Project on Trend Following Trading

Min Dai
National University of Singapore

Agenda

- Motivation
- Problem formulation
  - Long and Flat
  - Investors have options to switch between long and flat positions
  - The model resembles that of American option pricing
- Variational inequality equations and optimal strategies
- Numerical results and empirical analysis

Motivation: common investment strategies

- Buy and hold
  - Invest for the long run
- Contra-trend
  - Buy low and sell high: buy on falling and sell on rising
- Trend-following
  - Buy high and sell higher: buy on rising and sell on falling

Justifying "Buy and hold"

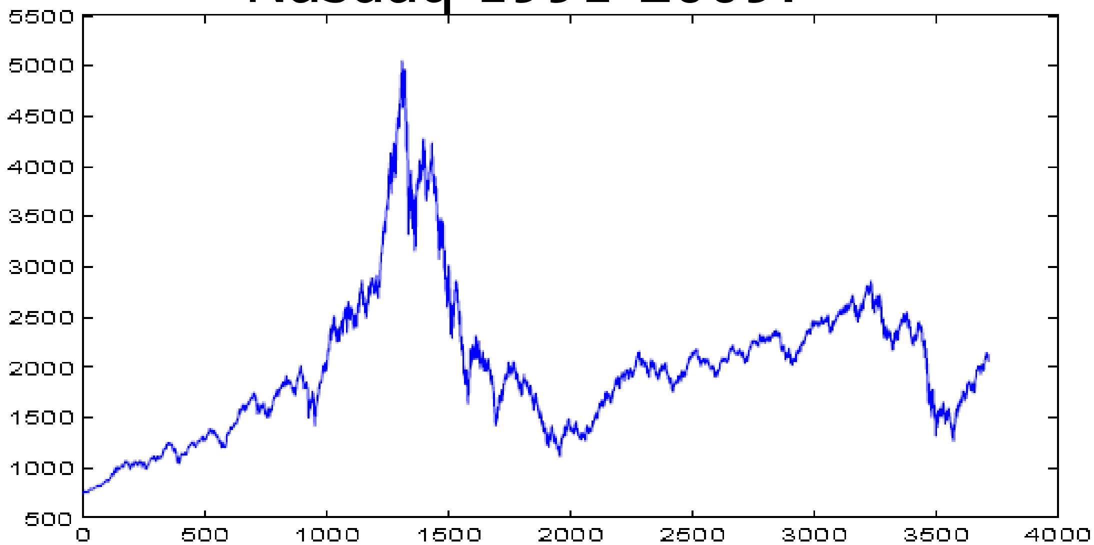
- justifiable if the average gain of stock market is above the bank rate.

Justifying "Contra-trend"

- justifiable if the investor is risk averse in a constant market
  - The 60/40 rule: 60% in stock and 40% in bond
- justifiable if the market has mean reverting properties:
  Q. Zhang and H.Q. Zhang (2008)

Justifying "trend following"

- We will show that it is justifiable if the market has trends – as we will make precise.
  - Dai, Zhang and Zhu (2010), SIAM J. Financial Math, where the investor is restricted to buy and sell one share of stock
  - Dai, Yang, Zhang and Zhu (2016), Optimal trend following trading rules Mathematics of Operations Research, 41(2), 626-642
- Nasdaq 1991-2009:



To capture the trends

-buy in early up-trend

-ride the up-trend

-sell in early down-trend

Modeling: index as a proxy for the market

Index value: $$\frac{dS_t}{S_t} = \mu(\alpha_t)dt + \sigma dB_t$$

Here B_t is a standard BM, $\sigma > 0$ the volatility,
$\alpha_t \in \{1, 2\}$ is a 2-state Markov chain independent of B_t,
$\mu(i) \equiv \mu_i$, $i = 1, 2$ the expected return rates, $\mu_1 > \mu_2$.

α_t represents the market trend: $\alpha_t = 1$ up; $\alpha_t = 2$ down.
Let $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$ denote the generator of α_t.
Typically, $1/\lambda_1 = 3$ years, $1/\lambda_2 = 0.5$ year.

Wonham filters (1965)

The market trend α_t is unobservable.

p_t: the conditional probability of $\alpha_t = 1$ given $\{S_u,\ u < t\}$,

$$dp_t = [-(\lambda_1 + \lambda_2)\, p_t + \lambda_2]\, dt + \frac{(\mu_1 - \mu_2) p_t (1 - p_t)}{\sigma} d\widehat{B}_t$$

where $\widehat{B}_t$ is another standard BM given by

$$d\widehat{B}_t = \frac{d \log(S_t) - [(\mu_1 - \mu_2) p_t + \mu_2 - \sigma^2/2] dt}{\sigma}.$$

So, $\quad dS_t = [(\mu_1 - \mu_2)\, p_t + \mu_2]\, S_t dt + \sigma S_t d\widehat{B}_t.$

Target

Given initial wealth w_0 and investment horizon T, our target is to maximize the expected return rate:

$$E\left[\log \frac{w_T}{w_0}\right],$$

where w_T is the terminal wealth.

Without loss of generality, we will take $w_0 = 1$.

Allowable trading decisions

Assume the net position at any time can be either
flat (i=0): no holding; or
long (i=1): all wealth in index.

Buy and sell decisions are determined by a sequence of stopping times $\{\tau_n, v_n\}$, $n = 1, 2, \cdots$. We buy at τ_n and sell at v_n.
If $i = 0$, then the decision sequence is $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \cdots)$.
If $i = 1$, then the decision sequence is $\Lambda_1 = (\overline{v}_1, \overline{\tau}_2, \overline{v}_2, \overline{\tau}_3, \cdots)$.

At maturity, the position is required to be flat. We denote the last sell time by v_m ($\overline{v}_{\overline{m}}$), and $\tau_{m+1} = \overline{\tau}_{\overline{m}+1} = T$.

Reward functions

Given $S_0 = S$, $p_0 = p$, and initial position $i = 0, 1$, the reward functions of the decision sequences Λ_0 and Λ_1 are given as follows:

$$J_i(p, \Lambda_i) = \begin{cases} E\left\{\log\left[\exp\left(\int_0^{\tau_1} \rho ds\right) \prod_{n=1}^{m} \left(\frac{S_{v_n}(1-\alpha)}{S_{\tau_n}(1+\theta)} \exp\left(\int_{v_n}^{\tau_{n+1}} \rho ds\right)\right)\right]\right\}, & \text{if } i = 0, \\ E\left\{\log\left[\frac{S_{\bar{v}_1}(1-\alpha)}{S_0} \exp\left(\int_{\bar{v}_1}^{\bar{\tau}_2} \rho ds\right) \prod_{n=2}^{\bar{m}} \left(\frac{S_{\bar{v}_n}(1-\alpha)}{S_{\bar{\tau}_n}(1+\theta)} \exp\left(\int_{\bar{v}_n}^{\bar{\tau}_{n+1}} \rho ds\right)\right)\right]\right\}, & \text{if } i = 1, \end{cases}$$

where ρ riskfree rate, α and θ the proportional costs.

Value functions

For $i = 0, 1$, define value functions

$$W_i(p,t) = \sup_{\Lambda_i} J_i(p,t,\Lambda_i)$$

It is easy to see

$$\begin{aligned} W_0(p,t) &= \sup_{\tau_1} E_t \left\{ \int_t^{\tau_1} \rho ds - \log(1+\theta) + W_1(p_{\tau_1}, \tau_1) \right\}, \\ W_1(p,t) &= \sup_{\overline{v}_1} E_t \left\{ \int_t^{\overline{v}_1} \widetilde{f}(p_s) ds + \log(1-\alpha) + W_0(p_{\overline{v}_1}, \overline{v}_1) \right\}, \end{aligned}$$

where

$$\widetilde{f}(p) = (\mu_1 - \mu_2)\, p + \mu_2 - \frac{\sigma^2}{2}$$

HJB equations

$$-\mathcal{A}W_0 = \rho \text{ if } W_0 > W_1 - \log(1+\theta)$$
$$-\mathcal{A}W_0 \geq \rho \text{ if } W_0 = W_1 - \log(1+\theta)$$

and

$$-\mathcal{A}W_1 = \widetilde{f} \text{ if } W_1 > W_0 + \log(1-\alpha)$$
$$-\mathcal{A}W_1 \geq \widetilde{f} \text{ if } W_1 = W_0 + \log(1-\alpha)$$

with $W_0(p,T) = 0$, $W_1(p,T) = \log(1-\alpha)$,

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2 \frac{\partial}{\partial p^2} + \left[-\left(\lambda_1+\lambda_2\right)p + \lambda_2\right]\frac{\partial}{\partial p}$$

A concise expression

$$\min\{-\mathcal{A}W_0 - \rho,\ W_0 - W_1 + \log(1+\theta)\} = 0$$

$$\min\left\{-\mathcal{A}W_1 - \widetilde{f},\ W_1 - W_0 - \log(1-\alpha)\right\} = 0$$

with $W_0(p,T) = 0,\ W_1(p,T) = \log(1-\alpha).$

An equivalent problem

Let $Z(p,t) = W_1(p,t) - W_0(p,t)$. Then Z is the unique solution to the double obstacle problem

$$\max\left\{\min\left\{-\mathcal{A}Z - f(p), Z - \log(1-\alpha)\right\}, Z - \log(1+\theta)\right\} = 0,$$

that is,

$$\begin{array}{rcl} -\mathcal{A}Z & = & f(p) \text{ if } \log(1-\alpha) < Z < \log(1+\theta) \\ -\mathcal{A}Z & \geq & f(p) \text{ if } Z = \log(1-\alpha) \\ -\mathcal{A}Z & \leq & f(p) \text{ if } Z = \log(1+\theta) \end{array}$$

in $p \in (0,1)$, with $Z(p,T) = \log(1-\alpha)$, where

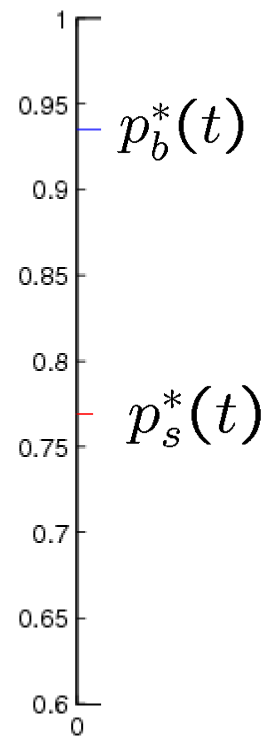
$$f(p) = \widetilde{f}(p) - \rho.$$

Optimal strategies

$$
\begin{aligned}
BR &= \{(p,t) \in (0,1) \times [0,T) : W_0 - W_1 + \log(1+\theta) = 0\}\,; \\
SR &= \{(p,t) \in (0,1) \times [0,T) : W_1 - W_0 - \log(1-\alpha) = 0\}\,; \\
NT &= (0,1) \setminus (BR \cup SR)\,.
\end{aligned}
$$

Theorem There exist $p_b^*(t)$, $p_s^*(t)$, $p_b^*(t) > p_s^*(t)$, such that

$$
\begin{aligned}
BR &= \{(p,t) \in (0,1) \times [0,T) : p \geq p_b^*(t)\} \\
SR &= \{(p,t) \in (0,1) \times [0,T) : p \leq p_s^*(t)\}
\end{aligned}
$$



Numerical solutions: penalty method

For the long/flat case,

$$-\mathcal{A}Z = \beta[\log(1-\alpha) - Z]^+ - \beta[Z - \log(1+\theta)]^+$$

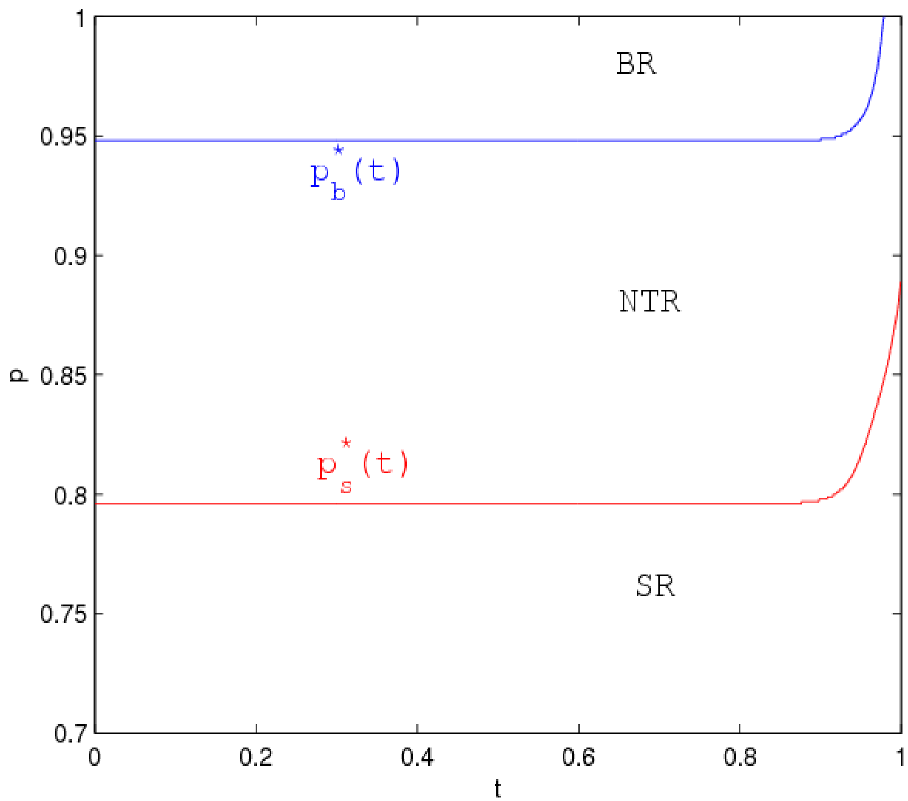
Penalty method: see Forsyth and Vetzal (2002) and Dai, Kwok and You (2007).

Numerical results

$\lambda_1 = 0.36$, $\lambda_2 = 2.53$, $\mu_1 = 0.18$, $\mu_2 = -0.77$, $\sigma = 0.184$, $\rho = 0.0679$, $\alpha = \theta = 0.1\%$ (statistics data of SP500)

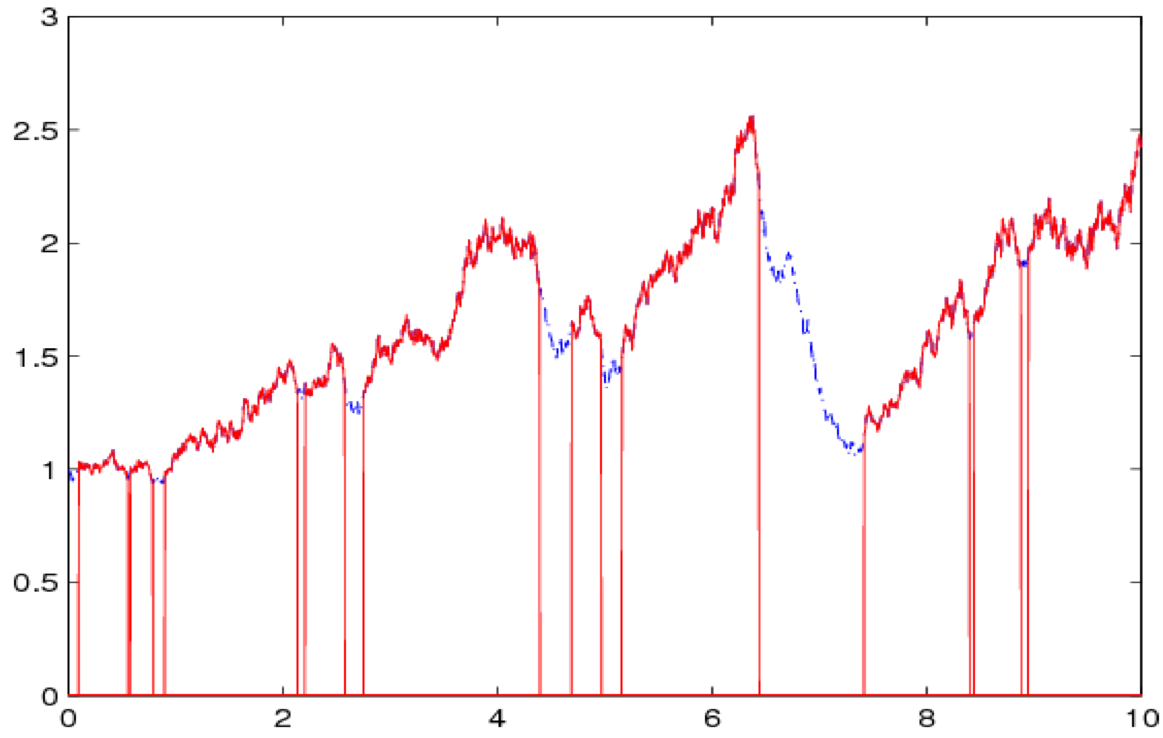
$$p_b^* = 0.948$$

$$p_s^* = 0.796$$



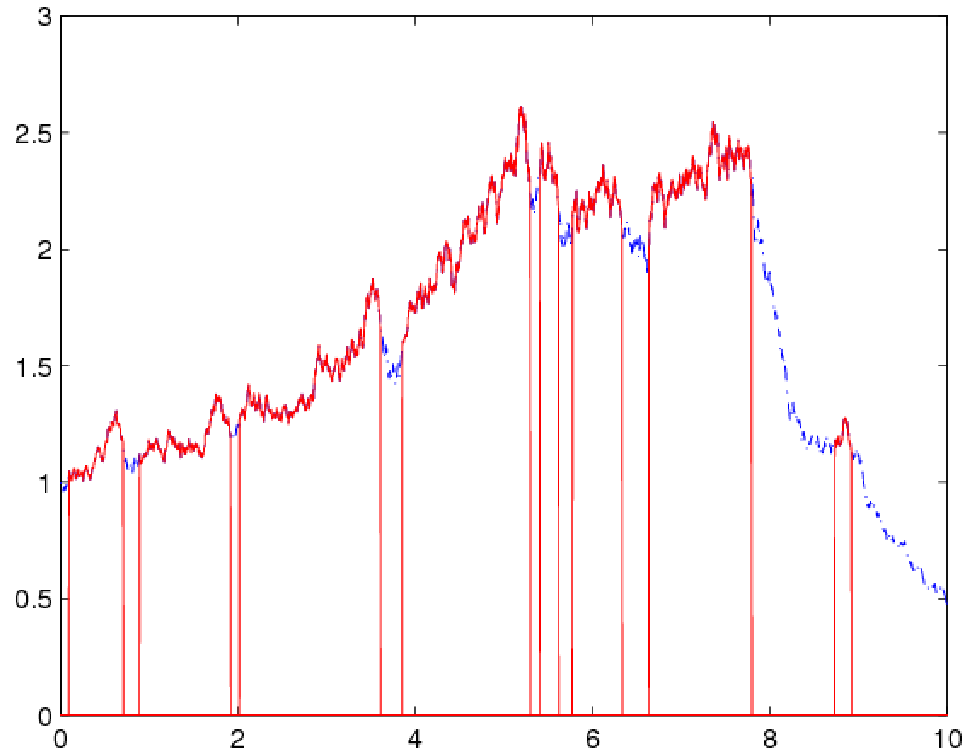
Continued: a simulation result

Red: holding;
blue: no positions



Continued: another sample

Red: holding;
blue: no positions



On average, the trend following strategy robustly outperforms the buy and hold strategy over 10 times in return.

A question

Why is the strategy trend-following?

Simulation results: average returns

TF	No. Trades	BH
78.71	41	5.86
77.17	40	5.72
72.48	41	5.92
72.52	41	5.73

Empirical analysis: statistics of markets

Index	λ_1	λ_2	μ_1	μ_2	σ_1	σ_2
SP500 (62-08)	0.353	2.208	0.196	-0.616	0.135	0.211
DJIA (62-08)	0.36	2.53	0.18	-0.77	0.144	0.223
NASDAQ (91-08)	2.158	2.3	0.875	-1.028	0.273	0.35

Continued

We compute p_s^* using σ_2 and p_b^* using σ_1.

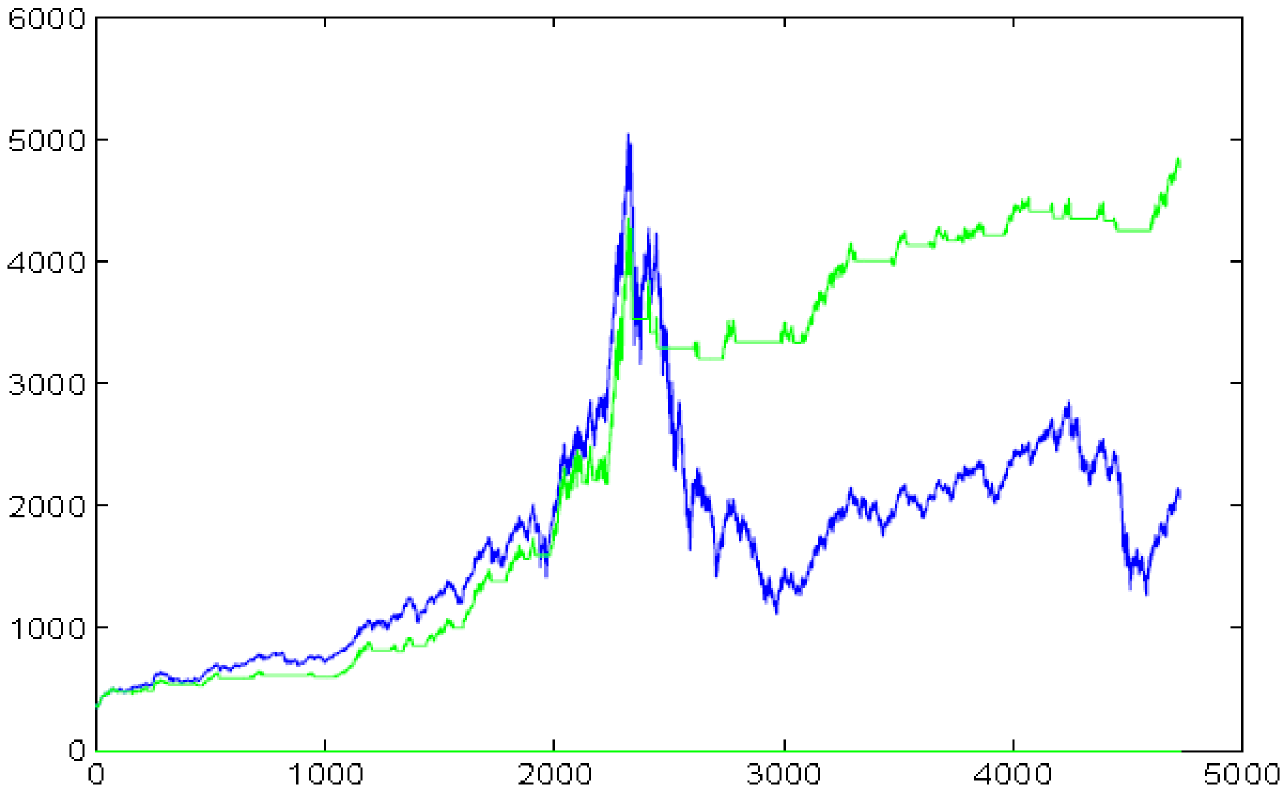
Index	p_s^*	p_b^*
SP500 σ_1	0.708	**0.923**
SP500 σ_2	**0.774**	0.922
DJIA σ_1	0.760	**0.948**
DJIA σ_2	**0.825**	0.950
NASDAQ σ_1	0.461	**0.716**
NASDAQ σ_2	**0.495**	0.711

Testing returns from real data

Index(time frame)	TF	BH	10y bonds	No. Trades
SP500 (1962-2008)	63.2	32.8	23.44	80
DJIA (1962-2008)	13.9	12.1	23.44	80
NASDAQ(1991-2008)	6.6	4.2	2.63	67

Legend: TF – trend following, BH – buy and hold

Trend following vs buy and hold: test on NASDAQ: 1991-2009



Conclusion

- We provide a model to justify the trend following strategy.
- Our model generates a trend following strategy, whose efficiency is demonstrated by simulations and empirical analysis.