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Optimal Trend Following Trading Rules

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This paper is concerned with the optimality of a trend following trading rule. The underlying market is modeled like a bull-bear switching market in which the drift of the stock price switches between two states: the uptrend (bull market) and the down trend (bear market). We consider the case when the market mode is not directly observable and model the switching process as a hidden Markov chain. This is a continuation of our earlier study reported in Dai et al. [Dai M, Zhang Q, Zhu Q (2010) Trend following trading under a regime-switching model. *SIAM J. Fin. Math.* 1:780–810] where a trend following rule is obtained in terms of a sequence of stopping times. Nevertheless, a severe restriction imposed in Dai et al. [Dai M, Zhang Q, Zhu Q (2010) trend following trading under a regime-switching model. *SIAM J. Fin. Math.* 1:780–810] is that only a single share can be traded over time. As a result, the corresponding wealth process is *not* self-financing. In this paper, we relax this restriction. Our objective is to maximize the expected log-utility of the terminal wealth. We show, via a thorough theoretical analysis, that the optimal trading strategy is trend following. Numerical simulations and backtesting, in support of our theoretical findings, are also reported.

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1. Introduction. Trading strategies can be classified into three categories: (i) buy and hold, (ii) contra-trend, and (iii) trend following. The buy-and-hold strategy is desirable when the average stock return is higher than the risk-free interest rate. Recently, Shiryaev et al. [16] provided a theoretical justification of the buy-and-hold strategy from the angle of maximizing the expected relative error between the stock selling price and the aforementioned maximum price. The contra-trend strategy, in contrast, focuses on taking advantages of mean reversion type of market behaviors. A contra-trend trader purchases a stock when its price falls to some low level and bets an eventual rebound. The trend following strategy tries to capture market trends. In contrast to the contra-trend investors, a trend- following believer often purchases shares when prices advance to a certain level and closes the position at the first sign of upcoming bear market.

There is an extensive literature devoted to contra-trend strategies. For instance, Merton [14] pioneered the continuous-time portfolio selection with utility maximization, which was subsequently extended to incorporate transaction costs by Magill and Constantinides [13] (see also Davis and Norman [6], Shreve and Soner [17], Liu and Loewenstein [12], Dai and Yi [2], and references therein). Assuming that there is no leverage or short-selling, the resulting strategies turn out to be contra-trend because the investor is risk averse and the stock market is assumed to follow a geometric Brownian motion with constant drift and volatility. Recently, Zhang and Zhang [21] showed that the optimal trading strategy in a mean reverting market is also contra-trend. Other work relevant to the contra-trend strategy includes Dai et al. [5], Song et al. [18], Zervos et al. [20], among others.

This paper is concerned with a trend-following trading rule. In practice, a trend-following trader often uses moving averages to determine the general direction of the market and generate trading signals. Related research along the line of statistical analysis in connection with moving averages can be found in, for example, Faber [7], among others. Nevertheless, rigorous mathematical analysis is absent. Recently, Dai et al. [4] provided a theoretical justification of the trend-following strategy in a bull-bear switching market and employed the conditional probability in the bull market to generate trade signals. However, the work imposed a less realistic assumption widely used in existing literature (e.g., Song et al. [18], Zervos et al. [20], and Zhang and Zhang [21]): only one share of stock is allowed to be traded, so the resulting wealth process is *not* self-financing. It is important to address how relevant the trading rule is to practice. It is the purpose of this paper to deal with more realistic self-financing trading strategies. Here, we adopt an objective function emphasizing the percentage gains. As a result, the corresponding payoff has to account for the gain/loss percentage of each trade, which is also desirable in actual trading. In contrast, these more realistic considerations make the model more technically involved than in the "single share" transaction considered in Dai et al. [4].

Most existing literature in trading strategies assumes that the investor can observe full market information (e.g., Jang et al. [8], Dai et al. [3]). In contrast, we follow Dai et al. [4] to model the trends in the markets using a geometric Brownian motion with regime switching and partial information. More precisely, two regimes are considered: the uptrend (bull market) and down trend (bear market), and the switching process is modeled like a two-state Markov chain, which is not directly observable. We consider a finite-horizon investment problem and aim to maximize the percentage gains. We assume that the investor trades all available funds in the form of either "all-in" (long) or "all-out" (flat). That is, when buying, one fills the position with the entire account balance, and when selling, one closes the entire position. We will show again that the optimal trading strategy is a trend-following system characterized by the conditional probability over time and its up and down crossings of two threshold curves. These thresholds can be obtained through solving a system of associated Hamilton-Jacobi-Bellman (HJB) equations. Such a trading strategy naturally generates entry time and exit time, which can be mathematically described by stopping times. We also carry out numerical simulations and market tests to demonstrate how the method works.

This work and Dai et al. [4] were initialized by an attempt to justify the technical analysis with moving average. A moving average trading strategy is generally in "all-in all-out" form but is difficult to justify theoretically. This motivates us to design and justify an alternative "all-in—all-out" strategy that is analogous to the moving average trading strategy. This work has been recently extended to the Merton's portfolio optimization problem by Chen et al. [1], where the investor may choose an optimal fraction of wealth invested in stock.

In contrast to Dai et al. [4], the present paper provides not only a more reasonable modeling but also a more thorough theoretical analysis. First, we remove a technical condition imposed in Dai et al. [4] when proving the verification theorem. The key step is to show that the optimal trading strategy incurs only a finite number of trades almost surely (Lemma 2). Second, since the solution to the resulting HJB equation is not smooth enough to use the Itô lemma, we employ an approximation approach to prove the verification theorem (Theorem 4). Third, we show that for the optimal trading strategy, the upper limit involved in defining the reward function is, in fact, a limit (Theorem 5). Hence the definition of the reward function makes sense in practice. Last but not least, we find that the theoretical characterization on the optimal trading strategy obtained in Dai et al. [4] remains valid for the present model (Theorem 1). We further present sufficient conditions to examine whether or not the optimal trading boundaries are attainable (Theorems 2 and 3). Although these conditions are not sharp, our result reveals that under certain scenarios, the optimal trading boundaries are always attainable for sufficiently small transaction costs.

The rest of the paper is arranged as follows. Following the problem formulation in the next section, §3 is devoted to a theoretical characterization of the optimal trading strategy in a regime-switching market. We report our simulation results and market tests in §4. We conclude in §5. Some technical proofs are given in the appendix.

2. Problem formulation. Consider a complete probability space $(\Omega, \mathcal{F}, P)$. Let S_r denote the stock price at time r satisfying the equation

$$dS_r = S_r[\mu(\alpha_r)\,dr + \sigma\,dB_r], \quad S_t = S, \quad t \le r \le T < \infty,$$

where $\alpha_r \in \{1, 2\}$ is a two-state Markov chain, $\mu(i) \equiv \mu_i$ is the expected return rate in regime $i = 1, 2$, $\sigma > 0$ is the constant volatility, B_r is a standard Brownian motion, and t and T are the initial and terminal times, respectively. We assume that the stock does not pay any dividends. No generality is lost because dividends, if they exist, can be reinvested in the stock, and thus reflected in the stock price.

The process α_r represents the market mode at each time r: $\alpha_r = 1$ indicates a bull market (uptrend) and $\alpha_r = 2$ a bear market (down trend). In this paper, we make a realistic assumption that α_r is not directly observable. Let

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},$$

$(\lambda_1 > 0,\ \lambda_2 > 0)$, denote the generator of α_r. Therefore λ_1 (λ_2) stands for the switching intensity from bull to bear (from bear to bull). We assume that $\{\alpha_r\}$ and $\{B_r\}$ are independent.

Due to the nonobservability of α_r, the decisions (of buying and selling) have to be based purely on the stock prices. Let $\mathcal{F}_t = \sigma\{S_r : r \le t\}$ denote the σ-algebra generated by the stock price. Let

$$t \le \tau_1^0 \le v_1^0 \le \tau_2^0 \le v_2^0 \cdots \le \tau_n^0 \le v_n^0 \le \cdots, \quad \text{a.s.},$$

denote a sequence of $\mathcal{F}_t$-stopping times. For each n, define

$$\tau_n = \min\{\tau_n^0, T\} \qquad \text{and} \qquad v_n = \min\{v_n^0, T\}.$$

A buying decision is made at τ_n if $\tau_n < T$ and a selling decision is made at v_n if $v_n < T$, $n = 1, 2, \ldots$. In addition, we require the liquidation of all long positions (if any) at the terminal time T.

We assume that the investor is taking an "all-in —all-out" strategy. This means that she is either long so that her entire wealth is invested in the stock, or flat so that all of her wealth is in a bank account that draws the risk-free interest rate. We use indicator $i = 0$ or 1 to signify the initial position to be flat or long, respectively. If initially the position is long (i.e., $i = 1$), the corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)$.

Let $0 < K_b < 1$ denote the percentage of slippage (or commission) per transaction with a buying order and $0 < K_s < 1$ with a selling order.

Let $\rho \ge 0$ denote the risk-free interest rate. Given the initial time t, initial stock price $S_t = S$, initial market trend $\alpha_t = \alpha$, and initial net position $i = 0, 1$, the reward functions of the decision sequences, Λ_0 and Λ_1, are the expected return rates of wealth:

$$J_i(S, \alpha, t, \Lambda_i) = \begin{cases} E_t\left\{\log\left(e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}}\right)\right\}, & \text{if } i = 0, \\ E_t\left\{\log\left(\left[\frac{S_{v_1}}{S} e^{\rho(\tau_2 - v_1)}(1-K_s)\right] \prod_{n=2}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}}\right)\right\}, & \text{if } i = 1, \end{cases}$$

where I_A represents the indicator function of A.

Remark 1. Note that different from the reward functions in Dai et al. [4], the above reward functions account for percentage gain/loss of each trade. Between trades, the entire balance is in a risk-free asset drawing interests at rate ρ. We only consider the control problem in the finite-time horizon $[0, T]$. This is signified by involving the indicator function $I_{\{\tau_n < T\}}$ in the payoff function J_i. The meaning of this indicator function is that a buying order at stopping time τ_n will be accounted only when $\tau_n < T$. If a long position remains at $t = T$, then it has to be sold at that time. Transactions at $t > T$ will not affect the payoff J_i.

It is clear that

$$J_i(S, \alpha, t, \Lambda_i) = \begin{cases} E_t\left\{\rho(\tau_1 - t) + \sum_{n=1}^{\infty}\left[\log \frac{S_{v_n}}{S_{\tau_n}} + \rho(\tau_{n+1} - v_n) + \log \frac{1-K_s}{1+K_b} I_{\{\tau_n < T\}}\right]\right\}, & \text{if } i = 0, \\ E_t\left\{\left[\log \frac{S_{v_1}}{S} + \log(1-K_s) + \rho(\tau_2 - v_1)\right]\right. \\ \quad \left. + \sum_{n=2}^{\infty}\left[\log \frac{S_{v_n}}{S_{\tau_n}} + \rho(\tau_{n+1} - v_n) + \log \frac{1-K_s}{1+K_b} I_{\{\tau_n < T\}}\right]\right\}, & \text{if } i = 1, \end{cases}$$

where the term $E_t \sum_{n=1}^{\infty} \xi_n$ for random variables ξ_n is interpreted as $\limsup_{N\to\infty} E_t \sum_{n=1}^{N} \xi_n$.

Remark 2. We will show in §3 that the optimal strategy can be given in terms of the conditional probability in a bull market and two threshold levels. A buying (selling) decision is triggered when the conditional probability in a bull market crosses these thresholds. Moreover, the optimal strategy involves only a finite number of trades (see Lemma 2).

It is easy to see that one should never buy a stock if the risk-free rate is greater than the log return rate of stock in the bull market, i.e., $\rho \ge \mu_1 - \sigma^2/2$, and never sell the stock if the risk-free rate is lower than the log return rate of stock in the bear market, i.e., $\rho \le \mu_2 - \sigma^2/2$. To exclude these trivial cases, we assume that

$$\mu_2 - \frac{\sigma^2}{2} < \rho < \mu_1 - \frac{\sigma^2}{2}. \tag{1}$$

Note that the market trend α_r is not directly observable. Thus, it is necessary to convert the problem into one that is observable. One way to accomplish this is to use the Wonham [19] filter..

Let $p_r = P(\alpha_r = 1 \mid \mathcal{F}_r)$ denote the conditional probability in a bull market $(\alpha_r = 1)$ given the filtration $\mathcal{F}_r = \sigma\{S_u\colon 0 \le u \le r\}$. Then, we can show (see Wonham [19]) that p_r satisfies the following stochastic differential equation:

$$dp_r = [-(\lambda_1+\lambda_2)p_r + \lambda_2]\,dr + \frac{(\mu_1-\mu_2)p_r(1-p_r)}{\sigma}\,d\hat{B}_r, \tag{2}$$

where $\hat{B}_r$ is the innovation process (a standard Brownian motion; see e.g., Øksendal [15]) given by

$$d\hat{B}_r = \frac{d\log(S_r) - [(\mu_1-\mu_2)p_r + \mu_2 - \sigma^2/2]\,dr}{\sigma}. \tag{3}$$

It is easy to see that S_r can be written in terms of $\hat{B}_r$:

$$dS_r = S_r[(\mu_1-\mu_2)p_r + \mu_2]\,dr + S_r\sigma\,d\hat{B}_r. \tag{4}$$

In view of this, the separation principle holds for the partially observed optimization problem.

The problem is to choose Λ_i to maximize the discounted return J_i subject to (2) and (4). We emphasize that this new problem is completely observable because p_r, the conditional probability in a bull market, can be obtained by using the stock price up to time r.

Note that the reward function J_i only accounts for the percentage gain/loss. For any given τ_n and v_n, we have

$$\log\frac{S_{v_n}}{S_{\tau_n}} = \int_{\tau_n}^{v_n} f(p_r)\,dr + \int_{\tau_n}^{v_n}\sigma\,d\hat{B}_r, \tag{5}$$

where

$$f(p_r) = (\mu_1-\mu_2)p_r + \mu_2 - \frac{\sigma^2}{2}. \tag{6}$$

Note also that

$$E_t\int_{\tau_n}^{v_n}\sigma\,d\hat{B}_r = 0. \tag{7}$$

Therefore the reward function J_i is independent of the initial stock price S. Consequently, given $p_t = p$, we can rewrite the reward function as

$$J_i = J_i(p, t, \Lambda_i).$$

For $i = 0, 1$, let $V_i(p,t)$ denote the value function with the state p at time t. That is,

$$V_i(p,t) = \sup_{\Lambda_i} J_i(p, t, \Lambda_i). \tag{8}$$

The following lemma gives the bounds of the value functions.

Lemma 1. *Let $V_i(p,t)$, $i = 1, 2$ be the value functions defined in (8). We have*

$$\rho(T-t) \le V_0(p,t) \le \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t)$$

and

$$\log(1-K_s) + \rho(T-t) \le V_1(p,t) \le \log(1-K_s) + \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t).$$

Proof. It is clear that the lower bounds for V_i follow from their definition. It remains to estimate their upper bounds. Using (5) and (7) and noticing $0 \le p_r \le 1$, we have

$$E_t\left(\log\frac{S_{v_n}}{S_{\tau_n}}\right) = E_t\left[\int_{\tau_n}^{v_n} f(p_r)\,dr\right] \le \left(\mu_1 - \frac{\sigma^2}{2}\right)\int_{\tau_n}^{v_n} dr = \left(\mu_1 - \frac{\sigma^2}{2}\right)(v_n - \tau_n).$$

Note that $\log(1-K_s)<0$ and $\log(1+K_b)>0$. It follows that

$$\begin{aligned} J_0(p,t,\Lambda_0) &\le E_t\left\{\rho(\tau_1-t)+\sum_{n=1}^{\infty}\left[\left(\mu_1-\frac{\sigma^2}{2}\right)(\upsilon_n-\tau_n)+\rho(\tau_{n+1}-\upsilon_n)\right]\right\} \\ &\le \max\left\{\rho,\mu_1-\frac{\sigma^2}{2}\right\}(T-t)=\left(\mu_1-\frac{\sigma^2}{2}\right)(T-t), \end{aligned}$$

where the last equality is due to (1). We then obtain the desired result. An upper bound for V_1 can be established similarly. $\square$

Next, we consider the associated HJB equations. Using the dynamic programming principle, one has

$$V_0(p,t)=\sup_{\tau_1\ge t}E_t\{\rho(\tau_1-t)-\log(1+K_b)+V_1(p_{\tau_1},\tau_1)\}$$

and

$$V_1(p,t)=\sup_{\upsilon_1\ge t}E_t\left\{\int_t^{\upsilon_1}f(p_s)\,ds+\log(1-K_s)+V_0(p_{\upsilon_1},\upsilon_1)\right\},$$

where $f(\cdot)$ is as given in (6). Let

$$L=\partial_t+\frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2\partial_{pp}+[-(\lambda_1+\lambda_2)p+\lambda_2]\partial_p$$

denote the generator of (t,p_t). Then the associated HJB equations are

$$\begin{cases}\min\{-LV_0-\rho,V_0-V_1+\log(1+K_b)\}=0,\\ \min\{-LV_1-f(p),V_1-V_0-\log(1-K_s)\}=0,\end{cases} \tag{9}$$

with the terminal conditions

$$\begin{cases}V_0(p,T)=0,\\ V_1(p,T)=\log(1-K_s).\end{cases} \tag{10}$$

Using the same technique as in Dai et al. [4], we can show that Problems (9)–(10) have a unique bounded strong solution (V_0,V_1), where $V_i\in W_q^{2,1}([\varepsilon,1-\varepsilon]\times[0,T])$ for any $\varepsilon\in(0,1/2)$, $q\in[1,+\infty)$. It should be pointed out that the differential operator L is degenerate at $p=0,1$ and the solution is only locally bounded in $W_q^{2,1}$.

Remark 3. In this paper, we restrict the state space of p to $(0,1)$ because $p=0$ and $p=1$ are entrance boundaries (see Karlin and Taylor [9] and Dai et al. [4] for definition and discussions).

Now, we define the buy region (BR), the sell region (SR), and the no-trading region (NT) as follows:

$$\begin{aligned}\mathrm{BR}&=\{(p,t)\in(0,1)\times[0,T]\colon V_1(p,t)-V_0(p,t)=\log(1+K_b)\},\\ \mathrm{SR}&=\{(p,t)\in(0,1)\times[0,T]\colon V_1(p,t)-V_0(p,t)=\log(1-K_s)\},\\ \mathrm{NT}&=(0,1)\times[0,T]\setminus(\mathrm{BR}\cup\mathrm{SR}).\end{aligned}$$

To study the optimal strategy, we only need to characterize these regions.

3. Main results.
In this section, we present the main theoretical results.

3.1. Characterization of the optimal trading strategy.
Let

$$p_0=\frac{\rho-\mu_2+\sigma^2/2}{\mu_1-\mu_2},\qquad a=\log\frac{1+K_b}{1-K_s}. \tag{11}$$

THEOREM 1. *There exist two monotonically increasing boundaries $p_s^*(t)$, $p_b^*(t)$: $[0, T) \to [0, 1]$ such that*

$$\text{SR} = \{(p, t) \in (0, 1) \times [0, T): p \le p_s^*(t)\}, \tag{12}$$

$$\text{BR} = \{(p, t) \in (0, 1) \times [0, T): p \ge p_b^*(t)\}. \tag{13}$$

Moreover,
(i) $p_b^*(t) \ge p_0 \ge p_s^*(t)$ *for all* $t \in [0, T)$;
(ii) $\lim_{t\to T^-} p_s^*(t) = p_0$;
(iii) *there is a* $\delta > a/(\mu_1 - \rho - \sigma^2/2)$ *such that* $p_b^*(t) = 1$ *for* $t \in (T - \delta, T)$;
(iv) $p_s^*(\cdot)$, $p_b^*(\cdot) \in C^\infty$ *if* $p_s^*(t), p_b^*(t) \in (0, 1)$.

PROOF. Denote $Z(p, t) \equiv V_1(p, t) - V_0(p, t)$. Similar to Dai et al. [4, Lemma 2.2], we can show that $Z(p, t)$ satisfies the following double obstacle problem:

$$\min\big\{\max\{-LZ - f(p) + \rho, Z - \log(1 + K_b)\}, Z - \log(1 - K_s)\big\} = 0, \tag{14}$$

in $(0, 1) \times [0, T)$ with the terminal condition $Z(p, T) = \log(1 - K_s)$, and

$$\begin{cases} -LV_0 = \rho + (-LZ - f(p) + \rho)^- = \rho I_{\{Z<\log(1+K_b)\}} + f(p) I_{\{Z=\log(1+K_b)\}}, \\ V_0(p, T) = 0, \end{cases} \tag{15}$$

$$\begin{cases} -LV_1 = f(p) + (-LZ - f(p) + \rho)^+ = f(p) I_{\{Z>\log(1-K_s)\}} + \rho I_{\{Z=\log(1-K_s)\}}, \\ V_1(p, T) = \log(1 - K_s). \end{cases} \tag{16}$$

Then, we can use the same argument as in the proof of Theorem 2.5 in Dai et al. [4] to obtain the desired results. □

We call $p_s^*(t)$ ($p_b^*(t)$) the optimal sell (buy) boundary. To see better how Theorem 1 works, we provide a numerical result for illustration. In Figure 1, we plot the optimal buy and sell boundaries against time, where the parameter values used are $\lambda_1 = 0.36$, $\lambda_2 = 2.53$, $\mu_1 = 0.18$, $\mu_2 = -0.77$, $\sigma = 0.184$, $K_b = K_s = 0.001$, $\rho = 0.0679$, and $T = 1$. It can be seen that $p_s^*(t)$ and $p_b^*(t)$ are almost flat except when t is close to T, where they sharply increase with t. Moreover, the sell boundary $p_s^*(t)$ approaches the theoretical value

$$p_0 = \frac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2} = \frac{0.0679 + 0.77 + 0.184^2/2}{0.18 + 0.77} \approx 0.9$$

as $t \to T = 1$. Between the two boundaries is the NT, above the buy boundary is the BR, and below the sell boundary is the SR. Also, we observe that there is a δ such that $p_b^*(t) = 1$ for $t \in [T - \delta, T]$, which indicates that it is never optimal to buy stock when t is very close to T. Using Theorem 1, the lower bound of δ is estimated as

$$\frac{a}{\mu_1 - \rho - \sigma^2/2} = \frac{\log(1.001/0.999)}{0.18 - 0.0679 - 0.184^2/2} \approx 0.021,$$

which is consistent with the numerical result.

The behavior of the thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$ when t approaches T is due to our technical requirement of liquidating all the positions at T. Interested in long-term investment, we will approximate these thresholds, as in Dai et al. [4] by constants $p_s^* = \lim_{T-t\to\infty} p_s^*(t)$ and $p_b^* = \lim_{T-t\to\infty} p_b^*(t)$. Assuming that the initial position is flat and the initial conditional probability $p(0) \in (p_s^*, p_b^*)$, our trading strategy can be described as follows: as p_t goes up to hit p_b^*, we take a long position, that is, investing all the wealth in the stock. We will close out the position only when p_t goes down and hits p_s^*. According to (2)–(3), we have

$$dp_r = g(p_r)\,dr + \frac{(\mu_1 - \mu_2) p_r (1 - p_r)}{\sigma^2}\, d\log S_r, \tag{17}$$

where

$$g(p) = -(\lambda_1 + \lambda_2) p + \lambda_2 - \frac{(\mu_1 - \mu_2) p_t (1 - p_t)((\mu_1 - \mu_2) p + \mu_2 - \sigma^2/2)}{\sigma^2}.$$

Relation (17) implies that p_t, the conditional probability in the bull market, increases (decreases) as the stock price goes up (down). Hence our optimal trading strategy buys when the stock price is going up and sells when the stock price declines. In other words, it is trend following in nature.

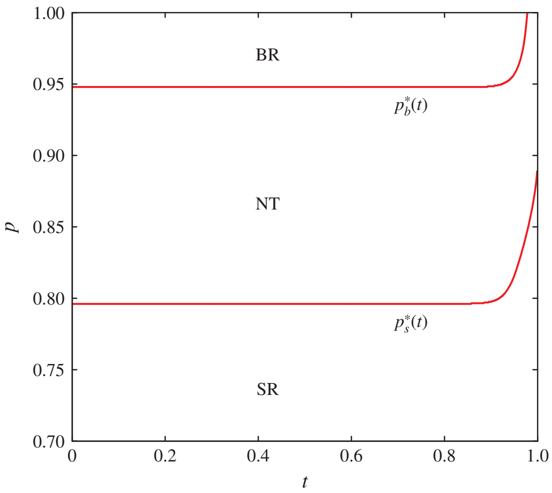

FIGURE 1. Optimal buy and sell boundaries.

Note. Parameter values: $\lambda_1 = 0.36$, $\lambda_2 = 2.53$, $\mu_1 = 0.18$, $\mu_2 = -0.77$, $\sigma = 0.184$, $K_b = K_s = 0.001$, $\rho = 0.0679$, $T = 1$.

We have seen from Proposition 1 that the buy and sell boundaries are increasing with time and that the buy (sell) boundary is bounded from below (above) by p_0. Note that $p = 0$ and $p = 1$ are entrance boundaries that cannot be reached from the interior of the state space (see Remark 2 in Dai et al. [4]). A natural question is whether or not the sell (buy) boundary can coincide with $p = 0$ ($p = 1$). The following theorem provides an affirmative answer and sufficient conditions.

THEOREM 2. *Let p_0 and a be given as in (11).*
(i) *If*

$$p_0 < \min\left\{\frac{1}{3}, \frac{\lambda_2}{6(\lambda_1+\lambda_2)}\right\} \tag{18}$$

and

$$\frac{p_0}{\lambda_2/(12(\mu_1-\mu_2)p_0) - (\lambda_1+\lambda_2)/(2(\mu_1-\mu_2))} \le a \le \frac{p_0}{9(\mu_1-\mu_2)/\sigma^2 + (2+6\lambda_1)/(\mu_1-\mu_2)}, \tag{19}$$

then

$$p_s^*(t) \equiv 0, \quad \forall\, t \le T - \frac{1}{p_0} - \frac{12p_0}{\lambda_2}.$$

(ii) *If $\lambda_1 > \lambda_2$ and*

$$p_0 \ge 1 - \min\left[\frac{1}{3}, \frac{\lambda_1-\lambda_2}{6(\lambda_1+\lambda_2)}, \frac{\sigma^2(\lambda_1+\lambda_2)}{18(\mu_1-\mu_2)^2}\right], \qquad a \ge \frac{\sigma^2(1-p_0)}{\mu_1-\mu_2}, \tag{20}$$

then

$$p_b^*(t) \equiv 1, \quad \forall\, t < T.$$

We present the proof of Theorem 2, which relies on a technical partial different equation approach, in the appendix.

Figure 2 illustrates situations where the parameter values satisfy the conditions in Theorem 2. In Figure 2(a), the sell boundary coincides with the entrance boundary $p = 0$ before $t = 0.98$. Hence one should never sell stock except when t is very close to 1. In Figure 2(b), the buy boundary remains at the entrance boundary $p = 1$, which means that one should never buy any stock.

Now, we present a sufficient condition to ensure that the sell boundary and the buy boundary are attainable when t is not close to the terminal time T.

THEOREM 3. *Let p_0 and a be as given in (11). If $p_0 < \frac{1}{3}$ and*

$$a \le \min\left\{\frac{p_0}{9(\mu_1-\mu_2)/\sigma^2 + (2+6\lambda_1)/(\mu_1-\mu_2)}, \frac{p_0}{8(\mu_1-\mu_2)/\sigma^2 + 16\lambda_2/((\mu_1-\mu_2)p_0)}\right\}, \tag{21}$$

then

$$p_s^*(t) > 0, \quad p_b^*(t) < 1, \qquad \forall\, t \le T - \frac{1}{p_0}.$$

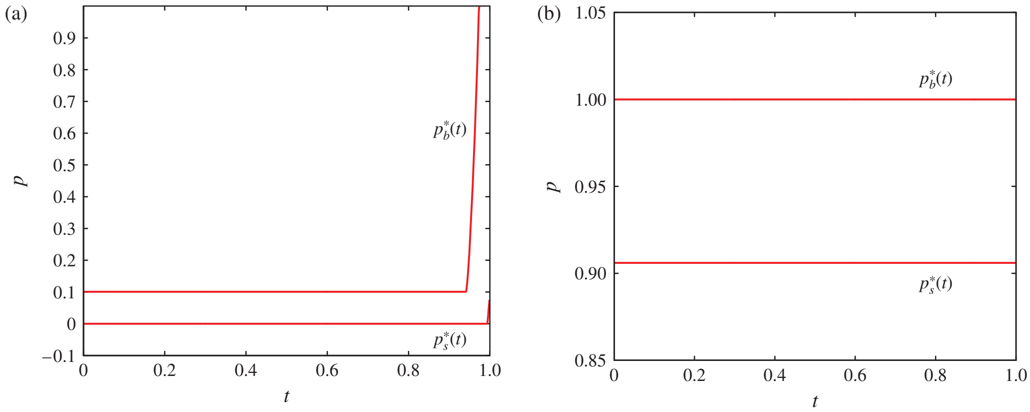

FIGURE 2. Scenarios of $p_s^*(t) = 0$, $p_b^*(t) \equiv 1$.

Notes. Parameter values. Case (a): $\lambda_1 = 0.2$, $\lambda_2 = 30$, $\mu_1 = 0.15$, $\mu_2 = 0.1$, $\sigma = 0.2$, $K_b = K_s = 0.0006$, $\rho = 0.085$, $T = 1$; Case (b): $\lambda_1 = 20$, $\lambda_2 = 1$, $\mu_1 = 0.2$, $\mu_2 = 0$, $\sigma = 0.45$, $K_b = K_s = 0.05$, $\rho = 0.08$, $T = 1$.

Again, we present the technical proof in the appendix.

The conditions in Theorem 3 are not sharp. However, condition (21) always holds if the transaction costs are sufficiently small. We also emphasize that the conditions presented in Theorem 3 are sufficient but not necessary. In fact, our numerical tests reveal that for reasonable parameter values, the buy and sell boundaries are strictly between $(0, 1)$ when t is not close to the terminal time T.

3.2. A verification theorem. We now present a verification theorem, indicating that the solutions V_0 and V_1 of problems (9)–(10) are equal to the value functions and sequences of optimal stopping times can be constructed by using (p_s^*, p_b^*).

THEOREM 4 (Verification Theorem). *Let $(w_0(p,t), w_1(p,t))$ be the unique solution to problems (9)–(10) and $p_b^*(t)$ and $p_s^*(t)$ be the associated free boundaries, where $w_i \in W_q^{2,1}([\varepsilon, 1-\varepsilon] \times [0,T])$, $i = 0, 1$ for any $\varepsilon \in (0, 1/2)$, $q \in [1, +\infty)$. Then, $w_0(p,t)$ and $w_1(p,t)$ are equal to the value functions $V_0(p,t)$ and $V_1(p,t)$, respectively.*

Moreover, let

$$\Lambda_0^* = (\tau_1^*, v_1^*, \tau_2^*, v_2^*, \dots),$$

where the stopping times are $\tau_1^* = T \wedge \inf\{r \ge t\colon p_r \ge p_b^*(r)\}$, $v_n^* = T \wedge \inf\{r \ge \tau_n^*\colon p_r \le p_s^*(r)\}$, and $\tau_{n+1}^* = T \wedge \inf\{r > v_n^*\colon p_r \ge p_b^*(r)\}$ for $n \ge 1$, and let

$$\Lambda_1^* = (v_1^*, \tau_2^*, v_2^*, \tau_3^*, \dots),$$

where the stopping times $v_1^* = T \wedge \inf\{r \ge t\colon p_r \le p_s^*(r)\}$, $\tau_n^* = T \wedge \inf\{r > v_{n-1}^*\colon p_r \ge p_b^*(r)\}$, and $v_n^* = T \wedge \inf\{r \ge \tau_n^*\colon p_r \le p_s^*(r)\}$ for $n \ge 2$. Then, Λ_0^* and Λ_1^* are optimal.

Note that in Theorem 4, we removed the technical condition $v_n^* \to T$ used in Dai et al. [4]. In addition, the solution to problems (9)–(10) is not smooth enough to use the Itô lemma. We will employ an approximation approach to overcome this difficulty. Note that one cannot directly use the results of Lamberton and Zervos [11], which are for a stationary problem.

Before proving Theorem 4, we introduce two lemmas. The first indicates that the optimal trading strategy incurs only a finite number of trades almost surely.

LEMMA 2. *Let v_n^*, τ_n^* be as given in Theorem 4. Define*

$$\mathcal{N} = \inf\{n\colon v_n^* = T \text{ or } \tau_{n+1}^* = T\} \qquad \text{and} \qquad \inf \varnothing = +\infty.$$

Then, there exists a constant C such that

$$\mathbb{E}(\mathcal{N}) \le C.$$

In particular, $\mathcal{N}(\omega)$ is finite almost surely. In other words, for a fixed path, $v_n^* = \tau_n^* = T$ when n is large enough.

Proof. Recalling $p_b^*(r) \geq p_0 \geq p_s^*(r)$, p_s^*, $p_b^* \in C^\infty$ (see Theorem 1), and

$$V_1(r, p_b^*(r)) - V_0(r, p_b^*(r)) = \log(1+K_b) > \log(1-K_s) = V_1(r, p_s^*(r)) - V_0(r, p_s^*(r)),$$

we deduce that $p_b^*(r) > p_s^*(r)$ and there is a $\delta > 0$ such that

$$p_b^*(r) - p_s^*(r) > 4\delta.$$

Denote

$$P_r^1 = p_t + \int_t^r [-(\lambda_1+\lambda_2)p_u + \lambda_2]\,du - p_s^*(r), \qquad P_r^2 = \int_t^r \frac{(\mu_1-\mu_2)p_u(1-p_u)}{\sigma}\,d\hat{B}_u,$$

where P^1 is an absolutely continuous-stochastic process and P^2 is a martingale. Apparently,

$$P_r^1 + P_r^2 = p_r - p_s^*(r). \tag{22}$$

Since stochastic process p_r has continuous paths, the definitions of p_s^*, p_b^* imply that

$$\begin{aligned}(P^1_{\tau_n^*} - P^1_{v_{n-1}^*}) + (P^2_{\tau_n^*} - P^2_{v_{n-1}^*}) &= (P^1_{\tau_n^*} + P^2_{\tau_n^*}) - (P^1_{v_{n-1}^*} + P^2_{v_{n-1}^*})\\ &= (p_{\tau_n^*} - p_s^*(\tau_n^*)) - (p_{v_{n-1}^*} - p_s^*(v_{n-1}^*))\\ &= p_b^*(\tau_n^*) - p_s^*(\tau_n^*) > 4\delta.\end{aligned}$$

Hence we deduce

$$\text{either } P^1_{\tau_n^*} - P^1_{v_{n-1}^*} > 2\delta, \text{ or } P^2_{\tau_n^*} - P^2_{v_{n-1}^*} > 2\delta. \tag{23}$$

In contrast, P^1 is clearly bounded since p_r, $p_s^*(r) \in [0,1]$. Owing to (22), we infer that P^2 is bounded as well. Hence we can choose a positive integer M such that

$$|P^2| \leq M\delta.$$

If $P^2_{\tau_n^*} - P^2_{v_{n-1}^*} > 2\delta$, then the continuity of P^2 implies that the martingale P^2 should cross upward at least one of the intervals $[i\delta, (i+1)\delta]$ $(i = -M, -M+1, \cdot, \cdot, \cdot, M-1)$ during $[v_{n-1}^*, \tau_n^*]$.

Hence, by virtue of (23), we deduce that

$$\mathcal{N} \leq \sum_{i=-M}^{M-1} \mathcal{U}_{[i\delta,(i+1)\delta]}(P^2) + \mathcal{U}_{2\delta}(P^1), \tag{24}$$

where $\mathcal{U}_{[i\delta,(i+1)\delta]}(P^2)$ denotes the number of crossings upward the interval $[i\delta, (i+1)\delta]$ for P^2 during $[0,T]$, and $\mathcal{U}_{2\delta}(P^1)$ denotes the number of crossing upward a 2δ-length interval for P^1 during $[0,T]$. In view of the inequality for crossing upward, we infer that

$$\mathbb{E}(\mathcal{U}_{[i\delta,(i+1)\delta]}(P^2)) \leq \frac{1}{\delta}(\mathbb{E}(|P^2|) + |i\delta|) \leq \frac{1}{\delta}\mathbb{E}(|P^2|) + M \leq \frac{C}{4M}, \tag{25}$$

where C is a constant large enough. Since $p_r \in [0,1]$ and p_s^* is increasing, it is easy to see

$$\mathcal{U}_{2\delta}(P^1) \leq \frac{C}{2}. \tag{26}$$

The combination of (24)–(26) yields the desired result. □

Our next lemma indicates that the solution to problems (9)–(10) has the same bounds as the value function (see Lemma 1).

Lemma 3. *Let $(w_0(p,t), w_1(p,t))$ be the solution to problem (9)–(10). Then,*

$$\rho(T-t) \leq w_0(p,t) \leq \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t)$$

and

$$\log(1-K_s) + \rho(T-t) \leq w_1(p,t) \leq \log(1-K_s) + \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t).$$

Proof. Clearly,

$$-L(w_0-\rho(T-t)) = -Lw_0-\rho \geq 0,$$

from which we immediately infer by the maximum principle $w_0 \geq \rho(T-t)$. Owing to $w_1-w_0-\log(1-K_s)\geq 0$, we have $w_1 \geq \log(1-K_s)+\rho(T-t)$.

To prove the right-hand side inequalities, we use (15) and (16) to get

$$-Lw_0 \leq \max\{\rho, f(p)\} \leq \mu_1-\frac{\sigma^2}{2},$$
$$-Lw_1 \leq \max\{\rho, f(p)\} \leq \mu_1-\frac{\sigma^2}{2}.$$

Again, by the maximum principle, the desired result follows. □

Now, we are ready to prove the verification theorem.

Proof of Theorem 4. First, we show that for any stopping times $\theta_2 \geq \theta_1 \geq t$,

$$E_t w_1(p_{\theta_1}, \theta_1) \geq E_t\left[\int_{\theta_1}^{\theta_2} f(p_r)\,dr + w_1(p_{\theta_2}, \theta_2)\right] = E_t\left[\log\frac{S_{\theta_2}}{S_{\theta_1}} + w_1(p_{\theta_2}, \theta_2)\right] \quad \text{a.s.} \tag{27}$$

Since w_1 is only *locally* bounded in $W_q^{2,1}((0,1)\times[0,T])$, we cannot directly use the Itô formula. To overcome the difficulty, we introduce the following stopping times:

$$\beta_m = \inf\{r \geq \theta_1\colon\, p_r \in (0, 1/m) \cup (1-1/m, 1)\} \wedge \theta_2, \quad m = 1, 2\ldots.$$

Note that $p=0$ and $p=1$ cannot be reached from the interior of $(0,1)$ (see Dai et al. [4, Remark 2]). We then infer that $\beta_m \to \theta_2$ as $m\to\infty$.

Due to $w_1 \in W_q^{2,1}([1/m, 1-1/m]\times[0,T])$, applying the Itô formula to $w_1(p_r, r)$ in $[\theta_1, \beta_m]$ yields (c.f. Krylov [10])

$$w_1(p_{\theta_1}, \theta_1) = w_1(p_{\beta_m}, \beta_m) - \int_{\theta_1}^{\beta_m} Lw_1(p_r, r)\,dr - \int_{\theta_1}^{\beta_m} \partial_p w_1(p_r, r)\frac{(\mu_1-\mu_2)p_r(1-p_r)}{\sigma}\,d\hat{B}_r \quad \text{a.e.}$$

By the Sobolev embedding theorem, $\partial_p w_1 \in C([1/m, 1-1/m]\times[0,T])$, which implies that the last term in the above equation is a martingale. Taking conditional expectation in the above equation, we deduce that

$$E_t w_1(p_{\theta_1}, \theta_1) = E_t\left[w_1(p_{\beta_m}, \beta_m) - \int_{\theta_1}^{\beta_m} Lw_1(p_r, r)\,dr\right]. \tag{28}$$

Since $w_0, w_1 \in W_{q,\mathrm{loc}}^{2,1}$, we can rewrite

$$\begin{aligned} Lw_1 &= -f(p)I_{\{w_1>w_0+\log(1-K_s)\}} + L(w_0+\log(1-K_s))I_{\{w_1=w_0+\log(1-K_s)\}} \\ &= -f(p)I_{\{w_1>w_0+\log(1-K_s)\}} - \rho I_{\{w_1=w_0+\log(1-K_s)\}}. \end{aligned}$$

Hence

$$E\left[\int_0^T |Lw_1(p_r, r)|\,dr\right] < \infty. \tag{29}$$

Sending $m\to\infty$ in (28) and using (29) and Lemma 3, we have by the dominated convergence theorem

$$E_t w_1(p_{\theta_1}, \theta_1) = E_t\left[-\int_{\theta_1}^{\theta_2} Lw_1(p_r, r)\,dr + w_1(p_{\theta_2}, \theta_2)\right] \quad \text{a.s.}$$

Using $-Lw_1 - f(p) \geq 0$, we then obtain (27). In a similar way, we can show

$$E_t w_0(p_{\theta_1}, \theta_1) \geq E_t[\rho(\theta_2-\theta_1) + w_0(p_{\theta_2}, \theta_2)] \quad \text{a.s.} \tag{30}$$

We next show for any Λ_1 and $k=1,2,\ldots,$

$$E_t w_0(p_{\upsilon_k}, \upsilon_k) \geq E_t\left[\rho(\tau_{k+1}-\upsilon_k) + \log\frac{S_{\upsilon_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{\upsilon_{k+1}}, \upsilon_{k+1}) + (\log(1-K_s)-\log(1+K_b))I_{\{\tau_{k+1}<T\}}\right]. \tag{31}$$

In fact, using (27) and (30) and noticing that

$$w_0 \geq w_1 - \log(1+K_b) \quad \text{and} \quad w_1 \geq w_0 + \log(1-K_s),$$

we have

$$\begin{aligned}
&E_t w_0(p_{v_k}, v_k)\\
&\quad \geq E_t[\rho(\tau_{k+1}-v_k) + w_0(p_{\tau_{k+1}}, \tau_{k+1})]\\
&\quad \geq E_t[\rho(\tau_{k+1}-v_k) + (w_1(p_{\tau_{k+1}}, \tau_{k+1}) - \log(1+K_b))I_{\{\tau_{k+1}<T\}}]\\
&\quad \geq E_t\left[\rho(\tau_{k+1}-v_k) + \left(\log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_1(p_{v_{k+1}}, v_{k+1}) - \log(1+K_b)\right)I_{\{\tau_{k+1}<T\}}\right]\\
&\quad \geq E_t\left[\rho(\tau_{k+1}-v_k) + \left(\log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}}, v_{k+1}) + \log(1-K_s) - \log(1+K_b)\right)I_{\{\tau_{k+1}<T\}}\right]\\
&\quad = E_t\left[\rho(\tau_{k+1}-v_k) + \log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}}, v_{k+1}) + (\log(1-K_s) - \log(1+K_b))I_{\{\tau_{k+1}<T\}}\right].
\end{aligned}$$

Note that the above inequalities also work when starting at t in lieu of v_1, i.e.,

$$w_0(p_t, t) \geq E_t\left[\rho(\tau_1 - t) + \log\frac{S_{v_1}}{S_{\tau_1}} + w_0(p_{v_1}, v_1) + (\log(1-K_s) - \log(1+K_b))I_{\{\tau_1<T\}}\right].$$

Use this inequality and iterate (31) with $k = 1, 2, \ldots,$ and note $w_0 \geq 0$ to obtain

$$w_0(p, t) \geq V_0(p, t).$$

Similarly, we can show that

$$w_1(p_t, t) \geq E_t\left[\log\frac{S_{v_1}}{S_t} + w_1(p_{v_1}, v_1)\right] \geq E_t\left[\log\frac{S_{v_1}}{S_t} + w_0(p_{v_1}, v_1) + \log(1-K_s)\right].$$

Use this and iterate (31) with $k = 1, 2, \ldots$ as above to obtain

$$w_1(p, t) \geq V_1(p, t).$$

By Lemma 2, we immediately obtain $v_k^*, \tau_k^* \to T$ as $k \to \infty$. It can be seen that the equalities hold when $\tau_k = \tau_k^*$ and $v_k = v_k^*$. This completes the proof. □

We conclude this section by showing that for the optimal trading strategy, the limsup in the reward function defined in §2 is, in fact, a limit. Hence the definition of the reward function makes sense in practice.

Theorem 5. *The limit of* $\mathbb{E}[\Theta(m)]$ *as* m *tends to infinity exists, where*

$$\Theta(m) = \sum_{n=1}^{m}\left[\log\frac{S_{v_n^*}}{S_{\tau_n^*}} + \rho(\tau_{n+1}^* - v_n^*) + \log\frac{1-K_s}{1+K_b}I_{\{\tau_n^*<T\}}\right].$$

Proof. Lemma 2 implies that for fixed path, $\tau_n = v_n = T$ for n large enough. Therefore the sum is finite a.s., and $\lim_{m\to\infty}\Theta(m)$ exists a.s.

Next, we estimate the bound of $\Theta(m)$. Similar to the proof of Lemma 1, we can obtain

$$\Theta(m) \leq \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t).$$

Using the same argument as in the proof of Lemma 1, we have

$$\sum_{n=1}^{m}\left[\log\frac{S_{v_n^*}}{S_{\tau_n^*}} + \rho(\tau_{n+1}^* - v_n^*)\right] \geq \left(\mu_2 - \frac{\sigma^2}{2}\right)(T-t).$$

Moreover, it is clear that

$$\sum_{n=1}^{m}\log\frac{1-K_s}{1+K_b}I_{\{\tau_n^*<T\}} \geq \log\frac{1-K_s}{1+K_b}\mathcal{N} \quad \text{for any } m.$$

Lemma 2 implies that

$$\mathbb{E}\left[\left(\mu_2 - \frac{\sigma^2}{2}\right)(T-t) + \log\frac{1-K_s}{1+K_b}\mathcal{N}\right]$$

exists. The convergence of $\mathbb{E}[\Theta(m)]$ follows from the Lebesgue dominated convergence theorem. □

Table 1. Parameter values.

λ_1	λ_2	μ_1	μ_2	σ	K	ρ
0.36	2.53	0.18	−0.77	0.184	0.001	0.0679

4. Simulation and market tests. In this section, we carry out numerical simulations and backtesting to examine the effectiveness of our trading strategy. To estimate p_t, the conditional probability in a bull market, we use a discrete version of the stochastic differential Equation (17) for $t = 0, 1, \ldots, N$ with $dt = 1/252$,

$$p_{t+1} = \min\left(\max\left(p_t + g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}\log(S_{t+1}/S_t), 0\right), 1\right), \tag{32}$$

where the price process S_t is determined by the simulated paths or the historical market data. The min and max are added to ensure that the discrete approximation p_t of the conditional probability in the bull market stays in the interval [0, 1].

4.1. Simulations. For simulation, we use the parameters given in Table 1. These numbers were used in Dai et al. [4]. The time horizon is 40 years.

We solve the HJB equations and derive $p_s^* = 0.796$ and $p_b^* = 0.948$. We run the 5,000 round simulations 10 times. Starting with \$1, the mean of the total/annualized return and the standard deviation are given in Table 2. The trend-following strategy clearly outperforms the buy and hold in terms of return. Moreover, the trend-following strategy has a monthly Sharp ratio of 0.22 while the return of the buy-and-hold strategy is lower than the risk-free rate $\rho = 0.0679$.

Comparing to the simulation results in Dai et al. [4], we only observe a slight improvement in terms of the ratio of mean return of the trend-following strategy to that of the buy-and-hold strategy. However, the improvement is not significant enough to distinguish statistically from the results in Dai et al. [4] despite theoretically, the present paper is more solid than Dai et al. [4]. Together with sensitivity tests on thresholds conducted in Dai et al. [4], this reveals that using the conditional probability in the bull market as trade signals is rather robust against the change of thresholds. It is analogous to the scenario when technical analysis is used: the effects of using 200-day moving average and 150-day moving average as trade signals are likely comparable.

The above simulation results are based on the average outcomes of large numbers of simulated paths. We now investigate the performance of our strategy with individual sample paths. Table 3 collects simulation results on 10 single paths using buy-and-sell thresholds $p_s^* = 0.795$ and $p_b^* = 0.948$ with the same data given in Table 1. We can see that the simulation is very sensitive to individual paths. Nevertheless, on large number of trials, our strategy clearly outperforms the buy-and-hold strategy statistically.

Table 2. Statistics of ten 5,000 path simulations.

Variable	Trend following	Buy and hold	No. of trades
Mean	75.76 (11.4%)	5.62 (4.4%)	41.16
Stdev	2.48	0.39	0.29

Table 3. Ten single path simulations.

Trend following	Buy and hold	No. of trades
67.0800	3.289200	36.000
24.804	2.2498	42.000
22.509	0.40591	42.000
1887.8	257.75	33.000
26.059	0.16373	48.000
60.267	1.5325	43.000
34.832	5.7747	42.000
8.6456	0.077789	46.000
128.51	30.293	37.000
224.80	29.807	40.000

Note that this observation is consistent with the measurement of an effective investment strategy in marketplace. For example, O'Neil's CANSLIM works during a period of time does not mean it works on each stock when applied. How it works is measured based on the overall average when applied to a group of stocks fitting the prescribed selection criteria.

4.2. Market tests. We now turn to the question whether the trend following trading strategy presented works in real markets. In view of the path sensitivity discussed in the end of the last section, we conduct our tests using a broad-based stock index, which reflects the aggregation of the behaviors of a large number of stocks. While ex post tests are employed in Dai et al. [4], we conduct the ex ante tests for the SP500 index—a broad-based index that has a set of accessible historical data reasonably long for our tests. Our goal is evaluating whether our theoretically optimal trend following strategy provides useful guidance in real market.

The historical data for SP500 have been available since 1962. We assume that any trading action will take place at the close of the market, and therefore will use the SP500 daily closing price for our test. We define an up trend to be rally at least 20% and a down trend decline at least 20%. For any given period of the SP500 historical data, say, 5 or 10 years, one can find several up and down trends. We can use the statistics of the duration and total appreciation/depreciation of these trends to empirically calibrate the parameters μ_i, λ_i, $i = 1, 2$ and σ. However, after quickly scanning several such periods of data we find that the empirical estimate of these parameters is quite different in different time periods. The change of the parameters, of course, is not unanticipated. Many social, economic, and technological factors contribute to such a change and make it difficult to precisely predict. However, these exogenous impacts on the parameters happen over time. Thus we make the following working assumptions: (a) the parameters gradually change over a long time horizon (say, 10 years) yet they are relatively stable in a short time horizon (say, 1 year) and (b) recent data is more relevant compared to the data in the distant past. Based on these assumptions, we determine the parameters by beginning with the statistical estimate of the 10-year data from 1962 to 1972 as follows: μ_1 and λ_1 are estimated as the average of annualized return and reciprocal of the length of the up trends, and μ_2 and λ_2 are the average of annualized return and reciprocal of the length of the down trends. We conduct the trend-following strategy using these parameters and the corresponding thresholds in the following year, and then update the parameters and the corresponding thresholds at the beginning of a new year, using the new data that become available if a new up or down trend is completed. To reflect assumption (b), we update the parameters using the so-called exponential average method, in which the update of the parameters is determined by the old parameters and new parameters with formula

$$\text{update} = (1 - 2/N)\ \text{old} + (2/N)\ \text{new},$$

where we chose $N = 6$ based on the number of up and down trends between 1962 and 1972. The exponential average allows us to overweigh the recent information, whereas avoiding unwanted abrupt changes due to dropping old information. Then, we use the yearly updated parameters to calculate the corresponding thresholds. Finally, we use these parameters and thresholds to test the SP500 index from 1972 to 2011. The equity curve of the trend following strategy is compared to the buy-and-hold strategy in the same period of time in Figure 3. The

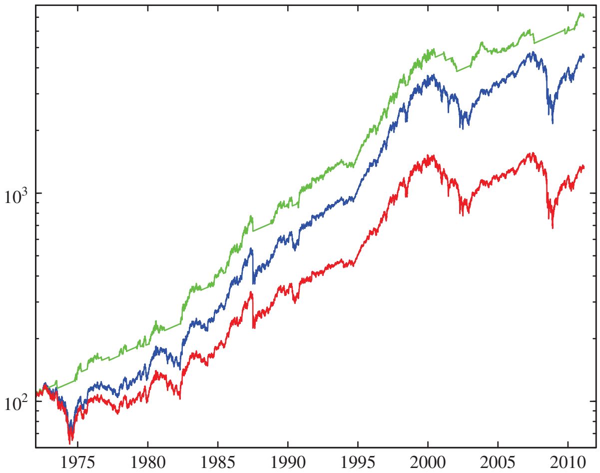

FIGURE 3. (Color online) Trend following trading of SP500 during 1972–2011 compared with buy and hold.

Table 4. Testing results for trend following trading strategies.

Index (timeframe)	TF (%)	TF sharpe	BH (%)	BH sharpe	10-year bonds (%)
SP500 (1972–2011)	11.03	0.217	9.8	0.128	6.79

upper, middle, and lower curves represent the equity curves of the trend following strategy, the buy-and-hold strategy, including dividend, and the SP500 index without dividend adjustment, respectively.

As we can see, the trend-following strategy not only outperforms the buy-and-hold strategy in total return, but also has a smoother equity curve, which means a higher Sharpe ratio; see Table 4.

The test result for SP500 here is, if not better, at least comparable to the ex pose test in Dai et al. [4] showing that trends indeed exist in the price movement of SP500. It is worthwhile pointing out that in Dai et al. [4], there is a mistake that the dividends are not treated as reinvestment. As a correction, the returns of the buy-and-hold strategy and the trend following strategy in Dai et al. [4] (Table 10) should be, respectively, 54.6 and 70.9 instead of 33.5 and 64.98, for SP500 (years 1962–2008).

We note that although an index such as the SP500 reflects the aggregation of the behavior of many individual stocks, trading it with the trend-following strategy could still experience an instability as observed in the end of last section. Using the trend-following strategy simultaneously on a large number of stocks should smooth out the fluctuation of the performance and achieve better stability. Although such tests belong to the area of developing proprietary trading strategies and do not fall in the scope of this paper, the testing methods used here are relevant and useful.

5. Conclusion. We have considered a finite-horizon investment problem in a bull-bear switching market, where the drift of the stock price switches between two parameters corresponding to an uptrend (bull market) and a down trend (bear market) according to an unobservable Markov chain. The goal is to maximize the expected log utility of the terminal wealth. We restricted attention to allowing flat and long positions only and used a sequence of stopping times to indicate the time of entering and exiting long positions. We have shown that the optimal trading strategy is trend following, characterized by the conditional probability in the uptrend crossing the buy and sell boundaries.

Regarding future research, it would be interesting to see how the approach works in models with more than two states, e.g., (bull, bear, sideways markets). In addition, substantial empirical tests on much broader selections of stocks will be useful to reveal when the trend following method works and when it fails in the marketplace.

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Appendix. Proofs of Theorems 2 and 3

Proof of Theorem 2. (i) First, we prove

$$Z(p,t) \equiv \log(1+K_b), \quad \forall\, p \ge 3p_0, \;\; 0 \le t \le T - 1/p_0. \tag{33}$$

Let us construct a function:

$$Z_1 = \begin{cases} -a[(p-2p_0)(T-t)-1]^2 + \log(1+K_b), & 2p_0 \le p \le \min\left\{2p_0 + \dfrac{1}{T-t}, 1\right\}; \\[2ex] \log(1+K_b), & \min\left\{2p_0 + \dfrac{1}{T-t}, 1\right\} < p \le 1. \end{cases}$$

We claim that Z_1 is a subsolution of (14) in $(2p_0, 1) \times (T - 1/p_0, T)$. Indeed,

$$Z_1\left(2p_0 + \frac{1}{T-t}, t\right) = \log(1+K_b), \qquad \partial_p Z_1\left(2p_0 + \frac{1}{T-t}, t\right) = 0, \quad \text{while } 2p_0 + \frac{1}{T-t} \le 1.$$

Therefore $Z_1 \in W_q^{2,1}((2p_0, 1) \times (0, T))$. Moreover, for $2p_0 \le p \le \min\{2p_0 + 1/(T-t), 1\}$ and $T - 1/p_0 \le t \le T$, we have

$$\begin{aligned}
-\mathcal{L}Z_1 &= -2a(p-2p_0)[(p-2p_0)(T-t)-1] + \frac{a(\mu_1-\mu_2)^2 p^2(1-p)^2(T-t)^2}{\sigma^2} \\
&\quad - 2a\lambda_1 p(T-t)[(p-2p_0)(T-t)-1] + 2a\lambda_2(T-t)[(p-2p_0)(T-t)-1](1-p) \\
&\le 2a + \frac{a(\mu_1-\mu_2)^2[(p-2p_0)^2 + 4p_0(p-2p_0) + 4p_0^2](1-p)^2(T-t)^2}{\sigma^2} \\
&\quad + 2a\lambda_1[(p-2p_0)+2p_0](T-t)[1-(p-2p_0)(T-t)],
\end{aligned}$$

where the inequality is due to $0 \le p - 2p_0 \le 1$ and $-1 \le (p-2p_0)(T-t) - 1 \le (1/(T-t))(T-t) - 1 = 0$. Noticing that $0 \le 1 - p \le 1$, $0 \le (p-2p_0)(T-t) \le 1$, and $0 \le p_0(T-t) \le 1$, we then deduce

$$\begin{aligned}
-\mathcal{L}Z_1 &\le 2a + \frac{a(\mu_1-\mu_2)^2(1+4+4)}{\sigma^2} + 2a\lambda_1(1+2) \\
&= \left[2 + \frac{9(\mu_1-\mu_2)^2}{\sigma^2} + 6\lambda_1\right]a \\
&\le (\mu_1-\mu_2)p_0,
\end{aligned}$$

where the last inequality is due to the right-hand side condition in (19). It is clear that for any $\min\{2p_0 + 1/(T-t), 1\} \le p \le 1$, we have

$$-\mathcal{L}Z_1 = -\mathcal{L}(\log(1+K_b)) = 0 \le (\mu_1-\mu_2)p_0.$$

In contrast, in the domain $\mathcal{M} \triangleq \{(p,t) \in [2p_0, 1) \times [T - 1/p_0, T]: Z(p,t) < \log(1+K_b)\}$, one has

$$-\mathcal{L}Z \ge f(p) - \rho \ge f(2p_0) - \rho = (\mu_1-\mu_2)p_0 \ge -\mathcal{L}Z_1.$$

Apparently,

$$Z_1(2p_0, t) = \log(1-K_s) \le Z(2p_0, t), \qquad Z_1(p, T) = \log(1-K_s) \le Z(p, T).$$

Using the maximum principle in the domain $\mathcal{M}$, we infer $Z \ge Z_1$ in $[2p_0, 1) \times [T - 1/p_0, T]$. In particular,

$$Z(3p_0, T - 1/p_0) \ge Z_1(3p_0, T - 1/p_0) = \log(1+K_b).$$

It is not hard to show that $Z(p,t)$ is decreasing with respect to t and increasing with respect to p. We then obtain (33).

Consider another function:

$$\underline{Z} = \log(1-K_s) + \frac{a}{6p_0}\left[p - 3p_0 + \frac{\lambda_2}{2}\left(T - \frac{1}{p_0} - t\right)\right] \text{ in } \mathcal{N},$$

where $\mathcal{N} \triangleq (0, 3p_0) \times (T - 1/p_0 - 12p_0/\lambda_2, T - 1/p_0)$. We now show that $\underline{Z}$ is a subsolution of (14) in $\mathcal{N}$. It is easy to verify

$$\partial_t \underline{Z} < 0, \qquad \partial_p \underline{Z} > 0, \qquad \underline{Z}(3p_0, T - 1/p_0 - 12p_0/\lambda_2) = \log(1+K_b), \qquad \underline{Z} < \log(1+K_b) \text{ in } \mathcal{N}.$$

Moreover,

$$-\mathcal{L}\underline{Z} = \frac{a}{6p_0}\left[(\lambda_1+\lambda_2)p - \frac{\lambda_2}{2}\right] \le \frac{a}{6p_0}\left[3(\lambda_1+\lambda_2)p_0 - \frac{\lambda_2}{2}\right].$$

In the domain $\{(p,t) \in \mathcal{N}: Z(p,t) < \log(1+K_b)\}$,

$$-\mathcal{L}Z \ge f(p) - \rho \ge -(\mu_1-\mu_2)p_0 \ge \frac{a}{6p_0}\left[3(\lambda_1+\lambda_2)p_0 - \frac{\lambda_2}{2}\right] \ge -\mathcal{L}\underline{Z},$$

where the third inequality is due to (18) and the left-hand side condition in (19). It is clear that

$$\begin{aligned}
&\underline{Z}(p, T - 1/p_0) \le \underline{Z}(3p_0, T - 1/p_0) = \log(1-K_s) \le Z(p, T - 1/p_0), \quad \forall p \in (0, 3p_0], \\
&\underline{Z}(3p_0, t) \le \log(1+K_b) = Z(3p_0, t), \quad \forall t \in (T - 1/p_0 - 12p_0/\lambda_2, T - 1/p_0).
\end{aligned}$$

Again, using the maximum principle, we deduce $\underline{Z} \le Z$ in the domain $\mathcal{N}$. In particular,

$$\begin{aligned}
Z(p,t) &\ge Z(p, T - 1/p_0 - 12p_0/\lambda_2) \ge \underline{Z}(p, T - 1/p_0 - 12p_0/\lambda_2) \\
&> \underline{Z}(0, T - 1/p_0 - 12p_0/\lambda_2) > \log(1-K_s), \quad \forall p > 0, \ t \le T - 1/p_0 - 12p_0/\lambda_2,
\end{aligned}$$

which yields the desired result.

(ii) From (20), we infer

$$p_0 \geq 2/3, \qquad (\lambda_1+\lambda_2)(3p_0-2)-\lambda_2 \geq \frac{\lambda_1+\lambda_2}{2};$$
$$\frac{4(\mu_1-\mu_2)^2(1-p_0)}{\sigma^2} \leq \frac{(\lambda_1+\lambda_2)}{2}, \qquad \frac{\sigma^2(\lambda_1+\lambda_2)}{18(\mu_1-\mu_2)} \geq (\mu_1-\mu_2)(1-p_0).$$

Construct the following function:

$$\bar{Z}(p,t)=\begin{cases}\log(1-K_s), & 0\leq p<3p_0-2,\\ \log(1-K_s)+\dfrac{\sigma^2[p-(3p_0-2)]^2}{9(\mu_1-\mu_2)(1-p_0)}, & 3p_0-2\leq p\leq 1.\end{cases}$$

It is easy to see that $\bar{Z}\geq\log(1-K_s)$ and $\bar{Z}\in W_q^{2,1}((0,1)\times[0,T))\cap C((0,1)\times[0,T])$ for any $q\geq 1$. For $0<p<3p_0-2$, we have

$$-L\bar{Z}=-L(\log(1-K_s))=0\geq f(3p_0-2)-\rho\geq f(p)-\rho.$$

For $3p_0-2\leq p\leq 2p_0-1$, we find

$$\begin{aligned}-L\bar{Z} &= \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\left\{\frac{-(\mu_1-\mu_2)^2p^2(1-p)^2}{\sigma^2}+2[(\lambda_1+\lambda_2)p-\lambda_2][p-(3p_0-2)]\right\}\\ &\geq -(\mu_1-\mu_2)(1-p_0)=f(2p_0-1)-\rho\geq f(p)-\rho.\end{aligned}$$

For $2p_0-1\leq p\leq 1$, we have

$$\begin{aligned}-L\bar{Z} &\geq \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\left[-\frac{(\mu_1-\mu_2)^2 4(1-p_0)^2}{\sigma^2}+(\lambda_1+\lambda_2)(1-p_0)\right]\\ &\geq \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\frac{(\lambda_1+\lambda_2)(1-p_0)}{2}\\ &\geq (\mu_1-\mu_2)(1-p_0)=f(1)-\rho\geq f(p)-\rho.\end{aligned}$$

Hence $\bar{Z}$ must be a supersolution of (14). We then deduce that

$$Z(p,t)\leq\bar{Z}(p,t)<\bar{Z}(1,t)=\log(1-K_s)+\frac{\sigma^2(1-p_0)}{\mu_1-\mu_2}\leq\log(1-K_s)+a=\log(1+K_b), \quad \forall\, p<1,$$

which implies that the BR does not exist. Therefore $p_b^*(t)\equiv 1$ for all t. □

Proof of Theorem 3. Consider an auxiliary function:

$$\bar{Z}=\begin{cases}\log(1-K_s)+a\left(\dfrac{4p}{p_0}-1\right)^2, & \dfrac{p_0}{4}\leq p\leq\dfrac{p_0}{2},\\ \log(1-K_s), & 0\leq p<p_0 4.\end{cases}$$

Clearly, $\bar{Z}\in W_q^{2,1}((0,p_0/2)\times(0,T))\cap C([0,p_0/2]\times[0,T])$ and

$$\bar{Z}\geq\log(1-K_s). \tag{34}$$

It is not hard to verify that for $p\in(p_0/4,p_0/2)$, we have

$$\begin{aligned}-L\bar{Z} &= \frac{a}{p_0^2}\left[\frac{-16(\mu_1-\mu_2)^2p^2(1-p)^2}{\sigma^2}+8(\lambda_1+\lambda_2)p(4p-p_0)-8\lambda_2(4p-p_0)\right]\\ &\geq -\left[\frac{4(\mu_1-\mu_2)^2}{\sigma^2}+\frac{8\lambda_2}{p_0}\right]a.\end{aligned}$$

Using (21), it follows

$$-L\bar{Z}\geq-(\mu_1-\mu_2)p_0/2=f(p_0/2)-\rho\geq f(p)-\rho \tag{35}$$

for $p\in(p_0/4,p_0/2)$. In the case $p\in(0,p_0/4)$,

$$-L\bar{Z}=-L(\log(1-K_s))=0\geq f(p)-\rho. \tag{36}$$

The combination of (34)–(36) yields

$$\min\{-L\bar{Z}-f(p)+\rho,\ \bar{Z}-\log(1-K_s)\}\geq 0$$

in $p \in (0, p_0/2)$, $t \in [0, T)$. Moreover, it is clear that

$$\bar{Z}(p, T) \geq \log(1 - K_s) = Z(p, T), \qquad \bar{Z}(p_0/2, t) = \log(1 + K_b) \geq Z(p_0/2, t).$$

Thus $\bar{Z}$ must be a supersolution of (14) in $[0, p_0/2] \times [0, T]$. By the maximum principle, we infer $\bar{Z} \geq Z$ in $[0, p_0/2] \times [0, T]$. Then, for $p < p_0/4$, we have

$$\log(1 - K_s) \leq Z \leq \bar{Z} \equiv \log(1 - K_s),$$

which implies $Z \equiv \log(1 - K_s)$ for $p < p_0/4$. Note that we can obtain (33) in terms of $p_0 < 1/3$ and (21). The desired result then follows. □

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