

DYNAMIC ROUTING AND JOCKEYING CONTROLS IN A TWO-STATION QUEUEING SYSTEM

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Abstract

This paper studies optimal routing and jockeying policies in a two-station parallel queueing system. It is assumed that jobs arrive to the system in a Poisson stream with rate λ and are routed to one of two parallel stations. Each station has a single server and a buffer of infinite capacity. The service times are exponential with server-dependent rates, μ_1 and μ_2 . Jockeying between stations is permitted. The jockeying cost is c_{ij} when a job in station i jockeys to station j , $i \neq j$. There is no cost when a new job joins either station. The holding cost in station j is h_j , $h_1 \leq h_2$, per job per unit time. We characterize the structure of the dynamic routing and jockeying policies that minimize the expected total (holding plus jockeying) cost, for both discounted and long-run average cost criteria. We show that the optimal routing and jockeying controls are described by three monotonically non-decreasing functions. We study the properties of these control functions, their relationships, and their asymptotic behavior. We show that some well-known queueing control models, such as optimal routing to symmetric and asymmetric queues, preemptive or non-preemptive scheduling on homogeneous or heterogeneous servers, are special cases of our system.

PARALLEL QUEUES; ROUTING; JOCKEYING; SCHEDULING; DYNAMIC PROGRAMMING;
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1. Introduction

This paper studies a queueing system that has two parallel stations, with a single server and a buffer of infinite capacity at each station. Jobs arrive to the system according to a Poisson process with rate λ . Service times at station j , $j = 1, 2$, are independent and identically distributed (i.i.d.) exponential random variables with rate μ_j , which are also independent of the service times at station i , $i \neq j$, $\mu_1 + \mu_2 > \lambda$. A new job, upon arrival, will be immediately routed to one of the two stations. The cost of routing a new job to any station is negligible. The cost of holding a job at station j per unit time is h_j (in Section 5 we generalize h_j to the class of increasing and convex functions). Without loss of generality, let $h_1 \leq h_2$.

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Jockeying between stations is permitted and instantaneous, i.e. the jockeying time is negligible. The jockeying cost c_{ij} is incurred when a job is switched from station i to station j , $i \neq j$. The objective is to find those dynamic routing and jockeying policies that minimize the expected total (holding plus jockeying) cost of the system, for both discounted and long-run average cost criteria.

This work is motivated by the wide applicability of routing and jockeying models in manufacturing, management of computer networks, telecommunications, and vehicular traffic flow. As a specific example, consider a multibeam satellite system serving earth-based stations that are organized into disjoint zones. Packets generated from earth zones arrive at the satellite that provides one or several buffers for the waiting packets. Packets are then sent to their destinations by multi-down-link beams. Effective packet routing and jockeying rules offer the possibility of improving the system performance by reducing the average packet delay on the satellite and the buffer overflow probability in the case of finite buffer size.

Studies of jockeying problems have been mainly concentrated on descriptive models, which evaluate the performance of a system under some proposed jockeying rule (Haight 1958, Disney and Mitchell 1971, Elsayed and Bastani 1985, Kao and Lin 1990, Zhao and Grassmann 1990, and Adan *et al.* 1991, Zhao and Grassmann 1995). In particular, Kao and Lin (1990) solved the problem of jockeying as soon as the difference between queue lengths exceeds one. They expressed their solution in terms of the eigenvalue of the rate matrix. Later Zhao and Grassmann (1990) provided an explicit solution to the problem. Nelson and Philips (1989) obtained the approximate response time for the shortest-queue jockeying rule. Zhao and Grassmann (1995) considered a jockeying model in which jobs are generated by a renewal process; jobs upon arrival join the shortest queue, and jockeying takes place as soon as the difference between the longest and the shortest queues exceeds a pre-set number. The authors proved that the equilibrium distribution of the queue length processes is a linear combination of geometric distributions and derived the expressions for performance measures.

Studies of optimal scheduling and routing in queueing systems are numerous. Here we shall only review those results closely related to our problem, and refer the reader to a comprehensive survey by Stidham and Weber (1993) for a summary of the research in this area. Winston (1977), Weber (1978), Whitt (1986), and Hordijk and Koole (1990), among others, studied the routing policy for symmetric queueing systems (a queueing system is said to be symmetric if parameters associated with different queues are identical). They showed that the optimal routing policy, under most circumstances, is the 'join-the-shortest-queue' policy. Davis (1977) and Abdel-Gawad (1984) considered the problem of admitting and routing jobs to two asymmetric queues (a queueing system is said to be asymmetric if parameters associated with different queues are different). They assumed that the two exponential queues are fed by a renewal arrival process and that the holding cost in each queue is an increasing and convex function of the number of jobs in the queue.

They proved that the optimal admission and routing policies have monotonicity properties: if rejecting a job is preferable to admitting it at a given congestion level, then it remains so when either queue is more crowded; if admitting to queue i is preferable to admitting to queue j , $i \neq j$, then it remains so when queue i is less crowded and/or queue j is more crowded. Hajek (1984) studied a more general system that incorporates some features of both parallel and series queues. A special case of his model is to route Poisson arrivals to two heterogeneous exponential servers. He, too, derived the monotonicity property for the optimal routing policy. Lin and Kumar (1984) and Walrand (1985) considered the problem of scheduling jobs on two heterogeneous servers and proved that the optimal scheduling policy is of threshold type. Xu (1994) proved that the optimal admission and scheduling policies in a two-server queueing system are characterized by two thresholds. Xu *et al.* (1993) considered the problem of scheduling two types of jobs to two service stations, with parallel servers at each station.

The problem addressed in this paper links together research in dynamic queueing control and evaluative studies. On the one hand, most work on dynamic control considers either the *routing* problem—in which customers are allocated to a queue at the time of their arrival—or the *scheduling* problem—in which customers are maintained in a single queue and allocated to servers when they become idle. Generally speaking, scheduling is considered a better control than routing, because it is more effective to allocate idle servers to customers than to allocate arriving customers to servers. However, in many realistic settings, a customer, upon arrival, must be routed to one of several queues, but can be reallocated later—possibly at certain cost—to a different queue. Intuitively, allowing jockeying will improve system performance, since the decision maker can alter his/her routing decisions when more information about the system becomes available, which results in a better use of the service resources. Indeed, the system with jockeying permitted always outperforms the system with jockeying forbidden, with other parameters being identical in both systems. In addition, for a two-station system, we may compare the performances of the systems with routing, routing/jockeying, and scheduling controls in the following sense. Suppose that $h_1 = h_2$ and preemptions are forbidden from either server. Then the system with scheduling control performs better than the system with routing/jockeying control (under which a waiting job is allowed to jockey at some positive jockeying cost), whereas the latter system performs better than the system with routing control. This is because in a scheduling problem all decisions are delayed until a server is available; in a routing/jockeying problem decisions must be made at job arrival times, but the decisions are restorative at certain cost at later times; and in a routing problem decisions are made upon job arrivals and they cannot be reversed. To our best knowledge, no authors have considered the routing/jockeying model as proposed in this paper. In Section 4, we illustrate that many aforementioned control models (e.g. the models studied by Winston, Weber, Lin and Kumar and Hajek) are special cases of our system. On

the other hand, most studies of descriptive jockeying models evaluate the performance of the system under specified jockeying rules without a clear understanding of which jockeying rule is the most effective for a given system configuration. Many proposed jockeying rules, such as the shortest-queue jockeying rule, are based on the common belief that the system efficiency is achieved by queue length balance. Our result indicates that this belief is often false, especially when jockeying is costly. This paper tries to redress this misconception and identify a class of effective routing/jockeying policies which justify the study of certain proposed jockeying policies (such as threshold policies) and/or point the way for future studies of descriptive jockeying models.

In this paper, we characterize the structure of the optimal routing and jockeying control policies that minimize the expected total (holding and jockeying) cost for both the discounted and the long-run average cost criteria. We also study the asymptotic behavior of the control functions. Our major findings are the following.

- (1) The optimal routing policy states that if it is optimal to route a job to station 1 when the state is (x_1, x_2) , it must be optimal to do the same when station 1 is less crowded (i.e. in state $(x_1 - m, x_2)$) or station 2 is more crowded (i.e. in state $(x_1, x_2 + m)$), where $m > 0$ and $x_j, j = 1, 2$, is the queue length in station j at the arrival instant of the job. This is equivalent to saying that the optimal *routing* policy is described by a monotonically non-decreasing function $F(x_1)$ such that a job, upon arrival, is routed to station 2 if $x_2 \leq F(x_1)$ and to station 1 otherwise.
- (2) The optimal jockeying policy states that if, in state (x_1, x_2) , it is optimal to move a job from station i to station j , $i \neq j$, then it must be optimal to do the same when station i is more crowded or station j is less crowded. This is equivalent to saying that the optimal *jockeying* policy is described by two monotonically non-decreasing functions $F_{12}(x_1)$ and $F_{21}(x_1)$, satisfying $F_{21}(x_1) > F_{12}(x_1)$, such that a job in station 1 jockeys to station 2 if $x_2 \leq F_{12}(x_1)$, a job in station 2 jockeys to station 1 if $x_2 \geq F_{21}(x_1)$, and no jockeying between stations occurs if $F_{12}(x_1) < x_2 < F_{21}(x_1)$.
- (3) If it is optimal to move a job from station i to station j in state (x_1, x_2) , then it must be optimal to route a new job to station j in the same state. This is equivalent to saying that the optimal routing function is bounded between the optimal jockeying functions: $F_{12} \leq F \leq F_{21}$.
- (4) If $h_1 = h_2$, the optimal jockeying policy, for either discounted or long-run average cost criteria, reduces to the threshold-type policy that lets a job in station i jockey to station j , $i \neq j$, if and only if station j is empty and the number of jobs in station i exceeds a threshold. If $h_1 < h_2$, then the optimal jockeying from station 1 to station 2 is of threshold type, whereas from station 2 to station 1 it is characterized by the increasing curve F_{21} .
- (5) The optimal control functions, for both discounted and long-run average

costs, often exhibit convergent behavior. For example, for the discounted cost, F approaches a finite asymptote (i.e. is bounded above by a finite constant) as $x_1 \rightarrow \infty$ if $h_1 < h_2$ or $h_1 = h_2$ and $c_{ij} = 0$ for some i, j ; otherwise F does not converge. For the long-run average case, F approaches a finite asymptote if and only if $h_1 < h_2$ and $c_{12} > 0$, or $h_1 = h_2$ and $c_{ij} = 0$ for some i, j ; otherwise F does not converge.

The result stated in (4) is interesting because it challenges the rationale of some popular jockeying rules such as the shortest-queue jockeying rule. Result (4) states that a job in the low-cost station will never jockey to the high-cost station when the latter is non-empty; if both stations have the same holding cost, then jockeying will take place ‘just-in-time’ (i.e. do not jockey to a non-empty station). It suggests that jockeying improves system performance mainly by efficiently using the fast or low-cost station rather than by balancing the work load among stations. The result stated in (5) has several implications. First, it implies that, when F possesses a finite asymptote, buffer sizes should be designed unbalanced, with the ‘good’ station (the low-holding-cost station or the zero-jockeying-cost station, if $h_1 = h_2$) having an unlimited capacity and the ‘bad’ station (the high-holding-cost station or the high-jockeying-cost station) a finite capacity. When the system congestion level is high, most arrivals will be routed to the good station and moved to the bad station later when necessary. Second, the computation of F needs to be carried out only for small or moderate x_1 , significantly reducing computational effort. Finally, it suggests that, in heavy traffic, the optimal routing and jockeying policies can be approximated by threshold-type policies (we elaborate on the above points further in Remarks 1–4).

The rest of the paper is organized as follows. In Section 2 we study the structure of the optimal routing and jockeying policies. In Section 3 we consider the limiting behavior of the control functions. In Section 4 we present some special cases of our model; some are well-studied in the literature. In Section 5 we discuss extensions of our model and suggest future research topics.

2. Optimal routing and jockeying policies

In this section we formulate the aforementioned queueing control problem as a Markov decision process and use dynamic programming to characterize the features of control policies. We shall concentrate on the discounted cost criterion, and sketch the analysis for the long-run average cost criterion.

2.1. Discounted cost. We use the state vector $X(t) = (X_1(t), X_2(t))$ to represent the system state at time t , where $X_j(t)$ is the number of jobs in station j at time t , including the job, if any, under service. Let $S = \{(x_1, x_2) : x_j \in \mathbb{Z}^+, j = 1, 2\}$ be the state space of the process, where $\mathbb{Z}^+ = \{0, 1, \dots\}$. Because the process has no memory, we only need to consider the class of stationary policies that depend only on the current state of the process at the decision epoch.

Decision epochs will be job arrival and departure times. At an arrival epoch, the system controller has the option to route the job to either station; at a departure epoch, s/he can move some jobs from one station to another, which is called jockeying. Our objective is to find the routing and jockeying policies that minimize the expected total (holding and jockeying) cost, continuously discounted at rate $\alpha > 0$ over an infinite time horizon.

Using uniformization as in Lippman (1975), we first convert the continuous-time process to an equivalent discrete-time process. Let a potential event be either an arrival or a service completion (real or fictitious) of a job. Since the expected inter-transition times are constant (due to uniformization), without loss of generality we let potential transitions occur at each unit of time; i.e. $\lambda + \mu_1 + \mu_2 = 1$.

Let $V_t(x_1, x_2)$ be the minimal expected t -period discounted cost with initial state (x_1, x_2) . Let $\bar{V}_t(x_1, x_2)$ be the minimal expected t -period discounted cost with initial state (x_1, x_2) , given that routing and jockeying will not occur until the next decision time. For any scalar a we let $[a]^+ = \max\{a, 0\}$. Then the dynamic programming optimality equation takes the form

$$(2.1) \quad \begin{aligned} \bar{V}_t(x_1, x_2) = & h_1 x_1 + h_2 x_2 + \alpha \mu_1 V_{t-1}([x_1 - 1]^+, x_2) + \alpha \mu_2 V_{t-1}(x_1, [x_2 - 1]^+) \\ & + \alpha \lambda \min \{V_{t-1}(x_1 + 1, x_2), V_{t-1}(x_1, x_2 + 1)\}, \end{aligned}$$

where

$$(2.2) \quad V_t(x_1, x_2) = \min \left\{ \begin{array}{l} \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 - k_2)\}, \\ \bar{V}_t(x_1, x_2), \\ \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + k_1)\} \end{array} \right\},$$

with the boundary conditions

$$\bar{V}_0(x_1, x_2) = 0, \quad V_0(x_1, x_2) = 0.$$

Since the infinite-period cost $V(x_1, x_2)$ or $\bar{V}(x_1, x_2)$ can be considered as the limit of the corresponding finite-period cost (Ross 1983), we have

$$V(x_1, x_2) = \lim_{t \rightarrow \infty} V_t(x_1, x_2),$$

$$\bar{V}(x_1, x_2) = \lim_{t \rightarrow \infty} \bar{V}_t(x_1, x_2).$$

The properties hold for V_t and \bar{V}_t remain true for V and \bar{V} , respectively.

From the optimality equations (2.1) and (2.2) one sees that the optimal decision is to do the following.

- (1) Route a new job to station 2 if and only if $\Delta(x_1, x_2) := V(x_1 + 1, x_2) - V(x_1, x_2 + 1) \geq 0$. In other words, an arrival will be sent to station 2 if and only if the resulting cost-to-go $V(x_1, x_2 + 1)$ is smaller than what the

cost-to-go would be (namely $V(x_1 + 1, x_2)$) if instead that arrival were sent to station 1.

- (2) Move k_1 jobs from station 1 to station 2 if and only if $k_1 c_{12} + \bar{V}(x_1 - k_1, x_2 + k_1) = V(x_1, x_2)$, move k_2 jobs from station 2 to station 1 if and only if $k_2 c_{21} + \bar{V}(x_1 + k_2, x_2 - k_2) = V(x_1, x_2)$, and do nothing if $\bar{V}(x_1, x_2) = V(x_1, x_2)$. In other words, k_1 (k_2) jobs will be moved from station 1 (station 2) to station 2 (station 1) if and only if the jockeying cost $k_1 c_{12}$ ($k_2 c_{21}$) plus the resulting cost-to-go $\bar{V}(x_1 - k_1, x_2 + k_1)$ ($\bar{V}(x_1 + k_2, x_2 - k_2)$) is the smallest among the cost associated with other jockeying decisions; otherwise jockeying will not take place in state (x_1, x_2) .

We now study the properties of V_t and \bar{V}_t . It is elementary to show, by induction on t , that $V_t(x_1, x_2)$ and $\bar{V}_t(x_1, x_2)$ are non-decreasing and convex for each of their arguments, with another argument fixed. Other properties of the dynamic equations are derived in Proposition 1.

Proposition 1.

- (a) $\bar{\Delta}_t(x_1, x_2) := \bar{V}_t(x_1 + 1, x_2) - \bar{V}_t(x_1, x_2 + 1)$ is non-decreasing in x_1 and non-increasing in x_2 .
- (b) $\Delta_t(x_1, x_2) := V_t(x_1 + 1, x_2) - V_t(x_1, x_2 + 1)$ is non-decreasing in x_1 and non-increasing in x_2 .

Proof. We derive (a)–(b) by induction on t . The proposition is trivially true for $t = 0$. We show that (a) and (b) hold for t , based on the hypotheses that they are true for $t - 1$. Denote the hypotheses associated with (a) and (b) by H_a and H_b , respectively.

Proof of (a) for t . From (2.1),

$$\begin{aligned}\bar{\Delta}_t(x_1, x_2) &= (h_1 - h_2) + \alpha\mu_1[V_{t-1}(x_1, x_2) - V_{t-1}([x_1 - 1]^+, x_2 + 1)] \\ &\quad + \alpha\mu_2[V_{t-1}(x_1 + 1, [x_2 - 1]^+) - V_{t-1}(x_1, x_2)] \\ &\quad + \alpha\lambda[\min\{V_{t-1}(x_1 + 2, x_2), V_{t-1}(x_1 + 1, x_2 + 1)\} \\ &\quad - \min\{V_{t-1}(x_1 + 1, x_2 + 1), V_{t-1}(x_1, x_2 + 2)\}].\end{aligned}$$

We now argue that each term in the above expression is monotonically non-decreasing in x_1 and non-increasing in x_2 . It is obviously true for the first (constant) term. Since $[x_1 - 1]^+ = x_1 - 1$ for $x_1 > 0$, by H_b the second term has the claimed monotonicity property for $x_1 > 0$. If $x_1 = 0$,

$$\begin{aligned}[V_{t-1}(1, x_2) - V_{t-1}(0, x_2 + 1)] - [V_{t-1}(0, x_2) - V_{t-1}(0, x_2 + 1)] \\ = V_{t-1}(1, x_2) - V_{t-1}(0, x_2) \geq 0,\end{aligned}$$

because V_{t-1} is non-decreasing in x_1 . Also

$$[V_{t-1}(0, x_2) - V_{t-1}(0, x_2 + 1)] - [V_{t-1}(0, x_2 + 1) - V_{t-1}(0, x_2 + 2)] \geq 0,$$

because $V_{t-1}(0, x_2)$ is convex in x_2 . This proves the monotonicity property for the second term. A similar argument establishes our assertion for the third term. As for the last term, we may rewrite it as (ignore the factor $\alpha\lambda$)

$$\begin{aligned} & \min \{V_{t-1}(x_1 + 2, x_2), V_{t-1}(x_1 + 1, x_2 + 1)\} \\ & - \min \{V_{t-1}(x_1 + 1, x_2 + 1), V_{t-1}(x_1, x_2 + 2)\} \\ & = \min \{0, V_{t-1}(x_1 + 2, x_2) - V_{t-1}(x_1 + 1, x_2 + 1)\} \\ & + \max \{0, V_{t-1}(x_1 + 1, x_2 + 1) - V_{t-1}(x_1, x_2 + 2)\} \\ & = \min \{0, \Delta_{t-1}(x_1 + 1, x_2)\} + \max \{0, \Delta_{t-1}(x_1, x_2 + 1)\}. \end{aligned}$$

Since, by H_b , both $\Delta_{t-1}(x_1 + 1, x_2)$ and $\Delta_{t-1}(x_1, x_2 + 1)$ are non-decreasing in their first argument and non-increasing in the their second argument, the same is true for the preceding expression. This completes the induction proof for (a).

Proposition 1(a) implies, for $x_1 \geq 1$ and $x_2 \geq 1$,

$$\bar{\Delta}_t(x_1 + k_2, x_2 - k_2) \geq \bar{\Delta}_t(x_1, x_2) \geq \bar{\Delta}_t(x_1 - k_1, x_2 + k_1), \quad \text{for } 0 \leq k_j \leq x_j, j = 1, 2,$$

and it further leads to

$$(2.3) \quad \begin{aligned} \bar{V}_t(x_1 + 1 + k_2, x_2 - k_2) - \bar{V}_t(x_1 + 1, x_2) & \geq \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2) \\ & - \bar{V}_t(x_1, x_2 + 1). \end{aligned}$$

Equation (2.3) is useful in the remaining proof of Proposition 1.

Proof of (b) for t. We only prove the ‘non-decreasing’ part; the proof for the ‘non-increasing’ part is the same. From (2.2),

$$(2.4) \quad V_t(x_1, x_2 + 1) = \min \left\{ \begin{array}{l} \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2)\}, \\ \bar{V}_t(x_1, x_2 + 1), \\ \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\} \end{array} \right\}.$$

To determine the domain of each term on the right-hand side (RHS) of (2.4), we select two terms on the RHS of (2.4) and consider their difference. The difference of the first and the second terms is

$$\begin{aligned} & \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_1 + 1 - k_2)\} - \bar{V}_t(x_1, x_2 + 1) \\ & = \min_{1 \leq k_2 \leq x_2+1} \left\{ k_2 c_{21} + \sum_{r=0}^{k_2-1} \{\bar{\Delta}_t(x_1 + r, x_2 - r)\} \right\}. \end{aligned}$$

By (a), the above is a non-decreasing function of x_1 . Similarly, it can be shown that the difference between the first and the third terms and the difference between the second and the third terms are all non-decreasing in x_1 . It implies that if the difference of any two selected terms is non-negative for (x_1, x_2) , then it remains so

when x_1 increases. Consequently, for a fixed x_2 , there exist numbers $p_{x_2}^*$ and $p_{x_2}^{**}$, where $\infty \leq p_{x_2}^* \leq p_{x_2}^{**} \leq \infty$, such that

(2.5)

$$V_t(x_1, x_2 + 1) = \begin{cases} \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2)\}, & \text{if } x_1 < p_{x_2}^*; \\ \bar{V}_t(x_1, x_2 + 1), & \text{if } p_{x_2}^* \leq x_1 \leq p_{x_2}^{**}; \\ \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\} & \text{if } x_1 > p_{x_2}^{**}. \end{cases}$$

Next we develop the corresponding expression for $V_t(x_1 + 1, x_2)$ and $\Delta_t(x_1, x_2)$, letting $V_t(x_1, x_2 + 1)$ assume one of the three expressions on the RHS of (2.5).

Case 1. $V_t(x_1, x_2 + 1) = \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2)\}$.

This case implies that at least one job in station 2 will be switched to station 1 in state $(x_1, x_2 + 1)$. Hence,

$$V_t(x_1, x_2 + 1) = c_{21} + V_t(x_1 + 1, x_2),$$

so,

$$\Delta_t(x_1, x_2) = V_t(x_1 + 1, x_2) - V_t(x_1, x_2 + 1) = -c_{21}.$$

Case 2. $V_t(x_1, x_2 + 1) = \bar{V}_t(x_1, x_2 + 1)$.

By (2.2), this case implies

$$(2.6) \quad \bar{V}_t(x_1, x_2 + 1) \leq \min \left\{ \begin{array}{l} \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2)\}, \\ \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\} \end{array} \right\}.$$

To find an expression for $V_t(x_1 + 1, x_2)$, we use (2.2),

$$(2.7) \quad V_t(x_1 + 1, x_2) = \min \left\{ \begin{array}{l} \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} + \bar{V}_t(x_1 + 1 + k_2, x_2 - k_2)\}, \\ \bar{V}_t(x_1 + 1, x_2), \\ \min_{1 \leq k_1 \leq x_1+1} \{k_1 c_{12} + \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \end{array} \right\}.$$

From (2.3) and (2.6), the difference of the first and the second terms of (2.7) satisfies

$$\begin{aligned} & \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} + \bar{V}_t(x_1 + 1 + k_2, x_2 - k_2)\} - \bar{V}_t(x_1 + 1, x_2) \\ & \geq \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2)\} - \bar{V}_t(x_1, x_2 + 1) \geq 0. \end{aligned}$$

In addition, by (2.6), the third term of (2.7) reduces to

$$\begin{aligned} & \min_{1 \leq k_1 \leq x_1+1} \{k_1 c_{12} + \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \\ &= \min \{c_{12} + \bar{V}_t(x_1, x_2 + 1), c_{12} + \min_{\substack{\text{text} \\ 1 \leq k_1 \leq x_1}} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\}\} \\ &= c_{12} + \bar{V}_t(x_1, x_2 + 1). \end{aligned}$$

The previous two expressions simplify (2.7) to

$$V_t(x_1 + 1, x_2) = \min \{\bar{V}_t(x_1 + 1, x_2), c_{12} + \bar{V}_t(x_1, x_2 + 1)\}.$$

Therefore,

$$\begin{aligned} \Delta_t(x_1, x_2) &= V_t(x_1 + 1, x_2) - V_t(x_1, x_2 + 1) \\ &= \min \{\bar{V}_t(x_1 + 1, x_2), c_{12} + \bar{V}_t(x_1, x_2 + 1)\} - \bar{V}_t(x_1, x_2 + 1) \\ &= \min \{\bar{\Delta}_t(x_1, x_2), c_{12}\}. \end{aligned}$$

Note that in this case $\bar{\Delta}_t(x_1, x_2) \geq -c_{21}$.

Case 3. $V_t(x_1, x_2 + 1) = \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\}$.

We prove that in this case the RHS of (2.7) is minimized by the third term. From (2.3), the difference of the first and the third terms is

$$\begin{aligned} & \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} + \bar{V}_t(x_1 + 1 + k_2, x_2 - k_2)\} - \min_{1 \leq k_1 \leq x_1+1} \{k_1 c_{12} + \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \\ &= \max_{1 \leq k_1 \leq x_1+1} \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} - k_1 c_{12} + \bar{V}_t(x_1 + 1 + k_2, x_2 - k_2) \\ &\quad - \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \\ &\geq \max_{1 \leq k_1 \leq x_1} \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} - k_1 c_{12} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2) - \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\} \\ &= \min_{1 \leq k_2 \leq x_2+1} \{k_2 c_{21} + \bar{V}_t(x_1 + k_2, x_2 + 1 - k_2) - \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\}\} \\ &\geq 0, \end{aligned}$$

where the last inequality holds for Case 3. Using (2.3) again, the difference of the second and the third terms satisfies

$$\begin{aligned} & \bar{V}_t(x_1 + 1, x_2) - \min_{1 \leq k_1 \leq x_1+1} \{k_1 c_{12} + \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \\ &\geq \bar{V}_t(x_1, x_2 + 1) - \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\} \geq 0, \end{aligned}$$

where the last inequality is true for Case 3. Thus (2.7) reduces to

$$\begin{aligned} V_t(x_1 + 1, x_2) &= \min_{1 \leq k_1 \leq x_1 + 1} \{k_1 c_{12} + \bar{V}_t(x_1 + 1 - k_1, x_2 + k_1)\} \\ &= c_{12} + \min \{\bar{V}_t(x_1, x_2 + 1), \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\}\} \\ &= c_{12} + \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{V}_t(x_1 - k_1, x_2 + 1 + k_1)\}. \end{aligned}$$

Therefore,

$$V_t(x_1 + 1, x_2) - V_t(x_1, x_2 + 1) = c_{12}.$$

Finally, we put Cases 1–3 together:

$$\tilde{\Delta}_t(x_1, x_2) = \begin{cases} -c_{21} & \text{if } x_1 < p_{x_2}^*; \\ \min \{\bar{\Delta}_t(x_1, x_2), c_{12}\} & \text{if } p_{x_2}^* \leq x_1 \leq p_{x_2}^{**}; \\ c_{12} & \text{if } x_1 > p_{x_2}^{**}. \end{cases}$$

From (a), the preceding expression is non-decreasing in x_1 . This completes the induction proof for (b).

Without loss of generality, we make the convention that a single job jockeying will occur in state (x_1, x_2) if and only if it is strictly advantageous to do so; i.e.

$$(2.8) \quad \bar{V}_t(x_1, x_2) - \bar{V}_t(x_1 - 1, x_2 + 1) - c_{12} = \bar{\Delta}_t(x_1 - 1, x_2) - c_{12} > 0,$$

or

$$(2.9) \quad \bar{V}_t(x_1, x_2) - \bar{V}_t(x_1 + 1, x_2 - 1) - c_{21} = -\bar{\Delta}_t(x_1, x_2 - 1) - c_{21} > 0.$$

Note that if no jockeying is preferable to a single-job jockeying, i.e. $\bar{\Delta}(x_1 - 1, x_2) - c_{12} \leq 0$, then no jockeying is preferable to the multiple-job jockeying. This is because

$$(2.10) \quad \bar{V}_t(x_1, x_2) - \bar{V}_t(x_1 - k_1, x_2 + k_1) - c_{12}k_1 = \sum_{i=0}^{k_1-1} [\bar{\Delta}_t(x_1 - 1 - i, x_2 + i) - c_{12}] \leq k_1 [\bar{\Delta}_t(x_1 - 1, x_2) - c_{12}] \leq 0,$$

where the first inequality follows from Proposition 1(a). It implies that no jockeying is at least as good as moving k_1 jobs from station 1 to station 2, $1 \leq k_1 \leq x_1$.

The structure of the optimal policy is a simple consequence of Proposition 1 and the above equations.

Theorem 1. Let (x_1, x_2) be the state of the system at a decision time.

- (a) There exists a non-decreasing function $F(x)$ such that it is optimal to route a new job to station 2 if and only if $x_2 \leq F(x_1)$.
- (b) There exist non-decreasing functions $F_{12}(x_1)$ and $F_{21}(x_1)$ such that a job in station 1 will jockey to station 2 if $x_2 \leq F_{12}(x_1)$ and a job in station 2 will jockey to station 1 if $x_2 \geq F_{21}(x_1)$.
- (c) $F_{12}(x_1) \leq F(x_1) \leq F_{21}(x_1)$. In addition, no jockeying will occur in state (x_1, x_2) if $F_{12}(x_1) < x_2 < F_{21}(x_1)$.

Proof. (a) We may define $F(x)$ by means of

$$(2.11) \quad F(x_1) = \sup \{x_2 : \Delta(x_1, x_2) \geq 0\}.$$

To prove that the optimal routing policy is characterized by the non-decreasing curve $F(x_1)$, it is sufficient to prove that if it is optimal to route a new job to station 2 in state (x_1, x_2) , then it remains optimal to do so in states $(x_1 + 1, x_2)$ and $(x_1, x_2 - 1)$. By Proposition 1(b),

$$V(x_1 + 2, x_2) - V(x_1 + 1, x_2 + 1) \geq V(x_1 + 1, x_2) - V(x_1, x_2 + 1),$$

$$V(x_1 + 1, x_2 - 1) - V(x_1, x_2) \geq V(x_1 + 1, x_2) - V(x_1, x_2 + 1),$$

thus the non-negativity of the right-hand sides implies the non-negativity of the left-hand sides of the previous expressions.

(b) From (2.8) and (2.9) we can let

$$(2.12) \quad F_{12}(x_1) = \sup \{x_2 : \bar{\Delta}(x_1 - 1, x_2) > c_{12}\},$$

$$(2.13) \quad F_{21}(x_1) = \inf \{x_2 : -\bar{\Delta}(x_1, x_2 - 1) > c_{21}\}.$$

We only prove the non-decreasing property of F_{12} , as the proof of the non-decreasing property of F_{21} is similar. Suppose that it is preferable to move a job to station 2 in state (x_1, x_2) , then (2.8) holds. By Proposition 1,

$$\bar{V}(x_1 + 1, x_2) - \bar{V}(x_1, x_2 + 1) = \bar{\Delta}(x_1, x_2) \geq \bar{\Delta}(x_1 - 1, x_2) > c_{12},$$

$$\bar{V}(x_1, x_2 - 1) - \bar{V}(x_1 - 1, x_2) = \bar{\Delta}(x_1 - 1, x_2 - 1) \geq \bar{\Delta}(x_1 - 1, x_2) > c_{12}.$$

Thus it is preferable to move a job to station 2 in states $(x_1 + 1, x_2)$ and $(x_1, x_2 - 1)$. This is equivalent to saying that $F_{12}(x_1)$ is non-decreasing.

(c) Let \mathcal{B} be the set of states for which a new job will be routed to station 2 under the optimal routing policy,

$$\mathcal{B} = \{(x_1, x_2) : \Delta(x_1, x_2) \geq 0\},$$

and let $\mathcal{B}_{12}(\mathcal{B}_{21})$ be the set of states for which a job in station 1 (station 2) will jockey to station 2 (station 1) under the optimal jockeying policy,

$$\mathcal{B}_{12} = \{(x_1, x_2) : \bar{\Delta}(x_1 - 1, x_2) > c_{12}\} = \{(x_1, x_2) : F_{12}(x_1) \geq x_2\}$$

$$\mathcal{B}_{21} = \{(x_1, x_2) : -\bar{\Delta}(x_1, x_2 - 1) > c_{21}\} = \{(x_1, x_2) : F_{21}(x_1) \leq x_2\}.$$

It suffices to show that

$$(2.14) \quad (x_1, x_2) \in \mathcal{B}_{12} \Rightarrow (x_1, x_2) \in \mathcal{B},$$

$$(2.15) \quad (x_1, x_2) \in \mathcal{B}_{21} \Rightarrow (x_1, x_2) \in \bar{\mathcal{B}},$$

where $\bar{\mathcal{B}}$ is the complement of \mathcal{B} .

For this purpose, let $(x_1, x_2) \in \mathcal{B}_{12}$. If $(x_1, x_2) \notin \mathcal{B}$, then a new job will be routed to station 1 in state (x_1, x_2) , resulting in state $(x_1 + 1, x_2)$. But by the monotonicity property of F_{12} , state $(x_1 + 1, x_2) \in \mathcal{B}_{12}$, because $(x_1, x_2) \in \mathcal{B}_{12}$. Hence the job will

jockey to station 2 as soon as it is routed to station 1. Since the cost of routing the job directly to station 2 can be no larger than that of routing the job to station 1 first and then immediately moving it to station 2, we have $(x_1, x_2) \in \mathcal{B}$ and (2.14) is proven.

Finally, if $F_{12}(x_1) < x_2 < F_{21}(x_1)$, then $(x_1, x_2) \notin \mathcal{B}_{12}$ and $(x_1, x_2) \notin \mathcal{B}_{21}$ and no jockeying is necessary.

2.2. Long-run average cost. In this section we prove that the monotonicity properties of the optimal control functions for the discounted cost problem remain true for the undiscounted, long-run average cost problem. We only sketch the analysis, as the technique to derive the properties of the long-run average cost problem via that of the discounted-cost problem is well developed and has become a standard procedure (Ross 1983, Walrand 1988, Borkar 1988, 1989).

The system we are dealing with is a Markov decision process with infinite states and unbounded costs. From the standard theory of dynamic programming, we can verify that our problem satisfies the sufficient conditions (see, e.g., p. 288 of Walrand (1988)) that guarantee the existence of the optimal routing and jockeying policies for the long-run average cost problem. Therefore, there exists a constant $g = \lim_{\alpha \rightarrow 0} (1 - e^{-\alpha}) V^\alpha(x_1, x_2)$ (g is interpreted as the minimal average cost that is independent of the initial state and we append α to function V to emphasize its dependency on α) and a bounded function $h(x_1, x_2)$ satisfying the average-cost version of the dynamic equations

$$g + h(x_1, x_2) = \min \left\{ \begin{array}{l} \min_{1 \leq k_2 \leq x_2} \{k_2 c_{21} + \bar{h}(x_1 + k_2, x_2 - k_2)\}, \\ \bar{h}(x_1, x_2), \\ \min_{1 \leq k_1 \leq x_1} \{k_1 c_{12} + \bar{h}(x_1 - k_1, x_2 + k_1)\} \end{array} \right\},$$

where \bar{h} is defined in (2.1) with \bar{V} replaced by \bar{h} and V replaced by h . Moreover, the policy that chooses the minimization actions is the average-cost optimal. It can be shown that

$$h(x_1, x_2) := \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2) - V^\alpha(0, 0)]$$

exists and is finite and is the non-negative solution to the average-cost dynamic equation.

Therefore, the relative costs h and \bar{h} inherit the structural forms of V and \bar{V} and Proposition 1 can be claimed with V and \bar{V} replaced by h and \bar{h} , respectively. Finally, Theorem 1 is true for the long-run average cost problem.

3. Asymptotic properties of the control functions

In this section we present some important properties of the optimal control functions. We identify necessary and sufficient conditions under which the optimal control functions possess finite asymptotic limits. The proof is done by a coupling

argument. We comment on the implications of the asymptotic behavior of the control functions to computational and system design issues.

3.1. Properties of the control functions for the discounted cost. We first present an interesting result, which states that jobs in the low-cost station (station 1) will never jockey to the high-cost station (station 2) unless the latter is empty. If both stations incur the same holding cost, then jockeying to station j , $j = 1, 2$, will take place only when station j is empty.

Theorem 2. Let $h_1 \leq h_2$. Then

$$(3.1) \quad \mathcal{B}_{12} = \{(x_1, 0) : x_1 \geq \bar{x}_1\}, \quad \text{for some } \bar{x}_1 \geq 0.$$

In addition, if $h_1 = h_2$,

$$(3.2) \quad \mathcal{B}_{21} = \{(0, x_2) : x_2 \geq \bar{x}_2\}, \quad \text{for some } \bar{x}_2 \geq 0.$$

Proof. We shall prove that if $x_1 \geq 0$, $x_2 > 0$, then $(x_1 + 1, x_2) \notin \mathcal{B}_{12}$. Then only the states of the form $(x_1 + 1, 0)$, $x_1 \geq 0$, can belong to \mathcal{B}_{12} . However, the monotonicity property of F_{12} (i.e. if $(x_1 + 1, 0) \in \mathcal{B}_{12}$, then $(x'_1 + 1, 0) \in \mathcal{B}_{12}$, for $x_1 \leq x'_1$) implies that there exists a number $\bar{x}_1 \geq 0$ such that (3.1) holds.

To this end, let the initial state be $(x_1 + 1, x_2)$, $x_1 \geq 0$, $x_2 > 0$. We call the $(x_1 + 1)$ st job in station 1 the *tagged job*. We prove that the tagged job will not jockey to station 2 if it is non-empty, under the optimal jockeying policy. From (2.8), we need to show that

$$(3.3) \quad \bar{V}(x_1 + 1, x_2) - \bar{V}(x_1, x_2 + 1) - c_{12} = \bar{\Delta}(x_1, x_2) - c_{12} \leq 0,$$

for all $x_1 \geq 0$, $x_2 > 0$.

To this end, let $X = (X_1(t), X_2(t))$ and $Y = (Y_1(t), Y_2(t))$ be the queue length processes with initial states $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2)$, and let process X follow the optimal routing and jockeying policies throughout, starting from the next period. Note that the tagged job is in station 2 in process X and in station 1 in process Y . Let $\delta_j^X(\delta_j^Y)$ be the first time that the queue length process $X_j(t)$ ($Y_j(t)$) becomes empty, $\delta_j^X = \min \{t : X_j(t) = 0\}$ ($\delta_j^Y = \min \{t : Y_j(t) = 0\}\}$, $j = 1, 2$.

We couple the two processes as follows. First, we let the tagged job in either process have the *last priority* to use the server in its station; i.e. the tagged job is served only when no other waiting jobs are in the station and is preempted as soon as another job arrives to the station during the service of the tagged job. Due to the memoryless property of the exponential distribution, the expected cost of either process subject to these ‘shufflings’ and ‘preemptions’ is identical to that of the process without the shufflings and preemptions. Second, we let each job in process Y , except the tagged job, follow the same (routing and jockeying) decision as its counterpart in process X and assume that each job has the same service time realization in both processes. Third, we let the tagged job in process X have the *first priority* to jockey to station 1. In other words, the first job to be switched from

station 2 to station 1 in process X is the tagged job, provided that the tagged job is still in station 2 when such an action takes place. Finally, let the tagged job in process Y behave as follows. It remains in station 1 until time $\sigma := \min\{\sigma_1, \delta_1^Y, \delta_2^Y\}$ and follows the same decision as its counterpart in X afterwards if its service has not been completed at σ , where σ_1 is the time that the tagged job in process X jockeys to station 1 ($\sigma_1 = \infty$ if the event never occurs). Note that σ is the time that either the tagged job in X jockeys to station 1 ($\sigma = \sigma_1$), or the tagged job in Y completes its service ($\sigma = \delta_1^Y$), or the tagged job in X becomes the first job in station 2 ($\sigma = \delta_2^Y$). Since the tagged job in Y will make the same decision as its counterpart in X after σ , it will remain in station 1 if $\sigma = \sigma_1$ and jockey to station 2 if $\sigma = \delta_2^Y$.

We compute the cost difference of the coupled processes. Because each job in process Y , except the tagged job, follows the same decision as its counterpart in process X , and their services are not affected by the presence of the tagged job (which has the last priority to access a server), the cost of any given job (except the tagged job) in both processes is the same. Thus the cost difference of the coupled processes is that of the tagged job in those two processes. Next we show that the tagged job in process Y is better off than its counterpart in process X under every sample path.

If $\sigma = \delta_1^Y$, then the tagged job in Y completes its service before its counterpart in X , and the cost difference of the tagged job is at least

$$c_{12} + (h_2 - h_1) \int_0^\sigma e^{-\alpha t} dt \geq 0.$$

If $\sigma = \sigma_1$, then the tagged job in process X jockeys to station 1 at time $\sigma = \sigma_1$ and the cost difference is

$$c_{12} + c_{21}e^{-\alpha\sigma} + (h_2 - h_1) \int_0^\sigma e^{-\alpha t} dt \geq 0.$$

If $\sigma = \delta_2^Y$, then the tagged job in process Y jockeys to station 2 at time $\sigma = \delta_2^Y$ and the cost difference is

$$c_{12}(1 - e^{-\alpha\sigma}) + (h_2 - h_1) \int_0^\sigma e^{-\alpha t} dt \geq 0.$$

This proves (3.3) and henceforth (3.1).

Finally, if $h_1 = h_2$, then we must have $h_1 \leq h_2$ and $h_2 \leq h_1$, and both (3.1) and (3.2) follow the just-established result.

Remark 1. The implications of Theorem 2 are twofold. First, it significantly reduces the computational effort in searching for the optimal jockeying functions, for \mathcal{B}_{12} now is determined by a single threshold \bar{x}_1 . If $h_1 = h_2$, \mathcal{B}_{21} is also determined by a single threshold \bar{x}_2 . Second, it challenges the wisdom of some widely adopted jockeying rules, such as the shortest-queue jockeying rule. Most proposed jockeying rules are based on the belief that the system efficiency is achieved through queue

length balance. Our result suggests that jockeying improves system performance by reducing server idleness for symmetric systems and by efficiently using the fast or the low-cost station for asymmetric systems. Thus although jockeying is a valuable option, it should be cautiously implemented to eliminate ineffective jockeying among the stations.

Our next theorem considers the asymptotic behavior of the optimal routing function F . It might be conceivable that F would increase without bound as x_1 approaches infinity such that a new job will inevitably be sent to station 2 when station 1 is too crowded. However, this intuition is not true for most cases. To understand the asymptotic behavior of F , we first define the convergence of a function f in the following sense.

Definition. We say that a function $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ possesses a *finite asymptote* if either of the following equations holds:

(i) there exists a finite x_1^* such that

$$(3.4) \quad f(x_1) = \infty \quad \text{for all } x_1 \geq x_1^*;$$

(ii) there exists a finite x_2^* such that

$$(3.5) \quad f(x_1) \leq x_2^* < \infty \quad \text{for all } x_1 \geq 0.$$

Otherwise, we say f does not have a finite asymptote; that is, f is finite when x_1 is finite and approaches infinity when x_1 does.

The next theorem states that there exists an optimal routing function F that possesses a finite asymptote except when both stations have the same holding cost and strictly positive jockeying costs.

Theorem 3.

- (a) If $h_1 < h_2$, then F possesses a finite asymptote; in fact, (3.5) holds.
- (b) If $h_1 = h_2$, then exists an F that has a finite asymptote if and only if either $c_{12} = 0$ or $c_{21} = 0$ or both.

Proof. Suppose a job upon its arrival observes state (x_1, x_2) . For convenience, we call the new job the *tagged job*.

(a) Note that if F_{21} satisfies (3.5), so does F , because $F \leq F_{21}$. Thus we may assume

$$(3.6) \quad \lim_{x_1 \rightarrow \infty} F_{21}(x_1) = \infty.$$

In other words, a job in station 2 will not jockey to station 1 when x_2 is finite and $x_1 \rightarrow \infty$.

Because $h_1 < h_2$, there exists a finite integer x'_2 such that, for $x_2 > x'_2$,

$$(3.7) \quad (h_2 - h_1) - h_2 \left(\frac{\mu_2}{\mu_2 + \alpha} \right)^{x_2 + 1} > 0.$$

For example, we can take

$$(3.8) \quad x'_2 = \left\lceil \frac{\ln [h_2/(h_2 - h_1)]}{\ln [(\mu_2 + \alpha)/\mu_2]} \right\rceil,$$

which guarantees (3.7) to be positive, where $\lfloor a \rfloor$ is the largest integer less than a .

To prove that F satisfies (3.5), it is sufficient to show that for $x_2 \geq x'_2$, where x'_2 is given in (3.8),

$$(3.9) \quad \liminf_{x_1 \rightarrow \infty} [V(x_1, x_2 + 1) - V(x_1 + 1, x_2)] > 0.$$

Let X , Y , δ_j^X and δ_j^Y , $j = 1, 2$, be defined as in the proof of Theorem 2. Let X follow the optimal routing and jockeying policies throughout. As before, we let the tagged job in either process have the last priority to use the server in its station and the tagged job in process X have the first priority to jockey to station 2 when it remains in the system. Let each job other than the tagged job in process Y follow the same (routing and jockey) decision as its counterpart in process X . Let the tagged job in process Y take no jockeying action throughout.

As we argued before, the cost difference of the coupled processes reduces to that of the tagged job in both processes; hence to prove (3.9) we only need to show that for $x_2 \geq x'_2$, the holding cost of the tagged job in process X is strictly greater than its counterpart in process Y , as $x_1 \rightarrow \infty$.

Since for any finite time $t < \infty$, $X_1(t) \rightarrow \infty$ as $x_1 \rightarrow \infty$ and $X_2(t) < \infty$ with probability 1, by (3.6) the tagged job in process X will not jockey to station 1 before t , $0 \leq t < \infty$. Let θ be the time that station 2 in process X finishes $(x_2 + 1)$ jobs, then θ is a gamma random variable with parameters $(x_2 + 1)$ and μ_2 . Since θ is finite with probability 1, the tagged job in X will not jockey to station 1 before θ . Therefore, the holding cost of the tagged job in X is at least

$$(3.10) \quad \begin{aligned} h_2 E \left[\int_0^\theta e^{-\alpha t} dt \right] &= \frac{h_2}{\alpha} [1 - E(e^{-\alpha \theta})] \\ &= \frac{h_2}{\alpha} \left[1 - \left(\frac{\mu_2}{\mu_2 + \alpha} \right)^{x_2 + 1} \right]. \end{aligned}$$

On the other hand, since the tagged job in Y remains in station 1 throughout, it will be completed at time δ_1^Y . Thus

$$(3.11) \quad h_1 E \left[\int_0^{\delta_1^Y} e^{-\alpha t} dt \right] \leq h_1 \int_0^\infty e^{-\alpha t} dt = \frac{h_1}{\alpha}.$$

Combining (3.10) and (3.11) and using (3.7), we reduce (3.9) to

$$\liminf_{x_1 \rightarrow \infty} [V(x_1, x_2 + 1) - V(x_1 + 1, x_2)] \geq \frac{h_2 - h_1}{\alpha} - \frac{h_2}{\alpha} \left(\frac{\mu_2}{\mu_2 + \alpha} \right)^{x_2 + 1} > 0.$$

(b) If jockeying from station 1 to station 2 is costless, $c_{12} = 0$, we can let $F = F_{12}$. Since, by (3.1), F_{12} satisfies

$$F(x_1) = F_{12}(x_1) < 1 \quad \text{for all } x_1 \geq 0,$$

(3.5) holds for F . Similarly, if $c_{21} = 0$, we can let $F = F_{21}$ and (3.4) holds for F , because (3.2) is true if $h_1 = h_2$.

Next we prove by contradiction that neither (3.4) nor (3.5) is valid if $h_1 = h_2$, with $c_{12} > 0$ and $c_{21} > 0$. Note that if we exchange coordinates x_1 and x_2 , (3.4) and (3.5) are symmetric (the exchange is justified because $h_1 = h_2$). Thus, we only consider (3.4), and we also assume that x_1^* is the minimum one satisfying the condition in (3.4). Under (3.4), we clearly have $V(x_1, x_2 + 1) - V(x_1 + 1, x_2) \leq 0$ for all (x_1, x_2) with $x_1 \geq x_1^*$ and $x_2 \geq 0$. Therefore,

$$(3.12) \quad \limsup_{x_2 \rightarrow \infty} [V(x_1, x_2 + 1) - V(x_1 + 1, x_2)] \leq 0, \quad \text{for } x_1 \geq x_1^*.$$

We shall prove inequality (3.12) is false. Let X , Y , δ_j^X , δ_j^Y , and θ be defined as in Theorem 2. We consider two cases: $\mathcal{B}_{21} \neq \phi$ and $\mathcal{B}_{21} = \phi$, where ϕ denotes the null set.

If $\mathcal{B}_{21} \neq \phi$ then, by Theorem 2, (3.2) holds for some finite \bar{x}_2 , because $h_1 = h_2$. We first show that δ_1^X must be finite with probability 1. By our definition of x_1^* , the optimal routing decisions in process X is to always route new jobs to station 2 as long as the number of jobs in station 1 is at least x_1^* . In addition, by Theorem 2, no job will jockey to station 1 before δ_1^X . Therefore, the queue length process $X_1(t)$ before δ_1^X is a birth-death process; its death rate is always μ_1 , and its birth rate may vary in time, but it is always less than or equal to λ and it equals zero if $X_1(t) \geq x_1^*$. Using stochastic dominant arguments (see Proposition 4.2.10 of Stoyan (1983) on stochastic ordering of a birth-death process), it is easily shown that $X_1(t)$ is ergodic, with δ_1^X , the first time $X_1(t)$ reaches 0, being finite with probability 1. Because δ_1^X is finite with probability 1, $X_2(\delta_1^X) > \bar{x}_2$ with probability 1 as $x_2 \rightarrow \infty$. Hence the tagged job in process X will jockey to station 1 at time δ_1^X and the remaining costs of the tagged job in both processes after δ_1^X are identical. Therefore,

$$\liminf_{x_2 \rightarrow \infty} [V(x_1, x_2 + 1) - V(x_1 + 1, x_2)] \geq c_{21} \mathbf{E}[e^{-\alpha \delta_1^X}] > 0.$$

If $\mathcal{B}_{21} = \phi$, then the tagged job will never jockey to station 1. In this case, the completion times of the tagged job in processes X and Y are δ_2^X and δ_2^Y , respectively. Since $\delta_2^X \geq \theta$ and $\theta \rightarrow \infty$ as $x_2 \rightarrow \infty$,

$$(3.13) \quad \begin{aligned} \liminf_{x_2 \rightarrow \infty} [V(x_1, x_2 + 1) - V(x_1 + 1, x_2)] &\geq \liminf_{x_2 \rightarrow \infty} \frac{h_1}{\alpha} [\mathbf{E}(e^{-\alpha \delta_2^Y}) - \mathbf{E}(e^{-\alpha \theta})] \\ &= \liminf_{x_2 \rightarrow \infty} \frac{h_1}{\alpha} \mathbf{E}(e^{-\alpha \delta_2^Y}), \quad \text{for } x_1 \geq x_1^*. \end{aligned}$$

Since process Y follows the optimal policy for process X , process $Y_1(t)$ is a

birth-death process: it starts at $Y_1(0) = x_1 + 1$, its death rate is μ_1 and its birth rate is no larger than λ when $Y_1(t) \leq x_1^*$ and vanishes when $Y_1(t) \geq x_1^* + 1$. Using stochastic dominant arguments, it is elementary to show that $Y_1(t)$ is an ergodic process and δ_1^Y , the first time $Y_1(t)$ reaches 0, is finite with probability 1. Therefore (3.13) is strictly positive. This contradicts (3.12) and establishes (b).

Corollary of Theorem 3(a). If $(h_2 - h_1)/\alpha - c_{21} > 0$, then $F_{21}(x_1)$ satisfies (3.5).

Proof. The proof essentially resembles that for Theorem 3(a); we briefly outline the proof. The same argument as in Theorem 3(a) leads us to

$$\liminf_{x_1 \rightarrow \infty} [\bar{V}(x_1, x_2 + 1) - \bar{V}(x_1 + 1, x_2) - c_{21}] \geq \left(\frac{h_2 - h_1}{\alpha} - c_{21} \right) - \frac{h_2}{\alpha} \left(\frac{\mu_2}{\mu_2 + \alpha} \right)^{x_2+1}.$$

The above expression will be strictly positive for $x_2 > x_2''$, where

$$x_2'' = \left[\frac{\ln [h_2/(h_2 - h_1 - \alpha c_{21})]}{\ln [(\mu_2 + \alpha)/\mu_2]} \right].$$

This proves the corollary.

Remark 2. The result derived in Theorem 3 can be used to aid buffer design. Theorem 3(a) implies that when the holding costs are non-identical, jobs will always be routed to the low-cost station if the number of jobs in the high-cost station is greater than a threshold number. Hence we may provide an infinite buffer to the low-cost station and a finite buffer (its capacity equals the threshold) to the high-cost station. Theorem 3(b) implies that when the holding costs are identical and at least one station has zero jockeying cost, it suffices to provide a single buffer of infinite capacity to the flexible station (in terms of free jockeying). Jobs will be routed to the flexible station unless the other station is idle and the number of jobs in the flexible station exceeds a threshold. In both cases, the station with infinite buffer capacity essentially serves as a cheap, temporary storage place; when the system congestion level is high, most jobs will be stored in this station and moved to the other station later when necessary. It is rather interesting to notice that the holding and jockeying costs, rather than the service rates, completely determine the asymptotic behavior of the optimal routing function.

3.2 Properties of the control functions for the long-run average cost. In this section we show that many properties that hold for the discounted cost problem can be extended to the long-run average cost problem. The following theorem is the counterpart of Theorem 2, for the long-run average cost criterion. Its proof resembles that of Theorem 2 and is omitted.

Theorem 4. Let $h_1 \leq h_2$. Then

$$\mathcal{B}_{12} = \{(x_1, 0) : x_1 \geq \bar{x}_1\}.$$

In addition, if $h_1 = h_2$,

$$\mathcal{B}_{21} = \{(0, x_2) : x_2 \geq \bar{x}_2\}.$$

Our next theorem states that for the long-run average cost, F converges if and only if either the holding costs are non-identical and jockeying to station 2 is permitted, or the holding costs are identical and at least one station has zero jockeying cost.

Theorem 5.

- (a) If $h_1 < h_2$, then $F(x_1)$ converges to a finite asymptote (as in (3.5)) as $x_1 \rightarrow \infty$ if and only if $c_{12} < \infty$.
- (b) If $h_1 = h_2$, then there exists an F that converges to a finite asymptote if and only if either $c_{12} = 0$ or $c_{21} = 0$ or both.

Proof. (a) We only sketch the proof, as it is similar to that of Theorem 3(a).

From our previous result, the optimal routing control is to route a job to station 1 if $h(x_1, x_2 + 1) - h(x_1 + 1, x_2) \geq 0$, and to station 2 otherwise, where

$$h(x_1, x_2) := \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2) - V^\alpha(0, 0)].$$

Thus it is sufficient to show that there exists a finite integer x'_2 such that, for $x_2 \geq x'_2$,

$$(3.14) \quad \liminf_{x_1 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2 + 1) - V^\alpha(x_1 + 1, x_2)] > 0.$$

Because $c_{12} < \infty$ and $h_1 < h_2$, there exists a finite integer x'_2 such that, for $x_2 \geq x'_2$,

$$(3.15) \quad \frac{(h_2 - h_1)x_2}{\mu_2} > c_{12}.$$

We prove that (3.14) holds for $x_2 > x'_2$.

Let X and Y be defined as in Theorem 2. Following the same argument as in the proof of Theorem 3(a), the tagged job in process X will not jockey to station 1 before time θ , where θ is a gamma random variable with parameters x_2 and μ_2 . Let the tagged job in process Y remain in station 1 until θ and then jockey to station 2. As we argued before, the cost difference of the process is that of the tagged job in both processes, which satisfies

$$\begin{aligned} \liminf_{x_1 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2 + 1) - V^\alpha(x_1 + 1, x_2)] &\geq (h_2 - h_1)E(\theta) - c_{12} \\ &= \frac{(h_2 - h_1)x_2}{\mu_2} - c_{12} > 0, \end{aligned}$$

where the last inequality is due to (3.15). Hence (3.14) holds for $x_2 > x'_2$.

Now suppose $c_{12} = \infty$, i.e. jockeying to station 2 is forbidden. We show that

neither (3.4) nor (3.5) can be true. Consider (3.4) first. Following the same argument as in the proof of Theorem 3(b), we only need to find a contradiction to the inequality

$$(3.16) \quad \limsup_{x_2 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2 + 1) - V^\alpha(x_1 + 1, x_2)] \leq 0, \quad \text{for } x_1 \geq x_1^*.$$

As before, we consider $\mathcal{B}_{21} \neq \phi$ and $\mathcal{B}_{21} = \phi$, separately. If $\mathcal{B}_{21} \neq \phi$, then because $F_{21} \geq F$, $F_{21}(x_1)$ satisfies (3.4); in other words, there exists a finite x_1^* such that $F_{21}(x_1^*) = \infty$ for all $x_1 \geq x_1^*$, where x_1^* is the minimum one satisfying the above condition. Because $\mathcal{B}_{21} \neq \phi$, $x_1^* > 0$.

Let $\bar{\delta}$ be the first time $X_1(t)$ reaches $x_1^* - 1$. It can be shown that $\bar{\delta}$ is finite with probability 1. Since $X_2(\bar{\delta}) \rightarrow \infty$ with probability 1 as $x_2 \rightarrow \infty$, the tagged job in process X will jockey to station 1 at time $\bar{\delta}$. This leads to

$$\liminf_{x_2 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2 + 1) - V^\alpha(x_1 + 1, x_2)] \geq c_{21} + (h_2 - h_1)E(\bar{\delta}) > 0,$$

which contradicts (3.16). If $\mathcal{B}_{21} = \phi$,

$$\liminf_{x_2 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1, x_2 + 1) - V^\alpha(x_1 + 1, x_2)] \geq \liminf_{x_2 \rightarrow \infty} [h_2 E(\delta_2^X) - h_1 E(\delta_1^Y)],$$

for $x_1 \geq x_1^*$. However, we can show that as $x_2 \rightarrow \infty$, $E(\delta_2^X) \rightarrow \infty$, and $E(\delta_1^Y) < \infty$. Thus, we reach an inequality from the above expression that contradicts inequality (3.16).

Finally, we need to show that (3.5) is false when $h_1 < h_2$ and $c_{12} = \infty$. For this purpose, we need to find a contradiction to the inequality

$$(3.17) \quad \limsup_{x_1 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1 + 1, x_2) - V^\alpha(x_1, x_2 + 1)] \leq 0, \quad \text{for } x_2 \geq x_2^*,$$

where x_2^* is the minimum one satisfying (3.5). Using the notation developed before, it suffices to show that the cost of the tagged job in X is strictly less than its counterpart in Y . Note that the tagged job in process Y will not jockey to station 2 because $c_{12} = \infty$. Let the tagged job in process X remain in station 2 until it is completed. Then the cost difference of the coupled process is

$$\liminf_{x_1 \rightarrow \infty} \lim_{\alpha \rightarrow 0} [V^\alpha(x_1 + 1, x_2) - V^\alpha(x_1, x_2 + 1)] \geq \liminf_{x_1 \rightarrow \infty} [h_1 E(\delta_1^X) - h_2 E(\delta_2^Y)].$$

However, it can be shown that $E[\delta_2^Y] < \infty$ and $E[\delta_1^X] \rightarrow \infty$ as x_1 does, which contradicts (3.17).

(b) Similar to the proof of Theorem 4(b). We omit the details.

The following corollary gives the sufficient condition under which the optimal jockeying function F_{21} satisfies (3.5). Its proof resembles that of Theorem 5(a) and is omitted.

Corollary of Theorem 5(a). If $h_1 < h_2$ and $c_{ij} < \infty$, $i, j = 1, 2$, and $i \neq j$, F_{21} satisfies (3.5).

Remark 3. It is rather surprising to see that the optimal routing functions for the long-run average problem with jockeying allowed and disallowed exhibit entirely different asymptotic behavior. Xu and Chen (1992) proved that, when jockeying is forbidden, the long-run average optimal routing function does not have a finite asymptote. However, Theorem 5 states that the optimal routing function has a finite asymptotic limit as long as the holding costs are not identical and jobs in the low-cost station are allowed to jockey to the high-cost station. Hence jockeying can be used as a tool to efficiently manage the work-in-process to reduce inventory costs.

Remark 4. The asymptotic limits of the optimal control functions also provide computational advantages. When an optimal control function approaches a finite limit, the optimal control, for large x_1 , will be characterized by a single threshold. The computation of F or F_{21} needs only to be carried out for small or moderate x_1 . It also suggests that, under heavy traffic conditions, simple threshold-type routing and jockeying policies, under which a job is routed to the high-cost station if and only if its queue length is less than a threshold number, and a job jockeys to the low-cost station if and only if the queue length of the high-cost station is greater than a threshold, should compare favorably with the optimal routing and jockeying policies.

4. Special cases

In this section we present some special cases of our model, including some well-known models extensively studied in the literature.

(1) *Symmetric queues with jockeying forbidden.* Suppose that the parameters associated with different queues are exchangeable: $h_1 = h_2$, $\mu_1 = \mu_2$ and $c_{12} = c_{21} = \infty$. Since jockeying is forbidden, our problem reduces to the problem of routing jobs to two identical stations to minimize the discounted or long-run average sojourn time. Routing jobs to symmetric (at least two) queues has been studied by many authors (e.g. Winston 1977, Weber 1978, Farrar 1992, Hordijk and Koole 1990). Very often, it is found that the ‘join-the-shortest-queue’ policy is optimal. In our case, the symmetry of parameters implies that the optimal routing function takes the form $\mathcal{B} = \{(x_1, x_2) : x_1 \geq x_2\}$, which corresponds to the ‘join-the-shortest-queue’ policy.

(2) *Symmetric queues with jockeying allowed.* Suppose that the system is the same as in (1), but $c_{12} = c_{21} < \infty$. By symmetry, the optimal routing policy is the ‘join-the-shortest-queue’ policy. By Theorems 3.3 and 3.5, the optimal jockey policy is of threshold type: there exists a critical number x^* such that a job in station i will jockey to queue j if and only if station j is empty and the number of jobs in station i exceeds x^* , $i \neq j$. If $c_{12} = c_{21} = 0$, then clearly $x^* = 0$. In this case the ‘jockey-to-the-empty-queue’ policy is optimal.

(3) *Asymmetric queues with jockeying forbidden.* Now suppose that holding costs and service rates are different, but jockeying is not allowed: $c_{12} = c_{21} = \infty$. In this case our problem reduces to a special case of the system studied by Hajek, who proved that the optimal routing policy is described by a monotonic switch-over curve. Xu and Chen proved that the optimal switch-over curve has a finite asymptotic limit for the discounted cost problem if and only if $h_1 < h_2$. For the long-run average cost, the switch-over curve does not have a finite asymptote.

(4) *Non-preemptive scheduling of asymmetric queues.* Lin and Kumar and Walrand studied the problem of non-preemptive scheduling of Poisson arrivals on two exponential servers with possibly different service rates. They showed that the optimal policy is to use the fast server whenever available and the slow server if and only if the fast server is busy and the number of customers in the system exceeds a threshold. Let us change the model somewhat by supposing that preemptions in the fast station (station 1) are allowed. The modified model is a special case of ours: let $h_1 = h_2 = h$, $\mu_1 \geq \mu_2$, $c_{12} = 0$ and $c_{21} = \infty$. Suppose that the system is initially empty. Since $h_1 = h_2$, by Theorems 2 and 4, a job will jockey to the slow station (station 2) only when station 2 is empty and the queue length in station 1 exceeds a threshold \bar{x}_1 . Since jockeying from station 1 to station 2 is costless, whereas from station 2 to station 1 is forbidden, we may let $F = F_{12}$, thus the optimal routing policy is to join station 1 unless station 2 is empty and the number of jobs in station 1 is at least \bar{x}_1 . Those routing and jockeying policies coincide with the optimal scheduling policy of Lin and Kumar and Walrand. However, because a job is never preempted from the fast server (i.e. jockeying will never occur when station 1 has a single job and station 2 is empty), it is also a legitimate policy for the original problem, in which preemptions from either station are forbidden. Hence, it must be optimal for that problem also.

(5) *Preemptive scheduling of asymmetric queues.* Suppose $h_1 = h_2$, $c_{12} = c_{21} = 0$ and $\mu_1 \geq \mu_2$. Here zero jockeying costs in both stations imply that preemptions are permitted. The objective is to minimize the sojourn time, within the class of preemptive schedules. The reader might easily be convinced that $\mathcal{B}_{12} = \{(x_1, 0), x_1 \geq 1\}$ and $\mathcal{B}_{21} = \{(x_2, 0), x_2 \geq 0\}$. On the other hand, since jockeying is costless, we may let $F = F_{12}$. The routing and jockeying policies correspond to the non-idling preemptive scheduling policy that give priority to the fast server.

5. Concluding remarks

In this paper we studied the problem of dynamic routing and jockeying in two interacting service stations. We have shown that the optimal routing and jockeying policies are described by non-decreasing functions. We also investigated the asymptotic behavior of control functions.

With some appropriate modifications, our analysis can be extended to the following situations.

- (i) *Increasing and convex holding cost functions.* We may assume that the cost of holding x_j jobs in station j is $h_j(x_j)$, $j = 1, 2$, where $h_j(x)$ is a non-decreasing and convex function of x_j . In this case, the term $h_1x_1 + h_2x_2$ in (2.1) is modified to $h_1(x_1) + h_2(x_2)$, and the term $h_1 - h_2$ in the $\bar{\Delta}$ function (see the first expression in the proof of Proposition 1) is modified to $[h_1(x_1 + 1) - h_1(x_1)] - [h_2(x_2 + 1) - h_2(x_2)]$, and our proofs in Section 2 can still go through. The results in Section 3 are valid under an extra assumption:

$$[h_1(x_1 + 1) - h_1(x_1)] - [h_2(x_2 + 1) - h_2(x_2)] \leq 0, \quad \text{for all } x_1 \geq 0 \text{ and } x_2 \geq 0.$$

i.e. the marginal cost of holding an extra job in station 1 is always less than that in station 2. This ensures that the cost of holding the tagged job in station 1 is no greater than that of holding it in station 2, regardless of the queue lengths in both stations.

- (ii) *Probabilistic routing.* Suppose that routing is not subject to control. A new job, upon arrival, will join station j with probability p_j , $p_1 + p_2 = 1$. However, jobs may jockey between stations. We can modify (2.1) as

$$\begin{aligned} \bar{V}(x_1, x_2) &= h_1x_1 + h_2x_2 + \alpha\mu_1 V([x_1 - 1]^+, x_2) + \alpha\mu_2 V(x_1, [x_2 - 1]^+) \\ &\quad + \alpha\lambda[p_1 V(x_1 + 1, x_2) + p_2 V(x_1, x_2 + 1)] \end{aligned}$$

where $\bar{V}(x_1, x_2)$ satisfies (2.2). Following our proof, it can be shown that the optimal jockeying curve F_{ij} preserves the monotonicity property. We believe that the results developed in this paper remain valid for other simple routing rules such as the ‘join-the-shortest-queue’ rule.

There are some important and interesting directions for future research. In this paper we assume that jockeying is instantaneous. This assumption may not be realistic in some situations. Optimal routing and jockeying policies with random jockeying time appear to be an interesting research topic. Another possibility is to study the system under heavy traffic conditions. The asymptotic behavior of the optimal control functions discovered in this paper seems to suggest that threshold-type routing and jockeying policies are asymptotically optimal under certain conditions (such as $h_1 < h_2$, $c_{12} < \infty$) when the arrival rate λ approaches the total service rate $\mu_1 + \mu_2$. In the case that the routing function F does not have a finite asymptote (e.g. $h_1 = h_2$ and both $c_{ij} > 0$), a natural question is: will F have a linear asymptote when $x_1 \rightarrow \infty$? An affirmative answer to this question can help us to find some simple routing and jockeying policies (e.g. linear control functions) that are asymptotically optimal.

Our results also suggest that it is meaningful to analyze performances of descriptive models under some threshold jockeying rules. For example, suppose that jobs upon arrival will join the shortest queue (or some other simple rules) and jockeying from station i to station j will occur when station j is empty and station i

has at least \bar{x}_i jobs. A queueing analysis of this system allows us to understand the impact of optimal jockeying rules on its performance.

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