2 Problem 2

We will investigate the stability of ETBS scheme applied to this IBVP

$$\frac{u_{k,l+1} - u_{k,l}}{h_t} = -a\frac{u_{k,l} - u_{k-1,l}}{h_x}, \quad (k \in \{1, 2, ..., N\}, l \in \{0, 1, 2, ...\}),$$

$$u_{k,0} = f(kh_x),$$

$$u_{0,l} = g(lh_t).$$

1. We consider spatial discretization only which yields an ODE. We can then apply Forward Euler.

$$u_t + a\frac{u_{k,l} - u_{k-1,l}}{h_x} = 0$$

$$\frac{\partial u_k}{\partial t} = -\frac{a}{h_x}(u_{k,l} - u_{k-1,l})$$

$$u_{0,l} = g(lh_t)$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \\ ... \\ \frac{\partial u_N}{\partial t} \end{bmatrix} = \frac{a}{h_x}\begin{bmatrix} -1 & 0 & 0 & ... & 0 \\ 1 & -1 & 0 & ... & 0 \\ 0 & 1 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_N \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ ... \\ g_N \end{bmatrix}$$

2. Stability of the system is determined by eigenvalues of the matrix A. $|1 - \frac{ah_t}{h_x}| < 1$ if $\lambda = \frac{ah_t}{h_x}$which leads to

$$0 < \lambda < 2$$

   Our CFL conditon region from ETBS using von Neumann stability analysis is subset of $0 < \lambda < 2$.

3. We simplify and set $a = 1$, $g(t) = 0$ and $\lambda = h_t/h_x = 3/2(h_t = 1, h_x = 2)$.

$$u_{l+1} = \frac{1}{h_x}(u_l + h_tAu_l) = \frac{1}{h_x}(I + h_tA)u_l.$$

   Hence we can write

$$B = (I/2 + h_tA/2) = -\frac{I}{2} + \frac{3E}{2},$$

   with

$$E = lower - traing(3/2, -3/2, 0).$$

Now using Binomial theorem we get B^n

$$B^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\frac{1}{2})^(n-j) (\frac{3}{2})^j E^j = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\frac{1}{2})^n 3^j E^j.$$

We start by

$$\|B^n\|_2 \geq \frac{\|B^n \xi\|}{\|\xi\|} \geq \frac{\| \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\frac{1}{2})^n 3^j E^j \xi \|_{L^2}}{\|\xi\|}.$$

$$(B^n \xi)_j = \sum_{j=0}^{n} \binom{n}{j} 3^j E^j (\frac{1}{2})^n$$

Next we consider L^2-norm of $B^n \xi$ where $\xi_j = (-1)^j$

$$\begin{aligned} \| \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\frac{1}{2})^n 3^j E^j \xi \|_2 &\leq \sqrt{ (\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\frac{1}{2})^n 3^j E^j \xi)^2 } \\ &\leq (\frac{1}{2})^n \frac{1}{\sqrt{N-n}} \sqrt{N} \|B^n\|_2 \end{aligned}$$

for $j > n$. Since $(\frac{n}{j})^j \leq \binom{n}{j}$.
and $\|E^j \xi_j\|_2 \leq \|E^j (-1)^j\| = \sqrt{N}$ for $N > n$. Hence

$$\|B^n\|_2 \geq 2^n \frac{\sqrt{N-n}}{\sqrt{N}}.$$

4. It fails because we do not know whether our matrix A is diagonalizable.