Math Review, Discrete-Time Signals, Fourier Analysis of Discrete-Time Signals

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Exercise 1

Part 1: Expressing x[n] in Cosine Form

Considering the discrete-time signal x, sampled with sampling period Ts:

$$x[n] = 2\sin\left(\omega_0 n T_s + \frac{\pi}{6}\right) + \cos(\omega_0 n T_s)$$

To express it in the form $x[n] = A\cos(\omega_0 n T_s + \phi)$*, the following identities must be used:*
$\gamma\sin(\alpha+\beta) = \gamma\sin(\alpha)\cos(\beta) + \gamma\cos(\alpha)\sin(\beta)$ *and* $\gamma\cos(\alpha+\beta) = \gamma\cos(\alpha)\cos(\beta) - \gamma\sin(\alpha)\sin(\beta)$*.*

Using the first identity $\gamma\sin(\alpha+\beta) = \gamma\sin(\alpha)\cos(\beta) + \gamma\cos(\alpha)\sin(\beta)$*, the portion of the signal* $2\sin\left(\omega_0 n T_s + \frac{\pi}{6}\right)$ *can be converted into*

$$2\sin\left(\omega_0 n T_s + \frac{\pi}{6}\right) = 2\sin(\omega_0 n T)\cos\left(\frac{\pi}{6}\right) + 2\cos(\omega_0 n T)\sin\left(\frac{\pi}{6}\right) = \sqrt{3}\sin(\omega_0 n T) + \cos(\omega_0 n T).$$

Substituting the equivalence back into the original equation for $x[n]$*:*

$$x[n] = 2\sin\left(\omega_0 n T_s + \frac{\pi}{6}\right) + \cos(\omega_0 n T_s) = \sqrt{3}\sin(\omega_0 n T) + \cos(\omega_0 n T) + \cos(\omega_0 n T_s) = \sqrt{3}\sin(\omega_0 n T) + 2\cos(\omega_0 n T).$$

Using the second identity $\gamma\cos(\alpha+\beta) = \gamma\cos(\alpha)\cos(\beta) - \gamma\sin(\alpha)\sin(\beta)$*, the constants in* $x[n]$ *can be assigned to the constants in the identity:*

$$\alpha = \omega_0 n T_s$$
$$-\gamma\sin(\beta) = \sqrt{3}$$
$$\gamma\cos(\beta) = 2$$

Isolating for γ the following equivalences can be obtaind: $\gamma = -\frac{\sqrt{3}}{\sin(\beta)}$ *and* $\gamma = \frac{2}{\cos(\beta)}$

Setting the γ equal, β can be solved for:

$$-\frac{\sqrt{3}}{\sin(\beta)} = \frac{2}{\cos(\beta)}$$

$$\tan(\beta) = -\frac{\sqrt{3}}{2}$$

$$\beta = \tan^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

Placing β back into the equation $\frac{2}{\cos(\beta)}$*:*

$$\gamma = \frac{2}{\cos\left(\tan^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)}$$

The values for amplitude A and phase ϕ can be obtained:

$$A = \frac{2}{\cos\left(\tan^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)} = \sqrt{(7)} \approx 2.64575$$

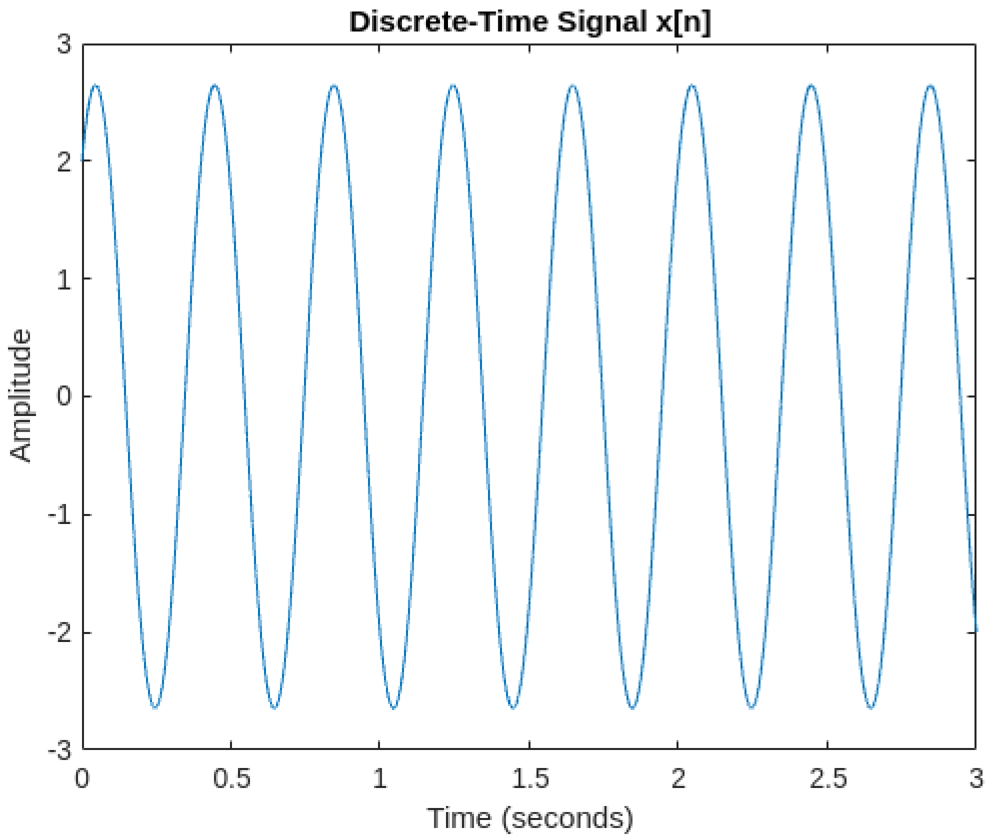
$$\phi = \tan^{-1}\left(-\frac{\sqrt{3}}{2}\right) \approx -0.71372$$

Part 2: Plotting the Signal

Assuming that $\omega_0 = 5\pi$ rad/s and Fs = 44100 Hz, plotting signal *x* within the time interval $[0, 3]$.

```
Fs = 44100; % Sampling frequency in Hz
Ts = 1/Fs; % Sampling period in seconds (interval between successive samples in a
discrete signal)
phi = atan(-sqrt(3)/2);
A = 2/(cos(phi));
w_0 = 5*pi; % rad/s
T = Fs;

n = linspace(0,3,3*T);
x = A*cos(w_0*n+phi);
plot(n, x);
title('Discrete-Time Signal x[n]');
xlabel('Time (seconds)');
ylabel('Amplitude');
```



Sources used to complete this section:

- https://numpy.org/doc/stable/reference/generated/numpy.linspace.html
- https://math.stackexchange.com/questions/645693/finding-amplitude-of-oscillation
- https://www.youtube.com/watch?v=It8R094MyPs&ab_channel=EricCytrynbaum
- https://www.youtube.com/watch?v=sfpvKAXRsnY&ab_channel=EricCytrynbaum

Exercise 2

Considering the Euler's formula, geometric series, and symmetry:

Part 1: Proofs

Showing that

$$\sum_{n=0}^{+\infty} \frac{\sin(nx)}{\alpha^n} = \frac{\alpha \sin(x)}{1 - 2\alpha\cos(x) + \alpha^2} \text{ with } x \in \mathbb{R}, \alpha \in \mathbb{R}^*$$

Using Euler's identities to convert sin *into its expoential equivalent and simplifying:*

$$\sum_{n=0}^{+\infty} \frac{\sin(nx)}{\alpha^n} = \sum_{n=0}^{+\infty} \frac{e^{jnx} - e^{-jnx}}{2j\alpha^n} = \sum_{n=0}^{+\infty} \frac{e^{jnx}}{2j\alpha^n} - \sum_{n=0}^{+\infty} \frac{e^{-jnx}}{2j\alpha^n} = \frac{1}{2j}\left[\sum_{n=0}^{+\infty}\left(\frac{e^{jx}}{\alpha}\right)^n - \sum_{n=0}^{+\infty}\left(\frac{e^{-jx}}{\alpha}\right)^n\right]$$

Using the geometric series $\sum_{n=0}^{+\infty}(a)^n = \dfrac{1}{1-a}$ *to replace the summations above:*

$$\frac{1}{2j}\left[\frac{1}{1-\frac{e^{jx}}{\alpha}}-\frac{1}{1-\frac{e^{-jx}}{\alpha}}\right]=\frac{1}{2j}\left(\frac{1-\frac{e^{-jx}}{\alpha}-1+\frac{e^{jx}}{\alpha}}{\left(1-\frac{e^{jx}}{\alpha}\right)\left(1-\frac{e^{-jx}}{\alpha}\right)}\right)=\frac{1}{2j}\left(\frac{\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}}{1-\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}+\frac{e^{jx-jx}}{\alpha^2}}\right)$$

Using the equivalence $sin(x)=\dfrac{e^{jx}-e^{-jx}}{2j}$ *and* $cos(x)=\dfrac{e^{jx}+e^{-jx}}{2}$ *to convert* $\dfrac{1}{2j}\left(\dfrac{\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}}{1-\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}+\frac{e^{jx-jx}}{\alpha^2}}\right)$*:*

$$\frac{1}{2j}\left(\frac{\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}}{1-\frac{e^{jx}}{\alpha}-\frac{e^{-jx}}{\alpha}+\frac{e^{jx-jx}}{\alpha^2}}\right)=\frac{1}{2j}\left(\frac{e^{jx}-e^{-jx}}{\alpha}\right)\left(\frac{1}{1-\frac{e^{jx}+e^{-jx}}{\alpha}+\frac{1}{\alpha^2}}\right)$$

$$=\frac{1}{\alpha}sin(x)\left(\frac{1}{1-\frac{e^{jx}+e^{-jx}}{\alpha}+\frac{1}{\alpha^2}}\right)=\frac{1}{\alpha}sin(x)\left(\frac{1}{1-\frac{2(e^{jx}+e^{-jx})}{2\alpha}+\frac{1}{\alpha^2}}\right)$$

$$=\frac{\alpha^2}{\alpha^2}\left(\frac{\frac{1}{\alpha}sin(x)}{1-\frac{2cos(x)}{\alpha}+\frac{1}{\alpha^2}}\right)=\left(\frac{\alpha sin(x)}{\alpha^2-2\alpha cos(x)+1}\right)=\frac{\alpha\sin(x)}{1-2\alpha\cos(x)+\alpha^2}$$

Part 2: Computing the allowable values for alpha

The equation $\dfrac{\alpha\sin(x)}{1-2\alpha\cos(x)+\alpha^2}$ *is undefined when the denominator is* 0*. This occurs when* $1-2\alpha\cos(x)+\alpha^2=0$*, which can be solved using the quadratic formula* $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ *in terms of* α*. Plugging the values* $a=1$*,* $b=-2cos(x)$*,* $c=1$ *into the formula, it can be shown that*

$$\alpha=\frac{--cos(x)\pm\sqrt{(cos(x))^2-4(1)(1)}}{2(1)}=\frac{2cos(x)\pm2\sqrt{(cos(x))^2-1}}{2}=cos(x)\pm\sqrt{(cos(x))^2-1}$$

Observing that the range of $\cos^2(x)$ *is* $[0,1]$*, this leaves the only possible value that will result in a non-negative value under square root to be 1. The value of x that are permitted are therefore* $x_1=0; x_2=\pi$*. Using those values, the square root is zero, leaving only the outside cosine and resulting in* $\alpha_1=1; \alpha_2=-1$ *which will lead to an undefined result.*

The allowable values are therefore $\alpha\in\mathbb{R}\setminus\{1,-1\}$.

Part 3: Odd Function

To show mathematically that the function is **odd***, the equivalence* $f(-x)=-f(x)$ *must be shown. The following identities will be used:* $cos(-x)=cos(x)$ *and* $sin(-x)=-sin(x)$*.*

$$f(x) = \frac{\alpha sin(x)}{1 - 2\alpha cos(x) + \alpha^2}$$

$$f(-x) = \frac{\alpha sin(-x)}{1 - 2\alpha cos(-x) + \alpha^2} = \frac{-\alpha sin(x)}{1 - 2\alpha cos(x) + \alpha^2} = -f(x)$$

Sources used to complete this section:

- https://amsi.org.au/ESA_Senior_Years/SeniorTopic2/2c/2c_2content_2.html#:~:text=Definitions,in%20the%20domain%20of%20f.
- https://www.cuemath.com/geometric-series-formula/
- https://www.purplemath.com/modules/quadform.htm
- https://www.varsitytutors.com/trigonometry-help/quadratic-formula-with-trigonometry

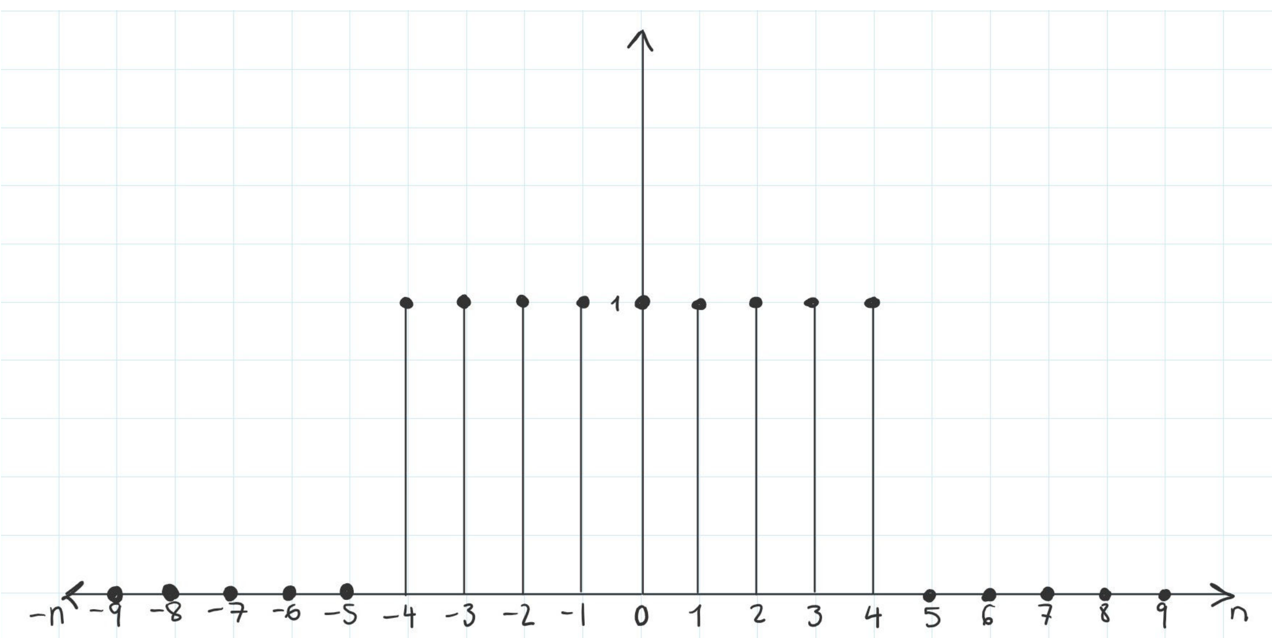
Exercise 3

Considering the signal x represented by the following discrete-time sequence:

$$x[n] = \begin{cases} 1 & \text{for } n \in [-N+1, N-1] \\ 0 & \text{for } n \notin [-N+1, N-1] \end{cases}$$

Part 1: Drawing the signal

Drawing the signal x for $N = 5$:



Part 2: Discrete-Time Fourier Transform (DTFT) of signal x

Computing $X(\omega)$, the Discrete-Time Fourier Transform (DTFT) of $x[n]$:

Using the linearity property, $x[n]$ can be split up into three different sections.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{-5} (0 \cdot e^{-j\omega n}) + \sum_{n=-4}^{4} (1 \cdot e^{-j\omega n}) + \sum_{n=5}^{\infty} (0 \cdot e^{-j\omega n}) = \sum_{n=-4}^{4} (1 \cdot e^{-j\omega n})$$

Using properties of geometric series:

$$\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a} = a\left(\sum_{n=0}^{N-1} a^n\right) = a\left(\frac{1 - a^{(N-1)+1}}{1 - a}\right) = a\left(\frac{1 - a^N}{1 - a}\right)$$

The equation can be expanded and converted to two terms containing exponentials. Note that from the original constrants $(-N+1, N-1)$ will be replaced with $(-M, M)$ for the purposes of simplying the calculations.

$$X(\omega) = \sum_{n=-M}^{M} (1 \cdot e^{j\omega n}) = \sum_{n=-M}^{M} e^{j\omega n} = e^{j\omega M} \sum_{n=0}^{2M} e^{-j\omega n} = e^{j\omega M}\left(\frac{1 - e^{-j\omega(2M+1)}}{1 - e^{-j\omega}}\right)$$

$$= e^{j\omega M}\left(\frac{e^{-j\omega\frac{2M+1}{2}} e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}} e^{-j\omega\frac{2M+1}{2}}}{e^{-j\omega\frac{1}{2}} e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}} e^{-j\omega\frac{1}{2}}}\right) = \left(\frac{e^{j\omega M} e^{-j\omega\frac{2M+1}{2}}}{e^{-j\omega\frac{1}{2}}}\right)\left(\frac{e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}}{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}}\right)$$

Using properties of exponentials $e^{j\omega} \cdot e^{-j\omega} = e^0 = 1$ and $e^{j\omega} = e^{j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}}$ to reduce the factorized term $\left(\frac{e^{j\omega M} e^{-j\omega\frac{2M+1}{2}}}{e^{-j\omega\frac{1}{2}}}\right)$:

$$\left(\frac{e^{j\omega M} e^{-j\omega\frac{2M+1}{2}}}{e^{-j\omega\frac{1}{2}}}\right) = \frac{e^{j\omega M}\left(e^{-j\omega\left(\frac{2M}{2}+\frac{1}{2}\right)}\right)}{e^{-j\frac{\omega}{2}}} = \frac{e^{j\omega M - j\omega\left(\frac{2M}{2}+\frac{1}{2}\right)}}{e^{-j\frac{\omega}{2}}} = e^{j\omega M - j\omega\left(\frac{2M}{2}+\frac{1}{2}\right) + j\frac{\omega}{2}} = e^{j\omega M - j\omega\frac{2M}{2} - j\frac{\omega}{2} + j\frac{\omega}{2}} = e^0 = 1$$

Using Euler's identity $\sin(\omega) = \dfrac{e^{j\omega} - e^{-j\omega}}{2j}$, the $\left(\frac{e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}}{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}}\right)$ can be additionally reduced:

$$\left(\frac{e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}}{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}}\right) = \left(\frac{\sin\left(\omega\left(M + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}\right)$$

Replacing back $M = N - 1$, the final solution of the DTFT is therefore:

$$X(\omega) = \frac{\sin\left[\omega\left(M + \frac{1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)} = \frac{\sin\left[\omega\left(N - \frac{1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}$$

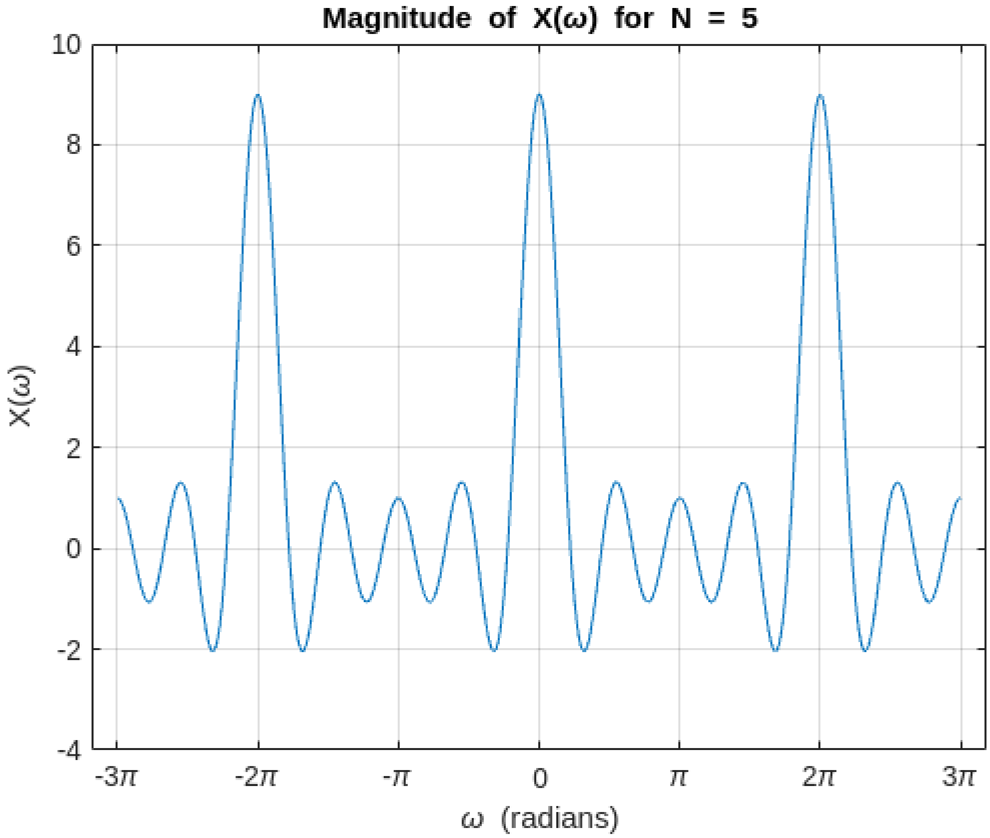
Part 3: Plotting the DTFT

Ploting $X(\omega)$**, for** $\omega \in [-3\pi, 3\pi]$ **and** $N = 5$

```
omega = linspace(-3*pi, 3*pi, 1000);
N = 5;
X_omega = sin(omega .* (N - 0.5)) ./ sin(omega / 2);

figure;
plot(omega, X_omega);
title('Magnitude of X(\omega) for N = 5');
xlabel('\omega (radians)');
ylabel('X(\omega)');

% Setting x-ticks to multiples of pi
xticks([-3*pi -2*pi -pi 0 pi 2*pi 3*pi]);
xticklabels({'-3\pi', '-2\pi', '-\pi', '0', '\pi', '2\pi', '3\pi'});
grid on;
```



Part 4: Discrete-Time Fourier Transform (DTFT) of signal y

Computing $Y(\omega)$, the DTFT of signal y:

$$y[n] = \begin{cases} 1 & \text{for } n \in [0, 2N-2] \\ 0 & \text{for } n \notin 0, 2N-2] \end{cases}$$

Using the shifting property of the DTFT the solution from part 2 can be modified by multiplying by an exponential with the shift amount in its expoent. The rectangular window function is now between $[0, 8]$ whereas the previous one was between $[-4, 4]$. It was therefore shifted by $-N+1$.

$$Y(\omega) = e^{-j\omega(-N+1)}\left(\frac{\sin\left(\omega\left(N-\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}\right) = e^{j\omega(N-1)}\left(\frac{\sin\left(\omega\left(N-\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}\right)$$

Sources used to complete this section:

- https://www.youtube.com/watch?v=GKKv9T-noO0&ab_channel=MichelvanBiezen
- https://home.engineering.iastate.edu/~julied/classes/ee524/LectureNotes/l5.pdf
- https://www.site.uottawa.ca/~jpyao/courses/ELG3120_files/ELG3125-Formula-Sheets.pdf
- https://www.mathworks.com/help/matlab/creating_plots/change-tick-marks-and-tick-labels-of-graph-1.html

Exercise 4

Computing the Inverse Discrete-Time Fourier Transform of a signal z, the expression of which is $Z(\omega)$:

$$Z(\omega) = \frac{A}{1 - e^{(\alpha + j(\omega - \omega_0))}} + Be^{-j\omega n_0} \quad \text{with} \quad \alpha < 0$$

$$z[n] = \mathcal{F}^{-1}(Z(\omega)) = \mathcal{F}^{-1}\left\{\frac{A}{1 - e^{(\alpha + j(\omega - \omega_0))}} + Be^{-j\omega n_0}\right\}$$

Linearity Property

$$z[n] = \mathcal{F}^{-1}\left\{\frac{A}{1 - e^{(\alpha + j(\omega - \omega_0))}}\right\} + \mathcal{F}^{-1}\left\{Be^{-j\omega n_0}\right\}$$

$$z[n] = (A)\mathcal{F}^{-1}\left\{\frac{1}{1 - e^{(\alpha + j(\omega - \omega_0))}}\right\} + (B)\mathcal{F}^{-1}\left\{e^{-j\omega n_0}\right\}$$

Shifting Property

$$z[n] = (A)\mathcal{F}^{-1}\left\{\frac{1}{1 - e^{(\alpha + j(\omega - \omega_0))}}\right\} + (B)\mathcal{F}^{-1}\{1\}[n - n_0]$$

Delta Dirac Replacement

$$z[n] = (A)\mathscr{F}^{-1}\left\{\frac{1}{1 - e^{\left(\alpha + j\left(\omega - \omega_0\right)\right)}}\right\} + (B)\{\delta[n]\}[n - n_0]$$

Applying Time Shift to the Delta Dirac

$$z[n] = (A)\mathscr{F}^{-1}\left\{\frac{1}{1 - e^{\left(\alpha + j\left(\omega - \omega_0\right)\right)}}\right\} + B\delta[n - n_0]$$

Frequency Shifting Property

$$z[n] = \left(Ae^{j\omega_0 n}\right)\mathscr{F}^{-1}\left\{\frac{1}{1 - e^{(\alpha + j\omega)}}\right\} + B\delta[n - n_0]$$

$$z[n] = \left(Ae^{j\omega_0 n}\right)\mathscr{F}^{-1}\left\{\frac{1}{1 - e^{\alpha}e^{j\omega}}\right\} + B\delta[n - n_0]$$

Unit Function Replacement

$$z[n] = Ae^{j\omega_0 n}e^{\alpha n}u[n] + B\delta[n - n_0]$$

$$z[n] = Ae^{n\left(j\omega_0 + \alpha\right)}u[n] + B\delta[n - n_0]$$

Sources used to complete this section:

- https://www.site.uottawa.ca/~jpyunit functionao/courses/ELG3120_files/ELG3125-Formula-Sheets.pdf
- https://www.site.uottawa.ca/~jpyao/courses/ELG3120_files/ch5.pdf
- https://tex.stackexchange.com/questions/113855/laplace-and-fourier-transforms