The cover time of a biased random walk on $G_{n,p}$

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Abstract

We analyse the cover time of a biased random walk on the random graph $G_{n,p}$. The walk is biased towards visiting vertices of low degree and this makes the cover time less than in the unbiased case.

1 Introduction

Let $G = (V, E)$ be a connected graph with n vertices and m edges. For $v \in V$, let C_v be the expected time for a simple random walk $\mathcal{W}_v$ on G starting at v, to visit every vertex of G. The *vertex cover time* $C(G)$ of G is defined as $C(G) = \max_{v\in V} C_v$. The vertex cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [3] that $C(G) \leq 2m(n-1)$. It was shown by Feige [9], [10], that for any connected graph G, the cover time satisfies $(1-o(1))n \log n \leq C(G) \leq (1+o(1))\frac{4}{27}n^3$. The asymptotic lower bound is obtained by the complete graph K_n. The asymptotic upper bound is obtained by the *lollipop* graph, which consists of a path of length $n/3$ joined to a clique of size $2n/3$.

The facts that the cover time of a simple random walk can be as large as $\Theta(n^3)$ for some classes of graphs, and that it is never $o(n \log n)$ for any graph encourage a study of modified random walks whose performance may be better (in order of magnitude), either in general or for specific classes of graphs. If we stick with reversible random walks then the lower bound cannot be improved. Recently, it was shown by David and Feige [7] that the upper bound can be reduced to $\Theta(n^2)$, which is best possible for reversible walks.

The properties of weighted random walk on an undirected graph are as follows, for more details see [2]. Each undirected edge $e = \{u, v\}$ has a positive weight $w_e = w_{v,u} = w_{u,v}$. The transition probability of the associated random walk at v is $p(v, u) = w_{v,u}/w_v$, where $w_v = \sum_{u\in N(v)} w_{v,u}$, and $N(v)$ are the neighbours of v in G. The stationary distribution of the walk at vertex v is $\pi(v) = w_v/w$, where $w = \sum_{e\in E} w_e$, and each edge is counted twice, once at each vertex. A Markov chain is reversible if $\pi(u)p(u, v) = \pi(v)p(v, u)$. Weighted walks are always reversible. For any edge $e = \{u, v\}$, $\pi(u)p(u, v) = w_e/w = \pi(v)p(v, u)$. As an example, for a simple random walk, $w_{u,v} = 1$, $w_v = d(v)$ the degree of vertex $v \in V$, and $p(u, v) = 1/d(v)$; the total weight $w = 2m$, and so $\pi(v) = d(v)/2m$.

An $O(n^2 \log n)$ upper bound on cover time for any connected n vertex graph G, was obtained by Ikeda, Kubo, Okumoto, and Yamashita [11] by using a weight $w_{u,v} = 1/\sqrt{d(u)d(v)}$ for edge $e = \{u, v\}$. The fact that the above edge weight is multiplicative makes the walk hard to analyse. The use of a simpler but related weight of $w(u, v) = 1/\min(d(u), d(v)) = \max(1/d(u), 1/d(v))$ was studied in [1]. Using this simplified weight David and Feige [7] proved an $O(n^2)$ upper bound on cover time for *any* connected n vertex graph G. As the cover time of paths and cycles by weighted walks is $\Theta(n^2)$ this result is best possible. Instead of choosing a uniform random neighbor, the walks of [1],[11] are biased towards lower degree vertices. In this way the walk tends to have a smaller cover time than the unbiased walk.

In this paper we study the cover time of the biased random walk first discussed in [1] on the random graph $G_{n,p}$. We let $\overline{C}(G)$ denote the cover time of the biased random walk and prove the following theorem.

THEOREM 1.1. *Let $G \sim G_{n,p}$ where $np = c \log n = \log n + \omega$ and where $\omega = (c-1)\log n \to \infty$. Then with high probability, the walk using edge weights $w(u, v) = \frac{1}{\min\{d(u),d(v)\}}$ for each edge $\{u, v\}$ has cover time $\overline{C}(G) \approx n \log n$.*

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This is to be compared with the performance of a simple random walk whose cover time is given by the following theorem (see [4]).

THEOREM 1.2. *Let $G \sim G_{n,p}$ where $np = c\log n = \log n + \omega$ where $\omega = (c-1)\log n \to \infty$. Then w.h.p. a simple random walk on G has cover time $C(G) \approx c\log\left(\frac{c}{c-1}\right) n\log n$.*

In this paper we use the following notation concerning the edge weights of a graph $G = (V, E)$:

1. We let $\psi(v,w) = \frac{1}{\min\{d(v),d(w)\}}$ for $\{v,w\} \in E$.
2. We let $\Psi(v) = \sum_{w\in N(v)} \psi(v,w)$.

The random walk $\mathcal{W}_u = (X_0 = u \in V, X_1, \dots, X_t, \dots)$ is then defined by

$$\mathsf{Pr}[X_{t+1} = w \mid X_t = v] = \begin{cases} \frac{\psi(v,w)}{\Psi(v)} & w \in N(v) \\ 0 & w \notin N(v) \end{cases}. \tag{1.1}$$

In Section 2 we state the central lemma for the proof of Theorem 1.1. The "first visit time lemma" bounds the probability that a vertex has not been visited in t steps after a suitably defined mixing time. For a proof of this lemma, in the stated form, see [5]. In Section 3, we describe relevant properties of $G_{n,p}$ that hold with high probability, and compute quantities necessary for applying the first visit lemma under these conditions. In Section 4 we prove Theorem 1.1.

For probabilistic inequalities we use the Chernoff bounds on the binomial $B(n,p)$:

$$\mathsf{Pr}[Bin(n,p) \le (1-\varepsilon)np] \le e^{-\varepsilon^2 np/2} \text{ for } 0 \le \varepsilon \le 1. \tag{1.2}$$

$$\mathsf{Pr}[Bin(n,p) \ge (1+\varepsilon)np] \le e^{-\varepsilon^2 np/3} \text{ for } 0 \le \varepsilon \le 1. \tag{1.3}$$

$$\mathsf{Pr}[Bin(n,p) \ge \alpha np] \le \left(\frac{e}{\alpha}\right)^{\alpha np} \text{ for } 0 < \alpha. \tag{1.4}$$

We sometimes write $A_n \approx B_n$ (resp. $A_n \lesssim B_n$) in place of $A_n = (1+o(1))B_n$ (resp. $A_n \le (1+o(1))B_n$) as $n \to \infty$.

Some further notation:

- For $S \subseteq V$ we let $N(S) = N_G(S) = \{w \notin S : \{v,w\} \in E\}$ be the disjoint neighborhood of S. Let $d(S) = \sum_{v\in S} d(v)$.
- We abbreviate $N(\{v\})$ to $N(v) = N_G(v)$ for $v \in V$. Thus, $d(v) = |N(v)|$.
- For $v \in V$ and positive integer k we let $N_k(v)$ denote the set of vertices within distance at most k of v.

2 The first visit time lemma

Let $G = (V, E)$ be a fixed graph, and let $u \in V$ be arbitrary. Let $\mathcal{W}_u$ denote the modified random walk defined in (1.1) starting with $X_0 = u$. The walk defines a reversible Markov chain with state space V. Let P be the matrix of transition probabilities, and π_v the stationary distribution of P. As previously mentioned, for $v \in V$,

$$\pi_v = \frac{\Psi(v)}{\sum_{u\in V}\Psi(u)}. \tag{2.5}$$

Considering the walk $\mathcal{W}_v$, starting at v, let $r_t = \mathsf{Pr}[\mathcal{W}_v(t) = v]$ be the probability that this walk returns to v at step $t \ge 0$, and let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate r_t. Our definition of return includes $r_0 = 1$. For $R(z)$ and given T let

$$R(T,z) = \sum_{j=0}^{T-1} r_j z^j.$$

We choose a value of T given by

$$T = L\log n, \tag{2.6}$$

and for this value of T let $R_v = R(T,1)$. In Lemma 3.2 we put a lower bound on the constant L which is sufficient to imply that T is the mixing time of $\mathcal{W}_u$, in a well-defined sense.

The following first visit time lemma bounds the probability a vertex has not been visited in time $T, T+1, \dots t$.

LEMMA 2.1. *[The first visit time lemma [5]]*
Let G be a graph satisfying the following conditions

(i) For all $t \ge T$, $\max_{u,x\in V} |P_u^{(t)}(x) - \pi_x| \le n^{-3}$

(ii) For some (small) constant $\theta > 0$ and some (large) constant $K > 0$,

$$\min_{|z|\le 1+\frac{1}{KT}} |R(T,z)| \ge \theta$$

(iii) $T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$

Let $\mathcal{A}_v(t)$ be the event that the random walk $\mathcal{W}_u$ on graph G does not visit vertex v at steps $T, T+1, \dots t$. Then, uniformly in v,

$$\mathsf{Pr}[\mathcal{A}_v(t)] = \frac{(1+O(T\pi_v))}{(1+p_v)^t} + O(T^2\pi_v e^{-t/KT})$$

where p_v is given by the following formula, with $R_v = R_v(T)$:

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}.$$

In our applications, $T = O(\log n), \pi_v = O(1/n)$ and $t = \Theta(n \log n)$. In which case we can write

$$\Pr[\mathcal{A}_v(t)] \approx e^{-t\pi_v/R_v}. \tag{2.7}$$

3 Properties of typical graphs $G \sim G_{n,p}, p = \frac{c \log n}{n}$

The following lemma defines a *typical* graph and shows that with high probability a graph $G \sim G_{n,p}, p = \frac{c \log n}{n}$ is typical. In Lemma 3.2 we show that T as given in Equation (2.6) is the mixing time for a typical graph, and in Lemma 3.3, we bound R_v for a typical graph.

LEMMA 3.1. *Let $\varepsilon > 0$ be an arbitrary small constant. Consider $G \sim G_{n,p}, p = \frac{c \log n}{n}$ with $c \geq 1$ and $\omega = (c-1)\log n \to \infty$. The graph G is "typical" if all of the following conditions are satisfied:*

1. *G is connected.*
2. *There are at most $n^{1-\varepsilon^2 c/4}$ vertices with degree less than $(1-\varepsilon)np$.*
3. *There are at most $n^{1-\varepsilon^2 c/4}$ vertices with degree more than $(1+\varepsilon)np$.*
4. *No vertex has degree more than $4np$.*
5. *Let V_k denote the vertices of degree k. Then $|V_k| \leq (3 \log n)^{k+1}$ for $k \leq \Lambda = \log \log n$.*
6. *Let A be the set of vertices with degree less than $np/100$. Then (i) $|A| < n^{17/12-c}$ and (ii) no vertex of A is within distance Λ of a cycle of size less than Λ and (iii) for all $u, v \in A$, the distance $dist(u,v) > \Lambda$.*
7. *Suppose that $np \leq n^\alpha$ for some constant $0 < \alpha < 1/2$. Then G contains no subgraphs H with number of vertices $v_H \leq 1/(2\alpha)$ and with number of edges $e_H \geq v_H + 1$.*
8. *For all $S \subseteq V = [n]$ such that $1/p \leq |S| \leq \frac{n}{2}$, $e(S, \overline{S}) \geq \frac{1}{2}|S|(n - |S|)p$ where $e(S, \overline{S})$ is the number of edges between the set S and its complement $\overline{S}$.*
9. *For all $S \subseteq V$ such that $|S| < 1/p$, $e(S,S) < |S|np/1000$, where $e(S,S)$ is the number of edges within the set S.*

With high probability, G is typical.

Proof (1) This is a standard result and follows from Erdős and Rényi [8].

(2) By the Chernoff bound (1.2),

$$\begin{aligned}
&\Pr[d(v) < (1-\varepsilon)c \log n] \\
&\quad \leq \Pr\left[Bin(n-1,p) \leq (1 - \varepsilon + o(n^{-1}))(n-1)p\right] \\
&\qquad \leq \exp\left(-\frac{\varepsilon^2 np}{2 + o(1)}\right) = n^{-\varepsilon^2 c/3}.
\end{aligned}$$

Let X be the number of vertices of degree less than $(1-\varepsilon)np$. The Markov inequality implies that

$$\begin{aligned}
&\Pr\left[X \geq n^{1-\varepsilon^2 c/4}\right] \\
&\quad < \Pr\left[X \geq n^{\varepsilon^2 c/12} \, \mathsf{E}[X]\right] < n^{-\varepsilon^2 c/12} = o(1).
\end{aligned} \tag{3.8}$$

(3) By the Chernoff bound (1.3),

$$\begin{aligned}
&\Pr[d(v) > (1+\varepsilon)np] \\
&\quad \leq \Pr[Bin(n,p) \geq (1+\varepsilon)np] \leq \exp\left(-\frac{\varepsilon^2 np}{3}\right) \\
&\qquad = n^{-\varepsilon^2 c/3}.
\end{aligned} \tag{3.9}$$

Let X be the number of vertices of degree greater than $(1+\varepsilon)np$. The Markov inequality implies that

$$\Pr\left[X \geq n^{1-\varepsilon^2 c/4}\right] < \Pr\left[X \geq n^{\varepsilon^2 c/12} \, \mathsf{E}[X]\right] < n^{-\varepsilon^2 c/12} = o(1). \tag{3.10}$$

(4) Let X be the number of vertices with degree greater than $4np$. Then

$$\Pr[X > 0] \leq \mathsf{E}[X] \leq n \Pr[Bin(n,p) > 4np] \leq n\left(\frac{e}{4}\right)^{4np} < n^{1-3c/2} = o(1). \tag{3.11}$$

(5) We have that for $k \leq \Lambda$ and $c \leq 2$,

$$\mathsf{E}[|V_k|] = n\binom{n-1}{k} p^k (1-p)^{n-k} \leq (2 \log n)^k.$$

The claim for $c \leq 2$ now follows from the Markov inequality. When $c > 2$, a first moment calculation shows that with high probability we have $V_k = \emptyset$ for $k \leq \Lambda$.

(6) First we observe that for any $a \in \{1,2,3\}$ and $b \in \{0,1,2\}$,

$$\begin{aligned}
&\Pr\left[Bin(n-a,p) < \frac{np}{100} - b\right] \\
&= \sum_{i=0}^{\frac{c\log n}{100}-b} \binom{n-a}{i} p^i (1-p)^{n-a-i} \\
&\lesssim \sum_{i=0}^{\frac{np}{100}} \left(\frac{ec\log n}{i}\right)^i n^{-c} \\
&\leq n^{1/3-c}.
\end{aligned}$$

To show (i), we give a probabilistic upper bound on the size of A. Since $d(v) \sim Bin(n-1,p)$, $\mathsf{E}[|A|] \leq n^{4/3-c}$. The Markov inequality implies $A = \emptyset$ with high probability if $c \geq 2$. Suppose then that $c \leq 2$. Then,

$$\Pr\left[|A| \geq n^{17/12-c}\right] < \Pr\left[X \geq n^{1/12}\,\mathsf{E}[|A|]\right] < n^{-1/12} = o(1). \tag{3.12}$$

Now let a cycle be small if has at most Λ vertices. Next we show (ii), that no vertex of A is part of a small cycle or within distance Λ of a small cycle. To show the former, let X_j be the number of cycles on j vertices that contain a vertex of A. Observe

$$\begin{aligned}
&\Pr[X_j > 0] \\
&\leq \mathsf{E}[X_j] \leq n^j p^j \Pr\left[Bin(n-3,p) < \frac{np}{100} - 2\right] \\
&\approx (c\log n)^j\, n^{1/3-c} = o(1)
\end{aligned}$$

for $j \leq \Lambda$.

Next, let $X_{j,\ell}$ be the number of structures that contain a k-cycle with path of length ℓ to a vertex in A. Observe

$$\begin{aligned}
\Pr[X_{j,\ell} > 0] &\leq \mathsf{E}[X_{j,\ell}] \leq \\
n^{j+\ell}\left(\frac{c\log n}{n}\right)^{j+\ell} &\Pr\left[Bin(n-2,p) < \frac{c\log n}{100} - 1\right] \\
&\approx (c\log n)^{j+\ell}\, n^{1/3-c} = o(1)
\end{aligned} \tag{3.13}$$

for $j, \ell \leq \Lambda$.

Finally we show (iii), that no two vertices of A are within distance Λ of each other. Let P be number of paths of length at most Λ with both ends in A.

$$\begin{aligned}
\Pr[P > 0] &\leq \mathsf{E}[P] \leq \\
&n^2\left(\Pr\left[Bin(n-2,p) < \frac{c\log n}{100} - 1\right]\right)^2 \\
&\times \sum_{i=0}^{\Lambda} n^i \left(\frac{c\log n}{n}\right)^{i+1} \\
&\leq 2(c\log n)^{\Lambda+1} n^{5/3-2c} = o(1).
\end{aligned} \tag{3.14}$$

(7) Let now X_j be the number of subgraphs H with $v_H = j$ and $e_H \geq v_H + 1$ in G. Let $\nu_j = O(2^{j^2})$ be the number of graphs H on vertex set $[j]$ vertices with $e_H \geq v_H + 1$. Then we have

$$\Pr[X_j > 0] \leq \mathsf{E}[X_j] \leq \nu_j n^j p^{j+1} \leq \nu_j n^{\alpha j} p = o(1)$$

for $j \leq \Lambda$.

(8) Let $s = |S|$. We apply the Chernoff bound (1.2) to obtain

$$\Pr\left[e(S,\overline{S}) < \tfrac{1}{2}s(n-s)p\right] \leq \exp\left(-\frac{s(n-s)p}{8}\right) = n^{-s(n-s)c/8n}. \tag{3.15}$$

Let now X_s be the number of subsets S of size s for which $e(S,\overline{S}) < \frac{1}{2}s(n-s)p$.

$$\Pr[X_s > 0] \leq \mathsf{E}[X_s] \leq \binom{n}{s} n^{s(n-s)c/8n} \leq \left(\frac{ne}{s}\cdot n^{-c/16}\right)^s \leq \left(ecn^{-c/16}\log n\right)^s = o(n^{-1}). \tag{3.16}$$

It follows that with high probability $X_s = 0$ for all $1/p \leq s \leq n/2$.

(9) Let $s = |S|$, and let now X_s be the number of sets of size s with $e(S,S) \geq snp/1000$. Then,

$$\begin{aligned}
\Pr[X_s > 0] \leq \mathsf{E}[X_s] &\leq \binom{n}{s}\binom{\binom{s}{2}}{snp/1000} p^{snp/1000} \\
&\leq \left(\frac{ne}{s}\right)^s \left(\frac{se}{2n/1000}\right)^{snp/1000} \\
&= \left(\frac{s}{n}\right)^{(-1+np/1000)s} \left(500e^{1+np/100}\right)^s.
\end{aligned} \tag{3.17}$$

Summing the RHS of (3.17) for $1 \leq s \leq n/\log n$ completes the proof. $\square$

The following claim describes the stationary distribution π_v of the modified walk on a typical graph G.

Claim 1. *Let G be a typical graph and let $\varepsilon > 0$ be an arbitrarily small constant. Let U be the set of vertices u such that the degree of u and the degrees of all its neighbors are in the range $((1-\varepsilon)c\log n, (1+\varepsilon)c\log n)$. Then $|U| \approx n$ and for $u \in U$,*

$$\frac{1-\varepsilon}{(1+\varepsilon)n} \lesssim \pi_u \lesssim \frac{1+\varepsilon}{(1-\varepsilon)n}.$$

Moreover, for $b_1 = \frac{1-\varepsilon}{1+\varepsilon}$ and $b_2 = \frac{401}{1+\varepsilon}$, for all $v \in V$,

$$\frac{b_1}{n} \lesssim \pi_v \lesssim \frac{b_2}{n}. \tag{3.18}$$

Proof We refer the reader to (2.5) for the value of $\pi_v, v \in V$. We first observe that

$$\Psi(v) \geq \sum_{w \in N(v)} \frac{1}{d(v)} = 1 \text{ for all } v \in V.$$

Since each vertex has degree at most $4c \log n$ and has at most one neighbor of degree less than $c \log n/100$,

$$\Psi(v) \in [1, 401] \text{ for } v \in V. \tag{3.19}$$

On the other hand, for $u \in U$, $\Psi(u) \in \left[1, \frac{1+\varepsilon}{1-\varepsilon}\right]$. Lemma 3.1 parts (2) and (3) imply that $|V \backslash U| = o(n)$. Therefore

$$n \leq \sum_{v \in V} \Psi(v) \lesssim \left(\frac{1+\varepsilon}{1-\varepsilon}\right) n,$$

and the statement follows. □

The following lemma shows that the mixing time of $\mathcal{W}_u, u \in V$ on a typical graph is $O(\log(n))$, as stated in (2.6).

Lemma 3.2. *Let $G = (V, E)$ be typical, and let u be an arbitrary vertex of G. Let $P_u^{(t)}(x)$ be the probability that $\mathcal{W}_u$ is at vertex x at time t. Then for b_1, b_2 as in Claim 1, $L = 10(8001 b_2)^2/b_1^2$, and $T = L \log n$,*

$$|P_u^{(T)}(x) - \pi_x| \leq n^{-3}.$$

Proof It is well known, see for example [13] that if λ_2 denotes the second largest absolute value of an eigenvalue of P, $\lambda_2 < 1$ and

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_2^t \leq (b_2/b_1)^{1/2} \lambda_2^t \leq 21 \lambda^t, \tag{3.20}$$

where the second inequality follows from (3.18) and the third assumes that ε is sufficiently small.

To compute the size of the second largest absolute value of an eigenvalue of P, we apply the Cheeger inequality. Recall the definition of conductance:

$$\Phi(G) = \min_{S \subset V(G), \pi(S) \leq \frac{1}{2}} \Phi(S),$$

where

$$\Phi(S) = \frac{\sum_{x \in S, y \in \overline{S}} \pi_x P_{x,y}}{\sum_{x \in S} \pi_x}.$$

Applying Claim 1 we observe

$$\Phi(S) = \frac{\sum_{x \in S, y \in \overline{S}} \pi_x P_{x,y}}{\sum_{x \in S} \pi_x} > \frac{\frac{b_1}{n} \sum_{x \in S} \frac{e(\{x\}, \overline{S})}{d(x)}}{|S| \frac{b_2}{n}} = \frac{b_1}{|S| b_2} \sum_{x \in S} \frac{e(\{x\}, \overline{S})}{d(x)}. \tag{3.21}$$

Define

$$D(S) := \sum_{x \in S} \frac{e(\{x\}, \overline{S})}{d(x)}.$$

We give a lower bound on $D(S)$ for all subsets S for which $\pi(S) \leq \frac{1}{2}$. By Claim 1, a set S for which $\pi(S) \leq \frac{1}{2}$ can have cardinality at most $\frac{n}{2} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \leq \frac{2n}{3}$, assuming ε is sufficiently small.

Case 1: $1/p \leq |S| \leq 2n/3$.
By Conditions 4 and 8 of Lemma 3.1

$$D(S) \geq \frac{e(S, \overline{S})}{4np} \geq \frac{|S|(n - |S|)}{8n}$$

and so

$$\Phi(S) \geq \frac{b_1(n - |S|)}{8n b_2} \geq \frac{b_1}{16 b_2}. \tag{3.22}$$

Case 2: $|S| < 1/p$.
To evaluate the case when $|S| < 1/p$, we consider two subcases. Let A be the set of vertices of degree less than $\frac{np}{100}$, as in Lemma 3.1.

Case 2a: $|A \cap S| < 3|S|/4$.
By Condition 9 of Lemma 3.1, we have

$$e(S, \overline{S}) \geq d(S) - 2e(S, S) \geq \frac{|S|}{4} \frac{np}{100} - 2|S| \frac{np}{1000}.$$

It follows that

$$D(S) \geq \frac{\frac{|S|}{4} \frac{np}{100} - 2|S| \frac{np}{1000}}{4c \log n} \gtrsim \frac{|S|}{8000}$$

and so

$$\Phi(S) \geq \frac{b_1}{8001 b_2}. \tag{3.23}$$

Case 2b: $|A \cap S| \geq 3|S|/4$.
Let $A^* \subseteq A \cap S$ be the vertices with no neighbors in $S \setminus A$. Since each vertex has at most one neighbor in A, $|A^*| \geq |S|/2$. We compute

$$D(S) \geq \sum_{x \in A^*} \frac{e(\{x\}, \overline{S})}{d(x)} = |A^*| > \frac{|S|}{2}$$

and so

$$\Phi(S) \geq \frac{b_1}{2 b_2}. \tag{3.24}$$

It follows from (3.22), (3.23), (3.24) that $\Phi(G) \geq \frac{b_1}{8001 b_2}$, and so by the Cheeger inequality

$$\lambda_2 \leq \left(1 - \frac{\Phi^2}{2}\right) \leq 1 - \frac{b_1^2}{2(8001 b_2)^2}.$$

Let $L = 10(8001b_2)^2/b_1^2 > -4/\log\left(1-\frac{b_1^2}{2(8001b_2)^2}\right)$. Letting $t = L\log n$ in (3.20), we see that

$$|P_u^{(t)}(x) - \pi_x| \le 21\left(1 - \frac{b_1^2}{2(8001b_2)^2}\right)^{-L\log n} < n^{-3}.$$

□

Finally, we give upper bounds on the values R_v, the expected number of times that $\mathcal{W}_v$ returns to v in T steps, where T is as defined in (2.6). Here we refer to the set of vertices within distance k of v as the $k-$neighborhood of a vertex v.

LEMMA 3.3. *Let G be typical. Let*

$$A = \left\{v : d(v) \le \frac{np}{100}\right\}.$$
$$B = \{v : N_{10}(v) \cap A = \emptyset\}.$$

(Thus $A \cap B = \emptyset$.)

Then $R_v \ge 1$ and

$$R_v \le \begin{cases} 1 + O\left(\frac{1}{\log n}\right) & v \in B \\ & \text{or } v \notin A \cup B, N(v) \cap A = \emptyset. \\ 1 + \frac{C}{d(v)} & v \in A \\ 1 + \frac{C}{d(v_1)}. & N(v) \cap A = \{v_1\}. \end{cases}$$

where C is an absolute constant.

Proof We estimate R_v by projecting the random walk in the neighborhood of v onto the nonnegative integers, with v corresponding to zero. Divide the vertices in the neighborhood of v into levels based on their distance from v. Let α be an upper bound on the probability the walk moves from level $i \le 4$ to level $i-1$, $\rho \ge \alpha$ is an upper bound on the probability the walk stays in level i, and $\beta = 1 - \alpha - \rho$ (a lower bound on the probability that the walk moves from level i to level $i+1$, that will soon be seen to be non-negative). We couple $\mathcal{W}_v$ with a random walk $\widehat{\mathcal{W}}$ on $\{0,1,2,3,4\}$ with parameters α, β, ρ so that $\widehat{\mathcal{W}} \le \mathcal{W}_v$ whenever $\mathcal{W}_v$ is within distance 4 of v. When $\widehat{\mathcal{W}}$ is at 4, it moves to 3 at the next step. Otherwise, if $\mathcal{W}_v$ is at level $0 < j \le 3$ and $\widehat{\mathcal{W}}$ is at $0 < i < j$ then $\widehat{\mathcal{W}}$ will move left or stay if $\mathcal{W}_v$ moves left. Also, if $\mathcal{W}_v$ is at level $0 < j \le 3$ and $\widehat{\mathcal{W}}$ is at $0 < i$ then $\widehat{\mathcal{W}}$ will move left if $\mathcal{W}_v$ moves left. If $\widehat{\mathcal{W}}$ is at 0 and $\mathcal{W}_v$ is at 1 and $\mathcal{W}_v$ moves left, then $\widehat{\mathcal{W}}$ will stay at 0. Here we use the assumption $\rho \ge \alpha$. We will in fact simplify matters by taking $\alpha = \rho$. It remains to define $\alpha = \rho, \beta$ and to estimate the expected number of times that $\widehat{\mathcal{W}}$ visits 0, if it starts there.

Let E_i be an upper bound on the expected number of times that $\widehat{\mathcal{W}}$, beginning at i, visits 0 in T steps. We have,

$$E_0 = 1 + \rho E_0 + (1-\rho)E_1$$

which implies that

$$E_0 \le 1 + 2\rho + E_1, \tag{3.25}$$

assuming that $\rho \le 1/2$.

More generally

$$E_i \le \alpha E_{i-1} + \rho E_i + \beta E_{i+1} \text{ for } 1 \le i \le 3. \tag{3.26}$$

Note that

$$E_3 \le T\alpha^2 \sum_{\ell \ge 0} (\alpha(1-\alpha))^\ell = \frac{T\alpha^2}{1 - \alpha(1-\alpha)} \le 2T\alpha^2. \tag{3.27}$$

since each time the walk moves left from 3 the chance of reaching zero before returning to 3 is less than $\alpha^2 \sum_{\ell \ge 0} (\alpha(1-\alpha))^\ell$ when $\alpha \le 1/2$. Here ℓ is the number of times that $\widehat{\mathcal{W}}$ moves 2,1,2 before finally moving to zero in two steps. (Included in 2,1,2 there might be some $x, x, \dots,$ where $x \in \{1,2\}$.) The second inequality uses our choice of $\alpha = \rho$ and the third inequality relies on $\alpha \le 1/4$.

Summing the inequalities in (3.26) for $i = 1,2,3$ yields

$$0 \le \alpha E_0 - \beta E_1 - \alpha E_2 + \beta E_3 \le \alpha E_0 - \beta E_1 + \beta E_3.$$

It follows that

$$E_0 \le 1 + 2\rho + E_1 \le 1 + 2\rho + \frac{\alpha E_0}{\beta} + E_3,$$

and so

$$E_0 \le \frac{1 + 2\rho + 2\alpha^2 T}{1 - \frac{\alpha}{\beta}}. \tag{3.28}$$

First we assume $np \le n^{1/100}$. We consider several cases.

Case 1: $v \in B$.

Consider u at distance $i \le 3$ from v. Condition 7 of Lemma 3.1 guarantees there are at most two edges from u to level $i-1$ and at most one edge from u to another vertex in level i. Since u and all its neighbors have degree at least $np/100$ we can take $\alpha = \rho = \frac{200}{np}$ and $\beta = 1 - 2\rho$. It follows from (3.28) that for $v \in B$ and $T = L\log n$,

$$R_v = E_0 \le \frac{1 + O(1/\log n)}{1 - O(1/\log n)} = 1 + O\left(\frac{1}{\log n}\right). \tag{3.29}$$

Case 2: $v \in A$.

We observe that if $\mathcal{W}_v$ is at $w \in N(v)$ then the probability it moves to v in one step can be bounded by

$$\frac{1/d(v)}{1/401 + 1/d(v)} = \frac{1}{1 + \frac{d(v)}{401}}. \tag{3.30}$$

Explanation: We have $\sum_{x \in N(w)\setminus v} \psi(w,x) \geq \frac{np/100-1}{4np} \geq 1/401$. This follows from Condition 4 of Lemma 3.1. Also, $\psi(w,v) = 1/d(v)$.

It follows that

$$\begin{aligned}(3.31) \quad R_v &\leq 1 + \frac{1}{d(v)} \sum_{w \in N(v)} \frac{R_w}{1 + \frac{d(v)}{401}} \\ &\leq 1 + \frac{1}{d(v)} \sum_{w \in N(v)} \frac{1+o(1)}{1 + \frac{d(v)}{401}} \leq 1 + \frac{402}{d(v)}.\end{aligned}$$

Explanation: For the first inequality, we see that when at v, $\mathcal{W}_v$ chooses a neighbor w uniformly from $N(v)$ and then each return to w yields a return to v with probability as given in (3.30). The second inequality follows from the case $v \in B$.

Case 3: $v \notin A \cup B$**.** It follows from Lemma 3.1 that the Λ-neighborhood of v is a tree ($\Lambda = \log\log n$), and there exists a single vertex in $a \in A$ in the 10-neighborhood of v. Let $N(v) = \{v_1, v_2, \ldots, v_d\}$ where the unique $a \in N_4(v)$ lies in the sub-tree rooted at v_1. Let α be the probability the walk moves from v to v_1. Let $\rho_{\overline{v}}$ be the probability that $\mathcal{W}_v$ returns to v if edge (v, v_1) is removed. Then we have

$$(3.32) \qquad R_v \leq 1 + \alpha R_v + \rho_{\overline{v}} R_v.$$

Explanation: For the first term assume that the walk always returns from v_1 and similarly, for the second term, we assume that the walk always returns to v, if it returns to $v_j, j \geq 2$.

We compute

$$(3.33) \qquad \alpha = \frac{\frac{1}{\min\{d(v), d(v_1)\}}}{\Psi(v)} \leq \frac{\frac{1}{\min\{d(v), d(v_1)\}}}{(d(v)-1)/d(v) + 1/d(v_1)}.$$

Furthermore, if $R_{\overline{v}}$ denotes the expected number of returns to v if edge (v, v_1) is removed then

$$1 + \rho_{\overline{v}} \leq R_{\overline{v}} \leq 1 + O\left(\frac{1}{\log n}\right)$$

implying that

$$(3.34) \qquad \rho_{\overline{v}} = O\left(\frac{1}{\log n}\right).$$

where we have used (3.29).

It follows from (3.32) that

$$(3.35) \qquad R_v \leq \frac{1}{1 - \alpha + O\left(\frac{1}{\log n}\right)}.$$

We now consider two subcases.

Case 3a: $v_1 \in A$**.** For $d(v_1) = 1$, it follows from (3.33) that $\alpha \approx 1/2$. For $d(v_1) \geq 2$, (3.33) implies $\alpha \leq \frac{1}{d(v_1)}$. Therefore (3.35) implies

$$(3.36) \qquad R_v \leq 1 + O\left(\frac{1}{d(v_1)}\right).$$

Case 3b: $v_1 \notin A$**.** It follows from (3.33) that $\alpha \leq \frac{100}{np}$. Therefore (3.35) implies

$$(3.37) \qquad R_v \leq \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Finally we consider $np > n^{1/100}$. The Chernoff bounds imply that with high probability $d(x) \approx np$ for all $x \in V$. Now,

$$(3.38) \qquad R_v \leq 1 + \frac{T}{\min_{w \in N(v)} d(w)} = 1 + o(1).$$

This is because there are at most T instances where $\mathcal{W}_v$ is at a neighbor of v and then the chance of moving to v on the next step is bounded by $\frac{1}{\min_{w \in N(v)} d(w)}$.

The lemma follows from (3.29), (3.31), (3.36), (3.37) and (3.38). $\square$

4 Cover time

In this section we prove upper and lower bounds on the cover time (Lemmas 4.2 and 4.3 respectively), which together imply Theorem 1.1.

4.1 The upper bound In the proof of the upper bound we apply the first visit time lemma. We rely on the following lemma to show Condition (ii) of the first visit time lemma.

LEMMA 4.1. *Equation (18) of [6] proves the following: Let v be a vertex of an arbitrary graph G. Let T be a mixing time satisfying Condition (i) of Lemma 2.1. If $T = o(n^3)$, $T\pi_v = o(1)$ and R_v is bounded above by a constant, then Condition (ii) of Lemma 2.1 holds for $\theta = 1/4$ and any constant $K \geq 3R_v$.*

LEMMA 4.2. *Let $G = G_{n,p}, p = \frac{c \log n}{n}$ with $c \geq 1$ and $\omega = (c-1)\log n \to \infty$. Then with high probability, $\overline{C}(G) \lesssim n \log n$.*

Proof With high probability G is typical, as stated in Lemma 3.1. Let u be an arbitrary vertex of G, let $T_G(u)$ be the time taken to visit all vertices by $\mathcal{W}_u$, and let U_s be the number of vertices that haven't been visited at step s. Let $\mathcal{A}_v(s)$ be the event that $\mathcal{W}_u$ does not visit v in $[T, s]$. For $t > T$,

$$\begin{aligned}(4.39) \quad \overline{C}(G) = \max_{u \in V} \mathsf{E}[T_G(u)] \leq \sum_{s>0} \mathsf{Pr}[U_s > 0] \leq \\ \sum_{s>0} \min\{1, \mathsf{E}[U_s]\} \leq t + \sum_{v \in V} \sum_{s>t} \mathsf{Pr}[\mathcal{A}_v(s)].\end{aligned}$$

Next we apply the first visit time lemma (Lemma 2.1) for $T = L\log n$. Lemma 3.2 guarantees Condition (i). The boundedness of R_v, proved in Lemma 3.3, implies the assumptions of Lemma 4.1, thereby guaranteeing Condition (ii) holds for $\theta = 1/4$ and $K > 12000L$. Note that $T\pi_v = O(\log n/n)$, therefore Condition (iii) holds. We apply the lemma and compute

$$\mathsf{Pr}[\mathcal{A}_v(t)] = (1+o(1))\, e^{-tp_v} + o(e^{-t/KT}) \approx e^{-\pi_v t/R_v}.$$

Let A, B be as defined in Lemma 3.3. By Claim 1, $1/\pi_v \le n\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ for all vertices. Then, by the bounds on R_v given in Claim 1,

1. $v \in B$ implies that $\frac{R_v}{\pi_v} \le (1+\theta)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) n$ for some $\theta = o(1)$.
2. $v \in D = \{v \notin A \cup B$ and $N(v) \cap A = \emptyset\}$ also implies that $\frac{R_v}{\pi_v} \le (1+\theta)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) n$.
3. $v \in V_k$ (vertices of degree k, see Lemma 3.1(5)) implies that $\frac{R_v}{\pi_v} \le \left(1+\frac{C}{k}\right)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) n$.

For ease of notation, let $\tau = (1+\theta)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ and let $t = \tau n \log n$. Applying Lemma 2.1 and using an approximation of e^{-x} we obtain $\theta_1, \theta_2 = o(1)$ such that

$$\begin{aligned}\overline{C}(G) &\le \tau n\log n + (1+\theta_1)\sum_{v\in V}\sum_{s>t} e^{-\pi_v s/R_v}\\ &= \tau n\log n + (1+\theta_2)\sum_{v\in V}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v}\\ &\le \tau n\log n + (1+\theta_2)|B\cup D|\left(\frac{1+\varepsilon}{1-\varepsilon}\right) + \\ &\qquad \sum_{k\ge1}\sum_{v\in V_k\setminus(B\cup D)}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v}\\ &\le \tau n\log n + O(n) + \sum_{k\ge1}\sum_{v\in V_k\setminus(B\cup D)}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v}.\end{aligned}$$

We complete the proof of the lemma by showing that

$$\sum_{k\ge1}\sum_{v\in V_k\setminus(B\cup D)}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v} = o(n) \tag{4.40}$$

and then letting $\varepsilon \to 0$.

Proof of (4.40)**:** If $k \le \Lambda$ then using condition 5 of Lemma 3.1, we have

$$\begin{aligned}&\sum_{v\in V_k\setminus(B\cup D)}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v} \le \\ &(3\log n)^{k+1}\times\left(1+\frac{C}{k}\right)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) n\times n^{-\tau/(2C+1)}\\ &\qquad\qquad \le n^{1-\Omega(1)}.\end{aligned}\tag{4.41}$$

If $k > \Lambda$ then using conditions 2, 3 and 6 of Lemma 3.1, we have

$$\begin{aligned}&\sum_{k>\Lambda}\sum_{v\in V_k\setminus(B\cup D)}\frac{R_v}{\pi_v}e^{-\pi_v t/R_v}\\ &\le 3n^{1-\varepsilon^2c/4}\times\left(1+\frac{2C}{\Lambda}\right)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) n\times n^{-\tau/(2C/\Lambda+1)}\\ &\qquad\qquad \le n^{1-\Omega(1)}.\end{aligned}\tag{4.42}$$

Here $2n^{1-\varepsilon^2c/4} + n^{17/12-c}\left(\frac{np}{100}\right) \le 3n^{1-\varepsilon^2c/4}$ is a bound on $|V\setminus(B\cup D)|$. Note that $A = \emptyset$ with high probability if $c \ge \log^2 n$ and this follows easily from the Chernoff bounds. Equation (4.40) follows from (4.41) and (4.42). □

4.2 The lower bound Finally, we give a lower bound on cover time. We observe that Feige's lower bound [9] is only claimed to hold for the simple random walk where each neighbor of the current vertex v is equally likely to be chosen as the next vertex to be visited. Furthermore, our results imply that with high probability $C_u(G) \approx n\log n$ for all $u \in V$.

LEMMA 4.3. *Let $G \sim G_{n,p}$ where $p = \frac{c\log n}{n}$ and $\omega = (c-1)\log n \to \infty$. Then with high probability, $\overline{C}(G) \gtrsim n\log n$.*

Proof Let $I = [(1-\varepsilon)np, (1+\varepsilon)np]$. Let S_0 be the set of vertices v such that

(P1) $d(v) \in I$.

(P2) $d(w) \in I$ for $w \in N_2(v)$.

(P3) $d(w) \le d(v)$ for $w \in N(v)$.

(P4) $v \in B$, where B is defined as in Lemma 3.3.

It follows from conditions 2, 3 and 6 of Lemma 3.1 that the number of vertices satisfying P1, P2 and P4 is $n - o(n)$.

CLAIM 2.
$$|S_0| \ge \frac{n}{5np}.$$

Proof Let $X_0 = \{v : d(v) > (1+\varepsilon)np\}$ and $Y_0 = X_0 \cup N(X_0)$. It follows from conditions 2,3 and 4 of Lemma 3.1 that $|Y_0| = o(n)$.

Next let $k_i = (1+\varepsilon)np - i$. Then let $X_1 = V_{k_1}\setminus Y_0$ and $Y_1 = X_1 \cup N(X_1)$. In general we let $X_{i+1} = V_{k_{i+1}} \setminus \bigcup_{j\le i} Y_j$ and $Y_{i+1} = X_{i+1}\cup N(X_{i+1})$.

We note that $x \in X_i, i \ge 1$ implies that $d(x) = k_i \ge d(w)$ for $w \in N(x)$. This is because x has no neighbors in V_ℓ for $\ell > k_i$.

Now Lemma 3.1 implies that $\sum_{i=1}^{2\varepsilon np} |X_i \cup Y_i| = n - o(n)$. And then our bound on the maximum degree of $4np$ implies that

$$|S_0| \geq \sum_{i=1}^{2\varepsilon np} |X_i| \geq \frac{n-o(n)}{4np}.$$

□

Given the claim, we divide the possible range for R_v into $\log^2 n$ sub-intervals and use the pigeon-hole principle to select a subset $S_1 \subseteq S_0$ of size $\Omega\left(\frac{n}{np\log^2 n}\right)$ such that

$$|R_u - R_v| \leq \frac{1}{\log^2 n} \text{ for } u, v \in S_1. \tag{4.43}$$

Next let S be a maximum size subset of vertices of S_1 such that no two vertices of S are within distance 10 of each other. We show that for ε sufficiently small and for $\delta = 3\varepsilon$, with high probability the set S will not be covered at time $t = (1-\delta)n\log n$.

We show next that a greedy algorithm applied to S_1 produces a set S of size at least $\frac{n-o(n)}{(4np)^{10}}$. After selecting k vertices from S_1, there will be at least $|S_1| - k(4np)^{10}$ vertices in S_0 available for the next choice of vertex for S. Therefore

$$|S| \geq \frac{|S_1|}{(4np)^{10}} \geq \frac{n}{(5np)^{11}}.$$

Let $S(t)$ denote the number of vertices in S that have not been visited by the random walk at step t. For $t > T$,

$$\mathsf{E}[S(t)] \geq -T + \sum_{v\in S} \mathsf{Pr}[A_v(t)].$$

We compute

$$\begin{aligned}\mathsf{Pr}[A_v(t)] &\approx \frac{1}{(1+\frac{\pi_v}{R_v})^t} \\ &\gtrsim \exp\left(-(1-o(1))\frac{t(1+\varepsilon)}{(1-\varepsilon)n}\right) \geq n^{-(1-\varepsilon/2)}.\end{aligned} \tag{4.44}$$

It follows

$$\mathsf{E}[S(t)] = \Omega\left(\frac{n^{\varepsilon/2}}{(5np)^{11}}\right) \to \infty,$$

assuming that

$$np \leq n^{\varepsilon/25}. \tag{4.45}$$

We make this assumption for now and deal with $c > n^{\varepsilon/25}$ in Section 4.3.

As in earlier papers, we apply the Chebyshev inequality to show that $S(t) \neq \emptyset$ with high probability. To estimate $\mathsf{E}[S(t)(S(t)-1)]$, we estimate the probability that two distinct vertices $u, v \in S$ have not been visited by time t. Let Γ be obtained from G by contracting u and v into a single vertex, which we call z.

Claim 3. *The probability a random walk $\overline{\mathcal{W}}_u$ in Γ doesn't visit z in t steps equals the probability that the random walk $\mathcal{W}_u$ in G visits neither u nor v in t steps.*

Proof Let $\omega = (u = v_0, v_1, \ldots, v_t)$ be a walk that does not visit u, v. Let p_W and $\widehat{p}_W$ be the probabilities that $\mathcal{W}_u$ follows W in ω and in Γ respectively. Then

$$p_W = \prod_{i=0}^{t-1} \frac{\psi(v_i, v_{i+1})}{\Psi(v_i)}, \tag{4.46}$$

The claim follows from (4.46) and the fact that if $\widehat{\psi}$ equals the induced values of ψ in Γ then

$$\widehat{\psi}(x,y) = \psi(x,y) \text{ for all } x, y \notin N(u) \cup N(v). \tag{4.47}$$

and

$$\widehat{\psi}(x,z) = \psi(x,u) \text{ for all } x \in N(u). \tag{4.48}$$

Equation (4.47) is clear and equation (4.48) follows from our condition P3. □

Note that (3.27) implies that the expected number of returns to v after reaching distance 4 from v is $O(1/\log^2 n)$. Therefore, since all paths between u to v contain vertices at distance at least four from u and v,

$$R_z = \frac{1}{2}R_u + \frac{1}{2}R_v + O\left(\frac{1}{\log^2 n}\right). \tag{4.49}$$

With respect to steady state probabilities, it follows from (4.47), (4.48) that we have

$$\pi_z = \pi_u + \pi_v. \tag{4.50}$$

It is straightforward to check that the conditions of Lemma 2.1 hold for Γ with $T = O(\log n)$. It follows from (2.7), (4.49) and (4.50) that

$$\begin{aligned}\mathsf{Pr}[A_z(t)] &\approx \exp\left(-\frac{(\pi_u+\pi_v)t}{\frac{1}{2}R_u + \frac{1}{2}R_v + O\left(\frac{1}{\log^2 n}\right)}\right) \\ &= \exp\left(-\frac{(\pi_u+\pi_v)t}{R_u + O\left(\frac{1}{\log^2 n}\right)}\right) \\ &\approx \exp\left(-\frac{\pi_u t}{R_u}\right) \times \exp\left(-\frac{\pi_v t}{R_v}\right) \\ &\approx \mathsf{Pr}[A_u(t)] \times \mathsf{Pr}[A_v(t)].\end{aligned} \tag{4.51}$$

It follows that

$$\mathsf{E}[S(t)(S(t)-1)] \lesssim \mathsf{E}[S(t)]^2$$

and so

$$\Pr[S(t)>0] \geq \frac{\mathsf{E}[S(t)]^2}{\mathsf{E}[S(t)^2]} = \frac{\mathsf{E}\left[S(t)^2\right]}{\mathsf{E}[S(t)(S(t)-1)]+\mathsf{E}[S(t)]} \geq \frac{1}{1+o(1)+\mathsf{E}[S(t)]^{-1}} = 1-o(1). \tag{4.52}$$

□

4.3 High average degree case We show how to amend the above argument for the case where $np \geq n^{\varepsilon/25}$. All vertices satisfy P1,P2 and P4 and we drop P3. We can however claim that with high probability

$$|d(v)-np| \leq \sqrt{10np\log n} \text{ for all } v \in V. \tag{4.53}$$

This follows from applying the Chernoff bounds to $d(v) \sim Bin(n-1,p)$.

We cannot claim (4.48), but because $d(z) \approx 2d(u)$, we have instead that with high probability

$$\widehat{\psi}(x,z) = \begin{cases} \psi(x,u) & d(x) \leq d(u). \\ \psi(x,u) - \frac{1}{d(u)} + \frac{1}{d(x)} & d(x) > d(u). \end{cases} \tag{4.54}$$

Now, (4.53) implies that

$$\frac{1}{d(u)} - \frac{1}{d(x)} = O\left(\frac{\log^{1/2} n}{(np)^2}\right).$$

It follows from this that instead of (4.50) we have

$$\pi(z) = (\pi(u)+\pi(v))\left(1+O\left(\frac{\log^{1/2} n}{np}\right)\right).$$

Going back to (4.51) we obtain

$$\Pr[A_z(t)] \approx \exp\left(-\frac{(\pi_u+\pi_v)\left(1+O\left(\frac{\log^{1/2} n}{np}\right)\right)t}{\frac{1}{2}R_u+\frac{1}{2}R_v+O\left(\frac{1}{\log^2 n}\right)}\right) \approx \exp\left(-\frac{(\pi_u+\pi_v)t}{R_u+O\left(\frac{1}{\log^2 n}\right)}\right)$$

and the proof continues as for the previous case. □

5 Conclusion

We have give an asymptotically tight analysis of the cover time of a biased random walk on $G_{n,p}$. It would certainly be of interest to consider other possible biased walks and also to consider the analysis of the walk in this paper on other models of a random graph. In particular, it would be of interest to analyse the performance of this walk on a preferential attachment graph.

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