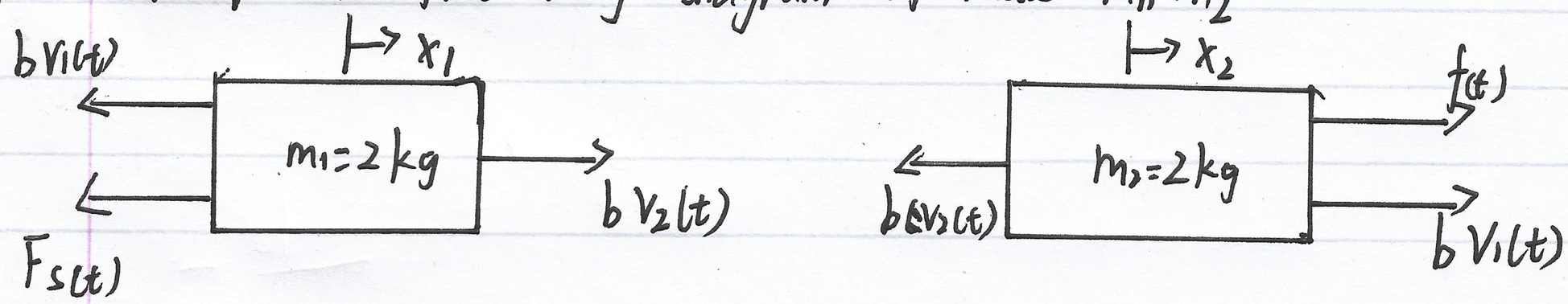
Part A.

Q1 The real life example: The suspension system for car

Answer

Q2. Draw the free body diagram of mass m_1, m_2



$m_1=2kg$, $m_2=2kg$, $b=2Ns/m$, $F_s=x_1^2 N$ have been known

① For m_1:

$$\sum F = m_1 a_1$$

$$bV_2(t)-bV_1(t) - F_s(t) = m_1 a_1$$

$$V=dx(t)/dt \quad a=d^2x(t)/dt^2$$

$$\begin{cases} V_2 = dx_2(t)/dt \\ V_1 = dx_1(t)/dt \\ a_1 = d^2x_1(t)/dt^2 \end{cases} \qquad \cancel{a_2 = d^2x_2(t)/dt^2}$$

So: $$2\frac{d^2x_2(t)}{dt} - 2\frac{dx_1(t)}{dt} - x_1^2(t) = m\frac{d^2x_1(t)}{dt^2}$$

$$2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + x_1^2(t) - 2\frac{dx_2(t)}{dt} = 0 \quad Eq(2)$$

② For m_2: $\sum F = m_2 a_2$

$$bV_1(t) + f(t) - bV_2(t) = m_2 a_2$$

$$m_2 a_2 - bV_1(t) + bV_2(t) = f(t)$$

$$\begin{cases} V_1(t) = dx_1(t)/dt \\ V_2(t) = dx_2(t)/dt \\ a_2 = d^2x_2(t)/dt^2 \end{cases}$$

So

$$2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t) \quad Eq(3)$$

Answer:

Q3. ~~Answer~~: Linearise the system for the Point $x_1(0)=1$, $x_2(0)=0$, $f(0)=0$

① The original equation is Eq(2) and Eq(3)

eq2: $$2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + x_1^2(t) - 2\frac{dx_2(t)}{dt} = 0$$

eq3: $$2\frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t)$$

The equilibrium point is $t = t_s = 0$

② Now Linearise differential equation for small excursions about equilibrium point

Let $t = t_s + \delta$, $t_s = 0$

So
$$\left.\begin{aligned} x_1(t) &= x_1(t_s+\delta) = x_1(0) + \delta x_1(t) = 1 + \delta x_1(t) \\ x_2(t) &= x_2(t_s+\delta) = x_2(0) + \delta x_2(t) = 0 + \delta x_2(t) \\ f(t) &= f(t_s+\delta) = f(0) + \delta f(t) = 0 + \delta f(t) \end{aligned}\right\} \text{Eq a}$$

And let $x_1^2(t) = F[x_1(t)]$

$$F[x_1(t)] = F[x_1(0) + \delta x_1(t)]$$
$$= F[x_1(0)] + c\,\delta x_1(t)$$

$$c = \left.\frac{dF[x_1(t)]}{dt}\right|_{t=0} = 2x_1(0) = 2$$

So $x_1^2(t) = 2\delta x_1(t) + 1$ Eq b

The $x_1(0)$, $x_2(0)$, $f(0)$ is constants, so its will be 0 in next step

③ Now substitute the Eq a, Eq b in to the original equation

So:
$$2\frac{d^2\delta x_1(t)}{dt^2} + 2\frac{d\delta x_1(t)}{dt} + (1 + 2\delta x_1(t)) - 2\frac{d\delta x_2(t)}{dt} = 0 \quad \text{Eq.(4)}$$

$$2\frac{d^2\delta x_2(t)}{dt^2} + 2\frac{d\delta x_2(t)}{dt} - 2\frac{d\delta x_1(t)}{dt} = \delta f(t) \quad \text{Eq(5)}$$

The result of linearising the system is Eq(4), Eq(5)

Part B:

Q4: Now apply Laplace transform on the linearised system in equation

Answer: Eq(6), Eq(7)

$$\text{Eq(6)}: \quad 2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + (1 + 2x_1(t)) - 2\frac{dx_2(t)}{dt} = 0$$

$$\text{Eq(7)}: \quad 2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t)$$

① Assuming the zero inital conditions is:

$x_1(0) = 0$ $\dot{x}_1(0) = 0$ $x_2(0) = 0$ $\dot{x}_2(0) = 0$

② Now apply the Laplace transfer for each item.

$$\left.\begin{aligned} \mathcal{L}\left(\frac{dx_1(t)}{dt}\right) &= \mathcal{L}(\dot{x}_1(t)) = sX_1(s) - x_1(0) = sX_1(s) + 0 \\ \mathcal{L}\left(\frac{d^2x_1(t)}{dt^2}\right) &= \mathcal{L}(\ddot{x}_1(t)) = s^2X_1(s) - sX_1(0) - \dot{x}(0) = s^2X_1(s) \\ \mathcal{L}\left(\frac{dx_2(t)}{dt^2}\right) &= \mathcal{L}(\dot{x}_2(t)) = sX_1(s) - x_2(0) = sX_2(s) \end{aligned}\right\} \text{Eq. c}$$

$f(s)$ to be $F(s)$

③ Substitute all the item (Eq.c) in to Equation Eq(6) Eq(7)

So Eq(6) to be:

$$2s^2X_1(s) + 2sX_1(s) + 2X_1(s) = 2sX_2(s)$$

result 1: $s^2X_1(s) + sX_1(s) + X_1(s) = sX_2(s)$

Eq(7) to be

result 2: $2s^2X_2(s) + 2sX_2(s) - 2sX_1(s) = F(s)$

Q5: Answer:

(a) The result of Q4 is:

$$\begin{cases} s^2X_1(s) + sX_1(s) + X_1(s) = sX_2(s) & \boxed{\text{Eq.(8)}} \\ 2s^2X_2(s) + 2sX_2(s) - 2sX_1(s) = F(s) & \boxed{\text{Eq.(9)}} \end{cases}$$

① The relationship between $X_1(s)$ and $X_2(s)$ can be obtined in Equation(8)

so

$$X_1(s) = \frac{1}{s^2+s+1} X_2(s) \qquad \boxed{\text{Eq.(9)}}$$

② Substitute the Eq(9) in to the Equation(8)

$$\left(2s^2 + 2s - \frac{2s^2}{s^2+s+1}\right) X_2(s) = F(s)$$

$$\frac{2s^4+4s^3+2s^2+2s}{s^2+s+1} X_2(s) = F(s)$$

③ $G(s) = \frac{\text{output}}{\text{input}} = \frac{X_2(t)}{f(t)} = \frac{X_2(s)}{F(s)} = \frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}$ is the transfer function

for the linearized system

(b) Answer:

Because the system order is the highest power of "s" in the denominator of the transfer function.

So: It is a 4th order systems

Q6 According to the Question 5, the transfer function is:

$$\frac{X_2(s)}{F(s)} = \frac{s^2+s+1}{2s^4+4s^3+2s^2+s} = \frac{s^2+s+1}{s(s^3+2s^2+s+1)} \quad Eq(7)$$

① The Poles are the roots of demominator:

So $s(s^3+2s^2+s+1)$ is the equation need to be solved

$s_1=0 \qquad s^3+2s^2+s+1=0$

② The cubic equation can be solved by the cardano's formula (get this method from BRILLIANT, the website is https://brilliant.org/wiki/cardano-method/)

So $s^3+2s^2+s+1=0 \quad a=1 \quad b=2 \quad c=1 \quad d=1 \; (ax^3+bx^2+cx+d=0)$

let $s=x$, has solution (can be clearer, avoid character repeat)

$$\begin{cases} x_1 = S+T-\frac{b}{3a} \\ x_2 = -\frac{S+T}{2}-\frac{b}{3a}+\frac{i\sqrt{3}}{2}(S-T) \\ x_3 = -\frac{S+T}{2}-\frac{b}{3a}-\frac{i\sqrt{3}}{2}(S-T) \end{cases}$$

where $S=\sqrt[3]{R+\sqrt{Q^3+R^2}}$, $T=\sqrt[3]{R-\sqrt{Q^3+R^2}}$

Let $x=y-\frac{2}{3}$, so the original equation can be $y^3+3Qy-2R=0$

In this equation, exists $\begin{cases} Q=(3ac-b^2)/9a^2 \\ R=\frac{9abc-27a^2d-2b^3}{54a^3} \end{cases}$

③ Substitute known data a,b,c,d into Q,R equation

$Q=(3ac-b^2)/9a^2=-\frac{1}{9}$

$R=\frac{9abc-27a^2d-2b^2}{54a^3}=-\frac{25}{54}$

④ substitute Q, R into S, T equation

$$S=\sqrt[3]{R+\sqrt{Q^3+R^2}}=-0.1140593 \quad T=\sqrt[3]{R-\sqrt{Q^3+R^2}}=-0.9741516$$

⑤ substitute S, T into x_1, x_2, x_3 equation

$$x_1=S+T-\frac{b}{3a}=-1.75488$$

$$x_2=-\frac{S+T}{2}-\frac{b}{3a}+\frac{j\sqrt{3}}{2}(S-T)$$

$$=-\frac{(-0.1140593-0.9741516)}{2}-\frac{2}{3}+\frac{\sqrt{3}}{2}(-0.1140593+0.9741516)j$$

$$=-0.12256+0.74486j$$

$$x_3=-\frac{S+T}{2}-\frac{b}{3a}-\frac{\sqrt{3}j}{2}(S-T)$$

$$=-\frac{(-0.1140593-0.9741516)}{2}-\frac{2}{3}-\frac{\sqrt{3}}{2}(-0.1140593+0.9741516)j$$

$$=-0.12256-0.74486j$$

⑥ Because having set $x=s$ at the start of calculation so, the ~~solution~~ roots of $s^3+2s^2+s+1=0$ is

$$\begin{cases} s_1=-1.75488 \\ s_2=-0.12256+0.74486j \\ s_3=-0.12256-0.74486j \end{cases}$$

so, ~~in~~ there are 4 poles: for $s^4+2s^3+s^2+s=0$

$$\begin{cases} s_1=0 \\ s_2=-1.75488 \\ s_3=-0.12256+0.74486j \\ s_4=-0.12256-0.74486j \end{cases}$$

⑦ The zero are roots of the molecules of transfer function $s^2+s+1=0$ can solve by quadratic formula ($ax^2+bx+c=0$)

(from website: https://www.cuemath.com/algebra/quadratic-equations/)

$\sqrt{\Delta}=\sqrt{b^2-4ac}$ $a=1, b=1, c=1$

so

$$s_1=\frac{-b+\sqrt{\Delta}}{2a}=-\frac{1}{2}+\frac{\sqrt{3}}{2}=-\frac{1}{2}+0.86602j$$

$$s_2=\frac{-b-\sqrt{\Delta}}{2a}=-\frac{1}{2}-\frac{\sqrt{3}}{2}=-\frac{1}{2}-0.86602j$$

⑧ In conclusion

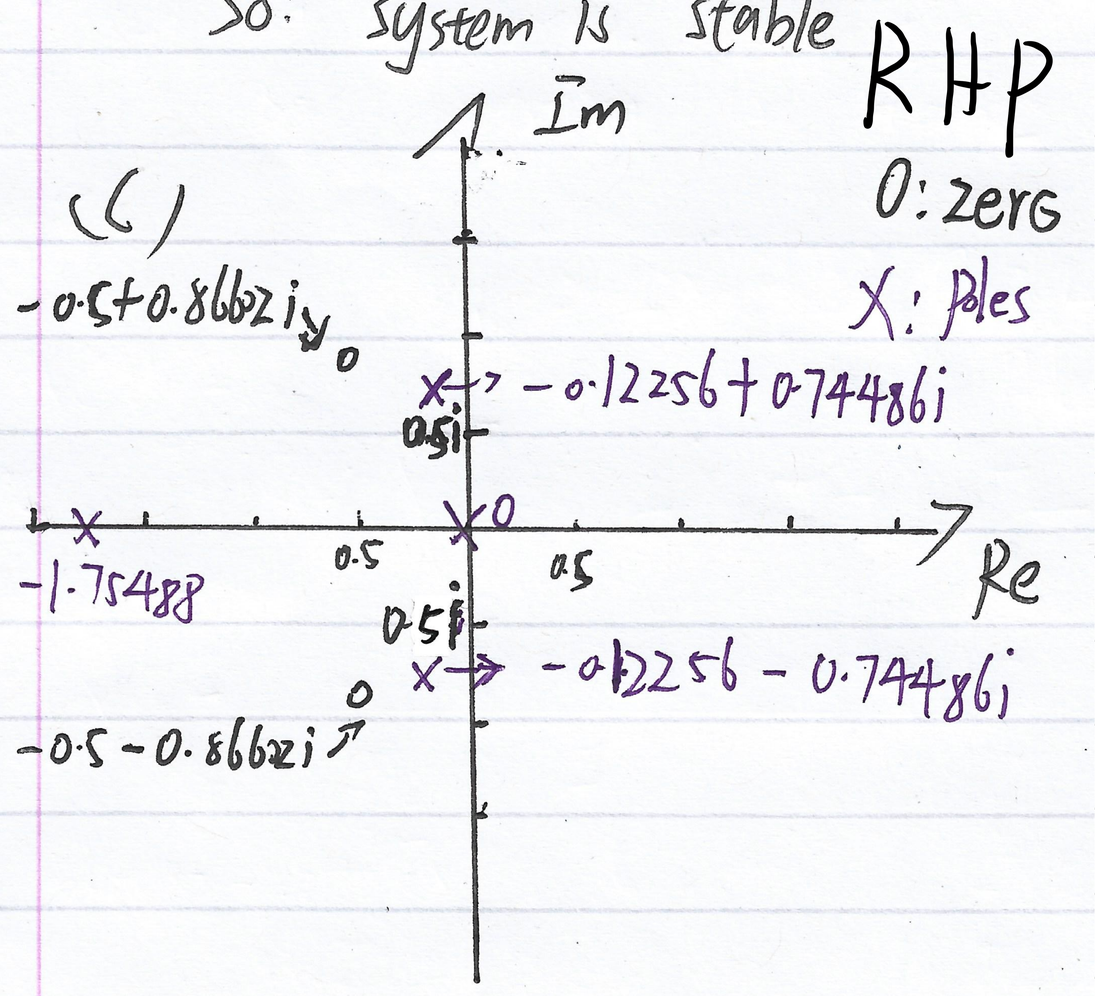
There are 2 zero $\begin{cases} s_1 = -0.5 + 0.86602i \\ s_2 = -0.5 - 0.86602i \end{cases}$

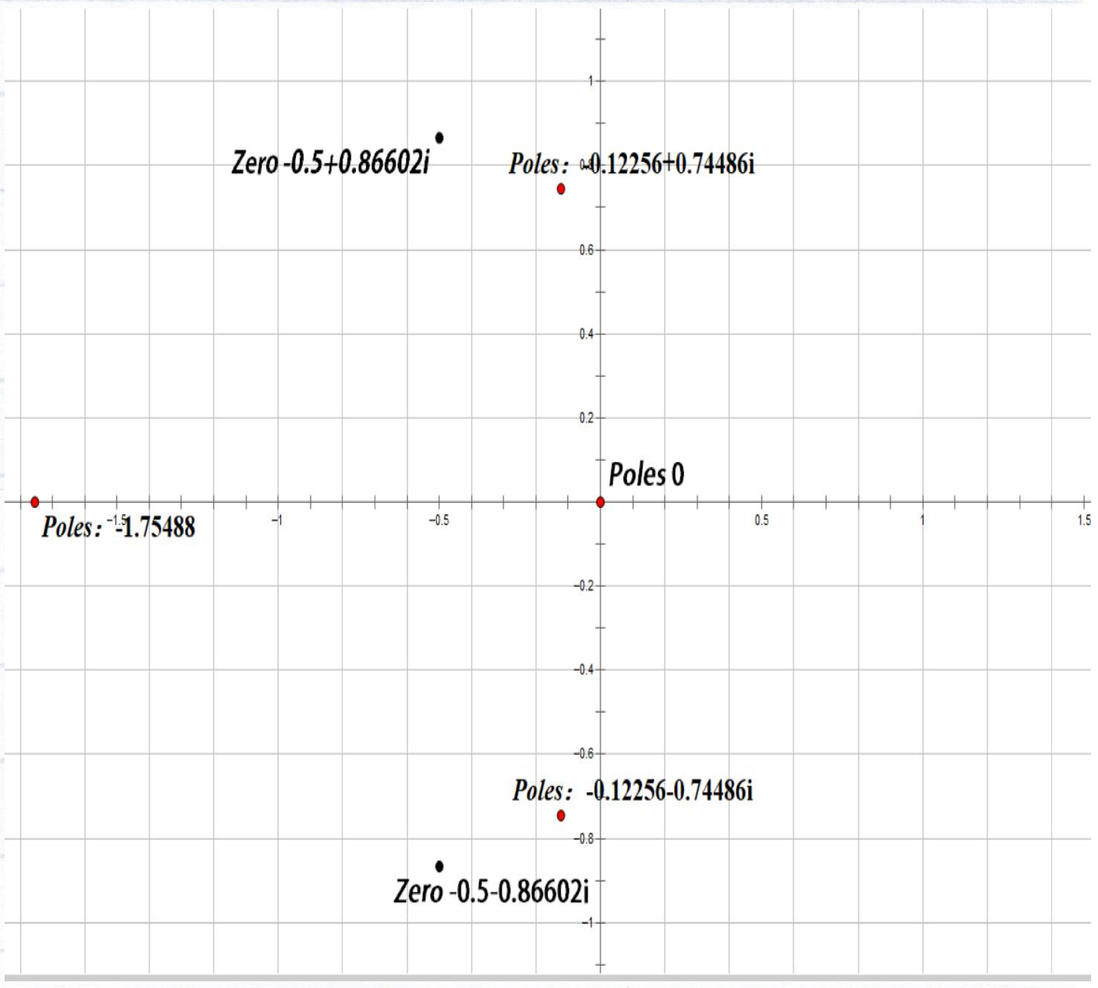
There are 4 poles $\begin{cases} s_1 = 0 \\ s_2 = -1.75488 \\ s_3 = -0.12256 + 0.74486i \\ s_4 = -0.12256 - 0.74486i \end{cases}$

(b) Because all the 4 poles have negative real part (0, -1.75488, -0.12256). If draw s-plane, the poles are all in left part.

So: system is stable

LHP (c)





Q7: Answer:

Final Value Theorem $\begin{cases} \text{La Place Transform}: \lim_{s \to 0} sF(s) \\ \text{Time function}: \lim_{t \to \infty} f(t) \end{cases}$

For unit impluse

$F(t) = \delta(t) \quad F(s) = 1$

So $\lim_{t \to \infty} X_2(t) = \lim_{s \to 0} [s X_2(s)]$

Because $s X_2(s) = F(s) \cdot G(s)$

$G(s) = \dfrac{s^2+s+1}{2s(s^3+2s^2+s+1)}$

So $sX_2(s) = \frac{s^2+s+1}{2(s^3+2s^2+s+1)}$

$$\lim_{t\to\infty}[x_2(t)] = \lim_{s\to 0}[sX_2(s)] = \frac{1}{2\times 1} = \frac{1}{2}$$

so, the final value is $\frac{1}{2}$

Q8 Eq(6): $2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + (1+2x_1(t)) - 2\frac{dx_2(t)}{dt} = 0$

Eq(7): $2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t)$

Answer:

Eq(6) to be: $2\ddot{x}_1 + 2\dot{x}_1 + (1+2x_1) - 2\dot{x}_2 = 0$ Eq(d)

Eq(7) to be: $2\ddot{x}_2 + 2\dot{x}_2 - 2\dot{x}_1 - f(t) = 0$ Eq(e)

let $\begin{cases} z_1 = x_1 \\ z_2 = \dot{x}_1 \\ z_3 = x_2 \\ z_4 = \dot{x}_2 \end{cases}$ So $\begin{cases} \dot{z}_1 = \dot{x}_1 = z_2 \\ \ddot{z}_2 = \ddot{x}_1 \\ \dot{z}_3 = \dot{x}_2 \\ \ddot{z}_4 = \ddot{x}_2 \end{cases}$

because the Equation Eq(d) and Eq(e)

so $\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_4 - z_1 - z_2 - \frac{1}{2} \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = -z_4 + z_2 + \frac{1}{2}f(t) \end{cases}$

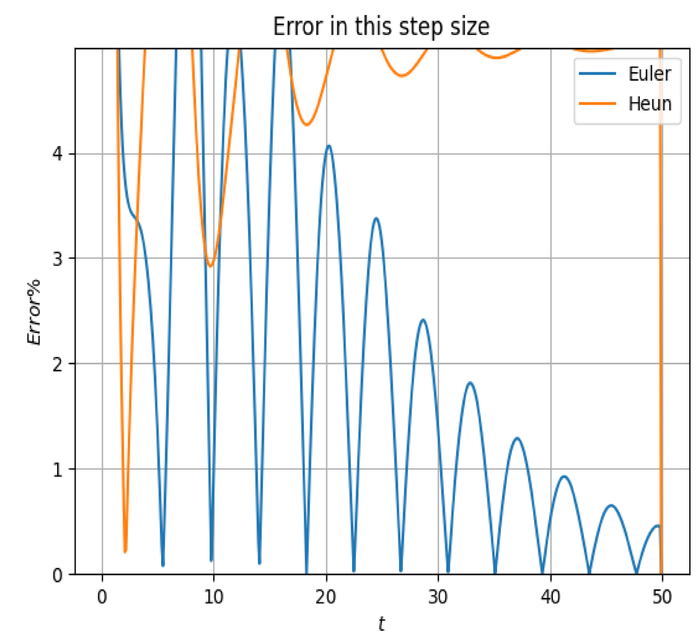
So $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} f(t) + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$

The Program see in zrx_model.py

(ii) Answer

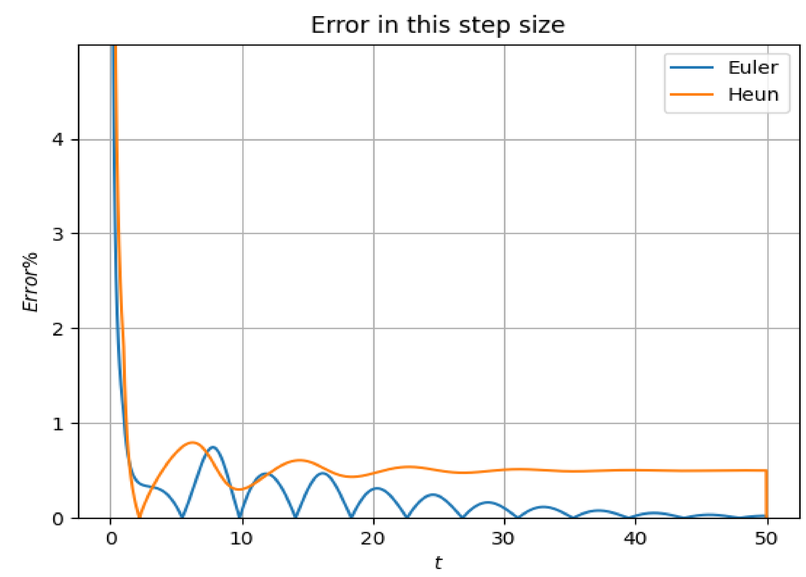
set $h=\Delta t=0.1$,

The mose of errors of two methods are more than 1%



set $h=\Delta t=0.01$

The mose of erros of two method less than 1%

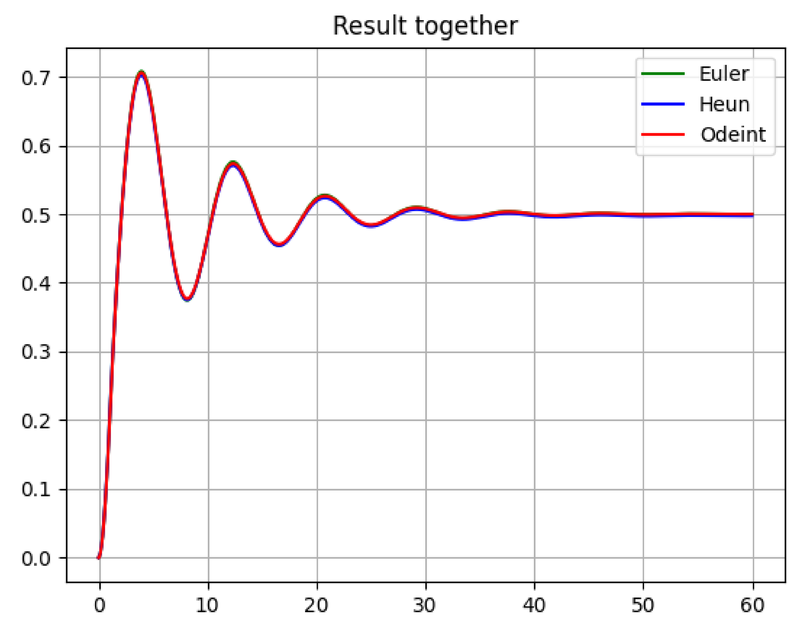


(iii) setting $t=60s$, the system was found start stabilising around 40s through simulation.

so set $t=40s+10s=50s$

To ensure system is stable.

Also, errors of both methods $<1\%$



(iv) Calculation by Python

Euler:

$E_{t\text{-}Euler} = \left|\frac{1}{2} - \text{result}\right| = 0.000788$

$|E_t|\% = \frac{E_{t\text{-}Euler}}{\frac{1}{2}} \times 100\% = 0.1577\%$

Heun:

$E_{t\text{-}Heun} = \left|\frac{1}{2} - \text{result}\right| = 0.003175$

$|E_t|\% = \frac{E_{t\text{-}Heun}}{\frac{1}{2}} \times 100\% = 0.6350\%$

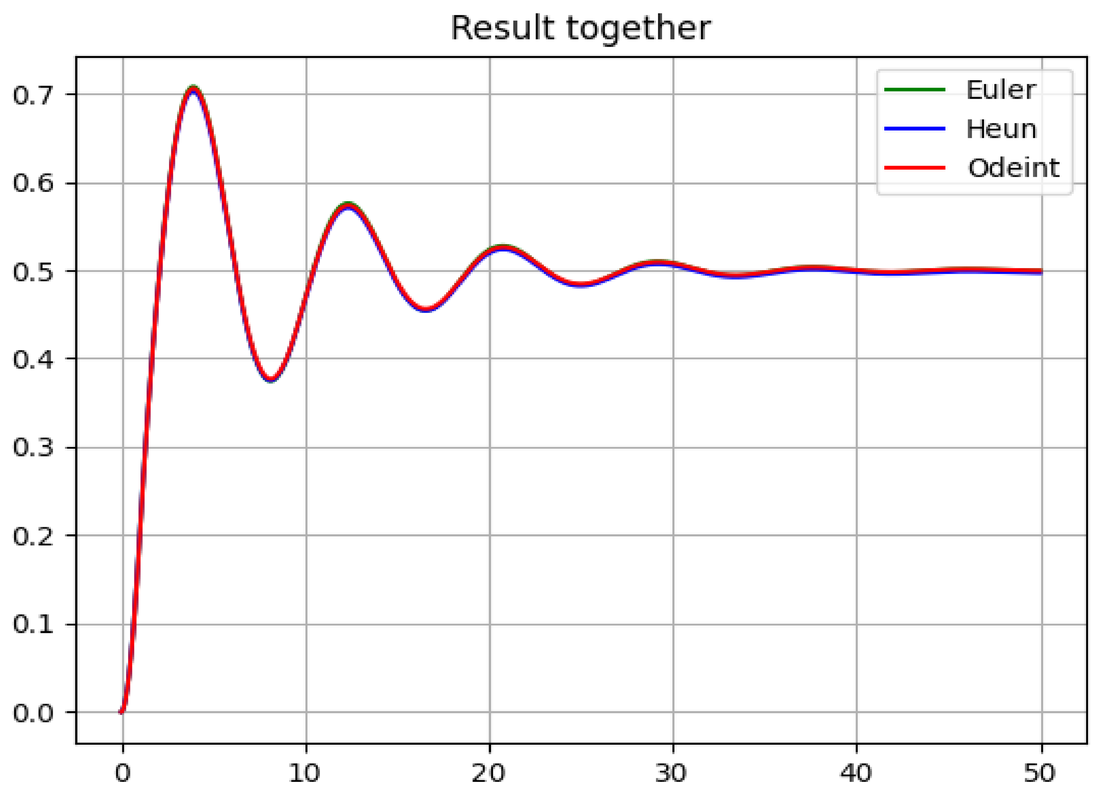
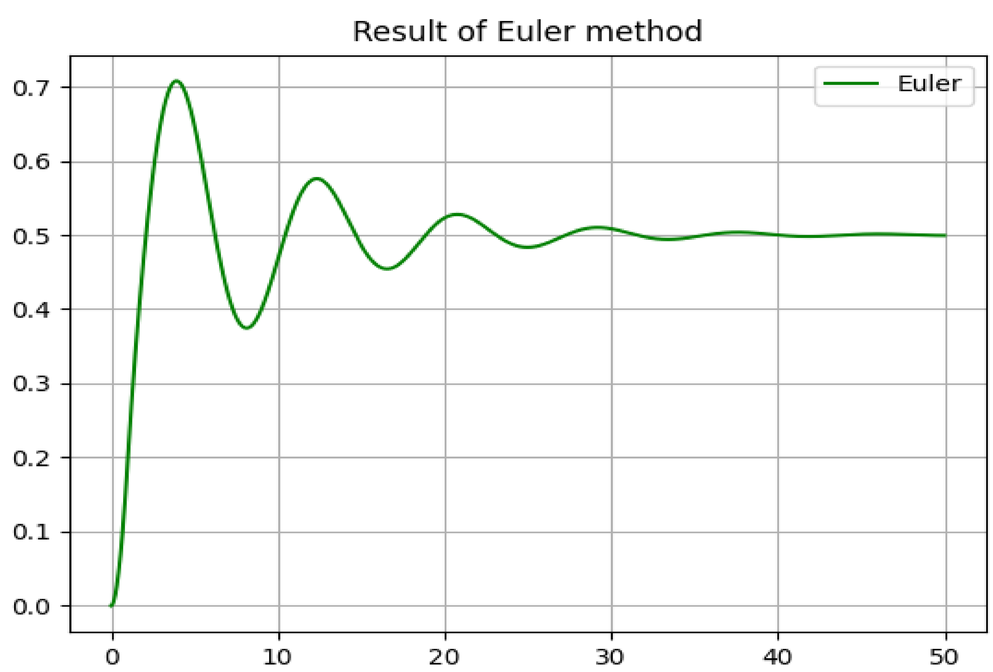
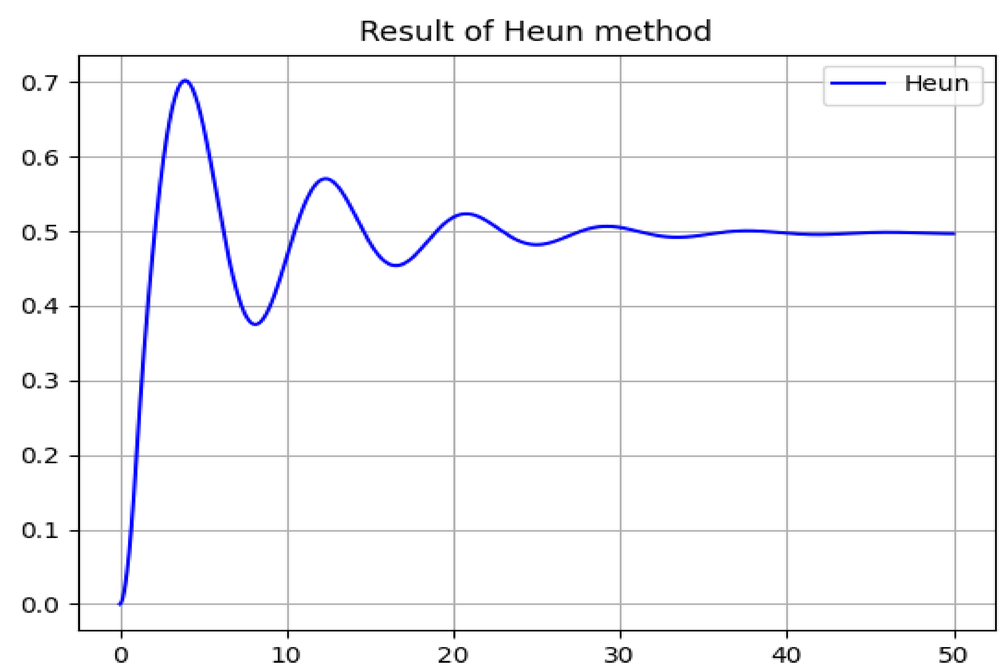
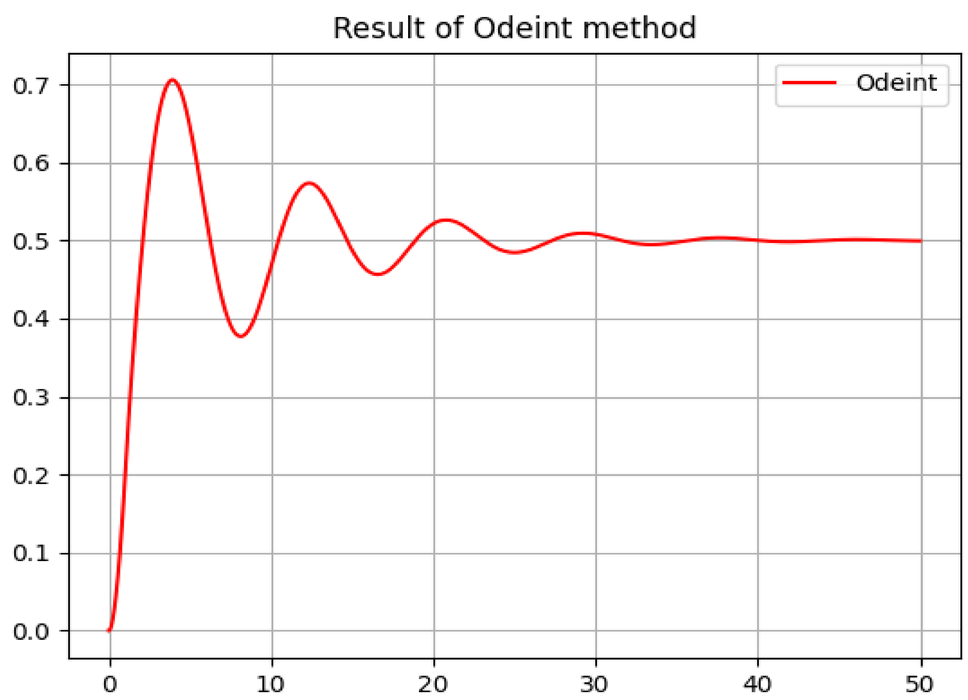
```python
# Error value calculation
E_Euler = abs(1/2-abs(Result_Euler[-1]))
E_Euler_percentage=E_Euler/0.5*100
print("The Euler method error is",E_Euler)
print("The Euler method error percentage is",E_Euler_percentage)


E_Heun = abs(1/2-abs(Result_Heun[-1]))
E_Heun_percentage=E_Heun/0.5*100
print("The Heun method error is",E_Heun)
print("The Heun method error percentage is",E_Heun_percentage)
```

问题 输出 终端 调试控制台

```
h=
0.1
PS C:\Users\PC200827\Desktop>  c:; cd 'c:\Users\PC200827\Desktop'; & 'C:\Program Files\Python37\python.exe' 'c:\Users\PC200827\.vscode\extensions\ms-python.python-2022.20.1\pythonFiles\lib\python\debugpy\adapter/../..\debugpy\launcher' '58392' '--' 'c:\Users\PC200827\Desktop\zrx_model.py'
h=
0.01
c:\Users\PC200827\Desktop\zrx_model.py:130: RuntimeWarning: invalid value encountered in double_scalars
  result=abs((y-s)/s*100)
c:\Users\PC200827\Desktop\zrx_model.py:130: RuntimeWarning: divide by zero encountered in double_scalars
  result=abs((y-s)/s*100)
The Euler method error is 0.0007886764514697386
The Euler method error percentage is 0.15773529029394773
The Heun method error is 0.003175249678478042 6
The Heun method error percentage is 0.6350499356956085
PS C:\Users\PC200827\Desktop>
```

Discuss: According to the result of error analysis, the Euler method has a smaller error. And when $t=50s$, $\Delta t=0.01$, error less than 0.1% for all method

(V): According to the figure results: the system is shown to be periodically damped and the damping amplitude is gradually reduce ~~and~~. Finally, reach the final value ($\frac{1}{2}$).

So: the system is an underdamped system