2 Problem 2

We are given the wave equation $u_{tt} = c^2\Delta u$. We will show that the energy

$$E = \int_{\mathbb{R}^d} [u_t^2 + c^2(\nabla u)^2]dx$$

is constant. We start by taking derivative of energy with respect to t.

$$\begin{aligned}\frac{dE}{dt} &= \partial_t \int_{\mathbb{R}^d} [u_t^2 + c^2(\nabla u)^2]dx = \int_{\mathbb{R}^d} 2u_t u_{tt} + c^2\nabla u \nabla u_t dx \\ &= \int_{\Omega_t} u_t u_{tt} + c^2[-\int_{\Omega_t} u_t\Delta u + \int_{\Omega_t} \frac{\partial u}{\partial \mu} u_t dS].\end{aligned}$$

Where the last expression was obtained by applying Green's formula. Now we observe that the last term will be zero, since we assume at time $t = 0$, $u = 0$ outside a bounded set. Hence $u_t = 0$. So we now get

$$\frac{dE}{dt} = \int_{\Omega_t} u_t(u_{tt} - c^2\Delta u) = 0,$$

since u is a solution to the wave equation. So derivative of energy with respect to time is zero, we may conclude that the energy is constant.