Z-transform and Frequency Response

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Exercise 1

The z-transform is computed for the following sequences. For each sequence, all the values of z that make the summation exist form a region of convergence (ROC) which is additionally provided.

Sequence 1:

$x[n] = u[n-5]$

Using the Shift/Delay Property:

$$X(z) = Z(x[n]) = Z(u[n-5]) = z^{-5}Z(u[n])$$

Using the z-transform table:

$$X(z) = z^{-5}Z(u[n]) = z^{-5}\frac{1}{1-z^{-1}} = \frac{z^{-5}}{1-z^{-1}} = \frac{1}{z^5 - z^4}$$

Since the shifting has no effect on the ROC, the region of convergence will therefore be $|z| > 1$.

Sequence 2:

$x[n] = (-0.6)^n u[n]$

Using the z-transform table:

$$X(z) = Z((-0.6)^n u[n]) = \frac{1}{1-(-0.6)z^{-1}} = \frac{1}{1+(0.6)z^{-1}} = \frac{z}{z+0.6}$$

The since the ROC is $|z| > |-0.6|$, *the region of convergence can be simplified to* $|z| > 0.6$.

Sequence 3:

$x[n] = (-0.6)^n(u[n] - u[n-5])$

Note that $u[n] - u[n-5]$ *is equal to* $\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

The $x[n]$ *can therefore be simplified to* $x[n] = (-0.6)^n(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$. *This equation shows that the* $x[n]$ *is a right-sided, finite sequence and its ROC is expected to be of the form* $|z| > r$.

Using the Linearity Property, the z-transform can be computed separately for $(-0.6)^n(u[n])$ *and* $-(-0.6)^n(u[n-5])$.

$$Z((-0.6)^n(u[n])) = \frac{z}{z+0.6}$$

Using the Shift/Delay Property $(-0.6)^n(u[n-5])$ *can be transformed into:*

$$Z(-(-0.6)^n(u[n-5])) = -z^{-5}(-0.6)^{n+5}u[n] = -z^{-5}(-0.6)^5(-0.6)^n u[n] = -z^{-5}\frac{(-0.6)^5 z}{z+0.6} = \frac{0.07776}{z^5 + 0.6z^4}$$

$$X(z) = \frac{z}{z+0.6} + \frac{0.07776}{z^5 + 0.6z^4}$$

Since the z-transform is a power series, it converges when $x[n]z^{-n}$ *is absolutely summable (in other words:* $\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$*). Plugging the signal into the formula, the final ROC is all values of* z *except for* $|z| \neq 0$. *The detailed derivation can be seen below.*

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

$$\sum_{n=-\infty}^{\infty} |(-0.6)^n(u[n] - u[n-5])z^{-n}| < \infty$$

$$\sum_{n=-\infty}^{\infty} |(-0.6z^{-1})^n(u[n] - u[n-5])| < \infty$$

$$\sum_{n=0}^{5} |(-0.6z^{-1})^n| < \infty$$

$$\sum_{n=0}^{5} |\left(-\frac{0.6}{z}\right)^n| < \infty$$

Sequence 4:

$x[n] = \alpha^n sin(\omega_0 n)u[n]$

Using the z-transform table:

$$X(z) = \frac{[\alpha sin(\omega_0)]z^{-1}}{1 - [2\alpha cos(\omega_0)]z^{-1} + \alpha^2 z^{-2}} = \frac{[\alpha sin(\omega_0)]z}{z^2 - [2\alpha cos(\omega_0)]z + \alpha^2}$$

The ROC for this z-transform is $|z| > \alpha$.

Sources used to complete this section:

- https://www.youtube.com/watch?v=Vu5PLKt2_LQ&t=183s&ab_channel=BarryVanVeen
- https://www.youtube.com/watch?v=4ZYIHTcdB8Q&ab_channel=BarryVanVeen
- https://www.slideshare.net/geethannadurai/transforms-190210047
- https://www.youtube.com/watch?v=XJRW6jamUHk&ab_channel=MATLAB
- https://www.tutorialspoint.com/signals_and_systems/z_transforms_properties.htm
- https://www.youtube.com/watch?v=8fFnBAX8aeg&ab_channel=IainExplainsSignals%2CSystems%2CandDigitalComms
- https://ocw.mit.edu/courses/6-003-signals-and-systhttps://timeseries.sci.muni.cz/index.php?pg=chapter-6--z-transformems-fall-2011/64490a008c1c5c25c86044351465abf7_MIT6_003F11_lec05.pdf
- https://timeseries.sci.muni.cz/index.php?pg=chapter-6--z-transform

Exercise 2

Below is a second-order FIR notch filter difference equation:

$$y[n] = x[n] - x[n-1] + x[n-2]$$

Part 1: Computing the Transfer Function

Evaluating for the transfer function expression (in the z-domain) of the second-order FIR notch filter.

$$y[n] = x[n] - x[n-1] + x[n-2]$$
$$Y(z) = X(z) - z^{-1}X(z) + z^{-2}X(z)$$
$$Y(z) = X(z)(1 - z^{-1} + z^{-2})$$
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1} + z^{-2}$$

Multiplying by $1 = \frac{z^2}{z^2}$ the transfer function becomes $H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 - z^1 + 1}{z^2}$.

Part 2: Computing the Poles and Zeros

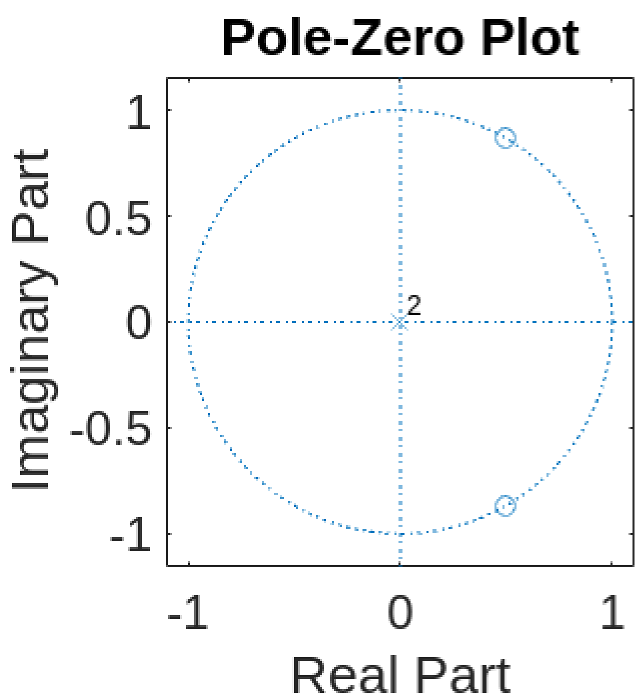
Computing the poles and zeros of $H(z) = \frac{z^2 - z^1 + 1}{z^2}$.

The zeros of a transfer function are the values of z for which $H(z) = 0$, while the poles are the values of z for which $H(z) = \infty$. The transfer function is zero when the numerator is zero such that $z^2 - z^1 + 1 = 0$. Using the formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the zeros of the $H(z)$ occur when $z = \frac{- -1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. The transfer function is ∞ when the denominator is $z^2 = 0$, so $H(z) = 0$ when $z = 0$.

Part 3: Drawing the Poles and Zeros in MATLAB

```
% Coefficients of the transfer function H(z)
zeros = [1, -1, 1]; % Numerator coefficients (zeros)
poles = [1, 0, 0]; % Denominator coefficients (poles)

figure('pos',[0,0,500,500],'DefaultAxesFontSize',18);
zplane(zeros, poles);
```



Part 4: Proof Frequency Response

Showing that the frequency response of the second-order FIR notch filter can be written as $H(e^{j\omega}) = e^{-j\omega}(2cos(\omega) - 1)$ *using the identity* $cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}$.

The frequency response of the second-order FIR notch filter is derived below:

$y[n] = x[n] - x[n-1] + x[n-2]$

$Y(\omega) = X(\omega) - e^{-j\omega}X(\omega) + e^{-2j\omega}X(\omega)$

$Y(\omega) = X(\omega)(1 - e^{-j\omega} + e^{-2j\omega})$

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = (1 - e^{-j\omega} + e^{-2j\omega})$

Multiplying $H(\omega)$ *by* $\frac{e^{-j\omega}}{e^{-j\omega}}$:

$H(\omega) = \frac{e^{-j\omega}}{e^{-j\omega}}(1 - e^{-j\omega} + e^{-2j\omega}) = e^{-j\omega}(\frac{1}{e^{-j\omega}} - \frac{e^{-j\omega}}{e^{-j\omega}} + \frac{e^{-2j\omega}}{e^{-j\omega}}) = e^{-j\omega}(e^{j\omega} - 1 + e^{-j\omega}) = e^{-j\omega}(e^{j\omega} + e^{-j\omega} - 1)$

Using $cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}$ *but slightly rearranged* $2cos(\omega) = e^{j\omega} + e^{-j\omega}$*:*

$$H(\omega) = e^{-j\omega}(e^{j\omega} + e^{-j\omega} - 1) = e^{-j\omega}(2cos(\omega) - 1)$$

Part 5: Proof Phase

Starting with the equation for the frequency response: $H(\omega) = 1 - e^{-j\omega} + e^{-2j\omega}$ *and using the identity* $e^{j\omega} = cos(\omega) + jsin(\omega)$*, the* $e^{j\omega}$ *values inside* $H(\omega)$ *can be replaced to show that* $\angle H(e^{j\omega}) = tan^{-1}\left(\frac{sin(\omega) - sin(2\omega)}{1 - cos(\omega) + cos(2\omega)}\right)$*. Additional properties such as the Even/Odd Property of trigonometric functions and the polar form are used.*

$$\begin{aligned}
H(\omega) = &\ 1 - e^{-j\omega} + e^{-2j\omega} \\
= &\ 1 - (\cos(-\omega) + j\sin(-\omega)) + (\cos(-2\omega) + j\sin(-2\omega)) \\
= &\ 1 - \cos(-\omega) - j\sin(-\omega) + \cos(-2\omega) + j\sin(-2\omega) \\
= &\ 1 - \cos(-\omega) + \cos(-2\omega) - j\sin(-\omega) + j\sin(-2\omega) \\
= &\ (1 - \cos(-\omega) + \cos(-2\omega)) + j(-\sin(-\omega) + \sin(-2\omega)) \\
= &\ (1 - \cos(\omega) + \cos(2\omega)) + j(\sin(\omega) - \sin(2\omega))
\end{aligned}$$

Using polar form to compute the angle where $1 - cos(-\omega) + cos(-2\omega)$ *is Real and* $j(sin(-\omega) - sin(-2\omega))$ *is Imaginary:*

$$\angle H(e^{j\omega}) = \theta = tan^{-1}\left(\frac{Imaginary}{Real}\right) = tan^{-1}\left(\frac{sin(\omega) - sin(2\omega)}{1 - cos(\omega) + cos(2\omega)}\right)$$

Part 6: Determining Phase at Different Omega Values

```
% Convert degrees to radians for the computation
omega1 = 59.999 * (pi / 180);
omega2 = 60.001 * (pi / 180);

% Compute phase angles in radians
phase1 = atan2(sin(omega1)-sin(2*omega1), 1-cos(omega1)+cos(2*omega1));
phase2 = atan2(sin(omega2)-sin(2*omega2), 1-cos(omega2)+cos(2*omega2));

% Convert phase angles from radians to degrees
phase1_deg = phase1 * (180 / pi);
phase2_deg = phase2 * (180 / pi);

% Compute the jump in phase near 60 degrees and check how close it is to 180 degrees
phase_jump = abs(phase2_deg-phase1_deg);
difference = abs(phase_jump-180);

fprintf('The phase at omega1 = %.3f degrees\n', phase1_deg);
```

```
The phase at omega1 = -59.999 degrees
```

```
fprintf('The phase at omega2 = %.3f degrees\n', phase2_deg);
```

```
The phase at omega2 = 119.999 degrees
```

```
fprintf('The jump in phase is %.3f degrees\n', phase_jump);
```

```
The jump in phase is 179.998 degrees
```

```
fprintf('The jump in phase is %.3f degrees close to 180\n', difference);
```

```
The jump in phase is 0.002 degrees close to 180
```

Part 7: Drawing the Magnitude and Phase of the Frequency Response

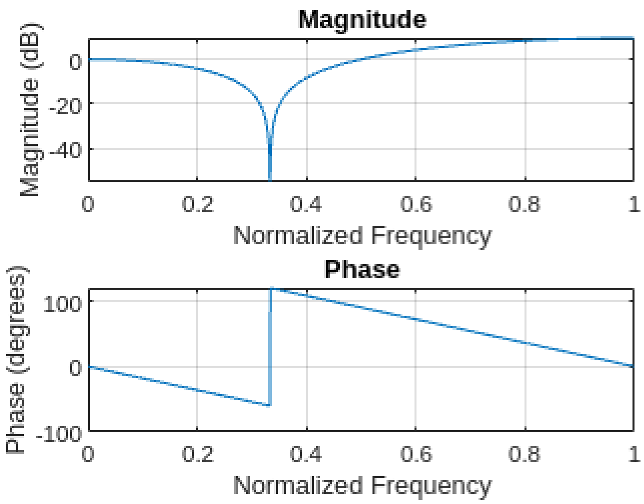
The magnitude and phase plots show the expected linear phase of a FIR filter, with a 180 phase jump when the frequency crosses the zero, similar to what was observed in the analytical results.

```
% Define the filter coefficients
b = [1 -1 1];
a = 1;

[h, w] = freqz(b, a, 1024);

% Plot the magnitude response in decibels (dB)
figure;
subplot(2,1,1);
plot(w/pi, 20*log10(abs(h))); % Normalize frequency by pi and convert magnitude to
dB
title('Magnitude');
xlabel('Normalized Frequency');
ylabel('Magnitude (dB)');
grid on;

% Plot the phase response in degrees
subplot(2,1,2);
plot(w/pi, angle(h)*(180/pi)); % Convert phase angle to degrees
title('Phase');
xlabel('Normalized Frequency');
ylabel('Phase (degrees)');
grid on;
```



Sources used to complete this section:

- https://www.youtube.com/watch?v=IJhyJGjeLvA&ab_channel=DrWaleedAl-Nuaimy
- https://www.youtube.com/watch?v=Vu5PLKt2_LQ&ab_channel=BarryVanVeen
- https://dsp.stackexchange.com/questions/9138/finding-the-frequency-response-of-a-filter-defined-as-a-z-domain-transfer-functi

Exercise 3

The following information is known about a linear time-invariant system:

a) The system is causal.

b) When the input signal x[n] is $x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$**, the output signal is** $y[n]$**.**

It is known that the z-transform of $y[n]$ **is** $Y(z) = \dfrac{1 - z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$.

Part 1: Z-Transform and ROC

Computing the $X(z)$**, the z-transform of** $x[n]$**, and specifying its region of converge below.**

Using the Linearity Property, the $-\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$ *and* $-\frac{4}{3}(2)^n u[-n-1]$ *parts of* $x[n]$ *can be used to compute the z-trasnform separately before combining.*

Taking the z-transform of $-\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$ *can be done by using the property* $Aa^n u[n] = \dfrac{A}{1 - az^{-1}} = \dfrac{Az}{z - a}$ *and ROC is* $|z| > |a|$*:*

$$Z(-\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} = \frac{-\frac{1}{3}z}{z-\frac{1}{2}} = -\frac{z}{3(z-\frac{1}{2})}$$ *with an ROC is* $|z| > |\frac{1}{2}|$

Taking the z-transform of $-\frac{4}{3}(2)^n u[-n-1]$ *can be done by using the property* $Aa^n u[-n-1] = \frac{-A}{1-az^{-1}} = \frac{-Az}{z-a}$ *and ROC is* $|z| < |a|$*:*

$$Z(-\frac{4}{3}(2)^n u[-n-1]) = -\frac{-\frac{4}{3}}{1-2z^{-1}} = \frac{\frac{4}{3}z}{z-2} = \frac{4z}{3(z-2)}$$ *with an ROC* $|z| < |2|$

The combined $X(z) = -\frac{z}{3(z-\frac{1}{2})} + \frac{4z}{3(z-2)}$ *with an ROC equal to* $|z| > |\frac{1}{2}| \cap |z| < |2|$ *so ROC is* $\frac{1}{2} < |z| < 2$.

Part 2: Inverse Z-Transform

Since the numerator degree is higher than the denominator, long division must be performed to convert the fraction into a form that can be simplified using partial fraction decomposition.

$$Y(z) = \frac{1-z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}$$

After performing long division the new $Y(z)$ *can be simplified further:*

$$Y(z) = -1 + 2\frac{1-\frac{5}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}$$

$$\frac{1-\frac{5}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$1-\frac{5}{4}z^{-1} = A(1-2z^{-1}) + B\left(1-\frac{1}{2}z^{-1}\right)$$

$$1-\frac{5}{4}z^{-1} = A - A2z^{-1} + B - B\frac{1}{2}z^{-1}$$

$$1 = A + B$$

$$-\frac{5}{4}z^{-1} = -A2z^{-1} - B\frac{1}{2}z^{-1} \iff 5 = 8A + 2B$$

The linear system can be solved for values of A and B:

$$\begin{bmatrix} 1 & 1 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$det(M) = 1 * 2 - 1 * 8 = -6$$
$$det(M_A) = 1 * 2 - 1 * 5 = -3$$
$$det(M_B) = 1 * 5 - 1 * 8 = -3$$
$$A = \frac{-3}{-6} = \frac{1}{2}$$
$$B = \frac{-3}{-6} = \frac{1}{2}$$

$$\frac{1 - \frac{5}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} = \frac{1}{2}\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{2}\frac{1}{1 - 2z^{-1}}$$
$$Y(z) = -1 + \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

Based on the new $Y(z)$, the possible values of ROC with there inverse z-transform are shown below:

ROC is $|z| > |\frac{1}{2}| \cap |z| > |2|$

$$y[n] = -\delta[n] + \left(\frac{1}{2}\right)^n u[n] + (2)^n u[n]$$

ROC is $|z| > |\frac{1}{2}| \cap |z| < |2|$

$$y[n] = -\delta[n] + \left(\frac{1}{2}\right)^n u[n] - (2)^n u[-n-1]$$

ROC is $|z| < |\frac{1}{2}| \cap |z| > |2|$

$$y[n] = -\delta[n] - \left(\frac{1}{2}\right)^n u[-n-1] + (2)^n u[n]$$

ROC is $|z| > |\frac{1}{2}| \cap |z| > |2|$

$$y[n] = -\delta[n] - \left(\frac{1}{2}\right)^n u[-n-1]] - (2)^n u[-n-1]$$

Since the system is causal, the ROC found in Part 1 for$X(z)$ must be the same for $Y(z)$. Identifying the inverse z-transform with an ROC of $|z| > |\frac{1}{2}| \cap |z| < |2|$ from the above formulas, the following solution would therefore be chosen to represent $y[n]$:

$$y[n] = -\delta[n] + \left(\frac{1}{2}\right)^n u[n] - (2)^n u[-n-1]$$

Part 3: Impulse Response

To compute the impulse response of the system, the corresponding transfer function $H(z) = \frac{Y(z)}{X(z)}$ *must first be derived using the calculations from the previous sections.*

Simplifying $X(z)$ *such that:*

$$X(z) = \frac{-\frac{1}{3}}{(1 - \frac{1}{2}z^{-1})} + \frac{\frac{4}{3}}{(1 - 2z^{-1})} = \frac{-\frac{1}{3}(1 - 2z^{-1})}{(1 - \frac{1}{2}z^{-1})} + \frac{\frac{4}{3}(1 - \frac{1}{2}z^{-1})}{(1 - 2z^{-1})} = \frac{-\frac{1}{3} + \frac{2}{3}z^{-1} + \frac{4}{3} - \frac{2}{3}z^{-1}}{((1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})|}$$

Computing $H(z)$*:*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1 - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})}}{\frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}} = 1 - z^{-2}$$

Using the inverse z-tranform table where $A = A\delta[n]$ *and* $Az^{-n_0} = A\delta[n - n_0]$ *the impulse response is therefore:*

$h[n] = 1 - z^{-2} = \delta[n] - \delta[n - 2]$

Sources used to complete this section:

- https://dsp.stackexchange.com/questions/87041/finding-the-inverse-z-transform
- https://www.symbolab.com/solver/polynomial-long-division-calculator/long%20division%20%5Cfrac%7B1-x%5E%7B2%7D%7D%7Bx%5E%7B2%7D-2.5x%2B1%7D?or=input
- https://www.maplesoft.com/content/EngineeringFundamentals/10/MapleDocument_10/Transfer%20Functions,%20Poles%20and%20Zeros.pdf
- https://www.quora.com/Is-the-impulse-response-of-a-function-the-same-as-the-inverse-of-the-Laplace-transform
- https://dsp.stackexchange.com/questions/87041/finding-the-inverse-z-transform