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Some extremal results on hypergraph Turán problems

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Abstract For two r-graphs $\mathcal{T}$ and $\mathcal{H}$, let $\mathrm{ex}_r(n,\mathcal{T},\mathcal{H})$ be the maximum number of copies of $\mathcal{T}$ in an n-vertex $\mathcal{H}$-free r-graph. The determination of the Turán number $\mathrm{ex}_r(n,\mathcal{T},\mathcal{H})$ has become the fundamental core problem in extremal graph theory ever since the pioneering work of Turán's theorem was published in 1941. Although we have some rich results for the simple graph case, only sporadic results have been known for the hypergraph Turán problems. In this paper, we mainly focus on the function $\mathrm{ex}_r(n,\mathcal{T},\mathcal{H})$ when $\mathcal{H}$ is one of two different hypergraph extensions of the complete bipartite graph $K_{s,t}$. The first extension is the complete bipartite r-graph $K_{s,t}^{(r)}$, which was introduced by Mubayi and Verstraëte (2004). Using the powerful random algebraic method, we show that if s is sufficiently larger than t, then $\mathrm{ex}_r(n,\mathcal{T},K_{s,t}^{(r)})=\Omega(n^{v-\frac{e}{t}})$, where $\mathcal{T}$ is an r-graph with v vertices and e edges. In particular, when $\mathcal{T}$ is an edge or some specified complete bipartite r-graph, we can determine their asymptotics. The second important extension is the complete r-partite r-graph $K_{s_1,s_2,\ldots,s_r}^{(r)}$, which has been widely studied. When $r=3$, we provide an explicit construction giving

$$\mathrm{ex}_3(n,K_{2,2,7}^{(3)})\geqslant\frac{1}{27}n^{\frac{19}{7}}+o(n^{\frac{19}{7}}).$$

Our construction is based on the norm graph, and improves the lower bound $\Omega(n^{\frac{73}{27}})$ obtained by the probabilistic method.

Keywords hypergraph Turán problem, random algebraic construction, explicit construction

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1 Introduction

In this paper, an r-graph is always an r-uniform hypergraph. Let $\mathcal{H}$ be an r-graph. An r-graph $\mathcal{G}$ is called $\mathcal{H}$-free if $\mathcal{G}$ contains no copy of $\mathcal{H}$ as a subhypergraph. Define $\mathrm{ex}_r(n,\mathcal{T},\mathcal{H})$ to be the maximum number of copies of $\mathcal{T}$ in an n-vertex $\mathcal{H}$-free r-graph. In particular, if $\mathcal{T}$ is a single edge, then $\mathrm{ex}_r(n,\mathcal{T},\mathcal{H})$ is equivalent to the classical Turán number $\mathrm{ex}_r(n,\mathcal{H})$. Moreover, when $r=2$, we usually use $\mathrm{ex}(n,T,H)$ rather than $\mathrm{ex}_2(n,T,H)$.

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The study of Turán numbers plays an important role in extremal graph theory. One of the oldest results on Turán numbers, which states that every graph on n vertices with more than $\frac{n^2}{4}$ edges contains a triangle, was proved by Mantel [30] in 1907. This result was generalized later to K_ℓ-free graphs by Turán [33]. Furthermore, the Erdős-Stone-Simonovits theorem [15,16] is an asymptotic version of a generalization of Turán's theorem, which gives the bound for the number of edges in an H-free graph, where H is a non-complete graph. Bollobás [5] described the Erdős-Stone-Simonovits theorem as the "fundamental theorem of extremal graph theory". The determination of the exact asymptotics for $\mathrm{ex}(n, H)$ is far from being solved when H is a bipartite graph. One of the important cases is the complete bipartite graph $K_{s,t}$. A well-known result of Kövari et al. [28] showed that $\mathrm{ex}(n, K_{s,t}) = O(n^{2-\frac{1}{s}})$ for any integers $t \geqslant s$. Erdős et al. [14] and Brown [6], respectively proved matching lower bounds for the cases where $s = 2$ and $s = 3$. For general values of s and t, Kollár et al. [27] first showed that $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$ when $t \geqslant s! + 1$. The bound on t was improved to $t \geqslant (s-1)! + 1$ by Alon et al. [1]. Based on some ideas in [4], Bukh [8] gave a new construction of $K_{s,t}$-free graphs which also yields a matched lower bound $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$, where t is sufficiently larger than s.

In contrast to the simple graph case, there are only a few results for the hypergraph Turán problems. For example, even the asymptotic value of $\mathrm{ex}_r(n, K_t^{(r)})$ is still unknown for any $t > r \geqslant 3$. In addition to complete r-graphs, some other cases were studied recently. Letting $K^{(r)}_{s_1,s_2,\ldots,s_r}$ be a complete r-partite r-graph with parts of sizes $s_1, s_2, \ldots, s_r$, Mubayi [31] conjectured that

$$\mathrm{ex}_r(n, K^{(r)}_{s_1,s_2,\ldots,s_r}) = \Theta(n^{r-\frac{1}{\prod_{i=1}^{r-1} s_i}}),$$

where $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_r$. In the same paper, the author verified this conjecture when $s_1 = s_2 = \cdots = s_{r-2} = 1$ and (i) $s_{r-1} = 2$, $s_r \geqslant 2$, (ii) $s_{r-1} = s_r = 3$, (iii) $s_{r-1} \geqslant 3, s_r > (s_{r-1} - 1)!$. Using the random algebraic method, Ma et al. [29] showed that if s_r is sufficiently larger than $s_1, s_2, \ldots, s_{r-1}$, then this conjecture is true.

For the function $\mathrm{ex}_r(n, \mathcal{T}, \mathcal{H})$, where $\mathcal{T}$ is not an edge, there are only sporadic results. When $r = 2$, it corresponds to the classical generalized Turán number $\mathrm{ex}(n, T, H)$, where H and T are graphs. In [2], Alon and Shikhelman studied $\mathrm{ex}(n, T, H)$ systematically and obtained many results on certain graphs such as complete graphs, complete bipartite graphs and trees. Later, Ma et al. [29] improved some of their results. They showed that for any positive integers $a < s$, $b \leqslant s$ and $t \geqslant f(a, b, s)$,

$$\mathrm{ex}(n, K_{a,b}, K_{s,t}) = \Theta(n^{a+b-\frac{ab}{s}}).$$

In the same paper, they also provided some bounds for $\mathrm{ex}_r(n, \mathcal{T}, K^{(r)}_{s_1,s_2,\ldots,s_{r-1},s_r})$ under certain conditions. For more extremal results of graphs and hypergraphs, we refer the readers to the surveys [20, 21, 26].

In 2004, Mubayi and Verstraëte [32] considered a hypergraph extension of the complete bipartite graph. In this paper, we call this extension a complete bipartite r-graph for simplicity.

Definition 1.1 (Complete bipartite r-graph)**.** Let $X_1, X_2, \ldots, X_t$ be t pairwise disjoint sets of size $r-1$, and let Y be a set of s elements, disjoint from $\bigcup_{i\in[t]} X_i$. Then $K^{(r)}_{s,t}$ denotes the complete bipartite r-graph with the vertex set $(\bigcup_{i\in[t]} X_i) \cup Y$ and the edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$.

In [32], Mubayi and Verstraëte showed some bounds for $\mathrm{ex}_r(n, K^{(r)}_{s,t})$ when $s \leqslant t$. They showed $\mathrm{ex}_3(n, K^{(3)}_{2,t}) = \Theta(n^2)$, and if $\frac{n}{3} \geqslant t \geqslant s \geqslant 3$, then $\mathrm{ex}_3(n, K^{(3)}_{s,t}) = O(n^{3-\frac{1}{s}})$. They also gave a construction which yields $\mathrm{ex}_3(n, K^{(3)}_{s,t}) = \Omega(n^{3-\frac{2}{s}})$ for $t > (s-1)! > 0$. In [17], Ergemlidze et al. determined the expression

$$g(t) = \lim_{n\to\infty} \frac{\mathrm{ex}_3(n, K^{(3)}_{2,t})}{\binom{n}{2}} = \Theta(t^{1+o(1)}) \quad \text{as } t \to \infty.$$

Note that $K^{(r)}_{s,t}$ and $K^{(r)}_{t,s}$ are nonisomorphic when $r \geqslant 3$ and $s \neq t$. Mubayi and Verstraëte [32] remarked that their results apply to both cases, so for simplicity they let $t \geqslant s$. In this paper, we focus on the other case where $s > t$ and $r \geqslant 3$.

Our first result gives a lower bound for $\mathrm{ex}_r(n, \mathcal{T}, K^{(r)}_{s,t})$ shown in the following theorem, where $\mathcal{T}$ is an arbitrary r-graph.

Theorem 1.2. *Let $r \geqslant 3$. For any positive integer t, and any r-graph $\mathcal{T}$ with v vertices and e edges, there exists some constant c which depends on r and t such that if $s \geqslant c$, then we have*

$$\mathrm{ex}_r(n, \mathcal{T}, K_{s,t}^{(r)}) = \Omega(n^{v-\frac{e}{t}}).$$

To obtain the lower bound in Theorem 1.2, our construction of $K_{s,t}^{(r)}$-free r-graphs is based on the random algebraic method which was introduced by Bukh [8]. Using the random algebraic method, Bukh and Conlon [9] verified the rational exponent conjecture which was presented in [12]. In recent years, the applications of the random algebraic method to various extremal problems have appeared in several papers [10, 19, 29].

We next show the following upper bound of the classical Turán number $\mathrm{ex}_r(n, K_{s,t}^{(r)})$ for $r \geqslant 3$ and $s \geqslant t \geqslant 2$, which is a generalization of the result of Mubayi and Verstraëte [32, Theorem 1.4].

Theorem 1.3. *Let $s \geqslant t \geqslant 2$. Then*

$$\mathrm{ex}_r(n, K_{s,t}^{(r)}) = O(n^{r-\frac{1}{t}}).$$

Let $\mathcal{T}$ in Theorem 1.2 be an edge. Combining Theorems 1.2 and 1.3, we can obtain the following asymptotic order for the Turán number of complete bipartite r-graphs.

Corollary 1.4. *Let $r \geqslant 3$. For any positive integer t, there exists some constant $c_{r,t}$ which depends on r and t such that when $s \geqslant c_{r,t}$, we have*

$$\mathrm{ex}_r(n, K_{s,t}^{(r)}) = \Theta(n^{r-\frac{1}{t}}).$$

If $\mathcal{T}$ is a complete bipartite r-graph $K_{a,b}^{(r)}$, where $a = 1$ and $b < t$, then we obtain the asymptotic bound for the generalized Turán number $\mathrm{ex}_r(n, K_{a,b}^{(r)}, K_{s,t}^{(r)})$.

Theorem 1.5. *Let $r \geqslant 3$. For any positive integer t, there exists some constant $c'_{r,t}$ which depends on r and t such that if $s \geqslant c'_{r,t}$, $a = 1$ and $b < t$, then we have*

$$\mathrm{ex}_r(n, K_{a,b}^{(r)}, K_{s,t}^{(r)}) = \Theta(n^{a+b(r-1)-\frac{ab}{t}}).$$

In the simple graph case, there were several results shown in [2, 18, 22, 23] concerning the generalized Turán problems. However, in the hypergraph case, much less is known about the Turán numbers. Corollary 1.4 determines the asymptotic order for Turán numbers of complete bipartite r-graphs $K_{s,t}^{(r)}$ when s is sufficiently larger than t. Moreover, the situation is even worse for the generalized hypergraph Turán problems, where we only know such tight results due to Ma et al. [29]. Hence Theorem 1.5 provides some new tight results on the generalized hypergraph Turán problems.

In addition to the results mentioned above, we also consider the case where $\mathcal{H}$ is a complete r-partite r-graph $K_{s_1,s_2,\dots,s_r}^{(r)}$, which can be seen as another extension of the complete bipartite graph. As we have mentioned, Mubayi [31] conjectured that

$$\mathrm{ex}_r(n, K_{s_1,s_2,\dots,s_r}^{(r)}) = \Theta(n^{r-\frac{1}{\prod_{i=1}^{r-1} s_i}}),$$

where $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_r$. In particular, when $s_1, s_2, \dots, s_r$ are relatively small, the case $\mathrm{ex}_r(n, K_{s_1,s_2,\dots,s_r}^{(r)})$ is more interesting. For example, Katz et al. [25] showed that $\mathrm{ex}_3(n, K_{2,2,2}^{(3)}) = \Omega(n^{\frac{8}{3}})$, which beats the lower bound from the probabilistic method. Next, we will show an improved lower bound for $\mathrm{ex}_3(n, K_{2,2,7}^{(3)})$ as follows.

Theorem 1.6. *The following holds*:

$$\mathrm{ex}_3(n, K_{2,2,7}^{(3)}) \geqslant \frac{1}{27} n^{\frac{19}{7}} + o(n^{\frac{19}{7}}).$$

The best previously known lower bound $\mathrm{ex}_3(n, K^{(3)}_{2,2,7}) = \Omega(n^{\frac{73}{27}})$ was obtained by probabilistic methods. Theorem 1.6 improves this by an explicit construction. Note that the upper bound part of Mubayi's conjecture was proven by Erdős [11], so the lower bound is the more interesting part.

The rest of this paper is organized as follows. In Section 2, we focus on the complete bipartite r-graphs. First, we prove Theorem 1.2 via the random algebraic construction. Then we give some general upper bounds to derive Corollary 1.4 and Theorem 1.5 in Section 3. In Section 4, we provide a new lower bound for $\mathrm{ex}_3(n, K^{(3)}_{2,2,7})$. Section 5 contains some remarks and the remaining problems on the main topics.

2 Constructions for $K^{(r)}_{s,t}$-free r-graphs, $s > t$

In this section, our goal is to prove Theorem 1.2 via the random algebraic construction.

2.1 Random algebraic construction

As far as we know, usually there are two types of constructions as follows:

(1) Randomized constructions with alternations, which are quite general and easy to apply, but usually do not give tight bounds.

(2) Algebraic constructions, which give tight bounds but appear to be somewhat magical and only work in certain special situations.

We briefly review the related work in the hypergraph Turán problem. Let $\mathcal{H}$ be an r-graph with v vertices and e edges. It was shown in [7] that

$$\mathrm{ex}_r(n, \mathcal{H}) = \Omega(n^{\frac{er-v}{e-1}}).$$

The above lower bound was obtained by a standard probabilistic argument. For example, when $\mathcal{H} = K^{(r)}_{s,t}$, the randomized construction gives a lower bound

$$\mathrm{ex}_r(n, K^{(r)}_{s,t}) = \Omega(n^{r-\frac{1}{t}-\frac{(r-1)t^2-rt+1}{st^2-t}}).$$

Recently, there is an interesting idea of Bukh [8] called the "random algebraic construction", which combines these two approaches. The idea is to construct a graph with the vertex set $V = \mathbb{F}_q^s \times \mathbb{F}_q^s$, just by choosing a random polynomial $f \in \mathbb{F}_q[x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_s]$ (within a certain family, say with bounded degree) and letting $(x, y) \in V$ be an edge if and only if $f(x, y) = 0$. The method aims to combine the advantages of both the flexibility of randomized constructions and the rigidity of algebraic constructions. Several papers [19, 24, 29] developed this method and generalized the idea to hypergraphs.

In order to apply the random algebraic method, our first task is to establish the relationship between polynomials and hypergraphs.

For given positive integers t and r with $r \geqslant 3$ and an r-graph $\mathcal{T}$ with v vertices and e edges, throughout this section, we always define $d = (r-1)t^2 - t + e + 1$. Let q be a sufficiently large prime power, and $\mathbb{F}_q$ be the finite field of order q.

Let $\boldsymbol{X}^i = (X^i_1, X^i_2, \ldots, X^i_t) \in \mathbb{F}_q^t$ for each $i \in [r]$. Consider polynomials $f \in \mathbb{F}_q[\boldsymbol{X}^1, \boldsymbol{X}^2, \ldots, \boldsymbol{X}^r]$ with rt variables over $\mathbb{F}_q$. We say such a polynomial f has degree at most td in $\boldsymbol{X}^i$, if each of its monomials has degree at most td with respect to $\boldsymbol{X}^i$, i.e., $(X^i_1)^{\alpha_1}(X^i_2)^{\alpha_2}\cdots(X^i_t)^{\alpha_t}$ satisfies $\sum_{j=1}^t \alpha_j \leqslant td$. Moreover, a polynomial f is called symmetric if exchanging $\boldsymbol{X}^i$ with $\boldsymbol{X}^j$ for every $1 \leqslant i \leqslant j \leqslant r$ does not affect the value of f. For convenience, we can view the domain of symmetric polynomials as the family $\binom{\mathbb{F}_q^t}{r}$. Then given a symmetric polynomial f, we can define an r-graph $\mathcal{G}_f$ as follows: the vertex set $V(\mathcal{G}_f)$ is a copy of $\mathbb{F}_q^t$, and every r-tuple

$$\{u^1, u^2, \ldots, u^r\} \in \binom{V}{r}$$

forms an edge of $\mathcal{G}_f$ if and only if $f(u^1, u^2, \ldots, u^r) = 0$.

Let $\mathcal{P} \subseteq \mathbb{F}_q[\boldsymbol{X}^1, \boldsymbol{X}^2, \ldots, \boldsymbol{X}^r]$ be the set of all the symmetric polynomials of degree at most td in $\boldsymbol{X}^i$ for every $1 \leqslant i \leqslant r$. Then we choose a polynomial f from $\mathcal{P}$ uniformly at random and let $\mathcal{G} = \mathcal{G}_f$ be the associated r-graph. Now we need to introduce two important lemmas from [8] and [29]. The first lemma is the key insight of the random algebraic construction, which provides very non-smooth probability distributions. While the second lemma will help us calculate the probability in certain situations.

Lemma 2.1 (See [8, Lemma 5]). *For every t and d, there exists a constant $c > 0$ such that the following holds: suppose that $f_1(Y), f_2(Y), \ldots, f_t(Y)$ are t polynomials on $\mathbb{F}_q^t$ of degree at most td, and consider the set*

$$W = \{y \in \mathbb{F}_q^t : f_1(y) = f_2(y) = \cdots = f_t(y) = 0\}.$$

Then either $|W| < c$ or $|W| \geqslant q - c\sqrt{q}$.

Lemma 2.2 (See [29, Lemma 2.2]). *Given a set $U \subseteq \binom{\mathbb{F}_q^t}{r}$, let $V \subseteq \mathbb{F}_q^t$ be the set consisting of all the points appearing as an element of an r-tuple in U. Suppose that*

$$\binom{|U|}{2} < q, \quad \binom{|V|}{2} < q \quad \text{and} \quad |U| \leqslant td.$$

If f is a random polynomial chosen from $\mathcal{P}$, then

$$\mathbb{P}[f(u^1, u^2, \ldots, u^r) = 0, \forall \{u^1, u^2, \ldots, u^r\} \in U] = q^{-|U|}.$$

With the above tools in hand, we are ready to prove Theorem 1.2.

2.2 Proof of Theorem 1.2

We choose a polynomial $f \in \mathcal{P}$ uniformly at random and let $\mathcal{G}$ be the associated r-graph $\mathcal{G}_f$. Let $n = q^t$ be the number of vertices in $\mathcal{G}$, where q is sufficiently large. Though this result only holds when q is a prime power and $n = q^t$, it is a simple matter to use Bertrand's postulate to show that the same conclusion holds for all positive integers n. We will show that on average this $\mathcal{G}$ contains many copies of $\mathcal{T}$ but very few copies of $K_{s,t}^{(r)}$, assuming s is sufficiently large. Then we can use the alteration argument to obtain a subhypergraph $\mathcal{G}'$ which is $K_{s,t}^{(r)}$-free and $\mathcal{G}'$ still contains the expected number of copies of $\mathcal{T}$.

Since $\mathcal{T}$ has v vertices and e edges, it is easy to check that

$$\binom{v}{2} < q, \quad \binom{e}{2} < q \quad \text{and} \quad e < t((r-1)t^2 - t + e + 1) = td.$$

Then by Lemma 2.2, for given v vertices, the probability that such v vertices form a copy of $\mathcal{T}$ is equal to $\frac{1}{q^e}$. Denote by X the number of copies of $\mathcal{T}$ in $\mathcal{G}$. Then the expectation

$$\mathbb{E}[X] = \Omega\left(\frac{1}{q^e}\binom{q^t}{v}\right) = \Omega(q^{tv-e}) = \Omega(n^{v-\frac{e}{t}}).$$

Let R be a fixed labeled copy of $K_{1,t}^{(r)}$, and we denote its vertices by a and u_j^i's for $1 \leqslant j \leqslant t$ and $i \in [r-1]$ such that $u_j^1, u_j^2, \ldots, u_j^{r-1}$ form t distinct $(r-1)$-tuples. Now fix any sequence of vertices w_j^i for $1 \leqslant j \leqslant t$ and $i \in [r-1]$ in $\mathcal{G}$. Let W be the family of copies of R in $\mathcal{G}$ such that w_j^i corresponds to u_j^i for all $1 \leqslant j \leqslant t$ and $i \in [r-1]$. It is difficult to estimate $|W|$ directly, and hence we consider the value of $|W|^d$. Note that $|W|^d$ counts the number of ordered collections of d copies of R from W, where these copies of R may be the same. So each member of such collections can be an element P in

$$\mathcal{K} := \{K_{1,t}^{(r)}, K_{2,t}^{(r)}, \ldots, K_{d,t}^{(r)}\}.$$

For given $P \in \mathcal{K}$, denote by $N_d(P)$ the total number of all the possible ordered collections of d copies of $R \in W$, which could appear in $\mathcal{G}$ as a copy of P. It is easy to see that $N_d(P) = O(n^{|P|-t(r-1)})$, where $|P|$ is the number of vertices in P. Since the number of edges $e(P) = t(|P| - t(r-1))$ of P is at most td and q is sufficiently large, by Lemma 2.2, the probability that a potential copy P appears in $\mathcal{G}$ is $q^{-e(P)}$. Through the above analysis, we have

$$\mathbb{E}[|W|^d] = \sum_{P \in \mathcal{K}} N_d(P) q^{-e(P)} = \sum_{P \in \mathcal{K}} O(q^{t(|P|-t(r-1))}) q^{-e(P)} = O(1).$$

Note that the set of unfixed vertices in W consists of vertices $x \in \mathbb{F}_q^t$ satisfying the system of t equations

$$f(w_j^1, w_j^2, \dots, w_j^{r-1}, x) = 0$$

for $1 \leqslant j \leqslant t$. Because $f(w_j^1, w_j^2, \dots, w_j^{r-1}, \cdot)$ has degree at most td, by Lemma 2.1, either $|W| < c$ or $|W| \geqslant q - c\sqrt{q} \geqslant \frac{q}{10}$, where the value of c depends on t and d. By Markov's inequality, we obtain

$$\mathbb{P}[|W| \geqslant c] = \mathbb{P}\left[|W| \geqslant \frac{q}{10}\right] = \mathbb{P}\left[|W|^d \geqslant \left(\frac{q}{10}\right)^d\right] \leqslant \frac{\mathbb{E}[|W|^d]}{(\frac{q}{10})^d} = \frac{O(1)}{q^d}.$$

A sequence of vertices w_j^i for $1 \leqslant j \leqslant t$ and $i \in [r-1]$ is called bad, if the corresponding set W satisfies $|W| \geqslant c$. Let B be the number of bad sequences in $\mathcal{G}$. It follows that

$$\mathbb{E}[B] \leqslant [t(r-1)]! \binom{n}{t(r-1)} \frac{O(1)}{q^d} = O(q^{(r-1)t^2-d}) = O(q^{t-e-1}).$$

Now we remove a vertex from each bad sequence to obtain a new hypergraph $\mathcal{G}'$, and clearly $\mathcal{G}'$ does not contain any bad sequences, so $\mathcal{G}'$ is $K_{s,t}^{(r)}$-free for $s \geqslant c$. Note that each vertex is in at most $O(n^{v-1})$ copies of $\mathcal{T}$ in $\mathcal{G}$, so the total number of copies of $\mathcal{T}$ removed is at most $O(n^{v-1}) \cdot B$. Hence the expected number of the remaining copies of $\mathcal{T}$ in $\mathcal{G}'$ is at least

$$\Omega(n^{v-\frac{e}{t}}) - \mathbb{E}[B] \cdot O(n^{v-1}) = \Omega(n^{v-\frac{e}{t}}).$$

It is easy to check that the expected number of remaining vertices is $n - O(n^{1-\frac{e+1}{t}}) = n - o(n)$.

Therefore, for any $s \geqslant c$, there exists a $K_{s,t}^{(r)}$-free r-graph with at most n vertices and $\Omega(n^{v-\frac{e}{t}})$ copies of $\mathcal{T}$. This completes the proof of Theorem 1.2.

3 Upper bound for $K_{s,t}^{(r)}$-free r-graphs

The result in Theorem 1.2 is intended to motivate our investigation of the matched upper bounds for some r-graphs $\mathcal{T}$. In this section, we will present matched upper bounds for $\mathrm{ex}_r(n, K_{s,t}^{(r)})$ and $\mathrm{ex}_r(n, K_{a,b}^{(r)}, K_{s,t}^{(r)})$ under the certain conditions.

3.1 Upper bound for $\mathrm{ex}_r(n, K_{s,t}^{(r)})$

This upper bound for $\mathrm{ex}_r(n, K_{s,t}^{(r)})$ can be seen as a generalization of the result of Mubayi and Verstraëte [32, Theorem 1.4]. Before we prove Theorem 1.3, we need the following useful lemma of Erdős and Kleitman [13].

Lemma 3.1 (See [13]). *Let $\mathcal{G}$ be an r-graph on rn vertices. Then $\mathcal{G}$ contains an r-partite subhypergraph $\mathcal{G}'$ with all the parts of size n, and $e(\mathcal{G}') \geqslant \frac{r!}{r^r} e(\mathcal{G})$.*

We write $z(n, K_{s,t}^{(r)})$ for the maximum number of edges in an r-partite $K_{s,t}^{(r)}$-free r-graph in which all the parts have size n. By Lemma 3.1, it suffices to prove that $z(n, K_{s,t}^{(r)}) = O(n^{r-\frac{1}{t}})$.

Proof of Theorem 1.3. Let $A_1, A_2, \dots, A_{r-1}, B$ be the r parts of size n of an r-partite $K_{s,t}^{(r)}$-free r-graph $\mathcal{H}$. Suppose that $\mathcal{H}$ has more than $c'_{s,t} n^{r-\frac{1}{t}}$ edges, where $c'_{s,t}$ is defined as the smallest integer for which every bipartite graph with the parts X and Y of the size n having more than $c'_{s,t} n^{2-\frac{1}{t}}$ edges contains a $K_{s,t}$ with t vertices in X and s vertices in Y. Clearly, $c'_{s,t}$ is independent of n by the Kövári-Sós-Turán bound [28]. Consider the complete $(r-1)$-partite $(r-1)$-uniform hypergraph $K_{n,n,\dots,n}^{(r-1)}$ on the vertex set $A_1 \times A_2 \times \cdots \times A_{r-1}$. It was shown in [3] that $K_{n,n,\dots,n}^{(r-1)}$ has a perfect matching decomposition, and hence we can partition the $(r-1)$-tuples of $A_1 \times A_2 \times \cdots \times A_{r-1}$ into $\frac{n^{r-1}}{n} = n^{r-2}$ matchings $M_1, M_2, \dots, M_{n^{r-2}}$. Let $\mathcal{H}_i$ be the subhypergraph of $\mathcal{H}$ induced by those edges that contain some $(r-1)$-tuples of M_i. By the pigeonhole principle, there exists some i such that $\mathcal{H}_i$ contains more than $c'_{s,t} n^{2-\frac{1}{t}}$ edges. Next, we construct an auxiliary bipartite graph G_i on the vertex set $A_{r-1} \cup B$ with the edge set

$$\{(a_{r-1}, b) : \exists\, (a_1, a_2, \dots, a_{r-2}),\, a_i \in A_i,\, (a_1, a_2, \dots, a_{r-1}, b) \in E(\mathcal{H}_i)\}.$$

Then by the choice of $c'_{s,t}$, we conclude that G_i contains a copy of $K_{s,t}$ with s vertices in B and t vertices in A_{r-1}, which extends via M_i to a $K_{s,t}^{(r)}$ in $\mathcal{H}$. The proof is finished. $\square$

3.2 Upper bound for $\mathrm{ex}_r(n, K_{a,b}^{(r)}, K_{s,t}^{(r)})$

We now complete the proof of Theorem 1.5 by proving the following lemma. The main idea of this proof is based on the ideas in [2].

Lemma 3.2. *If $b < t$, then we have*

$$\mathrm{ex}_r(n, K_{1,b}^{(r)}, K_{s,t}^{(r)}) = O(n^{b(r-1)-\frac{b}{t}+1}).$$

Proof. Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-graph with n vertices. For each vertex $v \in \mathcal{G}$, let $N(v)$ be the following set:

$$N(v) = \{(b_1, b_2, \ldots, b_{r-1}) \mid b_i \in V(\mathcal{G}),\ (v, b_1, b_2, \ldots, b_{r-1}) \in E(\mathcal{G})\}.$$

It is easy to see the number of $K_{1,b}^{(r)}$ in $\mathcal{G}$ is at most

$$\begin{aligned}
\sum_{v\in V(\mathcal{G})} \binom{|N(v)|}{b} &\leqslant \frac{1}{b!} \sum_{v\in V(\mathcal{G})} |N(v)|^b \\
&\leqslant \frac{1}{b!} n^{1-\frac{b}{t}} \left(\sum_{v\in V(\mathcal{G})} |N(v)|^t \right)^{\frac{b}{t}} \\
&\leqslant (1+o(1)) \frac{n^{1-\frac{b}{t}}}{b!} (((s-1)(t(r-1))! + (t(r-1)-1)!) n^{t(r-1)})^{\frac{b}{t}} \\
&= O(n^{b(r-1)-\frac{b}{t}+1}).
\end{aligned}$$

We need some basic facts in the above estimation. The first is that for any $0 < p \leqslant q$,

$$\sum_{i=1}^{m} x_i^p \leqslant m^{1-\frac{p}{q}} \left(\sum_{i=1}^{m} x_i^q \right)^{\frac{p}{q}}.$$

Moreover, we estimate $\sum_A \binom{|N(v)|}{t}$ via double counting, i.e., we take advantage of the formulation

$$\sum_A \binom{|N(v)|}{t} = \sum_{T\in\mathcal{T}_1} |N(T)| + \sum_{T\in\mathcal{T}_2} |N(T)|,$$

where $\mathcal{T}_1$ consists of all the t vertex disjoint $(r-1)$-tuples and $\mathcal{T}_2$ consists of the other t $(r-1)$-tuples. Moreover, $N(T)$ consists of the vertices which are adjacent to every $(r-1)$-tuple in T. Consider the first part $\sum_{T\in\mathcal{T}_1} |N(T)|$ for every $T \in \mathcal{T}_1$, if there are more than $(s-1)$ vertices in $N(T)$, then we can obtain a copy of $K_{s,t}^{(r)}$, which leads to a contradiction. For the second part $\sum_{T\in\mathcal{T}_2} |N(T)|$, note that $|N(T)| < n$ and the number of vertices in T is less than $t(r-1)$, and thus we have

$$\sum_{T\in\mathcal{T}_2} |N(T)| < (1+o(1))(t(r-1)-1)! n^{t(r-1)}.$$

The proof is finished. $\square$

4 An improved lower bound for $\mathrm{ex}_3(n, K_{2,2,7}^{(3)})$

Let $\mathcal{H}$ be an r-graph with v vertices and $e > 0$ edges. An application of the probabilistic method shows that $\mathrm{ex}(n, \mathcal{H}) > cn^{\alpha}$, where $\alpha = r - \frac{v-r}{e-1}$ and c is independent of n [7]. This yields

$$\mathrm{ex}(n, K_{2,2,7}^{(3)}) = \Omega(n^{\frac{73}{27}}).$$

In this section, we improve the exponent $\frac{73}{27}$ to $\frac{19}{7}$ by proving Theorem 1.6. Our construction is a variation of norm hypergraphs, and thus the construction is explicit.

Let $\mathbb{F}_q$ be a finite field, and $\mathbb{F}_{q^r}$ be a finite field extension of $\mathbb{F}_q$. The *norm* $\mathrm{Norm}_r(x)$ of $x \in \mathbb{F}_{q^r}$ over $\mathbb{F}_q$ is defined by

$$\mathrm{Norm}_r(x) = \prod_{i=0}^{r-1} x^{q^i}.$$

Then $\mathrm{Norm}_r(x) \in \mathbb{F}_q$. The following result can be found in [27].

Lemma 4.1 (See [27])**.** *If $(D_1, d_1), \ldots, (D_s, d_s)$ are distinct elements of $\mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$, then the system of s equations*

$$\mathrm{Norm}_{s-1}(D_i + X) = d_i x, \quad 1 \leqslant i \leqslant s$$

has at most $(s-1)!$ solutions $(X, x) \in \mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$.

We also need the following lemma.

Lemma 4.2. *Let m be a sufficiently large integer, $k = \lfloor \sqrt{m} \rfloor - 1$ and $\ell = \lfloor \frac{k}{2} \rfloor$. Let*

$$\begin{aligned} S_1 &= \{0, 1, 2, \ldots, \ell - 1\}, \\ S_2 &= \{0, k, 2k, \ldots, (\ell - 1)k\}, \\ S_3 &= \{0, k+1, 2(k+1), \ldots, (\ell - 1)(k+1)\} \end{aligned}$$

be additive sets in $\mathbb{Z}_m$. Then $|S_i + S_j| = |S_i||S_j| = \ell^2$ for $1 \leqslant i \neq j \leqslant 3$.

Proof. It is easy to see that $|S_1 + S_2| = \ell^2$.

For any $x \in \mathbb{Z}_m$, if $x = i + j(k+1)$, where $0 \leqslant i, j \leqslant \ell - 1$, then $x = (i+j) + jk$, and there is at most one solution for i and j. Hence $|S_1 + S_3| = \ell^2$. Similarly, we have $|S_2 + S_3| = \ell^2$. □

Proof of Theorem 1.6. Let q be an odd prime power, $k = \lfloor \sqrt{q-1} \rfloor - 1$ and $\ell = \lfloor \frac{k}{2} \rfloor$. Let

$$\begin{aligned} S_1 &= \{0, 1, 2, \ldots, \ell - 1\}, \\ S_2 &= \{0, k, 2k, \ldots, (\ell - 1)k\}, \\ S_3 &= \{0, k+1, 2(k+1), \ldots, (\ell - 1)(k+1)\} \end{aligned}$$

be additive sets in $\mathbb{Z}_m$. By Lemma 4.2, $|S_i + S_j| = |S_i||S_j| = \ell^2$ for $1 \leqslant i \neq j \leqslant 3$.

Let g be a primitive element of $\mathbb{F}_q$, and $B_i = \{g^j : j \in S_i\}$ for $1 \leqslant i \leqslant 3$. Let $\mathcal{G}$ be a 3-graph with parts $A_i = \mathbb{F}_{q^3} \times B_i$, $i = 1, 2, 3$. The vertices $(D_i, d_i) \in A_i$, $i = 1, 2, 3$ form an edge if

$$\mathrm{Norm}_3(D_1 + D_2 + D_3) = d_1 d_2 d_3.$$

Clearly, $\mathcal{G}$ has

$$n := 3\ell q^3 = \frac{3}{2} q^{\frac{7}{2}} + o(q^{\frac{7}{2}})$$

vertices, and it is easy to count that there are

$$\frac{(\ell q^3)^3}{q} \geqslant \frac{1}{27} n^{\frac{19}{7}} + o(n^{\frac{19}{7}})$$

edges.

We claim that $\mathcal{G}$ is $K_{2,2,7}^{(3)}$-free. Assume to the contrary, there exists a copy of $K_{2,2,7}^{(3)}$ in $\mathcal{G}$. Without loss of generality, suppose that $(D_i, d_i) \in A_1$, $(E_j, e_j) \in A_2$, $(X_k, x_k) \in A_3$, $i, j \in [2]$ and $k \in [7]$ form a copy of $K_{2,2,7}^{(3)}$. Let $T_{ij} = D_i + E_j$ and $t_{ij} = d_i e_j$. Then we have $\mathrm{Norm}_3(T_{ij} + X_k) = t_{ij} x_k$ for $i, j \in [2]$ and $k \in [7]$. This also implies that the system of the equations $\mathrm{Norm}_3(T_{ij} + X) = t_{ij} x$ for $i, j \in [2]$ has at least 7 solutions for (X, x).

By the definition of B_i, we have $|\{t_{ij} : i, j \in [2]\}| = 4$. Hence (T_{ij}, t_{ij}), $i, j \in [2]$ are distinct elements. By Lemma 4.1, there are at most 6 solutions for such a system of equations, which leads to a contradiction. Thus, $\mathcal{G}$ is $K_{2,2,7}^{(3)}$-free. □

Remark 4.3. We believe that the exponent $\frac{19}{7}$ can be improved, and hence we made no attempt to optimize the leading coefficient $\frac{1}{27}$ in the proof above.

5 Concluding remarks

In this paper, we have studied two extensions of hypergraph Turán problems of complete bipartite graphs. The first object is the complete bipartite r-graph. Mubayi and Verstraëte [32] introduced this structure and gave some general bounds and constructions for $\mathrm{ex}_r(n, K_{s,t}^{(r)})$. They also presented a conjecture for 3-graphs. Here, we generalize their conjecture for $r \geqslant 3$.

Conjecture 5.1. Let s and t be integers with $2 \leqslant s \leqslant t$. Then

$$\mathrm{ex}_r(n, K_{s,t}^{(r)}) = \Theta(n^{r-\frac{2}{s}}).$$

Though we still cannot verify this conjecture, there is some evidence that supports this conjecture. For example, Ergemlidze et al. [17] showed that

$$\mathrm{ex}_4(n, K_{2,t}^{(4)}) \geqslant (1 + o(1))\frac{t-1}{8}n^3.$$

Moreover, in [32], Mubayi and Verstraëte remarked that their results can apply for both $t \geqslant s$ and $s > t$, and hence for simplicity they let $t \geqslant s$. However, when s is sufficiently larger than t, to our surprise, we obtain the matched lower bounds for $\mathrm{ex}_r(n, K_{s,t}^{(r)}) = \Omega(n^{r-\frac{1}{t}})$ via the random algebraic construction.

We also obtain the lower bounds for the generalized Turán number $\mathrm{ex}_r(n, \mathcal{T}, K_{s,t}^{(r)})$, and we show the matched upper bounds when $\mathcal{T}$ is an edge or a complete bipartite r-graph $K_{1,b}^{(r)}$ with $b < t$. It will be interesting to find more examples reaching the lower bounds.

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