Homework 3

Zhanna Sakayeva

March 2022

1 Problem 1

We will analyze ‘multi-level‘ finite difference schemes. We are given ’leapfrog’ method

$$\frac{u_{k,l+1} - u_{k,l-1}}{2h_t} + a\frac{u_{k+1,l} - u_{k-1,l}}{2h_x} = 0.$$

(a) We start from the expression above

$$u_{k,l+1} - u_{k,l-1} + \lambda(u_{k+1,l} - u_{k-1,l}) = 0.$$

$$u_{k,l+1} - u_{k,l-1} = \lambda(u_{k+1,l} - u_{k-1,l}). \tag{1}$$

Now we apply Fourier inversion formula for u_l

$$u_{k,l} = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{ikh\phi}\hat{u}_l(\phi)d\phi.$$

And substitute into (1)

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{ikh\phi}\hat{u}_{l+1}(\phi)d\phi - \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{ikh\phi}\hat{u}_{l-1}(\phi)d\phi = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \lambda e^{ikh\phi}(e^{ih\phi} - e^{-ih\phi})\hat{u}_l(\phi)d\phi$$

If we combine all those expressions

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{ikh\phi}(\hat{u}_{l+1}(\phi) + 2i\lambda sin(\phi)\hat{u}_l(\phi) - \hat{u}_{l-1}(\phi))d\phi = 0$$

by noting that $(e^{ih\phi} - e^{-ih\phi}) = 2isin(\phi)$. By uniqueness of Fourier transform we say that the integrand must be equal to zero for each l. So we obtain

$$\hat{u}_{l+1}(\phi) + 2i\lambda sin(\phi)\hat{u}_l(\phi) - \hat{u}_{l-1}(\phi) = 0. \tag{2}$$

(b)

$$\hat{u}_0(\phi)\hat{g}(\phi)^{l+1} + 2i\lambda sin(\phi)\hat{u}_0(\phi)\hat{g}(\phi)^l - \hat{u}_0(\phi)\hat{g}(\phi)^{l-1} = 0$$

If we divide by $\hat{g}(\phi)^{l-1}$ we get

$$\hat{g}^2 + 2i\lambda sin(\phi)\hat{g} - 1 = 0. \tag{3}$$

In the non-degenerate case we can solve this quadratic equation to get two values of propagator $\hat{g}$.

$$\hat{g}_{\pm} = -i\lambda sin(\phi) \pm \sqrt{1-\lambda^2 sin^2(\phi)} \tag{4}$$

We can find $\hat{a}$ and $\hat{b}$ by considering $l=0$ and $l=1$ in this equation

$$\hat{u}_l(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^l + \hat{b}(\phi)\Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big]. \tag{5}$$

$$\hat{u}_0(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^0 + \hat{b}(\phi)\Big[\frac{\hat{g}_-(\phi)^0 - \hat{g}_+(\phi)^0}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big] = \hat{a}(\phi),$$

$$\hat{u}_1(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^1 + \hat{b}(\phi)\Big[\frac{\hat{g}_-(\phi)^1 - \hat{g}_+(\phi)^1}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big] = \hat{u}_0(\phi)\hat{g}_+(\phi) + \hat{b}(\phi).$$

Hence we have

$$\hat{a}(\phi) = \hat{u}_0(\phi),$$

$$\hat{b}(\phi) = \hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi)$$

for $\hat{a}(\phi)$ and $\hat{b}(\phi)$. Now we get

$$\hat{u}_l(\phi) = \hat{u}_0(\phi)\hat{g}_+(\phi)^l + (\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi))\Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big].$$

We see that it satisfies our main equation (2)

$$\hat{u}_0(\phi)\hat{g}_+(\phi)^{l+1} + (\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi))\Big[\frac{\hat{g}_-(\phi)^{l+1} - \hat{g}_+(\phi)^{l+1}}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big] +$$
$$+ 2i\lambda sin(\phi)(\hat{u}_0(\phi)\hat{g}_+(\phi)^l + (\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi))\Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big]) -$$
$$- \hat{u}_0(\phi)\hat{g}_+(\phi)^{l-1} - (\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi))\Big[\frac{\hat{g}_-(\phi)^{l-1} - \hat{g}_+(\phi)^{l-1}}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big] = 0,$$

since if we divide by $\hat{g}(\phi)^{l-1}$ we will get the same quadratic equation (3) in each term. For degenerate case

$$\hat{u}_l(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^l + \hat{b}(\phi)l\hat{g}_+(\phi)^{l-1} = \hat{u}_0(\phi)\hat{g}_+(\phi)^l + \Big(\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi)\Big)l\hat{g}_+(\phi)^{l-1}$$

We see that the general solution for degenerate case also satisfies initial condition and Leapfrog equation (2).

$$\hat{u}_0(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^0 + \hat{b}(\phi)0\hat{g}_+(\phi)^{-1} = \hat{a}(\phi),$$

$$\hat{u}_1(\phi) = \hat{a}(\phi)\hat{g}_+(\phi)^1 + \hat{b}(\phi)\hat{g}_+(\phi)^0 = \hat{u}_0(\phi)\hat{g}_+(\phi) + \hat{b}(\phi),$$

since we can represent it as a limit of

$$\hat{u}_l(\phi) = \hat{u}_0(\phi)\hat{g}_+(\phi)^l + (\hat{u}_1(\phi) - \hat{u}_0(\phi)\hat{g}_+(\phi))\Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big].$$

Particularly by L'Hopital rule with $\hat{g}_+ = \hat{g}_-$

$$\lim \Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big] \overset{\text{L-H}}{=} l\hat{g}_+(\phi)^{l-1}.$$

Hence, we can conclude that the general solution for degenerate case also satisfies Leapfrog equation, as the general solution for non-degenerate case does.

(c) We will now show L^2-stability of the scheme. First consider non-degenerate case and observe that

$$|\hat{g}_\pm(\phi)| \leq 1,$$

if λ is constant. From (4) with $|\lambda| \leq 1$ we have

$$|\hat{g}_+|^2 = |\hat{g}_-|^2 = 1 - (\lambda)^2 sin^2(\phi) + (\lambda sin(\phi)^2 = 1$$

If $|\lambda| > 1$ then $\phi = \pi/2$, we have (4) from

$$|\hat{g}_-(\pi/2)| = |\lambda| + \sqrt{(\lambda)^2 - 1} > 1$$

It means that this is unstable. Hence we consider $\lambda \leq 1$.
By using the bounds above L^2-norm of the solution will be

$$\begin{aligned}
\|u_l\|_2 = \frac{1}{2\pi}\|\hat{u}_l\|_2 &= \frac{1}{2\pi}\left(\|\hat{a}(\phi)\hat{g}_+(\phi)^l + \hat{b}(\phi)\Big[\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\Big]\|_2\right) \\
&\leq \frac{1}{2\pi}\left(\|\hat{a}(\phi)\|\|\hat{g}_+(\phi)^l\|_\infty + \|\hat{b}(\phi)\|\|\frac{\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\|_\infty\right) \\
&\leq \frac{1}{2\pi}\left(\|\hat{a}(\phi)\|\|\hat{g}_+(\phi)^l\|_\infty + \|\hat{b}(\phi)\|\|\frac{1}{\hat{g}_-(\phi) - \hat{g}_+(\phi)}\|_\infty \|\hat{g}_-(\phi)^l - \hat{g}_+(\phi)^l\|_\infty\right) \\
&\leq C
\end{aligned}$$

In degenerate case $\hat{g}_+(\phi)$ (and $\phi = \pm\frac{\pi}{2}$ equals to $\hat{g}_-(\phi)$ if only $|\lambda sin(\phi)| = 1$ but we already know that $|\lambda \leq 1|$ from the first case.

$$\begin{aligned}
\|u_l\|_2 = \frac{1}{2\pi}\|\hat{u}_l\|_2 &= \frac{1}{2\pi}\left(\|\hat{a}(\phi)\hat{g}_+(\phi)^l + \hat{b}(\phi)l\hat{g}_+(\phi)^{l-1}\|_2\right) \\
&\leq \frac{1}{2\pi}\left(\|\hat{a}(\phi)\|\|\hat{g}_+(\phi)^l\|_\infty + \|\hat{b}(\phi)\|\|l\hat{g}_+(\phi)^{l-1}\|_\infty\right) \\
&\leq \frac{1}{2\pi}\left(\|\hat{a}(\phi)\|\|\hat{g}_+(\phi)^l\|_\infty + \|\hat{b}(\phi)\|\|l\|\|\hat{g}_+(\phi)^{l-1}\|_\infty\right)
\end{aligned}$$

We can see that for the $\phi = \pm\frac{\pi}{2}$ the solution has the norm that grows linearly in l, and we cannot say that it is bounded for $|\lambda| = 1$. Therefore, the conclusion is that the scheme is unstable for $\lambda = 1$, hence only stable for $\lambda < 1$.

(d) We will compute the phase speed. First note that

$$\hat{g}_+(kh_x) = -i\lambda sin(kh_x) + \sqrt{1 - \lambda^2 sin^2(kh_x)} = e^{-i\omega(kh_x)h_t}$$

and

$$\hat{g}_+(kh_x) = |\hat{g}_+(kh_x)|e^{i\phi(kh_x)} = e^{\log|\hat{g}_+(kh_x)|+i\phi(kh_x)}\omega(kh_x).$$

The phase speed is given by

$$v_{ph}(kh_x) = -\frac{\phi(kh_x)}{kh_t}.$$

Where $\phi(kh_x) = arctan(-\frac{\lambda sin(kh_x)}{\sqrt{1-\lambda^2 sin^2(kh_x)}})$ is an argument of propagator $\hat{g}_+(\phi)$ and assume that $h_t = a\lambda h_x$ with $a = 1$. Hence

$$v_{ph}(kh_x) = -\frac{arctan(-\frac{\lambda sin(kh_x)}{\sqrt{1-\lambda^2 sin^2(kh_x)}})}{kh_x\lambda}.$$

The plot of the phase speed for $\lambda \in [0.25, 0.5, 0.75, 0.9]$ is as follows.

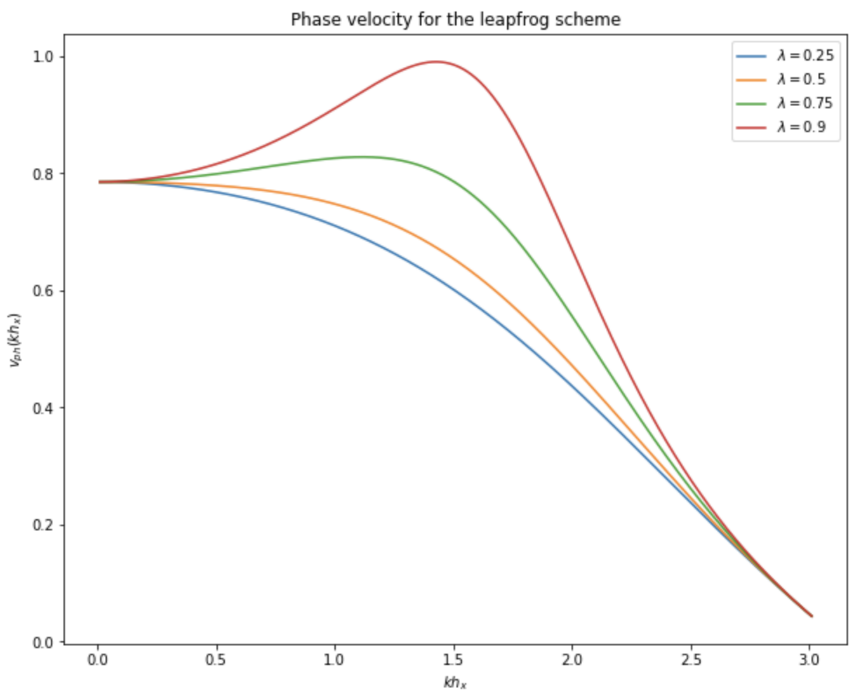

Figure 1: Phase speed for the leapfrog scheme