FACULTY OF SCIENCE - Course 2020

QUANTITATIVE FINANCE DEPARTMENT

QF 5202 Structure Product

Implementation of Trend Following Strategy Report

Abstract

This report is for the implementation of trend following strategy. The whole project consists of five sections. The first section is the solution to HJB Equations, which we utilize implicit finite difference grid method, upwind treatment and projected SOR to solve. The second section is estimation and initialization of parameters. In this section, we defined bull and bear market to estimate and initialize parameters. The third section is the implementation of trend following strategy on both SPY and CSI 300. The forth section is the sensitivity analysis. The fifth section is conclusion. We find that the trend following strategy can beat buy-and-hold strategy, given a long test period and frequent parameter estimation.

Changes & Improvement:

1. Emphasize that we consider the dividend and split (actually there is no split) in SPY in estimating the parameters and corresponding threshold, and plot the dividend payout graph (Figure (4));
2. Add standard deviation of trend following return (Volatility) in Table (2), (5), and (6), and analyze the effect;
3. For trend following strategy with long & short position, we not only consider the strategy that simply replaces flat position with short position, but also consider combine long, flat, and short position together.
4. Consider the trend following strategy with short & flat position in Appendix, and derive the HJB Equations and corresponding flat & short boundary (Figure (24)).

Contents

1 Solution to HJB Equations

1.1 HJB Equations

From Dai's paper, we consider the value functions and HJB equations.

A buying decision is made at τ_n if $\tau_n < T$ and a selling decision is made at v_n if $v_n < T, n = 1, 2 \ldots$ We assume that the investor is taking an "all-in and all-out" strategy. This means that she is either long so that her entire wealth is invested in the stock, or flat so that all of her wealth is in a bank account that draws the risk-free interest rate. We use indicator $i = 0$ or 1 to signify the initial position to be flat or long, respectively. If initially the position is long (i.e., $i = 1$), the corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots, t_2)$.

Define value functions, for $i = 0, 1$

$$W_i(p, t) = \sup_{\Lambda_i} J_i(p, t, \Lambda_i)$$

it is easy to see

$$W_0(p, t) = \sup_{\tau_1} E_t \left\{ \int_t^{\tau_1} \rho ds - log(1 + \theta) + W_1(p_{\tau_1}, \tau_1) \right\}$$

$$W_1(p, t) = \sup_{\overline{v}_1} E_t \left\{ \int_t^{\overline{v}_1} \widetilde{f}(p_s) ds + log(1 - \alpha) + W_0(p_{\overline{v}_1}, \overline{v}_1) \right\}$$

where

$$\widetilde{f}(p) = (\mu_1 - \mu_2)p + \mu_2 - \frac{\sigma^2}{2}$$

Then, let $Z(p, t) = W_1(p, t) - W_0(p, t)$. Then Z is the unique solution to the double obstacle problem

$$max \left\{ min \left\{ -\mathcal{A}Z - f(p), Z - log(1 - \alpha) \right\}, Z - log(1 + \theta) \right\} = 0$$

where

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2} \left(\frac{(\mu_1 - \mu_2)p(1 - p)}{\sigma} \right)^2 \frac{\partial}{\partial p^2} + [-(\lambda_1 + \lambda_2)p + \lambda_2] \frac{\partial}{\partial p}$$

$$f(p) = \widetilde{f}(p) - \rho$$

where $p \in [0, 1]$.

The upper boundary of Z value is $log(1+\theta)$ and the lower boundary of Z is $log(1-\alpha)$. We assume the value of Z larger than the upper boundary equal the upper boundary while smaller than the lower boundary equal the lower boundary, which is also the method to recognize the buy region and sell region. Moreover, as p is the conditional probability in the bull market, p equals 0 indicates short position and p equals 1 indicates long position. In addition, the position is required to be flat at maturity, so terminal condition of is $log(1 - \alpha)$.

Therefore, boundary condition and terminal condition are

$$Z(p, T) = log(1 - \alpha)$$
$$Z(0, t) = log(1 - \alpha)$$
$$Z(1, t) = log(1 + \theta)$$

Considering Dai's paper, we formulate the buy region(BR),the sell region(SR) and the no-trading region(NT) as follows:

$$BR = \{(p,t) \in (0,1) \times [0,T) : V_1(p,t) - V_0(p,t) = log(1+\theta)\}$$
$$SR = \{(p,t) \in (0,1) \times [0,T) : V_1(p,t) - V_0(p,t) = log(1-\alpha)\}$$
$$NT = (0,1) \times [0,T)\ (BR \cup SR).$$

1.2 Solution Process

Truncate the domain $p \in [0,1]$, $t \in [0,T]$ and use the finite difference grid:

$$\left\{\left(p_n^i, t_n\right) : p_n^i = i\Delta p, t_n = n\Delta t, i = 0,1,\dots,I, n = 0,1,\dots,N\right\}$$

and discretize the PDE at (p_n^i, t_n).

Let $\epsilon = \frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2$, and $b = -(\lambda_1+\lambda_2)p + \lambda_2$.

In the HJB Equations, as $p \to 0$ or $p \to 1$, $|b| >> |\epsilon|$, an **upwind treatment** should be applied to ensure the algorithm more stable.[1] The **upwind treatment** can be expressed as follow:

$$\left.\frac{\partial z}{\partial p}\right|_{(p_n^j,t_n)} = \begin{cases} \frac{z_n^{i+1}-z_n^i}{\Delta p} + O(\Delta p), & \text{if} \quad b \geq 0 \\ \frac{z_n^i - z_n^{i-1}}{\Delta p} + O(\Delta p), & \text{if} \quad b < 0 \end{cases}$$

For the other partial derivatives, use the following approximation:

$$\left.\frac{\partial z}{\partial t}\right|_{(p_n^i,t_n)} = \frac{z_{n+1}^i - z_n^i}{\Delta t} + O(\Delta t)$$
$$\left.\frac{\partial^2 z}{\partial p^2}\right|_{(p_n^i,t_n)} = \frac{z_n^{i-1} - 2z_n^i + z_n^{i+1}}{(\Delta p)^2} + O\left((\Delta p)^2\right)$$

After considering **upwind treatment**, dropping truncation error, collating and rearranging terms gives the PDE:

- if $b \geq 0$:

$$-\frac{\Delta t\epsilon}{(\Delta p)^2}Z_n^{i-1} + \left(1 + \frac{2t\epsilon}{(\Delta p)^2} + \frac{\Delta tb}{\Delta p}\right)Z_n^i + \left(-\frac{t\epsilon}{(\Delta p)^2} - \frac{\Delta tb}{\Delta p}\right)Z_n^{i+1} = \Delta t f(i\Delta p) + Z_{n+1}^i$$

- if $b < 0$:

$$\left(-\frac{\Delta t\epsilon}{(\Delta p)^2} + \frac{\Delta tb}{\Delta p}\right)Z_n^{i-1} + \left(1 + \frac{2t\epsilon}{(\Delta p)^2} - \frac{\Delta tb}{\Delta p}\right)Z_n^i - \frac{\Delta t\epsilon}{(\Delta p)^2}Z_n^{i+1} = \Delta t f(i\Delta p) + Z_{n+1}^i$$

with $i = 0, 1, \cdots, I$.

The "square" system of $I-1$ equations can be written in the matrix-vector form:

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{I-2} & b_{I-2} & c_{I-2} \\ & & & a_{I-1} & b_{I-1} \end{bmatrix} \begin{bmatrix} Z_n^1 \\ Z_n^2 \\ \vdots \\ Z_n^{I-2} \\ Z_n^{I-1} \end{bmatrix} = \begin{bmatrix} Z_{n+1}^1 \\ Z_{n+1}^2 \\ \vdots \\ Z_{n+1}^{I-2} \\ Z_{n+1}^{I-1} \end{bmatrix} + \begin{bmatrix} -a_1 Z_n^0 \\ \Delta t f(\Delta p) \\ \vdots \\ \Delta t f((I-1)\Delta p) \\ -c_{I-1}Z_n^I + \Delta t f(I\Delta p) \end{bmatrix}$$

[1]From the perspective of numerical computation, the upwind treatment can ensure that the coefficient matrix in implicit schemes is diagonally dominant, which makes the algorithm more stable.

which can be showed as $\mathbf{AZ_n = Z_{n+1} + F_n}$.
Solving every Z_n with **Projected SOR Method**.
Denote the matrix equation need to be solved as:

$$Ax = b$$

than decompose A as

$$A = D - L - U$$

where D is the diagonal part, L is the lower triangular part, U is the upper triangular part.

The equations can be solved when:

$$x_{i,gs}^{k+1} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k + b_i \right]$$

where

$$x_i^{k+1} = min\left(max\left((1-\omega)x_i^k + \omega x_{i,gs}^{k+1}, lower\right), upper\right)$$

The projected SOR algorithm:

Algorithm 1 Projected SOR

- **Input** A, b, V, n, ϵ, ω, $upper$, $lower$
- **Output** V_n
- converged = False
- While not converged

$$x_{i,gs}^{k+1} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k + b_i \right]$$

$$x_i^{k+1} = min\left(max\left((1-\omega)x_i^k + \omega x_{i,gs}^{k+1}, lower\right), upper\right)$$

If $\|x^{k+1} - x^k\| < \epsilon$, then set Converged=true, and $V_n = x_i^{k+1}$.

In the report, we set $\omega = 1.5$ and $\epsilon = 10^{-6}$

1.3 Test Example

We use the same parameters in Dai's paper to test the correctness of our solution:

λ_1	λ_2	μ_1	μ_2	σ	ρ	α	θ
0.360	2.530	0.180	-0.770	0.184	0.068	0.001	0.001

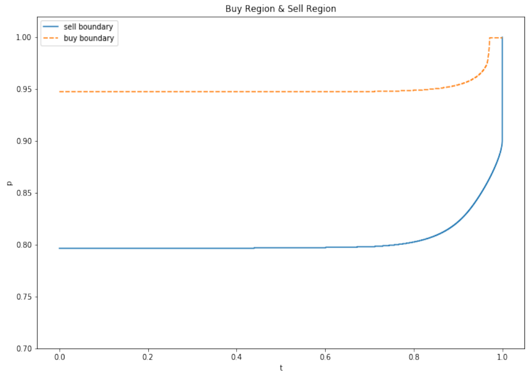


Figure 1: Test Example (SP500)

From Figure (1), we know both p_b and p_s are monotonically increasing and remain steady in early times, so we choose t large enough to get p_b is 0.9475 and p_s is 0.7965. This is accordance with our reference paper.

2 Estimation & Initialization of Parameters

2.1 Bull & Bear Market

Following Dai's paper, we define a bull market to be increased at least 25% and a bear market to decline at least 25%:[2]

Up Trend	Down Trend
25%	25%

Figure (2) and (3) illustrate the bull and bear market in SPY and CSI300, respectively.

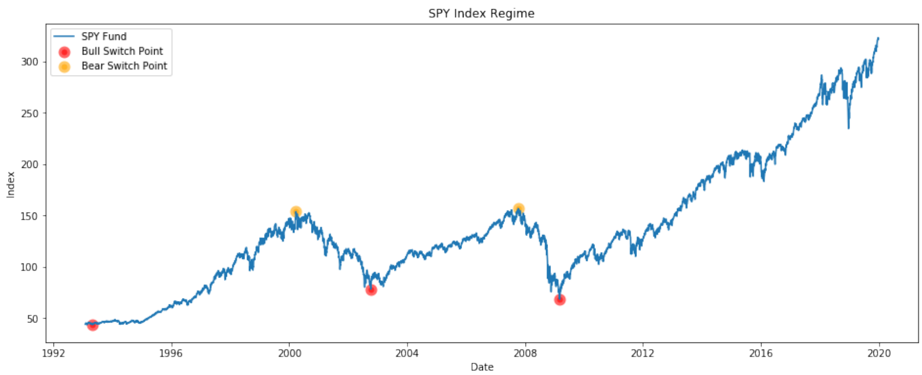


Figure 2: SPY Regime

[2] In Dai's paper, the percentage is 20%, but considering the data from 2001 to 2010, we think 25% fit the bull and bear trend better.

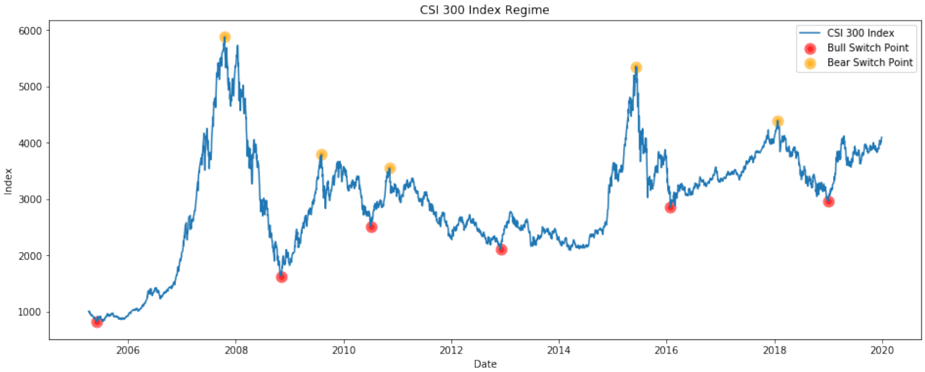


Figure 3: CSI300 Regime

For SPY, there are 2 bull and bear markets in the past 20 years; For CSI300, there are 5 bull and bear markets in the past 20 years. However, SPY experienced a 10-year bull market,[3] causing difficulty in updating parameters annually.

As for the available data for estimation, we will use the data from confirmed bull and bear markets. To be confirmed bull and bear markets from the observation of a specific date, for bull market, it needs to decline 25% from its highest point; For bear market, it needs to increase 25% from its lowest point. We will follow this rule in our parameters estimation part, or it will causes data leakage problem. [4]

SPY is an ETF, which means there will be dividend and split. We will therefore use adjusted stock price for dividend and split to estimate parameters and corresponding threshold. Figure (4) illustrate the dividend payout from the inception date, and we will include the dividend to obtain a more precise estimation. As for split, SPY haven't had split since the inception date. We only need to take dividend into consideration.

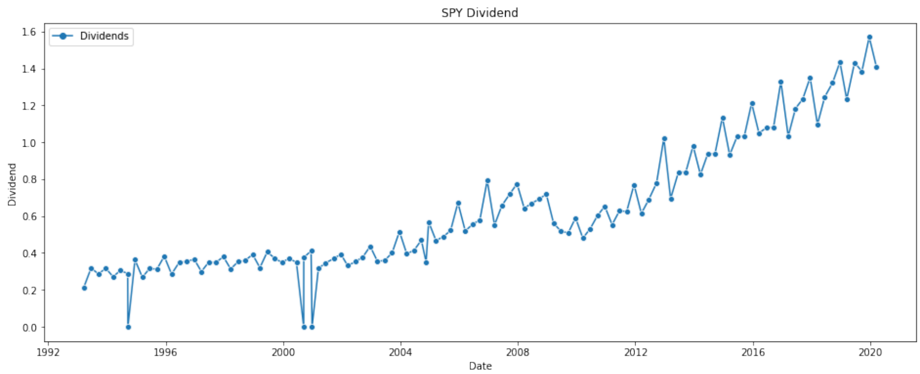


Figure 4: SPY Dividend

2.2 Estimation of Parameters: λ, μ, σ

After defining bull and bear market, we can use the trend data to estimate the λ, μ, and σ. For the estimation method, we use the statistics data of these trends to empirically calibrate the parameters. We avoid setting those parameters manually, since it's likely to cause overfitting problem and manipulate the Trend Following Strategy in the test period.

For λ, it stands for the switching intensity from bull to bear or from bear to bull, that

[3]This bull market ends with a big drop in March 2020.
[4]This means we use future data to do estimation.

can be estimated as average of reciprocal of duration during the bull or bear market:

$$\hat{\lambda}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1}\frac{1}{T_i^{bull}}, \quad \hat{\lambda}_2 = \frac{1}{N_2}\sum_{i=1}^{N_2}\frac{1}{T_i^{bear}}$$

where, $T_i^{bull}(T_i^{bear})$ means the duration of the i-th bull(bear) market period, and $N_1(N_2)$ means total number of bull(bear) market.

For μ and σ, they stand for the average annualized return and volatility during bull or bear market, respectively. We assume that the underlying market follows GBM, and the estimated $\hat{\mu}$ and $\hat{\sigma}$ can be expressed in this way:

$$\hat{\sigma} = \sqrt{\frac{1}{T}\Sigma(dlnS_t)^2}, \quad \hat{\mu} = \frac{\Sigma(dlnS_t)}{T} + \frac{\hat{\sigma}^2}{2}$$

2.3 Initialization of Parameters: p_0

After obtaining these estimated the λ, μ, and σ, we can initialize the p_0 and estimate p_t. To estimate p_t, the conditional probability in a bull market, we can consider the below stochastic differential equation:

$$dp_t = g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}dlogS_t$$

where,

$$g(p_t) = -(\lambda_1 + \lambda_2)p_t + \lambda_2 - \frac{(\mu_1 - \mu_2)p_t(1-p_t)((\mu_1 - \mu_2)p_t + \mu_2 - \frac{\sigma^2}{2})}{\sigma^2}$$

Discrete the SDE for $t = 0, 1, \cdots, N$ with $dt = 1/252$, and ensure p, the probability of bull market, stay in $[0, 1]$:

$$p_{t+1} = min\left(max\left(p_t + g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}log(\frac{S_{t+1}}{S_t}), 0\right), 1\right)$$

To initialize the p_0, firstly choosing a starting point for a bear market with a significant decline as the start date. [5]. Assuming the $p = 0$ at starting point to find p_t in 1st January, 2011. The estimate p_0 is presented below:

SPY	CSI 300
0.961	0.797

Moreover, changing the value of p at the start point would not affect the value of initialized p_0, since in 2008 financial crisis, the p hit 0 and all curves converge. This can be illustrated in Figure (5) and (6) below:

[5]For SPY and CSI300, we choose the date 16th October 2007. We will clarify there is no difference for starting value.

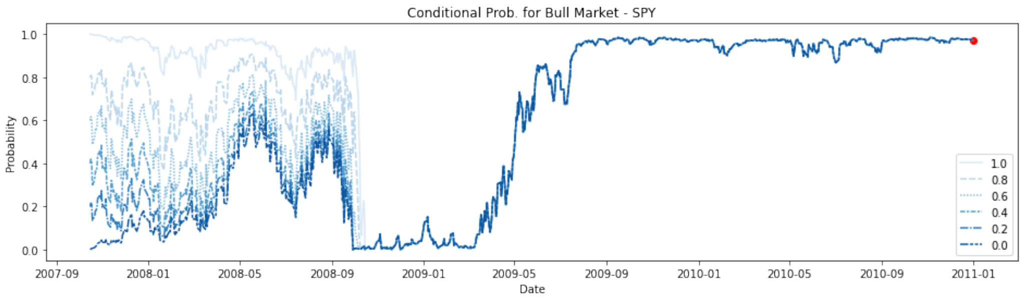


Figure 5: Initialization p_0 (SPY)

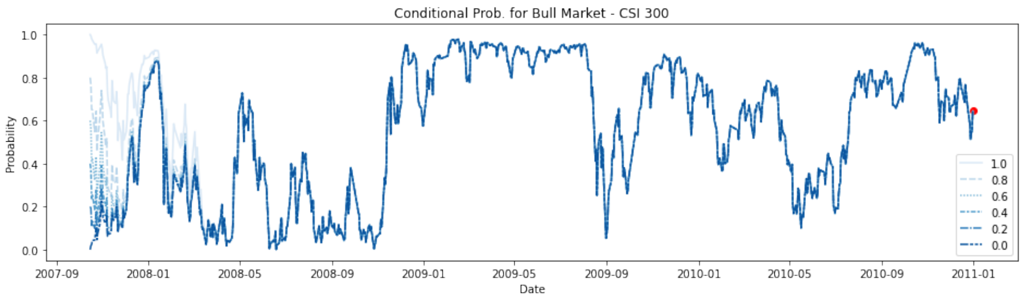


Figure 6: Initialization p_0 (CSI300)

3 Trend Following Strategy on SPY & CSI 300

Building the trend following strategy requires not only estimated parameters, but also some basic assumptions. According to the following assumptions, we implement trend following strategy on both SPY and CSI300 respectively.

3.1 Basic Assumptions

- The investment period is from **2011-01-01** to **2019-12-31**.
- Any trading action will take place at the **close** of the market.
- Use SPY and CSI 300 daily **closing price** for the tests.
- Assume the risk free rate $\rho^A = 2.86\%$ and $\rho^C = 3.89\%$.
- Assume the transaction cost $\alpha = 0.1\%$ and $\theta = 0.1\%$.
- Use σ^{bull} to calculate p_b^*, and σ^{bear} to calculate p_s^*.

3.2 SPY

Because SPY experienced a bull market from 2011 to 2020, there is no improvement for trend following strategy to conduct a rolling-base estimation.[6] To save computation time, we conduct one-for-all estimation, and use the estimated p_b and p_S for the period from

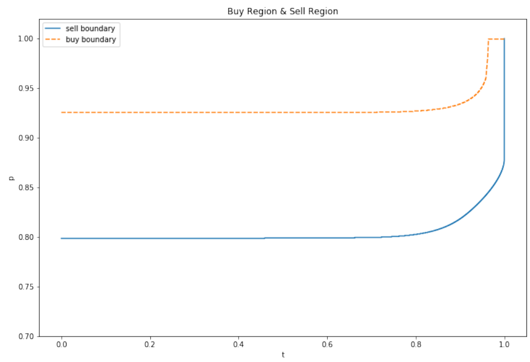
[6]This means the available data will always be before 2011. We used the data from 2001-01-01 to 2010-12-31 to estimate the parameters.

2011 to 2020

The estimated parameters for SPY, which calculate the buy & sell boundary $p_b^* = 0.9255$ and $p_s^* = 0.7985$:

λ_1	λ_2	μ_1	μ_2	σ	ρ	α	θ
0.200	0.706	0.143	-0.510	0.219	0.035	0.001	0.001

Table 1: Estimated Parameters (SPY)



We did two testings with different end dates for SPY, and show the return, sharp ratio and number of trade. The results are demonstrated in Table (2):

End Date	TF	Volatility	Sharp Ratio	BH	10y Bonds	No. Trade
2020-01-01	83.92%	17.47%	0.173	110.15%	22.49%	9
2020-03-19	53.74%	17.91%	0.084	47.22%	23.19%	10

Table 2: TF Strategy Return (SPY)

When the end date is at the beginning of 2020, Trend following strategy does not perform better than the buy-and-hold strategy. It's mainly because 10-year bull market with little drawback and recovery very quickly. Furthermore, we use fixed parameters rather than updated parameters and threshold. These parameters may be out-of-date after several years. However, if the test period extends to 2020-03-19, Trend following strategy outperforms SPY, because it partially avoids the big drop recently.

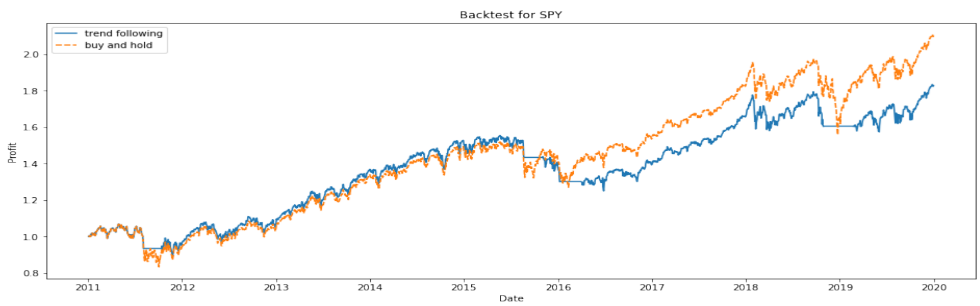


Figure 7: TF Strategy (2019-12-31)

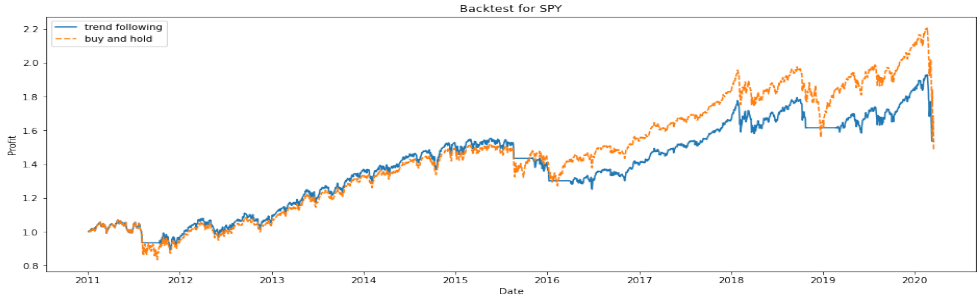


Figure 8: TF Strategy (2020-03-16)

3.3 CSI 300

For CSI 300, since there are several bull and bear markets in the past ten years, we have conducted Trend Following Strategy on rolling base, that means we updated the parameters and the corresponding thresholds at the beginning of a new year. And updating the parameters with exponential average method.

Rolling-Base Estimation:

- Determine the parameters by beginning with the statistical estimate of the **10-year data** [7].
- Update the parameters and the corresponding thresholds at the beginning of a new year [8].
- Use **exponential average method** to update the parameters:

$$update = (1 - 2/N)old + (2/N)new.$$

Estimated parameters and corresponding updated buy & sell boundaries for each year:

	λ_1	λ_2	μ_1	μ_2	σ_1	σ_2	σ
2011	1.829	5.241	0.959	-1.070	0.306	0.409	0.333
2012	2.096	5.241	0.967	-1.070	0.300	0.409	0.318
2013	2.096	5.241	0.967	-1.070	0.300	0.409	0.305
2014	2.096	4.051	0.967	-0.612	0.300	0.316	0.297
2015	2.096	4.051	0.967	-0.612	0.300	0.316	0.288
2016	1.785	4.165	0.797	-0.706	0.281	0.339	0.306
2017	2.123	4.165	0.788	-0.706	0.292	0.339	0.306
2018	2.291	5.948	0.658	-0.632	0.271	0.290	0.284
2019	2.070	5.948	0.504	-0.632	0.226	0.290	0.247

Table 3: Estimated Parameters (CSI 300)

From Table (3), λ, μ and σ of CSI 300 are much bigger than SPY, indicating that CSI 300 are easier to switch bull and bear regime and more volatile.

[7] If available data is less than 10 years, we use the available data to estimate the parameters

[8] using the new data that become available if a new up or down trend is completed.

	p_b^*	p_s^*	p_b^u	p_s^u
2011	0.726	0.501	0.726	0.501
2012	0.727	0.499	0.727	0.500
2013	0.727	0.499	0.727	0.499
2014	0.608	0.366	0.637	0.399
2015	0.608	0.366	0.608	0.366
2016	0.696	0.469	0.674	0.443
2017	0.694	0.468	0.694	0.468
2018	0.708	0.483	0.705	0.479
2019	0.748	0.531	0.738	0.519

Table 4: Buy & Sell Region (CSI 300)

We also did two backtests for CSI 300, with rolling parameters and fixed parameters respectively:

Estimation	TF	Volatility	Sharp Ratio	BH	10y Bonds	No. Trade
Rolling	42.2%	22.85%	0.030	-19.2%	36.96%	33
Fixing	8.47%	26.55%	-0.151	-19.2%	36.96%	43

Table 5: TF Strategy Return (CSI 300)

From Figure (9) and (10), we can observe that trend following strategy with rolling estimation outperforms the buy-and-hold strategy. However, if we use the fixed parameters and corresponding threshold,[9] it would lose the game to 10-year bond return.

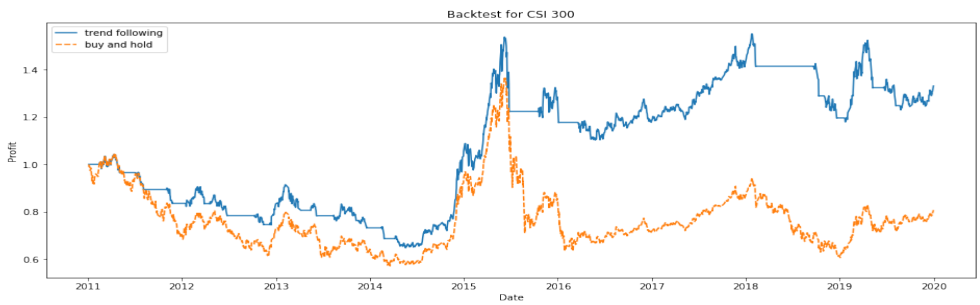


Figure 9: TF Strategy (Rolling)

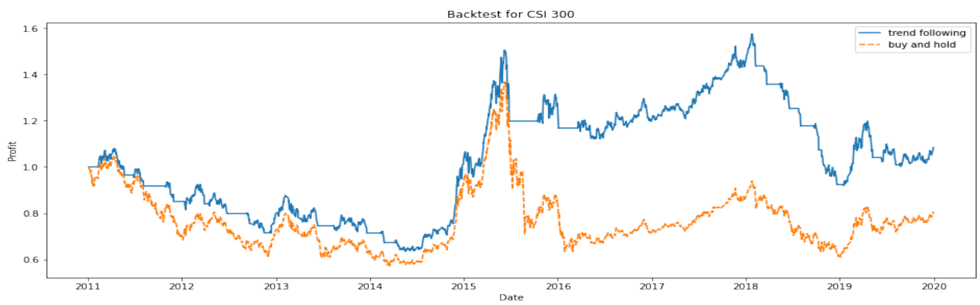


Figure 10: TF Strategy (Fixing)

[9]This means we use the parameters and corresponding threshold estimated by the CSI300 data from 2001 to 2010, and do not update those parameters and corresponding threshold.

From the above two figures, the return curves are similar in the first several years, and the main difference lies in the last 3 years. It's mainly because for rolling-base estimation, we updated parameters annually to keep pace with the CSI 300 index, but for fixed parameters, they may not fit the market after several years. It's self-evident that the success of the trend following strategy depends on how to update parameters and corresponding threshold in time.

Furthermore, if short position could be considered in the trend following strategy, the return curve can reach almost doublfold. However, This is not mathematical modelling since we don't change the underlying wealth process.

TF (short)	Volatility	Sharp Ratio (short)	TF (flat)	Volatility	Sharp Ratio (flat)
197.8%	38.10%	0.410	42.2%	26.55%	0.018

Table 6: TF Strategy with short position (CSI 300)

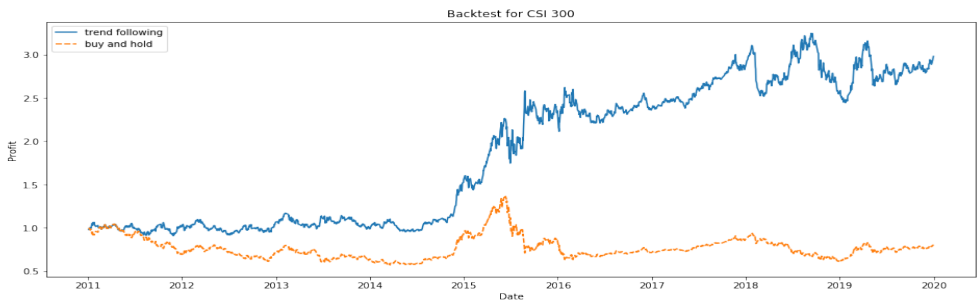


Figure 11: TF Strategy (Short)

Last but not least, to obtain a mathematical modelling that combine buy and short, since there is no solution to the model involving long, flat, and short position, we suppose we can formulate another HJB Equations with short and flat positions to calculate short and flat regions, and then combine the buy-and-flat boundary calculated from previous HJB Equations and short-and-flat boundary calculated from new HJB Equations to improve our trend following strategy.[10] However, the target is to maximize the wealth function, and combine two models may cause conflits in achieving the target. Moreover, we find the trend following strategy reacts slowly to the market regime switch. This model can beat the model that simply replaces the flat position with short position.

We derive the short-position model in Appendix.

4 Sensitivity Analysis

At final step, we did sensitivity analysis on **six parameters**: $\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1, \sigma_2$ respectively. Control the other five parameters stable when one is changed. Using estimated parameters in 2011 for CSI 300 and estimated parameters showed before for SPY as the basic value, and set the degree of change: $-15\%, -10\%, -5\%, 5\%, 10\%, 15\%$. In this case, we can observe the effect of parameter changes on the conditional possibility p_b^* and p_s^* and the strategy.

4.1 Sensitivity Test for SPY

For sensitivity of λ on buy & sell probability:

[10]This is more complicated modelling, and really need to carefully handle these four boundaries.

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9265	0.927	0.927	0.9275	0.9275	0.9275
p_s^*	0.8015	0.802	0.802	0.8025	0.803	0.803
$p_b^*\_per$	-0.054%	0%	0%	0.054%	0.054%	0.054%
$p_s^*\_per$	-0.125%	-0.062%	-0.062%	0%	0.062%	0.062%

Table 7: Sensitivity of λ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9275	0.927	0.927	0.927	0.927	0.927
p_s^*	0.8025	0.8025	0.8025	0.802	0.802	0.802
$p_b^*\_per$	0.054%	0%	0%	0%	0%	0%
$p_s^*\_per$	0%	0%	0%	-0.062%	-0.062%	-0.062%

Table 8: Sensitivity of λ_2

For sensitivity of λ on Strategy:

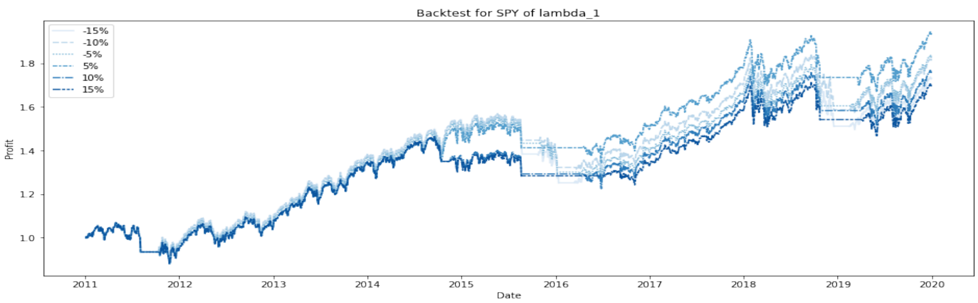


Figure 12: Sensitivity of λ_1

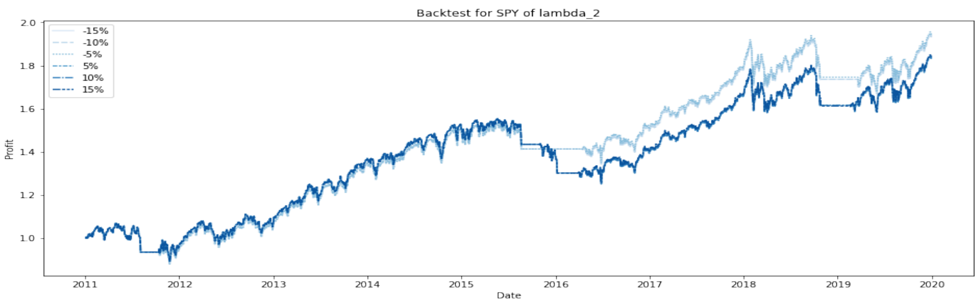


Figure 13: Sensitivity of λ_2

For sensitivity of μ on buy & sell probability:

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9485	0.9415	0.934	0.9205	0.9135	0.907
p_s^*	0.843	0.829	0.8155	0.79	0.7785	0.767
$p_b^*\_per$	2.319%	1.564%	0.755%	-0.701%	-1.456%	-2.157%
$p_s^*\_per$	5.047%	3.302%	1.620%	-1.558%	-2.991%	-4.424%

Table 9: Sensitivity of μ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.914	0.919	0.923	0.9305	0.934	0.937
p_s^*	0.7865	0.7925	0.7975	0.8065	0.811	0.8145
$p_b^*\_per$	-1.402%	-0.863%	-0.431%	0.378%	0.755%	1.079%
$p_s^*\_per$	-1.994%	-1.246%	-0.623%	0.498%	1.059%	1.495%

Table 10: Sensitivity of μ_2

For sensitivity of μ on strategy:

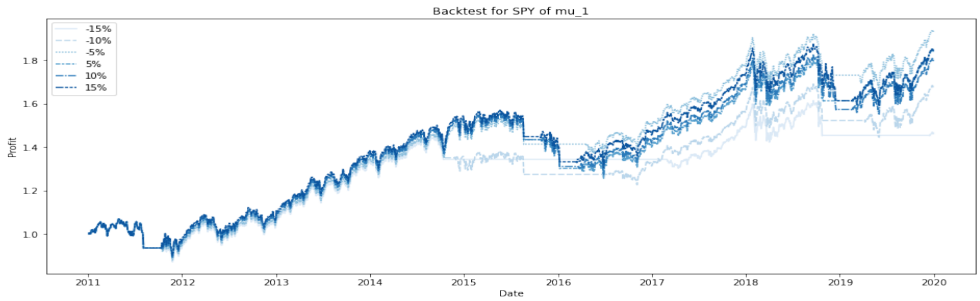


Figure 14: Sensitivity of μ_1

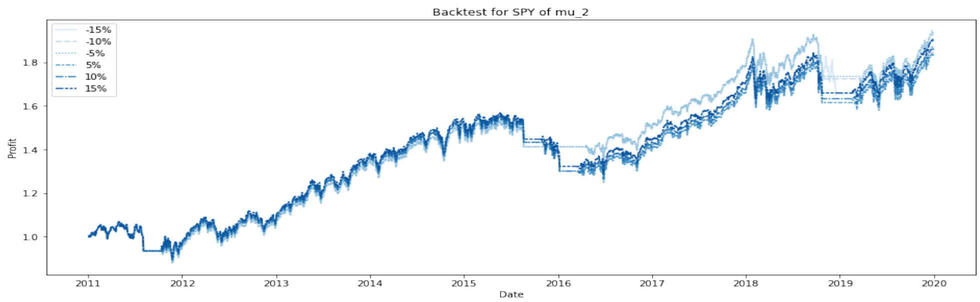


Figure 15: Sensitivity of μ_1

For sensitivity of σ on buy & sell probability:

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.924	0.9245	0.9255	0.929	0.931	0.933
p_s^*	0.8025	0.8025	0.8025	0.8025	0.8025	0.8025
$p_b^*\_per$	-0.324%	-0.270%	-0.162%	0.216%	0.431%	0.647%
$p_s^*\_per$	0%	0%	0%	0%	0%	0%

Table 11: Sensitivity of σ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.927	0.927	0.927	0.927	0.927	0.927
p_s^*	0.779	0.787	0.7945	0.8105	0.8185	0.8265
$p_b^*\_per$	0%	0%	0%	0%	0%	0%
$p_s^*\_per$	-2.928%	-1.931%	-0.997%	0.997%	1.994%	2.991%

Table 12: Sensitivity of σ_2

For sensitivity of σ on strategy:

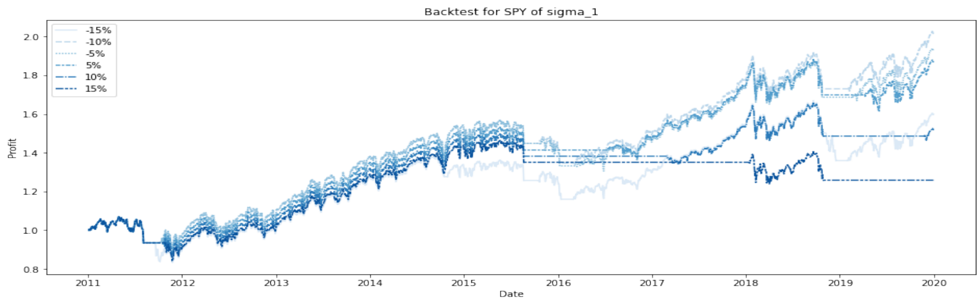


Figure 16: Sensitivity of σ_1

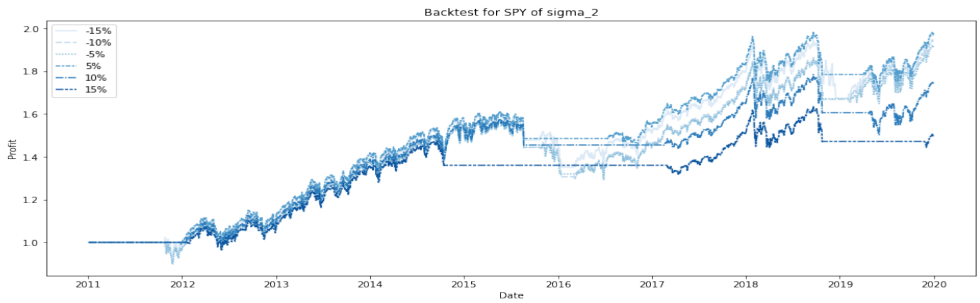


Figure 17: Sensitivity of σ_2

Remark 1. For example, for λ_1, the reciprocal of bull market's duration. When it becomes large, it means the bull market's duration is shorter and the boundary for buying and selling both become higher. And when it drop by 15%, it makes the strategy performed worst(the sarkest line). Parameter μ has a more significant influence than λ. And it is also reasonable for σ, if σ was too big, means the market risk is also big, in this case, the strategy would very likely to choose close out.

4.2 Sensitivity Test for CSI 300

For sensitivity of λ on buy & sell probability:

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.725	0.7255	0.7255	0.726	0.726	0.726
p_s^*	0.5005	0.5005	0.501	0.501	0.5015	0.5015
p_b^*_per	-0.069%	0%	0%	0.069%	0.069%	0.069%
p_s^*_per	-0.100%	-0.100%	0%	0%	0.100%	0.100%

Table 13: Sensitivity of λ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.7265	0.726	0.726	0.7255	0.725	0.725
p_s^*	0.502	0.502	0.5015	0.5005	0.5005	0.5
p_b^*_per	0.138%	0.069%	0.069%	0%	-0.069%	-0.069%
p_s^*_per	0.200%	0.200%	0.100%	-0.100%	-0.100%	-0.200%

Table 14: Sensitivity of λ_2

For sensitivity of λ on strategy:

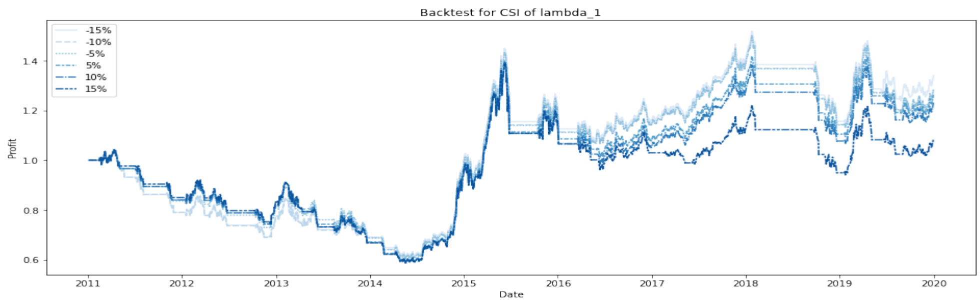


Figure 18: Sensitivity of λ_1

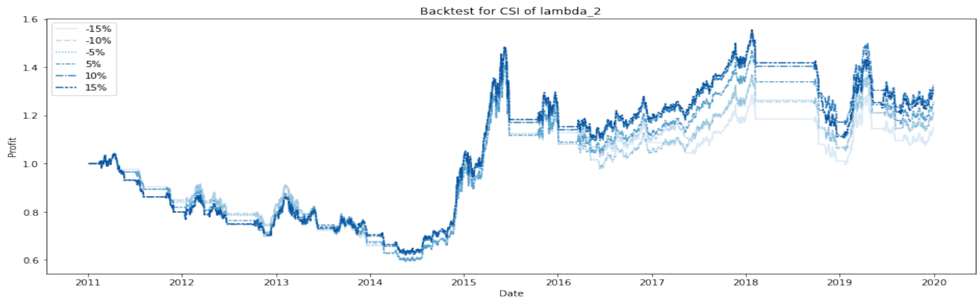


Figure 19: Sensitivity of λ_2

For sensitivity of μ on buy & sell probability:

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.751	0.742	0.7335	0.718	0.7115	0.705
p_s^*	0.5445	0.5295	0.515	0.488	0.4755	0.464
$p_b^*\_per$	3.515%	2.274%	1.103%	-1.034%	-1.930%	-2.826%
$p_s^*\_per$	8.683%	5.689%	2.794%	-2.595%	-5.090%	-7.385%

Table 15: Sensitivity of μ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.692	0.704	0.715	0.7355	0.7445	0.753
p_s^*	0.467	0.479	0.4905	0.511	0.5205	0.5295
$p_b^*\_per$	-4.618%	-2.963%	-1.447%	1.378%	2.619%	3.790%
$p_s^*\_per$	-6.786%	-4.391%	-2.096%	1.996%	3.892%	5.689%

Table 16: Sensitivity of μ_2

For sensitivity of μ on strategy:

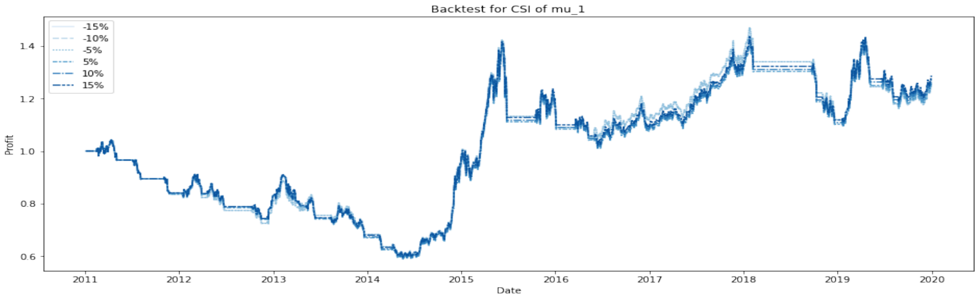


Figure 20: Sensitivity of μ_1

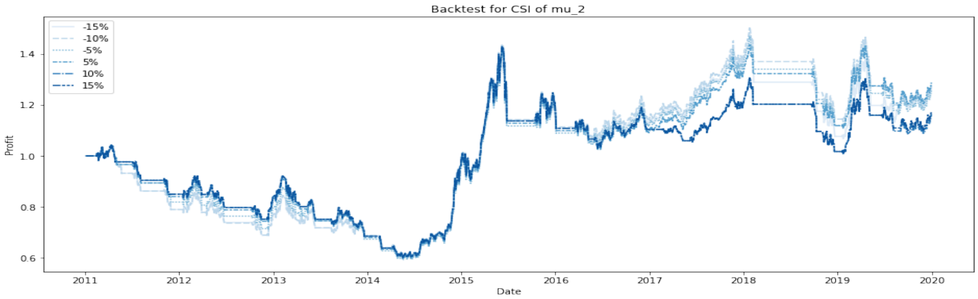


Figure 21: Sensitivity of μ_1

For sensitivity of σ on buy & sell probability:

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.748	0.7395	0.732	0.72	0.715	0.7105
p_s^*	0.501	0.501	0.501	0.501	0.501	0.501
$p_b^*\_per$	3.101%	1.930%	0.896%	-0.758%	-1.447%	-2.068%
$p_s^*\_per$	0%	0%	0%	0%	0%	0%

Table 17: Sensitivity of σ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.7255	0.7255	0.7255	0.7255	0.7255	0.7255
p_s^*	0.4895	0.493	0.497	0.5055	0.5105	0.5155
$p_b^*\_per$	0%	0%	0%	0%	0%	0%
$p_s^*\_per$	-2.295%	-1.597%	-0.798%	0.898%	1.896%	2.894%

Table 18: Sensitivity of σ_2

For sensitivity of σ on strategy:

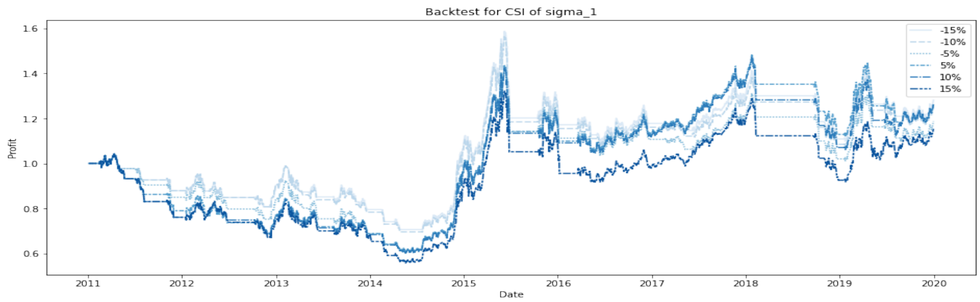


Figure 22: Sensitivity of σ_1

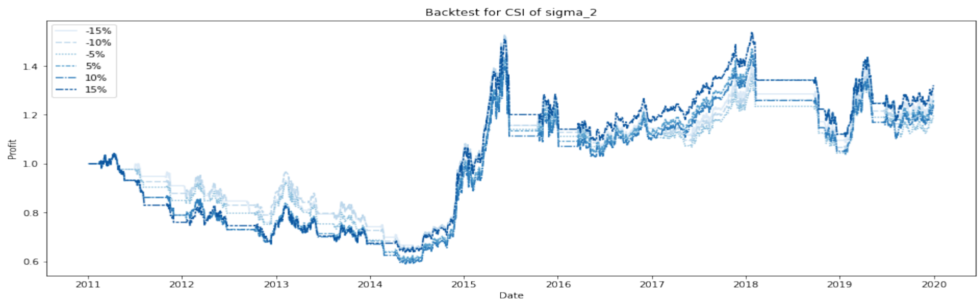


Figure 23: Sensitivity of σ_1

Remark 2. Compared to SPY, it seems not so sensitive for CSI 300, especially for parameter sigma. We think maybe SPY is in a mature market, which would be more sensitive with parameters. In this case, we think maybe increase the frequency of parameters estimation would help the strategy keep up with market better.

5 Conclusion

Limitation:

- Data Limitation: CSI 300 was released in 2005, there is no enough data for us to do parameters estimation.
- Parameter Estimation: The frequency of estimation is not high enough, maybe rolling parameters based on month would be better.
- Reaction Speed to Market: Compared with market, every switching between bull and bear is not quick enough in our strategy.

Contribution:

- We implemented a trend following strategy. The strategy could help investors to identify bull and bear market and avoid the plunge, which make the strategy performed better than buy-and-hold.
- The comparison of backtest on SPY and CSI 300 helps to realize that trend following strategy has different effects in mature and immature markets. In mature market, which is more sensitive with parameters, maybe the frequency of parameters estimation need to be higher.

Appendix

Consider trend following strategy for short:

A short decision is made at τ_n if $\tau_n < T$ and a flatten decision is made at v_n if $v_n < T, n = 1, 2 \ldots$ We assume that the investor is taking an "all-in and all-out" strategy. This means that she is either short so that her entire wealth is invested in the stock, or flat so that all of her wealth is in a bank account that draws the risk-free interest rate. We use indicator $i = 0$ or 1 to signify the initial position to be flat or short, respectively. If initially the position is short (i.e., $i = 1$), the corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots, t_2)$.

Given $S_0 = S$, $p_0 = p$, and initial position $i = 0$, 1, the reward functions of the decision sequences Λ_0 and Λ_1 are given as follows:

$$J_i^*(p, \Lambda_i) = \begin{cases} \mathbb{E}\left\{log\left[e^{\int_0^{\tau_1} \rho ds} \prod_{n=1}^{m} \left(\frac{S_{\tau_n}(1-\theta)}{S_{v_n}(1+\alpha)} e^{\int_{v_n}^{\tau_{n+1}} \rho ds}\right)\right]\right\}, & i = 0 \\ \mathbb{E}\left\{log\left[\frac{S_0}{S_{\bar{v}_1}(1+\alpha)} e^{\int_{\bar{v}_1}^{\bar{\tau}_2} \rho ds} \prod_{n=2}^{\bar{m}} \left(\frac{S_{\bar{\tau}_n}(1-\theta)}{S_{\bar{v}_n}(1+\alpha)} e^{\int_{\bar{v}_n}^{\bar{\tau}_{n+1}} \rho ds}\right)\right]\right\}, & i = 1 \end{cases}$$

Define value functions, for $i = 0, 1$

$$W_i^*(p, t) = \sup_{\Lambda_i} J_i^*(p, t, \Lambda_i)$$

it is easy to see

$$W_0^*(p, t) = \sup_{\tau_1} E_t \left\{ \int_t^{\tau_1} \rho ds + log(1-\theta) + W_1^*(p_{\tau_1}, \tau_1) \right\}$$

$$W_1^*(p, t) = \sup_{\bar{v}_1} E_t \left\{ -\int_t^{\bar{v}_1} \widetilde{f}(p_s) ds - log(1+\alpha) + W_0^*(p_{\bar{v}_1}, \bar{v}_1) \right\}$$

where

$$\widetilde{f}(p) = (\mu_1 - \mu_2)p + \mu_2 - \frac{\sigma^2}{2}$$

Then, let $Z^*(p, t) = W_1^*(p, t) - W_0^*(p, t)$. Then Z^* is the unique solution to the double obstacle problem

$$max\left\{min\left\{-\mathcal{A}Z^* + f^*(p), Z^* + log(1+\alpha)\right\}, Z^* + log(1-\theta)\right\} = 0$$

where

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2}\left(\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma}\right)^2 \frac{\partial}{\partial p^2} + [-(\lambda_1 + \lambda_2)p + \lambda_2]\frac{\partial}{\partial p}$$

$$f^*(p) = \rho + \widetilde{f}(p)$$

where $p \in [0, 1]$.

The upper boundary of Z^* value is $-log(1-\theta)$ and the lower boundary of Z is $-log(1+\alpha)$. Therefore, boundary condition and terminal condition are

$$Z(p, T) = -log(1-\theta)$$
$$Z(0, t) = -log(1+\alpha)$$
$$Z(1, t) = -log(1-\theta)$$

We formulate the Flat region(BR),the sell region(SR) and the no-trading region(NT) as follows:

$$FR = \{(p,t) \in (0,1) \times [0,T) : W_1^*(p,t) - W_0^*(p,t) = -log(1-\theta)\}$$
$$SR = \{(p,t) \in (0,1) \times [0,T) : W_1^*(p,t) - W_0^*(p,t) = -log(1+\alpha)\}$$
$$NT = (0,1) \times [0,T) \ (BR \cup SR).$$

We test this model using the parameters in test example. Figure (24) illustrate the numerical result:

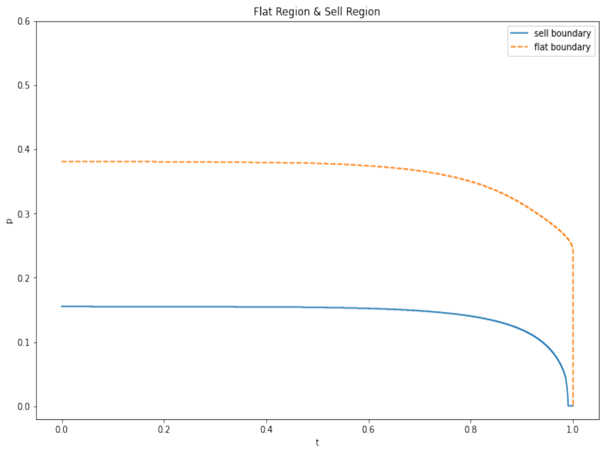


Figure 24: Test Example (Short)

Furthermore, we can use the flat and short boundary to conduct the trend following strategy for short.

However, there are many constraints on short selling in Chinese stock market, and the short selling cost is very high. It's hard to conduct the trend following strategy for short. [11]

[11] One way is to use future market to do short to improve the efficiency of our fund.