2 Problem 2

Consider the boundary value problem

$$\begin{aligned} -u'' + u' + u = f \quad \text{on } (0,1) \\ u'(0) = u'(1) = 0. \end{aligned}$$

1. Find a variational formulation of this problem over the space $H^1((0,1))$.

   Is your bilinear form $a(\cdot,\cdot)$ symmetric, i.e. does it satisfy $a(u,v) = a(v,u)$?

   It turns out that the symmetry assumption can be dropped from the Lax-Milgram theorem. If your form is non-symmetric, you may apply this version of the theorem. In the following, use Lax-Milgram to prove existence and uniqueness of the solution of the weak form of your BVP.
   We multiply by $v \in H(0,1)$ and integrate from 0 to 1

   $$\int_0^1 (-u'' + u' + u)v dx = \int_0^1 f v dx.$$

   Left hand side will be

   $$\begin{aligned} a(u,v) = \int_0^1 (-u'' + u' + u)v dx &= -vu'|_0^1 + \int_0^1 u'v' dx + \int_0^1 u'v dx + \int_0^1 uv dx \\ &= \int_0^1 (u'v' + u'v + uv) dx \end{aligned}$$

   by integration by parts and boundary conditions. Right hand side will be $F(v) = (f,v)$, which is continuous on $H^1(0,1)$. $a(u,v)$ is not symmetric since we have term $u'v$. So we have a variational problem: For $u \in H^1(0,1)$ such that $a(u,v) = F(v)$, $\forall v \in H^1(0,1)$ and Lax-Milgram Theorem guarantees both existence and uniqueness of the solution. Because we now show that even though bilinear form $a(u,v)$ is not symmetric, it is continuous and coercive.

2. Show that $a(\cdot,\cdot)$ is continuous (bounded). We consider the bound for $a(u,v)$ and get the following

   $$\begin{aligned} |a(u,v)| = |\int_0^1 (u'v' + u'v + uv)dx| &\le B\int_0^1 |u'||v'| + C\int_0^1 |u||v|dx + |\int_0^1 u'v dx| \\ &\le c\|u\|_{H^1}\|v\|_{H^1} + a\|u'\|_{L^1}\|v\|_{L^1} \\ &\le c\|u\|_{H^1}\|v\|_{H^1} \end{aligned}$$

   by triangle inequality and Schwarz' inequality.

3. Show that $a(\cdot,\cdot)$ is coercive. By definition of coercivity we consider

$$
\begin{aligned}
a(u,u) &= \int_0^1 (u'^2 + u'u + u^2)dx = \frac{1}{2}\int_0^1 (u'+u)^2 dx + \frac{1}{2}\int_0^1 (u'^2+u^2)dx \\
&\geq \frac{1}{2}\|u\|_{H^1}^2
\end{aligned}
$$

by $\|u\|_{H^1} = \sqrt{\int (u')^2 + u^2}$.

4. Now consider the related problem

$$
\begin{aligned}
-u'' + ku' + u = f \quad \text{on } [0,1] \\
u'(0) = u'(1) = 0,
\end{aligned}
$$

where k is some constant. Demonstrate that the Lax-Milgram theorem may not be applied to the variational formulation over the space $H^1((0,1))$.

We can similarly obtain the weak formulation

$$
\int_0^1 u'v'dx + k\int_0^1 u'vdx + \int_0^1 uvdx = \int_0^1 fvdx.
$$

$F(v)$ is continuous from part 1. Now we check the conditions for $a(u,v)$. Continuity holds in a similar way as in part 1

$$
\begin{aligned}
|a(u,v)| = |\int_0^1 (u'v' + ku'v + uv)dx| &\leq B\int_0^1 |u'||v'| + C\int_0^1 |u||v|dx + Ak|\int_0^1 u'vdx| \\
&\leq c\|u\|_{H^1}\|v\|_{H^1} + a\|u'\|_{L^1}\|v\|_{L^1} \\
&\leq c\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}
$$

However the coercivity assumption is violated.

$$
\begin{aligned}
a(u,u) &= \int_0^1 (u'^2 + ku'u + u^2)dx = \int_0^1 (u'^2+u^2)dx + k\int_0^1 u'u \\
&\geq \left(1 - \frac{|k|}{2}\right)\|u\|_{H^1}^2
\end{aligned}
$$

by $\|u\|_{H^1} = \sqrt{\int (u')^2 + u^2}$. So it would be coercive if $|k| < 2$ and would not be coercive for large values of k. For example if $v = x$, $a(v,v) = 1 + 1/3 + k/2 = 0$ will give us $k = -8/3$.