2 Problem 2

We consider IBVP for the advection equation given by

$$u_t + u_x = 0,\ u(x,0) = f(x),\ u(0,t) = u(1,t)$$

where $x \in [0,1]$ and $t \geq 0$.

- We start by substituting the Crank-Nicolson finite difference formulas for time and space into equation

  $$u_t = \frac{u_{k,l+1} - u_{k,l}}{h_t} \qquad \text{and} \qquad u_x = \frac{1}{2}\Big[\frac{u_{k+1,l+1} - u_{k-1,l+1}}{2h_x} + \frac{u_{k+1,l} - u_{k-1,l}}{2h_x}\Big].$$

  So we get

  $$\frac{u_{k,l+1} - u_{k,l}}{h_t} + \frac{1}{2}\Big[\frac{u_{k+1,l+1} - u_{k-1,l+1}}{2h_x} + \frac{u_{k+1,l} - u_{k-1,l}}{2h_x}\Big] = 0.$$

  We will simplify and get

  $$\frac{u_{k,l+1}}{h_t} - \frac{u_{k,l}}{h_t} + \frac{u_{k+1,l+1}}{4h_x} - \frac{u_{k-1,l+1}}{4h_x} + \frac{u_{k+1,l}}{4h_x} - \frac{u_{k-1,l}}{4h_x} = 0.$$

  Next, we multiply all terms by h_t and collect all the terms with u_{l+1} to the left hand side, and terms with u_l to the right hand side

  $$u_{k,l+1} + \frac{h_t}{4h_x}u_{k+1,l+1} - \frac{h_t}{4h_x}u_{k-1,l+1} = u_{k,l} - \frac{h_t}{4h_x}u_{k+1,l} + \frac{h_t}{4h_x}u_{k-1,l}.$$

  From here we observe that we can express the equation as $P_h u_{l+1} = Q_h u_l$. Where P_h and Q_h will take the following forms as a tridiagonal matrices.

  $$P_h = tridiag\Big(-\frac{h_t}{4h_x}, 1, \frac{h_t}{4h_x}\Big), \qquad Q_h = tridiag\Big(\frac{h_t}{4h_x}, 1, -\frac{h_t}{4h_x}\Big).$$

  Now we note that the periodic boundary condition $u_0 = u_3$ will allow us to write P_h and Q_h for a spatial grid containing $u_{0,l}, u_{1,l}, u_{2,l}, u_{3,l}$ as

  $$P_h = \begin{pmatrix} 1 & \frac{h_t}{4h_x} & -\frac{h_t}{4h_x} \\ -\frac{h_t}{4h_x} & 1 & \frac{h_t}{4h_x} \\ \frac{h_t}{4h_x} & -\frac{h_t}{4h_x} & 1 \end{pmatrix}, \quad Q_h = \begin{pmatrix} 1 & -\frac{h_t}{4h_x} & \frac{h_t}{4h_x} \\ \frac{h_t}{4h_x} & 1 & -\frac{h_t}{4h_x} \\ -\frac{h_t}{4h_x} & \frac{h_t}{4h_x} & 1 \end{pmatrix}.$$

  So we only get 3×3 matrices by periodic boundary condition.

  Then we solve for $u_{0,l+1}, u_{1,l+1}, u_{2,l+1}, u_{3,l+1}$. Our solution will take the form $u_{l+1} = P_h^{-1}Q_h u_l$. But we again will assume the periodic boundary conditions and get only three terms.

  $$\begin{pmatrix} u_{0,l+1} \\ u_{1,l+1} \\ u_{2,l+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{h_t}{4h_x} & -\frac{h_t}{4h_x} \\ -\frac{h_t}{4h_x} & 1 & \frac{h_t}{4h_x} \\ \frac{h_t}{4h_x} & -\frac{h_t}{4h_x} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -\frac{h_t}{4h_x} & \frac{h_t}{4h_x} \\ \frac{h_t}{4h_x} & 1 & -\frac{h_t}{4h_x} \\ -\frac{h_t}{4h_x} & \frac{h_t}{4h_x} & 1 \end{pmatrix} \begin{pmatrix} u_{0,l} \\ u_{1,l} \\ u_{2,l} \end{pmatrix}.$$

- In the similar manner, we start by substituting the Euler Backward scheme formulas into our advection equation

$$u_t = \frac{u_{k,l+1} - u_{k,l}}{h_t} \qquad \text{and} \qquad u_x = \Big[\frac{u_{k+1,l+1} - u_{k-1,l+1}}{2h_x}\Big].$$

So after simplification we get

$$u_{k,l+1} - u_{k,l} + \frac{h_t}{2h_x}u_{k+1,l+1} - \frac{h_t}{2h_x}u_{k-1,l+1} = 0.$$

Collecting the terms with u_{l+1} and u_l will give us

$$u_{k,l+1} - \frac{h_t}{2h_x}u_{k-1,l+1} + \frac{h_t}{2h_x}u_{k+1,l+1} = u_{k,l}.$$

From here we observe that P_h and Q_h will take the following forms

$$P_h = tridiag\Big(-\frac{h_t}{2h_x}, 1, \frac{h_t}{2h_x}\Big), \qquad Q_h = I.$$

By periodic boundary conditions, we get only 3×3 matrices in our case

$$P_h = \begin{pmatrix} 1 & \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} \\ -\frac{h_t}{2h_x} & 1 & \frac{h_t}{2h_x} \\ \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} & 1 \end{pmatrix}, \quad Q_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence we get

$$\begin{pmatrix} 1 & \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} \\ -\frac{h_t}{2h_x} & 1 & \frac{h_t}{2h_x} \\ \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{0,l+1} \\ u_{1,l+1} \\ u_{2,l+1} \end{pmatrix} = \begin{pmatrix} u_{0,l} \\ u_{1,l} \\ u_{2,l} \end{pmatrix}.$$

So if we find $u_{0,l+1}, u_{1,l+1}, u_{2,l+1}$

$$\begin{pmatrix} u_{0,l+1} \\ u_{1,l+1} \\ u_{2,l+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} \\ -\frac{h_t}{2h_x} & 1 & \frac{h_t}{2h_x} \\ \frac{h_t}{2h_x} & -\frac{h_t}{2h_x} & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_{0,l} \\ u_{1,l} \\ u_{2,l} \end{pmatrix}.$$