ (corresponding to any binary search tree which is a path), which occurs with probability $2^{n-1}/n!$. The smallest possible value (supposing, for simplicity, that $n = 2^k - 1$ for integer k) is $(k-2)2^k + 2 = N(\lg N - 2) + 2$ (corresponding to the perfect tree, in the terminology of [1, Section 3]); according to [1, Proposition 4.1], this value occurs with probability $\exp[-s(1)N + s(N+1)]$, where

$$s(\nu) := \sum_{j=1}^{\infty} 2^{-j}\ln(2^j\nu - 1).$$

Correspondingly, the largest possible value of $\widehat{Z}_n$ is

$$\lambda_n := \tfrac{n(n+7)}{2(n+1)} - 2H_n = (1+o(1))\tfrac{1}{2}N,$$

and the smallest is $-\sigma_n$, with

$$\sigma_n := -2H_N - \lg N - 2 = (2 - \tfrac{1}{\ln 2})\ln N + O(1).$$

The bound on $\mathbb{P}(\widehat{Z}_n > \lambda_n)$ is in fact also (by the same proof) a bound on the larger probability $\mathbb{P}(\widehat{Z}_n \ge \lambda_n)$, and equals

$$\exp\left\{-\tfrac{1}{2}N[\ln N + \ln\ln N - (2\ln 2 + 1) + o(1)]\right\},$$

whereas (using Stirling's formula) the truth is

$$\mathbb{P}(\widehat{Z}_n \ge \lambda_n) = \exp[-N\ln N + (1+\ln 2)N + O(\log N)].$$

The bound on $\mathbb{P}(\widehat{Z}_n \le -\sigma_n)$ equals

$$\exp\left[-e^{\ln N + O(1)}\right] = \exp[-\Omega(N)],$$

whereas (by [1, Proposition 4.1 and Table 1]) the truth is

$$\mathbb{P}(\widehat{Z}_n \le -\sigma_n) = \exp[-s(1)N + O(\log N)]$$

and (rounded to seven decimal places) $s(1) = 0.9457553$.

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