Homework 2

Zhanna Sakayeva

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1 Problem 1

We consider Lax-Wendroff scheme given by

$$u_{k,l+1} = u_{k,l} - \frac{h_t}{2h_x} c[u_{k+1,l} - u_{k-1,l}] + \frac{h_t^2}{2h_x^2} c^2 [u_{k+1,l} - 2u_{k,l} + u_{k-1,l}]$$

where $t \in \mathbb{R}_+$, space $x \in [0, L]$ and assume the periodic boundary conditions $u(x, 0) = u(x, L)$.
We need to establish the conditions under which the scheme is stable using von Neumann stability analysis. We let $\lambda = \frac{h_t}{h_x}$. Now we simplify our scheme by collecting terms with u_{l+1} and u_l

$$u_{k,l+1} = \Big(\frac{c}{2}\lambda + \frac{c^2}{2}\lambda^2\Big) u_{k-1,l} + \Big(1 - c^2\lambda^2\Big) u_{k,l} + \Big(\frac{c^2}{2}\lambda^2 - \frac{c}{2}\lambda\Big) u_{k+1,l}.$$

We observe that matrices P_h and Q_h will take the following forms

$$P_h = I \qquad Q_h = tridiag\Big(\frac{c}{2}\lambda + \frac{c^2}{2}\lambda^2, 1 - c^2\lambda^2, \frac{c^2}{2}\lambda^2 - \frac{c}{2}\lambda\Big).$$

More precisely,

$$p_k = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases} \qquad q_k = \begin{cases} \frac{c}{2}\lambda + \frac{c^2}{2}\lambda^2 & k = 1 \\ 1 - c^2\lambda^2 & k = 0 \\ \frac{c^2}{2}\lambda^2 - \frac{c}{2}\lambda & k = -1 \\ 0 & otherwise \end{cases}.$$

Next we calculate the Fourier transforms:

$$\hat{p}(\phi) = 1 \quad \text{and}$$

$$\hat{q}(\phi) = \Big(\frac{c}{2}\lambda + \frac{c^2}{2}\lambda^2\Big) e^{-i\phi} + \Big(1 - c^2\lambda^2\Big) + \Big(\frac{c^2}{2}\lambda^2 - \frac{c}{2}\lambda\Big) e^{i\phi}.$$

If we simplify, we get

$$\hat{q}(\phi) = 1 - ci\lambda sin(\phi) - c\lambda^2 + c^2\lambda^2 cos(\phi).$$

By $sin^2(\frac{\phi}{2}) = \frac{1-cos(\phi)}{2}$ we can express $\hat{q}(\phi)$ as

$$\hat{q}(\phi) = 1 - ci\lambda sin(\phi) - 2c^2\lambda^2 sin^2(\frac{\phi}{2}).$$

To study von Neumann stability analysis we need

$$max|s(\phi)|^2 = max\left|\frac{\hat{q}(\phi)}{\hat{p}(\phi)}\right|^2 = max|\hat{q}(\phi)|^2 \leq 1.$$

So if we simplify further

$$\begin{aligned}
|\hat{q}(\phi)|^2 &= (1 - 2c^2\lambda^2 sin^2(\frac{\phi}{2}))^2 + (c\lambda sin(\phi))^2 \\
&= (1 - 2c^2\lambda^2 sin^2(\frac{\phi}{2}))^2 + (2c\lambda sin(\frac{\phi}{2})cos(\frac{\phi}{2}))^2 \\
&= 1 - 4c^2\lambda^2 sin^2(\frac{\phi}{2}) + 4c^4\lambda^4 sin^4(\frac{\phi}{2}) + 4c^2\lambda^2 sin^2(\frac{\phi}{2})cos^2(\frac{\phi}{2}) \\
&= 1 - 4c^2\lambda^2 sin^2(\frac{\phi}{2})(1 - cos^2(\frac{\phi}{2})) + 4c^4\lambda^4 sin^4(\frac{\phi}{2}) \\
&= 1 - 4c^2\lambda^2 sin^4(\frac{\phi}{2}) + 4c^4\lambda^4 sin^4(\frac{\phi}{2}) \\
&= 1 - 4c^2\lambda^2(1 - c^2\lambda^2) sin^4(\frac{\phi}{2}).
\end{aligned}$$

From this part we see that $|\hat{q}(\phi)|^2 \leq 1$ if and only if $4c^2\lambda^2(1-c^2\lambda^2)sin^4(\frac{\phi}{2}) \geq 0$. It is clear that $4c^2\lambda^2$ and $sin^4(\frac{\phi}{2})$ are non-negative. So we check when this equation holds $(1 - c^2\lambda^2) \geq 0$. Hence it yields the following condition $|c\lambda| \leq 1$. Hence by von Neumann analysis we conclude that Lax-Wendroff scheme is stable if and only if $|c\lambda| \leq 1$, where $\lambda = \frac{h_t}{h_x}$.