Homework 5

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1 Problem 1

1. Let $\Omega \subset \mathbb{R}^2$ be bounded. Consider the space

$$H(\mathrm{div}, \Omega) := \{\mathbf{u} \in L^2(\Omega)^2 : \nabla^w \cdot \mathbf{u} \in L^2(\Omega)\},$$

where $\cdot^w$ indicates that the derivative is to be taken in the weak sense. Let the vector components of $\mathbf{u}$ be piecewise $C^1(\bar{\Omega})$ functions. You may assume that the boundaries between the pieces are smooth away from intersections. Show that $\mathbf{u} \in H(div, \Omega)$ if and only if for any smooth curve $\Gamma \subset \Omega$ we have $[\mathbf{u}] \cdot \hat{\mathbf{n}} = 0$ almost everywhere on Γ, where $[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$ represents the jump across the surface, and $\hat{\mathbf{n}}$ is the unit normal vector to Γ. Consider a divergence on a rectangle domain

$$\infty > \int_R div\, u = \int_R \nabla \cdot u = \int_{R_1(||)} \hat{n}\mathbf{u} + \int_{R_2(\perp)} \hat{n}\mathbf{u} = \lim_{h \to 0} \frac{\int_{R_1(||)} \hat{n}\mathbf{u}}{h} \to 0.$$

Hence $\mathbf{u}$ is a weak solution, or $\mathbf{u} \in H(div, \Omega)$.

2. Consider the model problem

$$\begin{aligned} -\Delta u &= f \quad (x \in \Omega), \\ u &= 0 \quad (x \in \partial\Omega), \end{aligned}$$

with convex Ω for $f \in L^2(\Omega)$ with the associated weak form

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad (v \in H_0^1(\Omega)).$$

Let $u \in H_0^1(\Omega)$ be the unique solution of this problem.
Clearly, $\nabla^w u \in L^2(\Omega)^2$. But is the weak divergence of $\nabla^w u$ also in $L^2(\Omega)$? Yes, if we let $v = div\, \nabla^w u$. Then by divergence theorem

$$\int_\Omega v\phi = -\int_\Omega \nabla^w u \nabla\phi,$$

for some $\phi \in C^\infty(\Omega)$, and $-\Delta u = -div\, \nabla^w u = f$ holds by the definition of weak derivative.

3. Now let $V_h \subset C^0(\bar{\Omega})$ be a finite element space and $u_h \in V_h$ be a solution of
$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h \quad (v_h \in V_h).$$
Is $\nabla^w u_h \in L^2(\Omega)$?.
Yes, it is by construction, since weak derivative of u, $\nabla^w u$ is in $L^2(\Omega)$.

In view of part 1, is the divergence of $\nabla^w u_h$ in $L^2(\Omega)$?
No, it does not hold in this case. Similarly we can start by definition
$$\int_\Omega v_h \phi = -\int_\Omega \nabla^w u_h \nabla \phi,$$
with $v_h = div\, \nabla^w u_h$.
$$-\int_\Omega \nabla^w u_h \nabla \phi = \int_\Omega div\, \nabla^w u_h \phi + \int_\Gamma [\nabla^w u_h] \cdot \hat{n} \phi,$$
which will be equal to $\int_\Omega v_h \phi$ if only $\int_\Gamma [\nabla^w u_h] \cdot \hat{n} \phi = 0$. Hence we cannot set $v_h = div\, \nabla^w u_h$ pointwise and the divergence of $\nabla^w u_h$ is not in $L^2(\Omega)$.