THE CHEBOTAREV DENSITY THEOREM AND ARTIN L-FUNCTIONS

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Contents

1. Introduction

The Chebotarev density theorem is a vast generalization of the prime number theorem in arithmetic progressions. Even proving ineffective versions of this theorem (i.e., with no error term) is hard. The purpose of this paper is to give the readers a rough idea of how to prove effective versions of this theorem with an explicit error term. The first such version was proved by Lagarias and Odlyzko in their 1977 paer [1],

Throughout this paper, let L/K be a Galois extension of number fields, with Galois group G. Let n_L,n_K denote the degress of the number fields, and let Δ_L, Δ_K denote the absolute value of their discriminants.

We set up some more notations: denote by $\mathfrak{p}$ a prime ideal in $\mathcal{O}_K$, and denote by $\mathfrak{P}$ a prime ideal in $\mathcal{O}_L$ lying above $\mathfrak{p}$. Let $D_{\mathfrak{P}}$ be the decomposition group of $\mathfrak{P}$, i.e., the set of elements in G that fix $\mathfrak{P}$. Then there is a surjective map $D_{\mathfrak{P}} \to \mathrm{Gal}(\mathcal{O}_L/\mathfrak{P} \mid \mathcal{O}_K/\mathfrak{p})$, whose kernel is by definition the inertia group $I_{\mathfrak{P}}$. Hence, there is an isomorphism $D_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}(\mathcal{O}_L/\mathfrak{P} \mid \mathcal{O}_K/\mathfrak{p})$, where the later group is generated by the Frobenius automorphism. Recall that $\mathfrak{p}$ is unramified if and only if $I_{\mathfrak{P}}$ is trivial. Thus, if $\mathfrak{p}$ is unramified, then there exists a unique element $\varphi_{\mathfrak{P}} \in D_{\mathfrak{P}}$ that is sent to the Frobenius automorphism under the isomorphism. The element $\varphi_{\mathfrak{P}}$ is what we call the Frobenius element of $\mathfrak{P}$. Different choices of $\mathfrak{P}$ lying above $\mathfrak{p}$ yield conjugate Frobenius elements. The corresponding conjugacy class of G is what we call the Artin symbol of $\mathfrak{p}$, written as $\left[\frac{L/K}{\mathfrak{p}}\right]$. If $\mathfrak{p}$ is ramified, then we get a Frobenius element in the quotient group $D_{\mathfrak{P}}/I_{\mathfrak{P}}$.

We now state the main theorem.

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Theorem 1.1 (Chebotarev Density Theorem)**.** *Let L/K be a finite Galois extension of number fields. Let $C \subset \mathrm{Gal}(L/K)$ be a conjugacy class, and define*

$$\pi_C(x) := \left\{ \mathfrak{p} \subset \mathcal{O}_K : N\mathfrak{p} \leq x, \mathfrak{p} \text{ unramified in } L, \ \left[\frac{L/K}{\mathfrak{p}}\right] = C \right\}.$$

Then

$$\pi_C(x) \sim \frac{|C|}{|G|} \operatorname{Li}(x) \qquad \text{as } x \to \infty,$$

where $\operatorname{Li}(x) = \int_2^x \frac{1}{\log t} dt$ is the logarithmic integral function.

Applying this theorem with $L = \mathbb{Q}(\zeta_m)$, i.e., the mth cyclotomic field, and $K = \mathbb{Q}$ yields the usual prime number theorem in arithmetic progressions (PNTAP).

The goal of this paper is to sketch a proof of effective versions of Theorem 1.1 with explicit error terms. About half of the proof will be similar to the proof of the usual prime number theorem in arithmetic progressions, but there will be a few key differences. We will focus on these differences in our exposition.

2. Artin L-functions

The key tool to studying prime ideals and Frobenius elements is the Artin L-function, which we now define. These L-functions encodes important arithmetic data of the Galois extension.

If $D \to \mathrm{GL}(V)$ is any representation of a group D, and if $I \subset D$ is any normal subgroup, we let V^I be the set of elements of V that are fixed by I. Then D acts on V^I because I is normal, and since I acts trivially, the action factors through D/I. Hence, there is a representation $D/I \to \mathrm{GL}(V^I)$.

Definition 2.1. Let $\mathrm{Gal}(L/K) \to \mathrm{GL}(V)$ be a representation with character ϕ. The associated Artin L-function is

$$\mathrm{L}(s, \phi, L/K) = \prod_{\mathfrak{p}} \left(\det(1 - \varphi_{\mathfrak{P}}(N\mathfrak{p})^{-s}; V^{I_{\mathfrak{P}}} \right)^{-1}, \tag{1}$$

where $\mathfrak{P}$ is any prime ideal lying over $\mathfrak{p}$, and $\varphi_{\mathfrak{P}}$ is the corresponding Frobenius element in $D_{\mathfrak{P}}/I_{\mathfrak{P}}$, understood as a linear endomorphism of $V^{I_{\mathfrak{P}}}$ and so its determinant is defined. Note that this is independent of the choice of $\mathfrak{P}$ because different choices yield simultaneous conjugate Frobenius elements (as well as the decomposition group and the inertia group).

Theorem 2.2 (Properties of Artin L-functions)**.** *Let L/K be a Galois extension of number fields.*

(i) $\mathrm{L}(s, \mathbb{1}_G, L/K) = \zeta_K(s)$, *where $\mathbb{1}_G$ is the one-dimensional trivial representation of G.*

(ii) *If χ, χ' are two characters of G, then*

$$\mathrm{L}(s, \chi + \chi', L/K) = \mathrm{L}(s, \chi, L/K) \cdot \mathrm{L}(s, \chi', L/K).$$

(iii) *For two Galois extensions $L' \supseteq L \supseteq K$, and a character χ of $\mathrm{Gal}(L/K)$ one has*

$$\mathrm{L}(s, \chi, L'/K) = \mathrm{L}(s, \chi, L/K),$$

where χ is also understood as a character of $\mathrm{Gal}(L'/K)$ via the quotient map $\mathrm{Gal}(L'/K) \to \mathrm{Gal}(L/K)$

(iv) *If M is an intermediate field, $L \supseteq M \supseteq K$, and χ is a character of* $\mathrm{Gal}(L/M) \subset \mathrm{Gal}(L/K)$*, then*

$$\mathrm{L}(s, \chi, L/M) = \mathrm{L}(s, \chi^*, L/K),$$

where $\chi^* = \mathrm{Ind}_H^G \chi$ is the character on G induced from χ.

(v) *If G is abelian, then there exists a primitive Hecke character $\widetilde{\chi}$ of K such that* $\mathrm{L}(s, \chi, L/K)$ *equals the Hecke L-function associated to $\widetilde{\chi}$. Thus, in this case, the Artin L-function is entire (unless χ is trivial) of order* 1 *and admits an analytic continuation and functional equation.*

Proof. This is proved in [3], Chapter VII, section 11. ∎

Remark 2.3. We remark that the last statement about abelian Artin L-functions is extremely non-trivial and its proof hinges on deep results from class field theory, namely the Artin reciprocity law. Details can be found in [3], Chapter VII, section 10.

As an application of (iv), take the trivial subgroup $\{1\}$ of G and the trivial representation on $\{1\}$; the induced representation on G is the regular representation, which admits a decomposition

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi,$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of G. Therefore, using (i) to (iv) of Lemma 2.2, we obtain the important identity:

$$\zeta_L(s) = \prod_{\chi} \mathrm{L}(s, \chi, L/K)^{\chi(1)} = \zeta_K(s) \prod_{\chi \neq \mathbb{1}_G} \mathrm{L}(s, \chi, L/K)^{\chi(1)}.$$

If, in addition, $\mathrm{Gal}(L/K)$ is abelian, then $\chi(1) = 1$ for all χ, and the Artin L-function associated to each χ is a Hecke L-function.

The Artin conductor of χ, denoted by $\mathfrak{f}_\chi$, is an ideal in $\mathcal{O}_K$ that is canonically defined in terms of χ and the ramification groups at each prime ideal, generalizing the notion of conductor of a Dirichlet character. It is beyond the scope of this paper to define $\mathfrak{f}_\chi$. In the abelian case, $\mathfrak{f}_\chi$ equals the conductor of $\widetilde{\chi}$ as a Hecke character. This was proved in [3, p.538]. Furthermore, we have the following

Theorem 2.4 (Conductor-Discriminant Formula)**.** *For an arbitrary Galois extension L/K of number fields, one has*

$$\Delta_{L/K} = \prod_{\chi} \mathfrak{f}_\chi^{\chi(1)},$$

where χ varies over the irreducible characters of $\mathrm{Gal}(L/K)$ *and $\Delta_{L/K}$ is the relative discriminant.*

Proof. This is proved in [3], Chapter VII, Theorem 11.9. ∎

In order to state the functional equation, we define

$$A_\chi = \Delta_K^{\chi(1)} N_{K/\mathbb{Q}}(\mathfrak{f}_\chi).$$

The quantity A_χ will be the *analytic conductor* of our L-function. Note that by the conductor-discriminant formula, we have:

$$\prod_\chi A_\chi^{\chi(1)} = \left(\prod_\chi \Delta_K^{\chi(1)^2}\right) N_{K/\mathbb{Q}}\left(\prod_\chi \mathfrak{f}_\chi^{\chi(1)}\right) = \Delta_K^{[L:K]} N_{K/\mathbb{Q}}(\Delta_{L/K}) = \Delta_L,$$

because

$$\sum_\chi \chi(1)^2 = |G| = [L : K].$$

Now, we state the following important result:

Theorem 2.5. *Let χ be an irreducible character of $G = \mathrm{Gal}(L/K)$. Then there exist two nonnegative integers $a = a(\chi)$ and $b = b(\chi)$ satisfying $a + b = n_K\chi(1)$, such that if we define*

$$\gamma_\chi(s) = \left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right]^a \left[\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right]^b$$

and

$$\xi(s, \chi) = [s(s-1)]^{\delta(\chi)} A_\chi^{s/2}\gamma_\chi(s)\mathrm{L}(s, \chi),$$

then $\xi(s, \chi)$ is meromorphic on $\mathbb{C}$ and satisfies the functional equation

$$\xi(1 - s, \bar{\chi}) = W(\chi)\xi(s, \chi),$$

where $W(\chi)$ is some complex number of absolute value 1. Furthermore, if G is abelian, then $\xi(s, \chi)$ is entire of order 1.

Proof. This was proved in [3], Chapter VII, Section 12. We remark that the treatment there is classical. A more modern treatment is Tate's thesis [7]. ∎

The above theorem is the most fundamental tool enabling us to study the distribution of Frobenius elements among prime ideals in $\mathcal{O}_K$. It is worth emphasizing that we do not know whether $\xi(s, \chi)$ (and hence $\mathrm{L}(s, \chi)$ for $s \neq 1$) is entire unless G is abelian. It is conjectured that the same is true for all finite groups. This is known as the *Artin holomorphy conjecture*:

Conjecture 2.6 (Artin's Holomorphy Conjecture)**.** *Let L/K be a Galois extension of number fields with Galois group G, not necessary abelian. Then for every irreducible character χ of G, the function $\mathrm{L}(s, \chi)$ is entire (unless χ is trivial, in which case there is a pole at $s = 1$).*

3. Relating Artin L-functions to the Chebotarev density theorem

After a brief introduction to Artin L-functions, we now come back and start proving the Chebotarev density theorem. Fix a conjugacy class $C \subset G$. We would like to study the counting function $\pi_C(x)$. Using simple partial summation methods, one easily relates $\pi_C(x)$ to the partial sums of the generalized von Mangoldt function:

$$\psi_C(x) := \sum_{\substack{N\mathfrak{p}^m \leq x \\ \mathfrak{p} \text{ unramified} \\ \left[\frac{L/K}{\mathfrak{p}}\right]^m = C}} \log(N\mathfrak{p}). \tag{2}$$

We will cook up a Dirichlet series such that the partial sums of it coefficients equals ψ_C, up to finitely many ramified primes. This involves the logarithmic derivatives of Artin L-functions, which we compute now.

Fix an irreducible representation of G and let ϕ be its character. Define

$$\phi_K(\mathfrak{p}^m) = \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \phi(\varphi_{\mathfrak{P}}^m \alpha),$$

where $\mathfrak{P}$ is any prime ideal lying above $\mathfrak{p}$. Note that this is well defined, for if a different $\mathfrak{P}'$ above $\mathfrak{p}$ was chosen, then $\varphi_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ are simultaneously conjugate to $\varphi_{\mathfrak{P}'}$ and $I_{\mathfrak{P}'}$, say by $g \in G$, and so

$$\sum_{\alpha \in I_{\mathfrak{P}'}} \phi(\varphi_{\mathfrak{P}'}^m \alpha) = \sum_{\alpha \in I_{\mathfrak{P}}} \phi((g\varphi_{\mathfrak{P}} g^{-1})^m g\alpha g^{-1}) = \sum_{\alpha \in I_{\mathfrak{P}}} \phi(g\varphi_{\mathfrak{P}}^m \alpha g^{-1}) = \sum_{\alpha \in I_{\mathfrak{P}}} \phi(\varphi_{\mathfrak{P}}^m \alpha)$$

because ϕ is a class function.

Lemma 3.1. *Let L/K and ϕ be as above. Then for* $\mathrm{Re}(s) > 1$*, we have*

$$-\frac{\mathrm{L}'}{\mathrm{L}}(s, \phi, L/K) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \phi_K(\mathfrak{p}^m) \log(N\mathfrak{p})(N\mathfrak{p})^{-ms}. \tag{3}$$

Proof. Parts of the proof were modified from [5]. It suffices to compute the logarithmic derivative of the characteristic polynomial

$$f(s) = \det(1 - \varphi_{\mathfrak{P}}(N\mathfrak{p})^{-s}; V^{I_{\mathfrak{P}}})$$

for each prime ideal $\mathfrak{p}$. Since any matrix can be upper triangularized, we see that

$$f(s) = \prod_{i=1}^{r} (1 - \lambda_i (N\mathfrak{p})^{-s}),$$

where λ_i are the eigenvalues of $\varphi_{\mathfrak{P}}$ on $V^{I_{\mathfrak{P}}}$. It follows that

$$\begin{aligned} -\frac{f'}{f}(s) &= \sum_{i=1}^{r} \frac{\lambda_i (N\mathfrak{p})^{-s} \log(N\mathfrak{p})}{1 - \lambda_i (N\mathfrak{p})^{-s}} \\ &= \log(N\mathfrak{p}) \sum_{i=1}^{r} \sum_{m=1}^{\infty} \lambda_i^m (N\mathfrak{p})^{-ms}. \end{aligned}$$

Therefore, it suffices to show that for any $m \geq 1$,

$$\sum_{i=1}^{r} \lambda_i^m = \phi_K(\mathfrak{p}^m).$$

Since λ_i^m are precisely the eigenvalues of $\varphi_{\mathfrak{P}}^m$, the left hand side equals $\mathrm{tr}(\varphi_{\mathfrak{P}}^m)$. The statement now follows from the following purely representation theoretic lemma, applied with $D = D_{\mathfrak{P}}$, $I = I_{\mathfrak{P}}$, and $g = \varphi_{\mathfrak{P}}$. ∎

Lemma 3.2. *Let D be a finite group acting on a finite dimensional vector space V. Let I be a normal subgroup, so that D/I (and hence D) acts on V^I as well. Then for any $g \in D$, one has*

$$\mathrm{tr}(g|_{V^I}) = \frac{1}{|I|} \sum_{\alpha \in I} \mathrm{tr}(g\alpha).$$

Proof. Choose a basis $\{v_1, \dots, v_r\}$ of V^I and extend it to a basis $\{v_1, \dots, v_n\}$ of V, where $r \le n$. It suffices to compute the trace of $h = g\left(\sum_{\alpha \in I} \alpha\right)$. For v_i $(1 \le i \le r)$, $hv_i = |I|gv_i$, and so the action of h on V^I is just $|I|$ times the action of $g|_{V^I}$. On the other hand, for v_j $(r+1 \le j \le n)$, the vector

$$\sum_{\alpha \in I} \alpha v_j$$

clearly lies in V^I, and so $hv_j \in V^I$. This means that the matrix representation of h is of the form

$$\begin{pmatrix} |I| \cdot g|_{V^I} & * \\ 0 & 0 \end{pmatrix},$$

whose trace is therefore $|I| \cdot \mathrm{tr}(g|_{V^I})$. This proves the desired formula. ∎

Now, using character orthogonality, a suitable linear combination of (3) allows us to extract prime ideals with a given Artin symbol. More specifically, the indicator function of $C \subset G$ can be decomposed as

$$\frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C)\phi(y) = \begin{cases} 1 & \text{if } y \in C, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of G. Hence, if we take

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C) \frac{\mathrm{L}'}{\mathrm{L}}(s, \phi, L/K), \tag{4}$$

then for $\mathrm{Re}(s) > 1$:

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \underbrace{\frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C)\phi_K(\mathfrak{p}^m)}_{\theta(\mathfrak{p}^m)} \log(N\mathfrak{p})(N\mathfrak{p})^{-ms}.$$

Let $\theta(\mathfrak{p}^m)$ be the under-braced expression. Then

$$\theta(\mathfrak{p}^m) = \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C)\phi(\varphi_{\mathfrak{P}}^m \alpha).$$

The quantity $\frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C)\phi(\varphi_{\mathfrak{P}}^m \alpha)$ either 0 or 1, so $0 \le \theta(\mathfrak{p}^m) \le 1$. However, if $\mathfrak{p}$ is unramified, then I is trivial, and so

$$\theta(\mathfrak{p}^m) = \frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C)\phi(\varphi_{\mathfrak{P}}^m) = \begin{cases} 1 & \text{if } \left[\frac{L/K}{\mathfrak{p}}\right]^m = C, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the partial sums of the coefficients of $F_C(s)$, namely $\theta(\mathfrak{p}^m)\log(N\mathfrak{p})$, give us the function $\psi_C(x)$ defined in (2) up to finitely many ramified primes, which can be easily dealt with since their contribution is microscopic compared to the error term we will eventually obtain.

4. Reduction to the abelian case

Since ψ_C is essentially the partial sum of the Dirichlet coefficients of $F_C(s)$, a standard application of the inverse Mellin transform allows us to express ψ_C as the following truncated integral

$$I_C(x,T) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} F_C(s)\frac{x^s}{s}ds \tag{5}$$

plus a relatively small error term. The techniques used here are identical the ones used in the proof of the prime number theorem in arithmetic progressions, so we omit them.

At this point, we would like to use the "old plan" that we had in the proof of the usual prime number theorem in arithmetic progressions. Namely, we want to push the integration contour to the left, picking up the residues of L'/L at the zeros and poles of L, expecting that the main contribution comes from the pole at $s = 1$. This, however, would not work well because for non-abelian extensions L/K, we do not know whether the Artin L-functions have poles other than at $s = 1$. In other words, we only have a meromorphic continuation of $\mathrm{L}(s,\phi,L/K)$ to the left of $\mathrm{Re}(s) > 1$. We did not have such a trouble when dealing with Dirichlet L-functions. This significantly impacts our ability to estimate the integrals.

A clever argument due to Serre in [4, Section 2.7] is now employed to reduce the non-abelian Artin L-functions to the abelian case. The goal is to re-express (4) as a linear combination of characters of some subextension of L/K. To this end, we shall use the following lemma:

Lemma 4.1. *Let C be a conjugacy class of G, and let H be any subgroup of G intersecting C. Pick an element $u \in C \cap H$, and let C_H be the conjugacy class in H that contains u. Then*

$$\sum_{\phi\in \mathrm{Irr}(G)} \bar{\phi}(C)\phi = \sum_{\chi \in \mathrm{Irr}(H)} \bar{\chi}(C_H)\chi^*, \tag{6}$$

as class functions on G, where $\chi^* = \mathrm{Ind}_H^G \chi$ is the character of G induced by χ.

Proof. Let $\delta_C : G \to \{0,1\}$ and $\delta_{C_H} : H \to \{0,1\}$ be the indicator functions of C and C_H, respectively. By character orthogonality, we have

$$\delta_C = \frac{|C|}{|G|}\sum_{\phi\in\mathrm{Irr}(G)}\bar{\phi}(C)\phi \quad \text{and} \quad \delta_{C_H} = \frac{|C_H|}{|H|}\sum_{\chi\in\mathrm{Irr}(H)}\bar{\chi}(C_H)\chi. \tag{7}$$

On the other hand, consider the induced class function $\delta^*_{C_H} = \mathrm{Ind}_H^G(\delta_{C_H})$. If $g \in G\setminus C$, then any conjugate of g is not in C, and hence not in C_H. Therefore, $\delta^*_{C_H}(g) = 0$. This shows that $\delta^*_{C_H}$ vanishes outside C. Since $\delta^*_{C_H}$ is a class function on G, it must be a constant multiple of δ_C, say $\delta^*_{C_H} = \lambda\delta_C$. By Frobenius reciprocity and definition of the inner products on G and H, we have

$$\lambda\frac{|C|}{|G|} = \langle \delta^*_{C_H}, \mathbb{1}_G\rangle = \langle \delta_{C_H}, \mathbb{1}_H\rangle = \frac{|C_H|}{|H|}.$$

It follows that

$$\lambda = \frac{|C_H|}{|H|}\frac{|G|}{|C|},$$

and so

$$\frac{|H|}{|C_H|}\delta^*_{C_H} = \frac{|G|}{|C|}\delta_C.$$

Comparing this with (7) yields the result. ∎

Lemma 4.1 holds for any subgroup H intersecting the conjugacy class C. Now, we take H to be an abelian subgroup intersecting C. (For example, take any element $g \in C$, and let H be the cyclic subgroup generated by g.) Inserting (6) into the equation

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\phi}(C) \phi(\varphi_{\mathfrak{P}}^m \alpha) \log(N\mathfrak{p})(N\mathfrak{p})^{-ms}, \tag{8}$$

we obtain

$$\begin{aligned} F_C(s) &= \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \frac{|C|}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \bar{\chi}(C_H) \chi^*(\varphi_{\mathfrak{P}}^m \alpha) \log(N\mathfrak{p})(N\mathfrak{p})^{-ms} \\ &= -\frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(H)} \bar{\chi}(C_H) \frac{\mathrm{L}'}{\mathrm{L}}(s, \chi^*, L/K) \\ &= -\frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(H)} \bar{\chi}(C_H) \frac{\mathrm{L}'}{\mathrm{L}}(s, \chi, L/E), \end{aligned}$$

where $E = L^H$ is the fixed field of H. The last step uses (iv) of Theorem 2.2. Since H is abelian, we have expressed $F_C(s)$ as a linear combination of Hecke L-functions, which admit analytic (rather than meromorphic) continuations to $\mathbb{C}$. All of the above equalities hold for $\mathrm{Re}(s) > 1$, and hence they hold for all $s \in \mathbb{C} \setminus \{1\}$.

We remark that this reduction step comes with certain costs. If we knew the Artin conjecture for G, then we could work directly with the L-functions of L/K, and all error terms will involve invariants of K (such as n_K and Δ_K). However, since we are working with the Artin L-functions of L/E instead, all estimates will depend on E, which in general results in weaker error terms.

5. Remaining steps of the proof

Now that we have expressed $F_C(s)$ in terms of abelian (Hecke) L-functions, it remains to estimate the integral in (5), which is broken into pieces of the form

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} -\frac{\mathrm{L}'}{\mathrm{L}}(s, \chi, L/E) \frac{x^s}{s} ds.$$

This is strongly reminiscent of the proof of the prime number theorem in arithmetic progressions, where Dirichlet L-functions have been replaced by Hecke L-functions. The remaining parts of the analysis proceed similarly. In other words, one now pushes the integration contour to the left of the critical strip, while picking up the residues at the zeros of $\mathrm{L}(s, \chi)$. An explicit formula for $\psi_C(x)$ now follows, where the main term comes from the pole of $\mathrm{L}(s, \mathbb{1}_G) = \zeta_K(s)$ at $s = 1$. To estimate the residues coming from the nontrivial zeroes , a zero free region is needed for each $\mathrm{L}(s, \chi)$. However, in light of the relation

$$\zeta_L(s) = \prod_{\chi \in \mathrm{Irr}(H)} \mathrm{L}(s, \chi),$$

it suffices to prove a zero free region for ζ_L, which is much easier to accomplish. This relatively cheap soluation yields weaker error terms, but it is what was done in section 8 of Lagarias-Odlyzko [1]. The proof does not exclude the existence of an exceptional Siegel zero β_1 of ζ_L

near $s = 1$, which must be real and simple if it exists. The best known upper bound for β_1 is due to Stark [6], in which it was proved that

$$1 - \beta_1 \gg (n_L^{n_L} \log D_L + D_L^{1/n_L})^{-1}.$$

Substituting the upper bound for β_1 into the explicit formula, we eventually obtain the desired theorems, which we state below.

Theorem 5.1 (Lagarias-Odlyzko)**.** *There exists an effectively computable positive absolute constant c_1 such that if GRH holds for the Dedekind zeta function of L, then for every $x > 2$,*

$$\left|\pi_C(x, L/K) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq c_1 \left\{\frac{|C|}{|G|} x^{\frac{1}{2}} \log\left(\Delta_L x^{n_L}\right) + \log \Delta_L\right\}.$$

Theorem 5.2 (Lagarias-Odlyzko)**.** *There exist absolute effectively computable constant c_2 and c_3 such that if*

$$x \geq \exp(10 n_L (\log \Delta_L)^2),$$

then

$$\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_1}) + c_2 x \exp(-c_3 n_L^{-\frac{1}{2}} (\log x)^{\frac{1}{2}}),$$

where the β_1 is the possible exceptional zero of ζ_L.

The second theorem stated above has quite weak error terms. For example, it is not strong enough to recover the Siegel-Walfisz theorem on primes in arithmetic progressions. Several substantial improvements have been made since then. Interested readers may refer to [8] for more details.

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