

LOCAL WELL-POSEDNESS FOR GENERALIZED SEMILINEAR BLACK-SCHOLES EQUATIONS

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ABSTRACT. We consider a semilinear parabolic equation which is based on a variety of nonlinear generalizations to the Black-Scholes equation of mathematical finance. In particular, we prove local well-posedness for a Cauchy problem with initial data in $W^{s,p}$ -type Sobolev spaces.

1. INTRODUCTION

The Black-Scholes equation, proposed by Black and Scholes in [4], is the linear parabolic partial differential equation

$$(1) \quad V_\tau + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_S - rV = 0.$$

Here, $V(S, \tau) : [0, \infty) \times [0, T] \longrightarrow \mathbb{R}$ is function which represents the monetary value of an *option*, which is a contract to buy or sell an asset at a specified price, called the *exercise price*, at a future time T . The value V is assumed to depend on the current time $\tau \leq T$, and the current price $S \geq 0$ of the underlying asset. The constant $\sigma > 0$ represents the *volatility*, which is a measure of the fluctuation of the value of the asset, while r represents an interest rate for a bank account, in which investors could deposit their money with a guaranteed rate of return instead of buying the option. This equation has become an important tool in the field of financial modeling, and more recently, the insurance industry [12].

In recent years, more sophisticated models of options pricing have appeared which have led to nonlinear generalizations of equation (1). Some relevant examples are:

- The Barles-Soner Model [1]:

$$V_\tau + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_S - rV = \frac{\sigma^2}{2} S^2 V_{SS} G(e^{r(T-\tau)} a^2 S^2 V_{SS}),$$

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where a is a parameter and G is a function which is a solution to a certain ordinary differential equation.

- The risk-adjusted Black-Scholes equation (RABS) [10, 7]:

$$(2) \quad V_\tau + \frac{\sigma^2}{2} S^2 (1 - \mu |SV_{SS}|^{-2/3} V_{SS}) V_{SS} + rSV_S - rV = 0,$$

with μ a parameter.

- The Çetin-Jarrow-Protter (CJP) model [5, 6]:

$$(3) \quad V_\tau + \frac{\sigma^2}{2} S^2 V_{SS} (1 + 2\rho SV_{SS}) = 0,$$

where ρ is a parameter.

- The Platen-Schweizer (PS) model [13, 6]:

$$V_\tau + \frac{\sigma^2}{2} \frac{S^2}{(1 - \rho SV_{SS})^2} V_{SS} = 0.$$

These equations are fully nonlinear equations, which can often be quite difficult to study.

Due to the difficulty in analyzing fully nonlinear partial differential equations, many authors have proposed semilinear models, which are significantly easier to study. Some examples of such semilinear models can be found in [8, 11, 2, 3, 9]. Since the appearance of these results, some authors have begun considering generalized Black-Scholes models. One example of interest to us is the generalized model of Sowrirajan and Balachandran [14], who considered generalized models of the form

$$V_\tau + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_S - rV = F(V)$$

where F is a Lipschitz function.

In the present work, we will consider more generalized Black-Scholes models which are based on the results listed above. The first model is an extension of the work of Sowrirajan and Balachandran given by the equation

$$(4) \quad u_\tau + AS^2u_{SS} + BSu_S + Cu = F_1(u)$$

with coefficients $A, B, C \in \mathbb{R}$, and with nonlinearities F which are Lipschitz functions; that is, they satisfy an inequality of the form

$$|F_1(u) - F_1(v)| \leq C|u - v|$$

for some constants $C > 0$.

To motivate the second equation we will study, recall the risk-adjusted Black-Scholes equation (2), and the Çetin-Jarrow-Protter model from equation (3). By introducing the new function $u = SV_{SS}$ as was done

in [7], it is easy to see that equations (2) and (3) can be rewritten in a modified Black-Scholes form as

$$\begin{aligned} u_\tau + A_1 S^2 u_{SS} + B_1 S u_S - B_1 u &= D_1 S(u^{4/3})_S + E_1 S^2(u^{4/3})_{SS}, \\ u_\tau + A_2 S^2 u_{SS} + B_2 S u_S - C_2 u + D_2 S(u^2)_S + E_2 S^2(u^2)_{SS} &= 0, \end{aligned}$$

respectively, for appropriate constants A_i, B_i, C_i, D_i , and E_i . Note that the coefficients on the left side of each equation differ from those in the original Black-Scholes equation (1). Regardless of this, these two equations have a similar structure, namely in that they are both of the form

$$u_\tau + A S^2 u_{SS} + B S u_S + C u = D S(F(u))_S + E S^2(F(u))_{SS}$$

for some coefficients $A, B, C, D, E \in \mathbb{R}$ and where $F(u) = u^p$ for some exponent $p > 1$. The presence of the second derivative in the right-hand side makes these equations quasilinear. If we assume that $E = 0$ we then obtain the semilinear equation

$$(5) \quad u_\tau + A S^2 u_{SS} + B S u_S + C u = S(F(u))_S.$$

This is the second equation which will be studied below.

To formulate our main results, we first rewrite equations (4) and (5) as nonlinear heat equations. To do this, we use a standard change of variables which is used in the study of Black-Scholes equations. Let

$$x = \ln S, \quad \text{and} \quad t = A(T - \tau).$$

A simple computation will show that, if $A \neq 0$, we may rewrite equations (4) and (5) as

$$\begin{aligned} u_t - u_{xx} &= A_1 u_x + A_2 F_1(u), \\ u_t - u_{xx} &= B_1 u + B_2 (F_2(u))_x, \end{aligned}$$

respectively. Note that the variable x can now be negative, in contrast to the variable S . In addition, the constants A_i and B_i are largely irrelevant to the analysis. We will thus set them all to be equal to 1.

It should be noted that in applications of Black-Scholes equations, the interest is primarily in the final value problem; that is, we want to find a function $u(S, \tau)$ such that $u(S, T) = f(S)$ for some specified function f . The change of variables above not only converts the equation into a heat-type equation, but it also converts the final value problem into an initial value problem, for which many standard tools apply. Thus, the above change of variables for τ is not just an artificial change which puts the equation in a standard form. With this observation in mind, we may now state the main results to be proved.

Theorem 1. *Let $F_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function. Then the Cauchy problem*

$$(6) \quad \begin{aligned} u_t - u_{xx} &= u_x + F_1(u) \\ u(x, 0) &= f(x) \end{aligned}$$

is locally well-posed in L^p for any $p \geq 1$ to be determined later, as well. That is, for all initial conditions $f \in L^p$, there is a $T > 0$ such that there exists a unique solution $u \in C([0, T]; L^p)$ to equation (6). Moreover, the solution map $f \mapsto u$ is continuous.

Theorem 2. *Let $F_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function. Then the Cauchy problem*

$$(7) \quad \begin{aligned} u_t - u_{xx} &= u + (F_2(u))_x \\ u(x, 0) &= f(x) \end{aligned}$$

is locally well-posed in L^p for any $p \geq 1$ to be determined later, as well. That is, for all initial conditions $f \in L^p$, there is a $T > 0$ such that there exists a unique solution $u \in C([0, T]; L^p)$ to equation (7). Moreover, the solution map $f \mapsto u$ is continuous.

The proofs of both of these theorems will involve a standard fixed-point argument. In section 2, we will introduce the necessary tools for the proofs. Then, in sections 3 and 4, we will use the results in section 2 to prove Theorems 1 and 2, respectively.

2. PRELIMINARIES

To set up the argument, we first recall some basic facts about the linear heat equation. Consider the linear Cauchy problem

$$\begin{aligned} u_t - u_{xx} &= G(x, t), \\ u(0) &= f. \end{aligned}$$

for $x \in \mathbb{R}$, $t \geq 0$. The solution to this problem is given by the well-known formula

$$(8) \quad u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t) f(y) dy + \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds,$$

where the function $\Phi(x, t)$ is the so-called *heat kernel*, given by

$$(9) \quad \Phi(x, t) = \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

To apply the expression in equation (8) to a nonlinear problem of the form

$$(10) \quad \begin{aligned} u_t - u_{xx} &= G(u, u_x), \\ u(0) &= f, \end{aligned}$$

define a function Ψ on $C([0, T]; L^p)$ by

$$\Psi(u) = \int_{\mathbb{R}} \Phi(x - y, t) f(y) dy + \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds,$$

where $G(y, s) = G(u(y, s), u_x(y, s))$ here. It is easy to see that the function $w = \Psi(u)$ satisfies

$$\begin{aligned} w_t - w_{xx} &= G(u, u_x), \\ w(0) &= f, \end{aligned}$$

If we can prove the existence of a fixed point u for the operator Ψ , then this fixed point will satisfy the nonlinear problem in equation (10).

The proof of the existence of such fixed points will follow from a standard contraction mapping argument. For this, we will need the following estimates on the heat kernel Φ :

Lemma 3. *The heat kernel $\Phi(x, t)$ satisfies the identities*

$$\begin{aligned} \bullet \quad \|\Phi(\cdot, t)\|_{L^p} &= \frac{C_p}{(4\pi t)^{\frac{1}{2}(1-1/p)}} \\ \bullet \quad \|\Phi_x(\cdot, t)\|_{L^p} &= \frac{C_p}{(4\pi t)^{1-1/2p}} \end{aligned}$$

for some constants C_p and all $t > 0$.

These can be proved by direct integration.

Next, we need the following estimates for the linear and nonlinear portion of the solutions:

Lemma 4. *Let $f \in L^p$ and $G \in C([0, T]; L^p(\mathbb{R}))$. Then the following estimates hold:*

$$(11) \quad \left\| \int_{\mathbb{R}} \Phi(x - y, t) f(y) dy \right\|_{L_t^\infty L_x^p} \leq \|f\|_{L^p}$$

$$(12) \quad \left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds \right\|_{L_t^\infty L_x^p} \leq CT \|G\|_{L_t^\infty L_x^p}$$

for some constant $C > 0$. If we further assume that G is differentiable, then

$$(13) \quad \left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G_x(y, s) dy ds \right\|_{L_t^\infty L_x^p} \leq CT^{1/2} \|G\|_{L_t^\infty L_x^p}$$

for some constant $C > 0$.

Proof. To begin the proof, we first observe that

$$\int_{\mathbb{R}} \Phi(x - y, t) f(y) dy = \Phi(\cdot, t) * f(x),$$

where $*$ denotes convolution in the spatial variable. Thus

$$\left\| \int_{\mathbb{R}} \Phi(x - y, t) f(y) dy \right\|_{L_x^p} = \|\Phi(\cdot, t) * f(x)\|_{L_x^p}.$$

By Young's inequality, we have

$$\|\Phi(\cdot, t) * f(x)\|_{L_x^p} \leq \|\Phi(\cdot, t)\|_{L_x^1} \|f\|_{L^p}.$$

By Lemma 3, this reduces to

$$\|\Phi(\cdot, t) * f(x)\|_{L_x^p} \leq \|f\|_{L^p}.$$

The desired result follows by taking the L^∞ norm in the time variable.

To prove equation (12), we first write

$$\int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds = \int_0^t \Phi(\cdot, t - s) * G(\cdot, s)(x) ds$$

By Minkowski's inequality, Young's inequality and Lemma 3, we have

$$\begin{aligned} \left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds \right\|_{L_x^p} &\leq \int_0^t \|\Phi(\cdot, t - s)\|_{L_x^1} \|G(\cdot, s)\|_{L_x^p} ds \\ &\leq \int_0^t \|G(\cdot, s)\|_{L_x^p} ds \\ &\leq t \|G\|_{L_t^\infty L_x^p}. \end{aligned}$$

Taking the L^∞ norm in time yields

$$\left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) G(y, s) dy ds \right\|_{L_t^\infty L_x^p} \leq T \|G\|_{L_t^\infty L_x^p},$$

as desired.

For the proof of estimate (13), we observe that

$$\begin{aligned} \int_{\mathbb{R}} \Phi(x - y, t - s) G_x(y, s) dy &= \Phi(\cdot, t - s) * G_x(\cdot, s)(x) \\ &= \Phi_x(\cdot, t - s) * G(\cdot, s)(x) \end{aligned}$$

by standard properties of convolutions. If we again apply Minkowski's inequality, Young's inequality, and Lemma 3, we obtain

$$\left\| \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) G_x(y, s) dy \right\|_{L_x^p} \leq C t^{1/2} \|G\|_{L_t^\infty L_x^p}.$$

Taking the L^∞ norm in time yields equation (13), as desired. \square

3. PROOF OF THEOREM 1

In this section, we consider equation (10) in the case

$$G(u, u_x) = u_x + F_1(u)$$

for some Lipschitz function F_1 with $F_1(0) = 0$. By Lemma 4, we have that

$$\Psi : C([0, T]; L^p(\mathbb{R})) \longrightarrow C([0, T]; L^p(\mathbb{R})).$$

Thus we need only show it is a contraction. This requires an estimate of the form

$$\|\Psi(u) - \Psi(v)\|_{L_t^\infty L_x^p} \leq C \|u - v\|_{L_t^\infty L_x^p}$$

for some constant $0 < C < 1$. To obtain such an estimate, we first observe that, for fixed $f \in L^p$, we have

$$\begin{aligned} \Psi(u) - \Psi(v) &= \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) [u_x(y, s) - v_x(y, s)] dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) [F_1(u(y, s)) - F_1(v(y, s))] dy ds. \end{aligned}$$

If we apply the estimate in equation (13), we see that

$$\begin{aligned} \left\| \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) [u_x(y, s) - v_x(y, s)] dy ds \right\|_{L_t^\infty L_x^p} \\ \leq C T^{1/2} \|u - v\|_{L_t^\infty L_x^p}. \end{aligned}$$

If we apply the estimate in equation (12), it is easy to see that

$$\begin{aligned} \left\| \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) [F_1(u(y, s)) - F_1(v(y, s))] dy ds \right\|_{L_t^\infty L_x^p} \\ \leq C T \|F_1(u) - F_1(v)\|_{L_t^\infty L_x^p} \end{aligned}$$

Since F_1 is Lipschitz, it follows that

$$\|F_1(u) - F_1(v)\|_{L_t^\infty L_x^p} \leq C \|u - v\|_{L_t^\infty L_x^p}$$

Putting all this together, we have shown that

$$\|\Psi(u) - \Psi(v)\|_{L_t^\infty L_x^p} \leq C(T + T^{1/2}) \|u - v\|_{L_t^\infty L_x^p}$$

for some constant $C > 0$. If we choose T small enough so that

$$C(T + T^{1/2}) < 1,$$

we will then have that Ψ is a contraction on $C([0, T]; L^p(\mathbb{R}))$.

Next, we prove that the solution map $f \mapsto u$ is continuous. Let $f, g \in L^p$, and let u_f and u_g be the corresponding fixed points. By applying estimates (11), (12), and (13), we may obtain

$$\begin{aligned} \|u_f - u_g\|_{L_t^\infty L_x^p} &= \|\Psi(u_f) - \Psi(u_g)\|_{L_t^\infty L_x^p} \\ &\leq \|f - g\|_{L_x^p} + C(T + T^{1/2})\|u_f - u_g\|_{L_t^\infty L_x^p}, \end{aligned}$$

which implies that

$$\|u_f - u_g\|_{L_t^\infty L_x^p} \leq \frac{1}{1 - C(T + T^{1/2})} \|f - g\|_{L_x^p}.$$

It follows that, for T as chosen above, the solution map is continuous. In fact, the computation above implies the solution map is Lipschitz.

4. PROOF OF THEOREM 2

We now consider equation (10) for

$$G(u, u_x) = u + (F_2(u))_x,$$

where F_2 is a differentiable Lipschitz function. As before, it suffices to prove an estimate of the form

$$\|\Psi(u) - \Psi(v)\|_{L_t^\infty L_x^p} \leq C\|u - v\|_{L_t^\infty L_x^p}$$

for some $0 < C < 1$. Proceeding exactly as we did in the proof of Theorem 1, we first observe that equation (12) implies that

$$\left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s)[u(y, s) - v(y, s)] dy ds \right\|_{L_t^\infty L_x^p} \leq CT\|u - v\|_{L_t^\infty L_x^p}.$$

Similarly, estimate (13) implies that

$$\begin{aligned} \left\| \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s)[(F_2(u(y, s)) - F_2(v(y, s)))_x] dy ds \right\|_{L_t^\infty L_x^p} \\ \leq CT^{1/2}\|F_2(u) - F_2(v)\|_{L_t^\infty L_x^p}. \end{aligned}$$

Applying the fact that F_2 is Lipschitz, we see that

$$\|F_2(u) - F_2(v)\|_{L_t^\infty L_x^p} \leq \|u - v\|_{L_t^\infty L_x^p}.$$

Combining these facts together, we have that

$$\|\Psi(u) - \Psi(v)\|_{L_t^\infty L_x^p} \leq C(T + T^{1/2})\|u - v\|_{L_t^\infty L_x^p}.$$

If we again choose T so that

$$C(T + T^{1/2}) < 1,$$

then we again have that Ψ is a contraction. The existence of a unique fixed point then follows from the Banach fixed point theorem. The proof that the solution map is Lipschitz (and therefore continuous) is analogous to that of Theorem 1, and so we will be omitted here.

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10 D. DA SILVA, K. IGIBAYEVA, A. KHOROSHEVSKAYA, AND ZH. SAKAYEVA

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