Recent results on permutations without short cycles

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Abstract. The density, denoted by $\kappa(n, r)$, of permutations having no cycles of length less than $r + 1$ in a symmetric group S_n is explored. New asymptotic formulas for $\kappa(n, r)$ are obtained using the saddle-point method when $5 \leq r < n$ and $n \to \infty$.

Keywords: symmetric group, long cycles, Buchstab's function, Dickman's function, saddle-point method

The probability $\kappa(n, r)$ that a permutation sampled from the symmetric group S_n uniformly at random has no cycles of length less than $r + 1$, where $1 \leq r < n$ and $n \to \infty$, is explored. New asymptotic formulas valid in specified regions are obtained using the saddle-point method. One of the results is applied to show that estimate of the total variation distance for permutations can be expressed only through the function $\nu(n, r)$ which is a probability that a permutation sampled from the S_n uniformly at random has no cycles of length greater than r.

To address the problem, we need recollect the following functions. Buchstab's function $\omega(v)$ is defined as a solution to difference-differential equation

$$(v\omega(v))' = w(v - 1)$$

for $v > 2$ with the initial condition $\omega(v) = 1/v$ if $1 \leq v \leq 2$. Dickman's function $\varrho(v)$ is the unique continuous solution to the equation

$$v\varrho'(v) + \varrho(v - 1) = 0$$

for $v > 1$ with initial condition $\varrho(v) = 1$ if $0 \leq v \leq 1$.

The interest to the problem begins with the classical example of derangements

$$\kappa(n, 1) \;\; = \;\; \sum_{j=0}^{n} \frac{(-1)^j}{j!} = \mathrm{e}^{-1} + O\left(\frac{1}{n!}\right)$$

and the trivial case $\kappa(n, r) = 1/n$ if $n/2 \leq r < n$. There was a series of works concerning general asymptotic formulas of the probability $\kappa(n, r)$ the strongest of which are presented here as Proposition 1 and Proposition 2.

Proposition 1 *For* $1 \le r < n$, *we have*

$$\kappa(n,r) = e^{-H_r+\gamma}\omega(n/r) + O\left(\frac{1}{r^2}\right).$$

See [6, Theorem 3].

Proposition 2 *Let* $u = n/r$. *For* $1 \le r \le n/\log n$,

$$\kappa(n,r) = e^{-H_r} + O\left(\frac{(u/e)^{-u}}{r^2}\right).$$

If $r \ge 3$, *we can replace* e *by 1 in the error term.*

See [12, Proposition 2]. Together these propositions provide stronger estimates of $\kappa(n,r)$ than those in [2], [3], [4]. New results are the following theorems:

Theorem 1 *For* $\sqrt{n\log n} \le r < n$, *we have*

$$\kappa(n,r) = e^{-H_r+\gamma}\omega(n/r) + O\left(\frac{\varrho(n/r)}{r^2}\right).$$

Proof. The result is a corollary of Theorem 1 in [7]. It is obtained from the probability generating function using saddle-point method, the technique is elaborated in [11].

Theorem 2 *For* $(\log n)^4 \le r < n$, *we have*

$$\kappa(n,r) = e^{-H_r} + O\left(\frac{\varrho(n/r)}{r}\right).$$

Proof. The saddle-point method is applied to the Cauchy's integral representation of $\kappa(n,r)$, as in the proof of Theorem 1. However, there are some other technical difficulties one must to overcome.

Theorem 3 *For* $5 \le r < n$, *we have*

$$\kappa(n,r) = e^{-H_r} + O\left(\frac{\nu(n,r)}{r}\right).$$

Proof. Quite the same technique to that used in the proof of Theorem 2 is employed, just a different approximation of the saddle point is taken and Corollary 5 of [8] is applied.

Theorem 1 and Theorem 2 (see also Corollary 2.3 in [5]) improve on Proposition 1 and Proposition 2. Theorem 3 is of separate interest; as we see, it can be useful in formulas where both probabilities $\kappa(n,r)$ and $\nu(n,r)$ are involved. Here is an example.

Let $k_j(\sigma)$ equal the number of cycles of length j in a permutation $\sigma \in \mathrm{S}_n$, $\overline{k}(\sigma) = (k_1(\sigma), k_2(\sigma), \ldots, k_n(\sigma))$, and $\overline{Z} = (Z_1, Z_2, \ldots, Z_n)$, where Z_j are Poisson random variables such that $\mathbf{E}Z_j = 1/j$, $j \in \mathrm{N}$. Thus, if $5 \leq r < n$, we have (see Lemma 3.1 on p. 69 of [1])

$$
\begin{aligned}
d_{TV}(n, r) &= \sup_{V \subseteq \mathrm{Z}_+^r} \left| \frac{\#\{\sigma : \overline{k}(\sigma) \in V\}}{n!} - \Pr(\overline{Z} \in V) \right| \\
&= \frac{1}{2} \sum_{m=0}^{\infty} \nu(m, r) \left| \kappa(n - m, r) - \mathrm{e}^{-H_r} \right| \\
&= \frac{\mathrm{e}^{-H_r}}{2} \sum_{m=n-r}^{\infty} \nu(m, r) + \frac{1}{2}\nu(n, r) + O\left(\frac{1}{r} \sum_{m=0}^{n-r-1} \nu(m, r)\nu(n - m, r) \right).
\end{aligned}
$$

Consequently, only results on the probability $\nu(n, r)$ are needed attempting to improve on the order of notable estimate for $d_{TV}(n, r)$ in [2].

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