THE SMOOTH SERRE-SWAN THEOREM

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Abstract. In this expository paper, we give a detailed proof of the smooth Serre-Swan theorem. That is, we prove that the category of vector bundles over a connected smooth manifold M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$, the ring of smooth functions on M. We will also present some of the consequences of this important theorem, including the fact the global section functor preserves exterior powers.

Contents

1. Introduction

Vector bundles are of fundamental importance in modern differential geometry. Its philosophy is similar to that of a manifold: roughly speaking, a vector bundle is a space that locally looks like the product space with $\mathbb{R}^k$ but globally might have different structures.

An extremely important theme in geometry is the relation between vector bundles and modules. The slogan is that modules can be thought of algebraic generalizations of vector bundles, and they have better formal properties than vector bundles.

In 1955, Serre proved in [3] a correspondence between vector bundles over an affine variety and finitely generated projective modules over its coordinate ring. In 1962, Swan proved in [4] an analog of this result for topological vector bundles over compact Hausdorff spaces. As one might have guessed, there is yet another analog of this result in differential geometry, which concerns smooth vector bundles over arbitrary smooth manifolds. This is known as the smooth Serre-Swan theorem, which is the subject of this paper. Unfortunately, I do not know who is the first person to write down this theorem in a fully rigorous fashion, but it seems that the book [2] by Nestruev is one of the earliest and most common references of this result. It should

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be noted that the name Nestruev refers a collective of Russian authors. We shall mostly follow Chapter 12 of this book.

The smooth Serre-Swan theorem states that for every connected manifold M, there is an equivalence of categories between the category of vector bundles over M and the category of finitely generated projective modules over $C^\infty(M)$, the ring of smooth real-valued functions on M. It is quite a remarkable fact that this theorem holds in this level of generality: there is no compactness assumption on the base manifold, which is necessary in the topological case. Hence, their proofs are quite different from each other, and the smooth case is arguably more complicated (and also more interesting). In this paper, we shall present a rigorous, concise, and self-contained proof of this beautiful theorem and some of its consequences.

We shall assume that the readers are reasonably familiar with the basics of vector bundles. More specifically, we assume that the readers have seen the following concepts:

(i) definitions of vector bundles, subbundles, morphisms, sections, and frames;
(ii) basic operations on vector bundles, such as direct sums and tensor products (or in general, any smooth functor);
(iii) other things such as pullback of vector bundles and fiber metrics.

We encourage the readers to first read Chapter 10 of [1] if they are not already familiar with these materials. Nevertheless, to make things as self-contained as possible, we shall state these definitions in section 2.

This paper is organized as follows: In section 2 we shall review some basic facts about vector bundles and state without proof a few elementary facts that we will keep using throughout the paper. We will also provide a setup for the smooth Serre-Swan theorem and formally state the result.

In section 3, we introduce a key construction: the tautological vector bundle over the Grassmannian. This is a special type of vector bundles that will play a crucial role in the eventual proof of the smooth Serre-Swan theorem.

In section 4, we return to general vector bundles and prove a few more facts about them. They will come in handy later on.

In section 5, we give our promised proof of the smooth Serre-Swan theorem. The proof falls into two parts: one where we prove that the global section functor is essentially surjective, and the other where we prove that it is fully faithful. The two parts use different techniques, and both involve some very interesting tricks.

In section 6, we present one important consequence of the smooth Serre-Swan theorem. We shall show that the global section functor is monoidal, i.e., it preserves tensor products. In fact, we will show that it preserves all exterior powers.

In section 7, we present one more elegant corollary of the theorem. We shall describe a natural relationship between pullbacks of vector bundles and extensions of scalars in modules. This is where all of our hard work pays off and we are able to prove some very juicy results.

2. Review of the basics of vector bundles

We follow [1] in this section. For convenience, we shall assume that all vector bundles are of constant rank. In fact, let us assume without loss of generality that all vector bundles are over some *connected* base manifold. We now begin our very brief review of the basic theory of vector bundles.

Definition 2.1. A *vector bundle* over a manifold M with typical fiber V (a finite dimensional real vector space) is a manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions.

(i) For every $q \in M$, the fiber $E_q := \pi^{-1}(q)$ is a vector space isomorphic to V.
(ii) For every $q \in M$, there exists an open neighborhood U of q and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times V$ such that $\varphi(E_{q'}) = \{q'\} \times V$ for all $q' \in U$. In other words, the following diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times V \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes; furthermore, each map on fibers $\varphi|_{E_{q'}} : E_{q'} \to \{q'\} \times V$ is an isomorphism of vector spaces.

In this case, E is called the total space; M is called the base; $\varphi : \pi^{-1}(U) \to U \times V$ is called a local trivialization. The dimension of V is called the rank of E.

Definition 2.2. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over the same manifold M. Then a morphism of vector bundles is a smooth map $f : E \to F$ so that (i) for each $q \in M$, $f(E_q) \subseteq F_q$, i.e.,

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

commutes, and (ii) each map on fibers $f|_{E_q} : E_q \to F_q$ is linear. It's easy to check that vector bundles over a fixed manifold M form a category VB_M.

Definition 2.3. Given a vector bundle $\pi : E \to M$, a subbundle of E is a submanifold D of E such that

(i) for each $p \in M$, the subset $D_p = D \cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p;
(ii) the restriction $\pi_D := \pi|_D$ makes $\pi_D : D \to M$ into a vector bundle over M.

Definition 2.4. Given a vector bundle $\pi : E \to M$ and a smooth map $f : N \to M$, we take the pullback (i.e., fiber product) of the diagram and obtain:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

Because π is obviously a submersion, it can be shown using transversality that f^*E is an embedded submanifold of $N \times E$ and that the map $f^*\pi : f^*E \to N$ makes f^*E into a vector bundle over N. This is called the *pullback* of E along f, which is a fundamental operation of vector bundles.

Definition 2.5. A *section* of a vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. We write $\Gamma(E)$ or simply $\Gamma(E)$ for the set of all sections. The set of sections $\Gamma(E)$, or $\Gamma(E)$, is a module over $C^\infty(M)$. A local section of $E \xrightarrow{\pi} M$ is a section $s : U \to E|_U := \pi^{-1}(U)$ for some open set $U \subseteq M$.

Definition 2.6. A *local frame* of a vector bundle $E \to M$ over an open set $U \subseteq M$ is a collection of local sections $\{s_i : U \to E|_U\}_{i=1}^k$ such that for each $x \in U$, the set $\{s_1(q), \dots, s_k(q)\}$ is a basis of E_q. If $U = M$, then it is said to be a *global frame*.

We state without proof the following elementary but highly useful lemma.

Lemma 2.7 (Lemma 10.32 in [1]). *Let $\pi : E \to M$ be a smooth vector bundle, and suppose that for each $p \in M$ we are given an m dimensional linear subspace $D_p \subseteq E_p$. Then $D = \bigcup_{p\in M} D_p \subseteq E$ is a smooth subbundle of E if and only if the following condition is satisfied: Each point of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \dots, \sigma_m : U \to E$ with the property that $\sigma_1(q), \dots, \sigma_m(q)$ form a basis for D_q at each $q \in U$.*

Finally, there is an obvious functor $\Gamma : \mathsf{VB}_M \to \mathsf{Mod}_{C^\infty(M)}$, defined on objects by taking a vector bundle $E \to M$ to its global section $\Gamma(E)$. On morphisms, it $f : E \to F$ to the corresponding map between modules of sections $\Gamma(E) \to \Gamma(F)$ defined by $s \mapsto f \circ s$. We call this functor Γ, the global section functor. The smooth Serre-Swan theorem states that this functor is fully faithful, and that its essential image is the full subcategory $\mathsf{FinProj}_{C^\infty(M)}$ consisting of all finitely generated projective modules over $C^\infty(M)$. Hence, vector bundles and modules are really the "same" thing.

3. Tautological vector bundles over the Grassmannian

Definition 3.1. Let V be an n-dimensional vector space. Let $\mathrm{Gr}_k(V)$ be the set of k-dimensional linear subspaces of V. This is called the k-dimensional *Grassmannian* of V. If $V = \mathbb{R}^n$, then we may write $\mathrm{Gr}_k(\mathbb{R}^n)$ as $\mathrm{Gr}_{n,k}$.

It is well known that $\mathrm{Gr}_k(V)$ is a manifold. We sketch the construction by showing how to introduce coordinate charts on $\mathrm{Gr}_k(V)$.

Let $V = P \oplus Q$ be a decomposition of V into the direct sum of two subspaces, where $\dim P = k$. Let $\mathrm{Hom}(P, Q)$ be the vector space of all linear maps from P to Q. For every $f \in \mathrm{Hom}(P, Q)$, the graph of f, namely,

$$G(f) := \{v + f(v) : v \in P\},$$

is a k-dimensional subspace of V. Clearly $G(f) \cap Q = \{0\}$.

Conversely, any subspace W of V that intersects Q trivially is the graph of a unique linear map $f \in \mathrm{Hom}(P, Q)$. Uniqueness is obvious because two different linear maps must have different graphs. For existence, let $\pi_P : V \to P$ and $\pi_Q : V \to Q$ be the projections determined by the direct sum decomposition. Then $\pi_P|_W : W \to P$ is an isomorphism by hypothesis. For every $v \in P$, let $f(v) = (\pi_Q|_W) \circ (\pi_P|_W)^{-1}(v)$. Then W is the graph of f.

Let $U_Q \subset \mathrm{Gr}_k(V)$ be the subset consisting of all k-dimensional subspaces that intersect Q trivially. Then the map

$$G : \mathrm{Hom}(P, Q) \to U_Q,$$

defined by sending a linear map f to its graph, is a bijection. Since $\mathrm{Hom}(P, Q)$ is a vector space of dimension $k(n-k)$, the map is the (inverse of the) coordinate chart we want.

It remains to check that these coordinate charts overlap smoothly, which amounts to some complicated but routine calculations. We omit the details here and refer the readers to Example 1.36 in [1]. Now, we start constructing the tautological bundles over these Grassmann manifolds.

Lemma 3.2. *The subset*

$$\Theta_k(V) := \{(S, v) \in \mathrm{Gr}_k(V) \times V : v \in S\}$$

is a rank-k subbundle of the trivial bundle $\pi : \mathrm{Gr}_k(V) \times V \to \mathrm{Gr}_k(V)$.

Proof. By Lemma 2.7, it suffices to find local frames for $\Theta_k(V)$. Consider the open set $U_Q \subset \mathrm{Gr}_k(V)$ defined above. Pick a complementary subspace P of Q, and pick a basis $\{v_1, \dots, v_k\}$ of P. Let $\varphi : \mathrm{Hom}(P, Q) \to U_Q$ be the chart defined above. For $i = 1, \dots, k$, define $s_i : U_Q \to \Theta_k(V)$ by

$$s_i(G(f)) = (G(f), v_i + f(v_i)).$$

Since $\{v_i\}$ form a basis of P, $\{v_i + f(v_i)\}$ form a basis of $G(f)$. It follows that $\{s_1, \dots, s_k\}$ is a local frame of the bundle $\Theta_k(V)$, once we verify that s_i are *smooth* sections. Indeed, we already defined s_i in terms of the coordinate charts Γ. Since s is a map into the product space $\mathrm{Gr}_k(V) \times V$, and the first component is the identity map, we only need to verify that the second component of s_i is smooth. But this is obvious because the assignment $\mathrm{Hom}(P, Q) \to V$, $f \mapsto v_i + f(v_i)$, is clearly smooth because f is linear. ∎

Lemma 3.3. *Let M be a manifold and V be an n-dimensional real vector space. Suppose $f_1, \dots, f_k : M \to V$ are smooth functions such that $f_1(x), \dots, f_k(x)$ are linearly independent for all $x \in M$. Then the map $\varphi : M \to \mathrm{Gr}_k(V)$ defined by $\varphi(x) = \mathrm{span}(f_1(x), \dots, f_k(x))$ is smooth.*

Proof. We argue that this map is smooth locally. Since $\{f_i(x)\}_{i=1}^k$ are linearly independent, in particular $f_i(x)$ never vanish. By continuity of f_i, for every point in M there exist a neighborhood U and an $(n-k)$-dimensional subspace $Q \subset V$ such that $\mathrm{span}(f_1(x), \dots, f_k(x))$ intersects Q trivially for all $x \in U$. Pick a complement subspace P of Q in V. Then $\varphi(U)$ is contained in the chart domain $G(\mathrm{Hom}(P, Q))$. It remains to check that $G^{-1} \circ \varphi : U \to \mathrm{Hom}(P, Q)$ is smooth. Indeed, $G^{-1}(\varphi(x))$ is the linear map determined by

$$G^{-1}(\varphi(x))(v) = (\pi_Q|_{\varphi(x)}) \circ (\pi_P|_{\varphi(x)})^{-1}(v),$$

which depends smoothly on x. ∎

Remark 3.4. The statement would be false for nontrivial bundles. Indeed, if the ambient bundle is not $M \times \mathbb{R}^n$, then there is no way to identify each fiber with $\mathbb{R}^n$ in a canonical way because it involves a choice of local trivializations.

4. More facts about vector bundles

We now collect a few facts about vector bundles that might be handy later on, and we will give complete proofs here. For a point $x \in M$, let $\mathfrak{m}_x$ be the maximal ideal of all $f \in C^\infty(M)$ such that $f(x) = 0$.

Lemma 4.1. *Suppose $s \in \Gamma(E)$ is such that $s(x) = 0$ for some $x \in M$. Then there exists $f_1, \dots, f_n \in \mathfrak{m}_x \subset C^\infty(M)$ and $s_1, \dots, s_n \in \Gamma(E)$ such that $s = \sum_{i=1}^n f_i s_i$.*

Proof. Choose a local frame $\{\sigma_1, \dots, \sigma_k\}$ around an open neighborhood U of x. Then $s|_U = \sum_{i=1}^k g_i \sigma_i$ for some $g_i \in C^\infty(U)$, with $g_i(x) = 0$. Choose a bump function ψ with $\psi(x) = 1$ and $\operatorname{supp} \psi \subset U$. Then ψg_i are well defined smooth functions on M, and $\psi \sigma_i$ are well defined smooth global sections. Then

$$\psi^2 s = \sum_{i=1}^k (\psi g_i)(\psi \sigma_i),$$

from which it follows that

$$s = (1 - \psi^2)s + \sum_{i=1}^k (\psi g_i)(\psi \sigma_i),$$

as desired. ■

Remark 4.2. The preceding lemma shows that the submodule of sections that vanish at a point x is precisely $\mathfrak{m}_x \Gamma(E)$. In particular, there is a short exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{m}_x \Gamma(E) \longrightarrow \Gamma(E) \longrightarrow E_x \longrightarrow 0,$$

where the third arrow is evaluation at the point x.

Lemma 4.3. *Let $\{s_1, \dots, s_n\} \subset \Gamma(E)$ be a set of sections. Then TFAE.*

(i) *For every $x \in M$, the set $\{s_1(x), \dots, s_n(x)\}$ spans the fiber E_x.*
(ii) *The set $\{s_1, \dots, s_n\}$ generates $\Gamma(E)$ as a $C^\infty(M)$-module.*

Proof. (ii) implies (i): this is easy because for every $p \in M$ and every $v \in E_x$, there exists a section $s \in \Gamma(E)$ such that $s(x) = v$.

(i) implies (ii): Let k be the rank of E. For every subset $I = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$ consisting of k elements, let U_I be the set of $x \in M$ for which $\{s_{i_1}(x), \dots, s_{i_k}(x)\}$ is linearly independent (hence a basis) at the fiber E_x. By continuity, each U_I is open, and by hypothesis, U_I cover M. Pick a partition of unity $\{\rho_I\}$ subordinate to the open cover $\{U_I\}$. Then for any $s \in \Gamma(E)$,

$$s|_{U_I} = \sum_{i \in I} f_{I,i} s_i,$$

for some $f_{I,i} \in C^\infty(U_I)$. Now, note that

$$s = \sum_I \rho_I \cdot s|_{U_I} = \sum_I \rho_I \sum_{i \in I} f_{I,i} s_i = \sum_I \sum_{i \in I} (\rho_I f_{I,i}) s_i,$$

where $\rho_I f_{I,i}$ is a well defined smooth function on M by construction. ■

Theorem 4.4. *A vector bundle $E \to M$ is trivial if and only if $\Gamma(E)$ is a free $C^\infty(M)$-module.*

Proof. ($\Rightarrow$) If $E \to M$ is trivial, then $\Gamma(E) \cong C^\infty(M)^k$, which is free.

($\Leftarrow$) Suppose $\Gamma(E)$ is free, with basis $\{s_1, \dots, s_n\}$. We claim that $\{s_1, \dots, s_n\}$ is a global frame. By Lemma 4.3, these sections span the fiber at every point. It remains to show that they are linearly independent on each fiber. Suppose not. Then there exist $x \in M$ and $\lambda_i \in \mathbb{R}$, not all zero, such that $\lambda_1 s_1(x) + \ldots + \lambda_n s_n(x) = 0$ in E_x. Assume without loss of generality that $\lambda_1 \neq 0$. Then $\sum_{i=1}^n \lambda_i s_i$ is a section vanishing at x. By Lemma 4.1, there exist $f_j \in \mathfrak{m}_x$ and $t_j \in \Gamma(E)$ such that

$$\lambda_1 s_1 + \ldots + \lambda_n s_n = f_1 t_1 + \ldots + f_r t_r.$$

Since $\{s_1, \dots, s_n\}$ is a basis of $\Gamma(E)$, we may express each t_j in terms of s_i. Since $\mathfrak{m}_x$ is an ideal, the resulting coefficients are still in $\mathfrak{m}_x$. Hence, there exist $g_i \in \mathfrak{m}_x$ such that

$$\lambda_1 s_1 + \ldots + \lambda_n s_n = g_1 s_1 + \ldots + g_n s_n.$$

Thus,

$$(\lambda_1 - g_1)s_1 + \ldots + (\lambda_n - g_n)s_n = 0.$$

Note that $\lambda_1 - g_1(x) = \lambda_1 \neq 0$. Hence, $\{s_1, \dots, s_n\}$ is linearly dependent in $\Gamma(E)$, a contradiction. ∎

Remark 4.5. Lemma 4.3 would be false if the word "span" is replaced by "linearly independent." More precisely, if $\{s_1(x), \dots, s_r(x)\}$ is linearly independent in each fiber E_x, then $\{s_1, \dots, s_r\}$ is $C^\infty(M)$-linearly independent in $\Gamma(E)$, but the converse is false. Consider the trivial bundle $M \times \mathbb{R} \to M$. Let $x \in M$ be a point, and let $f \in C^\infty(M)$ be a function whose zero set is exactly $\{x\}$. Regarding f as an element of $\Gamma(M \times \mathbb{R})$, we see that f is linearly independent in this module because if $g \in C^\infty(M)$ is such that $gf = 0$, then $g = 0$ on $M \setminus \{x\}$, whence $g = 0$ on all of M. However, $f(x) = 0$, which is not linearly independent in $\mathbb{R}$.

Lemma 4.6. *Let g be a morphism of bundles over possibly different bases covering f as follows*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\xi_1"'] & F \arrow[d, "\xi_2"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{4.1}$$

Then g gives rise to a morphism over M

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "\xi_1"'] & & f^*F \arrow[dl, "f^*\xi_2"] \\ & M & \end{tikzcd} \tag{4.2}$$

Furthermore, the original morphism is a fiberwise isomorphism if and only if the latter morphism is an isomorphism.

Proof. The universal property of pullback directly implies that there is a smooth map ϕ:

$$\begin{tikzcd}
E \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\xi_1"'] & & \\
& f^*F \arrow[r] \arrow[d, "f^*\xi_2"'] & F \arrow[d, "\xi_2"] \\
& M \arrow[r, "f"'] & N
\end{tikzcd}$$

If $p \in M$, then $(f^*F)_p = \{p\} \times F_{f(p)}$. The maps on fibers induced by ϕ is exactly the same as $g|_{E_p} : E_p \to F_{f(p)}$, so ϕ is linear fiberwise. Hence, ϕ is a morphism of vector bundles.

If g in (4.1) is a fiberwise isomorphism, then so is ϕ. Since ϕ is a morphism of vector bundles covering the identity map on the same base, ϕ must be an isomorphism. Conversely, if ϕ in (4.2) is an isomorphism, then it induces isomorphism on each fiber, so g is a fiberwise isomorphism. ∎

5. Proof of the smooth Serre-Swan theorem

Theorem 5.1. *Every vector bundle can be obtained as the pullback of the tautological bundle over some Grassmannian.*

Proof. Let $E \to M$ be a rank k vector bundle, and let $m = \dim M$. Recall that for any vector space V, the tangent space T_0V can also be naturally identified with V itself. Hence, for every $z \in M$, we may identify $E_z \simeq T_{(z,0)}E_z \subset T_{(z,0)}E$.

By the Whitney immersion theorem, there exists an immersion $\phi : E \to \mathbb{R}^N$, where we may take $N = 2(m+k)$ if we want. The differential of ϕ is injective at every point, and hence $d_{(z,0)}\phi(T_{(z,0)}E_z)$ is a k-dimensional subspace of $\mathbb{R}^{2(m+k)}$, where we again identify $T_p\mathbb{R}^N \simeq \mathbb{R}^N$ for any $p \in \mathbb{R}^N$. We define the Gauss map $g : M \to \mathrm{Gr}_{N,k}$ by

$$g(z) = d_{(z,0)}\phi(T_{(z,0)}E_z).$$

Furthermore, there is a morphism of vector bundles $\gamma : E \to \Theta_{N,k}$ covering g, defined on each fiber by

$$\gamma|_{E_z} = (d_{(z,0)}\phi)|_{E_z},$$

where we implicitly identify $E_z \simeq T_{(z,0)}E_z$ as discussed before. Since ϕ is an immersion, γ is a fiberwise isomorphism by construction. Once we check that g and γ are smooth, it would follow immediately from Lemma 4.6 that $E \simeq g^*(\Theta_{N,k})$.

To see that g is smooth, choose $U \subset M$ over which E is trivial. Choose a local frame $\{s_1, \dots, s_k\}$ of E over U. Then for each s_i, there are maps

$$U \longrightarrow E|_U \longrightarrow TE \xrightarrow{T\phi} T\mathbb{R}^N \simeq \mathbb{R}^N \times \mathbb{R}^N \xrightarrow{\pi_2} \mathbb{R}^N,$$

where the first map is s_i; the second map is the inclusion $(x, v) \mapsto ((x, 0), v)$, where $v \in E_x \simeq T_{(x,0)}E_x \subset T_{(x,0)}E$; the third map is the map of tangent bundles induced by ϕ, and the fourth map is projection on the the second factor. Each map is smooth, so the composition is also smooth, call it f_i. Furthermore, since $\{s_i\}$ is a local frame over U, we have $g(x) =$

$\operatorname{span}(f_1(x),\ldots,f_k(x))$. By Lemma 3.3, g is smooth. Similarly, γ on $E|_U$ is of the form $\gamma(x,a_1s_1(x),\ldots,a_ks_k(x))=(g(x),a_1f_1(x),\ldots,a_kf_k(x))$, γ must be smooth as well. ■

Corollary 5.2. *Let $E\to M$ rank k vector bundle over an m-dimensional manifold. Then $\Gamma(E)$ admits a set of generators consisting of $k\binom{2(m+k)}{k}$ elements. In particular, $\Gamma(E)$ is a finitely generated $C^\infty(M)$-module.*

Proof. We just showed that E can be obtained as the pullback of $\Theta_{2(m+k),k}\to \mathrm{Gr}_{2(m+k),k}$. By construction, $\mathrm{Gr}_{2(m+k),k}$ admits a covering by $n=\binom{2(m+k)}{k}$ trivializing open sets. It follows that M can also be covered by n trivializing open sets, since E is the pullback of $\Theta_{2(m+k),k}$. Call these open sets $\{U_1,\ldots,U_n\}$. For each $1\le i\le n$, choose a local frame $\{s_{ij}\}_{1\le j\le k}$ of E over U_i. Choose a partition of unity $\{\phi_1,\ldots\phi_n\}$ dominated by $\{U_1,\ldots,U_n\}$, i.e., $\operatorname{supp}\phi_i\subset U_i$ for all i. Define

$$\tilde{s}_{ij}=\begin{cases}\phi_i s_{ij}, & \text{on } U_i,\\ 0 & \text{on } M\setminus\operatorname{supp}\phi_i.\end{cases}$$

For each i, let $V_i\subseteq M$ be the open set on which $\phi_i>0$. Because the ϕ_i's sum to 1, $\{V_1,\ldots,V_n\}$ is also an open cover of M. The $\{\tilde{s}_{ij}\}_{1\le j\le k}$ are smooth global sections of E, with the property that their restrictions to V_i span the fibers over V_i. By Lemma 4.3, $\{\tilde{s}_{ij}:1\le i\le n,1\le j\le k\}$ generate $\Gamma(E)$. ■

Remark 5.3. The bound $k\binom{2(m+k)}{k}$ is not optimal. In fact, using some more sophisticated dimension theory arguments one can show that M can be covered by $m+1$ trivializing open sets.

Corollary 5.4. *Let $E\to M$ be a vector bundle. Then the $C^\infty(M)$-module $\Gamma(E)$ is projective.*

Proof. We just saw in corollary 5.2 that $\Gamma(E)$ is finitely generated. To show that it is projective, let $g:M\to\mathrm{Gr}_{N,k}$ be the Gauss map constructed in Theorem 5.1. Since the tautological bundle $\Theta_{N,k}$ is a subbundle of the trivial bundle $\mathrm{Gr}_{N,k}\times\mathbb{R}^N$, we may pick a complementary subbundle H such that $\mathrm{Gr}_{N,k}\times\mathbb{R}^N=\Theta_{N,k}\oplus H$. Since the pullback of a trivial bundle is trivial, we obtain

$$M\times\mathbb{R}^N=g^*(\Theta_{N,k})\oplus g^*H=E\oplus g^*H.$$

Hence, E is a direct summand of the trivial bundle $M\times\mathbb{R}^N$, whence $\Gamma(E)$ is a direct summand of the free module $\Gamma(M\times\mathbb{R}^N)\cong C^\infty(M)^N$, hence projective. ■

Theorem 5.5 (Essential surjectivity)**.** *Let P be a module over $C^\infty(M)$, where M is a connected smooth manifold. Then P is isomorphic to $\Gamma(E)$ for some vector bundle $E\to M$ if and only if P is finitely generated and projective.*

Proof. By corollary 5.2 and 5.4, we know that if P is isomorphic to $\Gamma(E)$, then P must be finitely generated and projective, so one direction is clear.

Conversely, let P be a finitely generated projective module over $C^\infty(M)$. Up to isomorphism, we may assume that P is a direct summand of of a free module, i.e., assume $C^\infty(M)^N=P\oplus Q$ for some submodule Q. Also note that $\Gamma(M\times\mathbb{R}^N)\cong C^\infty(M)^N$, so both P and Q can be viewed as submodules of sections of the trivial bundle $\pi:M\times\mathbb{R}^N\to M$.

For each $z\in M$, define $P_z=\{p(z):p\in P\}$, and define Q_z similarly. Denote the fiber of π over z by π_z. Then we claim that $\pi_z=P_z\oplus Q_z$ for all $z\in M$. Indeed, for every $v\in\pi_z$, there is $s \in \Gamma(M \times \mathbb{R}^N)$ such that $s(z) = v$. Writing $s = p + q$, then $s(z) = p(z) + q(z)$, whence $\pi_z = P_z + Q_z$. On the other hand, suppose that $v = p(z) = q(z)$, then $p - q$ is a section vanishing at z. By Lemma 4.1, there exist functions $f_i \in \mathfrak{m}_z$ and sections $s_i \in \Gamma(M \times \mathbb{R}^N)$ such that $p - q = \sum_i f_i s_i$. For each s_i, we may write $s_i = p_i + q_i$ for some unique $p_i \in P$ and $q_i \in Q$. Then

$$p - q = \sum_i f_i p_i + \sum_i f_i q_i,$$

whence

$$p - \sum_i f_i p_i = \sum_i f_i q_i - q.$$

Since the left-hand side is in P, and the right-hand side is in Q, we conclude that $p - \sum_i f_i p_i = 0$. Evaluating this at z, we find that $v = p(z) = 0$ because $f_i(z) = 0$ for all i. Hence, $\pi_z = P_z \oplus Q_z$ for all $z \in M$.

Now we argue that the dimensions of the fibers P_z are constant. Suppose $\dim P_z = r$ for some point $z \in M$. Then there exist global sections $s_1, \ldots, s_r \in \Gamma(M \times \mathbb{R}^N)$ such that $\{s_1(z), \ldots, s_r(z)\}$ form a basis of P_z. By continuity, the set remains linearly independent in P_y for all y in some open neighborhood U of z. It follows that $\dim P_y \geq \dim P_z$ for all $y \in U$. Similarly, $\dim Q_y \geq \dim Q_z$ for all $y \in U$. But since $\dim P_y + \dim Q_y$ is constant, we conclude that all P_y have the same dimension for all $y \in U$. Since M is connected, all fibers have constant dimensions. This also shows that $E = \bigcup_{z \in M} P_z$ is a subbundle of $M \times \mathbb{R}^N$ because the sections $s_1, \ldots, s_r$ form a local frame on U.

Finally, it remains to show that $\Gamma(E)$ is isomorphic to P. Indeed, any element in P is a section of E by construction. Conversely, if s is a section of E, then it is in particular a section of the trivial bundle $M \times \mathbb{R}^N$. If $s \notin P$, then $s = p + q$ for some $p \in P$ and nonzero $q \in Q$. Hence, there exist $z \in M$ for which $q(z) \neq 0$, whence $s(z) = p(z) + q(z) \notin P_z$. Therefore, s cannot be a section of E, a contradiction, and so $\Gamma(E) \cong P$. ■

It remains for us to prove that Γ is fully faithful. This is the content of Theorem 5.6 below.

Theorem 5.6 (Fullness and faithfulness)**.** *Let E and F be vector bundles over M. Then for every $C^\infty(M)$-linear map $\varphi : \Gamma(E) \to \Gamma(F)$, there exists a unique morphism of vector bundles $\tilde{\varphi} : E \to F$ such that $\tilde{\varphi} \circ s = \varphi(s)$ for all $s \in \Gamma(E)$.*

Proof. We argue that φ acts *pointwise* in the following sense: if $s_1, s_2 \in \Gamma(E)$ agree at a point x, then $\varphi(s_1), \varphi(s_2)$ must also agree at x. Indeed, since $s_1 - s_2$ vanishes at x, Lemma 4.1 implies that

$$s_1 - s_2 = \sum_i f_i t_i,$$

for some $t_i \in \Gamma(E)$ and $f_i \in \mathfrak{m}_x$. Then

$$\varphi(s_1) - \varphi(s_2) = \sum_i f_i \varphi(t_i),$$

because φ is $C^\infty(M)$-linear. Hence, $\varphi(s_1) - \varphi(s_2)$ also vanishes at x, as desired.

Now, we define $\tilde{\varphi} : E \to F$ as follows: given any $x \in M$ and $v \in E_x$, take a section $s \in \Gamma(E)$ with $s(x) = v$, and let $\tilde{\varphi}(v) = \varphi(s)(x)$. This is well defined by the discussion above. The map $\tilde{\varphi}$ clearly preserves base points and is linear on each fiber. It remains to show that $\tilde{\varphi}$ is smooth. As always, we check this locally. Choose $U \subset M$ over which both bundles are trivial, and let $\{\sigma_i\}$ and $\{\sigma'_j\}$ be the associated local frames over U. Shrinking U if necessary, we may assume that there exist global frames $\{\tilde{\sigma}_i\}$ that agree with $\{\sigma_i\}$ on U. Since $\{\sigma'_j\}$ is a local frame, there are smooth functions $A_i^j \in C^\infty(U)$ such that $\varphi(\tilde{\sigma}_i)|_U = \sum_j A_i^j \sigma'_j$. For any $q \in U$ and $v \in E_q$, we can write $v = \sum_i v_i \sigma_i(q)$ for some real numbers $(v_1, \ldots, v_k)$, and then

$$\tilde{\varphi}\left(\sum_i v_i \sigma_i(q)\right) = \varphi\left(\sum_i v_i \tilde{\sigma}_i\right)(q) = \sum_i v_i \varphi(\tilde{\sigma}_i)(q) = \sum_{i,j} v_i A_i^j(q) \sigma'_j(q),$$

because $\sum_i v_i \tilde{\sigma}_i$ is a global smooth section of E whose value at q is v. Under the local trivialization, we see that $v \in E_q$ is essentially the same as the numbers $(v_1, \ldots, v_k)$, and the formula above shows that $\tilde{\varphi}$ depends smoothly on v, as required. ∎

Remark 5.7. We remark that the property of being $C^\infty(M)$-linear is sometimes referred to as being "tensorial." In other words, a tensorial map should act *pointwise* like a tensor, instead of depending on the global structure of the sections. There are lots of maps that are not tensorial. For example, a *connection* on a vector bundle E is a map $\nabla : \mathfrak{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ that is tensorial in the first input, but not in the second one; instead, ∇ satisfies the Leibniz rule in the second input. This is the foundation of Riemannian geometry, so we won't go into the details here.

Finally, the Serre-Swan theorem follows from combining Theorems 5.5 and 5.6.

Theorem 5.8 (Serre-Swan)**.** *For every connected manifold M, the global section functor Γ is an equivalence of categories between the category of vector bundles over M and the category of finitely generated projective modules over $C^\infty(M)$.*

As a simple application of the Serre-Swan theorem, we prove the following fact, which will come in handy in Section 7 later:

Theorem 5.9. *Let P, Q be finitely generated projective modules over $C^\infty(M)$. A morphism of modules $\varphi : P \to Q$ is an isomorphism if and only if the induced map*

$$\varphi_x : P/\mathfrak{m}_x P \to Q/\mathfrak{m}_x Q$$

is an isomorphism for each $x \in M$.

Proof. Since P, Q are finitely generated and projective, they are isomorphic to $\Gamma(E), \Gamma(F)$ for some vector bundles E, F over M. Furthermore, $\varphi : P \to Q$ corresponds uniquely to a morphism of vector bundles $\Phi : E \to F$.

By the remark after Lemma 4.1, we have $E_x \cong \Gamma(E)/\mathfrak{m}_x\Gamma(E) \cong P/\mathfrak{m}_x P$. Hence, the above statement follows immediately from the fact that a morphism of vector bundles is an isomorphism if and only if it is a fiberwise isomorphism. ∎

6. The global section functor is monoidal

There are lots of *very* interesting and important corollaries of the Serre-Swan theorem. In this section, we prove that the global section functor Γ is monoidal. We only check that Γ preserves tensor products, and the other compatibility conditions are left to the readers.

Let $E, F \to M$ be vector bundles. Given sections $s \in \Gamma(E)$ and $t \in \Gamma(F)$, one can form $s \otimes t \in \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$. On the other hand, the symbol $s \otimes t$ can be understood as a section of the tensor bundle $E \otimes F$ via $(s \otimes t)(x) = s(x) \otimes t(x)$. The following theorem shows that these notations are really the same.

Theorem 6.1. *The global section functor Γ preserves tensor products in the sense that if E, F are vector bundle over M, then*

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F).$$

Proof. We present a quick proof. A different proof can be found in Theorem 12.39 in [2]. For convenience, we write $\Gamma(E) \otimes \Gamma(F)$ for $\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

We shall use the following facts

(i) Every vector bundle E is isomorphic to its double dual E^{**} via the fiberwise canonical identification.

(ii) $\Gamma(\underline{\mathrm{Hom}}(E, F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(F))$, where $\underline{\mathrm{Hom}}(E, F)$ is the Hom-bundle. In particular, we have $\Gamma(E^*) \cong \Gamma(E)^*$

(iii) If P, Q are finitely generated projective A-modules, then $\mathrm{Hom}_A(P, Q)$ is (naturally) isomorphic to $P^* \otimes_A Q$.

All three statements can be easily checked. We sketch the proof of (ii). By the Serre-Swan theorem, it suffices to check that sections of $\underline{\mathrm{Hom}}(E, F)$ correspond to bundle morphisms $E \to F$. Given a section s of the bundle $\underline{\mathrm{Hom}}(E, F)$, at each point $x \in M$ we obtain a linear map $s(x) \in \mathrm{Hom}(E_x, F_x)$; piecing these maps together yields a map $E \to F$, which is fiberwise linear by definition. Conversely, given a bundle morphism $E \to F$, we obtain a linear map $E_x \to F_x$ at each fiber. This gives us a section of $\underline{\mathrm{Hom}}(E, F)$.

Now, we have

$$\begin{aligned}\Gamma(E \otimes F) \cong \Gamma(E^{**} \otimes F) \cong \Gamma(\underline{\mathrm{Hom}}(E^*, F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E^*), \Gamma(F)) \cong \Gamma(E^*)^* \otimes \Gamma(F) \\ \cong \Gamma(E^{**}) \otimes \Gamma(F) \cong \Gamma(E) \otimes \Gamma(F).\end{aligned}$$

All isomorphisms above are canonically defined. Indeed, their composition gives rise to the map we described earlier: $\Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F)$, $s \otimes t \mapsto (x \mapsto s(x) \otimes t(x))$. ■

Now, we show that Γ is even better than being monoidal, in the sense that Γ actually preserves all exterior powers. First, we make some general remarks on exterior powers of modules over arbitrary rings. Let R be a ring such that $1/k! \in R$. For example, take $R = \mathbb{Q}$ or any ring containing $\mathbb{Q}$. Then for every R-module M, there is an embedding

$$\bigwedge^k(M) \to M^{\otimes k},$$

defined by

$$v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\mathrm{sign}(\sigma)} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}. \tag{6.1}$$

The map is well defined because of the universal property of $\bigwedge^k(M)$. Composing this map with the projection $M^{\otimes k} \to \bigwedge^k(M)$ yields the identity.

Theorem 6.2. *For every vector bundle $E \to M$, there is an isomorphism*

$$\bigwedge^k(\Gamma(E)) \cong \Gamma(\bigwedge^k E),$$

where the map is given by

$$A(s_1 \wedge \cdots \wedge s_k)(x) = s_1(x) \wedge \cdots \wedge s_k(x). \tag{6.2}$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
\Gamma(E)^{\otimes k} \arrow[r, "B"] \arrow[d, shift right, "f"'] & \Gamma(E^{\otimes k}) \arrow[d, shift right, "g"'] \\
\bigwedge^k(\Gamma(E)) \arrow[r, "A"'] \arrow[u, shift right, "f'"'] & \Gamma(\bigwedge^k(E)) \arrow[u, shift right, "g'"']
\end{tikzcd}$$

where the maps are defined as follows:

(i) A is the map defined in (6.2);
(ii) B is the isomorphism of tensor powers, implied by Theorem 6.1;
(iii) f is the projection map from tensor power to exterior power;
(iv) g is the map of sections induced by the fiberwise projections $E^{\otimes k} \to \bigwedge^k(E)$.
(v) f' is the embedding of exterior power into tensor power, as in (6.1).
(vi) g' is the map of sections induced by the fiberwise embedding $\bigwedge^k(E) \to E^{\otimes k}$.

We claim that the map $f \circ B^{-1} \circ g'$ is the inverse of A. First, we prove that

$$f \circ B^{-1} \circ g' \circ A : \bigwedge^k(\Gamma(E)) \to \bigwedge^k(\Gamma(E))$$

is the identity map. Since every map is $C^\infty(M)$-linear, it suffices to consider a simple wedge product of the form $s_1 \wedge \cdots \wedge s_k$. Then we have

$$\begin{aligned}
s_1 \wedge \cdots \wedge s_k &\overset{A}{\longmapsto} (x \mapsto s_1(x) \wedge \cdots \wedge s_k(x)) \\
&\overset{g'}{\longmapsto} \left(x \mapsto \frac{1}{k!} \sum_\sigma (-1)^{\mathrm{sign}(\sigma)} s_{\sigma(1)}(x) \otimes \cdots \otimes s_{\sigma(k)}(x) \right) \\
&\overset{B^{-1}}{\longmapsto} \frac{1}{k!} \sum_\sigma (-1)^{\mathrm{sign}(\sigma)} s_{\sigma(1)} \otimes \cdots \otimes s_{\sigma(k)} \\
&\overset{f}{\longmapsto} s_1 \wedge \cdots \wedge s_k,
\end{aligned}$$

as required.

Now, we prove that

$$\gamma := A \circ f \circ B^{-1} \circ g' : \Gamma(\textstyle\bigwedge^k E) \to \Gamma(\textstyle\bigwedge^k E)$$

is also the identity map. Note that this is a map of $C^\infty(M)$-modules. By the Serre-Swan theorem, in particular Theorem 5.6, it is induced by a unique morphism of bundles $h : \bigwedge^k E \to \bigwedge^k E$. We shall show that h is the identity map, which implies that γ is the identity map as well.

Recall that h can be constructed in the following way: given any $p \in M$ and any $\alpha \in (\bigwedge^k E)|_p$, choose a section $s \in \Gamma(\bigwedge^k E)$ such that $s(p) = \alpha$, and define $h(\alpha) = \gamma(s)(p)$. The fact that γ is $C^\infty(M)$-linear implies that h is well defined, i.e., independent of the choice of s.

Hence, fix $p \in M$ and $\alpha \in (\bigwedge^k E)|_p$. We may assume that $\alpha = v_1 \wedge \cdots \wedge v_k$, where $v_i \in E|_p$. Choose sections $s_i \in \Gamma(E)$ such that $s_i(p) = v_i$. Then $s(x) = s_1(x) \wedge \cdots \wedge s_k(x)$ is a section of

$\bigwedge^k E$ with $s(p) = \alpha$. Then $h(p) = \gamma(s)(p)$. Note that $s = A(s_1 \wedge \cdots \wedge s_k)$. Hence,

$$\gamma(s) = A \circ (f \circ B^{-1} \circ g' \circ A)(s) = A(s_1 \wedge \cdots \wedge s_k) = s,$$

since we already proved that $f \circ B^{-1} \circ g' \circ A$ is the identity map on $\bigwedge^k(\Gamma(E))$. Therefore, $h(\alpha) = \alpha$, as desired. ■

Remark 6.3. Alternatively, one could prove the surjectivity of A as follows: pick generators of $\Gamma(E)$, say $s_1, \ldots, s_n$. Then for any $p \in M$ and any $v \in E|_p$, v can be expressed as a linear combination of $s_1(p), \ldots, s_n(p)$. It follows that any $\alpha \in (\bigwedge^k E)|_p$ can be expressed as a linear combination of $s_{i_1}(p) \wedge \cdots \wedge s_{i_k}(p)$, there the indices run over $1 \leq i_1 < \cdots < i_k \leq n$. Therefore, the collection of sections

$$\mathcal{E} = \{s_{i_1} \wedge \cdots \wedge s_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\} \subset \Gamma(\textstyle\bigwedge^k E)$$

is spanning at each fiber of $\bigwedge^k E$. By Lemma 4.3, $\mathcal{E}$ is a finite set of generators of $\Gamma(\bigwedge^k E)$. Each element of $\mathcal{E}$ is in the image of A, so A must be surjective.

7. Pullbacks and extensions of scalars

In this section, we prove yet another remarkable property of the global section functor that relates the topological structure of pullback bundles and the algebraic structure of modules. We will see that pullback bundles and extensions of scalars are really the “same” thing. Before that, let us recall some definitions from algebra:

Definition 7.1. Let $f : A \to B$ be a morphism of commutative rings.

(i) If M is a B-module, then the “restriction of scalar” of M along f is the A-module structure on M defined by $a \cdot m = f(a)m$, for $a \in A$ and $m \in M$.

(ii) If M is an A-module, then the “extension of scalar” of M along f is the B-module $B \otimes_A M$ defined by $b' \cdot (b \otimes m) = bb' \otimes m$, for $b, b' \in B$ and $m \in M$.

As the readers may readily check, the restrictions and extensions of scalars (along f) can be made into functors, and extension is left adjoint to restriction. Furthermore, if P is a projective (resp. finitely generated) A-module, then $B \otimes_A P$ is a projective (resp. finitely generated) B-module.

Since extensions of scalars preserves projectivity and the property of being finitely generated, it is natural to ask whether it relates to vector bundles. Such a natural connection does exists, and it is an intriguing interplay between algebra and geometry. This is the content of the following theorem.

Theorem 7.2. *Let M, N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Let $E \to M$ be a vector bundle, and consider the pullback bundle:*

$$\begin{tikzcd} \varphi^* E \arrow[r] \arrow[d, "\varphi^*\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "\varphi"'] & M \end{tikzcd}$$

Then $\Gamma(\varphi^* E)$ is isomorphic to the extension of scalars of $\Gamma(E)$ along the induced map $\varphi^* : C^\infty(M) \to C^\infty(N)$.

Proof. Let $A = C^\infty(M)$ and $B = C^\infty(N)$.

Given a section $s \in \Gamma(E)$, the map $\widehat{\varphi}(s) := s \circ \varphi$ is a section of $\varphi^* E$. Now, consider the map

$$\nu : B \otimes_A \Gamma(E) \to \Gamma(\varphi^* E),$$
$$\nu(f \otimes s) = f \cdot \widehat{\varphi}(s).$$

We need to show that ν is an isomorphism. Since $B \otimes_A \Gamma(E)$ is a finitely generated projective $C^\infty(N)$-module, it is isomorphic to the module of sections of some vector bundle over N. By Theorem 5.9, it suffices to show that for all $w \in N$, the map

$$\nu_w : (B \otimes_A \Gamma(E))/(\mathfrak{m}_w \otimes \Gamma(E)) \to (\varphi^* E)_w,$$
$$\nu_w([g \otimes s]) = g(w) s(\varphi(w)),$$

is an isomorphism. Clearly, ν_w is onto, because every point in the vector bundle E is in the range of some section of E. It suffices to show injectivity.

Suppose $\sum g_i \otimes s_i \in B \otimes_A \Gamma(E)$ is such that

$$\sum_i g_i(w) s_i(\varphi(w)) = 0. \tag{7.1}$$

Let $\bar{g}_i \in C^\infty(N)$ be defined by $\bar{g}_i = g_i - g_i(w)$. Then $\bar{g}_i(w) = 0$, and so $\bar{g}_i \in \mathfrak{m}_w$. Furthermore, (7.1) implies that the section $\sum_i g_i(w) s_i$ of E vanishes at the point $\varphi(w)$. Therefore, Lemma 4.1 implies that

$$\sum_i g_i(w) s_i = \sum_j f_j t_j,$$

for some $f_j \in \mathfrak{m}_{\varphi(w)}$ and $t_j \in \Gamma(E)$. Therefore, we have

$$\begin{aligned} \sum_i g_i \otimes s_i &= \sum_i \bar{g}_i \otimes s_i + 1 \otimes \sum_i g_i(w) s_i \\ &= \sum_i \bar{g}_i \otimes s_i + 1 \otimes \sum_j f_j t_j \\ &= \sum_i \bar{g}_i \otimes s_i + \sum_j \varphi^* f_j \otimes t_j. \end{aligned}$$

Since $\bar{g}_i$ and $\varphi^* f_j$ are both in $\mathfrak{m}_w$, this shows that $\sum_i g_i \otimes s_i \in \mathfrak{m}_w \otimes \Gamma(E)$. Hence, ν_w is injective. ■

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