# Discrete-Time Systems, Frequency Response

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## Exercise 1

- *Stable/Unstable: every bounded input produces a bounded output for all time in a stable system (BIBO), meaning that from* $-\infty$ *to* $+\infty$ *the amplitude of the signal has to be finite.*
- *Causal/Anti-Causal: the output depends only on the current and past inputs, not on future inputs. An anti-causal system is one where the output depends only on future inputs.*
- *Linear/Non-Linear: obeys the superposition and scalabiliy or in other words called the Law of Additivity and Law of Homogeneity.*
- *Time-Variant/Time-Invariant: the system's response to inputs do not change with time (a shift in the input produces a shift in the output).*
- *Memoryless/Not-Memoryless: the output at a given time is dependent only on the input at that same time.*

### System 1:

$y[n] = e^{nx[n]}$

1. ***Unstable:*** *increasing/decreasing the* $n$ *value causes the output amplitude to increase until* $\pm\infty$.
2. ***Causal:*** *the output at any time* $n$ *depends on current and past values of* $x[n]$.

3. ***Non-Linear:*** *the system does not obey the superposition property. For example, setting* $x[n] = x_1[n] + x_2[n]$ and plugging into $y[n]$, it is expected to get $y_1[n] + y_2[n]$. The derivation shows that $e^{n(x_1[n]+x_2[n])} = e^{nx_1[n]+nx_2[n]} = e^{nx_1[n]} * e^{nx_2[n]} \neq y_1[n] + y_2[n]$.
4. ***Time Variant:*** *shift the* $y[n]$ *by* $n_0$ *such that* $y[n - n_0] = e^{(n-n_0)x[n-n_0]}$*. Shift the input of to* $x$ *such that* $x[n - n_0]$ *and plug back into equation* $e^{nx[n-n_0]}$*. The two shifts are not equivalent and the system is therefore time variant.*
5. ***Memoryless:*** *the output at any time* $n$ *depends only on the input at that same time* $n$*. For example, at* $n = 0,\ y[1] = e^{1x[1]}$ *the output depends on the present value of* $x[n]$*.*

## System 2:

$y[n] = x[n] - u[n + 1]$

1. ***Stable:*** *since* $u[n + 1]$ *is bounded, if the input* $x[n]$ *is bounded, the output will also be bounded.*
2. ***Anti-Causal:*** *output at time* $n$ *depends on the future value at* $n + 1$*.*
3. ***Non-Linear:*** *the system does not obey the superposition property. For example, setting* $x[n] = x_1[n] + x_2[n]$ *and plugging into* $y[n]$*, it is expected to get* $y_1[n] + y_2[n] = (x_1[n] - u[n + 1]) + (x_2[n] - u[n + 1]) = x_1[n] + x_2[n] - 2u[n + 1]$*. The derivation shows that* $(x_1[n] + x_2[n]) - u[n + 1] = x_1[n] + x_2[n] - u[n + 1] \neq y_1[n] + y_2[n]$*.*
4. ***Time Variant:*** *shift the* $y[n]$ *by* $n_0$ *such that* $y[n - n_0] = x[n - n_0] - u[(n - n_0) + 1]$*. Shift the input of to* $x$ *such that* $x[n - n_0]$ *and plug back into equation* $x[n - n_0] - u[n + 1]$*. The two shifts are not equivalent and the system is therefore time variant.*
5. ***Memoryless:*** *the output at any time* $n$ depends only on a present value since for $u[n + 1]$ it is known that $u[n]$ will be 1 when $n \geq 0$and will be 0 for $n < 0$. Therefore, $u[n - 1]$ will be 1 when $n \geq -1$such that $y[n] = x[n] - 1$ and $u[n - 1]$ will be 0 when $n < -1$ such that $y[n] = x[n] - 0$.

## System 3:

$y[n] = ax[n] + b$

1. ***Stable:*** *if* $a$, $b$ *are finite and* $x[n]$ *is bounded.*
2. ***Causal:*** *the output at any time* $n$ *depends on current and past values of* $x[n]$*.*
3. ***Non-Linear:*** *the system does not obey the superposition property. For example, setting* $x[n] = x_1[n] + x_2[n]$ *and plugging into* $y[n]$*, it is expected to get* $y_1[n] + y_2[n] = (ax_1[n] + b) + (ax_2[n] + b)$*. The derivation shows that* $a(x_1[n] + x_2[n]) + b = ax_1[n] + ax_2[n] + b \neq y_1[n] + y_2[n]$*. HOWEVER, the system can be* ***Linear*** *if the constant b = 0.*
4. ***Time Invariant:*** *the system is performing time shifting so it will most likely be time invariant as a result. Firstly, shift the* $y[n]$ *by* $n_0$ *such that* $y[n - n_0] = ax[n - n_0] + b$*. Secondly, shift the inputs to* $x$ *such that* $x[n - n_0]$ *and plug back into equation* $x[n - n_0] + b$*. The two are equivalent and the system is therefore time invariant.*

5. ***Memoryless:*** *the output at any time* $n$ *depends only on the input at that same time* $n$*. For example, at* $n = 0,\ y[0] = ax[0] + b$ *the output depends on the present value of* $x[n]$.

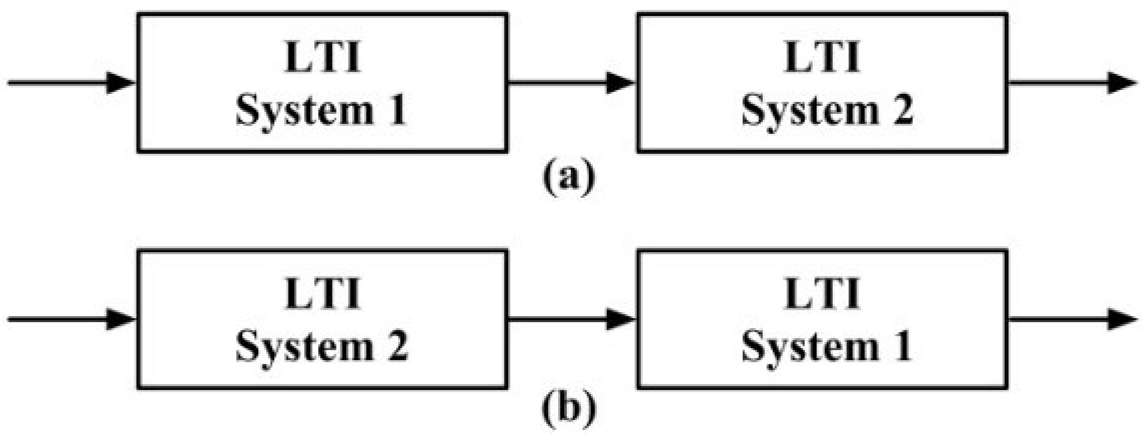
**Sources used to complete this section:**

- https://www.youtube.com/watch?v=ngJ2QWvMPbI&ab_channel=NesoAcademy
- https://www.youtube.com/watch?v=IdwpoS7YeJA&ab_channel=NesoAcademy
- tutorialspoint.com/signals-and-systems-stable-and-unstable-system
- https://math.stackexchange.com/questions/3270918/the-system-yn-c-cdot-xn-d-is-given-is-the-system-linear-time-invarian
- https://www.tutorialspoint.com/signals-and-systems-causal-non-causal-and-anti-causal-signals
- https://www.youtube.com/watch?v=mqwUtn5cip8&ab_channel=NesoAcademy
- https://www.youtube.com/watch?v=wOQDGvCLOs8&ab_channel=NesoAcademy
- https://www.youtube.com/watch?v=LezLNMznZm4&ab_channel=NesoAcademy
- https://www.youtube.com/watch?v=0sRAMfPUQ58&ab_channel=NesoAcademy

# Exercise 2

## Part 1

**Proving that the block diagram in Figure 1.a is equivalent to the one in Figure 1.b. In other words, proving the commutativity of the convolution sum.**




*System 1 =* $s_1[n]$

*System 2 =* $s_2[n]$

$$(s_1 * s_2)[n] = s_1[n] * s_2[n] = \sum_{k=-\infty}^{\infty} s_1[k]s_2[n-k]$$

*Using the following equivalences:*

$$K = n - k \text{ such that } k = n - K$$
$$k = \infty \text{ then } K = -\infty$$
$$k = -\infty \text{ then } K = \infty$$

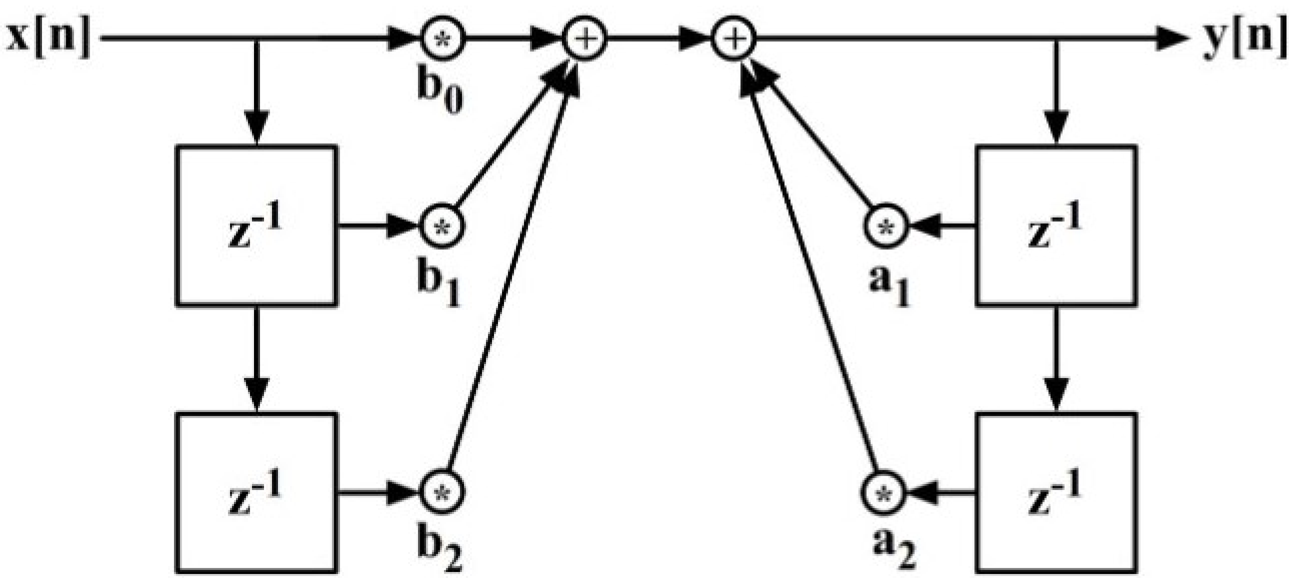
*The original convolution sum can be rewritten by substituting* $k = n - K$*:*

$$y[n] = \sum_{K=-\infty}^{\infty} s_1[n-K]s_2[n-(n-K)] = \sum_{K=-\infty}^{\infty} s_2[K]s_1[n-K]$$

*Observing that the range of* $k$ *and* $K$*are the same, the following equivalence can be derived giving that* $s_1 * s_2 = s_2 * s_1$*:*

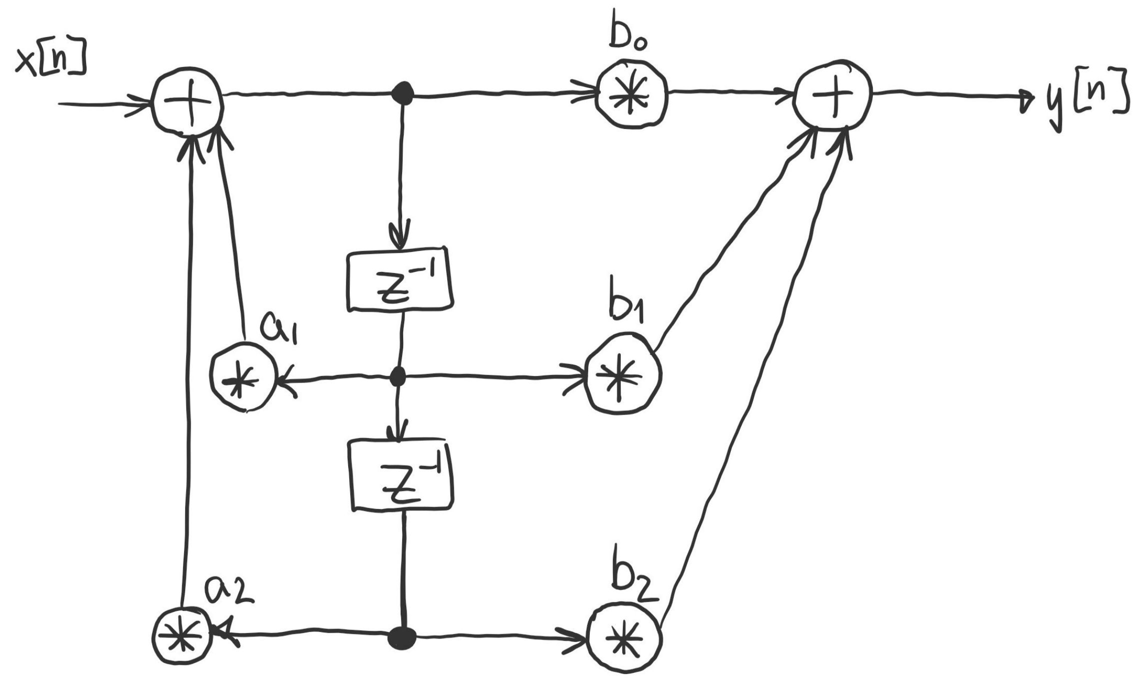
$$y[n] = \sum_{K=-\infty}^{\infty} s_2[K]s_1[n-K] = \sum_{k=-\infty}^{\infty} s_2[k]s_1[n-k] = s_2[n] * s_1[n]$$

### Part 2




$$\begin{aligned}
y[n] = \; & b_0x[n] + b_1x[n-1] + b_2x[n-2] + a_1y[n-1] + a_2y[n-2] \\
= \; & (x[n])b_0 + (x[n] * z^{-1})b_1 + (x[n] * z^{-1} * z^{-1})b_2 + (y[n] * z^{-1})a_1 + (y[n] * z^{-1} * z^{-1})a_2 \\
= \; & (x[n])b_0 + (z^{-1} * x[n])b_1 + (z^{-1} * z^{-1} * x[n])b_2 + (z^{-1} * y[n])a_1 + (z^{-1} * z^{-1} * y[n])a_2 \\
= \; & (x[n])b_0 + \left(z^{-1} * b_1x[n]\right) + \left(z^{-1} * z^{-1} * b_2x[n]\right) + \left(z^{-1} * a_1y[n]\right) + \left(z^{-1} * z^{-1} * a_2y[n]\right) \\
= \; & (x[n])b_0 + \left(z^{-1} * \left(b_1x[n] + a_1y[n] + \left(z^{-1} * b_2x[n]\right) + \left(z^{-1} * a_2y[n]\right)\right)\right) \\
= \; & (x[n])b_0 + \left(z^{-1} * \left(b_1x[n] + a_1y[n] + z^{-1} * (b_2x[n] + a_2y[n])\right)\right) \\
= \; & b_0x[n] + b_1x[n-1] + a_1y[n-1] + b_2x[n-2] + a_2y[n-2] \\
= \; & b_0x[n] + b_1x[n-1] + b_2x[n-2] + a_1y[n-1] + a_2y[n-2]
\end{aligned}$$

*Applying the commutativity of the convolution sum, the simplified flow diagram that uses only two 1-sample delays is displayed below.*




### Part 3

*The frequency response* $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$ *of the system*

$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$ *is derived below:*

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$$
$$Y(\omega) - a_1e^{-j\omega}Y(\omega) - a_2e^{-j2\omega}Y(\omega) = b_0X[\omega] + b_1e^{-j\omega}X[\omega] + b_2e^{-j2\omega}X\omega$$
$$Y(\omega)(1 - a_1e^{-j\omega} - a_2e^{-j2\omega}) = X\omega(b_0 + b_1e^{-j\omega} + b_2e^{-j2\omega})$$
$$H[\omega] = \frac{Y(\omega)}{X\omega} = \frac{(b_0 + b_1e^{-j\omega} + b_2e^{-j2\omega})}{(1 - a_1e^{-j\omega} - a_2e^{-j2\omega})}$$

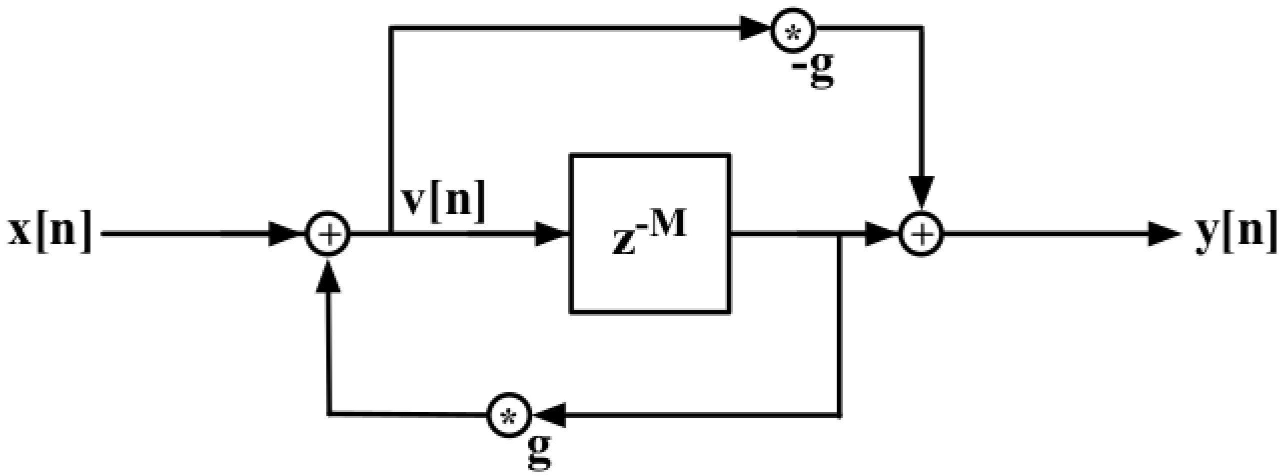
**Sources used to complete this section:**

- https://www.dsprelated.com/freebooks/filters/Direct_Form_II.html
- https://www.youtube.com/watch?v=ap1qXBTKU8g&list=PLbqhA-NKGP6Afr_KbPUuy_yIBpPR4jzWo&index=8&ab_channel=AkashMurthy
- https://www.youtube.com/watch?v=mRKPGt97ytw&ab_channel=Engg-Course-Made-Easy
- https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Signal_Processing_and_Modeling/Signals_and_Systems_(Baraniuk_et_al.)/04%3A_Time_Domain_Analysis_of_Discrete_Time_Systems/4.04%3A_Properties_of_Discrete_Time_Convolution
- https://httpsachinbagul.files.wordpress.com/2017/12/lec-10.pdf
- https://www.youtube.com/watch?v=yuS86,oimY5Y&ab_channel=ECAcademy
- https://www.youtube.com/watch?v=Jl6dapilt58&ab_channel=JohnBuck

## Exercise 3

## Part 1: Computing the Frequency Response

**The difference equations** $y[n] = v[n-M] - gv[n]$ **and** $v[n] = x[n] + gv[n-M]$ **can be used to describe the system in the figure below. Using the two equations, the frequency response** $H(\omega) = \frac{Y(\omega)}{X(\omega)}$ **can be computed.**




*The transfer function* $H(\omega) = \frac{Y(\omega)}{X(\omega)}$*, describes the steady-state response of a system to sinusoidal inputs of varying frequencies. In other words it is how a sinusoidal signal of a given frequency is affected by the filter.*

*Rearranging* $v[n] = x[n] + gv[n-M]$ *to get* $v[n] - gv[n-M] = x[n]$*.Taking the DTFT on both sides of the equation:*

$$
\begin{aligned}
V(\omega) - ge^{-jM\omega}V(\omega) &= X(\omega) \\
V(\omega)(1 - ge^{-jM\omega}) &= X(\omega) \\
V(\omega) &= \frac{X(\omega)}{(1 - ge^{-jM\omega})}
\end{aligned}
$$

*Taking the DTFT on both sides of the equation* $y[n] = v[n-M] - gv[n]$*:*

$$
\begin{aligned}
Y(\omega) &= e^{-jM\omega}V(\omega) - gV(\omega) \\
Y(\omega) &= V(\omega)(e^{-jM\omega} - g)
\end{aligned}
$$

*Substituting* $V(\omega)$ *into* $Y(\omega)$ *to obtain* $H(\omega)$*:*

$$Y(\omega) = \frac{X(\omega)}{(1 - ge^{-jM\omega})}(e^{-jM\omega} - g)$$

$$Y(\omega) = X(\omega)\frac{(e^{-jM\omega} - g)}{(1 - ge^{-jM\omega})}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(e^{-jM\omega} - g)}{(1 - ge^{-jM\omega})}$$

### Part 2: Plotting the Frequency and Phase Responses

*The frequency response appears to have positive magnitude for all omega inputs which suggests this may be an all-pass filter.*

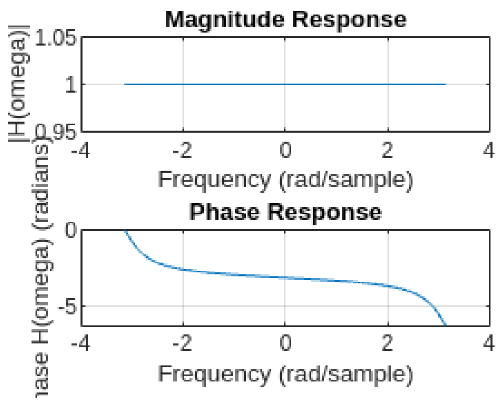
```
% Given values
g = 0.7;
M = 1000;
omega = linspace(-pi, pi, M); % the frequency range

% Calculate the frequency response H(omega) based on the computed transfer function
in Part 1
Y_omega = exp(-1j*M*omega) - g;
X_omega = 1 - g*exp(-1j*M*omega);
H_omega = Y_omega ./ X_omega;

% Calculate magnitude and phase response
magnitude_response = abs(H_omega);
phase_response = unwrap(angle(H_omega)); % Unwrap the phase response

% Plot magnitude response
figure;
subplot(2, 1, 1);
plot(omega, magnitude_response);
title('Magnitude Response');
xlabel('Frequency (rad/sample)');
ylabel('|H(omega)|');
grid on;
ylim([0.95 1.05]); % Adjust the y-axis range to better visualize the magnitude
response

% Plot phase response
subplot(2, 1, 2);
plot(omega, phase_response);
title('Phase Response');
xlabel('Frequency (rad/sample)');
ylabel('Phase H(omega) (radians)');
grid on;
```




## Part 3: Filter Implementation

*The filter implementation function a2_filter can be found in the Appendix.*

## Part 4: Impulse Response of Filter

*Passing an impulse through the filter produces a sort of clicking sound that is slightly delayed (there are small silences) and also dampened with time. The best analogy is a small round object bouncing to a stop.*

```
% Given values
g = 0.7;
M = 1000;

% Impulse
duration = get_impulse_duration(M, g);
x = get_impulse(duration);

% Evaluate the filter
y = a2_filter(x, M, g);
sound(y);
```

## Part 5: Cascading Filters

*Passing an impulse through the filter produces a clicking sound that is damped and low pitched but has smaller silences between sounds.The best analogy is either a very thick rubber band being flicked or a Vargan instrument oscillating.*

```
% Given values
g = 0.7;
Ms = [1051, 337, 113];

% Calculate durations
durations = arrayfun(@(M) get_impulse_duration(M, g), Ms);
duration = get_impulse_duration(Ms(1), g);

% Create impulse
x = get_impulse(duration);
```

```
% Cascade filters
y1 = a2_filter(x, Ms(1), g);
y2 = a2_filter(y1, Ms(2), g);
y3 = a2_filter(y2, Ms(3), g);
sound(y3);
```

### Part 6: Sound File Through Filter

*Passing a guitar.wav sound file through the a2_filter system produces a sound that is identical to the input. The system is therefore an* **all-pass filter***.*

```
% Given values
g = 0.7;
[x, Fs] = audioread('guitar.wav');
M = length(x);
sound(x, Fs); % original sound

y = a2_filter(x, M, g);
sound(y, Fs); % filtered sound
```

**Sources used to complete this section:**

- https://www.mathworks.com/matlabcentral/answers/1756050-how-to-plot-frequency-response-phase-response-from-transfer-function
- https://www.projectrhea.org/rhea/index.php/Zachary_Curosh_-_Frequency_Response_and_Difference_Equations_ECE301Fall2008mboutin
- https://www.youtube.com/watch?v=cNwuwoGRuH8&ab_channel=StephenMendes
- https://www.mathworks.com/matlabcentral/answers/44770-logarithm-of-base-other-than-e-10-and-2
- https://www.youtube.com/watch?v=Utvvsr_d02c&ab_channel=EdD

## Exercise 4

**The goal of this exercise is to find the transient response of a causal Linear Time Invariant (LTI) system when the input is a causal signal, i.e. a signal that starts at time n = 0. The idea is to understand what happens during the attack of a sound in common situations where the sound is generated by a sudden activation of the production system.**

**The general strategy is to get the transient response $y_{tr}[n]$ by comparing $y_c[n]$ and $y[n]$, the causal and non-causal responses of the system respectively. Two strategies are going to be used: first strategy in parts 1-3, second strategy in parts 4 and 5.**

### Part 1

**Considering a causal LTI system *H* characterized by its impulse response $h[n] = \frac{1}{3}\sum_{l=0}^{l=2} \delta[n-l]$, then**

**$y_c[n]$ will be the response of system *H* to a 'causal sinusoidal' input $x_c[n] = 2cos(\frac{\pi}{3}n)u[n]$, where $u[n]$**

**is the unit-step signal. Using the *fold-shift-add* method of convolution, a mathematical expression for $y_c[n] = x_c[n] * h[n]$ is obtained below.**

*Manually drawing each signal,* $h[n]$ *looks like* $h[n] = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$ for $n \in \mathbb{N}$ *and* $x_c[n]$ *looks like* $x_c[n] = \{2, 1, -1, -2, -1, 1, 2, 1, -1, -2, -1, 1, \ldots\}$ for $n \in \mathbb{N}$. *Convolving the two signals together:*

$$
\begin{aligned}
&\frac{1}{3}(2) = \frac{2}{3}\\
&\frac{1}{3}(1) + \frac{1}{3}(2) = 1\\
&\frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(2) = \frac{2}{3}\\
&\frac{1}{3}(-2) + \frac{1}{3}(-1) + \frac{1}{3}(1) = -\frac{2}{3}\\
&\frac{1}{3}(-1) + \frac{1}{3}(-2) + \frac{1}{3}(-1) = -\frac{4}{3}\\
&\frac{1}{3}(1) + \frac{1}{3}(-1) + \frac{1}{3}(-2) = -\frac{2}{3}\\
&\frac{1}{3}(2) + \frac{1}{3}(1) + \frac{1}{3}(-1) = \frac{2}{3}\\
&\frac{1}{3}(1) + \frac{1}{3}(2) + \frac{1}{3}(1) = \frac{4}{3}\\
&\frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(2) = \frac{2}{3}\\
&\frac{1}{3}(-2) + \frac{1}{3}(-1) + \frac{1}{3}(1) = -\frac{2}{3}\\
&\ldots
\end{aligned}
$$

*Based on the above derivation, a general formula for the convolution* $y_c[n]$ *can be derived:*

$$
y_c[n] = \begin{cases}
\frac{1}{3}\left(2\cos\left(\frac{\pi}{3}n\right) + 2\cos\left(\frac{\pi}{3}(n-1)\right) + 2\cos\left(\frac{\pi}{3}(n-2)\right)\right), & \text{for } n \geq 2\\
\frac{1}{3}\left(2\cos\left(\frac{\pi}{3}n\right) + 2\cos\left(\frac{\pi}{3}(n-1)\right)\right), & \text{for } n = 1\\
\frac{1}{3}\left(2\cos\left(\frac{\pi}{3}n\right)\right), & \text{for } n = 0\\
0, & \text{for } n < 0
\end{cases}
$$

*After simplifying the equations, the final convolution* $y_c[n]$ *is obtained below:*

$$y_c[n] = \begin{cases} \frac{2}{3}\left(\cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}(n-1)\right) + \cos\left(\frac{\pi}{3}(n-2)\right)\right), & \text{for } n \geq 3 \\ \frac{2}{3}, & \text{for } n = 2 \\ 1, & \text{for } n = 1 \\ \frac{2}{3} & \text{for } n = 0 \\ 0, & \text{for } n < 0 \end{cases}$$

## Part 2

**Computing the frequency response $H(\omega)$ of the LTI system *H* defined in the beginning of this section. Deriving the specific expression of $H(\omega)$ when $\omega = \omega_0 = \frac{\pi}{3}$ to describe the action of *H* in terms of delay and amplification. The expression for the response $y[n]$ is expressed in terms of the signal $x[n] = 2cos(\frac{pi}{3}n)$. The part below computes the expression for $y_{tr}[n]$ by comparing the corresponding $y[n]$ to the already obtained expression of $y_c[n]$ through $y_{tr}[n] = y_c[n] - y[n]$.**

$$h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$

$$H(\omega) = \frac{1}{3}(1 + e^{-\omega j} + e^{-2\omega j})$$

$$H\left(\frac{\pi}{3}\right) = \frac{1}{3}\left(1 + e^{\frac{-\pi}{3}j} + e^{\frac{-2\pi}{3}j}\right)$$

$$H\left(\frac{\pi}{3}\right) = \frac{1}{3}\left(1 + \frac{1}{2} - \frac{\sqrt{3}}{2}j - \frac{1}{2} - \frac{\sqrt{3}}{2}j\right)$$

$$H\left(\frac{\pi}{3}\right) = \frac{1}{3}(1 - \sqrt{3}\,j)$$

$$H\left(\frac{\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{3}j$$

$$\angle H\left(\frac{\pi}{3}\right) = \tan\left(\frac{\frac{-\sqrt{3}}{3}}{\frac{1}{3}}\right) = \tan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

$$|H\left(\frac{\pi}{3}\right)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{3}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

*From the above result the $y[n]$ can be deduced to be* $\frac{4}{3}\cos\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)$ which results in the following equation for $y_{tr}[n]$:

$$y_{tr}[n] = \begin{cases} \frac{2}{3}\left(\cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}(n-1)\right) + \cos\left(\frac{\pi}{3}(n-2)\right)\right) - \frac{4}{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right), & \text{for } n \geq 2 \\ \frac{2}{3}\left(\cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}(n-1)\right)\right) - \frac{4}{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right), & \text{for } n = 1 \\ \frac{2}{3}\left(\cos\left(\frac{\pi}{3}n\right)\right) - \frac{4}{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right), & \text{for } n = 0 \\ 0, & \text{for } n < 0 \end{cases}$$

*After simplifying the equations, the final convolution* $y_{tr}[n]$ *is obtained below:*

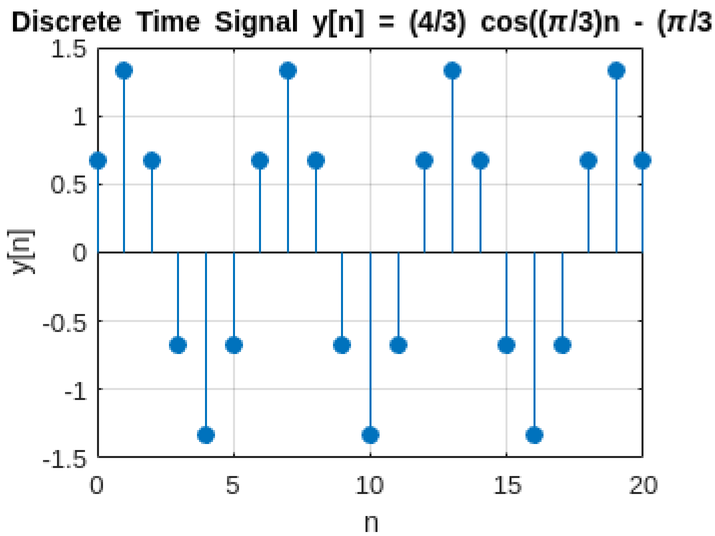
$$y_{tr}[n] = \begin{cases} \frac{2}{3}\left(\cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}(n-1)\right) + \cos\left(\frac{\pi}{3}(n-2)\right)\right) - \frac{4}{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right), & \text{for } n \geq 3 \\ 0, & \text{for } n = 2 \\ -\frac{1}{3}, & \text{for } n = 1 \\ 0, & \text{for } n = 0 \\ 0, & \text{for } n < 0 \end{cases}$$

## Part 3

*Plotting the steady-state response* $y[n]$.

```
% Plotting the computed y[n]
n = 0:20; % Define the range
y_n = (4/3) * cos((pi/3) * n - (pi/3));

% Plot the signal
stem(n, y_n, 'filled');
title('Discrete Time Signal y[n] = (4/3) cos((\pi/3)n - (\pi/3))');
xlabel('n');
ylabel('y[n]');
grid on;
```

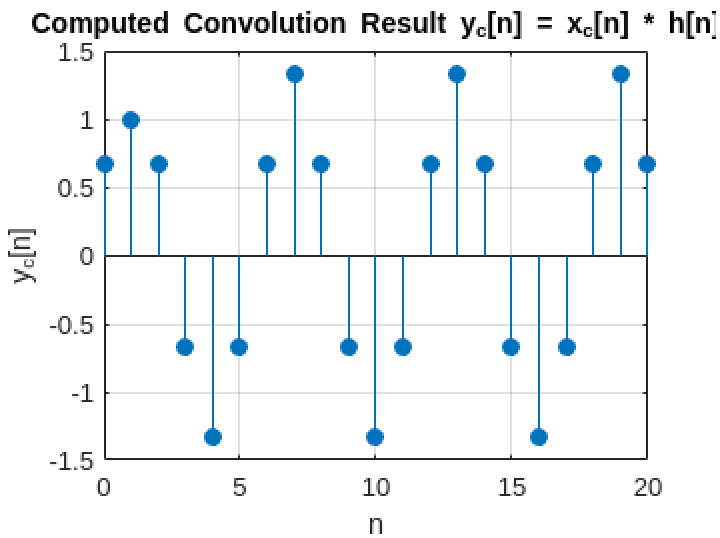


*Verifying $y_c[n]$ by first computing the convolution and plotting the result, then plotting the obtained mathematical expression from Part 1. The two graphs are the same.*

```
% Plotting the computed y_c[n]
n = 0:20; % Define the range
% Computing the convolution y_c[n] = x_c[n] * h[n] for verification
h = (1/3)*ones(1,3); % Define the impulse response h[n]
x_c = 2*cos(pi/3*n); % Define the input signal x_c[n]

% Zero-pad x_c to make sure the convolution result is the same length as n
x_c_padded = [x_c, zeros(1, length(h) - 1)];
y_c = conv(x_c_padded, h ); % y_c[n] = x_c[n] * h[n]

% Plot the signal
figure;
stem(n, y_c(1:length(n)), 'filled');
title('Computed Convolution Result y_{c}[n] = x_{c}[n] * h[n]');
xlabel('n');
ylabel('y_{c}[n]');
grid on;
```

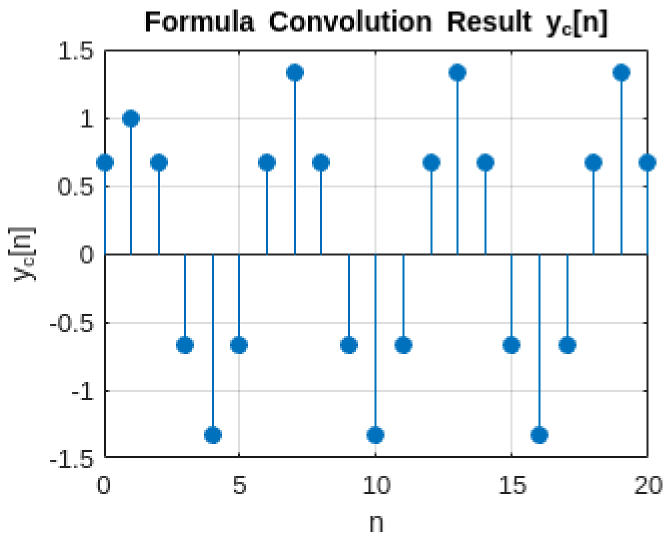



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the formula y_c[n] obtained in Part 1
x_c = 2*cos(pi/3*n);

% Calculate y[n] for each value of n using the piecewise function
y = zeros(1, length(n));
for i = 1:length(n)
    if i >= 3
        y(i) = (1/3) * (x_c(i) + x_c(i-1) + x_c(i-2));
    elseif i == 2
        y(i) = (1/3) * (x_c(i) + x_c(i-1));
    elseif i == 1
```

```
        y(i) = (1/3) * x_c(i);
    end
end

% Plot the signal
stem(n, y, 'filled');
title('Formula Convolution Result y_{c}[n]');
xlabel('n');
ylabel('y_{c}[n]');
grid on;
```



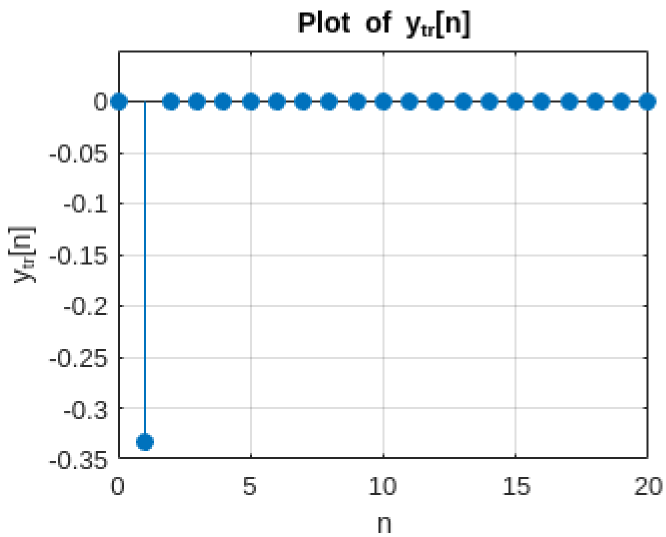

*Plotting the transient response* $y_{tr}[n] = y_c[n] - y[n]$.

```
% Plotting thecomputed y_tr[n]
n = 0:20;  % Assuming we want to plot from n=0 to n=20

% Calculate y_tr[n] for each value of n using the piecewise function
y_tr = zeros(1, length(n));
for i = 1:length(n)
    if n(i) >= 2
        % for n >= 2
        y_tr(i) = (2/3)*(cos(pi/3*n(i)) + cos(pi/3*(n(i)-1)) + cos(pi/3*(n(i)-2)))
- (4/3)*cos(pi/3*n(i) - pi/3);
    elseif n(i) == 1
        % for n = 1
        y_tr(i) = (2/3)*(cos(pi/3*n(i)) + cos(pi/3*(n(i)-1))) - (4/3)*cos(pi/3*n(i)
- pi/3);
    elseif n(i) == 0
        % for n = 0
        y_tr(i) = (2/3)*cos(pi/3*n(i)) - (4/3)*cos(pi/3*n(i) - pi/3);
    end
end

% Plot the signal
stem(n, y_tr, 'filled');
```

```
title('Plot of y_{tr}[n]');
xlabel('n');
ylabel('y_{tr}[n]');
grid on;
```




## Part 4

**This section will now derive a general expression for the transient response $y_{tr}[n]$ of a causal LTI system, for which the impulse response is $h[n]$ when the input is a 'causal sinusoid' xc, i.e. a sinusoid that starts at time n = 0. Starting from the expression of the output of the LTI as a convolution sum $\sum_{k=-\infty}^{k=+\infty} x[k]h[n-k]$, it is shown that the transient response can be expressed as:**

$$y_{tr}[n] = -\sum_{k=n+1}^{k=+\infty} h[k]cos(\omega(n-k)) \text{ for } n \geq 0$$

*Using the definition of convolution:*

$$y_c[n] = \sum_{k=-\infty}^{\infty} h[k]\cos(\omega(n-k))u[n-k]$$

*Applying the $u[n]$ step function:*

$$y_c[n] = \sum_{k=0}^{\infty} h[k]\cos(\omega(n-k))$$

*Assuming that the filter $h$ has a length of $L$ and using the formula for the convolution of infinite sequences with a finite filter:*

$$y_c[n] = \sum_{k=0}^{\min(n,L)} h[k]\cos(\omega(n-k))$$

*At steady state, $n \geq L$ and the above expression can be simplified to express the steady state response using the following equation:*

$$y[n] = \sum_{k=0}^{L} h[k]\cos(\omega(n-k))$$

*Using the formula for the transient response $y_{tr}[n] = y_c[n] - y[n]$:*

$$y_{tr}[n] = \sum_{k=0}^{\min(n,L)} h[k]\cos(\omega(n-k)) - \sum_{k=0}^{L} h[k]\cos(\omega(n-k))$$

*Since both summations have the same $h[k]cos(\omega(n-k))$, the first few terms cancel (depending on what is the $n$) and the equation is left with the values at $n+1$, such the lower bound of the summation of the second term gets updated to $k = n+1$and the first term gets removed. Leaving the following equation:*

$$y_{tr}[n] = -\sum_{k=n+1}^{L} h[k]\cos(\omega(n-k))$$

*Since the length of the filter is not known, it can be assumed that$L = \infty$:*

$$y_{tr}[n] = -\sum_{k=n+1}^{\infty} h[k]\cos(\omega(n-k))$$

### Part 5

$$\begin{aligned} y_{tr}[n] &= -\sum_{k=n+1}^{\infty} h[k]\cos(\omega(n-k)) \\ &= -\sum_{k=n+1}^{\infty}\sum_{l=0}^{2}\frac{1}{3}\delta[k-l]2\cos(\frac{\pi}{3}(n-k)) \\ &= -\frac{2}{3}\sum_{k=n+1}^{\infty}\sum_{l=0}^{2}\delta[k-l]\cos(\frac{\pi}{3}(n-k)) \\ &= -\frac{2}{3}\sum_{k=n+1}^{2}\cos(\frac{\pi}{3}(n-k)) \end{aligned}$$

**Sources used to complete this section:**

- https://www.reddit.com/r/DSP/comments/w1ajdp/found_this_super_easy_way_of_doing_convolution/
- https://dsp.stackexchange.com/questions/23988/why-is-the-output-of-an-lti-system-expressed-as-the-convolution-of-the-input-wit
- https://eceweb1.rutgers.edAssuming that the filter h as a length of :u/~orfanidi/intro2sp/2e/orfanidis-isp2e-1up.pdf

## Appendix

### *Exercise 3 Part 3*

```matlab
function y = a2_filter(x, M, g)
    % Args:
    %   x: Input signal
    %   M: Delay length
    %   g: Gain
    %
    % Returns:
    %   y: Output signal (vector) after processing through the filter.
    %
    % The filter implements the following difference equations:
    %   v[n] = x[n] + g*v[n-M]
    %   y[n] = v[n-M] - g*v[n]

    % Filter 1
    % v[n] = x[n] + g*v[n-M]
    b_1 = 1; % feedforward coefficients
    a_1 = [1, zeros(1, M-1), -g]; % feedbackward coefficient

    % Filter 2
    % y[n] = v[n-M] - g*v[n]
    b_2 = [-g, zeros(1, M-1), 1]; % feedforward coefficients
    a_2 = 1; % feedbackward coefficient

    v = filter(b_1, a_1, x);
    y = filter(b_2, a_2, v);
end
```

### *Helper Functions*

```matlab
function duration = get_impulse_duration(M, g)
    duration = round(M*(1 - ((log10(1000*(1-g.^2))) ./ (log10(abs(g))))));
end

function impulse = get_impulse(duration)
    impulse = [1; zeros(duration-1, 1)];
end
```