FIR & IIR Filters

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Exercise 1

A *zero-phase* filter is a special case of a linear-phase filter in which the phase slope is 0. For a real-coefficient filter, the phase response only takes values of either 0 or π. Below, it is shown that the impulse response $h[n] = b_1\delta[n+1] + b_0\delta[n] + b_1\delta[n-1]$ leads to a *zero-phase* frequency response when $(b_0, b_1) \in \mathbb{R}^2$.

Starting out with the impulse response $h[n] = b_1\delta[n+1] + b_0\delta[n] + b_1\delta[n-1]$, the z-transform table can be used to convert it to the frequency response $H(z) = b_1 z + b_0 + b_1 z^{-1}$.

Input in $e^{j\omega}$ for z to find the frequency response:

$$H\left(e^{j\omega_c}\right) = b_1 e^{j\omega_c} + b_0 + b_1 e^{-j\omega_c}$$

Using Euler's identity $cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$:

$$H\left(e^{j\omega_c}\right) = b_0 + b_1\left(e^{j\omega_c} + e^{-j\omega_c}\right) = b_0 + b_1 2cos(\omega_c)$$

The real part of the equation is $b_0 + b_1 2cos(\omega_c)$ and the imaginary part is 0. Using the formula for computing the phase response:

$$\angle H(e^{j\omega}) = tan^{-1}\left(\frac{-0}{b_0 + b_1 2cos(\omega_c)}\right) = tan^{-1}(0) = 0$$

Sources used to complete this section:

- https://en.wikipedia.org/wiki/Euler%27s_formula

Exercise 2

Figure 1 is used for parts 1 to 4 and contains 3 graphs:

1. $x[n]$**, a signal of the form** $x[n] = (A + Bcos(\omega_0 n))u[n]$**, where** $u[n]$ **is the unit step, and** ω_0 **is the normalized angular speed.**
2. $y[n]$**, the output of an FIR filter** H **of order** M **when** $x[n]$ **is the input to the filter.**
3. **A zoomed-in display of y[n].**

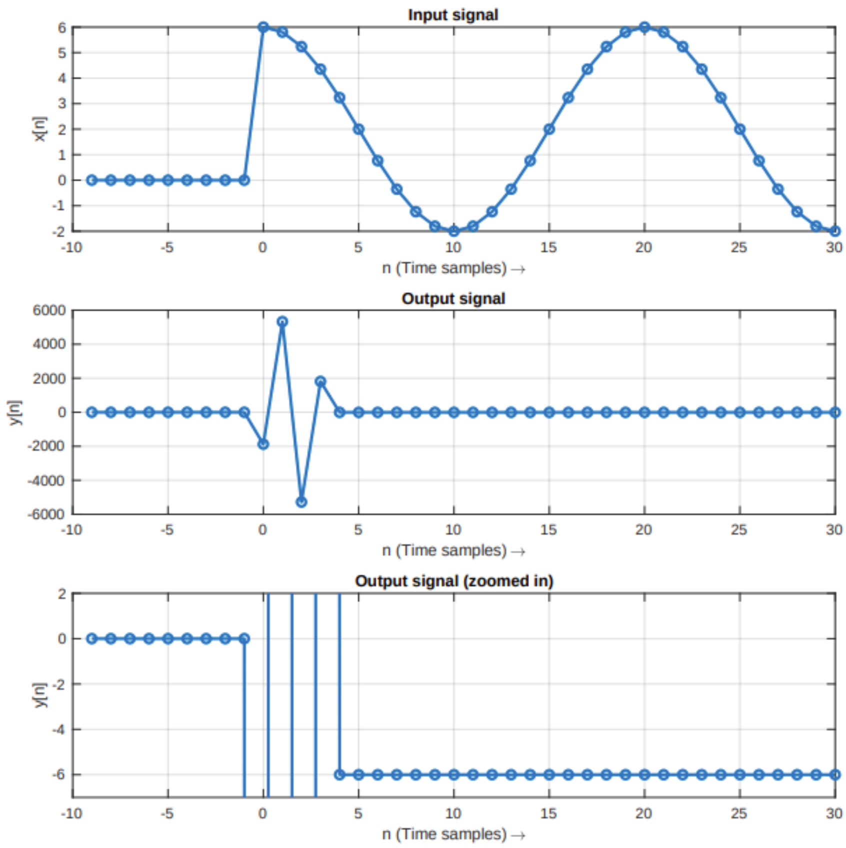


Figure 1: Input $x[n]$ and two views of the corresponding output $y[n]$ after filtering by system H.

Part 1: A, B, and Angular Speed

Using Figure 1, A is the DC offset which is the value around which the signal evolves (average of the signal). B is the amplitude of the signal which is the value after subtracting the DC offset. ω_0 is the unnormalized frequency which can be obtained by computing the relationship between the period obtained from the graph and the frequency $f = \frac{1}{T}$ and multiplied by 2π to normalize.

$A = \frac{6 + -2}{2} = 2$

$B = 6 - 2 = 4$

$\omega_0 = \frac{2\pi}{20} = \frac{pi}{10} \approx 0.314$

Part 2: Order of the Filter

The order of an FIR is equivalent to the *warm-up* period duration (transient response). Using the Figure 1 to find the order M of the filter H, it is determined that $y[n]$ settles at the 4th sample. The filter is therefore of order $M = 4$.

Part 3: Zeros of the Transfer Function

The zeros of the transfer function $H(z)$ on the unit circle correspond to the locations where the frequency is filtered out. Observing the $y[n]$ output, the initial frequency is filtered output when the steady-state response is achieved. On the unit circle, there is therefore one zero and its conjugate. In polar form, the zero can be found at $re^{j\omega}$, where $r = 1$ because the zero is directly on the unit circle and $\omega = \frac{\pi}{10}$. The final value for the zero is therefore $(1)e^{\frac{j\pi}{10}}$. There is also the conjuage zero which is in the same location but reflected over the Re axis such that $(1)e^{-\frac{j\pi}{10}}$. The two zeros are additionally expressed in cartesian form below.

First zero: $e^{j\omega} = cos(\omega) + jsin(\omega) = cos(\frac{\pi}{10}) + jsin(\frac{\pi}{10}) = 0.951 + 0.309j$

Second zero: $e^{-j\omega} = cos(-\omega) + jsin(-\omega) = cos(-\frac{\pi}{10}) + jsin(-\frac{\pi}{10}) = 0.951 - 0.309j$

Part 4: The DC Gain

The DC gain of a filter is the value of its frequency response where $\omega = 0$. In other words, DC gain is the ratio of the magnitude of the response to the steady-state step to the magnitude of the step input.

Magnitude of the steady-state

$\lim_{n\to\infty} y_{\text{steady}}[n] = -6$

Using the result found in part 1, where the $A = 2$, the ratio can be written as the below formula

$DC_{\text{gain}} = \frac{1}{A} \lim_{n\to\infty} y_{\text{steady}}[n] = \frac{-6}{2} = -3$

Sources used to complete this section:

- https://www.music.mcgill.ca/~gary/618/week1/node6.html
- https://brianmcfee.net/dstbook-site/content/ch12-ztransform/PoleZero.html

- https://www.youtube.com/watch?v=esZ_6n-qHuU&ab_channel=DavidDorran
- https://eceweb1.rutgers.edu/~orfanidi/intro2sp/2e/orfanidis-isp2e-1up.pdf
- https://www.electrical4u.com/dc-gain-transfer-function/

Exercise 3

Damping and bandwidth of a two-pole resonator are determined by the magnitude r of its complex poles $p = re^{j\omega_c}$, and its complex conjugate $\overline{p}$. For example, it is possible to get the bandwidth B (in Hertz) of a resonator in terms of r with $B = -\frac{ln(r)}{\pi T_s}$ where $T_s = \frac{1}{F_s}$ is the sampling period. This allows in return to calculate r in terms of bandwidth B, i.e. $r = e^{-\pi BT_s}$.

Part 1:

Considering a general two-pole filter defined by its resonant frequency $f_c = \frac{F_s}{4}$, and its bandwidth $B = 0.01F_s$.

Magnitude r

The magnitude r can be computed by plugging in the given values for B and T_s.

$$r = e^{-\pi BT_s} = e^{-\pi 0.01F_sT_s} = e^{-\pi 0.01F_s\frac{1}{F_s}} = e^{-\pi 0.01} = e^{-\pi 0.01F_sT_s} = e^{-\pi 0.01} \approx 0.969$$

Angular Frequency ω_c

$$\omega_c = \frac{2\pi F_s}{F_s} = \frac{2\pi}{4} = \frac{\pi}{2} \approx 1.571\frac{radians}{sample}$$

Transfer Function $H(z)$

$$H(z) = \frac{b_0}{(1 - pz^{-1})(1 - \overline{p}z^{-1})} = \frac{b_0}{(1 - re^{j\omega_c}z^{-1})(1 - re^{-j\omega_c}z^{-1})} = \frac{b_0}{1 - 2rcos(\omega_c)z^{-1} + r^2z^{-2}} = \frac{b_0}{1 - 2e^{\pi 0.01}cos(\frac{\pi}{2})z^{-1} + e^{-\pi 0.02}z^{-2}} = \frac{\quad}{1 -}$$

Since $\omega_c = \frac{\pi}{2}$, $\cos(\omega_c) = 0$, the equation can be further simplified:

$$H(z) = \frac{b_0}{1 - 1.938cos(\frac{\pi}{2})z^{-1} + 0.939z^{-2}} = \frac{b_0}{1 + 0.939z^{-2}}$$ where $a_1 = 0$ and $a_2 = 0.939$.

Part 2:

$$H(e^{j\omega_c}) = \frac{b_0}{1 + 0.939(e^{j\omega_c})^{-2}} = 1$$

$$b_0 = 1 + 0.939(e^{-2j\omega_c}) = 1 + 0.939(e^{-2j\frac{\pi}{2}}) = 1 + 0.939(e^{-j\pi}) = 1 + 0.939(cos(-\pi) + jsin(-\pi)) = 1 + 0.939(cos(-\pi)) = 1 - 0.93$$

Part 3:

The frequency response of this filter is

$$H(e^{j\omega}) = \frac{0.0610}{1 + 0.939e^{-2j\omega}} = \frac{0.0610}{1 + 0.939(\cos(-2\omega) + j\sin(-2\omega))} = \frac{0.0610}{(1 + 0.939\cos(2\omega)) + (-0.939\sin(2\omega))j}$$

Magnitude Response

$$|H(e^{j\omega})| = \left|\frac{0.0610}{(1 + 0.939\cos(2\omega)) + (-0.939\sin(2\omega))j}\right| = \frac{0.0610}{\sqrt{(1 + 0.939\cos(2\omega))^2 + (-0.939\sin(2\omega))^2}}$$

inputing the $\omega = \omega_c = \frac{\pi}{2}$, the magnitude is 1 which align with part 2

Phase Response

$$\angle H(e^{j\omega}) = \angle\left(\frac{0.0610}{(1 + 0.939\cos(2\omega)) + (-0.939\sin(2\omega))j}\right)$$
$$= \angle(0.0610) - \angle((1 + 0.939\cos(2\omega)) + (-0.939\sin(2\omega))j)$$
$$= 0 - \arctan\left(\frac{-0.939\sin(2\omega)}{1 + 0.939\cos(2\omega)}\right)$$
$$= -\arctan\left(\frac{-0.939\sin(2\omega)}{1 + 0.939\cos(2\omega)}\right)$$

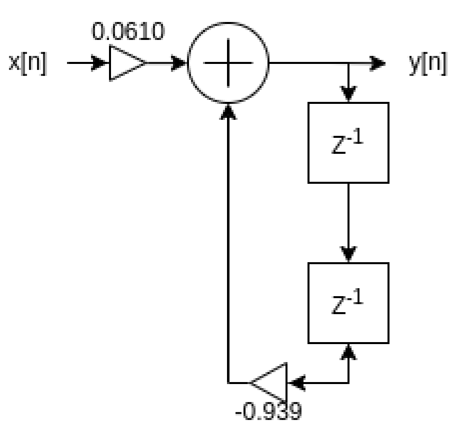
inputing the $\omega = \omega_c = \frac{\pi}{2}$, the angle of 0 which align with part 2

Part 4:

The two-pole filter can be defined by the following difference equation: $y[n] = b_0x[n] - a_1y[n-1] - a_2y[n-2]$.
Where $a_1 = 0$ and $a_2 = 0.939$ so the difference equation becomes $y[n] = 0.0610x[n] - 0.939y[n-2]$.

Part 5:

System diagram in direct form 1:



Sources used to complete this section:

- https://www.dsprelated.com/freebooks/filters/Two_Pole.html
- https://ccrma.stanford.edu/~jos/fp/Two_Pole.html
- https://eceweb1.rutgers.edu/~orfanidi/intro2sp/2e/orfanidis-isp2e-1up.pdf