Homework 4

Zhanna Sakayeva

April 2022

1 Problem 1

We will find the formula for the general entropy solution of

$$u_t + f(u)_x = 0,$$

$$u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases}$$

where $f''(u) > 0$ and $u_l \neq u_r$.

We consider two cases: $u_r < u_l$ and $u_r > u_l$.

1. If $u_r < u_l$ then we have an entropy solution is a shock wave

   $$u(x,t) = \begin{cases} u_l & x/t < s, \\ u_r & x/t > s. \end{cases}$$

   Where

   $$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}.$$

2. If $u_r > u_l$ then we have entropy solution is a rarefaction wave

   $$u(x,t) = \begin{cases} u_l & x/t < f'(u_l), \\ (f'(u))^{-1}(x/t) & f'(u_l) < x/t < f'(u_r), \\ u_r & x/t > f'(u_r). \end{cases}$$

   Now we will show that these solutions satisfy entropy condition. In the first case when we have a shock wave, it can be derived from Rankine-Hugoniot condition. Since it is $u_r < u_l$, it also automatically satisfies the entropy condition.
   In the second case when we have a rarefaction wave, we can assume that $u(x,t) = (f'(u))^{-1}(x/t)$. In that case our equation will be

   $$u_t + f(u)_x = \frac{1}{f''}(\frac{x}{t})\frac{1}{t}\Big(f'\big((f')^{-1}\big) - \frac{x}{t}\Big)\Big) = 0.$$

This will give us $f'((f')^{-1}) = \frac{x}{t}$, that is a weak solution. Next we check the entropy condition in the region $f'(u_l) < x/t < f'(u_r)$. $\forall x, y$ such that $f'(u_l)t \leq x < x + y \leq f'(u_r)t$, $\exists 0 < y^* < y$ such that

$$u(x+y,t) - u(x,t) = (f')^{-1}\big(\frac{x+y}{t}\big) - (f')^{-1}\big(\frac{x}{t}\big) = \frac{1}{f''}\big(\frac{x+y^*}{t}\big)\frac{z}{t} \leq 0,$$

by convexity of f. Hence, both cases give us an entropy solution.

2 Problem 2

Consider a flux function $f \in C^\infty(\mathbb{R})$ with $f''(u) > 0$.

1. The Roe flux can be written as

$$f^*(u,v) = \begin{cases} f(u) & a(u,v) \geq 0, \\ f(v) & a(u,v) < 0, \end{cases}$$

where

$$a(u,v) = \frac{f(v) - f(u)}{v - u}.$$

We will prove that the Roe flux is locally Lipschitz continuous, i.e. that there exists an $\epsilon > 0$ and an L such that

$$|f^*(u,v) - f^*(\bar{u},\bar{v})| \leq L(|u - \bar{u}| + |v - \bar{v}|)$$

for $|u - \bar{u}| + |v - \bar{v}| < \epsilon$. We consider the following cases:

(a) If $a(u,v) \geq 0$ and $a(\bar{u},\bar{v}) \geq 0$:

$$|f^*(u,v) - f^*(\bar{u},\bar{v})| = |f(u) - f(\bar{u})| \leq L_1|u - \bar{u}|,$$

since f is continuously differentiable and it is locally Lipschitz.

(b) If $a(u,v) < 0$ and $a(\bar{u},\bar{v}) < 0$

$$|f^*(u,v) - f^*(\bar{u},\bar{v})| = |f(v) - f(\bar{v})| \leq L_2|v - \bar{v}|,$$

since f is continuously differentiable and it is locally Lipschitz.

(c) If $a(u,v) \geq 0$ and $a(\bar{u},\bar{v}) < 0$

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |f(u) - f(\bar{v})| \\ &= |f(u) - f(\bar{u}) + f(\bar{u}) - f(v) + f(v) - f(\bar{v})| \\ &\leq |f(u) - f(\bar{u})| + |f(v) - f(\bar{v})| + |f(\bar{u}) - f(v)| \\ &\leq L_1|f(u) - f(\bar{u})| + L_2|f(v) - f(\bar{v})| \\ &\leq \max(L_1, L_2)(|u - \bar{u}| + |v - \bar{v}|), \end{aligned}$$

by triangle inequality and since f is locally Lipschitz, for a large constant, we can remove the last term.

(d) If $a(u,v) < 0$ and $a(\bar{u},\bar{v}) \geq 0$

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |f(v) - f(\bar{u})| \\ &= |f(v) - f(\bar{v}) + f(\bar{v}) - f(u) + f(u) - f(\bar{u})| \\ &\leq |f(v) - f(\bar{v})| + |f(u) - f(\bar{u})| + |f(\bar{v}) - f(u)| \\ &\leq L_1|f(v) - f(\bar{v})| + L_2|f(u) - f(\bar{u})| \\ &\leq \max(L_1, L_2)(|v - \bar{v}| + |u - \bar{u}|), \end{aligned}$$

by triangle inequality and since f is locally Lipschitz, for a large constant, we can remove the last term.

2. Next we prove that the Godunov flux is locally Lipschitz continuous. Godunov flux can be written as

$$f^*(u,v) = \begin{cases} \min_{[u,v]} f(w) & u \le v, \\ \max_{[u,v]} f(w) & u > v. \end{cases}$$

Similarly I consider the following cases.
If $u \le v$ and $\bar{u} \le \bar{v}$:

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |\min_{[u,v]} f(w) - \min_{[\bar{u},\bar{v}]} f(w)| \\ &= |\min(f(u), f(v)) - \min(f(\bar{u}), f(\bar{v}))|. \end{aligned}$$

We will get four possible cases from here depending on the values and functions

$$|f(u) - f(\bar{u})|, \quad |f(v) - f(\bar{v})|, \quad |f(u) - f(\bar{v})|, \quad |f(v) - f(\bar{u})|.$$

The first two cases are simple, since we have f locally Lipschitz

$$|f(u) - f(\bar{u})| \le L_1 |u - \bar{u}|, \quad |f(v) - f(\bar{v})| \le L_2 |v - \bar{v}|.$$

The last two cases can be similarly bounded as in Part 1 (Case 3) and Part 1 (Case 4) for Roe flux.
If $u > v$ and $\bar{u} > \bar{v}$:

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |\max_{[u,v]} f(w) - \max_{[\bar{u},\bar{v}]} f(w)| \\ &= |\max(f(u), f(v)) - \max(f(\bar{u}), f(\bar{v}))|. \end{aligned}$$

Similarly, we have four cases and get similar result as in the previous case.
If $u \le v$ and $\bar{u} > \bar{v}$:

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |\min_{[u,v]} f(w) - \max_{[\bar{u},\bar{v}]} f(w)| \\ &= |\min(f(u), f(v)) - \max(f(\bar{u}), f(\bar{v}))|. \end{aligned}$$

If $u > v$ and $\bar{u} \le \bar{v}$:

$$\begin{aligned} |f^*(u,v) - f^*(\bar{u},\bar{v})| &= |\max_{[u,v]} f(w) - \min_{[\bar{u},\bar{v}]} f(w)| \\ &= |\max(f(u), f(v)) - \min(f(\bar{u}), f(\bar{v}))|. \end{aligned}$$

These last two cases also can similarly be treated as previous cases.